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Γ-CONVERGENCE OF NONCONVEX UNBOUNDED INTEGRALS IN STRONGLY CONNECTED SETS

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Abstract. We study Γ-convergence of nonconvex integrals of the calculus of variations in strongly connected sets when the integrands have not polynomial growth and can take infinite values. Application to homogenization of unbounded integrals in strongly perforated sets is also developed.

Contents

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Key words and phrases. Γ-convergence, strong connectedness, perforated sets, nonconvex unbounded integrals, homogenization.

1. Introduction

Asymptotic analysis for boundary value problems in perforated sets was studied for the first time by Cioranescu and Saint Jean Paulin (see [CSJP99] and the reference therein) and Khruslov and Marchenko (see [MK06] and the reference therein). The approach of Cioranescu and Saint Jean Paulin is based upon multiscale methods like formal two-scale asymptotic expansions, compensated compactness and oscillating test functions and two-scale convergence (see [JKO94, CD99, CPS07]) while the one of Khruslov and Marchenko uses variational analysis like Γ-convergence (see [DM93, BD98]). The common point of the two approaches is the use of extension theorems for passing from perforated to non-perforated sets (see [CP79, ACPDMP92]). In this paper we consider the variational approach which consists of computing the Γ-limit as $\varepsilon \to 0$ of integral functionals of type

$$\int_O W_\varepsilon(x, \nabla u(x))\mathbb{1}_{O_\varepsilon}(x)dx, \tag{1.1}$$

where $u \in W^{1,p}(O;\mathbb{R}^m)$ with $p > 1$ and, for each $\varepsilon > 0$, $W_\varepsilon : O \times \mathbb{M} \to [0,\infty]$ is a not necessarily convex Borel measurable function with $\mathbb{M}$ denoting the space of $m \times N$ matrices and $O \backslash O_\varepsilon$ represents the holes at the scale ε in the bounded open set $O \subset \mathbb{R}^N$. Following Khruslov and Marchenko, the Γ-convergence is taken with respect to the $L^p(O_\varepsilon, O)$-convergence, i.e.

$$u_\varepsilon \overset{L^p(O_\varepsilon,O)}{\longrightarrow} u \text{ if and only if } \lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{L^p(O_\varepsilon;\mathbb{R}^m)} = 0,$$

and our object is to deal with the problem of finding a $\Gamma(L^p(O_\varepsilon, O))$-limit of type

$$\int_O W_{\lim}(x, \nabla u(x))dx \tag{1.2}$$

with $W_{\lim} : O \times \mathbb{M} \to [0,\infty]$ an integrand for which we wish to give a formula depending on W_ε. When W_ε has p-growth, i.e. there exist $\alpha, \beta > 0$, which does not depend on ε, such that for every $(x,\xi) \in O \times \mathbb{M}$,

$$\alpha|\xi|^p\mathbb{1}_{O_\varepsilon}(x) \leqslant W_\varepsilon(x,\xi)\mathbb{1}_{O_\varepsilon}(x) \leqslant \beta(1 + |\xi|^p)\mathbb{1}_{O_\varepsilon}(x),$$

the problem was treated (for general periodic perforated sets) by Acerbi, Chiadò Piat, Dal Maso and Percivale in the scalar case (see [ACPDMP92]) and by Braides and Chiadò Piat in the vector-valued case (see [BCP94]). In the present paper we consider the case where $p > N$ and W_ε has G-growth, i.e. there exists a Borel measurable and p-coercive function $G : \mathbb{M} \to [0,\infty]$ and there exist $\alpha, \beta > 0$, which does not depend on ε, such that for every $(x,\xi) \in O \times \mathbb{M}$,

$$\alpha G(\xi)\mathbb{1}_{O_\varepsilon}(x) \leqslant W_\varepsilon(x,\xi)\mathbb{1}_{O_\varepsilon}(x) \leqslant \beta(1 + G(\xi))\mathbb{1}_{O_\varepsilon}(x),$$

which allows to W_ε to take infinite values. Note that, as in the p-growth case, since G is p-coercive, there exists $C > 0$, which does not depend on ε, such that for every $(x,\xi) \in O \times \mathbb{M}$,

$$W_\varepsilon(x,\xi)\mathbb{1}_{O_\varepsilon}(x) \geqslant C|\xi|^p\mathbb{1}_{O_\varepsilon}(x). \tag{1.3}$$

Such a unbounded case is of interest in nonlinear elasticity where a fundamental open problem is to develop variational techniques to deal with energy densities that can take infinite values and verify the two basic conditions of nonlinear elasticity, namely the noninterpenetration of the matter and the neccessity of an infinite amount of energy to compress a finite volume into zero volume. The results of this paper give some improvments in this direction in the framework of perforated nonlinear elastic materials.

The plan of the paper is as follows. In §2.1 we recall the definition of $L^p(O_\varepsilon, O)$-convergence (see Definition 2.1) and the one of strongly connected set (see Definition 2.3). Note that a weaker notion of connected set exists (see [CSJP99, Chapter 1, §2.5 pp. 40] and [BD98, Chapter 19 pp. 167] for more details). This weak notion allows to consider more general perforated sets but it is not considered here due to the fact that our Γ-convergence method does not apply in such a situation. In fact, the Γ-convergence of unbounded integrals in weakly connected sets is an open problem. When O_ε is strongly connected, bounded sequences in $W^{1,p}(O_\varepsilon;\mathbb{R}^m)$ are relatively compact with respect to the $L^p(O_\varepsilon, O)$-convergence (see Theorem 2.4). This makes that $\Gamma(L^p(O_\varepsilon, O))$-convergence (see Definition 2.5 in §2.2) is well adapted to deal with variational problems involving integral functionals of type (1.1) satisfying (1.3) (see Proposition 2.7 in §2.2).
Our main result, which establishes the $\Gamma(L^p(O_\varepsilon, O))$-convergence of (1.1) to (1.2), is stated in §3.1 and proved in §5.3, see Theorem 3.6 and also Proposition 3.7 which makes more precise the formula of the limit integrand $W_{\lim}$ in (1.2). Classically, its proof is a consequence of Proposition 3.4 (the lower bound) and Proposition 3.5 (the upper bound).
The proofs of Propositions 3.4 and 3.5 are given in §5.1 and §5.2 respectively. In §2.3 we recall the concept of (family of) ru-usc[1] integrand(s) and its main properties which are used in the proof of both Propositions 3.4 and 3.5 (and in Proposition 3.7). The proof of Proposition 3.5 also needs the use of the Vitali envelope of a set function which is recalled in §4.1.
Finally, application to homogenization of unbounded integrals in strongly perforated sets is developed in §3.2, see Theorem 3.11. This homogenization theorem is proved in §5.4 by using an extension theorem (see Theorem 3.8) and a subadditive theorem (see Theorem 4.4 in §4.2).

Notation. Let $\mathbb{M}$ be the space of $m \times N$ matrices, for $A \subset \mathbb{M}$ we denote the interior and the closure of A by $\mathrm{int}(A)$ and $\overline{A}$ respectively.
The symbol $⨍$ stands for the mean-value integral with respect to the Lebesgue measure $\mathscr{L}^N$ on $\mathbb{R}^N$, i.e. $⨍_Q = \frac{1}{\mathscr{L}^N(Q)} \int_Q$.

2. Preliminaries

In what follows, $m, N \geqslant 1$ are two integers and $p > 1$ is a real number.

2.1. $L^p(O_\varepsilon, O)$**-convergence and strong connectedness.** Let $O \subset \mathbb{R}^N$ be a bounded open set and, for each $\varepsilon > 0$, let $O_\varepsilon \subset O$ be an open set. In what follows we consider the following two conditions:

(C$_1$) $\lim\limits_{\varepsilon\to 0} \mathscr{L}^N(O \backslash O_\varepsilon) = 0$;

[1]The abbreviation ru-usc means radially uniformly upper semicontinuous.

(C_2) there exists $C > 0$ such that for all $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(O;\mathbb{R}^m)$ there exists $\{\widehat{u}_\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(O;\mathbb{R}^m)$ such that for all $\varepsilon > 0$,

$$\begin{cases} \widehat{u}_\varepsilon = u_\varepsilon \text{ on } O_\varepsilon \\ \|\widehat{u}_\varepsilon\|_{L^p(O;\mathbb{R}^m)} \leqslant C\|u_\varepsilon\|_{L^p(O_\varepsilon;\mathbb{R}^m)} \\ \|\nabla\widehat{u}_\varepsilon\|_{L^p(O;\mathbb{R}^m)} \leqslant C\|\nabla u_\varepsilon\|_{L^p(O_\varepsilon;\mathbb{R}^m)}. \end{cases}$$

The following two definitions are due to Khruslov and Marchenko (see [MK06, Chapter 4, Definition 4.5 pp. 114 and Definition 4.7 pp. 116]).

Definition 2.1. We say that $\{u_\varepsilon\}_{\varepsilon>0} \subset L^p(O;\mathbb{R}^m)$ is $L^p(O_\varepsilon, O)$-convergent if there exists $u \in L^p(O;\mathbb{R}^m)$ such that

$$\lim_{\varepsilon\to 0}\|u_\varepsilon - u\|_{L^p(O_\varepsilon;\mathbb{R}^m)} = 0, \text{ i.e. } \lim_{\varepsilon\to 0}\int_{O_\varepsilon}|u_\varepsilon - u|^p dx = 0.$$

We then write $u_\varepsilon \overset{L^p(O_\varepsilon,O)}{\longrightarrow} u$.

Remark 2.2. Under (C_1) if $\{u_\varepsilon\}_{\varepsilon>0} \subset L^p(O;\mathbb{R}^m)$ is $L^p(O_\varepsilon, O)$-convergent then its limit is unique.

Definition 2.3. When (C_1)–(C_2) hold we say that $\{O_\varepsilon\}_{\varepsilon>0}$ is *p-strongly connected.*

Khruslov and Marchenko have also proved the following compactness result (see [MK06, Chapter 4, Theorem 4.8 pp. 116]).

Theorem 2.4. *Assume that $\{O_\varepsilon\}_{\varepsilon>0}$ is p-strongly connected. If $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(O;\mathbb{R}^m)$ and if $\sup_{\varepsilon>0}\|u_\varepsilon\|_{W^{1,p}(O_\varepsilon;\mathbb{R}^m)} < \infty$ then, up to a subsequence, there exists $u \in W^{1,p}(O;\mathbb{R}^m)$ such that $u_\varepsilon \overset{L^p(O_\varepsilon,O)}{\longrightarrow} u$.*

2.2. **$\Gamma(L^p(O_\varepsilon, O))$-convergence.** We begin with the definition of $\Gamma(L^p(O_\varepsilon, O))$-convergence. (For more details on the theory of Γ-convergence we refer to [DM93].)

Definition 2.5. By the $\Gamma(L^p(O_\varepsilon, O))$-limit of $I_\varepsilon : W^{1,p}(O;\mathbb{R}^m) \to [0,\infty]$ as $\varepsilon \to 0$ we mean a functional $I_{\lim} : W^{1,p}(O;\mathbb{R}^m) \to [0,\infty]$ such that:

Γ-$\underline{\lim}$: for every $u \in W^{1,p}(O;\mathbb{R}^m)$, $\Gamma(L^p(O_\varepsilon, O))\text{-}\varliminf_{\varepsilon\to 0} I_\varepsilon(u) \geqslant I_{\lim}(u)$ with

$$\Gamma(L^p(O_\varepsilon, O))\text{-}\varliminf_{\varepsilon\to 0} I_\varepsilon(u) := \inf\left\{\varliminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) : u_\varepsilon \overset{L^p(O_\varepsilon,O)}{\longrightarrow} u\right\},$$

or equivalently, for every $u \in W^{1,p}(O;\mathbb{R}^m)$ and every $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(O;\mathbb{R}^m)$ such that $u_\varepsilon \overset{L^p(O_\varepsilon,O)}{\longrightarrow} u$,

$$\varliminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \geqslant I_{\lim}(u);$$

Γ-$\overline{\lim}$: for every $u \in W^{1,p}(O;\mathbb{R}^m)$, $\Gamma(L^p(O_\varepsilon, O))\text{-}\varlimsup_{\varepsilon\to 0} I_\varepsilon(u) \leqslant I_{\lim}(u)$ with

$$\Gamma(L^p(O_\varepsilon, O))\text{-}\varlimsup_{\varepsilon\to 0} I_\varepsilon(u) := \inf\left\{\varlimsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) : u_\varepsilon \overset{L^p(O_\varepsilon,O)}{\longrightarrow} u\right\},$$

or equivalently, for every $u \in W^{1,p}(O;\mathbb{R}^m)$ there exists $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(O;\mathbb{R}^m)$ such that $u_\varepsilon \overset{L^p(O_\varepsilon,O)}{\longrightarrow} u$ and

$$\varlimsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \leqslant I_{\lim}(u).$$

We then write $I_{\lim} = \Gamma(L^p(O_\varepsilon, O))\text{-}\lim\limits_{\varepsilon\to 0} I_\varepsilon$.

Remark 2.6. It is easy to see that $\Gamma(L^p(O_\varepsilon, O))\text{-}\underline{\lim}_{\varepsilon\to 0} I_\varepsilon$ and $\Gamma(L^p(O_\varepsilon, O))\text{-}\overline{\lim}_{\varepsilon\to 0} I_\varepsilon$ are lsc[2] with respect to the $L^p(O;\mathbb{R}^m)$-convergence.

Theorem 2.4 makes that $\Gamma(L^p(O_\varepsilon, O))$-convergence is well adapted to deal with variational problems involving integral functionals of type (1.1) satisfying (1.3). This is summarized in the following proposition whose proof is left to the reader.

