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Abstract

A proper n-coloring of a graph G is an assignment of colors from $\{1, \ldots, n\}$ to its vertices such that no two adjacent vertices get assigned the same color. The chromatic number of G, denoted by $\chi(G)$, refers to the smallest n such that G admits a proper n-coloring. This notion naturally extends to edge-colorings (resp. total-colorings) when edges (resp. both vertices and edges) are to be colored, and this provides other parameters of G: its chromatic index $\chi'(G)$ and its total chromatic number $\chi''(G)$.

These coloring notions are among the most fundamental ones of the graph coloring theory. As such, they gave birth to hundreds of studies dedicated to several of their aspects, including generalizations to more general structures such as oriented graphs. They include notably the notions of oriented n-colorings and oriented n-arc-colorings, which stand as natural extensions of their undirected counterparts, and which have been receiving increasing attention.

Our goal is to introduce a missing piece in this line of work, namely the oriented counterparts of proper n-total-colorings and total chromatic number. We first define these notions and show that they share properties and connections with oriented (arc) colorings that are reminiscent of those shared by their undirected counterparts. We then focus on understanding the oriented total chromatic number of particular types of oriented graphs, such as oriented forests, cycles, and some planar graphs. Finally, we establish a full complexity dichotomy for the problem of determining whether an oriented graph is totally k-colorable.

Throughout this work, each of our results is compared to what is known regarding the oriented chromatic number and oriented chromatic index. We also disseminate some directions for further research on the oriented total chromatic number.

Keywords: oriented coloring; oriented arc-coloring; total-coloring; oriented graph.

1. Introduction

In this article, we introduce and study a generalization of *proper total-colorings* of (undirected) graphs to *oriented graphs*. Previous work on related topics proved that there are usually several possible ways of generalizing coloring-related notions, and this still holds in the current context. Our main source of motivation for introducing proper total-colorings of oriented graphs the way we do, is that their connections with *oriented colorings* and *oriented arc-colorings* of oriented graphs are reminiscent of the connections between the more classical notions of proper total-colorings, proper colorings, and proper edge-colorings of graphs.

We assume that the reader is familiar with the most standard notions and terminology on graphs; for anything not defined here, we refer them to any monograph on the topic.

Definitions. Given a graph[1] G, an *orientation* $\overrightarrow{G}$ of G is obtained by assigning a direction to every edge joining any two vertices u and v, resulting in an *arc* either uv from u to v (v being an *out-neighbor* of u, u being an *in-neighbor* of v, and uv being *in-coming* to v and *out-going* from u), or vu from v to u (and *vice versa*). Such an orientation $\overrightarrow{G}$ is also called an *oriented graph* with *underlying graph G*. Alternatively, an oriented graph is a directed graph without any directed cycle of length 1 or 2. For an oriented graph $\overrightarrow{G}$, we denote by $V(\overrightarrow{G})$ and $A(\overrightarrow{G})$ its vertex and arc sets.

Oriented total-colorings. As mentioned earlier, there are multiple ways to generalize classical coloring notions from graphs to oriented graphs. The coloring notions for oriented graphs we investigate throughout this work are based on the following simple notion of elements flowing to each other, which, although not classical, provides a general framework encapsulating all these upcoming oriented coloring notions. Let uv and vw be two distinct arcs of an oriented graph. We say that the vertex u *flows* to the arc uv and to the vertex v, and that the arc uv *flows* to the vertex v and to the arc vw. Conversely, we say that the vertex w *flows* from the vertex v and from the arc vw, and that the arc vw *flows* from the vertex v and from the arc uv.

Through this notion of elements flowing to each other, we are now ready to recall and introduce the three main coloring notions to be investigated in this paper. Let us start with the new, most general, notion we introduce in this paper (see below for reasons why we introduce it in this precise way). Let $\overrightarrow{G}$ be an oriented graph, and let $n \geqslant 1$. An *oriented n-total-coloring* of $\overrightarrow{G}$ is a function $\tau : V(\overrightarrow{G}) \cup A(\overrightarrow{G}) \to \{1, \dots, n\}$ satisfying the following conditions:

1. if $a, b \in V(\overrightarrow{G}) \cup A(\overrightarrow{G})$ are such that a flows to b, then $\tau(a) \neq \tau(b)$;

2. if $a, b, c, d \in V(\overrightarrow{G}) \cup A(\overrightarrow{G})$ are such that a flows to b and c flows to d, then $\tau(a) = \tau(d)$ implies $\tau(b) \neq \tau(c)$.

The *oriented total chromatic number* of $\overrightarrow{G}$, denoted by $\chi''_o(\overrightarrow{G})$, is the minimum n such that $\overrightarrow{G}$ admits an oriented n-total-coloring. Now, if, in the definition above, we restrict the domain of the function τ to $V(\overrightarrow{G})$, then what we get is exactly the definition of *oriented n-colorings* of oriented graphs [17]. To make it clear, note that an oriented coloring is a proper assignment of colors to the vertices (i.e., no two adjacent vertices get assigned the same color) such that if we focus on the vertices being assigned any pair of distinct colors i and j, then we have the property that all arcs joining such vertices always go the same direction, i.e., either from the vertices colored i to those colored j, or *vice versa*. Similarly, if we restrict the domain of τ to $A(\overrightarrow{G})$, then we obtain exactly the definition of *oriented n-arc-colorings* of oriented graphs [12]. Recall that the *oriented chromatic number* $\chi_o(\overrightarrow{G})$ of $\overrightarrow{G}$ is the smallest n such that $\overrightarrow{G}$ admits oriented n-colorings, while the *oriented chromatic index* $\chi'_o(\overrightarrow{G})$ of $\overrightarrow{G}$ is the smallest n such that $\overrightarrow{G}$ admits oriented n-arc-colorings. Thus, as suggested earlier, our definition of oriented total-colorings actually encapsulates that of oriented colorings and oriented arc-colorings, which can be seen as a combination of the two. This will be made even more apparent through later connections, which we postpone to Section 2 for the sake of keeping the current introduction short.

Related work. Recall that oriented colorings were introduced as early as in the 1990s, as they emerged as a generalization of proper colorings of graphs to oriented graphs in the famous series of works led by Courcelle on monadic second order logic (see e.g. [6]). Soon after, the work [13], dedicated to the topic, generated a lot of interest into this new area which, since then, has been intensively explored by several authors, see [2, 3, 4, 5, 8, 9, 1, 17]. This all opened the way for many later works generalizing other related classical coloring notions to the realm of oriented graphs, such as the clique number [11], the chromatic index [12], the achromatic number [16], and the list chromatic number [18], to name a few. In a sense, our main intention in this work takes place in that line of research, since, as far as we are aware, no attempt to generalize proper total-colorings to oriented graphs has been made to date.

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[1] Throughout this work, the term *graph* refers to a simple undirected graph.

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Organization of the paper. We start off in Section 2 by giving an alternative definition of oriented colorings using the so-called graph homomorphisms, which also leads to a better illustration of how the three kinds of colorings relate to each other. As a warm up, we then investigate oriented graphs with small oriented total chromatic number, as well as easy classes of oriented graphs (namely oriented forests and cycles) in Section 3. We then go more general in Section 4, by looking for connections between the oriented total chromatic number and other graph chromatic parameters. Next, in Section 5, we prove that determining whether a given oriented graph $\overrightarrow{G}$ verifies $\chi_o''(\overrightarrow{G}) = k$ is polynomial-time solvable for every $k \leqslant 5$ and NP-complete for every $k \geqslant 6$. Lastly, in Section 6, we focus on the case where the underlying graph is planar, with possibly some restrictions such as having a large girth or being outerplanar.

As mentioned earlier, oriented (arc) colorings already received an enormous amount of attention over the years, as attested notably by the survey [17]. As a result, many directions can be taken to study the total version we introduce. We disseminate some throughout the paper, and summarize them in the concluding Section 7.

