Dynamics of Generalized Conic Trajectories

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Abstract: a class of generalized conic curves is obtained generalizing a property of an initial conic, linked to its radius of curvature. The class possesses a large variety of well-known curves and accepts as a limiting case the logarithmic spiral curve. The determination of corresponding law of central force allows to link in the same relation the restoring, inverse square and inverse cubic forces. A geometric transformation inside the class is highlighted as a method to construct the tangents. Results are linked with contemporary preoccupations of mathematical Physics and celestial mechanics.

Keywords: Central force; Celestial Mechanics; Generalized conic trajectories; Power law of potentials;

1. **Introduction**

It is well known that the study of plane curves and trajectories described by the following general equation

$$r(\theta) = \frac{a}{(1 + e\cos k\theta)^{\frac{1}{m}}} \qquad (1)$$

Where
a is a length
e is a positive number
m and k are unidimensional numbers with $m \neq 0$
θ is the polar angle

is an old and important topic of classical physics, and particularly of celestial mechanics (see for example [1]). Indeed, depending on the choice of constants, this equation can describe a wide variety of fundamental curves. Let us quote for example, of course the ellipses but also famous curves such as the lemniscates, cardioids, etc.. Moreover the study of the cases corresponding to $k = 1,\ m = 1$ (ellipse described from one of its foci, keplerian orbit) and to $k = 2, m = 2$ (ellipse described from its center, two-dimensional harmonic oscillator) as well as the question of their reciprocal transformation, naturally remains a major concern of physicists, from the founding fathers [2] to the present day (see for example [3]). Finally, let us quote, for example, the cases corresponding to $m = 1$, $k \neq 1$ which represents the trajectories studied in the famous "revolving theorem" [4].

However the study of the curves resulting from this equation presents another interest, this time in one of the fields of geometry. It is indeed possible to extract from it classes of generalized conic curves, as it is for example presented in reference [5]. This approach, as the geometrical study of these classes of curve, represent another important topic of mathematics, with historically the works, among others, of famous physicists and mathematicians such as Descartes, Maxwell, etc.

The paper can be seen as a contribution to both topics, and presents several original results. One of them concerns the establishment of a class of generalized conic. The common geometric property of these curves is related to the direction of their radius of curvature, as we present it in the first part of the article. The class of curves accepts several fundamental lines, such cardioïds, Lemniscates, and as a limiting case, the logarithmic spirals.

A second interest of the paper concerns the geometrical study of this class of curve. In particular, several graphic methods (which often seem original) to draw the radii of curvatures and the tangents to the curves are detailed. Moreover, relations of geometric transformations and symetries are highlighted, with the establishement of relations of inversion.

In the next part of the paper we adopt this time a physical point of view, and no longer simply geometric. We indeed consider these curves as the possible trajectories of a point particle subjected to a central force field. This field of force is determined, using a classical approach, known as Sciacci's theorem. It is shown that these forces can be described by a single and simple formula, which connects three fundamental potentials of Physics. Note this result is probably the most interesting and important of the article.

Finally, in the last part of the article, we discuss our results and present possible contemporary applications. Indeed, we suggest several of them, in particular in celestial mechanics, with the presentation of a precession phenomena due to weak modifications of the central force field. We also link our work with several recent publications, in the field of astrodynamics.

As can be seen, this work is therefore a contribution to classical mechanics and applied mathematics, which sheds original light on some of its historical concerns. For this reason, the reader will regularly recognize great authors of classical physics and will find references, which we will try to highlight as best as possible.

2 – Geometric

As we presented it, the class of curves is defined by a generalization of a property of the classical conic curve. This property concerns the intersection I of its radius of curvature with its semi major axis (Figure 1).

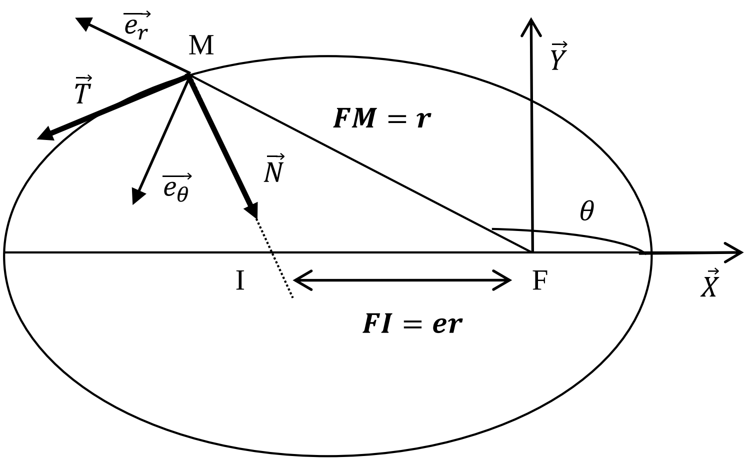


Figure 1. A property of Conic Trajectories

In fact, this relation is simply given with

$$FI = er \qquad (2)$$

This property is naturally already known, and a geometric demonstration can be found in reference [6]. However, we present an analytical proof, to illustrate further reasoning.

2.1 Proof

Consider a moving point M which describes a plane arbitrary curve around the origin of a polar system of coordinate $(F;\overrightarrow{e_r};\overrightarrow{e_\theta})$. The angle θ is definite by

$$(\vec{X};\overrightarrow{e_r}) = \theta$$

Where $(F;\vec{X};\vec{Y})$ is a second system of coordinate, Cartesian and fix this time.
coordinate are thus given by $\overrightarrow{FM} = r\overrightarrow{e_r}$

Using the classical change of variable we obtain

$$\begin{Bmatrix} u = \dfrac{1}{r} \\ u' = \dfrac{d}{d\theta}u \end{Bmatrix}$$

We note that a vector $\vec{T}$ tangential to the curve can be given with

$$\vec{T} = -u'\overrightarrow{e_R} + u\overrightarrow{e_\theta}$$

To describe the curve we write the following condition: Normal vector $\vec{N}$ is directed toward a point I located on $F\vec{X}$ axis such

$$\overrightarrow{FI} = -\frac{e}{u}\vec{X}$$

In the polar system of coordinate, we obtain now

$$\overrightarrow{FI} = -\frac{e}{u}\vec{X} = -\frac{e}{u}(\cos\theta\,\overrightarrow{e_R} - \sin\theta\,\overrightarrow{e_\theta})$$

And, consequently

$$\overrightarrow{IM} = \overrightarrow{IF} + \overrightarrow{FM} = \frac{1}{u}[1 + e\,cos\,\theta)\,\overrightarrow{e_R} - e\,sin\,\theta\,\overrightarrow{e_\theta}]$$

We write thus the scalar product

$$\overrightarrow{IM}.\vec{T} = 0$$

And we obtain the differential equation

$$u'(1 + e\,cos\,\theta) + ue\,sin\,\theta = 0 \qquad (3)$$

Whose solving is

$$u = C_1(1 + ecos\theta)$$

Trajectory is well a conic, whose point F is one of the foci and e its eccentricity.

2.2 Generalized conic curves

To obtain classes of curves we make a precession on the initial ellipse of an angle α around one of its foci. We generalize previously property considering this time that point I is located on a rotating vector $\vec{x}$ (Figure 2).

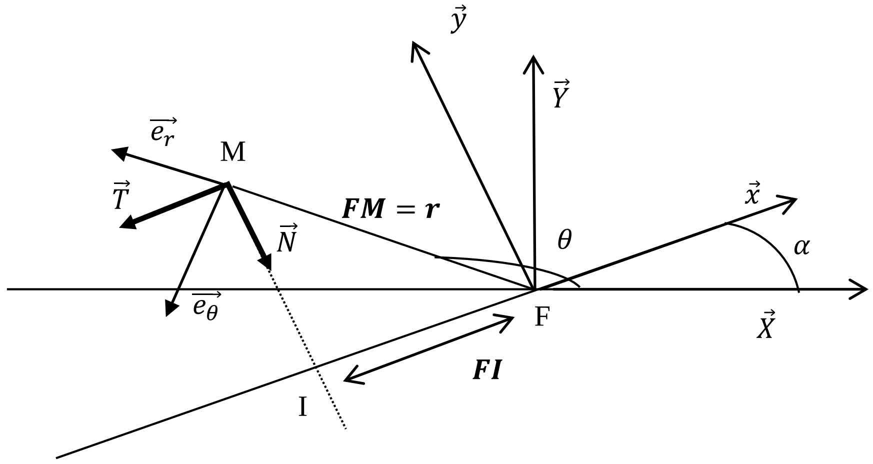


Figure 2. Precession of angle α

We definite the system of coordinate $(F; \vec{x}; \vec{y})$ by

$$\left(\vec{X}, \vec{x}\right) = \alpha$$

And we introduce a scalar relation

$$FI = e'r$$

Where e' is given, if the system doesn't precess, with the relation

$$e' = e$$

We obtain thus

$$\overrightarrow{FI} = -\frac{e'}{u}\vec{x} = -\frac{e'}{u}(\cos(\theta-\alpha)\,\overrightarrow{e_R} - \sin(\theta-\alpha)\,\overrightarrow{e_\theta})$$

And using

$$\overrightarrow{IM} = \overrightarrow{IF} + \overrightarrow{FM}$$

It comes

$$\overrightarrow{IM} = \frac{e'}{u}(\cos(\theta-\alpha)\,\overrightarrow{e_R} - \sin(\theta-\alpha)\,\overrightarrow{e_\theta}) + \frac{e'}{u}\overrightarrow{e_R}$$

Thus

$$\overrightarrow{IM} = \frac{e'}{u}\left(\left[\frac{1}{e'} + \cos(\theta-\alpha)\right]\overrightarrow{e_R} - \sin(\theta-\alpha)\,\overrightarrow{e_\theta}\right)$$

$\vec{T}$ is again

$$\vec{T} = -u'\overrightarrow{e_R} + u\overrightarrow{e_\theta}$$

Writing as previously the condition of perpendicularity

$$\overrightarrow{IM}.\vec{T} = 0$$

We obtain after simplification the differential equation

$$u' + u'e'(\cos\theta\cos\alpha + \sin\theta\sin\alpha) + ue'(\cos\alpha\sin\theta - \sin\alpha\cos\theta) = 0 \qquad (4)$$

Note it is the same differential equation if $\alpha = 0$.