Proposition 2.7. *Assume that $\{O_\varepsilon\}_{\varepsilon>0}$ is p-strongly connected, $\Gamma(L^p(O_\varepsilon, O))\text{-}\lim_{\varepsilon\to 0} I_\varepsilon = I_{\lim}$ and there exists $C > 0$ such that for every $\varepsilon > 0$ and $u \in W^{1,p}(O;\mathbb{R}^m)$,*

$$I_\varepsilon(u) \geqslant C\int_{O_\varepsilon} |\nabla u(x)|^p dx.$$

Let $f \in L^q(O;\mathbb{R}^m)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and, for each $\varepsilon > 0$, set

$$\theta_\varepsilon := \inf\left\{ I_\varepsilon(u) - \int_{O_\varepsilon} f(x)u(x)dx : u \in W^{1,p}(O;\mathbb{R}^m)\right\}.$$

Then, every minimizing sequence $\{u_\varepsilon\}_{\varepsilon>0}$ for the variational problems θ_ε is relatively compact with respect to the $L^p(O_\varepsilon, O)$-convergence, and every $L^p(O_\varepsilon, O)$-cluster point $\overline{u}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ is such that

$$\lim_{\varepsilon\to 0}\theta_\varepsilon = I_{\lim}(\overline{u}) - \int_O f(x)\overline{u}(x)dx = \inf\left\{ I_{\lim}(u) - \int_O f(x)u(x)dx : u \in W^{1,p}(O;\mathbb{R}^m)\right\}.$$

2.3. **Ru-usc property.** We begin by recalling the concept of ru-usc function which was introduced in [AH10] (see also [AHM14] and [AHM11, §3.1]).

2.3.1. *Ru-usc function.* Let $O \subset \mathbb{R}^N$ be an open set and $L : O \times \mathbb{M} \to [0,\infty]$ be a Borel measurable function, where $\mathbb{M}$ denotes the space of $m \times N$ matrices. For each $x \in O$, we denote the effective domain[3] of $L(x,\cdot)$ by $\mathbb{L}_x$ and, for each $a \in L^1_{\rm loc}(O;]0,\infty])$, we consider $\Delta^a_L : [0,1] \to]-\infty,\infty]$ defined by

$$\Delta^a_L(t) := \sup_{x\in O}\sup_{\xi\in\mathbb{L}_x}\frac{L(x,t\xi) - L(x,\xi)}{a(x) + L(x,\xi)}. \tag{2.1}$$

Definition 2.8. We say that $L : O \times \mathbb{M} \to [0,\infty]$ is ru-usc if there exists $a \in L^1_{\rm loc}(O;]0,\infty])$ such that

$$\overline{\lim_{t\to 1^-}}\,\Delta^a_L(t) \leqslant 0. \tag{2.2}$$

The interest of Definition 2.8 comes from the following theorem. (For a proof we refer to [AHM11, Theorem 3.5] and also [AHM12, §4.2]) Let $\widehat{L} : O \times \mathbb{M} \to [0,\infty]$ be defined by

$$\widehat{L}(x,\xi) := \underline{\lim_{t\to 1^-}}\, L(x,t\xi).$$

[2]The abbreviation lsc means lower semicontinuous.

[3]Given a set X and a function $f : X \to [0,\infty]$, by the effective domain of f we mean the set of $z \in X$ such that $f(z) < \infty$.

Theorem 2.9. *If $L : O \times \mathbb{M} \to [0,\infty]$ is ru-usc and if for every $x \in O$,*

$$t\overline{\mathbb{L}_x} \subset \mathrm{int}(\mathbb{L}_x) \text{ for all } t \in]0,1[, \tag{2.3}$$

then:

(a) $\widehat{L}$ *is ru-usc*;
(b) $\widehat{L}(x,\xi) = \lim\limits_{t\to 1^-} L(x,t\xi)$ *for all* $(x,\xi) \in O \times \mathbb{M}$.

If moreover, for every $x \in O$, $L(x,\cdot)$ is lsc on $\mathrm{int}(\mathbb{L}_x)$ then:

(c) $$\widehat{L}(x,\xi) = \begin{cases} L(x,\xi) & \text{if } \xi \in \mathrm{int}(\mathbb{L}_x) \\ \lim\limits_{t\to 1^-} L(x,t\xi) & \text{if } \xi \in \partial\mathbb{L}_x \\ \infty & \text{otherwise}; \end{cases}$$

(d) *for every $x \in O$, $\widehat{L}(x,\cdot)$ is the lsc envelope of $L(x,\cdot)$.*

2.3.2. *Family of ru-usc functions.* The following definition generalizes Definition 2.8 to the case of a family of functions.

Definition 2.10. For each $\varepsilon > 0$, let $L_\varepsilon : O \times \mathbb{M} \to [0,\infty]$ be a Borel measurable function. We say that the family $\{L_\varepsilon\}_{\varepsilon>0}$ is ru-usc if there exist $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(O;]0,\infty])$ and $a_0 \in L^1(O;]0,\infty])$ such that:

$$\lim_{\varepsilon\to 0}\int_A a_\varepsilon(x)dx = \int_A a_0(x)dx \text{ for all } A \in \mathcal{O}(O);$$
$$\overline{\lim_{t\to 1^-}}\sup_{\varepsilon>0}\Delta^{a_\varepsilon}_{L_\varepsilon}(t) \leqslant 0.$$

The following lemma will be useful for dealing with $\Gamma(L^p(O_\varepsilon,O))$-convergence. (For a proof we refer to [AHM21, Lemma 2.21].)

Lemma 2.11. *For each $\varepsilon > 0$, let $L_\varepsilon : O \times \mathbb{M} \to [0,\infty]$ be a Borel measurable function and, for each $\rho > 0$, let $\mathscr{H}^\rho[L_\varepsilon] : O \times \mathbb{M} \to [0,\infty]$ be defined by*

$$\mathscr{H}^\rho[L_\varepsilon](x,\xi) := \inf\left\{ \fint_{Q_\rho(x)} L_\varepsilon(y,\xi+\nabla v(y))dy : v \in W^{1,p}_0(Q_\rho(x);\mathbb{R}^m)\right\} \tag{2.4}$$

with $Q_\rho(x) := x+]-\frac{\rho}{2},\frac{\rho}{2}[^N$. If $\{L_\varepsilon\}_{\varepsilon>0}$ is ru-usc with $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(O;]0,\infty])$ and $a_0 \in L^\infty(O;]0,\infty])$ then

$$L_\infty := \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\mathscr{H}^\rho[L_\varepsilon] : O \times \mathbb{M} \to [0,\infty]$$

is ru-usc with the constant function $\|a_0\|_{L^\infty}$.

For application to homogenization (see §3.2) we will need the following result. (For a proof we refer to [AHM21, Lemma 2.24].)

Lemma 2.12. *Let $L : \mathbb{R}^N \times \mathbb{M} \to [0,\infty]$ be Borel measurable function such that $L(\cdot,\xi)$ is 1-periodic for all $\xi \in \mathbb{M}$, i.e. for every $(x,z) \in \mathbb{R}^N \times \mathbb{Z}^N$, $L(x+z,\xi) = L(x,\xi)$, and, for each $\varepsilon > 0$, let $L_\varepsilon : O \times \mathbb{M} \to [0,\infty]$ be defined by*

$$L_\varepsilon(x,\xi) := L\left(\frac{x}{\varepsilon},\xi\right).$$

Let $a \in L^1_{\rm loc}(\mathbb{R}^N;]0,\infty])$ be a 1-periodic function and, for each $\varepsilon > 0$, let $a_\varepsilon \in L^1_{\rm loc}(\mathbb{R}^N;]0,\infty])$ be defined by

$$a_\varepsilon(x) := a\left(\frac{x}{\varepsilon}\right).$$

If L is ru-usc with the function a then $\{L_\varepsilon\}_{\varepsilon>0}$ is ru-usc with the family of functions $\{a_\varepsilon\}_{\varepsilon>0}$ and the constant function $\langle a\rangle := \int_{]0,1[^N} a(y)dy$.

3. Main results

3.1. **The $\Gamma(L^p(O_\varepsilon, O))$-convergence result.** Let $O \subset \mathbb{R}^N$ be a bounded open set and, for each $\varepsilon > 0$, let $O_\varepsilon \subset O$ be an open set and consider the following condition:

(C$_0$) for all $\varepsilon > 0$ and all open cube $Q \subset O$, if $\mathscr{L}^N(O_\varepsilon \cap Q) > 0$ then $\mathscr{H}^{N-1}(O_\varepsilon \cap \partial Q) > 0$;

Let $\mathbb{M}$ denote the space of $m \times N$ matrices and let $G : \mathbb{M} \to [0,\infty]$ be a Borel measurable function satisfying the following conditions:

(A$_0$) $0 \in \mathrm{int}(\mathbb{G})$, where $\mathbb{G}$ denotes the effective domain of G, i.e. $\mathbb{G} := \{\xi \in \mathbb{M} : G(\xi) < \infty\}$;

(A$_1$) there exists $\gamma > 0$ such that for every $\xi, \zeta \in \mathbb{M}$ and every $t \in]0,1[$,

$$G(t\xi + (1-t)\zeta) \leqslant \gamma(1 + G(\xi) + G(\zeta));$$

(A$_2$) G is p-coercive, i.e. there exists $c > 0$ such that for every $\xi \in \mathbb{M}$,

$$G(\xi) \geqslant c|\xi|^p.$$

Remark 3.1. If (A$_1$) is satisfied then $\mathbb{G}$ is convex, but G is not necessarily convex (see [AHMZ15, Sect. 9]). So, if moreover (A$_0$) holds then

$$t\overline{\mathbb{G}} \subset \mathrm{int}(\mathbb{G}) \text{ for all } t \in]0,1[,$$

and there exists $r > 0$ such that

$$\sup_{|\xi|\leqslant r} G(\xi) < \infty,$$

see [AHM12, Lemma 4.1].

Let $G_\infty : O \times \mathbb{M} \to [0,\infty]$ be defined by

$$G_\infty(x,\xi) := \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\mathscr{H}^\rho[G\mathbb{1}_{O_\varepsilon}](x,\xi) \tag{3.1}$$

with $\mathscr{H}^\rho[G\mathbb{1}_{O_\varepsilon}]$ given by (2.4) with $L_\varepsilon = G\mathbb{1}_{O_\varepsilon}$ where $G\mathbb{1}_{O_\varepsilon} : O \times \mathbb{M} \to [0,\infty]$ is defined by $G\mathbb{1}_{O_\varepsilon}(x,\xi) = G(\xi)\mathbb{1}_{O_\varepsilon}(x)$. Denote the effective domain of $G_\infty(x,\cdot)$ by $\mathbb{G}_{\infty,x}$. We further suppose that:

(A$_3$) for every $u \in W^{1,p}(O;\mathbb{R}^m)$, if $\int_O G_\infty(x,\nabla u(x))dx < \infty$ and if $\nabla u(x) \in \mathrm{int}(\mathbb{G}_{\infty,x})$ for $\mathscr{L}^N$-a.a. $x \in O$, then $\int_O G(\nabla u(x))dx < \infty$;

(A$_4$) for every $x \in O$, $G_\infty(x,\cdot)$ is lsc on $\mathrm{int}(\mathbb{G}_{\infty,x})$.

Remark 3.2. (i) For every $(x,\xi) \in O \times \mathbb{M}$, $G_\infty(x,\xi) \leqslant G(\xi)$, and so $\mathbb{G} \subset \mathbb{G}_{\infty,x}$ for all $x \in O$.

(ii) Defining $\mathscr{G}, \mathscr{G}_\infty : W^{1,p}(O;\mathbb{R}^m) \to [0,\infty]$ by $\mathscr{G}(u) := \int_O G(\nabla u(x))dx$ and $\mathscr{G}_\infty(u) := \int_O G_\infty(x, \nabla u(x))dx$ and denoting their effective domains by $\mathrm{dom}(\mathscr{G})$ and $\mathrm{dom}(\mathscr{G}_\infty)$, we see that (A$_3$) means that

$$\Big\{u \in \mathrm{dom}(\mathscr{G}_\infty) : \nabla u(x) \in \mathrm{int}(\mathbb{G}_{\infty,x}) \text{ for } \mathscr{L}^N\text{-a.a. } x \in O\Big\} \subset \mathrm{dom}(\mathscr{G}).$$

(iii) If either $\mathrm{dom}(\mathscr{G}_\infty) = \mathrm{dom}(\mathscr{G})$ or $\mathscr{G}(u) < \infty$ for all $u \in W^{1,p}(O;\mathbb{R}^m)$ such that $\nabla u(x) \in \mathrm{int}(\mathbb{G}_{\infty,x})$ for $\mathscr{L}^N$-a.a. $x \in O$, then (A$_3$) can be dropped.

(iv) If G satisfies (A$_1$) then G_∞ verifies the same condition, i.e. for every $x \in O$, every $\xi, \zeta \in \mathbb{M}$ and every $t \in]0,1[$,

$$G_\infty(x, t\xi + (1-t)\zeta) \leqslant \gamma(1 + G_\infty(x,\xi) + G_\infty(x,\zeta)),$$

and so $\mathbb{G}_{\infty,x}$ is convex for all $x \in O$. Hence, under (A$_0$)–(A$_1$), for every $x \in O$,

$$t\overline{\mathbb{G}_{\infty,x}} \subset \mathrm{int}(\mathbb{G}_{\infty,x}) \text{ for all } t \in]0,1[.$$

(v) If G is convex then $G_\infty(x,\cdot)$ is convex for all $x \in O$, and so (A$_4$) can be dropped. More Generally, if for every $x \in O$, $G_\infty(x,\cdot)$ is quasiconvex, i.e. for every $\xi \in \mathbb{M}$, $G_\infty(x,\xi) = \inf\Big\{\int_{]0,1[^N} G_\infty(x, \xi + \nabla\varphi(y))dy : \varphi \in W_0^{1,\infty}(]0,1[^N;\mathbb{R}^m)\Big\}$, then (A$_4$) can be dropped (see [Fon88]).

For each $\varepsilon > 0$, let $W_\varepsilon : O \times \mathbb{M} \to [0,\infty]$ be a Borel measurable function with the following G-growth condition:

(A$_5$) there exist $\alpha, \beta > 0$ such that for every $\varepsilon > 0$ and every $(x,\xi) \in O \times \mathbb{M}$,

$$\alpha G(\xi)\mathbb{1}_{O_\varepsilon}(x) \leqslant W_\varepsilon(x,\xi)\mathbb{1}_{O_\varepsilon}(x) \leqslant \beta(1 + G(\xi))\mathbb{1}_{O_\varepsilon}(x).$$

We further assume that

(A$_6$) $\{W_\varepsilon \mathbb{1}_{O_\varepsilon}\}_{\varepsilon>0}$ is ru-usc with $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(O;]0,\infty])$ and $a_0 \in L^\infty(O;]0,\infty])$, where $W_\varepsilon \mathbb{1}_{O_\varepsilon} : O \times \mathbb{M} \to [0,\infty]$ is defined by $(W_\varepsilon \mathbb{1}_{O_\varepsilon})(x,\xi) := W_\varepsilon(x,\xi)\mathbb{1}_{O_\varepsilon}(x)$.