2. Preliminaries

2.1. Colorings as graph homomorphisms

A common way to study oriented colorings is through the lens of homomorphisms of oriented graphs, a notion we recall now. A *homomorphism* of an oriented graph $\overrightarrow{G}$ to an oriented graph $\overrightarrow{H}$ is a vertex-mapping $f : V(\overrightarrow{G}) \to V(\overrightarrow{H})$ such that for every arc uv of $\overrightarrow{G}$, the arc $f(u)f(v)$ is also in $\overrightarrow{H}$. If $\overrightarrow{G}$ admits a homomorphism f to $\overrightarrow{H}$, then we denote it by $f : \overrightarrow{G} \to \overrightarrow{H}$, or more generally by $\overrightarrow{G} \to \overrightarrow{H}$. Note that a homomorphism of $\overrightarrow{G}$ to $\overrightarrow{H}$ provides a coloring (which we call an $\overrightarrow{H}$-*coloring*) of the vertices of $\overrightarrow{G}$ with colors from $V(\overrightarrow{H})$. Actually, it can be checked from the definition that any $\overrightarrow{H}$-coloring of $\overrightarrow{G}$ is an oriented coloring. In particular, any oriented n-coloring of $\overrightarrow{G}$ is an $\overrightarrow{H}$-coloring for some oriented graph $\overrightarrow{H}$ with $|V(\overrightarrow{H})| = n$, and $\chi_o(\overrightarrow{G})$ is thus the smallest n such that $\overrightarrow{G}$ admits a homomorphism to an oriented graph with order n.
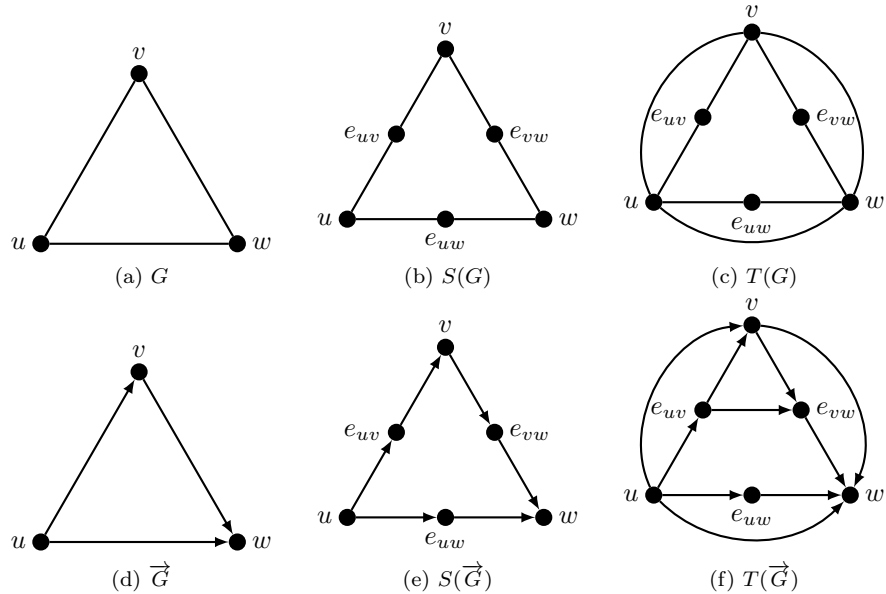
Figure 1: Constructing the subdivision and total versions of a graph or oriented graph.

These observations extend to oriented arc-colorings and oriented total-colorings through the following transformations (see Figure 1). Recall that a k-*path* in a graph is a path of length k, and that an orientation of a k-path is a *directed k-path* if its arcs are oriented from one end to the other. Similarly, a k-*cycle* C_k is a cycle of length k, and a *directed k-cycle* $\overrightarrow{C_k}$ is an orientation of a k-cycle in which all arcs follow the same direction. For an oriented graph $\overrightarrow{G}$, its *subdivision*

digraph $S(\overrightarrow{G})$ is obtained when replacing each of its arcs uv by a directed 2-path $ue_{uv}v$, where e_{uv} is a new vertex. Now, the *total digraph* of $\overrightarrow{G}$, denoted by $T(\overrightarrow{G})$, is obtained from $S(\overrightarrow{G})$ by adding an arc from any vertex u to any vertex v if, in $S(\overrightarrow{G})$, there is a directed path of length at most 2 from u to v. In $S(\overrightarrow{G})$ and $T(\overrightarrow{G})$, the e_{uv}'s are called *new vertices* while the other vertices are called *original vertices*. Note that the subgraph of $T(\overrightarrow{G})$ induced by the original vertices is $\overrightarrow{G}$. Moreover, the subgraph of $T(\overrightarrow{G})$ induced by the new vertices is the *line digraph* $L(\overrightarrow{G})$ of $\overrightarrow{G}$ (see e.g. [12]).

Remark now that an oriented arc-coloring of $\overrightarrow{G}$ can be seen as a homomorphism $L(\overrightarrow{G}) \to \overrightarrow{H}$ for some oriented graph $\overrightarrow{H}$ (which yields an $\overrightarrow{H}$-*arc-coloring* of $\overrightarrow{G}$), while an oriented total-coloring of $\overrightarrow{G}$ can be seen as a homomorphism $T(\overrightarrow{G}) \to \overrightarrow{H}$ for some oriented graph $\overrightarrow{H}$ (which yields an $\overrightarrow{H}$-*total-coloring* of $\overrightarrow{G}$). Now, as for the oriented chromatic number, the oriented chromatic index $\chi'_o(\overrightarrow{G})$ of $\overrightarrow{G}$ is the smallest n such that $\overrightarrow{G}$ admits an $\overrightarrow{H}$-arc-coloring for some oriented graph $\overrightarrow{H}$ of order n, while the oriented total chromatic number $\chi''_o(\overrightarrow{G})$ of $\overrightarrow{G}$ is the smallest n such that $\overrightarrow{G}$ admits an $\overrightarrow{H}$-total-coloring for an oriented graph $\overrightarrow{H}$ of order n.

It is worth reminding that these connections between the three coloring notions are reminiscent of similar connections that exist between their undirected counterparts. Indeed, proper n-colorings of G can also be expressed as homomorphisms of G to graphs H of order n, where, recall, a homomorphism $f : V(G) \to V(H)$ of G to H is a vertex-mapping of $V(G)$ to $V(H)$ that preserves all edges. In turn, proper n-edge-colorings and proper n-total-colorings of G can be expressed as homomorphisms of graphs obtained from G to graphs H of order n. Namely, from G, one can obtain its subdivision graph $S(G)$ by replacing every edge uv by a 2-path $ue_{uv}v$, its total graph $T(G)$ by adding edges between vertices at distance 2 in $S(G)$, and its line graph $L(G)$ by considering the subgraph of $S(G)$ induced by the e_{uv}'s. As previously, proper n-edge-colorings and proper n-total-colorings can now be seen as homomorphisms of $L(G)$ and $T(G)$ to graphs of order n, respectively.

Using this alternate definition, we can state the following property, whose proof is an easy adaptation of the corresponding undirected statement.

Proposition 2.1. *If $\overrightarrow{G}$ and $\overrightarrow{H}$ are two oriented graphs such that $\overrightarrow{G} \to \overrightarrow{H}$, then $\chi''_o(\overrightarrow{G}) \leqslant \chi''_o(\overrightarrow{H})$.*

2.2. Cliques in oriented graphs

The *clique number* $\omega(G)$ of a graph G, defined as the size of a largest clique (or complete subgraph) in G, stands as an obvious lower bound for the chromatic number, i.e., $\omega(G) \leqslant \chi(G)$ always holds. This notion was extended to oriented graphs as follows. Let $\overrightarrow{G}$ be an oriented graph. We say that a subset R of vertices of $\overrightarrow{G}$ is a *relative oriented clique* if the vertices of R must all be assigned distinct colors in every oriented coloring of $\overrightarrow{G}$. As will be made apparent in upcoming Lemma 2.2, R being a relative oriented clique does not mean that $\chi_o(\overrightarrow{G}[R]) = |R|$, since vertices of R may be forced to receive distinct colors because of vertices and arcs of $\overrightarrow{G}$ not in $\overrightarrow{G}[R]$. This justifies the introduction of the distinct notion of *absolute oriented cliques*, which are subsets S of vertices inducing an oriented graph with oriented chromatic number $|S|$. We denote by $\omega_a(\overrightarrow{G})$ and $\omega_r(\overrightarrow{G})$ the size of the largest absolute oriented cliques and relative oriented cliques of $\overrightarrow{G}$, respectively. Note that we get from the definitions that $\omega_a(\overrightarrow{G}) \leqslant \omega_r(\overrightarrow{G}) \leqslant \chi_o(\overrightarrow{G})$ always holds, which generalized the aforementioned naive lower bound.

In this work, we will only need the following characterization of relative oriented cliques. For more details on all these notions, we refer the interested reader to e.g. [14].

Lemma 2.2 ([11]). *For an oriented graph $\overrightarrow{G}$, a subset $R \subseteq V(\overrightarrow{G})$ is a relative oriented clique if and only if every two of its vertices are either adjacent or joined by a directed 2-path (in $\overrightarrow{G}$).*

3. Oriented graphs of small oriented total chromatic number

In this section, we provide a description of all oriented graphs with oriented total chromatic number at most 5. We handle the totally 5-chromatic oriented graphs for later, and start with the totally 4-colorable ones.