2.3 Solving

To solve equation (4) we need to introduce a relationships which links angles θ and α : A correct and simple solution, is given with

$$\alpha = n\theta + \beta$$

Where n is a rational number and β a constant angle. Indeed, with this solution the solving leads for $n = 0$ to the initial conic, with a phase shift β. As we introduced it, doing variations around this expression, we obtained a class of curves, using

$$e' = f(e, n) \qquad (5)$$

2.4 Class of generalized conic trajectories

With previously relation we obtained two families of solutions, depending on n :

2.4.1 General case : For $n \neq 1$: solutions are given with the relation

$$u(\theta) = C_1(1 + e'\cos((n-1)\theta + \beta)))^{\frac{1}{1-n}} \qquad (6)$$

To simplify we choose $\beta = 0$, which leads to

$$r(\theta) = \frac{C'_1}{(1 + e'\cos\theta(n-1))^{\frac{1}{1-n}}} \qquad (7)$$

The function is periodic. Knowing our initials conditions we identify the constants C'_1 and e'

$$\begin{cases} r_{min} = a(1-e) = \dfrac{C'_1}{(1+e')^{\frac{1}{1-n}}} \\ r_{max} = a(1+e) = \dfrac{C'_1}{(1-e')^{\frac{1}{1-n}}} \end{cases}$$

And obtain

$$\begin{cases} e' = \dfrac{(1+e)^{1-n} - (1-e)^{1-n}}{(1+e)^{1-n} + (1-e)^{1-n}} & (8.1) \\ C'_1 = a(1-e^2)\left[\dfrac{2}{(1+e)^{1-n} + (1-e)^{1-n}}\right]^{1/_{1-n}} & (8.2) \end{cases}$$

2.4.2 particular case : For $n = 1$ and $\beta \neq 0$: Solution is a circle or a logarithmic spiral given by

$$u(\theta) = C_2 e^{\frac{e\prime \sin\beta}{1+e\cos\beta}\theta}$$

We can again determine the constant using the initial condition

$$r(\theta = 0) = a(1-e) = C_2$$

2.4.3 mathematical expression

The class of generalized conic can thus be written

$$\left\{ \begin{array}{lll} n \neq 1 & r(\theta) = \dfrac{C'_1}{(1 + e' \cos(\theta(n-1) + \beta))^{\frac{1}{1-n}}} & (9.1) \\ n = 1 & r(\theta) = a(1-e)e^{\frac{-e' \sin\beta}{1+e\cos\beta}\theta} & (9.2) \end{array} \right\}$$

Where C'_1 and e' are defined in the general case by (8).

2.5 geometric consequences

The polar curves corresponding to the equation (9.1) have a common geometric property. As presented in section 2.2, it concerns a relationship of proportionality connecting the radial distance to the location of the intersection of the radius with a rotating axis, defined by the relation

$$\alpha = n\theta + \beta$$

the angles being defined as presented previously. The distance between the origin O of the system of coordinate and the point of intersection I is given by the relation

$$OI = e'OM$$

Where e' is definite with relation (8). This geometric property leads to the possibility of graphically determining the direction of the radius of curvature, and therefore of the tangent to the curve. It is obviously a fundamental geometric property, which does not seem to have always been noticed, in most of the cases. To illustrate it, we present in this part some examples. Let us first consider the simplest cases, corresponding to the eccentricities of the initial conic equal to 1 ($e = e' = 1$). An interesting case is of course the cardioid curve [7] which we present in figure 3.1 :

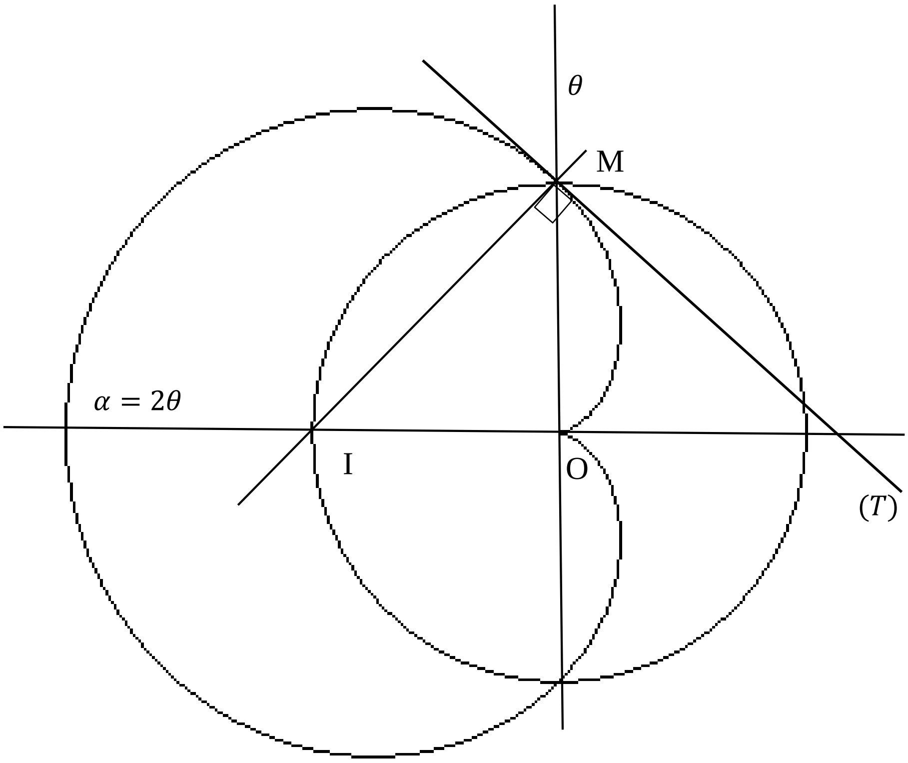


Fig 3.1. Cardioïd. $n = 2$. The perpendicular to the tangent (T) to the curve at M intersects at I such that: $OI = OM$ and $\alpha = 2\theta$

The axis of the radius of curvature (IM) is here simply drawn by reporting the distance OM on the axis corresponding on $\alpha = 2\theta$, and the direction of the tangent in M is directly deduced therefrom. This construction method can be compared with other methods that we encounter in the literature (see for example reference [8]) and seems, at least to our knowledge, original.

A similar result can also be presented for the prestigious Lemniscate of Bernouilli [9], considering this time the relation $\alpha = 5\theta$ (Figure 3.2).

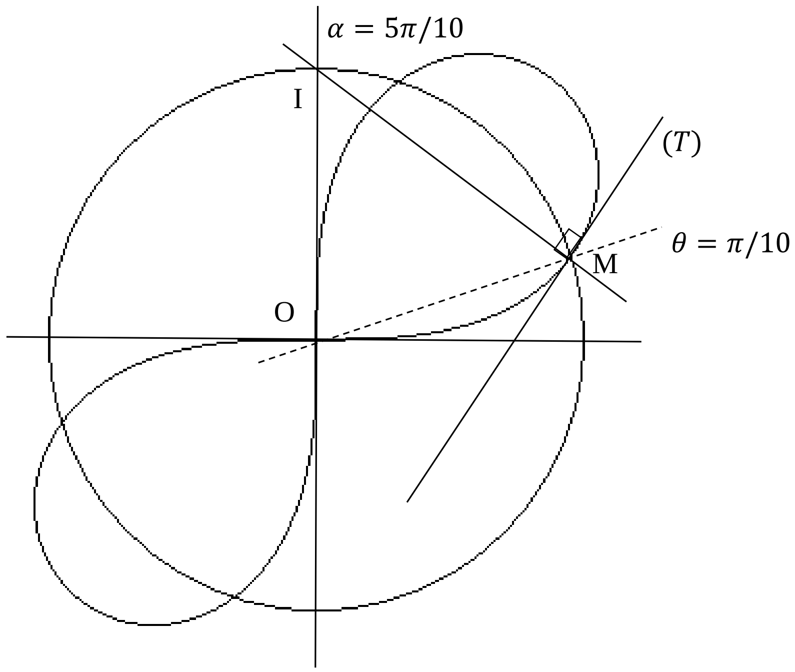


Fig 3.2 . Lemniscate de Bernouilli. $n = 5$. The perpendicular to the tangent (T) to the curve at M intersects at I such that: $OI = e'OM = OM$ and $\alpha = 5\theta$

However, This result can be extended to all initial eccentricities between 0 and 1, but considering that the distance is no longer equal. An example is detailed in figure 3.3, corresponding to an ellipse, and where the two circles of radius $OM$and $e'OM$ are drawn :

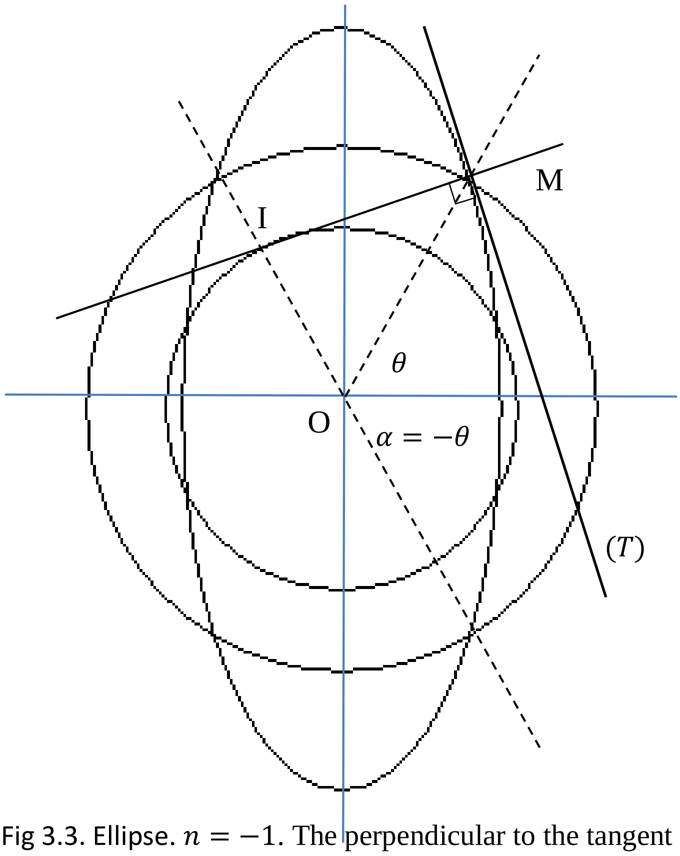


Fig 3.3. Ellipse. $n = -1$. The perpendicular to the tangent (T) to the curve at M intersects at I such that: $OI = e'OM$ and $\alpha = -\theta$

Note again that others methods of graphical constructions corresponding on this curve can be found in reference [10].