Remark 3.3. As $\Delta^{a_\varepsilon}_{W_\varepsilon \mathbb{1}_{O_\varepsilon}}(t) \leqslant \max\{0, \Delta^{a_\varepsilon}_{W_\varepsilon}(t)\}$ for all $\varepsilon > 0$ and all $t \in [0,1]$, if $\{W_\varepsilon\}_{\varepsilon>0}$ is ru-usc with $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(O;]0,\infty])$ and $a_0 \in L^\infty(O;]0,\infty])$, then also is $\{W_\varepsilon \mathbb{1}_{O_\varepsilon}\}_{\varepsilon>0}$.

For each $\varepsilon > 0$ and each $\rho > 0$, let $\mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}] : O \times \mathbb{M} \to [0,\infty]$ be defined by (2.4) with $L_\varepsilon = W_\varepsilon \mathbb{1}_{O_\varepsilon}$, i.e.

$$\mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,\xi) := \inf\left\{ \fint_{Q_\rho(x)} W_\varepsilon(y, \xi + \nabla v(y))\mathbb{1}_{O_\varepsilon}(y)dy : v \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)\right\} \tag{3.2}$$

and consider the following assumption:

(A$_7$) for every $x \in O$ and every $\xi \in \mathrm{int}(\mathbb{G}_{\infty,x})$,

$$\overline{\lim_{\rho\to 0}}\ \underline{\lim_{\varepsilon\to 0}}\ \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,\xi) \geqslant \overline{\lim_{\rho\to 0}}\ \overline{\lim_{\varepsilon\to 0}}\ \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,\xi).$$

For each $\varepsilon > 0$, let $I_\varepsilon : W^{1,p}(O;\mathbb{R}^m) \to [0,\infty]$ be defined by

$$I_\varepsilon(u) := \int_{O_\varepsilon} W_\varepsilon(x, \nabla u(x))dx = \int_O W_\varepsilon(x,\nabla u(x))\mathbb{1}_{O_\varepsilon}(x)dx.$$

Here are the main results of the paper.

Proposition 3.4. *Assume that $p > N$. Under* (C_1)–(C_2) *and* (A_0)–(A_6) *we have*

$$\Gamma(L^p(O_\varepsilon, O))\text{-}\varliminf_{\varepsilon\to 0} I_\varepsilon(u) \geqslant \int_O \varliminf_{t\to 1^-} \varlimsup_{\rho\to 0} \varliminf_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x, t\nabla u(x))dx$$

for all $u \in W^{1,p}(O;\mathbb{R}^m)$.

Proposition 3.5. *Assume that $p > N$. Under* (C_0)–(C_2) *and* (A_0)–(A_6) *we have*

$$\Gamma(L^p(O_\varepsilon, O))\text{-}\varlimsup_{\varepsilon\to 0} I_\varepsilon(u) \leqslant \int_O \varliminf_{t\to 1^-} \varlimsup_{\rho\to 0} \varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x, t\nabla u(x))dx$$

for all $u \in W^{1,p}(O;\mathbb{R}^m)$.

As a consequence of Propositions 3.4 and 3.5 we have

Theorem 3.6. *Assume that $p > N$. Under* (C_0)–(C_2) *and* (A_0)–(A_7) *we have*

$$\Gamma(L^p(O_\varepsilon, O))\text{-}\lim_{\varepsilon\to 0} I_\varepsilon(u) = \int_O W_{\lim}(x, \nabla u(x))dx$$

for all $u \in W^{1,p}(O;\mathbb{R}^m)$ with $W_{\lim} : O \times \mathbb{M} \to [0,\infty]$ given by

$$W_{\lim}(x,\xi) := \varliminf_{t\to 1^-} \varlimsup_{\rho\to 0} \varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x, t\xi).$$

Let $W_\infty : O \times \mathbb{M} \to [0,\infty]$ be defined by

$$W_\infty(x,\xi) := \varlimsup_{\rho\to 0} \varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x, \xi).$$

Let $\widehat{W}_\infty : O \times \mathbb{M} \to [0,\infty]$ be given by

$$\widehat{W}_\infty(x,\xi) := \varliminf_{t\to 1^-} W_\infty(x, t\xi)$$

and, for each $x \in O$, let $\overline{W}_\infty(x,\cdot)$ denotes the lsc envelope of $W_\infty(x,\cdot)$. The following proposition makes more precise the formula of the limit integrand $W_{\lim}$ in Theorem 3.6.

Proposition 3.7. *Assume that* (A_0)–(A_1) *and* (A_5)–(A_6) *hold.*

(i) *For every $x \in O$,*

$$\widehat{W}_\infty(x,\xi) = \lim_{t\to 1^-} W_\infty(x, t\xi) = \begin{cases} \lim\limits_{t\to 1^-} W_\infty(x,t\xi) & \text{if } \xi \in \overline{\mathbb{G}_{\infty,x}} \\ \infty & \text{otherwise.} \end{cases}$$

So, in Theorem 3.6 we have $W_{\lim} = \widehat{W}_\infty$.

(ii) *Suppose furthermore that for each $x \in O$, $W_\infty(x,\cdot)$ is lsc on $\mathrm{int}(\mathbb{G}_{\infty,x})$. Then*

$$\widehat{W}_\infty(x,\xi) = \overline{W}_\infty(x,\xi) = \begin{cases} W_\infty(x,\xi) & \text{if } \xi \in \mathrm{int}(\mathbb{G}_{\infty,x}) \\ \lim\limits_{t\to 1^-} W_\infty(x,t\xi) & \text{if } \xi \in \partial\mathbb{G}_{\infty,x} \\ \infty & \text{otherwise.} \end{cases} \tag{3.3}$$

In such a case, in Theorem 3.6, $W_{\lim}$ is given by (3.3).

Proof of Proposition 3.7. From (A_6) and Lemma 2.11, we can assert that W_∞ is ru-usc. Moreover, by (A_5) it is easily seen that for every $x \in O$, the effective domain of $W_\infty(x, \cdot)$ is equal to $\mathbb{G}_{\infty,x}$. So, taking (A_0)–(A_1) (see Remark 3.2(iii)) into account, Proposition 3.7 follows from Theorem 2.9. ■

3.2. **Homogenization of unbounded integrals in strongly perforated sets.** Let $Y := [0, 1[^N$, let $H \subset Y$ be an open set with Lipschitz boundary, which represents the holes in Y, such that

(H_0) $\overline{H} \cap \partial Y = \varnothing$, i.e. the holes do not intersect the boundary ∂Y, and $Y \backslash H$ is connected.

Let $E := Y\backslash H + \mathbb{Z}^N$ and, for each $\varepsilon > 0$, we denote the ε-homothetic set of E by E_ε, i.e.

$$E_\varepsilon := \left\{ x \in \mathbb{R}^N : \frac{x}{\varepsilon} \in E \right\},$$

and we set $O_\varepsilon := O \cap E_\varepsilon$. (Note that E is 1-periodic, i.e. for every $(x, z) \in \mathbb{R}^N \times \mathbb{Z}^N$, $\mathbb{1}_E(x + z) = \mathbb{1}_E(x)$.) We further assume that

(H_1) for every $\varepsilon > 0$, $(\mathbb{R}^N\backslash E_\varepsilon) \cap \partial O = \varnothing$, i.e. the holes do not intersect the boundary ∂O.

In this framework, it is clear that (C_1) in §2.1 and (C_0) in §3.1 are satisfied. Moreover, we have the following extension result due to Cioranescu and Saint Jean Paulin [CP79] (see also [CSJP99, §2.3 pp. 25]).

Theorem 3.8. *If* (H_0)–(H_1) *hold then* (C_2) *in* §2.1 *holds.*

Let $\mathbb{M}$ denote the space of $m \times N$ matrices and let $G : \mathbb{M} \to [0, \infty]$ be a p-coercive convex function and let $G_\infty : O \times \mathbb{M} \to [0, \infty]$ be defined by (3.1). We suppose that:

(H_2) for every $x \in O$, $\mathrm{int}(\mathbb{G}_{\infty,x}) \subset \mathbb{G}$;

(H_3) for every $u \in W^{1,p}(O; \mathbb{R}^m)$, if $\int_O G_\infty(x, \nabla u(x))dx < \infty$ and if $\nabla u(x) \in \mathrm{int}(\mathbb{G})$ for $\mathscr{L}^N$-a.a. $x \in O$, then $\int_O G(\nabla u(x))dx < \infty$.

Remark 3.9. Under (H_2) we have $\mathrm{int}(\mathbb{G}_{\infty,x}) = \mathrm{int}(\mathbb{G})$ and $\overline{\mathbb{G}_{\infty,x}} = \overline{\mathbb{G}}$ for all $x \in O$.

Let $W : \mathbb{R}^N \times \mathbb{M} \to [0, \infty]$ be a Borel measurable function satisfying the following conditions:

(H_4) there exist $\alpha, \beta > 0$ such that for every $(x, \xi) \in \mathbb{R}^N \times \mathbb{M}$,

$$\alpha G(\xi)\mathbb{1}_E(x) \leqslant W(x, \xi)\mathbb{1}_E(x) \leqslant \beta(1 + G(\xi))\mathbb{1}_E(x);$$

(H_5) for every $\xi \in \mathbb{M}$, $W(\cdot, \xi)$ is 1-periodic, i.e. for every $(x, z) \in \mathbb{R}^N \times \mathbb{Z}^N$,

$$W(x + z, \xi) = W(x, \xi);$$

(H_6) $W\mathbb{1}_E$ is ru-usc with a 1-periodic function $a \in L^1_{\mathrm{loc}}(\mathbb{R}^N;]0, \infty])$, where $W\mathbb{1}_E : \mathbb{R}^N \times \mathbb{M} \to [0, \infty]$ is defined by $(W\mathbb{1}_E)(x, \xi) := W(x, \xi)\mathbb{1}_E(x)$.

Remark 3.10. As for a family of functions (see Remark 3.3), if W is ru-usc with a 1-periodic function $a \in L^1_{\mathrm{loc}}(\mathbb{R}^N;]0, \infty])$, then also is $W\mathbb{1}_E$.

For each $\varepsilon > 0$, let $J_\varepsilon : W^{1,p}(O; \mathbb{R}^m) \to [0, \infty]$ be defined by

$$J_\varepsilon(u) := \int_{O_\varepsilon} W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx = \int_O W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) \mathbb{1}_{O_\varepsilon}(x)dx.$$

As a consequence of Theorem 3.6 and Proposition 3.7(i) we have the following homogenization result.

Theorem 3.11. *Assume that $p > N$. Under* (H_0)–(H_6) *we have*

$$\Gamma(L^p(O_\varepsilon, O))\text{-}\lim_{\varepsilon\to 0} J_\varepsilon(u) = \int_O \widehat{W}_{\mathrm{hom}}(\nabla u(x))dx$$

for all $u \in W^{1,p}(O;\mathbb{R}^m)$ with $\widehat{W}_{\mathrm{hom}} : \mathbb{M} \to [0,\infty]$ given by

$$\widehat{W}_{\mathrm{hom}}(\xi) = \begin{cases} \lim\limits_{t\to 1^-} W_{\mathrm{hom}}(t\xi) & \text{if } \xi \in \overline{\mathbb{G}} \\ \infty & \text{otherwise,} \end{cases}$$

where $W_{\mathrm{hom}} : \mathbb{M} \to [0,\infty]$ is defined by

$$W_{\mathrm{hom}}(\xi) := \inf_{k\in\mathbb{N}^*} \frac{1}{k^N} \inf\left\{ \int_{]0,k[^N\cap E} W(y, \xi + \nabla\varphi(y))dy : \varphi \in W_0^{1,p}(]0,k[^N;\mathbb{R}^m) \right\}.$$

4. Auxiliary results

4.1. **Integral representation of the Vitali envelope of a set function.** Let $O \subset \mathbb{R}^N$ be a bounded open set and let $\mathcal{O}(O)$ be the class of open subsets of O. For each $\delta > 0$ and each $A \in \mathcal{O}(O)$, we denote the class of countable families $\{Q_i = Q_{\rho_i}(x_i) := x_i +]-\frac{\rho_i}{2}, \frac{\rho_i}{2}[^N\}_{i\in I}$ of disjoint open cubes of A with $x_i \in A$ and $\rho_i \in]0,\delta[$ and such that $\mathscr{L}^N(A\setminus \cup_{i\in I} Q_i) = 0$ by $\mathscr{V}_\delta(A)$.

Definition 4.1. Given $\mathcal{S} : \mathcal{O}(O) \to [0,\infty]$, for each $\delta > 0$ we define $\mathcal{S}^\delta : \mathcal{O}(O) \to [0,\infty]$ by

$$\mathcal{S}^\delta(A) := \inf\left\{ \sum_{i\in I} \mathcal{S}(Q_i) : \{Q_i\}_{i\in I} \in \mathscr{V}_\delta(A) \right\}.$$

By the Vitali envelope of $\mathcal{S}$ we call the set function $\mathcal{S}^* : \mathcal{O}(O) \to [-\infty,\infty]$ defined by

$$\mathcal{S}^*(A) := \sup_{\delta>0} \mathcal{S}^\delta(A) = \lim_{\delta\to 0} \mathcal{S}^\delta(A).$$

The interest of Definition 4.1 comes from the following integral representation result. (For a proof we refer to [AHM18, §3.3] or [AHCM17, §A.4].)

Theorem 4.2. *Let $\mathcal{S} : \mathcal{O}(O) \to [0,\infty]$ be a set function satisfying the following two conditions*:

(i) *there exists a finite Radon measure ν on O which is absolutely continuous with respect to $\mathscr{L}^N$ such that $\mathcal{S}(A) \leqslant \nu(A)$ for all $A \in \mathcal{O}(O)$;*
(ii) *$\mathcal{S}$ is subadditive, i.e. $\mathcal{S}(A) \leqslant \mathcal{S}(B) + \mathcal{S}(C)$ for all $A, B, C \in \mathcal{O}(O)$ with $B, C \subset A$, $B \cap C = \varnothing$ and $\mathscr{L}^N(A\setminus(B\cup C)) = 0$.*

Then $\lim_{\rho\to 0} \frac{\mathcal{S}(Q_\rho(\cdot))}{\rho^N} \in L^1(O)$ and for every $A \in \mathcal{O}(O)$, one has

$$\mathcal{S}^*(A) = \int_A \lim_{\rho\to 0} \frac{\mathcal{S}(Q_\rho(x))}{\rho^N} dx.$$

4.2. **A subadditive theorem.** Let $\mathcal{O}_{\rm b}(\mathbb{R}^N)$ be the class of all bounded open subsets of $\mathbb{R}^N$. We begin with the following definition.