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Recall that, in an oriented graph, a *source* is a vertex incident to no in-coming arcs, while a *sink* is a vertex incident to no out-going arcs.

Theorem 3.1. *For every oriented graph $\overrightarrow{G}$, we have the following:*

1. $\chi_o''(\overrightarrow{G}) \leqslant 1$ *if and only if $V(\overrightarrow{G})$ is an independent set.*

2. $\chi_o''(\overrightarrow{G}) \notin \{2, 4\}$.

3. $\chi_o''(\overrightarrow{G}) \leqslant 3$ *if and only if every vertex of $\overrightarrow{G}$ is either a source or a sink.*

Proof. Let $\overrightarrow{G}$ be an oriented graph. If $\overrightarrow{G}$ has no arcs, then assigning the same color to every vertex yields an oriented 1-total-coloring of $\overrightarrow{G}$. Otherwise, $\overrightarrow{G}$ contains a directed 1-path $\overrightarrow{P}$ as a subgraph. This means that $\overrightarrow{P} \to \overrightarrow{G}$ and by Proposition 2.1, we get $\chi_o''(\overrightarrow{G}) \geqslant \chi_o''(\overrightarrow{P}) = 3$.

Now, if $\overrightarrow{G}$ contains a directed 2-path $\overrightarrow{P}$, then we again have $\overrightarrow{P} \to \overrightarrow{G}$, and thus $\chi_o''(\overrightarrow{G}) \geqslant \chi_o''(\overrightarrow{P}) = 5$. Otherwise, $\overrightarrow{G}$ does not contain any directed 2-path. In particular, every vertex is either a source or a sink, and assigning color 1 to the sources and isolated vertices, color 2 to the arcs, and color 3 to the sinks yields an oriented 3-total-coloring of $\overrightarrow{G}$.

Note that the previous arguments imply the second item of the statement holds. $\square$

Note that Theorem 3.1 indicates that some values cannot appear as the oriented total chromatic number of any oriented graph. As a byproduct of the proof of Theorem 5.1 (and more precisely Lemma 5.3), we know that 2 and 4 are actually the only such values.

We now focus on totally 5-chromatic oriented graphs. We first need the following definitions. Let $\overrightarrow{G}$ be an oriented graph and $\gamma = v_0 \cdots v_k v_0$ be a closed walk in the underlying graph G. For every $i \in \{0, \ldots, k\}$, we say that $v_i v_{i+1}$ (where, by convention, $v_{k+1} = v_0$) is a *forward edge* if $v_i v_{i+1} \in A(\overrightarrow{G})$ and a *backward edge* otherwise. Note that some edges of G can be counted as forward and backward if γ goes several times through them. The *signature* of γ is the absolute value of the difference between the number of forward and backward edges of γ.

Theorem 3.2. *An oriented graph $\overrightarrow{G}$ verifies $\chi_o''(\overrightarrow{G}) \leqslant 5$ if and only if one of the following holds:*

- $\overrightarrow{G}$ *does not contain a directed 3-path.*

- *The signature of every closed walk of G is a multiple of 5.*

We say that $\overrightarrow{G}$ is *consistent* if it fulfils one of the two conditions. Note that it is equivalent to require that 5 divides the signature of every cycle of G.
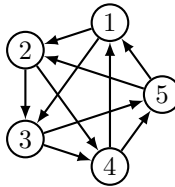

Figure 2: The tournament $\overrightarrow{T_5}$.

Proof of Theorem 3.2. Assume that there exists an oriented 5-total-coloring f of $\overrightarrow{G}$ and that $\overrightarrow{G}$ contains a directed 3-path $wxyz$. Without loss of generality, in $T(\overrightarrow{G})$ we may assume that $w, e_{wx}, x, e_{xy}, y, e_{yz}, z$ have colors 1 to 5 in this order. Moreover, we must have $f(e_{yz}) = f(w) = 1$ and $f(z) = f(e_{wx}) = 2$. In particular, f is actually a $\overrightarrow{T_5}$-coloring of $T(\overrightarrow{G})$ where $\overrightarrow{T_5}$ is depicted on Figure 2. Also, if uv is an arc, then $f(v) - f(u) = 2 \bmod 5$. By summing this relation for each edge of a closed walk in G, we get $2\sigma = 0 \bmod 5$ where σ is the signature of the closed walk. Therefore, σ is a multiple of 5 and $\overrightarrow{G}$ is consistent.

Conversely, let $\overrightarrow{G}$ be a consistent oriented graph. If $\overrightarrow{G}$ does not contain any directed 3-path, then we color each source with 1, each sink with 5, and the remaining vertices (which are neighbors

of precisely one source and one sink) with 3. Now each arc uv gets color $\frac{c_u+c_v}{2}$ where c_u and c_v are the colors of u and v. This yields an oriented 5-total-coloring of $\overrightarrow{G}$ (and more precisely a homomorphism from $T(\overrightarrow{G})$ to the only orientation of K_5 without directed cycle).

We may thus assume that $\overrightarrow{G}$ contains a directed 3-path and that the signature of every closed walk of G is a multiple of 5. We will construct a $\overrightarrow{T_5}$-total-coloring of $\overrightarrow{G}$ in two steps: first, we color the vertices with colors from $\{1,\ldots,5\}$ such that, for every arc xy, if x gets color i then y gets color $i+2 \bmod 5$; second, we extend this coloring to arcs.

For the first step, we choose any vertex v of $\overrightarrow{G}$ arbitrarily and color it with 1. Then, for every other vertex w of $\overrightarrow{G}$, we consider a path $\gamma = v_0\cdots v_k$ from $v_0 = v$ to $v_k = w$ in G (which exists, since G can be assumed to be connected). We denote by f_γ the number of *forward arcs* $v_iv_{i+1} \in A(\overrightarrow{G})$ and by b_γ the number of *backward arcs* $v_{i+1}v_i \in A(\overrightarrow{G})$. We color w with $1 + 2(f_\gamma - b_\gamma) \bmod 5$. Note that this is independent of the choice of γ. Indeed, if γ' is another such path, then following γ and then γ' in reverse order yields a closed walk in G, whose signature is $|f_\gamma - b_\gamma - (f_{\gamma'} - b_{\gamma'})|$. Since $\overrightarrow{G}$ is consistent, we get that $1 + 2(f_\gamma - b_\gamma) = 1 + 2(f_{\gamma'} - b_{\gamma'}) \bmod 5$. Therefore, this oriented coloring is well defined.

Now for each arc xy of $\overrightarrow{G}$, we know that if x has color i, then y is colored with $i+2 \bmod 5$. We color xy with $i+1 \bmod 5$. This eventually yields a $\overrightarrow{T_5}$-total-coloring of $\overrightarrow{G}$. $\square$

We now apply these results to provide a full classification of oriented forests and cycles with respect to their oriented total chromatic number.

Recall, see e.g. [17], that $\chi'_o(\overrightarrow{F}) \leqslant \chi_o(\overrightarrow{F}) \leqslant 3$ for every oriented forest $\overrightarrow{F}$. Regarding the oriented total chromatic number, earlier Theorems 3.1 and 3.2 directly yield the following.

Theorem 3.3. *If $\overrightarrow{F}$ is an oriented forest, then:*

1. $\chi''_o(\overrightarrow{F}) \leqslant 1$ *if and only if $V(\overrightarrow{F})$ is an independent set;*

2. $\chi''_o(\overrightarrow{F}) \leqslant 3$ *if and only if $\overrightarrow{F}$ contains no directed 2-path;*

3. $\chi''_o(\overrightarrow{F}) = 5$ *otherwise, i.e., if and only if $\overrightarrow{F}$ contains at least one directed 2-path.*

Recall, see [17], that $\chi'_o(\overrightarrow{C}) \leqslant \chi_o(\overrightarrow{C}) \leqslant 5$ for every oriented cycle $\overrightarrow{C}$ (actually, $\chi'_o(\overrightarrow{C}) = \chi_o(\overrightarrow{C}) = 5$ only for $\overrightarrow{C}$ being the directed 5-cycle). Regarding the oriented total chromatic number, we prove the following.