To conclude this part, we consider as a final example the case of a more exotic curve, which is drawn in figure 3.4 :

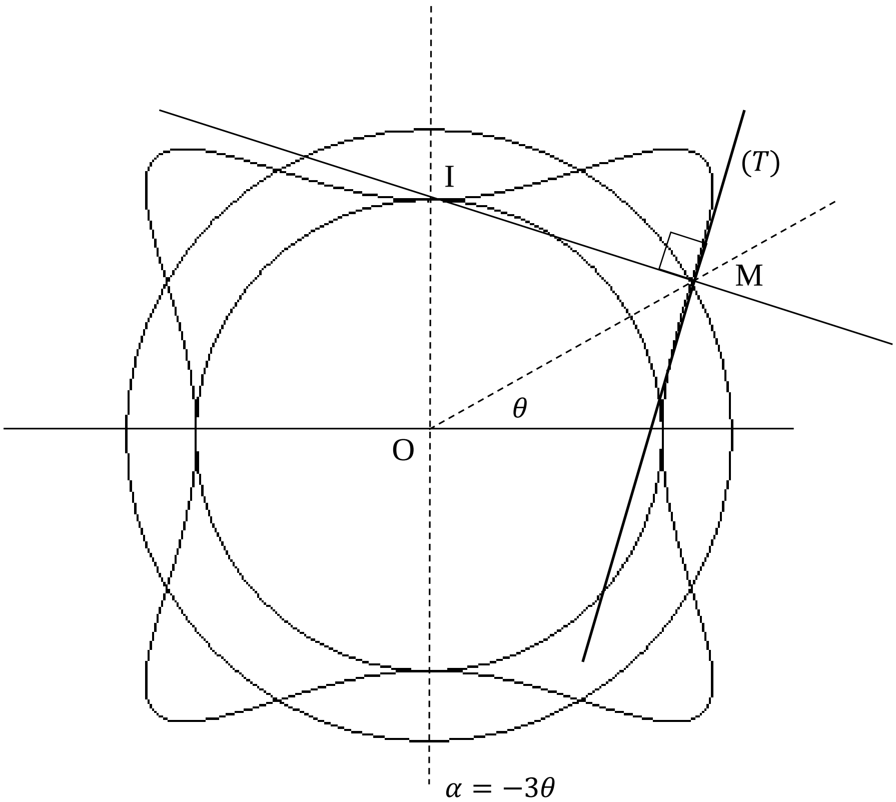


Fig 3.4. $n = -3$, $e = 0.25$, $e' = 0.47$ The perpendicular to the tangent (T) to the curve at M intersects at I such that: $OI = e'OM$ and $\alpha = -3\theta$

2.5.1 Generalisation of the property

By considering these results, it is possible to propose their generalization and to present a geometrical method of construction of the tangents of this class of curves. According to results presented in section 2.2, it consists to draw on the same graph the curve describes by the point I. This associated curve is thus the locus of these points of intersection and should be not confused with the evolute of the curve. The expression of this associated curve is given by :

$$r(\theta) = \frac{-e'C'_1}{\left(1 + e' \cos\left(\theta \frac{(n-1)}{n}\right)\right)^{\frac{1}{1-n}}} \qquad (10)$$

This method is particularly simple when the eccentricity of the initial conic is equal to 1. In this case, the circle of radius OM (in dotted line in following figures) intersects with the associated curve at I. (IM) is the direction of the radius of curvature. Consider again, for example, the case of a cardioïd curve C_1 given by

$$r = 1 - \cos\theta$$

Curve C_2 (associated curve) is given in this case by the relation

$$r' = 1 - \cos\frac{\theta}{2}$$

As presented in Figure 3.5 :

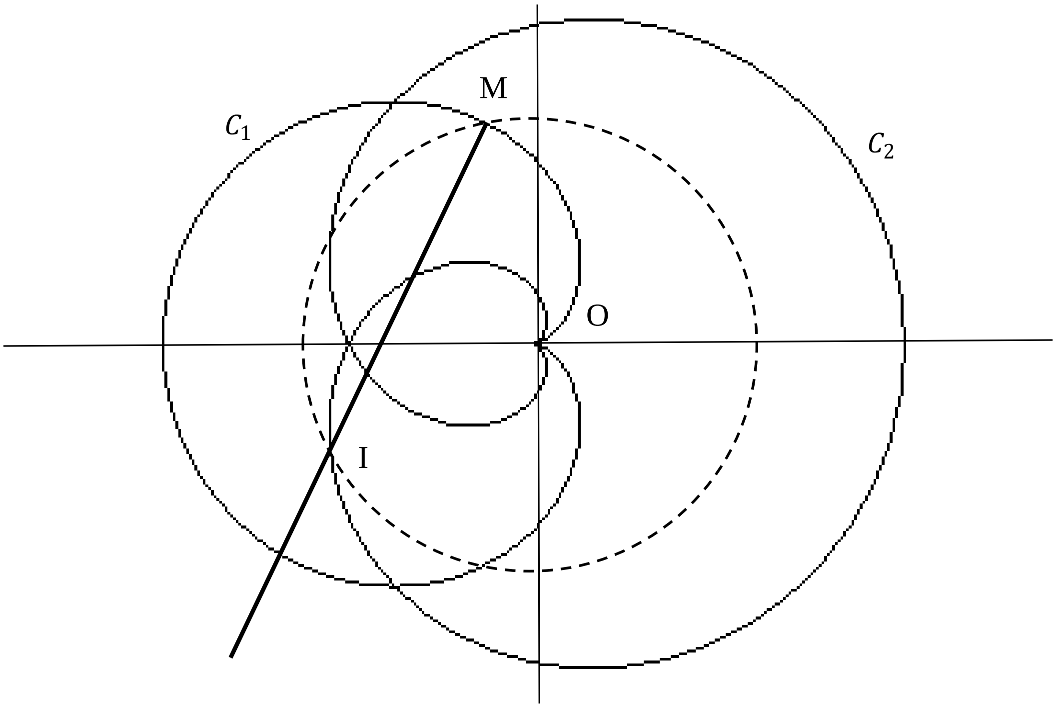


Fig.3.5. $n = 2$. The cardioïd C_1 is drawn with its associated curve C_2. $OM = OI$.

(IM) is the direction of the radius of curvature of C_1

Another remarkable and extremely simple example is presented in the following figure, and this time concerns the curve of the circles passing through the origin (case $n = 3$). Corresponding curves are respectively given with

$$\begin{cases} r = \sqrt{1 - \cos 2\theta} \\ r = \sqrt{1 - \cos \dfrac{2\theta}{3}} \end{cases}$$

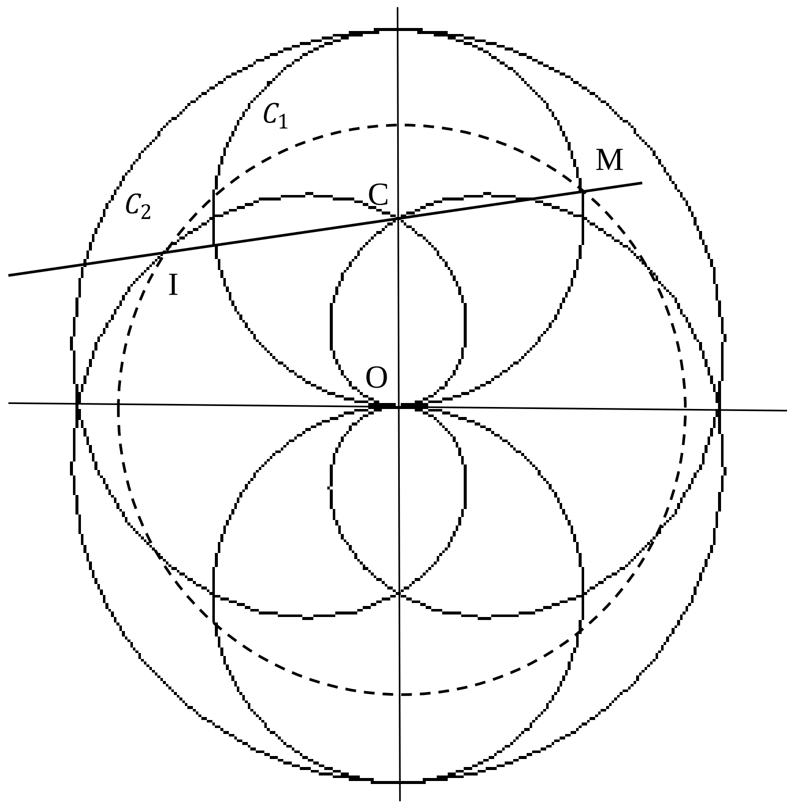


Fig.3.6. $n = 3$. The double circle C_1passing through the origine O is drawn with its associated curve C_2. $OM = OI$.

(IM) is the direction of the radius of curvature of C_1, C its center.

The radius of curvature obviously passing through the center of the circle. Consequently, as visible, the three points M, C and I are aligned.

When initial eccentricity isn't equal to 1, a solution is to draw a third curve corresponding on $|e'|C'_1$, i.e. an homothetic curve of center 0. An example of the method is presented in Figure 3.7.