Definition 4.3. Let $\mathcal{S} : \mathcal{O}_{\rm b}(\mathbb{R}^N) \to [0,\infty]$ be a set function.

(i) We say that $\mathcal{S}$ is subadditive if
$$\mathcal{S}(A) \leqslant \mathcal{S}(B) + \mathcal{S}(C)$$
for all $A, B, C \in \mathcal{O}_{\rm b}(\mathbb{R}^N)$ with $B, C \subset A$, $B \cap C = \varnothing$ and $\mathcal{L}^N(A\backslash(B \cup C)) = 0$.

(ii) We say that $\mathcal{S}$ is $\mathbb{Z}^N$-invariant if
$$\mathcal{S}(A + z) = \mathcal{S}(A)$$
for all $A \in \mathcal{O}_{\rm b}(\mathbb{R}^N)$ and all $z \in \mathbb{Z}^N$.

Let $\mathrm{Cub}(\mathbb{R}^N)$ be the class of all open cubes in $\mathbb{R}^N$. The following theorem is due to Akcoglu and Krengel (see [AK81] and also [LM02] and [AHM11, Theorem 3.11]).

Theorem 4.4. *Let $\mathcal{S} : \mathcal{O}_{\rm b}(\mathbb{R}^N) \to [0,\infty]$ be a subadditive and $\mathbb{Z}^N$-invariant set function for which there exists $C \in]0,\infty[$ such that for every $A \in \mathcal{O}_{\rm b}(\mathbb{R}^N)$,*
$$\mathcal{S}(A) \leqslant C\mathcal{L}^N(A).$$
Then, for every $Q \in \mathrm{Cub}(\mathbb{R}^N)$,
$$\lim_{\varepsilon\to 0} \frac{\mathcal{S}\left(\frac{1}{\varepsilon}Q\right)}{\mathcal{L}^N\left(\frac{1}{\varepsilon}Q\right)} = \inf_{k\geqslant 1} \frac{\mathcal{S}(]0,k[^N)}{k^N}.$$

5. Proofs

5.1. **Proof of the lower bound.** Here we prove Proposition 3.4.

Proof of Proposition 3.4. Let $u \in W^{1,p}(O;\mathbb{R}^m)$ and let $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(O;\mathbb{R}^m)$ be such that
$$u_\varepsilon \overset{L^p(O_\varepsilon,O)}{\longrightarrow} u, \text{ i.e. } \lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{L^p(O_\varepsilon;\mathbb{R}^m)} = 0. \tag{5.1}$$
We have to prove that
$$\varliminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \geqslant \int_O \varliminf_{t\to 1^-} \varlimsup_{\rho\to 0} \varliminf_{\varepsilon\to 0} \mathcal{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x, t\nabla u(x))dx. \tag{5.2}$$
Without loss of generality we can assume that $\varliminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) < \infty$, and so
$$\sup_{\varepsilon>0} I_\varepsilon(u_\varepsilon) < \infty. \tag{5.3}$$
From (5.3) it follows that
$$\sup_{\varepsilon>0} \int_O W_\varepsilon(x, \nabla u_\varepsilon(x))\mathbb{1}_{O_\varepsilon}(x)dx < \infty. \tag{5.4}$$
From (5.1) and (5.4) together with (A_2) and the left inequality in (A_3), we have
$$\begin{cases} \sup_{\varepsilon>0} \|u_\varepsilon\|_{L^p(O_\varepsilon;\mathbb{R}^m)} < \infty \\ \sup_{\varepsilon>0} \|\nabla u_\varepsilon\|_{L^p(O_\varepsilon;\mathbb{R}^m)} < \infty. \end{cases} \tag{5.5}$$

By (C_2) there exist $C > 0$ and $\{\widehat{u}_\varepsilon\}_{\varepsilon>0} \subset W^{1,p}(O;\mathbb{R}^m)$ such that for every $\varepsilon > 0$,

$$\widehat{u}_\varepsilon = u_\varepsilon \text{ on } O_\varepsilon \tag{5.6}$$

and

$$\begin{cases} \|\widehat{u}_\varepsilon\|_{L^p(O;\mathbb{R}^m)} \leqslant C\|u_\varepsilon\|_{L^p(O_\varepsilon;\mathbb{R}^m)} \\ \|\nabla\widehat{u}_\varepsilon\|_{L^p(O;\mathbb{R}^m)} \leqslant C\|\nabla u_\varepsilon\|_{L^p(O_\varepsilon;\mathbb{R}^m)}. \end{cases} \tag{5.7}$$

From (5.5) and (5.7) we deduce that $\sup_{\varepsilon>0}\|\widehat{u}_\varepsilon\|_{W^{1,p}(O;\mathbb{R}^m)} < \infty$, and so, up to a subsequence, there exists $\widehat{u} \in W^{1,p}(O;\mathbb{R}^m)$ such that:

$$\lim_{\varepsilon\to 0}\|\widehat{u}_\varepsilon - \widehat{u}\|_{L^p(O;\mathbb{R}^m)} = 0; \tag{5.8}$$
$$\nabla\widehat{u}_\varepsilon \rightharpoonup \nabla\widehat{u} \text{ in } L^p(O;\mathbb{R}^m).$$

But (5.1), (5.6) and (5.8) implies that $\widehat{u} = u$, and consequently

$$\lim_{\varepsilon\to 0}\|\widehat{u}_\varepsilon - u\|_{L^p(O;\mathbb{R}^m)} = 0; \tag{5.9}$$
$$\nabla\widehat{u}_\varepsilon \rightharpoonup \nabla u \text{ in } L^p(O;\mathbb{R}^m). \tag{5.10}$$

As $p > N$, from (5.9) and (5.10) we can assert that, up to a subsequence,

$$\lim_{\varepsilon\to 0}\|\widehat{u}_\varepsilon - u\|_{L^\infty(O;\mathbb{R}^m)} = 0.$$

On the other hand, taking (5.6) into account, from the left inequality in (A_3) we deduce that

$$\sup_{\varepsilon>0}\int_O G(\nabla\widehat{u}_\varepsilon(x))\mathbb{1}_{O_\varepsilon}(x)dx < \infty,$$

hence $G(\nabla\widehat{u}_\varepsilon(x))\mathbb{1}_{O_\varepsilon}(x) < \infty$ for all $\varepsilon > 0$ and $\mathscr{L}^N$-a.a. $x \in O$, and so, taking (A_0) into account,

$$\nabla\widehat{u}_\varepsilon(x)\mathbb{1}_{O_\varepsilon}(x) \in \mathbb{G} \text{ for all } \varepsilon > 0 \text{ and } \mathscr{L}^N\text{-a.a. } x \in O. \tag{5.11}$$

As $\mathbb{G}$ is convex, see (A_1) and Remark 3.1, from (C_1), (5.10) and (5.11) it follows that

$$\nabla u(x) \in \overline{\mathbb{G}} \text{ for } \mathscr{L}^N\text{-a.a. } x \in O. \tag{5.12}$$

Step 1: localization. For every $\varepsilon > 0$, we define the (positive) radon measure μ_ε on O by

$$\mu_\varepsilon := W_\varepsilon(\cdot,\nabla\widehat{u}_\varepsilon(\cdot))\mathbb{1}_{O_\varepsilon}(\cdot)\mathscr{L}^N. \tag{5.13}$$

From (5.4) we see that $\sup_{\varepsilon>0}\mu_\varepsilon(O) < \infty$, and so there exists a (positive) Radon measure μ on O such that, up to a subsequence, $\mu_\varepsilon \rightharpoonup \mu$ weakly. By Lebesgue's decomposition theorem, we have $\mu = \mu^a + \mu^s$ where μ^a and μ^s are (positive) Radon measures on O such that $\mu^a \ll \mathscr{L}^N$ and $\mu^s \perp \mathscr{L}^N$. Thus, to prove (5.2) it suffices to show that

$$\mu^a \geqslant \varliminf_{t\to 1^-}\varlimsup_{\rho\to 0}\varliminf_{\varepsilon\to 0}\mathscr{H}^\rho[W_\varepsilon\mathbb{1}_{O_\varepsilon}](\cdot,t\nabla u(\cdot))\mathscr{L}^N. \tag{5.14}$$

From Radon-Nikodym's theorem we have $\mu^a = f(\cdot)\mathscr{L}^N$ with

$$f(\cdot) := \lim_{\rho\to 0}\frac{\mu(Q_\rho(\cdot))}{\mathscr{L}^N(Q_\rho(\cdot))} \in L^1(O;[0,\infty[), \tag{5.15}$$

and so to prove (5.14) it is sufficient to establish that for $\mathscr{L}^N$-a.e. $x_0 \in O$,

$$f(x_0) = \lim_{\rho\to 0}\frac{\mu(Q_\rho(x_0))}{\mathscr{L}^N(Q_\rho(x_0))} \geqslant \varliminf_{t\to 1^-}\varlimsup_{\rho\to 0}\varliminf_{\varepsilon\to 0}\mathscr{H}^\rho[W_\varepsilon\mathbb{1}_{O_\varepsilon}](x_0,t\nabla u(x_0)). \tag{5.16}$$

Fix $x_0 \in O\backslash N$ where $N \subset O$ is a suitable set such that $\mathscr{L}^N(N) = 0$. As $\mu(O) < \infty$, without loss of generality we can assume that $\mu(\partial Q_\rho(x_0)) = 0$ for all $\rho > 0$, which implies, by Alexandrov's theorem, that $\mu(Q_\rho(x_0)) = \lim_{\varepsilon\to 0} \mu_\varepsilon(Q_\rho(x_0))$. Consequently, to prove (5.16) it suffices to show that

$$\lim_{\rho\to 0}\lim_{\varepsilon\to 0} \fint_{Q_\rho(x_0)} W_\varepsilon(x, \nabla\widehat{u}_\varepsilon(x))\mathbb{1}_{O_\varepsilon}(x)dx \geqslant \varliminf_{t\to 1^-} \varlimsup_{\rho\to 0} \varliminf_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x_0, t\nabla u(x_0)). \tag{5.17}$$

On the other hand, as $\mathbb{G}$ is convex, see (A_1) and Remark 3.1, and $0 \in \mathrm{int}(\mathbb{G})$, see ($\mathrm{A}_0$), from (5.11) we can assert for every $t \in]0, 1[$,

$$t\nabla\widehat{u}_\varepsilon(x)\mathbb{1}_{O_\varepsilon}(x) \in \mathbb{G} \text{ for all } \varepsilon > 0 \text{ and for } \mathscr{L}^N\text{-a.a. } x \in O.$$

Hence, given any $t \in]0, 1[$, we see that for every $\varepsilon > 0$ and every $\rho > 0$,

$$\begin{aligned}\fint_{Q_\rho(x_0)} W_\varepsilon(x, t\nabla\widehat{u}_\varepsilon(x))\mathbb{1}_{O_\varepsilon}(x)dx \;&\leqslant\; \big(1 + \Delta(t)\big)\fint_{Q_\rho(x_0)} W_\varepsilon(x, \nabla\widehat{u}_\varepsilon(x))\mathbb{1}_{O_\varepsilon}(x)dx \\ &\quad + \Delta(t)\fint_{Q_\rho(x_0)} a_\varepsilon(x)dx\end{aligned}$$

with $\Delta(t) := \sup_{\varepsilon>0} \Delta^{a_\varepsilon}_{W_\varepsilon \mathbb{1}_{O_\varepsilon}}(t)$, where $\Delta^{a_\varepsilon}_{W_\varepsilon \mathbb{1}_{O_\varepsilon}}(t)$ is given by (2.1) with $L(x, \xi) = W_\varepsilon(x, \xi)\mathbb{1}_{O_\varepsilon(x)}$ and $a = a_\varepsilon$. Letting $\varepsilon \to 0$ and $\rho \to 0$ we obtain

$$\begin{aligned}\varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \fint_{Q_\rho(x_0)} W_\varepsilon(x, t\nabla\widehat{u}_\varepsilon(x))\mathbb{1}_{O_\varepsilon}(x)dx \;&\leqslant\; \big(1 + \Delta(t)\big) \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \fint_{Q_\rho(x_0)} W_\varepsilon(x, \nabla\widehat{u}_\varepsilon(x))\mathbb{1}_{O_\varepsilon}(x)dx \\ &\quad + \varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \Delta(t)\fint_{Q_\rho(x_0)} a_\varepsilon(x)dx.\end{aligned}$$

But, from (A_6) (see also Definition 2.10) we have:

$$\begin{aligned}&\varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \fint_{Q_\rho(x_0)} a_\varepsilon(x)dx = \varlimsup_{\rho\to 0} \fint_{Q_\rho(x_0)} a_0(x)dx = a_0(x_0) \in [0, \infty[;\\ &\varlimsup_{t\to 1^-} \Delta(t) \leqslant 0,\end{aligned}$$

hence

$$\varlimsup_{t\to 1^-} \varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \Delta(t)\fint_{Q_\rho(x_0)} a_\varepsilon(x)dx \leqslant 0,$$

and consequently

$$\varlimsup_{t\to 1^-} \varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \fint_{Q_\rho(x_0)} W_\varepsilon(x, t\nabla\widehat{u}_\varepsilon(x))\mathbb{1}_{O_\varepsilon}(x)dx \leqslant \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \fint_{Q_\rho(x_0)} W_\varepsilon(x, \nabla\widehat{u}_\varepsilon(x))\mathbb{1}_{O_\varepsilon}(x)dx.$$

Thus, to prove (5.17) it is sufficient to show that

$$\varlimsup_{t\to 1^-} \varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \fint_{Q_\rho(x_0)} W_\varepsilon(x, t\nabla\widehat{u}_\varepsilon(x))\mathbb{1}_{O_\varepsilon}(x)dx \geqslant \varliminf_{t\to 1^-} \varlimsup_{\rho\to 0} \varliminf_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x_0, t\nabla u(x_0)). \tag{5.18}$$