Theorem 3.4. *If $\overrightarrow{C}$ is an oriented cycle, then:*

1. $\chi''_o(\overrightarrow{C}) \leqslant 3$ *if and only if $\overrightarrow{C}$ contains no directed 2-path;*

2. $\chi''_o(\overrightarrow{C}) \leqslant 5$ *if and only if $\overrightarrow{C}$ is consistent;*

3. $\chi''_o(\overrightarrow{C}) \leqslant 6$ *if and only if $\overrightarrow{C}$ is neither the directed 4-cycle $\overrightarrow{C_4}$ nor the directed 7-cycle $\overrightarrow{C_7}$;*

4. $\chi''_o(\overrightarrow{C}) = 7$ *if and only if $\overrightarrow{C} = \overrightarrow{C_7}$;*

5. $\chi''_o(\overrightarrow{C}) = 8$ *if and only if $\overrightarrow{C} = \overrightarrow{C_4}$.*

Proof. The first item follows from Theorem 3.1, while the second item follows from Theorem 3.2. The fifth item follows from the fact that $T(\overrightarrow{C_4})$ is an absolute oriented clique on 8 vertices.

We now show that $\chi''_o(\overrightarrow{C_7}) = 7$. An oriented 7-total-coloring of $\overrightarrow{C_7}$ is given on Figure 3(e). Moreover, observe that $T(\overrightarrow{C_7})$ has exactly 14 vertices, and that, for any three of them, at least two are adjacent or linked by a directed 2-path. Therefore, in any oriented total-coloring of $\overrightarrow{C_7}$, each color class has size at most 2. Hence $\chi''_o(\overrightarrow{C_7}) \geqslant \frac{14}{2} = 7$, which proves that the oriented total chromatic number of $\overrightarrow{C_7}$ is exactly 7.

It now remains to show that every oriented cycle other than $\overrightarrow{C_4}$ and $\overrightarrow{C_7}$ has an oriented 6-total-coloring. Let $\overrightarrow{C}$ be an oriented cycle which is not directed. In particular, $\overrightarrow{C}$ contains a source u and a sink v. Let us denote by $\overrightarrow{P}$ and $\overrightarrow{P'}$ the two arc-disjoint oriented paths with ends u and v. By Theorem 3.3, we know that there exist homomorphisms $f : T(\overrightarrow{P}) \to \overrightarrow{T_5}$ and $f' : T(\overrightarrow{P'}) \to \overrightarrow{T_5}$. Furthermore, since $\overrightarrow{T_5}$ is vertex-transitive, we can assume that $f(u) = 1 = f'(u)$. Now, to obtain an oriented 6-total-coloring of $\overrightarrow{C}$, we can just assign color 6 to v and color the other vertices of $T(\overrightarrow{C})$ with their color from f or f'.

We are left with the case of directed cycles $\overrightarrow{C_n}$ with $n \notin \{4, 7\}$. Note that we can write $n = 3a + 5b$ for some non-negative integers a and b. Moreover, Figures 3(a) and (c) handle the cases $n \in \{3, 5\}$, and, hence, we may assume that $a + b \geqslant 2$. We partition the arcs of $\overrightarrow{C_n}$ into a directed 3-paths and b directed 5-paths, each one having its ends coinciding with those of other directed paths. We now color each directed path using one of the following two ways.

Observation 3.5. *For every directed* 3*-path* $v_0 v_1 v_2 v_3$*, setting* $\tau(v_i) = 2i + 1 \bmod 6$ *and* $\tau(v_i v_{i+1}) = 2i + 2 \bmod 6$ *for every* i *yields an oriented* 6*-total-coloring satisfying* $\tau(v_0) = \tau(v_3) = 1$.

Observation 3.6. *For every directed* 5*-path* $v_0 v_1 v_2 v_3 v_4 v_5$*, setting* $\tau(v_i) = 2i + 1 \bmod 5$ *and* $\tau(v_i v_{i+1}) = 2i + 2 \bmod 5$ *for every* i *yields an oriented* 6*-total-coloring satisfying* $\tau(v_0) = \tau(v_5) = 1$.

Note that these total-colorings are compatible, in the sense that there cannot be an arc between vertices colored i and j in one path, and between vertices colored j and i in another. Therefore, combining these colorings yields an oriented 6-total-coloring of $\overrightarrow{C_n}$, which concludes. □
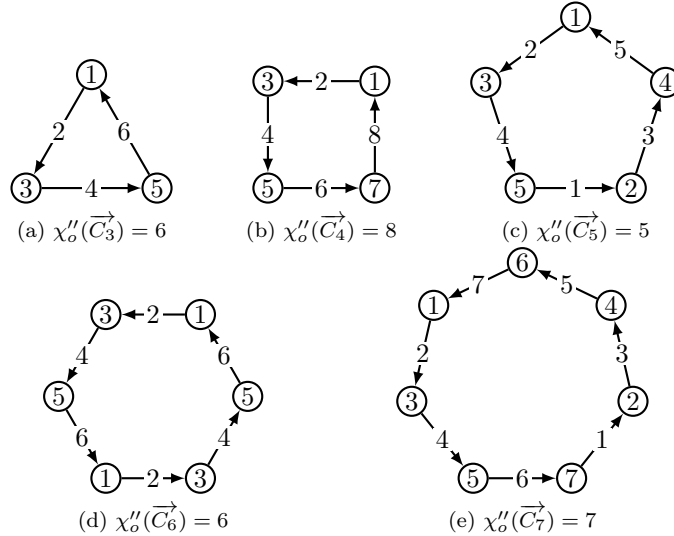

Figure 3: The unique optimal oriented total-colorings of small directed cycles.

In the case of small directed cycles, we can even go a bit beyond Theorem 3.4, by showing that, up to permuting and renaming colors, such oriented cycles admit a unique oriented total-coloring assigning the least number of colors possible. This actually holds for directed cycles of length at most 6, while, for longer directed cycles, one can check that this is not true.

Theorem 3.7. *For every* $n \in \{3, 4, 5, 6\}$*, there is a unique oriented total-coloring of* $\overrightarrow{C_n}$ *with* $\chi_o''(\overrightarrow{C_n})$ *colors.*

Proof. For each of the directed cycles mentioned in the statement, an oriented total-coloring assigning the stated number of colors is depicted in Figure 3. These colorings are indeed unique:

- For $n \in \{3, 4\}$, the total digraph $T(\overrightarrow{C_n})$ is an absolute oriented clique, which concludes in this case.

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- For $n = 5$, note that the line digraph $L(\overrightarrow{C_5})$ is a directed 5-cycle. Thus, both $\overrightarrow{C_5}$ and $L(\overrightarrow{C_5})$ are absolute oriented cliques, meaning that both the five vertices and the five arcs of $\overrightarrow{C_5}$ must receive distinct colors by any oriented 5-total-coloring of $\overrightarrow{C_5}$. Furthermore, since every original vertex of $T(\overrightarrow{C_5})$ is either adjacent or connected by a directed 2-path to all but one vertex, there is a unique way to totally 5-color $\overrightarrow{C_5}$.

- For $n = 6$, observe that $\chi_o''(\overrightarrow{C_6}) = 6$, and moreover that no three vertices of $T(\overrightarrow{C_6})$ can get the same color by an oriented total-coloring since, for any three vertices of $T(\overrightarrow{C_6})$, at least two of them are adjacent or joined by a directed 2-path. Since $T(\overrightarrow{C_6})$ has exactly 12 vertices, each color must thus be assigned to exactly two vertices in any oriented 6-coloring of $T(\overrightarrow{C_6})$. Now, assume that some color appears on two elements u, v that are not antipodal in $\overrightarrow{C_6}$. Let X (resp. X') be the set of vertices on some directed path from u to v (resp. from v to u) in $T(\overrightarrow{C_6}) - \{u, v\}$. Note that $X, X', \{u, v\}$ is a partition of the vertices of $T(\overrightarrow{C_6})$.

  Since u and v are not antipodal but at distance at least 3 in $T(\overrightarrow{C_6})$, we have $|X| = 4$ and $|X'| = 6$ (up to swapping u and v). But now some color must appear twice in X', which is not possible since either it ends up on vertices at distance at most 2 in $T(\overrightarrow{C_6})$ or on one vertex seeing u but seen by v, a contradiction since u and v get the same color. Therefore, each color class is a pair of antipodal elements, which concludes. $\square$

4. Connections between the oriented total chromatic number and other parameters

In this section, we explore how the oriented total chromatic number of an oriented graph relates to others of its chromatic parameters, such as its oriented chromatic number, its oriented chromatic index, or the acyclic chromatic number of its underlying graph.

4.1. Connections between the three oriented chromatic parameters

As a starting point, let us recall that, for any given graph G, one can obtain a proper $(n+m)$-total-coloring of G by combining a proper n-coloring and a proper m-edge-coloring of G in the obvious way. In other words, $\chi''(G) \leqslant \chi(G) + \chi'(G)$ holds for every graph G.