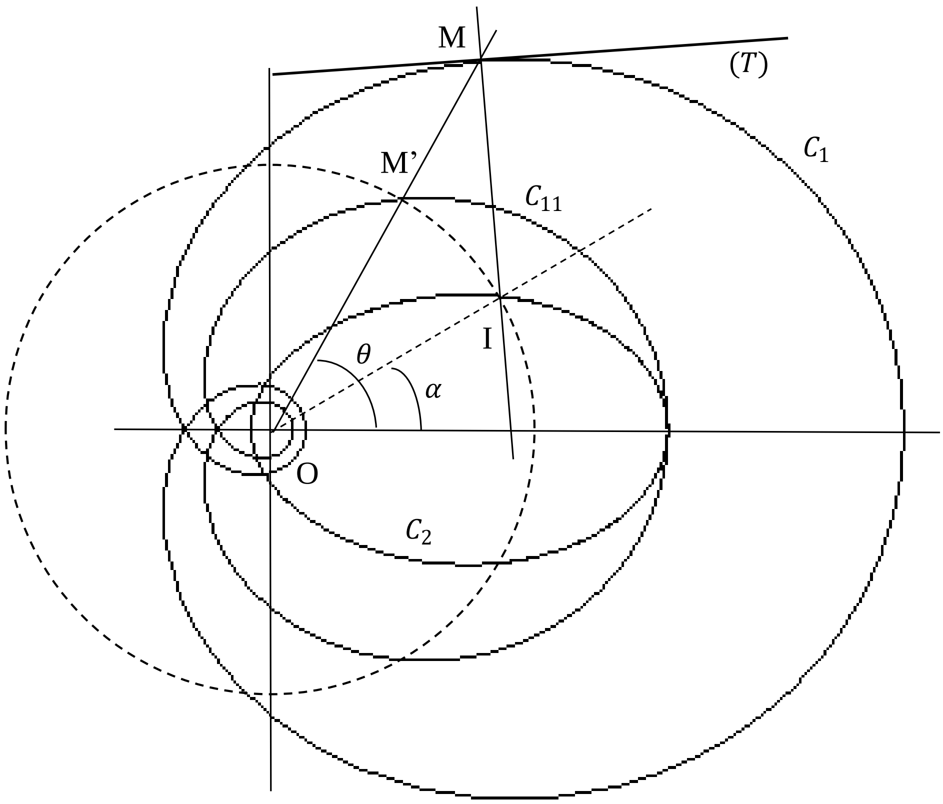


Fig 3.7. .$n = 0.5$, $e = 0.9$, $e^{'} = 0.628$. C_1 is drawn with its homothetic curve C_{11} and its associated curve C_2. $OM' = OI$. $2\alpha = \theta$.

(IM) is the direction of the radius of curvature of C_1, (T) the tangent at M.

A remarkable case concerns the ellipse define from its center ($n = -1$). Consider the curve C_1 given by

$$r = \frac{1}{\sqrt{1 + e' \cos 2\theta}}$$

Eccentricity of this ellipse e_H is linked to e' and e by

$$\left\{ \begin{array}{l} e' = \dfrac{{e_H}^2}{2 - {e_H}^2} \\ e = \dfrac{1 - \sqrt{1 - {e_H}^2}}{1 - \sqrt{1 - {e_H}^2}} \end{array} \right\} \qquad (11)$$

Associated curve C_2 is here define using (10)

$$r = \frac{e'}{\sqrt{1 + e' \cos 2\theta}}$$

i.e. a second ellipse of the same eccentricity, which thus is confused whith the homothetic curve C_{11}. Drawing them on the same graph (Figure 3.8)

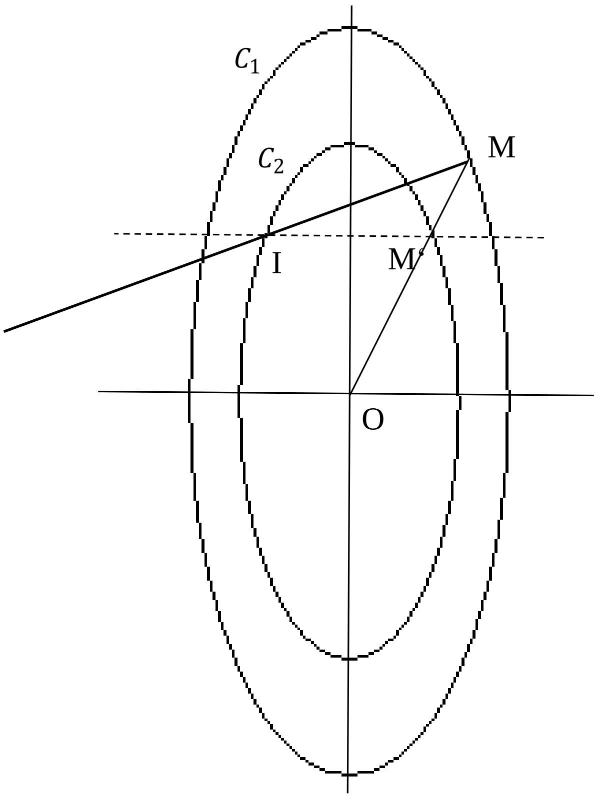


Fig.3.8. $n = -1$, $e = 0.4$, $e^{'} = 0.689$ the ellipse C_1 is drawn with its associated curve C_2. $OM" = OI$.

(IM) is the direction of the radius of curvature of C_1

2.6 Geometric transformation

Another fundamental concern of the study of curve classes is the determination of symmetry relations and geometric transformations. In our case, we were able to highlight a relation of transformation between the curves whose values of n are symmetric with respect to 1. Indeed, consider two generalized conics of parameters $GC_1\{C_{11}, n_1, e_{11}\}$ and $GC_2\{C_{12}, n_2, e_{12}\}$, where parameters of intial conic $\{a, e\}$ are conserved and which are linked with the relation

$$1 - n_1 = n_2 - 1$$

Which is therefore equivalent to saying that n_1 and n_1 are symmetric around the central value $n = 1$. We can rewrite it

$$n_1 = 2 - n_2 \qquad (12)$$

Writing the expression of these generalized conics

$$\left\{\begin{aligned} r_1(\theta) &= \frac{C'_{11}}{(1+e'_{11}\cos\theta(n_1-1))^{\frac{1}{1-n_1}}} \\ r_2(\theta) &= \frac{C'_{12}}{(1+e'_{12}\cos(\theta(1-n_2))^{\frac{1}{1-n_2}}} \end{aligned}\right\}$$

Consider now the parameters $e'$of the two generalized conics

$$e'_{11} = \frac{(1+e)^{1-n_1}-(1-e)^{1-n_1}}{(1+e)^{1-n_1}+(1-e)^{1-n_1}} = \frac{(1+e)^{n_2-1}-(1-e)^{n_2-1}}{(1+e)^{n_2-1}+(1-e)^{n_2-1}} = -e'_{12}$$

They are conserved, apart from sign. Having now a look to the second parameter,

$$C'_{11} = a(1-e^2)\left[\frac{2}{(1+e)^{1-n_1}+(1-e)^{1-n_1}}\right]^{1/1-n_1}$$

$$= a(1-e^2)\left[\frac{2}{(1+e)^{n_2-1}+(1-e)^{n_2-1}}\right]^{1/n_2-1}$$

Finally

$$C'_{11} = \frac{a^2(1-e^2)}{C'_{12}}$$

Synthetizing the results

$$\begin{bmatrix} n_1-1 = -(n_2-1) \\ e'_1 = -e'_2 \\ C'_{11} = \dfrac{a^2(1-e^2)}{C'_{12}} \end{bmatrix}$$

It indicates the two associated conics GC_1 and GC_2 are linked with two successive geometric transformations, that we detail in the following.

2.6.1 Inversion

First of them is an inversion, which we can describe by

$$r \to \frac{a^2(1-e^2)}{r}$$

i.e. an inversion whose center is the origin of the system of coordinate. Calling it O , the image M'_2 of a point M_1 located on one of the generalized conic GC_1 can thus be defined with

$$OM_1.OM'_2 = a^2(1-e^2)$$

And is consequently the inverse of M_1 with respect to the reference circle $(O, r = a\sqrt{(1-e^2)})$. We note this transformation of inversion as follows

$$I(O,\sqrt{(1-e^2)})$$

M'_2 is located on an intermediary generalized conic $GC'_2\{C_{12}, n_2, e_{11}\}$, given thus with

$$r'_2(\theta) = \frac{1}{C'_{11}} \frac{a^2(1-e^2)}{(1+e'_{11}\cos\theta(n_1-1))^{\frac{-1}{1-n_1}}} = \frac{C'_{12}}{(1-e'_{12}\cos\theta(1-n_2))^{\frac{1}{1-n_2}}}$$

An illustration of the transformation

$$GC_1 \xrightarrow{I(0,a\sqrt{1-e^2})} GC'_2$$

Can be found in Figure 4:

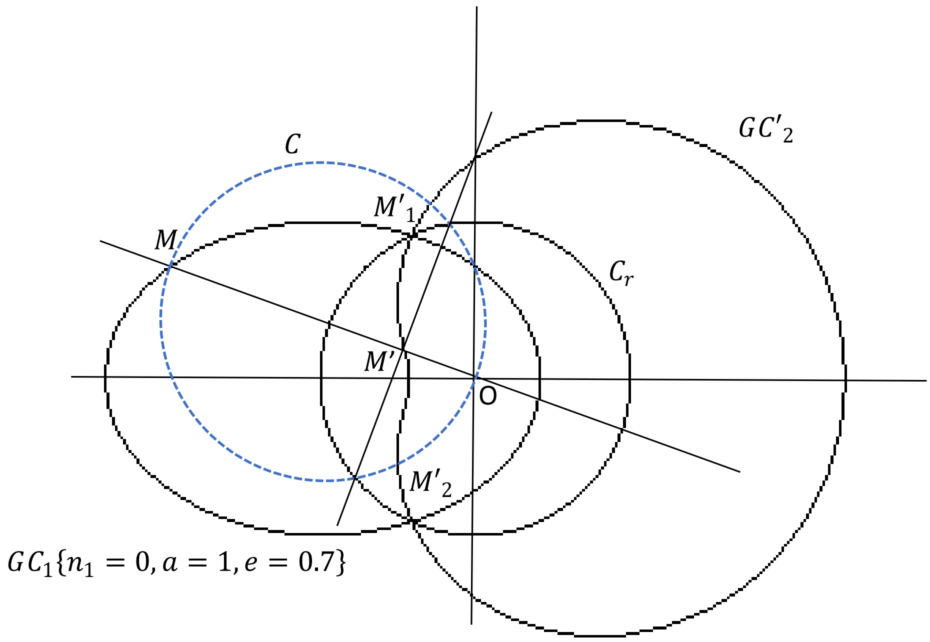
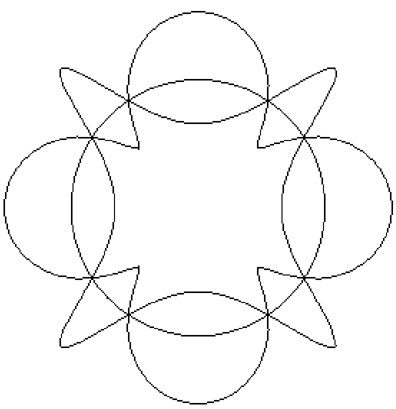