Step 2: cut-off method. Fix any $\varepsilon > 0$, any $t \in]0, 1[$, any $\sigma \in]t, 1[$, any $\lambda \in]0, 1[$ and any $\rho > 0$. Let $\varphi \in C^\infty(O)$ be a cut-off function for the pair $(O\backslash Q_\rho(x_0), \overline{Q}_{\lambda\rho}(x_0))$, i.e. $\varphi(x) \in [0, 1]$ for all $x \in O$, $\varphi(x) = 0$ for all $x \in O\backslash Q_\rho(x_0)$ and $\varphi(x) = 1$ for all $x \in \overline{Q}_{\lambda\rho}(x_0)$, such that $\|\nabla\varphi\|_{L^\infty} \leqslant \frac{\theta}{\rho(1-\lambda)}$ for some $\theta > 0$ (which does not depend on ρ and λ). Define $v_\varepsilon \in W^{1,p}(O;\mathbb{R}^m)$ by

$$v_\varepsilon := \varphi\widehat{u}_\varepsilon + (1-\varphi)u_{x_0} = \varphi(\widehat{u}_\varepsilon - u_{x_0}) + u_{x_0}$$

with $u_{x_0}(\cdot) := u(x_0) + \nabla u(x_0)(\cdot - x_0)$. Then

$$tv_\varepsilon - tu_{x_0} \in W_0^{1,p}(Q_\rho(x_0);\mathbb{R}^m) \tag{5.19}$$

and

$$t\nabla v_\varepsilon = \begin{cases} t\nabla\widehat{u}_\varepsilon & \text{in } \overline{Q}_{\lambda\rho}(x_0) \\ \frac{t}{\sigma}\big(\varphi\sigma\nabla\widehat{u}_\varepsilon + (1-\varphi)\sigma\nabla u(x_0)\big) + \big(1-\frac{t}{\sigma}\big)\,\Psi_{\varepsilon,\rho} & \text{in } Q_\rho(x_0)\backslash\overline{Q}_{\lambda\rho}(x_0) \end{cases} \tag{5.20}$$

with $\Psi_{\varepsilon,\rho} := \frac{t}{1-\frac{t}{\sigma}}\nabla\varphi\otimes(\widehat{u}_\varepsilon - u_{x_0})$. Using the right inequality in (A_5) it follows that

$$\begin{aligned} \fint_{Q_\rho(x_0)} W_\varepsilon(x,t\nabla v_\varepsilon)\mathbb{1}_{O_\varepsilon}dx &= \frac{1}{\mathscr{L}^N(Q_\rho(x_0))}\int_{\overline{Q}_{\lambda\rho}(x_0)} W_\varepsilon(x,t\nabla\widehat{u}_\varepsilon)\mathbb{1}_{O_\varepsilon}dx \\ &\quad + \frac{1}{\mathscr{L}^N(Q_\rho(x_0))}\int_{Q_\rho(x_0)\backslash\overline{Q}_{\lambda\rho}(x_0)} W_\varepsilon(x,t\nabla v_\varepsilon)\mathbb{1}_{O_\varepsilon}dx \\ &\leqslant \fint_{Q_\rho(x_0)} W_\varepsilon(x,t\nabla\widehat{u}_\varepsilon)\mathbb{1}_{O_\varepsilon}dx + \beta(1-\lambda^N) \\ &\quad + \frac{\beta}{\mathscr{L}^N(Q_\rho(x_0))}\int_{Q_\rho(x_0)\backslash\overline{Q}_{\lambda\rho}(x_0)} G(t\nabla v_\varepsilon)\mathbb{1}_{O_\varepsilon}dx. \end{aligned} \tag{5.21}$$

On the other hand, taking (5.20) into account and using (C_1) and the left inequality in (A_5), we have

$$\begin{aligned} G(t\nabla v_\varepsilon)\mathbb{1}_{O_\varepsilon} &\leqslant c_1\left(1 + G(\sigma\nabla\widehat{u}_\varepsilon) + G(\sigma\nabla u(x_0)) + G(\Psi_{\varepsilon,\rho})\right)\mathbb{1}_{O_\varepsilon} \\ &\leqslant c_1\left(1 + \frac{1}{\alpha}W_\varepsilon(x,\sigma\nabla\widehat{u}_\varepsilon)\mathbb{1}_{O_\varepsilon} + G(\sigma\nabla u(x_0)) + G(\Psi_{\varepsilon,\rho})\right) \end{aligned} \tag{5.22}$$

with $c_1 := 2(\gamma+\gamma^2) > 0$. Note that from ($A_0$) and (5.12) we can assert that $\sigma\nabla u(x_0) \in \mathbb{G}$, and so

$$G(\sigma\nabla u(x_0)) < \infty.$$

Moreover, it is easy to see that

$$\begin{aligned} \|\Psi_{\varepsilon,\rho}\|_{L^\infty(Q_\rho(x_0);\mathbb{M})} &\leqslant \frac{\theta t}{(1-\frac{t}{\sigma})(1-\lambda)}\frac{1}{\rho}\|u - u_{x_0}\|_{L^\infty(Q_\rho(x_0);\mathbb{R}^m)} \\ &\quad + \frac{\theta t}{\rho(1-\frac{t}{\sigma})(1-\lambda)}\|\widehat{u}_\varepsilon - u\|_{L^\infty(O;\mathbb{R}^m)}, \end{aligned}$$

where

$$\lim_{\rho\to 0}\frac{\theta t}{(1-\frac{t}{\sigma})(1-\lambda)}\frac{1}{\rho}\|u - u_{x_0}\|_{L^\infty(Q_\rho(x_0);\mathbb{R}^m)} = 0 \tag{5.23}$$

because $\lim_{\rho\to 0}\frac{1}{\rho}\|u - u_{x_0}\|_{L^\infty(Q_\rho(x_0);\mathbb{R}^m)} = 0$ since $p > N$, and

$$\lim_{\varepsilon\to 0}\frac{\theta t}{\rho(1-\frac{t}{\sigma})(1-\lambda)}\|\widehat{u}_\varepsilon - u\|_{L^\infty(O;\mathbb{R}^m)} = 0 \tag{5.24}$$

by (5.9). From (A_0) and (A_1) there exists $r > 0$ such that

$$c_2 := \sup_{|\xi|\leqslant r} G(\xi) < \infty$$

(see Remark 3.1). By (5.23) there exists $\bar\rho > 0$ such that $\frac{\theta t}{(1-\frac{t}{\sigma})(1-\lambda)}\frac{1}{\rho}\|u-u_{x_0}\|_{L^\infty(Q_\rho(x_0);\mathbb{R}^m)} < \frac{r}{2}$ for all $\rho \in]0,\bar\rho[$. Fix any $\rho \in]0,\bar\rho[$. Taking (5.24) into account we can assert that there exists $\varepsilon_\rho > 0$ such that

$$G(\Psi_{\varepsilon,\rho}) \leqslant c_2 \text{ for all } \varepsilon \in]0,\varepsilon_\rho[. \tag{5.25}$$

Thus, from (5.21), (5.22) and (5.25) we deduce that

$$\fint_{Q_\rho(x_0)} W_\varepsilon(x, t\nabla v_\varepsilon)\mathbb{1}_{O_\varepsilon}dx \leqslant \fint_{Q_\rho(x_0)} W_\varepsilon(x, t\nabla \widehat{u}_\varepsilon)\mathbb{1}_{O_\varepsilon}dx + c_3(\sigma)(1-\lambda^N) + \frac{\beta c_1}{\alpha}\Gamma_{\varepsilon,\rho,\lambda,\sigma}$$

for all $t \in]0, t_\rho[$ with:

$$c_3(\sigma) := \beta c_1\left(1 + \frac{1}{c_1} + G(\sigma\nabla u(x_0)) + c_2\right) \in]0,\infty[\ ;$$

$$\Gamma_{\varepsilon,\rho,\lambda,\sigma} := \frac{1}{\mathscr{L}^N(Q_\rho(x_0))}\int_{Q_\rho(x_0)\setminus\overline{Q}_{\lambda\rho}(x_0))} W_\varepsilon(x, \sigma\nabla\widehat{u}_\varepsilon)\mathbb{1}_{O_\varepsilon}dx.$$

But, taking (5.19) into account, we see that

$$\mathscr{H}^\rho[W_\varepsilon\mathbb{1}_{O_\varepsilon}](x_0, t\nabla u(x_0)) \leqslant \fint_{Q_\rho(x_0)} W_\varepsilon(x, t\nabla v_\varepsilon)\mathbb{1}_{O_\varepsilon}dx,$$

hence, for every $\rho > 0$, every $\varepsilon \in]0,\varepsilon_\rho[$, every $\lambda \in]0,1[$, every $t \in]0,1[$ and every $\sigma \in]t,1[$, we have

$$\mathscr{H}^\rho[W_\varepsilon\mathbb{1}_{O_\varepsilon}](x_0, t\nabla u(x_0)) \leqslant \fint_{Q_\rho(x_0)} W_\varepsilon(x, t\nabla \widehat{u}_\varepsilon)\mathbb{1}_{O_\varepsilon}dx + c_3(\sigma)(1-\lambda^N) + \frac{\beta c_1}{\alpha}\Gamma_{\varepsilon,\rho,\lambda,\sigma}. \tag{5.26}$$

Step 3: passing to the limit. Letting $\varepsilon \to 0$, $\rho \to 0$, $\lambda \to 1^-$, $\sigma \to 1^-$ and $t \to 1^-$ in (5.26), we obtain

$$\begin{aligned}\varliminf_{t\to1^-}\varlimsup_{\rho\to0}\varliminf_{\varepsilon\to0}\mathscr{H}^\rho[W_\varepsilon\mathbb{1}_{O_\varepsilon}](x_0, t\nabla u(x_0)) \ &\leqslant\ \varlimsup_{t\to1^-}\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\fint_{Q_\rho(x_0)} W_\varepsilon(x, t\nabla \widehat{u}_\varepsilon)\mathbb{1}_{O_\varepsilon}dx \\ &\quad + \frac{\beta c_1}{\alpha}\varlimsup_{\sigma\to1^-}\varlimsup_{\lambda\to1^-}\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\Gamma_{\varepsilon,\rho,\lambda,\sigma}.\end{aligned} \tag{5.27}$$

Substep 3-1: proving that $\boldsymbol{\varlimsup_{\sigma\to1^-}\varlimsup_{\lambda\to1^-}\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\Gamma_{\varepsilon,\rho,\lambda,\sigma} = 0}$. For every $\varepsilon \in]0,\varepsilon_\rho[$, we have

$$\begin{aligned}\Gamma_{\varepsilon,\rho,\lambda,\sigma} \ &\leqslant\ \big(1+\Delta(\sigma)\big)\frac{\mu_\varepsilon\left(Q_\rho(x_0)\setminus\overline{Q}_{\lambda\rho}(x_0))\right)}{\mathscr{L}^N(Q_\rho(x_0))} \\ &\quad +\Delta(\sigma)\frac{1}{\mathscr{L}^N(Q_\rho(x_0))}\int_{Q_\rho(x_0)\setminus\overline{Q}_{\lambda\rho}(x_0)} a_\varepsilon(x)dx.\end{aligned} \tag{5.28}$$

But from (A_6) (see also Definition 2.10) we have $\overline{\lim}_{\sigma\to1^-}\Delta(\sigma)\leqslant 0$ and

$$\begin{aligned}\overline{\lim_{\lambda\to1^-}}\,\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\frac{1}{\mathscr{L}^N(Q_\rho(x_0))}\int_{Q_\rho(x_0)\setminus\overline{Q}_{\lambda\rho}(x_0)}a_\varepsilon(x)dx &\leqslant \overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}-\!\!\!\!\!\!\int_{Q_\rho(x_0)}a_\varepsilon(x)dx\\ &= \overline{\lim_{\rho\to0}}-\!\!\!\!\!\!\int_{Q_\rho(x_0)}a_0(x)dx=a_0(x_0)\in[0,\infty[\end{aligned}$$

with $\frac{1}{\mathscr{L}^N(Q_\rho(x_0))}\int_{Q_\rho(x_0)\setminus\overline{Q}_{\lambda\rho}(x_0)}a_\varepsilon(x)dx\geqslant 0$, hence

$$\overline{\lim_{\sigma\to1^-}}\,\overline{\lim_{\lambda\to1^-}}\,\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\Delta(\sigma)\frac{1}{\mathscr{L}^N(Q_\rho(x_0))}\int_{Q_\rho(x_0)\setminus\overline{Q}_{\lambda\rho}(x_0)}a_\varepsilon(x)dx\leqslant 0. \tag{5.29}$$

As $\mu_\varepsilon\rightharpoonup\mu$ weakly and $\overline{Q}_\rho(x_0)\setminus Q_{\lambda\rho}(x_0)$ is compact, by Alexandrov's theorem, we have

$$\overline{\lim_{\varepsilon\to0}}\,\mu_\varepsilon\left(\overline{Q}_\rho(x_0)\setminus Q_{\lambda\rho}(x_0)\right)\leqslant\mu\left(\overline{Q}_\rho(x_0)\setminus Q_{\lambda\rho}(x_0)\right),$$

hence

$$\overline{\lim_{\varepsilon\to0}}\,\mu_\varepsilon\left(Q_\rho(x_0)\setminus\overline{Q}_{\lambda\rho}(x_0)\right)\leqslant\mu\left(\overline{Q}_\rho(x_0)\right)-\mu\left(Q_{\lambda\rho}(x_0)\right),$$

and consequently

$$\overline{\lim_{\varepsilon\to0}}\frac{\mu_\varepsilon\left(Q_\rho(x_0)\setminus\overline{Q}_{\lambda\rho}(x_0)\right)}{\mathscr{L}^N(Q_\rho(x_0))}\leqslant\frac{\mu\left(\overline{Q}_\rho(x_0)\right)}{\mathscr{L}^N(\overline{Q}_\rho(x_0))}-\lambda^N\frac{\mu\left(Q_{\lambda\rho}(x_0)\right)}{\mathscr{L}^N(Q_{\lambda\rho}(x_0))}.$$

It follows that

$$\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\frac{\mu_\varepsilon\left(Q_\rho(x_0)\setminus\overline{Q}_{\lambda\rho}(x_0)\right)}{\mathscr{L}^N(Q_\rho(x_0))}\leqslant\left(1-\lambda^N\right)f(x_0)$$

with $f\in L^1(O;[0,\infty[)$ given by (5.15), and so

$$\lim_{\lambda\to1^-}\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\frac{\mu_\varepsilon\left(Q_\rho(x_0)\setminus\overline{Q}_{\lambda\rho}(x_0)\right)}{\mathscr{L}^N(Q_\rho(x_0))}=0.$$

with $\frac{\mu_\varepsilon\left(Q_\rho(x_0)\setminus\overline{Q}_{\lambda\rho}(x_0)\right)}{\mathscr{L}^N(Q_\rho(x_0))}\geqslant 0$. Consequently, by using (A_6),

$$\overline{\lim_{\sigma\to1^-}}\lim_{\lambda\to1^-}\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\left(1+\Delta(\sigma)\right)\frac{\mu_\varepsilon\left(Q_\rho(x_0)\setminus\overline{Q}_{\lambda\rho}(x_0)\right)}{\mathscr{L}^N(Q_\rho(x_0))}\leqslant 0. \tag{5.30}$$

From (5.28), (5.29) and (5.30) we deduce that

$$\overline{\lim_{\sigma\to1^-}}\,\overline{\lim_{\lambda\to1^-}}\,\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\Gamma_{\varepsilon,\rho,\lambda,\sigma}=0. \tag{5.31}$$

Substep 3-3: end of the proof. Combining (5.31) with (5.27) we obtain (5.18), and the proof of the lower bound is complete. ■

5.2. **Proof of the upper bound.** Here we prove Proposition 3.5.