(a) An oriented 3-coloring   (b) An oriented 3-arc-coloring   (c) Not an oriented 6-total-coloring
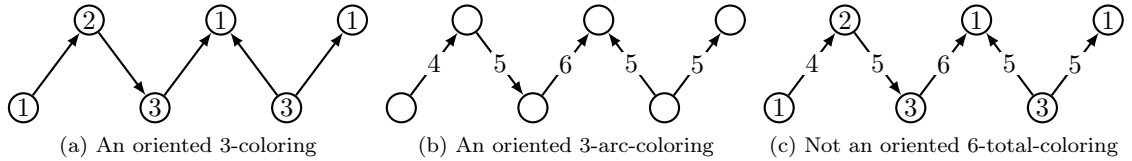
Figure 4: Combining an oriented coloring (a) and an oriented arc-coloring (b) assigning distinct colors in an oriented graph does not have to result in an oriented total-coloring (c).

As depicted in Figure 4, the same argument does not hold in the oriented setting: we provide an oriented graph colored so that colors $\{1, 2, 3\}$ induce an oriented 3-coloring and colors $\{4, 5, 6\}$ induce an oriented 3-arc-coloring, but the associated 6-total-coloring is not oriented (since colors 3 and 5 appear on two pairs of elements that flow to each other in opposite directions).

Due to such issues, it is not clear that the inequality $\chi'' \leqslant \chi + \chi'$ still holds in the oriented setting. We leave this as an open question.

Question 4.1. *Do we have $\chi_o''(\overrightarrow{G}) \leqslant \chi_o(\overrightarrow{G}) + \chi_o'(\overrightarrow{G})$ for every oriented graph $\overrightarrow{G}$?*

Note that the oriented graph $\overrightarrow{G}$ from Figure 4 does not stand as a counterexample since $\chi_o(\overrightarrow{G}) = \chi_o'(\overrightarrow{G}) = 3$ and $\chi_o''(\overrightarrow{G}) = 5$.

Another interesting problem is to determine how good the upper bound in Question 4.1 would be (if it holds). In other words, how large can be the difference $|\chi_o + \chi_o' - \chi_o''|$. As already shown by Theorem 3.4, the directed 5-cycle has all its oriented chromatic parameters equal to 5, hence the difference can be at least 5. Through a generalisation of this example, we provide a construction

showing that the difference $|\chi_o(\overrightarrow{G}) + \chi'_o(\overrightarrow{G}) - \chi''_o(\overrightarrow{G})|$ is not bounded by an absolute constant for every oriented graph $\overrightarrow{G}$.

Theorem 4.2. *For every $k \geqslant 1$, there exists an oriented graph $\overrightarrow{G}$ satisfying $\chi_o(\overrightarrow{G}) = \chi'_o(\overrightarrow{G}) = \chi''_o(\overrightarrow{G}) = 5k$, and thus $|\chi_o(\overrightarrow{G}) + \chi'_o(\overrightarrow{G}) - \chi''_o(\overrightarrow{G})| = 5k$.*

Proof. Let $k \geqslant 1$ be fixed, and let $\overrightarrow{G}$ be the oriented graph constructed as follows. Start from the disjoint union of k copies $\overrightarrow{X_1}, \ldots, \overrightarrow{X_k}$ of the directed 5-cycle $\overrightarrow{C_5}$. Now, for every $1 \leqslant i < j \leqslant k$, add an arc from every vertex of $\overrightarrow{X_i}$ to every vertex of $\overrightarrow{X_j}$.

Note that every two vertices of $\overrightarrow{G}$ are either adjacent or joined by a directed 2-path. Similarly, every two arcs ab and cd either are at distance at most 2 in some $\overrightarrow{X_i}$, or they belong to different $\overrightarrow{X_i}$'s in which case bc or da is an arc. From this, we deduce that $\overrightarrow{G}$ is an absolute oriented clique, and thus $\chi_o(\overrightarrow{G}) = 5k$, and that the arcs of the $\overrightarrow{X_i}$'s induce a relative oriented clique in $L(\overrightarrow{G})$, and thus $\chi'_o(\overrightarrow{G}) \geqslant 5k$. The fact that $\chi_o(\overrightarrow{G}) = 5k$ implies also that $\chi''_o(\overrightarrow{G}) \geqslant 5k$.

To see now that we also have $\chi'_o(\overrightarrow{G}) = \chi''_o(\overrightarrow{G}) \leqslant 5k$, consider the following $5k$-total-coloring of $\overrightarrow{G}$. For every directed 5-cycle $\overrightarrow{X_i}$, start from any vertex, and go along its elements (vertices and arcs) in order, assigning colors $1_i, 2_i, 3_i, 4_i, 5_i, 1_i, 2_i, 3_i, 4_i, 5_i$ as traversing them, just as depicted in Figure 3(c). Now, for every arc xy such that x belongs to some $\overrightarrow{X_i}$ and y belongs to some $\overrightarrow{X_j}$ with $i < j$, assign to xy the same color as that assigned to $x'y$, where $x'y$ is the unique arc in-coming to y in $\overrightarrow{X_j}$. It can be checked that the resulting $5k$-total-coloring of $\overrightarrow{G}$ is oriented. Thus, $\chi''_o(\overrightarrow{G}) \leqslant 5k$, and the result follows since $\chi'_o(\overrightarrow{G}) \leqslant \chi''_o(\overrightarrow{G})$. $\square$

It is worth recalling that, for an oriented graph $\overrightarrow{G}$, we always have $\chi'_o(\overrightarrow{G}) \leqslant \chi_o(\overrightarrow{G})$. Indeed, one can always convert an oriented coloring f into an oriented arc-coloring f' by setting $f'(uv) = f(u)$ for every arc uv. Thus, a positive answer to Question 4.1 would imply that we always have $\chi''_o(\overrightarrow{G}) \leqslant 2 \cdot \chi_o(\overrightarrow{G})$. We show that this bound holds through an *ad hoc* proof.

Theorem 4.3. *For every oriented graph $\overrightarrow{G}$, we have $\chi_o(\overrightarrow{G}) \leqslant \chi''_o(\overrightarrow{G}) \leqslant 2 \cdot \chi_o(\overrightarrow{G})$.*

Proof. The lower bound being obvious, we focus on proving the upper bound. Set $\chi_o(\overrightarrow{G}) = n$, and let f be an oriented n-coloring of $\overrightarrow{G}$ assigning colors in $\{1, \ldots, n\}$. Consider τ, the $2n$-total-coloring of $\overrightarrow{G}$ obtained by setting $\tau(u) = f(u)$ for all $u \in V(\overrightarrow{G})$ and $\tau(uv) = f(u) + n$ for all $uv \in A(\overrightarrow{G})$.

We conclude by showing that τ is actually an oriented total-coloring. Observe that by construction, τ never assigns the same color to a vertex and an arc. Therefore, if two elements x, y satisfy $\tau(x) = \tau(y)$, then either x, y are both arcs, with sources x', y' and $f(x') = \tau(x') - n = \tau(y') - n = f(y')$, or they are vertices and setting $x' = x$ and $y' = y$ yields again $f(x') = \tau(x) = \tau(y) = f(y')$. Moreover, if x flows to y then x' flows to y'.

Let a, b, c, d be elements of $\overrightarrow{G}$ such that a flows to b and c flows to d.

- Assume that $\tau(a) = \tau(b)$. Then $f(a') = f(b')$ and a' flows to b', a contradiction since f is an oriented coloring.

- Assume that $\tau(a) = \tau(d)$ and $\tau(b) = \tau(c)$. Again, we get vertices a', b', c', d' such that $f(a') = f(d')$ and $f(b') = f(c')$. One can also observe that either $a = b'$ (if a is a vertex and b is an arc) or a' flows to b' (otherwise). Similarly, either $c = d'$ or c' flows to d'. In each case, we get again a contradiction. $\square$

The directed cycles $\overrightarrow{C_4}$ and $\overrightarrow{C_5}$ show that the bounds in Theorem 4.3 are tight, since their oriented chromatic numbers are respectively 4 and 5 (see [17]), while their oriented total chromatic numbers are respectively 8 and 5 by Theorem 3.4.

We now establish a connection between the oriented total chromatic number of an oriented graph and its oriented chromatic index. It is worth mentioning that the proof technique in the next proof builds upon one developed by Ochem, Pinlou, and Sopena in [12], which allowed them to express bounds on the oriented chromatic number in terms of oriented chromatic index.