Fig 4. M' is the image of M with the tranformation of inversion I. C_r is the circle of reference centered at O, C the circle of diameter OM. GC_1 is an ellipse whose foci is located at O, GC'_2 its image (generalized cardioïd). M_1 and M_2 are invariable points and located at the intersections of GC_1, GC'_2 and C_r

To illustrate it, we suggest in following Figures, the representation of certain associated generalized conics, with the circle of reference:



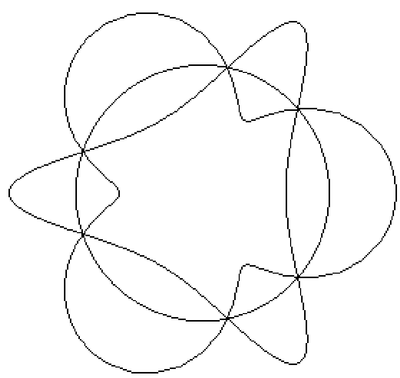


Fig 4.1 $\{e = 0.4, n_1 = -3, n_2 = 5\}$ $\{e = 0.4, n_1 = -2, n_2 = 4\}$

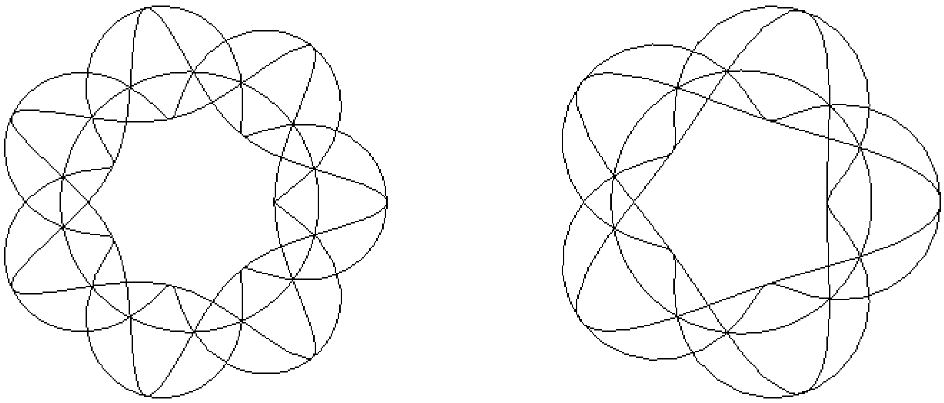


Fig 4.2 $\{e = 0.4, n_1 = -2.5, n_2 = 4.5\}$ $\{e = 0.4, n_1 = -1.5, n_2 = 3.5\}$

And, for the fundamental trajectories

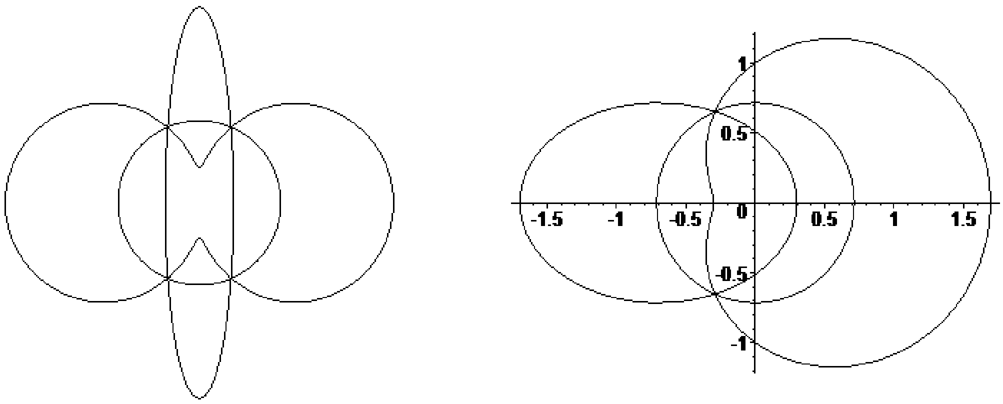


Fig 4.3 $\{e = 0.7, n_1 = -1, n_2 = 3\}$ $\{e = 0.7, n_1 = 0, n_2 = 2\}$

Ellipse of Hooke and generalized circle Keplerian orbit and generalized cardioid

Several points can be highlighted on these figures. In particular, the curves are bounded between two radius, depending on the choice of the parameters of the initial conic, and corresponding on $a(1-e) \leq r \leq a(1+e)$. Moreover, we can notice that the two associated generalized conics and the reference circle intersect at the same point, which is a property of this geometrical transformation.

In fact, these results were known when e' is equal to 1. For example, it is well known that “the image under inversion of a line not through the center of inversion is a circle passing through” this center, which corresponding here to the associated curves $(n_1 = -1, n_2 = 2)$ when $e = 1$. Another classical case is the Lemniscate of Bernouilli, whose image is an hyperbolae $(n_1 = -3, n_2 = 5)$ [9]. But it appears that the general cases have not been detailed (corresponding on $e \neq 1$).

2.6.2 Rotation

To obtain GC_2 we need to make a second geometric transformation, this time a rotation around O and of angle given by

$$\frac{\pi}{n_1 - 1} = \frac{\pi}{1 - n_2}$$

Indeed, with this transformation we obtain

$$\frac{C'_{12}}{(1 - e'_{12}\cos(\theta(1-n_2)))^{\frac{1}{1-n_2}}} \rightarrow \frac{C'_{12}}{(1 + e'_{12}\cos(\theta(1-n_2)))^{\frac{1}{1-n_2}}} = r_2$$

Noting this geometric transformation R_{O,n_1-1} we obtain thus

$$GC'_2 \xrightarrow{R_{O,n_1-1}} GC_2$$

2.6.3 Synthesis

Finally, the associated conic can be linked by the two successive transformations

$$GC_1 \xrightarrow{I} GC'_2 \xrightarrow{R} GC_2$$

We present an illustration in Figure 5.

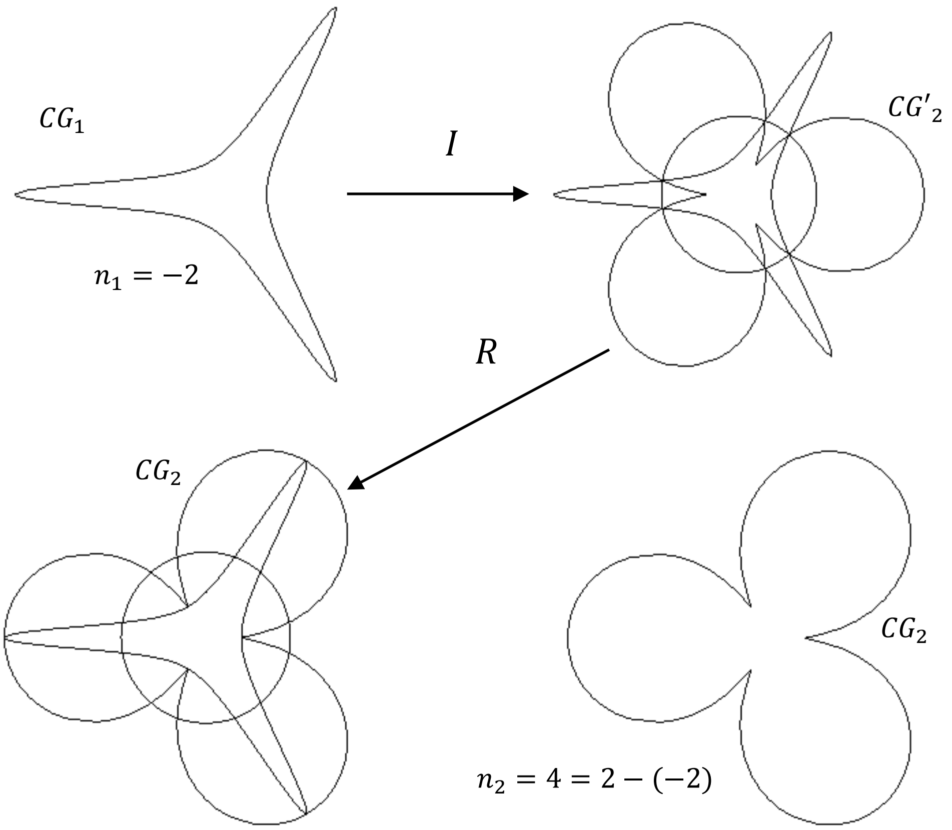


Figure 5. Initial conic $GC_1\{a = 1, e = 0.7, n = -2\}$ is modified by two successive geometric transformations in $GC_2\{a = 1, e = 0.7, n = 4\}$

2.7 Limiting case : $n = 1$. Circle and Logarithmic spiral

As presented in (9.2) the solution is a logarithmic spiral, also called “equiangular spiral”, because its slope angle is constant. In our case the expression is given by

$$r(\theta) = a(1-e).e^{\frac{e''\sin\beta}{1+e'\cos\beta}\theta}$$

As previously, we can illustrate the geometric properties contained in differential equation (4). The leading coefficient of this curve is

$$\gamma = \frac{r}{r'} = \frac{1+e'\cos\beta}{e'\sin\beta}$$

If we now have a look on this coefficient in the general case defined by (7) we obtain

$$\gamma = -\frac{1+e'\cos(n-1)\theta}{e'\sin(n-1)\theta}$$

Consequently, radius of curvature are parallel when

$$\beta = (1-n)\theta$$

To illustrate it we choose the simple case $n = 0$ and $\beta = {}^{\pi}/_{2}$ (Figure 6.1). We plot thus an ellipse (curve C_0) and the corresponding spiral (C_1)

