Introductory Note on the draft paper "Generalized affine signal analysis with time-delay thresholds" by J. W. Dash, A. Grossmann, and T. Paul

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Thierry Paul and Jan W. Dash

Personal recollections of Alex Alex was a delight and a genius. His family and ours were close friends. He spoke many languages; since my office door at the CNRS Marseille-Luminy was across from his, I could hear him responding to telephone calls in callers' native languages. Alex mentored me on some of the mathematics in this paper. He also took time to learn to play the clarinet and I was privileged to return the favor and be his teacher! After we left France, Alex and DK visited us regularly in the US and we visited them in France. He once visited me at Bell Labs and gave a seminar. Alex was also interested in the quantitative finance and risk management issues in which I was later involved, and insightfully helped me with a difficult point involving my correlation-based expansion of n-dimensional Gaussian integrals.

Alex is sorely missed.

JWD

Je vous parle d'un temps que les moins de vingt ans ne peuvent pas connaitre...
La Bohême, J. Plante-C. Aznavour

The paper, an unpublished draft, is dated May 1986, a time when LaTex didn't exist. Thirty-six years later, we only have a hard copy with annotations by Dash probably written at the same period than the draft itself. The "source file" was not found[1].

The subject of the paper is to place some previously introduced useful functions - most notably containing a time-delay parameter - within the context of wavelet theory.

In this introductory note we discuss historical background for the paper and comment on the content for perspective. The text in italics and numbered references are from the original.

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[1] We don't remember very well why the draft was never completed and sent to a journal. Jan found it when he was moving in 2021. In a letter to Alex and Thierry dated at the time of the 1986 draft, Jan suggested after completion "not sending this to a mathematical journal but rather a journal like the European Signal Processing Journal or the IEEE version of it."

1 Wavelets in the mid-1980s

The paper was written in May 1986, after discussions between the three authors in 1985-1986 and earlier. These years were a time of a full explosion of wavelets theory.

1.1 The group-theoretic view of wavelets

Attention focused at that time on the group-theoretic view of wavelets - i.e. orbits of functions by a representation of a (non-unimodular) group. The interest in this group theoretic aspect was a bit unjustly diminished by the suspicion that the approach might not help solve practical problems. The coup de grace was given by the discovery of orthogonal bases of wavelets, especially those having compact support. Nevertheless, the group theoretic approach for wavelets provided a precious tool for a multidimensional extension. The group theoretic approach is also applicable in the case discussed in the paper, since an orthogonal basis for these functions has yet to be found.

1.2 Overcompletness can be a virtue

The discretization of a continuous wavelet transform was supplanted by the use of an orthonormal basis of (compactly supported) wavelets. Orthogonality has the advantage of providing exact inversion formulae. This procedure however has the disadvantage of being rigid, compared to the freedom provided by discretizing the grid and specifying other parameters for an overcomplete set of functions[2]. This specification can be done using least squares or other techniques. Only a few of these more complicated functions have to be employed to get a description of complex phenomena, as opposed to having to use a large number of standard wavelets. This is a main point of the approach.

1.3 Comments on the first sentence: "A class of functions recently introduced by Dash and Paul . . . "

The functions introduced by Dash contain multiple parameters; as mentioned in the text they were useful for some applications in particle physics (Ref. 8), signal analysis (Ref. 7), and - as proposed much later - finance[3]. The continuous wavelet introduced in the thesis of Paul (Ref. 9) consists in taking the Fourier transform of an exponential function with support on the positive half line, and dilating and translating it as wavelet philosophy suggests[4]. The overcomplete set of functions used in the draft are precisely these exponential functions, multiplied by a Heaviside function, and then dilated and translated in time (first formula of the paper).

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[2] See Chapter 52, Dash, Jan W. Quantitative Finance and Risk Management, a Physicist's Approach, 2nd Ed. (ISBN 9789814571234); copyright 2016 by World Scientific Publishing Co. See $https://www.worldscientific.com/doi/abs/10.1142/9789814571241_0052$ This chapter had a "plea" to mathematicians to find a complete subset of these overcomplete functions.