Theorem 4.4. *Every oriented graph $\overrightarrow{G}$ of oriented chromatic index k satisfies $k \leqslant \chi_o''(\overrightarrow{G}) \leqslant k+2^k$.*

Proof. The lower bound being obvious, let us focus on proving the upper bound. Let $g : A(\overrightarrow{G}) \to S$ be an oriented k-arc-coloring of $\overrightarrow{G}$ where $|S| = k$. We color each vertex with the set of the colors of the arcs leaving it. More precisely, for each $u \in V(\overrightarrow{G})$, let $f(u) = \{g(uv) : uv \in A(\overrightarrow{G})\}$.

We claim that f is an oriented coloring of $\overrightarrow{G}$. Indeed, consider any two arcs wx and yz of $\overrightarrow{G}$. Note that every arc xv must satisfy $g(xv) \neq g(wx)$. In particular, $g(wx) \in f(w)$ but $g(wx) \notin f(x)$, hence $f(w) \neq f(x)$. Assume now that $f(w) = f(z)$ and $f(x) = f(y)$. Then $g(wx) \in f(w) = f(z)$, and hence there exists a vertex u such that $g(zu) = g(wx)$. Similarly, there exists a vertex v such that $g(xv) = g(yz)$. Now wx flows to xv and yz flows to zu, a contradiction to g being an oriented arc-coloring.

Let us now color vertices of $\overrightarrow{G}$ with f and arcs with g. We claim that the resulting assignment τ is an oriented total-coloring. This would conclude since τ uses at most $k + 2^k$ colors. Let $a, b, c, d \in V(\overrightarrow{G}) \cup A(\overrightarrow{G})$ be such that a flows to b and c flows to d. Since f and g are oriented colorings and τ assigns colors from disjoint sets to vertices and arcs, we already have that $\tau(a) \neq \tau(b)$. Assume now that $\tau(a) = \tau(d)$ and $\tau(b) = \tau(c)$. In this case, a and d (resp. b and c) must be either both vertices or both arcs. If all of a, b, c, d are vertices (resp. arcs), then we reach a contradiction as f (resp. g) is an oriented (resp. arc) coloring. By symmetry, assume that a and d are both vertices and b and c are both arcs. In particular, b is out-going from a and c is in-coming to d. Since $g(b) \in f(a) = \tau(a) = \tau(d) = f(d)$, there exists an arc de with $\tau(de) = g(b) = \tau(b)$. But then c flows to de and $\tau(c) = \tau(b) = \tau(de)$, which is impossible. $\square$

The lower bound in Theorem 4.4 is tight, as shown by the directed 5-cycle: $\chi_o'(\overrightarrow{C_5}) = \chi''(\overrightarrow{C_5}) = 5$. Regarding the upper bound, we leave open the question of whether it is tight.

Question 4.5. *Can we have $\chi_o''(\overrightarrow{G}) = k + 2^k$ for an oriented graph $\overrightarrow{G}$ with $\chi_o'(\overrightarrow{G}) = k$?*

4.2. Connections with the acyclic chromatic number

Recall that an *acyclic k-coloring* of a graph G is a proper k-coloring such that the vertices of G assigned any two distinct colors i, j induce a forest. The *acyclic chromatic number* $\chi_a(G)$ of G is the minimum k such that G admits an acyclic k-coloring.

We now establish connections between the oriented total chromatic number of an oriented graph and the acyclic chromatic number of its underlying graph. Our main result reads as follows:

Theorem 4.6. *For every oriented graph $\overrightarrow{G}$ with underlying graph G of acyclic chromatic number k, we have $\chi_o''(\overrightarrow{G}) \leqslant k \cdot 2^{k-1} + 2k(k-1)$.*

Note that if Question 4.1 had a positive answer, then it would imply the theorem, by using two bounds from [17], namely: $\chi_o(\overrightarrow{G}) \leqslant k \cdot 2^{k-1}$ and $\chi_o'(\overrightarrow{G}) \leqslant 2k(k-1) - \lfloor k/2 \rfloor$ for every oriented graph $\overrightarrow{G}$ with $\chi_a(G) = k$. In our case, we show that the constructions from [17] of a coloring and an arc-coloring satisfying these bounds can be combined into an oriented total-coloring.

Before proving Theorem 4.6, we first need two observations.

Observation 4.7. *Let $\overrightarrow{F}$ be an oriented forest. Then we can extend any proper 2-coloring c of F into an oriented 4-coloring $f : V(\overrightarrow{F}) \to \{1, 2\} \times \{0, 1\}$ such that projecting f on the first component yields c.*

Proof. We can simply color $\overrightarrow{F}$ recursively: if y is a leaf in F with neighbor x colored $f(x) = (c(x), \alpha)$, we set $f(y) = (c(y), \beta)$ where $\beta \in \{0, 1\}$ satisfies $\alpha = \beta$ if and only if $c(x) < c(y)$. $\square$

Through coloring again arcs by the color of their source vertex, we obtain the following arc version of previous Observation 4.7.

Observation 4.8. *Let $\overrightarrow{F}$ be an oriented forest. Then we can extend any proper 2-coloring c of F into an oriented 4-arc-coloring $g : A(\overrightarrow{F}) \to \{1, 2\} \times \{0, 1\}$ such that for every arc xy, projecting $g(xy)$ on the first component yields $c(x)$.*

Recall that even though forests are of unbounded chromatic index in general, all oriented forests are orientedly 3-arc-colorable [17]. Observation 4.8 thus means that up to paying one more color, one can "synchronise" an oriented arc-coloring with a bipartition of the underlying graph.

Now we are ready to prove Theorem 4.6.

Proof of Theorem 4.6. Let c be an acyclic coloring of G with colors from $\{1, \ldots, k\}$. For $1 \leqslant i < j \leqslant k$, we denote by $\overrightarrow{F_{i,j}}$ the oriented forest induced, in $\overrightarrow{G}$, by $c^{-1}(\{i, j\})$.

For all distinct $i, j \in \{1, \ldots, k\}$, take $f_{i,j} = (f_{i,j}^1, f_{i,j}^2)$ and $g_{i,j}$ the oriented 4-coloring and oriented 4-arc-coloring of $\overrightarrow{F_{i,j}}$ obtained through Observations 4.7 and 4.8. Our goal is to obtain an oriented total-coloring of $\overrightarrow{G}$ by combining the $f_{i,j}$'s and $g_{i,j}$'s in some fashion.

Note that for each arc of $\overrightarrow{G}$, there exists a unique $i < j$ such that $xy \in A(\overrightarrow{F_{i,j}})$. In that case we assign color $\tau(xy) = (g_{i,j}(xy), i, j)$ to xy. We then color each vertex x such that $c(x) = i$ with

$$\tau(x) = \left(c(x), f_{1,i}^2(x), f_{2,i}^2(x), \ldots, f_{(i-1),i}^2(x), f_{i,(i+1)}^2(x), \ldots, f_{i,k}^2(x) \right).$$

Note that there are $k \cdot 2^{k-1}$ possible colors for vertices and $4 \times \binom{k}{2}$ possible colors for arcs. It remains to prove that τ is indeed an oriented total-coloring of $\overrightarrow{G}$.

Let a, b, c, d be elements of $\overrightarrow{G}$ such that a flows to b and c flows to d. If $\tau(a) = \tau(b)$, then a, b are both vertices or both arcs since τ assigns disjoint colors to vertices and arcs. The former cannot happen for otherwise $c(a) = c(b)$ but a and b are adjacent in G. In the latter case, we get $g_{i,j}(a) = g_{j,k}(b)$ where i, j, k are the colors under c of the ends of a and b. By projecting on the first component, we get $i = j$, which is impossible since the ends of a are colored with i and j under c. Therefore $\tau(a) \neq \tau(b)$.

Similarly, assume that $\tau(a) = \tau(d)$ and $\tau(b) = \tau(c)$. Then, a and d (resp. b and c) must be either both vertices or both arcs. We consider several cases:

- If a, b, c, d are all vertices, then we get $c(a) = c(d) = i$ and $c(b) = c(c) = j$ for some $i \neq j$. Thus $f_{i,j}^1(a) = c(a) = c(d) = f_{i,j}^1(d)$ and $f_{i,j}^2(a) = f_{i,j}^2(d)$, hence $f_{i,j}(a) = f_{i,j}(d)$. Similarly, we get $f_{i,j}(b) = f_{i,j}(c)$, which is not possible since $f_{i,j}$ is an oriented coloring of $\overrightarrow{F_{i,j}}$.

- If a, b, c, d are all arcs, then we get $g_{i,j}(a) = g_{i,j}(d)$ and $g_{i',j'}(b) = g_{i',j'}(c)$ for some $i \neq j$ and $i' \neq j'$. Since a flows to b and c to d, then we must have $i = i'$ and $j = j'$. But this is not possible, since $g_{i,j}$ is an oriented arc-coloring of $\overrightarrow{F_{i,j}}$.