Proof of Proposition 3.5. In what follows, $\mathscr{O}(O)$ denotes the class of all open subsets of O. For each $u \in W^{1,p}(O;\mathbb{R}^m)$, let $\overline{\mathrm{m}}_u : \mathscr{O}(O) \to [0,\infty]$ be defined by

$$\overline{\mathrm{m}}_u(A) := \varlimsup_{\varepsilon\to 0} \mathrm{m}_u^\varepsilon(A).$$

with, for each $\varepsilon > 0$, $\mathrm{m}_u^\varepsilon : \mathscr{O}(O) \to [0,\infty]$ given by

$$\begin{aligned} \mathrm{m}_u^\varepsilon(A) &= \inf\left\{\int_{O_\varepsilon\cap A} W_\varepsilon(x,\nabla v(x))dx : v-u \in W_0^{1,p}(A;\mathbb{R}^m)\right\} \\ &= \inf\left\{\int_A W_\varepsilon(x,\nabla v(x))\mathbb{1}_{O_\varepsilon}(x)dx : v-u \in W_0^{1,p}(A;\mathbb{R}^m)\right\}. \end{aligned} \tag{5.32}$$

For each $\delta > 0$ and each $A \in \mathscr{O}(O)$, we denote the class of countable families $\{Q_i := Q_{\rho_i}(x_i)\}_{i\in I}$ of disjoint open cubes of A with $x_i \in A$ and $\rho_i \in]0,\delta[$ such that $\mathscr{L}^N(A\setminus \cup_{i\in I} Q_i) = 0$ by $\mathscr{V}_\delta(A)$, and we consider $\overline{\mathrm{m}}_u^\delta : \mathscr{O}(O) \to [0,\infty]$ given by

$$\overline{\mathrm{m}}_u^\delta(A) := \inf\left\{\sum_{i\in I} \overline{\mathrm{m}}_u(B_i) : \{B_i\}_{i\in I} \in \mathscr{V}_\delta(A)\right\},$$

and we define $\overline{\mathrm{m}}_u^* : \mathscr{O}(O) \to [0,\infty]$ by

$$\overline{\mathrm{m}}_u^*(A) := \sup_{\delta>0} \overline{\mathrm{m}}_u^\delta(A) = \lim_{\delta\to 0} \overline{\mathrm{m}}_u^\delta(A).$$

The set function $\overline{\mathrm{m}}_u^*$ is called the Vitali envelope of $\overline{\mathrm{m}}_u$ (see §4.1).

Step 1: link between Γ-$\overline{\lim}$ and Vitali envelope. Let $u \in W^{1,p}(O;\mathbb{R}^m)$. We are going to prove that

$$\Gamma(L^p(O_\varepsilon,O))\text{-}\varlimsup_{\varepsilon\to 0} I_\varepsilon(u) \leqslant \overline{\mathrm{m}}_u^*(O). \tag{5.33}$$

Without loss of generality we can assume that $\overline{\mathrm{m}}_u^*(O) < \infty$. Fix any $\delta > 0$. By definition of $\overline{\mathrm{m}}_u^\delta(O)$ there exists $\{Q_i\}_{i\in I} \in \mathscr{V}_\delta(O)$ such that

$$\sum_{i\in I} \overline{\mathrm{m}}_u(Q_i) \leqslant \overline{\mathrm{m}}_u^\delta(O) + \frac{\delta}{2}. \tag{5.34}$$

Fix any $\varepsilon > 0$. For each $i \in I$, by definition of $\mathrm{m}_u^\varepsilon(Q_i)$ there exists $v_\varepsilon^i \in W^{1,p}(O;\mathbb{R}^m)$ such that $v_\varepsilon^i - u \in W_0^{1,p}(Q_i;\mathbb{R}^m)$ and

$$\int_{O_\varepsilon\cap Q_i} W_\varepsilon\left(x,\nabla v_\varepsilon^i(x)\right)dx \leqslant \mathrm{m}_u^\varepsilon(Q_i) + \frac{\delta\mathscr{L}^N(Q_i)}{2\mathscr{L}^N(O)}. \tag{5.35}$$

Define $u_\varepsilon^\delta : O \to \mathbb{R}^m$ by

$$u_\varepsilon^\delta := \begin{cases} u & \text{in } O\setminus \underset{i\in I}{\cup} Q_i \\ v_\varepsilon^i & \text{in } Q_i. \end{cases}$$

Then $u_\varepsilon^\delta - u \in W_0^{1,p}(O;\mathbb{R}^m)$. Moreover, we have $\nabla u_\varepsilon^\delta(x) = \nabla v_\varepsilon^i(x)$ for $\mathscr{L}^N$-a.a. $x \in Q_i$. From (5.35) we see that

$$I_\varepsilon(u_\varepsilon^\delta) \leqslant \sum_{i\in I} \mathrm{m}_u^\varepsilon(Q_i) + \frac{\delta}{2},$$

hence $\overline{\lim}_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon^\delta) \leqslant \overline{\mathrm{m}}_u^\delta(O) + \delta$ by using (5.34), and consequently

$$\overline{\lim_{\delta\to 0}}\,\overline{\lim_{\varepsilon\to 0}} I_\varepsilon(u_\varepsilon^\delta) \leqslant \overline{\mathrm{m}}_u^*(O). \tag{5.36}$$

On the other hand, we have

$$\|u_\varepsilon^\delta - u\|^p_{L^p(O_\varepsilon;\mathbb{R}^m)} = \int_{O_\varepsilon} \left|u_\varepsilon^\delta - u\right|^p dx = \sum_{i\in I} \int_{O_\varepsilon\cap Q_i} \left|v_\varepsilon^i - u\right|^p dx.$$

Taking (C_0) into account and since $\rho_i \in]0,\delta[$ for all $i \in I$, by Poincaré's inequality (see [DD12, Exercise 2.9 pp. 106]) we have

$$\sum_{i\in I} \int_{O_\varepsilon\cap Q_i} \left|v_\varepsilon^i - u\right|^p dx \leqslant C\delta^p \sum_{i\in I} \int_{O_\varepsilon\cap Q_i} \left|\nabla v_\varepsilon^i - \nabla u\right|^p dx$$

with $C > 0$ (which only depends on p) and so

$$\|u_\varepsilon^\delta - u\|^p_{L^p(O_\varepsilon;\mathbb{R}^m)} \leqslant 2^{p-1} C\delta^p \left(\sum_{i\in I} \int_{O_\varepsilon\cap Q_i} |\nabla v_\varepsilon^i|^p dx + \int_{O_\varepsilon\cap Q_i} |\nabla u|^p dx \right). \tag{5.37}$$

Taking the left inequality in (A_3), (5.35) and (5.34) into account, from (5.37) we deduce that

$$\overline{\lim_{\varepsilon\to 0}} \|u_\varepsilon^\delta - u\|^p_{L^p(O;\mathbb{R}^m)} \leqslant 2^{p-1} C\delta^p \left(\frac{1}{\alpha}\left(\overline{\mathrm{m}}_u^\delta(O) + \delta\right) + \int_O |\nabla u|^p dx \right)$$

with $\alpha > 0$ given by (A_3), which gives

$$\lim_{\delta\to 0} \overline{\lim_{\varepsilon\to 0}} \|u_\varepsilon^\delta - u\|^p_{L^p(O_\varepsilon;\mathbb{R}^m)} = 0 \tag{5.38}$$

because $\lim_{\delta\to 0} \overline{\mathrm{m}}_u^\delta(O) = \overline{\mathrm{m}}_u^*(O) < \infty$ and $u \in W^{1,p}(O;\mathbb{R}^m)$. According to (5.36) and (5.38), by diagonalization there exists a mapping $\varepsilon \mapsto \delta_\varepsilon$, with $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, such that:

$$\lim_{\varepsilon\to 0} \|w_\varepsilon - u\|^p_{L^p(O_\varepsilon;\mathbb{R}^m)} = 0; \tag{5.39}$$

$$\overline{\lim_{\varepsilon\to 0}} I_\varepsilon(w_\varepsilon) \leqslant \overline{\mathrm{m}}_u^*(O) \tag{5.40}$$

with $w_\varepsilon := u_\varepsilon^{\delta_\varepsilon}$. By (5.39) we have $\Gamma(L^p(O_\varepsilon, O))\text{-}\overline{\lim}_{\varepsilon\to 0} I_\varepsilon(u) \leqslant \overline{\lim}_{\varepsilon\to 0} I_\varepsilon(w_\varepsilon)$, and (5.33) follows from (5.40).

Step 2: differentiation with respect to $\mathscr{L}^N$. Let $u \in W^{1,p}(O;\mathbb{R}^m)$ be such that $\mathscr{G}(u) := \int_O G(\nabla u(x))dx < \infty$. We are going to prove that

$$\overline{\mathrm{m}}_u^*(O) = \int_O \lim_{\rho\to 0} \frac{\overline{\mathrm{m}}_u(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} dx. \tag{5.41}$$

According to Theorem 4.2, to prove (5.41) it suffices to establish that $\overline{\mathrm{m}}_u$ is subadditive and there exists a finite Radon measure ν on O which is absolutely continuous with respect to $\mathscr{L}^N$ such that

$$\overline{\mathrm{m}}_u(A) \leqslant \nu(A) \tag{5.42}$$

for all $A \in \mathcal{O}(O)$. For each $\varepsilon > 0$, from the definition of m_u^ε in (5.32), it is easy to see that for every $A, B, C \in \mathcal{O}(O)$ with $B, C \subset A$, $B \cap C = \varnothing$ and $\mathscr{L}^N(A\backslash(B \cup C)) = 0$, one has

$$\mathrm{m}_u^\varepsilon(A) \leqslant \mathrm{m}_u^\varepsilon(B) + \mathrm{m}_u^\varepsilon(C),$$

and so

$$\overline{\lim_{\varepsilon\to 0}}\,\mathrm{m}_u^\varepsilon(A) \leqslant \overline{\lim_{\varepsilon\to 0}}\,\mathrm{m}_u^\varepsilon(B) + \overline{\lim_{\varepsilon\to 0}}\,\mathrm{m}_u^\varepsilon(C),$$

i.e.

$$\overline{\mathrm{m}}_u(A) \leqslant \overline{\mathrm{m}}_u(B) + \overline{\mathrm{m}}_u(C),$$

which shows the subadditivity of $\overline{\mathrm{m}}_u$. On the other hand, given any $\varepsilon > 0$, by using the right inequality in (A_5) we have

$$\mathrm{m}_u^\varepsilon(A) \leqslant \int_A \beta(1 + G(\nabla u(x)))dx$$

for all $A \in \mathscr{O}(O)$. Thus (5.42) holds with the Radon measure $\nu := \beta(1+G(\nabla u(\cdot)))\mathscr{L}^N$ which is necessarily finite since $\mathscr{G}(u) < \infty$.

Step 3: cut-off method. Let $t \in]0, 1[$, let $\sigma \in]t, 1[$ and let $u \in W^{1,p}(O;\mathbb{R}^m)$ be such that $\mathscr{G}(\sigma u) < \infty$. We are going to prove that for $\mathscr{L}^N$-a.e. $x \in O$,

$$\overline{\lim_{\rho\to 0}}\,\frac{\overline{\mathrm{m}}_{tu}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} \leqslant \overline{\lim_{\rho\to 0}}\,\frac{\overline{\mathrm{m}}_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} \tag{5.43}$$

with $u_x(\cdot) := u(x) + \nabla u(x)(\cdot - x)$.