[3] The application in particle physics (ref. 8) was the description of some data in high-energy diffractive scattering. Here the functions are pieces of the imaginary part of the scattering amplitude, time is the logarithm of the beam energy, and the time thresholds correspond to the successive production in energy of particle with different flavors (strangeness, charm...), with successively increasing of mass scales.

The application in signal analysis (ref. 7) was the description of the measured response of some equipment in spatially separated locations to input transcient electromagnetic fields. The time-threshold behaviour occurred because different parts of the equipment reacted to the electromagnetic fields at different times.

The proposed application in finance (ref. preceding footnote) is that these functions could form part of the macro component of financial markets operating over long time scales, and possibly also on shorter micro time scales for trading.

[4] One of the main interest in TP's thesis was to provide a Hilbert space representation of quantum mechanics where the Schrödinger equation for the hydrogen atom was explicitly solvable, in the sense that the eigenvalue problem becomes a first order differential equation. Equivalent to the Bargmann space for the harmonic oscillator, this setting qualitatively "explained" why the Bohr quantization rules are exact for the hydrogen atom. The "Cauchy" wavelet was also used extensively some years later for detecting singularities of functions.

1.4 Phase

As we just mentioned, this is the Fourier transform of our functions which is translated. This generates a phase in our wavelets themselves, a situation totally different from the usual wavelets. The study of the phase was another feature of interest in wavelets at that time. Note that the phase has regained interest in the last few years. One can look at the contribution to this memorial volume by S. Mallat et al. The functions can also have overall phase factors, which is useful in practice (Ref. 7).

2 Specific comments on the text

The action of the Weyl-Heisenberg group is direct, not acting on the Fourier transform. All the machinery of orthogonality relations still works in this case, as explained in Section II of the paper: we are dealing with the same group but through another representation.

A few proposed uses of these functions discussed at the time (e.g. speech recognition) remain to be explored. We have always felt that these functions could be generally useful with many applications - in addition to the applications cited above.

The use of least squares to determine parameters in the 1980s could be upgraded in principle with machine learning.

A technical appendix was planned for the paper but was never written.

3 Le Baron de Prony's overcomplete set of "Wavelets à la Neanderthal"

Gaspard de Prony[5] invented a method that constitutes to our knowledge the first example (in 1795!) of a signal expansion in terms of an overcomplete set of continuous functions along with a prescription for the expansion using a discrete number of constraints from a set of measurements in time. Prony's method has practical applications. However it assumes one pre-specified initial time, and a large number of terms - numerically unstable in practice - have to be used to simulate a time-delay threshold (this observation actually was the origin of the work in Ref. 7). De Prony functions are a special case of our functions.

4 Alex

This paper seems to both of us representative of the spirit of Alex "at work", as it touches a large landscape of the sciences, history, and general knowledge. This paper shows also how Alex was interested in applications of mathematics in many areas. For example Alex wrote a paper on numerical simulations[6] where the role of geometry was pointed out. He also worked in the biological sciences incorporating combinatorics[7].... Alex was interested and curious about optimization issues, and always kept attached to keeping the maximum of freedom in wavelets analysis. Although he was present at the very early beginning of the "discrete" story of wavelets[8], TP remembers very well how he was always more interested in discretizing the continuous than fixing a priori discrete sets of parameters.

5 Addendum/Corrigendum

- p. 6, l. 2-3: Replace Refs. [xxx] by: Refs. [5, 6, 9].
- The pagination "7" is correct.
- p. 7, l. 13: A point is missing after $\psi(t)$.

[5] Gaspard de Prony (1755-1839) was an influential and very productive French engineer. He lived during the period in which many scientific institutions were born in France. He was Professor at the Ecole Polytechnique. See $https://en.wikipeda.org/wiki/Gaspard_de_Prony$. A description of de Prony's method is here: $https://en.wikipedia.org/wiki/Prony\%27_s method$. de Prony was quickly re-baptized "Le Baron" by Alex, and we all were amused by quoting a paper from 1795; see Ref. 1, which can be found through the link $http://users.polytech.unice.fr/leroux/PRONY.pdf$.