- Otherwise, assume by symmetry that a and d are vertices and b and c are arcs. In particular, a is the source vertex of b and d is the target vertex of c. From $\tau(a) = \tau(d)$ we get $c(a) = c(d)$, and from $\tau(b) = \tau(c)$ we get that the ends of b and c are colored with two colors i, j (note that $c(a)$ is one of them), and moreover, $g_{i,j}(b) = g_{i,j}(c)$. Projecting $\tau(a) = \tau(d)$ on the correct component, we get $f_{i,j}(a) = f_{i,j}(d)$. Moreover, denoting by c' the source of c (so that $c = c'd$ and c' flows to d), we get by definition of $g_{i,j}$ that $f_{i,j}(a) = g_{i,j}(b) = g_{i,j}(c) = f_{i,j}(c')$. But this is a contradiction since $f_{i,j}$ is an oriented coloring of $\overrightarrow{F_{i,j}}$. □

Again, we leave open the question of whether the bound in Theorem 4.6 is tight in general.

Question 4.9. *Can we have $\chi_o''(\overrightarrow{G}) = k \cdot 2^{k-1} + 2k(k-1)$ for an oriented graph $\overrightarrow{G}$ with underlying graph G of acyclic chromatic number k?*

5. Complexity dichotomy

We here consider the decision problem OTC_k, which asks whether a given oriented graph $\overrightarrow{G}$ admits an oriented k-total-coloring. Note that OTC_k is clearly in NP for every k. Our main result in this section is a full dichotomy for the complexity of this problem, namely:

Theorem 5.1. OTC_k *is* NP*-complete for all $k \geqslant 6$ and is polynomial-time solvable otherwise.*

As reported in [17], it is known that the problem of deciding whether a given oriented graph has oriented chromatic number or oriented chromatic index k is polynomial-time solvable for every $k \leqslant 3$ and NP-complete otherwise, i.e., if $k \geqslant 4$.

We start by proving the polynomial cases of Theorem 5.1, which actually can be established from results we have proved earlier.

Theorem 5.2. OTC_k *is polynomial-time solvable for every* $k \leqslant 5$. *Moreover, in each case, an oriented k-total-coloring can be found in polynomial time when it exists.*

Proof. Let $\overrightarrow{G}$ be an oriented graph. Remind that, through Theorem 3.1, we have given a characterization of oriented graphs with oriented total chromatic number in $\{1, 2, 3, 4\}$. Particularly, remind that the oriented total chromatic number of an oriented graph can never be equal to 2 or 4. Now, to check if $\chi_o''(\overrightarrow{G}) = 1$, it suffices to check whether $\overrightarrow{G}$ has no arcs, while, to check if $\chi_o''(\overrightarrow{G}) = 3$, it suffices to check whether $\overrightarrow{G}$ has arcs and all vertices are sources and sinks. These properties can clearly be checked in polynomial time; actually, note that, in these cases, even constructing an oriented 1-total-coloring or an oriented 3-total-coloring can be done in polynomial time (using the constructive proof of Theorem 3.1).

Now, by Theorem 3.2, we have $\chi_o''(\overrightarrow{G}) \leqslant 5$ if and only if $\overrightarrow{G}$ is consistent, meaning that either it has no directed 3-paths, or the signatures of the closed walks of G are multiples of 5. Note that determining the length of the longest directed paths of an oriented graph can clearly be done in polynomial time. If the longest directed paths of $\overrightarrow{G}$ have length 2, then an oriented 5-total-coloring can be obtained in polynomial time by coloring $\overrightarrow{G}$ as described in the proof of Theorem 3.2. Otherwise, checking whether $\overrightarrow{G}$ admits oriented 5-total-colorings is equivalent to finding a $\overrightarrow{T_5}$-coloring of $T(\overrightarrow{G})$, which is again equivalent to finding a homomorphism $f : \overrightarrow{G} \to \overrightarrow{T_5}$ where each arc xy is mapped to an arc $i(i+2)$ in $\overrightarrow{T_5}$.

Such a coloring f can be obtained as follows. Choose an arbitrary vertex v and a spanning tree T of G rooted at v. Now set $f(v) = 1$, and color the remaining vertices as follows: consider an edge xy of T such that y is colored but x is not, and color x with $f(y) - 2 \bmod 5$ if $xy \in A(\overrightarrow{G})$ and $f(y) + 2 \bmod 5$ if $yx \in A(\overrightarrow{G})$. Now, let xy be an arc in $\overrightarrow{G}$ but not in T and let σ be the signature of the closed walk of G formed by taking the edge xy and then the path in T between y and x. By summing $f(a) - f(b)$ for each edge ab in this walk, we end up with $0 = f(y) - f(x) + 2\sigma - 2$ hence either $f(y) = f(x) + 2 \bmod 5$ or σ is not a multiple of 5, and we can conclude that $\chi_o''(\overrightarrow{G}) > 5$. Therefore, in polynomial time, either we find an obstruction and can answer negatively, or we correctly construct f. $\square$

We now focus on proving the NP-hardness of OTC_k for every $k \geqslant 6$. For that, we need to consider the decision problem OTC_k^*, which is the restriction of OTC_k to oriented graphs $\overrightarrow{G}$ that have neither sources nor sinks. Indeed, it turns out that we have the following connections between the OTC_k and OTC_k^* problems:

Lemma 5.3. *If* OTC_k^* *is NP-hard for some k, then* OTC_{k+1}, OTC_{k+2} *and* OTC_{k+3}^* *are NP-hard.*

Proof. Assume OTC_k^* is NP-hard for some k. The result follows from the following three dedicated polynomial-time reductions, performed from an oriented graph $\overrightarrow{G}$ being an instance of OTC_k^*.

- Regarding OTC_{k+1}, consider the following construction from $\overrightarrow{G}$. Let $\overrightarrow{H}$ be the oriented graph obtained from $\overrightarrow{G}$ by adding a new vertex s dominating all other vertices (i.e., sx is an arc for every $x \in V(\overrightarrow{G})$). We claim that $\overrightarrow{G}$ admits an oriented k-total-coloring if and only if $\overrightarrow{H}$ admits an oriented $(k+1)$-total-coloring.

  Assume first that $\overrightarrow{H}$ admits an oriented $(k+1)$-total-coloring τ. Because s dominates all other vertices, note that the color $\tau(s)$ must be assigned to s only. This means that the restriction of τ to $\overrightarrow{H} - s$, thus to $\overrightarrow{G}$, must be an oriented k-total-coloring.

  Assume now that $\overrightarrow{G}$ admits an oriented k-total-coloring τ. We extend τ to an oriented $(k+1)$-total-coloring τ' of $\overrightarrow{H}$, by assigning color $k+1$ to s, and for each arc sx, by defining

$\tau'(sx) = \tau(yx)$ where y is a vertex of $\overrightarrow{G}$ such that $yx \in A(\overrightarrow{G})$ (such vertex exists since $\overrightarrow{G}$ has no source).

- For OTC_{k+2}, consider $\overrightarrow{H}$ the oriented graph obtained from $\overrightarrow{G}$ by adding a vertex s dominating all vertices of $\overrightarrow{G}$ and a vertex t dominated by all these vertices. That is, for every $x \in V(\overrightarrow{G})$, both sx and xt are arcs of $\overrightarrow{H}$. Arguing in the same way as in the previous item, $\overrightarrow{G}$ admits an oriented k-total-coloring if and only if $\overrightarrow{H}$ admits an oriented $(k+2)$-total-coloring.