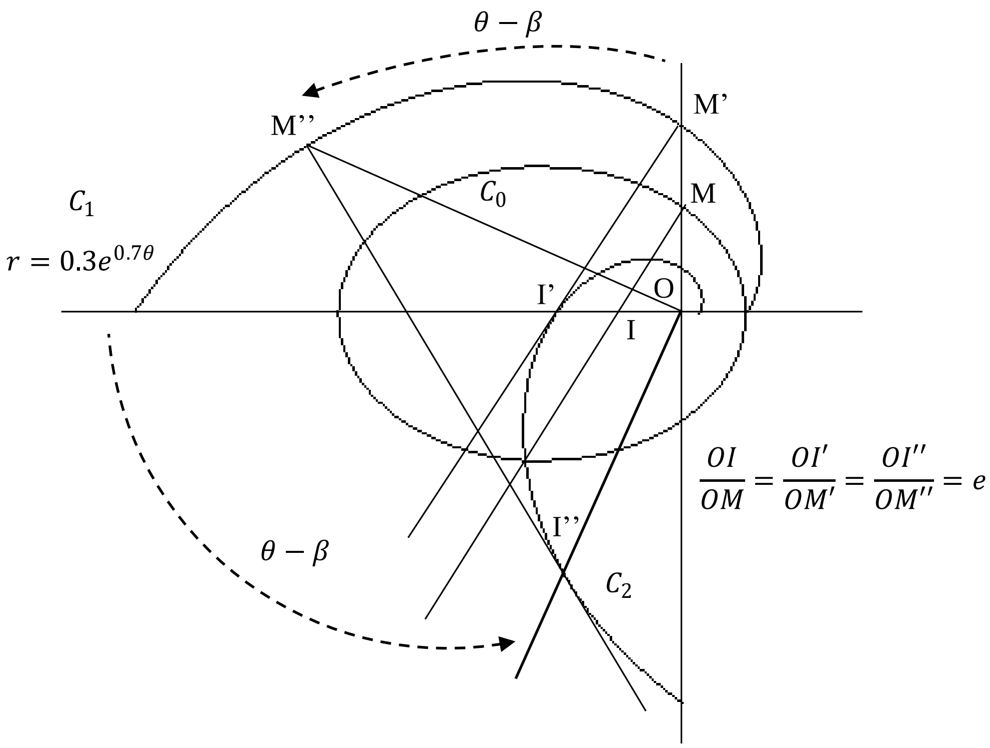


Fig. . $e = 0.7$, $\beta = {}^{\pi}/_{2}$. $(M'I')$ and (MI) are parallel because $\theta = {}^{\pi}/_{2}$.

(IM) is the direction of the radius of curvature of C_O, $(I^{'}M^{'})$ and $(I^{''}M^{''})$ of C_1.

Consider now a precession of the abscissa axis on an angle $(\theta - \beta)$ (axe (OI') on the Figure). The intersection of the radius of curvature of C_1 intersects with this axis as such manner that relation of proportionality is maintained, and is therefore the geometric consequence of equation (4). These points of intersection I, I', I''.. draw to their whole a logarithmic spiral C_2 given by

$$r = -e(1-e)e^{0.7(\pi-\beta+\theta)}$$

By considering these results, we can again suggest an original method of construction of the tangents to this kind of curves: Considering a logarithmic spiral C_1 given by

$$r = ae^{k\theta}$$

We seek to determine the corresponding angle β. So we pose the equation

$$k = \frac{\sin\beta}{1+\cos\beta}$$

Solving leads to

$$\sin\beta = \frac{2k}{1+k^2}$$

Associated curve C_2 is given by

$$r = -ae^{k(\pi-\beta+\theta)}$$

i.e. a second logarithmic curve. Drawing the two curves on the same graph, we deduce the direction of the radius of curvature as previously. Consider for example the case $a = 1$, $k = 2$ (Figure 6.2).
We obtain $\sin\beta = 4/5$. We draw thus

$$r = -e^{2(\pi-\sin^{-1}(4/5)+\theta)}$$

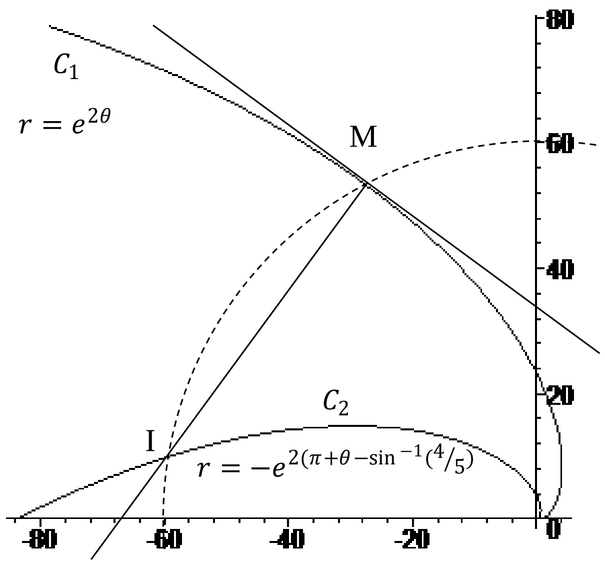


Fig.6.2. the spiral C_1 is drawn with its associated curve C_2. $OM = OI$.
(IM) is the direction of the radius of curvature of C_1

To conclude this geometrical study, we have a brief look on the relation of transformation. We know that the transform of the spiral by the inversion is another spiral given here by

$$r = a(1+e)e^{-\frac{e'\sin\beta}{1+e'\cos\beta}\theta}$$

It is interesting to note that the curves corresponding on the central value $n = 1$ have consequently an unique property inside the class : they are the only curves whose nature isn't modified by the relation of inversion. Moreover, they are the only curves to which correspond associated curves of the same nature**.**

3. Dynamics

As we have evoked it in the introduction, we adopt in this second part a physical point of view. Indeed, we consider now the previously curves as possible trajectories of a point particle submitted to a field of central force.

3 .1 Law of central Forces

Central forces can be determined using the laws of dynamics, but it is generally easier to determine the forces knowing the trajectory than the inverse. It exists several methods based on this approach. In particular, historically, Newton detailed several methods based on geometrical considerations [11]. In the 19th the Italian mathematician and ballistician Siacci wrote a theorem [12] (recently generalized [13]), which provides that the central acceleration can be written

$$\overrightarrow{a_r} = -\frac{C^2}{p^3}\frac{dp}{dr}\overrightarrow{e_r}$$

Where C is the constant of area given by

$$C = r^2\dot{\theta}$$

And p is the pedal curve of the trajectory, given by

$$p = \frac{r^2}{\sqrt{r^2 + r'^2}}$$

With

$$r' = \frac{dr}{d\theta}$$

We consider the polar curve corresponding on

$$r(\theta) = \frac{C'_1}{(1 + e'\cos\theta(n-1))^{\frac{1}{1-n}}}$$

Pedal curve is

$$p(\theta) = C'_1 \frac{(1 + e'\cos(n-1)\theta)^{n/_{n-1}}}{\sqrt{(1 + e'^2 + 2e'\cos n\theta)}}$$

Using the relation

$$\frac{dp}{dr} = \frac{d\theta}{dr}\frac{dp}{d\theta} = \frac{p'}{r'}$$

We obtain the central acceleration, which after reorganization of the terms, becomes

$$\overrightarrow{F_r} = -C^2 {C'_1}^{2n-2}\left[(1+n)\frac{{C'_1}^{1-n}}{r^{2+n}} - n\frac{\left(1-{e'}^2\right)}{r^{2n+1}}\right]\overrightarrow{e_r} \quad (13)$$

And central forces (per mass unity) can finally be rewritten under the form

$$F = \left[-(1+n)\frac{A}{r^{n+2}} + n\frac{B}{r^{2n+1}}(1-e'^2)\right] \qquad (14)$$

Where A and B are two positive constant. We see the value $n = 1$ isn't here forbidden and is corresponding to an inverse cubic force, which is natural because the logarithmic spiral is one of the curve of Cotes [14].

3.2 Particular cases

The previously law of central forces seems not have been obtained until today, at least at our knowledge. It is interesting to note it is the sum of two central forces except for three fundamentals cases, corresponding on n =-1, $n = 0$ and $n = 1$. Forces are thus conservative for these three cases, such we can write a power law of potentials using

$$V(r) = -\int F dr$$

Thus

$$V(r) = -\frac{A}{r^{1+n}} + \frac{1}{2}\frac{B}{r^{2n}}\left(1-{e'}^2\right) \qquad (15)$$

Listing them for the three fundamental cases in tab 1.

$n = -1$	Harmonic oscillator	$V(r) = \frac{1}{2}Kr^2$	$r(\theta) = \frac{a}{\sqrt{(1+e'\cos 2\theta)}}$
$n = 0$	Keplerian orbit	$V(r) = -\frac{A}{r}$	$r(\theta) = \frac{a}{1+e\cos\theta}$
$n = 1$	Logarithmic Spiral trajectory	$V(r) = \frac{B}{r^2}$	$r(\theta) = ae^{k\theta}$

Tab 1. The three fundamental and conservative potentials

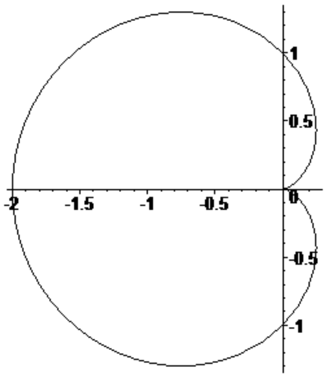
Formulae (15) allows thus to link in only one relation these three fundamental potentials of Physics. In the others cases we know that, with respect for the Bertrand's theorem [15], the combination of central forces is not conservative, except for $e = 1$. Indeed, using (8.1) we see that in this case $e'$is also equal to 1 and the law of force becomes simply

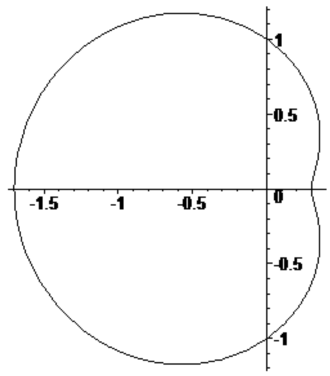
$$F = -(1+n)\frac{A}{r^{n+2}}$$

In these limiting cases we can thus write the corresponding potentials

$$V(r) = -\frac{A}{r^{1+n}} \quad (15)$$

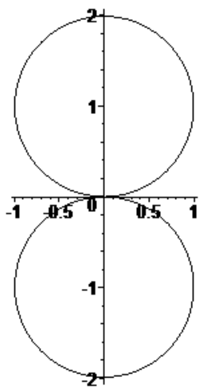
We reobtain here a well-known series of trajectories, where we can distinguish, for example, the cardioid, circle passing through the origin, Lemnicaste.. We can consider these cases as generalized parabola of the class of curves. Note that this time this law of potential was already known (formulae valuable except for limiting case $n = -1$ [1]).