[6] A. Grossmann, R. Coquereaux and B. Lautrup, "Iterative Method for Calculation of the Weierstrass Elliptic Function", IMA Journal of Numerical Analysis 10(1), 119-128, 1990.

[7] See, e.g. G. Didier, E. Corel, I. Laprevotte, A. Grossmann, and C. Landès-Devauchelle. Variable length local decoding and alignment-free sequence comparison. Theoretical Computer Science, **462**:1– 11, 2012

[8] A. Grossmann, I. Daubechies and Y. Meyer, "Painless nonorthogonal expansions", J. Math. Phys. 27, 1271,1986

- p. 7: The last three lines hand written in red read the following:
The set of overlap integrals forms another Hilbert space H, this time of functions $\hat{f}$ *on the group G, with scalar product* $(\hat{f}_1, \hat{f}_2) = \int d\mu(g) \hat{f}_1^*(g) \hat{f}_2^*(g)$.
- p. 9, l. 7: The scalar product here is the one we just mentioned: $(e_g, \hat{h}) = \int d\mu(g')[e_g(g')]^* \hat{h}(g'))$.
- p. 9, first formula: f_0 is the function f defined p. 3.
- p. 9, l. 3: Let us recall that the orthogonality relations for square integrable representations of non-unimodular groups read

$$\int d\mu(g) |U(g) f_1\rangle\langle U(g) f_2| = Cst \times \text{Identity}$$

are valid also when $f_1 \neq f_2$ (as soon as they obey the "admissibility condition", see [Ref. 6] in the draft (precised below in the last item) . Therefore, the reproducing kernel K built up with different values of t_0 still "reproduces".
- p. 9, last formula: Let us remark that we "find" this formula as a consequence of the decomposition $\mathrm{F}_{\mathrm{INT}}^{(\mathrm{N})}(g) = \sum_{i=1}^{N} b_i e_{g_i}(g)$, which is of course not valid in general but which is the ansatz we take. Letting this ansatz satisfy the constraints gives immediately that $b_i = (K^{-1}\zeta)_i$ so that, as written, $\mathrm{F}_{\mathrm{INT}}^{(\mathrm{N})}(g) = \sum_{i=1}^{N} K(g, g_i)(K^{-1}\zeta)_i$. Writing now $\zeta = KK^{-1}\zeta$ and using the Cramer's formula, we find exactly the last formula of page 9, after expanding the numerator determinant by the first row and pushing (in the jth co-factor) the first column to the jth position.
- p.11, l. 5: $\langle ij\rangle$ stands for $(g^{(g_i)}, g^{(g_j)})_{L^2(\mathbf{R})}$ and likewise $\langle j\psi\rangle = (g^{(g_j)}, \psi)_{L^2(\mathbf{R})}$.
- p. 14, l. 11: Replace xx by "4th previous equation".
- The missing information, denoted by xxx, in the references are the following.
- Ref (6): The second reference in Ref (6) is A. Grossmann, J. Morlet, and T. Paul, "Transforms associated to square integral group representations" I. General results, J. Math. Phys. , vol 26, 2473-2479, 1985, and II. Examples, Ann. Inst. Henri Poincaré, Phys. Théor., vol 45, 293, 1986. Another related reference is A. Grossmann and T. Paul, "Wave functions on subgroups of the group of affine canonical transformations" in "Resonances, models and phenomena". Lect. Notes in Phys, 211, Springer-Verlag, 1984.
- Ref(8): Another reference is J.W. Dash and S.T. Jones, "Flavoring, RFT and In^{2}s Physics at the SPS Collider", Physics Letters B157, p. 229 (1985).
- Ref(9): T. Paul, Thèse d'état, Université d'Aix-Marseille, 1985.
- Ref(13): B. Roy Frieden, "VIII Evaluation, Design and Extrapolation Methods for Optical Signals, Based on Use of the Prolate Functions", Progress in Optics **9** Ch. VIII, 311-407, 1971.
- Ref(14): E.S. Abers, B.W. Lee, "Gauge theories", Physics Reports **9**, Issue 1, 1-141 1973.