Remark 5.1. For $\mathscr{L}^N$-a.e. $x \in O$, one has

$$\overline{\lim_{\rho\to 0}}\,\frac{\overline{\mathrm{m}}_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} = \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x, t\nabla u(x)).$$

Remark 5.2. If $\mathscr{G}(tu) < \infty$ then $\mathscr{G}(tu_x) < \infty$ for $\mathscr{L}^N$-a.a. $x \in O$, and so, by the step 2,

$$\overline{\lim_{\rho\to 0}}\,\frac{\overline{\mathrm{m}}_{tu}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} = \lim_{\rho\to 0}\frac{\overline{\mathrm{m}}_{tu}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} \text{ and } \overline{\lim_{\rho\to 0}}\,\frac{\overline{\mathrm{m}}_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} = \lim_{\rho\to 0}\frac{\overline{\mathrm{m}}_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}.$$

Fix any $\varepsilon > 0$, any $\lambda \in]0, 1[$, any $\rho > 0$ and any $\delta > 0$. By definition of $\mathrm{m}_{tu_x}^\varepsilon(Q_{\lambda\rho}(x))$ in (5.32), there exists $w \in W^{1,p}(O;\mathbb{R}^m)$ such that

$$tw - tu_x \in W_0^{1,p}(Q_{\lambda\rho}(x);\mathbb{R}^m) \tag{5.44}$$

and

$$\int_{Q_{\lambda\rho}(x)} W_\varepsilon(y, t\nabla w(y))dy \leqslant \mathrm{m}_{tu_x}^\varepsilon(Q_{\lambda\rho}(x)) + \delta\mathscr{L}^N(Q_{\lambda\rho}(x)). \tag{5.45}$$

Let $\varphi \in C^\infty(O)$ be a cut-off function for the pair $(O\backslash Q_\rho(x), \overline{Q}_{\lambda\rho}(x))$, i.e. $\varphi(y) \in [0, 1]$ for all $y \in O$, $\varphi(y) = 0$ for all $y \in O\backslash Q_\rho(x)$ and $\varphi(y) = 1$ for all $y \in \overline{Q}_{\lambda\rho}(x)$, such that

$$\|\nabla\varphi\|_{L^\infty} \leqslant \frac{\theta}{\rho(1-\lambda)}$$

for some $\theta > 0$ (which does not depend on ρ and λ). Define $v \in W^{1,p}(O;\mathbb{R}^m)$ by

$$v := \varphi u_x + (1-\varphi)u = \varphi(u_x - u) + u.$$

Then

$$tv - tu \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m) \tag{5.46}$$

and

$$t\nabla v = \begin{cases} t\nabla u(x) & \text{in } \overline{Q}_{\lambda\rho}(x_0) \\ \frac{t}{\sigma}\big(\varphi\sigma\nabla u(x) + (1-\varphi)\sigma\nabla u\big) + \big(1-\frac{t}{\sigma}\big)\,\Psi_\rho & \text{in } Q_\rho(x_0)\backslash\overline{Q}_{\lambda\rho}(x_0) \end{cases} \tag{5.47}$$

with $\Psi_\rho := \frac{t}{1-\frac{\tau}{\sigma}}\nabla\varphi\otimes(u_x-u)$. From (5.44) and (5.46) we have

$$tv + (tw - tu_x) - tu \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m),$$

and so, noticing that $\nabla(tw-tu_x)(y) = t\nabla w - t\nabla u_x = 0$ for $\mathscr{L}^N$-a.a. $y\in Q_\rho(x)\backslash\overline{Q}_{\lambda\rho}(x)$,

$$\begin{aligned}
\frac{\mathrm{m}^\varepsilon_{tu}(Q_\rho(x))}{\mathscr{L}^N(Q_{\lambda\rho}(x))} &\leqslant \frac{1}{\mathscr{L}^N(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)} W_\varepsilon\left(y, t\nabla v + t\nabla w - t\nabla u_x\right)\mathbb{1}_{O_\varepsilon}(y)dy \\
&= \frac{1}{\mathscr{L}^N(Q_{\lambda\rho}(x))}\int_{\overline{Q}_{\lambda\rho}(x)} W_\varepsilon\left(y, t\nabla u(x) + t\nabla w - t\nabla u(x)\right)\mathbb{1}_{O_\varepsilon}(y)dy \\
&\quad + \frac{1}{\mathscr{L}^N(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)\backslash\overline{Q}_{\lambda\rho}(x)} W_\varepsilon(y, t\nabla v)\mathbb{1}_{O_\varepsilon}(y)dy \\
&= \frac{1}{\mathscr{L}^N(Q_{\lambda\rho}(x))}\int_{Q_{\lambda\rho}(x)} W_\varepsilon\left(y, t\nabla w\right)\mathbb{1}_{O_\varepsilon}(y)dy \\
&\quad + \frac{1}{\mathscr{L}^N(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)\backslash\overline{Q}_{\lambda\rho}(x)} W_\varepsilon(y, t\nabla v)\mathbb{1}_{O_\varepsilon}(y)dy.
\end{aligned}$$

From (5.45) and the right inequality in (A_5) it follows that

$$\begin{aligned}
\frac{\mathrm{m}^\varepsilon_{tu}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} \leqslant \frac{\mathrm{m}^\varepsilon_{tu}(Q_\rho(x))}{\mathscr{L}^N(Q_{\lambda\rho}(x))} &\leqslant \frac{\mathrm{m}^\varepsilon_{tu_x}(Q_{\lambda\rho}(x))}{\mathscr{L}^N(Q_{\lambda\rho}(x))} + \delta + \frac{\beta(1-\lambda^N)}{\lambda^N} \\
&\quad + \frac{\beta}{\mathscr{L}^N(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)\backslash\overline{Q}_{\lambda\rho}(x)} G(t\nabla v)dy.
\end{aligned} \tag{5.48}$$

On the other hand, taking (5.47) into account and using (A_1), we have

$$G(t\nabla v) \leqslant c_1\left(1 + G(\sigma\nabla u(x)) + G(\sigma\nabla u) + G(\Psi_\rho)\right) \tag{5.49}$$

with $c_1 := 2(\gamma+\gamma^2) > 0$. Moreover, it is easy to see that

$$\|\Psi_\rho\|_{L^\infty(Q_\rho(x);\mathbb{M})} \leqslant \frac{\theta t}{(1-\frac{t}{\sigma})(1-\lambda)}\frac{1}{\rho}\|u-u_x\|_{L^\infty(Q_\rho(x);\mathbb{R}^m)}$$

with

$$\lim_{\rho\to 0}\frac{\theta t}{(1-\frac{t}{\sigma})(1-\lambda)}\frac{1}{\rho}\|u-u_x\|_{L^\infty(Q_\rho(x);\mathbb{R}^m)} = 0 \tag{5.50}$$

because $\lim_{\rho\to0}\frac{1}{\rho}\|u-u_x\|_{L^\infty(Q_\rho(x);\mathbb{R}^m)} = 0$ since $p > N$. From (A_0) and (A_1) there exists $r > 0$ such that

$$c_2 := \sup_{|\xi|\leqslant r} G(\xi) < \infty$$

(see Remark 3.1). By (5.50) there exists $\bar\rho>0$ such that $\frac{\theta t}{(1-\frac{t}{\sigma})(1-\lambda)}\frac{1}{\rho}\|u-u_x\|_{L^\infty(Q_\rho(x);\mathbb{R}^m)}<r$ for all $\rho\in]0,\bar\rho[$. Fix any $\rho\in]0,\bar\rho[$. We then have

$$G(\Psi_\rho)\leqslant c_2. \tag{5.51}$$

From (5.49) and (5.51) it follows that

$$\begin{aligned}\frac{\beta}{\mathscr{L}^N(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)\setminus\overline{Q}_{\lambda\rho}(x)}G(t\nabla v)dy &\leqslant \beta c_1\big(1+G(\sigma\nabla u(x))+c_2\big)\frac{1-\lambda^N}{\lambda^N}\\ &\quad+\frac{\beta c_1}{\mathscr{L}^N(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)\setminus\overline{Q}_{\lambda\rho}(x)}G(\sigma\nabla u(y))dy.\end{aligned}$$

But

$$\begin{aligned}\int_{Q_\rho(x)\setminus\overline{Q}_{\lambda\rho}(x)}G(\sigma\nabla u(y))dy &\leqslant \mathscr{L}^N(Q_\rho(x))⨍_{Q_\rho(x)}|G(\sigma\nabla u(y))-G(\sigma\nabla u(x))|dy\\ &\quad+\mathscr{L}^N(Q_\rho(x)\setminus\overline{Q}_{\lambda\rho}(x))G(\sigma\nabla u(x)),\end{aligned}$$

hence

$$\begin{aligned}\frac{\beta}{\mathscr{L}^N(Q_{\lambda\rho}(x))}\int_{Q_\rho(x)\setminus\overline{Q}_{\lambda\rho}(x)}G(t\nabla v)dy &\leqslant \beta c_1\big(1+2G(\sigma\nabla u(x))+c_2\big)\frac{1-\lambda^N}{\lambda^N}\\ &\quad+\frac{\beta c_1}{\lambda^N}⨍_{Q_\rho(x)}|G(\sigma\nabla u(y))-G(\sigma\nabla u(x))|dy. \end{aligned}\tag{5.52}$$

From (5.48) and (5.52) we deduce that

$$\begin{aligned}\frac{\mathrm{m}^\varepsilon_{tu}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} &\leqslant \frac{\mathrm{m}^\varepsilon_{tu_x}(Q_{\lambda\rho}(x))}{\mathscr{L}^N(Q_{\lambda\rho}(x))}+\delta\\ &\quad+\beta c_1\left(1+\frac{1}{c_1}+2G(\sigma\nabla u(x))+c_2\right)\frac{1-\lambda^N}{\lambda^N}\\ &\quad+\frac{\beta c_1}{\lambda^N}⨍_{Q_\rho(x)}|G(\sigma\nabla u(y))-G(\sigma\nabla u(x))|dy. \end{aligned}\tag{5.53}$$

As $\mathscr{G}(\sigma u)<\infty$, i.e. $G(\sigma\nabla u(\cdot))\in L^1(O)$, we can assert that:

$$G(\sigma\nabla u(x))<\infty; \tag{5.54}$$

$$\lim_{\rho\to 0}⨍_{Q_\rho(x)}|G(\sigma\nabla u(y))-G(\sigma\nabla u(x))|dy=0. \tag{5.55}$$

Moreover, we have:

$$\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\frac{\mathrm{m}^\varepsilon_{tu}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}=\overline{\lim_{\rho\to0}}\frac{\overline{\mathrm{m}}_{tu}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}; \tag{5.56}$$

$$\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\frac{\mathrm{m}^\varepsilon_{tu_x}(Q_{\lambda\rho}(x))}{\mathscr{L}^N(Q_{\lambda\rho}(x))}\leqslant\overline{\lim_{\rho\to0}}\,\overline{\lim_{\varepsilon\to0}}\frac{\mathrm{m}^\varepsilon_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}=\overline{\lim_{\rho\to0}}\frac{\overline{\mathrm{m}}_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))}. \tag{5.57}$$

Letting $\varepsilon \to 0$, $\rho \to 0$ and $\lambda \to 1^-$ in (5.53) and using (5.54), (5.55), (5.56) and (5.57) we conclude that

$$\varlimsup_{\rho\to 0} \frac{\overline{\mathrm{m}}_{tu}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} \leqslant \varlimsup_{\rho\to 0} \frac{\overline{\mathrm{m}}_{tu_x}(Q_\rho(x))}{\mathscr{L}^N(Q_\rho(x))} + \delta,$$

and (5.43) follows by letting $\delta \to 0$.

Conclusion of the steps 1, 2 and 3. As a direct consequence of (5.33), (5.41) and (5.43) together with Remarks 5.1 and 5.2, we have the following lemma.

Lemma 5.3. *For every $t \in]0,1[$ and every $u \in W^{1,p}(O;\mathbb{R}^m)$ such that $\mathscr{G}(tu) < \infty$ and $\mathscr{G}(\sigma u) < \infty$ for some $\sigma \in]t,1[$, one has*

$$\Gamma(L^p(O_\varepsilon,O))\text{-}\varlimsup_{\varepsilon\to 0} I_\varepsilon(tu) \leqslant \int_O \varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,t\nabla u(x))dx.$$

Step 4: end of the proof. Let $u \in W^{1,p}(O;\mathbb{R}^m)$. We have to prove that

$$\Gamma(L^p(O_\varepsilon,O))\text{-}\varlimsup_{\varepsilon\to 0} I_\varepsilon(u) \leqslant \int_O \varliminf_{t\to 1^-}\varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,t\nabla u(x))dx. \tag{5.58}$$

Without loss of generality we can assume that

$$\int_O \varliminf_{t\to 1^-}\varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,t\nabla u(x))dx =: I_{\lim}(u) < \infty. \tag{5.59}$$

Then, by Proposition 3.7(i) we have

$$\nabla u(x) \in \overline{\mathbb{G}_{\infty,x}} \text{ for } \mathscr{L}^N\text{-a.a. } x \in O \tag{5.60}$$

and, for $\mathscr{L}^N$-a.e. $x \in O$,

$$\varliminf_{t\to 1^-}\varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,t\nabla u(x)) = \lim_{t\to 1^-}\varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,t\nabla u(x)). \tag{5.61}$$

Substep 4-1: proving (5.58) under the constraint $\boldsymbol{\nabla u(x) \in \mathrm{int}(\mathbb{G}_{\infty,x})}$ for $\boldsymbol{\mathscr{L}^N}$-a.a. $\boldsymbol{x \in O}$. Assume that

$$\nabla u(x) \in \mathrm{int}(\mathbb{G}_{\infty,x}) \text{ for } \mathscr{L}^N\text{-a.a. } x \in O. \tag{5.62}$$

Then, since (A_0)–(A_1) (see Remark 3.2(iii)) implies that $t\nabla u(x) \in \mathrm{int}(\mathbb{G}_{\infty,x})$ for all $t \in]0,1[$ and for $\mathscr{L}^N$-a.a. $x \in O$, by (A_4) we have

$$\varliminf_{t\to 1^-} G_\infty(x,t\nabla u(x)) \geqslant G_\infty(x,\nabla u(x)) \text{ for } \mathscr{L}^N\text{-a.a. } x \in O. \tag{5.63}$$

Using (5.63) and the left inequality in (A_5) we see that

$$\frac{1}{\alpha} I_{\lim}(u) \geqslant \int_O \varliminf_{t\to 1^-} G_\infty(x,t\nabla u(x))dx \geqslant \int_O G_\infty(x,\nabla u(x))dx =: \mathscr{G}_\infty(u),$$

hence, by (5.59), $\mathscr{G}_\infty(u) < \infty$, and so taking (A_3) into account, from (5.62) it follows that

$$\mathscr{G}(u) < \infty. \tag{5.64}$$

But, by (A_1) we see that for every $t \in]0,1[$, $\mathscr{G}(tu) \leqslant \gamma \mathscr{L}^N(O)(1+G(0)) + \gamma \mathscr{G}(u)$, hence, by ($A_0$) and (5.64), $\mathscr{G}(tu) < \infty$ for all $t \in]0,1[$, and so, by Lemma 5.3 we have

$$\Gamma(L^p(O_\varepsilon, O))\text{-}\overline{\lim_{\varepsilon\to 0}}\, I_\varepsilon(tu) \leqslant \int_O \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x, t\nabla u(x))dx \text{ for all } t \in]0,1[. \tag{5.65}$$

On the other hand, from the right inequality in (A_5) we see that for every $t \in]0,1[$,

$$\begin{aligned} \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x, t\nabla u(\cdot)) &\leqslant \beta\big(1 + G_\infty(x, t\nabla u(\cdot))\big) \\ &\leqslant \beta\big(1 + G(t\nabla u(\cdot))\big), \end{aligned}$$

and consequently, by using (A_1),

$$\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x, t\nabla u(\cdot)) \leqslant \beta\big(1 + G(0) + G(\nabla u(\cdot)\big) := f(\cdot) \text{ for all } t \in]0,1[$$

with $f \in L^1(O)$ by (A_0) and (5.64). Taking (5.61) into account, from Lebesgue's dominated convergence theorem we deduce that

$$\underline{\lim_{t\to 1^-}} \int_O \overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x, t\nabla u(x))dx = \int_O \underline{\lim_{t\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x, t\nabla u(x))dx.$$

From (5.65) we conclude that

$$\underline{\lim_{t\to 1^-}}\,\Gamma(L^p(O_\varepsilon, O))\text{-}\overline{\lim_{\varepsilon\to 0}}\, I_\varepsilon(tu) \leqslant \int_O \underline{\lim_{t\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x, t\nabla u(x))dx,$$

and (5.58) follows because $\Gamma(L^p(O_\varepsilon, O))\text{-}\overline{\lim}_{t\to\infty} I_\varepsilon$ is lsc with respect to the $L^p(O;\mathbb{R}^m)$-convergence and $tu \to u$ in $L^p(O;\mathbb{R}^m)$ as $t \to 1^-$.