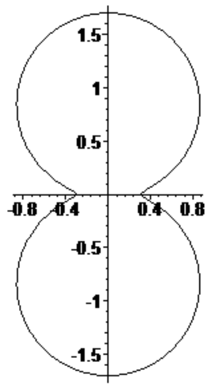




$$F = -\frac{A}{r^4} \qquad\qquad F = -\frac{A}{r^4} + \frac{B}{r^5}$$

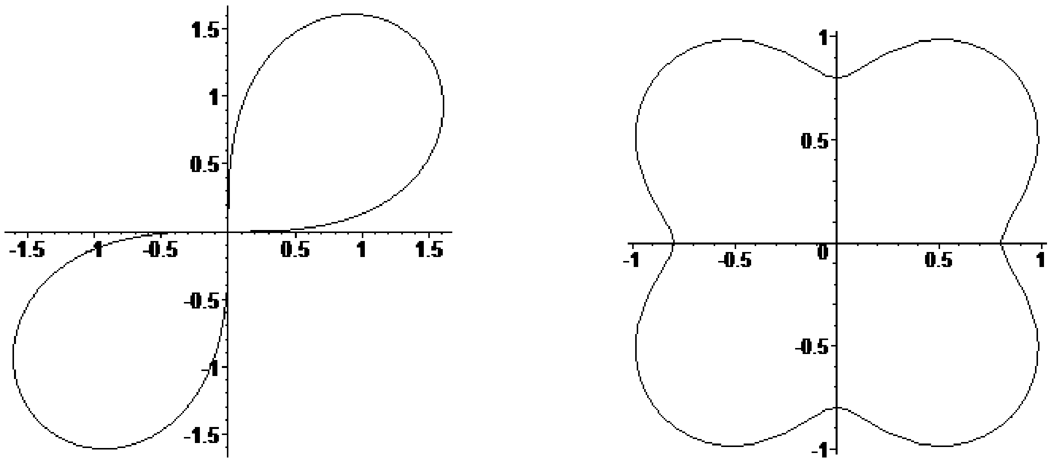
Fig 7.1 cardioîd and its genenalized form ($n = 2$)





$$F = -\frac{A}{r^5} \qquad\qquad F = -\frac{A}{r^5} + \frac{B}{r^7}$$

Fig. 7.2 Circular orbits passing through the point of attraction and its generalized form ($n = 3$). The solution was mentioned by Newton, in corollary 1 to proposition VII of the Principia.



$$F = -\frac{A}{r^7} \qquad\qquad F = -\frac{A}{r^7} + \frac{B}{r^{11}}$$

Fig 7.3 Lemniscate of Bernouilli and its generalized form ($n = 5$)

3.3 Associated curves

We have also investigated the taw of central force corresponding on the associated curves define by relation (10), using again the Siacci's theorem. The law of central we obtained can this time be written as follows:

$$\vec{F} = -\frac{C^2 {C'_1}^{2n-2}}{n^2}\left[(1+n)\frac{{C'_1}^{1-n}}{r^{2+n}} - n\frac{\left(1-{e'}^2\right)}{r^{1+2n}}(n^2-1) + \frac{{C'_1}^{2-2n}}{r^3}\right]\overrightarrow{e_r} \qquad (16)$$

That we can rewrite using three constants A', B', C' linked together

$$\vec{F} = -\frac{1}{n^2}\left[(1+n)\frac{B'}{r^{2+n}} - n(1-e^2)\frac{C'}{r^{1+2n}} + (n^2-1)\frac{A'}{r^3}\right]\overrightarrow{e_r} \qquad (17)$$

This second law of central force excludes thus the value $n = 0$, which is natural since there are no curves associated to the Keplerian orbit. Note that for $n = -1$, we reobtain an attractive restoring force, which is an agreement with the results presented in section 2.5.1 (the associated curve of an ellipse defined with respect to its center, i.e. a two-dimensional harmonic oscillator, is another ellipse of same eccentricity).

Moreover, having a look on previously law of central force (13), we note the difference is basically due to the addition of an inverse cubic force.

4. Discussion and Physical applications

As we introduced it, the paper can be viewed as multidisciplinary and compatible with several areas of Physics, naturally mathematical Physics but also general Physics, classical and celestial Mechanics. In particular, it presents advances in the field of geometry, with the presentation of a class of generalized conics and some of its properties. But some results can also be related to different physical concerns. In this discussion, we want consider a part of them and place them into the framework of modern investigations.

4.1 Combinations of central forces

The results we have obtained can in particular be linked to a popular problem of study, which led to the investigation of certain types of force laws for which differential equation is solvable in terms of known functions. We think in particular to the work of Whittaker [16], who have studied the cases of central forces given with

$$F = ar^n$$

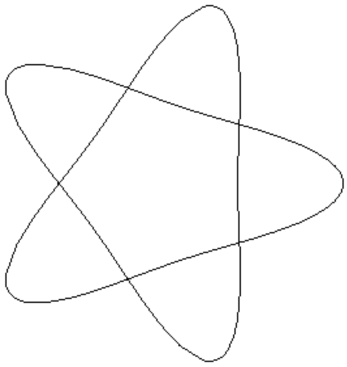
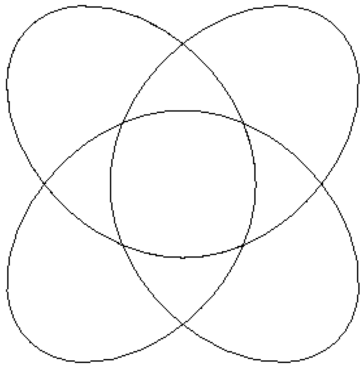
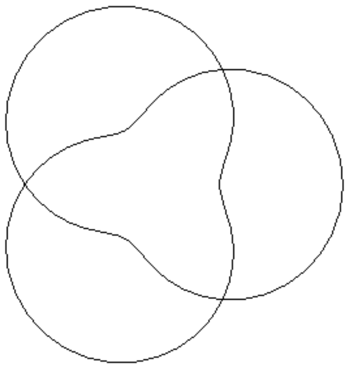
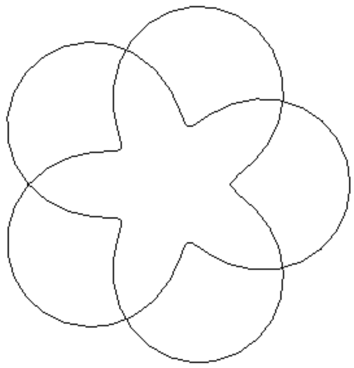
And who established that corresponding differential equations are solvable in terms of circular functions if $n \in \{1, -2, -3\}$ (that we also reobtain) and by elliptic functions if

$$n \in \left\{7, -5, -4, 0, 3, 5, -\frac{3}{2}, -\frac{5}{2}, -\frac{1}{3}, -\frac{5}{3}, -\frac{7}{3}\right\}$$

Later, others works examined the question of linear combinations of this kind of central forces and, after a generalization of Whittaker's results, obtained 6 combinations of forces which can be integrated in terms of known functions [17], [18]. The results which are presented in the paper can thus be linked to these classical studied, and provide the original result that the general law of combination of central forces given with

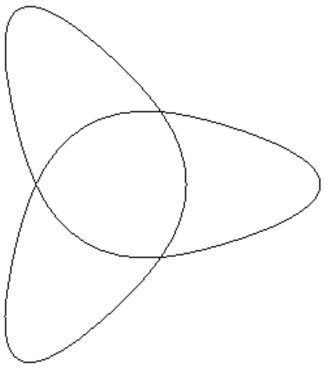
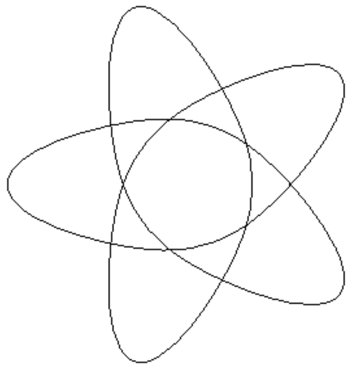
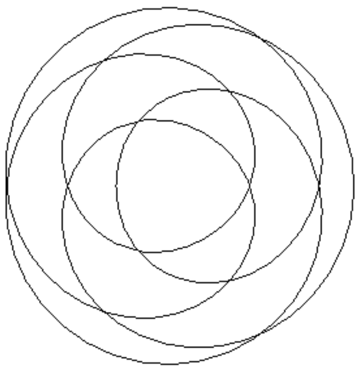
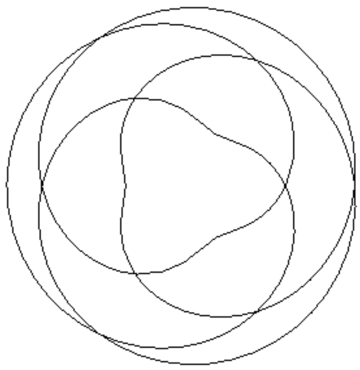
$$F = -C^2 {C'_1}^{2n-2}\left[(1+n)\frac{{C'_1}^{1-n}}{r^{2+n}} - n\frac{\left(1-{e'}^2\right)}{r^{2n+1}}\right]$$

(where C'_1 is a length, C the constant of areas) leads to the class of generalized conic trajectories presented in (9). A large part of new solutions can thus be added to the already known solutions. See for example certain remarkable trajectories in the following tab 2.

$n = -1.5$ $e = 0.4$  $F = \frac{A}{\sqrt{r}} - Br^2$	$n = -{}^{1}\!/_{3}$ $e = 0.5$  $F = -\frac{A}{r^{5/3}} - \frac{B}{r^{1/3}}$
$n = 2.5$ $e = 0.5$  $F = -\frac{A}{r^{9/2}} + \frac{B}{r^6}$	$n = 3.5$ $e = 0.5$  $F = -\frac{A}{r^{11/2}} + \frac{B}{r^8}$

Tab 2. Several closed generalized conic trajectories and corresponding central forces.

Others solutions concern the second class of curves we have obtained, defined in relation (10) and (16). We present certain trajectories in following table:

$n=-2$ $e=0.5$  $F=-A+Br^3-\frac{C}{r^3}$	$n=-1.5$ $e=0.5$  $F=-\frac{A}{\sqrt{r}}+B-\frac{C}{r^3}$
$n=2.5$ $e=0.5$  $F=-\frac{A}{r^{9/2}}+\frac{B}{r^6}-\frac{C}{r^3}$	$n=4$ $e=0.5$  $F=-\frac{A}{r^6}+\frac{B}{r^9}-\frac{C}{r^3}$

Tab 3. Several associated curves and central forces

4.2 Astrodynamics

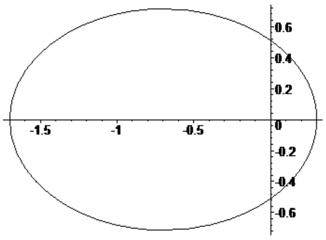
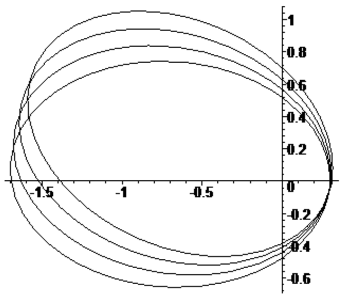
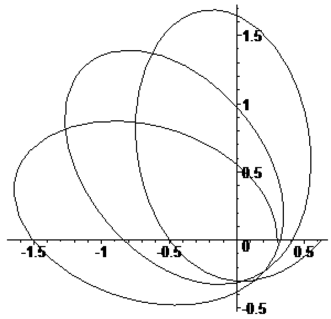
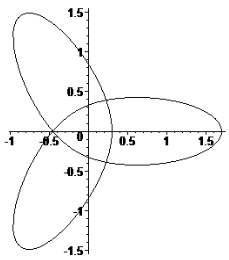
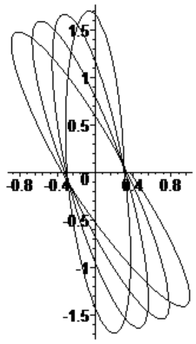
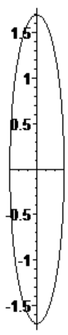
The determination of central forces depending on the radial distance isn't "only" a theoretical concern but is a major preoccupation of the astrodynamics. Indeed, the determination of exact analytical solutions making it possible to describe orbits represents progress in this field. For example, this kind of results is relevant for the determination of spacecraft trajectories propelled by thrust system. The thrust required to follow this trajectory can then, in certain cases, be computed. For this reason, works are regularly published in the literature devoted to astrodynamics. See for example, among them, references [17] but also [19] and [20].

About our work, some trajectories present an interest in this field. Indeed, comparable trajectories have already been studied, for example the logaritmic spiral [21] and more generally the curves of Cotes and their generalization [19]. A study of another fundamental case, evoked in this paper, the generalized cardioïd, can be found in [20], But, if we have reobtained a part of these trajectories, we have, in addition, presented a large number of other exact solutions. In particular, the trajectories we obtained offer a wide variety of possibilities. For example, they partly depend on the eccentricity of the initial conic and can therefore be quasi-circular or parabolic. Moreover, some of them are closed and are therefore periodic. For these reasons, This class of curves can present a progress in the field, despite the fact that we cannot know which trajectory exactly could be particularly interesting.

4.3 Celestial Mechanics: Precession phenomenon

Another interest concerns the precession of orbits of celestial bodies. The study of these precessions is naturally a major preoccupation of the celestial mechanics. For example, considering an historical point of view, the attempts to understand the precession of the Moon's orbits led to the establishment of the famous revolving theorem, which is still a contemporary preoccupation of the scientific community [4]. Another classical example concerns naturally the precession of Mercury and the validation of General Gravity. More recently, this phenomenon remains crucial to test the large varieties of alternative theories of Gravity, which have been published these last decades. It is the reason for why, nowadays, papers are regularly published about this question. For example, references [18] concerns the modern investigations about consequences of alternative potential of gravity (for example logarithmic potential, or others) on trajectories, especially on precession phenomenon.

This preoccupation is also present in our work. Indeed, we have showed that a precession around one of the foci of a Keplerian conic is due to a modification of the law of central force. For example, in Tab.4 we present the precession corresponding on the progressive transformation of $n\ \ 0 \to -1$:

$n = 0$  Initial conic - Keplerian orbit $a = 1, \quad e = 0.7$ $F = -\frac{A}{r^2}$	$n = -0.02$  precession around the foci of the ellipse under the action of modified central force	$n = -0.1$ 
$n = -0.5$  $F = -\frac{A}{r^{3/2}} - B$	$n = -0.9$  Precession this time around the center of the ellipse	$n = -1$  Harmonic oscillator $F = -Ar$ Eccentricity e_H of the conic is linked to e by (11)

Consider now the expression of the force given by

$$F = -C^2 {C'_1}^{2n-2}\left[(1+n)\frac{{C'_1}^{1-n}}{r^{2+n}} - n\frac{(1-e'^2)}{r^{2n+1}}\right]$$

And make a series for $n \to 0$ at the first order. We obtain an expression of the central force which can be written under the form

$$F = -\frac{A}{r^2} + n\frac{B}{r^2}\left[Ln\frac{r}{C'_1} + \frac{r}{C'_1}(1-e'^2)\right]$$

The expression is thus corresponding on the precession presented in the paper, when this kind of small perturbation is added to the classical inverse square force.

5. Conclusion and perspective

Several original results have been established. Generalizing a property of an initial conic, we have obtained the mathematical expression of an original class of curves, among them certain well known cases. This class of curves presents interesting geometric properties, in particular an internal inversion relationship. Moreover, it is possible to define a family of associated curves, which describes the fundamental geometric properties of the curves.

Knowing the mathematical expressions of the curves, we have extracted two laws of central forces which are, at our knowledge, unknown in the general case. One of them allows to link in only one formulae three fundamental potentials. These results contribute to improve the knowledge of the relationships between trajectories and central forces. They may have physical applications, particularly in the fields of celestial mechanics and astrodynamics. But more generally, they can represent progress in some fundamental areas such applied mathematics, geometry and general Physics.

In this general paper, we have not study these trajectories comprehensively. Later works should develop the mathematical study of the class of curves, with likely a particular attention to the geometric transformation. The study of the dynamics of trajectories could also be deepened, with particular attention to the possible physical applications.

References

[1] D.Lynden-Bell, S.Jin, *Analytic central orbits and their transformation group* MNRAS on. Not. R. Astron. Soc. 000, 1–12 (2002) ; [
[2] R.W.Hall, K.S.Josi, *Planetary Motion and the Duality of Force Laws*, SIAM Review, Vol 42 N°1 pp.115-124 (2000)
[3] P.Collas, *Equivalent potentials in classical Physics* Journal of mathematical Physics 22, 2512 (1981)
[4] D.Lynden-Bell, S.Jin, *On the shapes of Newton's Revolving Orbits* Not. R. Astron. Soc., 51 (2) 195-198 (1997)
[5] T.M.Apostol, M.A.Mnatsakanian, *New Horizons in Geometry*. The Mathematical Association of America. (2012)
[6] C. lebossé, C. Hémery , Géométrie, classe de mathématiques, Fernand Nathan, 1965, 23e leçon : foyers et directrices, propriétés diverses, n° 603 page 409
[7] Cardioide (mathcurve.com)
[8] Construction de la tangente en un point – GeoGebra
[9] Lemniscate de Bernoulli (mathcurve.com)
[10] Construction à la règle et au compas de la tangente à une ellipse, sans utiliser les foyers(pagesperso-orange.fr) http://alain.pichereau.pages.perso-orange.fr
[11] M.Nauenberg, « Newton's graphical method for central force orbit » American Journal of Physics 86, 765 (2018)
[12] Siacci.F. *moto per linea piana* Atti. R. Accad. Sci. Torino Vol.14 750 – 760, 946 – 951 (1879)
[13] Casey J. *Siacci's resolution of the acceleration vector for a space curve* Meccanica 46: 471–476 (2011)
[14] Fundamental of celestial mechanics J.M.A Danby.MacMillan, New York 1962
[15] [1] Bertrand J C.R.Acad. Sci. Paris (1873) 77 849
[16] Whittaker, E. T., A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, fourth edition, Dover, New York, 1944.

[17] R. Broucke « *Notes on the central forces* r^n » Astrophysics and Space Science 72 : 33 – 53 (1981) ;
[18] F.M. Mahomed, F. Vawda, « *Applications of Symmetries to Central Force Problems* » Nonlinear Dynamics 21 : 307 – 315 (2000)
[19] Roa, J., Pelàez J., Senent, *"New analytic solution with continuous thrust: Generalized logarithmic spirals"* Journal of Guidance, Control, and Dynamics, 39 (10), 2336-2351 (2016)
[20] Roa, J., ' *Nonconservative extension of Keplerian integrals and a new class of integrable* system' MNRAS, Vol. 463, Issue 3, Pages 3204–3219 (2016)
[21] Ralph Hoyt Bacon "*Logarithmic spiral: An ideal trajectory for the interplanetary vehicle with engines of low sustained thrust"*, American Journal of Physics 27, 164 (1959)
[22] G. S. Adkins and J. McDonnell ``*Orbital precession due to central-force perturbations''*, Phys. Rev. D75 (2007), 082001 ;