Substep 4-2: proof of (5.58). First of all, from (A_6) and Lemma 2.11 we can assert that $W_\infty := \overline{\lim}_{\rho\to 0}\overline{\lim}_{\varepsilon\to 0}\mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}]$ is ru-usc with the constant function $\|a_0\|_{L^\infty}$. Moreover, by (A_5) we see that for every $x \in O$, the effective domain of $W_\infty(x,\cdot)$ is equal to $\mathbb{G}_{\infty,x}$. Taking (A_0)–(A_1) (see Remark 3.2(iii)) into account, from Theorem 2.9(ii) it follows that

$$\widehat{W}_\infty := \underline{\lim_{t\to 1^-}}\,\overline{\lim_{\rho\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\,\mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}] \text{ is ru-usc with the constant function } \|a_0\|_{L^\infty}. \tag{5.66}$$

From (5.59) we see that $\nabla u(x) \in \widehat{\mathbb{W}}_{\infty,x}$ for $\mathscr{L}^N$-a.a. $x \in O$, where $\widehat{\mathbb{W}}_{\infty,x}$ denotes the effective domain of $\widehat{W}_\infty(x,\cdot)$. Hence, for every $t \in]0,1[$,

$$\begin{aligned} \int_O \widehat{W}_\infty(x, t\nabla u(x))dx &\leqslant \big(1 + \Delta^{\|a_0\|_{L^\infty}}_{\widehat{W}_\infty}(t)\big)\int_O \widehat{W}_\infty(x, \nabla u(x))dx \\ &\quad + \Delta^{\|a_0\|_{L^\infty}}_{\widehat{W}_\infty}(t)\|a_0\|_{L^\infty}\mathscr{L}^N(O) \end{aligned}$$

with $\Delta^{\|a_0\|_{L^\infty}}_{\widehat{W}_\infty}(t) := \sup_{x\in O}\sup_{\xi\in\widehat{\mathbb{W}}_{\infty,x}} \frac{\widehat{W}_\infty(x,t\xi) - \widehat{W}_\infty(x,\xi)}{\|a_0\|_{L^\infty} + \widehat{W}_\infty(x,\xi)}$, i.e.

$$I_{\lim}(tu) \leqslant \big(1 + \Delta^{\|a_0\|_{L^\infty}}_{\widehat{W}_\infty}(t)\big) I_{\lim}(u) + \Delta^{\|a_0\|_{L^\infty}}_{\widehat{W}_\infty}(t)\|a_0\|_{L^\infty}\mathscr{L}^N(O) \tag{5.67}$$

for all $t \in]0,1[$. Using (5.59) we see that

$$I_{\lim}(tu) < \infty \text{ for all } t \in]0,1[. \tag{5.68}$$

On the other hand, from (5.60) and (A_0)–(A_1) (see Remark 3.2(iii)) we deduce that

$$\nabla(tu)(x) \in \mathrm{int}(\mathbb{G}_{\infty,x}) \text{ for all } t \in]0,1[\text{ and } \mathscr{L}^N\text{-a.a. } x \in O. \tag{5.69}$$

According to (5.69) and (5.68), from the substep 4-1 we can assert that

$$\Gamma(L^p(O_\varepsilon, O))\text{-}\varlimsup_{\varepsilon\to 0} I_\varepsilon(tu) \leqslant I_{\lim}(tu)$$

for all $t \in]0,1[$, and so, taking (5.67) into account,

$$\Gamma(L^p(O_\varepsilon, O))\text{-}\varlimsup_{\varepsilon\to 0} I_\varepsilon(tu) \leqslant \big(1 + \Delta^{\|a_0\|_{L^\infty}}_{\widehat{W}_\infty}(t)\big) I_{\lim}(u) + \Delta^{\|a_0\|_{L^\infty}}_{\widehat{W}_\infty}(t)\|a_0\|_{L^\infty}\mathscr{L}^N(O) \tag{5.70}$$

for all $t \in]0,1[$. Moreover, by (5.66) we have $\varlimsup_{t\to 1^-} \Delta^{\|a_0\|_{L^\infty}}_{\widehat{W}_\infty}(t) \leqslant 0$. Hence, letting $t \to 1^-$ in (5.70) we conclude that

$$\varliminf_{t\to 1^-} \Gamma(L^p(O_\varepsilon, O))\text{-}\varlimsup_{\varepsilon\to 0} I_\varepsilon(tu) \leqslant I_{\lim}(u),$$

and (5.58) follows because $\Gamma(L^p(O_\varepsilon, O))\text{-}\varlimsup_{\varepsilon\to 0} I_\varepsilon$ is lsc with respect to the $L^p(O;\mathbb{R}^m)$-convergence and $tu \to u$ in $L^p(O;\mathbb{R}^m)$ as $t \to 1^-$. ■

5.3. **Proof of the $\Gamma(L^p(O_\varepsilon, O))$-convergence theorem.** Here we prove Theorem 3.6.

Proof of Theorem 3.6. By (A_5) we see that

$$\alpha G_\infty(x,\xi) \leqslant \varlimsup_{\rho\to 0}\varliminf_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,\xi) \leqslant \varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,\xi) \leqslant \beta\big(1 + G_\infty(x,\xi)\big)$$

for all $(x,\xi) \in O \times \mathbb{M}$. So, for every $x \in O$, one has

$$\mathrm{dom}\left(\varlimsup_{\rho\to 0}\varliminf_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,\cdot)\right) = \mathrm{dom}\left(\varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,\cdot)\right) = \mathbb{G}_{\infty,x}, \tag{5.71}$$

where $\mathrm{dom}\left(\varlimsup_{\rho\to 0}\varliminf_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,\cdot)\right)$ and $\mathrm{dom}\left(\varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,\cdot)\right)$ denotes the effective domain of $\varlimsup_{\rho\to 0}\varliminf_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,\cdot)$ and $\varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,\cdot)$ respectively. Let $(x,\xi) \in O \times \mathbb{M}$. If $\xi \notin \overline{\mathbb{G}_{\infty,x}}$ then there exists $t_\xi \in]0,1[$ such that $t\xi \notin \mathbb{G}_{\infty,x}$ for all $t \in [t_\xi, 1[$. Hence:

- if $\xi \notin \overline{\mathbb{G}_{\infty,x}}$ then, by (5.71),

$$\varlimsup_{\rho\to 0}\varliminf_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,t\xi) = \varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,t\xi) = \infty \text{ for all } t \in [t_\xi, 1[;$$

- if $\xi \in \overline{\mathbb{G}_{\infty,x}}$ then, from (A_0)–(A_1) (see Remark 3.2(iv)), we have $t\xi \in \mathrm{int}(\mathbb{G}_{\infty,x})$ for all $t \in]0,1[$, and so, by (A_7),

$$\varlimsup_{\rho\to 0}\varliminf_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,t\xi) \geqslant \varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,t\xi) \text{ for all } t \in]0,1[.$$

It follows that

$$\varliminf_{t\to 1^-}\varlimsup_{\rho\to 0}\varliminf_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,t\xi) \geqslant \varliminf_{t\to 1^-}\varlimsup_{\rho\to 0}\varlimsup_{\varepsilon\to 0} \mathscr{H}^\rho[W_\varepsilon \mathbb{1}_{O_\varepsilon}](x,t\xi)$$

for all $(x,\xi)\in O\times\mathbb{M}$. From Propositions 3.4 and 3.5 we deduce that

$$\begin{aligned}\Gamma(L^p(O_\varepsilon,O))\text{-}\varliminf_{\varepsilon\to0} I_\varepsilon(u) &\geqslant \int_O \varliminf_{t\to1^-}\varlimsup_{\rho\to0}\varliminf_{\varepsilon\to0}\mathscr{H}^\rho[W_\varepsilon\mathbb{1}_{O_\varepsilon}](x,t\nabla u(x))dx\\ &\geqslant \int_O \varliminf_{t\to1^-}\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\mathscr{H}^\rho[W_\varepsilon\mathbb{1}_{O_\varepsilon}](x,t\nabla u(x))dx\\ &\geqslant \Gamma(L^p(O_\varepsilon,O))\text{-}\varlimsup_{\varepsilon\to0} I_\varepsilon(u).\end{aligned}$$

for all $u\in W^{1,p}(O;\mathbb{R}^m)$. Hence

$$\Gamma(L^p(O_\varepsilon,O))\text{-}\lim_{\varepsilon\to0} I_\varepsilon(u)=\int_O W_{\lim}(x,\nabla u(x))dx$$

for all $u\in W^{1,p}(O;\mathbb{R}^m)$. ■

5.4. **Proof of the homogenization theorem.** Here we prove Theorem 3.11.

Proof of Theorem 3.11. Taking (H_0)–(H_1) and Theorem 3.8 into account, we see that (C_0)–(C_2) hold. Since G is p-coercive convex function satisfying (H_2)-(H_3), it is clear that (A_0)–(A_4) hold. For each $\varepsilon>0$, let $W_\varepsilon:O\times\mathbb{M}\to[0,\infty]$ be defined by

$$W_\varepsilon(x,\xi):=W\left(\frac{x}{\varepsilon},\xi\right).$$

From (H_4) we can assert that for every $\varepsilon>0$ and every $(x,\xi)\in\mathbb{R}^N\times\mathbb{M}$,

$$\alpha G(\xi)\mathbb{1}_E\left(\frac{x}{\varepsilon}\right)\leqslant W\left(\frac{x}{\varepsilon},\xi\right)\mathbb{1}_E\left(\frac{x}{\varepsilon}\right)\leqslant\beta(1+G(\xi))\mathbb{1}_E\left(\frac{x}{\varepsilon}\right),$$

i.e.

$$\alpha G(\xi)\mathbb{1}_{E_\varepsilon}(x)\leqslant W_\varepsilon(x,\xi)\mathbb{1}_{E_\varepsilon}(x)\leqslant\beta(1+G(\xi))\mathbb{1}_{E_\varepsilon}(x),$$

and so (A_5) is verified. By (H_5) we see that for each $\xi\in\mathbb{M}$, $x\mapsto W(x,\xi)\mathbb{1}_E(x)$ is 1-periodic because, by assumption, $\mathbb{1}_E$ is 1-periodic. Hence, taking (H_6) into account, from Lemma 2.12 we deduce that (A_6) holds. So, it remains to prove that (A_7) is satisfied, i.e. according to (H_2), for every $x\in O$ and every $\xi\in\mathrm{int}(\mathbb{G})$,

$$\varlimsup_{\rho\to0}\varliminf_{\varepsilon\to0}\frac{\mathcal{S}^\xi\left(\frac1\varepsilon Q_\rho(x)\right)}{\mathscr{L}^N\left(\frac1\varepsilon Q_\rho(x)\right)}\geqslant\varlimsup_{\rho\to0}\varlimsup_{\varepsilon\to0}\frac{\mathcal{S}^\xi\left(\frac1\varepsilon Q_\rho(x)\right)}{\mathscr{L}^N\left(\frac1\varepsilon Q_\rho(x)\right)}\tag{5.72}$$

with $\mathcal{S}^\xi:\mathcal{O}_\mathrm{b}(\mathbb{R}^N)\to[0,\infty]$ defined by

$$\mathcal{S}^\xi(A):=\inf\left\{\int_A W(x,\xi+\nabla\varphi(x))\mathbb{1}_E(x)dx:\varphi\in W^{1,p}_0(A;\mathbb{R}^m)\right\},$$

where $\mathcal{O}_\mathrm{b}(\mathbb{R}^N)$ denotes the class of all bounded open subsets of $\mathbb{R}^N$. Fix $x\in O$ and $\xi\in\mathrm{int}(\mathbb{G})$. First of all, it is clear that $\mathcal{S}^\xi$ is subadditive. Then, by using (H_5) and the fact that $\mathbb{1}_E$ is 1-periodic, it easy to see that $\mathcal{S}^\xi$ is $\mathbb{Z}^N$-invariant. Finally, from the right inequality in (H_4) we deduce that for every $A\in\mathcal{O}_\mathrm{b}(\mathbb{R}^N)$, $\mathcal{S}^\xi(A)\leqslant C_\xi\mathscr{L}^N(A)$ with $C_\xi:=\beta(1+G(\xi))<\infty$ because $\xi\in\mathrm{int}(\mathbb{G})$. From Theorem 4.4 it follows that for every $\rho>0$,

$$\lim_{\varepsilon\to0}\frac{\mathcal{S}^\xi\left(\frac1\varepsilon Q_\rho(x)\right)}{\mathscr{L}^N\left(\frac1\varepsilon Q_\rho(x)\right)}=\inf_{k\geqslant1}\frac{\mathcal{S}^\xi(]0,k[^N)}{k^N},$$

which implies (5.72). Thus (C_0)–(C_2) and (A_0)–(A_7) are satisfied, and the theorem follows from Theorem 3.6 and Proposition 3.7(i) (with Remark 3.9) in noticing that $\inf_{k\geqslant 1} \frac{\mathcal{S}^\xi(]0,k[^N)}{k^N} = W_{\rm hom}(\xi)$. ■

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