- We finally consider the problem OTC^*_{k+3}. This time, to obtain $\overrightarrow{H}$ from $\overrightarrow{G}$, we just proceed as in the previous case, i.e., add s and t as above, and also add the arc ts. By similar arguments as earlier, $\overrightarrow{G}$ admits an oriented k-total-coloring if and only if $\overrightarrow{H}$ admits an oriented $(k+3)$-total-coloring. Note that $\overrightarrow{H}$, here, has neither sources nor sinks. $\qquad\square$

Due to Lemma 5.3, Theorem 5.1 boils down to prove that OTC^*_6 is NP-hard, which is the goal of the rest of this section. We first need to introduce some gadgets and to point out some of their properties. The first two gadgets, the *target gadget* $\overrightarrow{G_{\text{target}}}$ and the *target* $\overrightarrow{T}$, are depicted in Figure 5. Throughout what follows, we deal with their vertices and arcs through the terminology given in the figure.
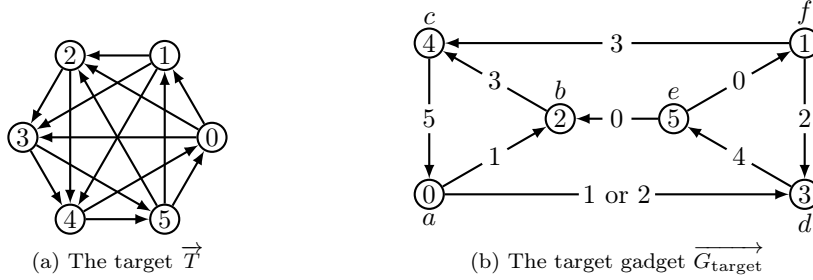

(a) The target $\overrightarrow{T}$ (b) The target gadget $\overrightarrow{G_{\text{target}}}$

Figure 5: The target $\overrightarrow{T}$ and the target gadget $\overrightarrow{G_{\text{target}}}$, with its only possible oriented 6-total-colorings.

Lemma 5.4. *Every oriented 6-total-coloring of $\overrightarrow{G_{\text{target}}}$ must be a $\overrightarrow{T}$-total-coloring. Besides, up to renaming the vertices of $\overrightarrow{T}$, such a coloring is depicted on Figure 5(b).*

Proof. Assume $\overrightarrow{G_{\text{target}}}$ admits an oriented 6-total-coloring τ, and that τ is an $\overrightarrow{H}$-total-coloring for some oriented graph $\overrightarrow{H}$ on 6 vertices (where $V(\overrightarrow{H}) = \{0,1,2,3,4,5\}$). By Theorem 3.7, recall that a directed triangle, up to renaming the colors, is uniquely orientedly 6-total-colorable. Consequently, we may assume that $\tau(a) = 0$, $\tau(ab) = 1$, $\tau(b) = 2$, $\tau(bc) = 3$, $\tau(c) = 4$, and $\tau(ca) = 5$. This determines the direction of all the arcs of $\overrightarrow{H}$ (which are as in $\overrightarrow{T}$, in Figure 5(a)), except for the edges $03, 14$ and 25 that are yet not oriented.

Since ad is an arc and $\tau(a) = 0$, we must have $\tau(d) \in \{1,2,3\}$. Actually, we cannot have $\tau(d) = 2$ because de and eb are arcs and $\tau(b) = 2$. Moreover, we cannot have $\tau(d) = 1$ as otherwise there would not be any possible color for ad. Hence $\tau(d) = 3$ and $\overrightarrow{H}$ contains the arc 03. This implies that $\tau(e) \in \{4,5\}$, and $\tau(e) \neq 4$ since eb is an arc with $\tau(b) = 2$. Therefore $\tau(e) = 5$ and $\overrightarrow{H}$ contains the arc 52. Finally, because of the arc fc and fd, we must have $\tau(f) \in \{1,2\}$. But if $\tau(f) = 2$, then fd cannot be colored, hence $\tau(f) = 1$, and due to the arc fc, $\overrightarrow{H}$ contains the arc 14 and $\overrightarrow{H} = \overrightarrow{T}$.

It can then easily be checked that there is no more choice left for the colors of the arcs except for ad, as shown in Figure 5(b). $\qquad\square$

Based on the in-degrees and out-degrees in $\overrightarrow{T}$, we say that colors $0, 1, 5$ are *weak* while colors $2, 3, 4$ are *strong*. Furthermore, we say that a pair $(c, c') \in V(\overrightarrow{T}) \times V(\overrightarrow{T})$ of colors is *good* if c is weak, c' is strong, and $c = c' + 3 \bmod 6$. Note that the only good pairs are thus $(0, 3)$, $(1, 4)$ and $(5, 2)$ (here and further, operations over vertices of $\overrightarrow{T}$ are understood modulo 6).
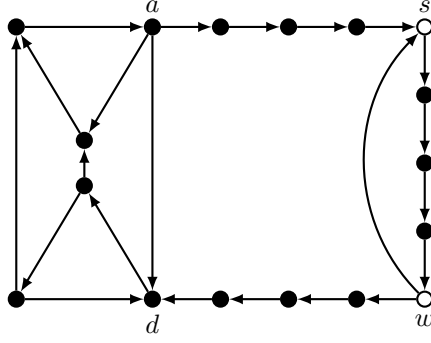
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Figure 6: The vertex gadget $\overrightarrow{V}$.

The last gadget we need is the *vertex gadget* $\overrightarrow{V}$, depicted in Figure 6. Note that $\overrightarrow{V}$ is obtained from the target gadget $\overrightarrow{G_{\text{target}}}$ by attaching some structure to the vertices a and d. Besides, $\overrightarrow{V}$ has two particular vertices, a *strong vertex* (denoted s in the figure), and a *weak one* (denoted w).

Note that this gadget is obtained by adding directed paths to $\overrightarrow{G_{\text{target}}}$; hence, before stating properties of $\overrightarrow{V}$, we state a result about $\overrightarrow{T}$-total-colorings of directed paths.

Lemma 5.5. *Let* $\overrightarrow{P} = s_1 s_2 s_3 s_4 s_5$ *be the directed 4-path, and* τ *be a* $\overrightarrow{T}$*-total-coloring of* $\overrightarrow{P}$. *We have the following:*

1. *if* $\tau(s_1)$ *is weak or 4, then* $\tau(s_5) - \tau(s_1) \in \{2, 3, 4\}$;

2. *if* $\tau(s_1)$ *is strong but not 4, then* $\tau(s_5) - \tau(s_1) \in \{2, 3\}$;

3. *if* $\tau(s_5)$ *is strong or 5, then* $\tau(s_5) - \tau(s_1) \in \{2, 3, 4\}$;

4. *if* $\tau(s_5)$ *is weak but not 5, then* $\tau(s_5) - \tau(s_1) \in \{2, 3\}$.

Proof. Note that it suffices to prove the first two items, as reversing the direction of all arcs of $\overrightarrow{T}$ swaps weak and strong colors. To this end, it suffices to apply several times one of the following observations.

Assume $\tau(s_i)$ is weak. Then $\tau(s_i s_{i+1})$ and $\tau(s_{i+1})$ lie in $\{\tau(s_i)+1, \tau(s_i)+2, \tau(s_i)+3\}$, but $s_i s_{i+1}$ flows to s_{i+1} and hence we can have neither $\tau(s_i s_{i+1}) = \tau(s_i) + 3$ nor $\tau(s_{i+1}) = \tau(s_i) + 1$. Hence, we have $\tau(s_i s_{i+1}) \in \{\tau(s_i)+1, \tau(s_i)+2\}$ and $\tau(s_{i+1}) \in \{\tau(s_i)+2, \tau(s_i)+3\}$. Otherwise, if $\tau(s_i)$ is strong, then by the same argument we must get $\tau(s_i s_{i+1}) = \tau(s_i) + 1$ and $\tau(s_{i+1}) = \tau(s_i) + 2$. $\square$

We may now state the desired properties of $\overrightarrow{V}$.

Lemma 5.6. *Every oriented 6-total-coloring* τ *of* $\overrightarrow{V}$ *is a* $\overrightarrow{T}$*-total-coloring such that* $(\tau(w), \tau(s))$ *is good. Conversely, for every good pair* (c, c'), *there exists a* $\overrightarrow{T}$*-total-coloring* τ *of* $\overrightarrow{V}$ *where:*

- $\tau(w) = c$ *and* $\tau(s) = c'$;

- *the only arc* xw *verifies* $\tau(xw) = c - 1$;

- *the only arc* sx *verifies* $\tau(sx) = c' + 1$.

Proof. Let τ be an oriented 6-total-coloring of $\overrightarrow{V}$. By Lemma 5.4, since $\overrightarrow{V}$ contains $\overrightarrow{G_{\text{target}}}$, τ is actually a $\overrightarrow{T}$-total-coloring. In particular, $\tau(a) = 0$ and $\tau(d) = 3$. Now, since a and s are joined by a directed 4-path, and similarly for w and d, by Lemma 5.5 we deduce that $\tau(s) \in \{2, 3, 4\}$ is strong and $\tau(w) \in \{0, 1, 5\}$ is weak. Moreover, we have:

- If $\tau(s) \in \{2, 3\}$, then, because ws is an arc, we must have $\tau(s) - \tau(w) \in \{2, 3\}$. By Lemma 5.5 applied to the directed 4-path linking s to w, we must have $\tau(w) - \tau(s) \in \{2, 3\}$. This implies that we must have $\tau(w) = \tau(s) + 3$.
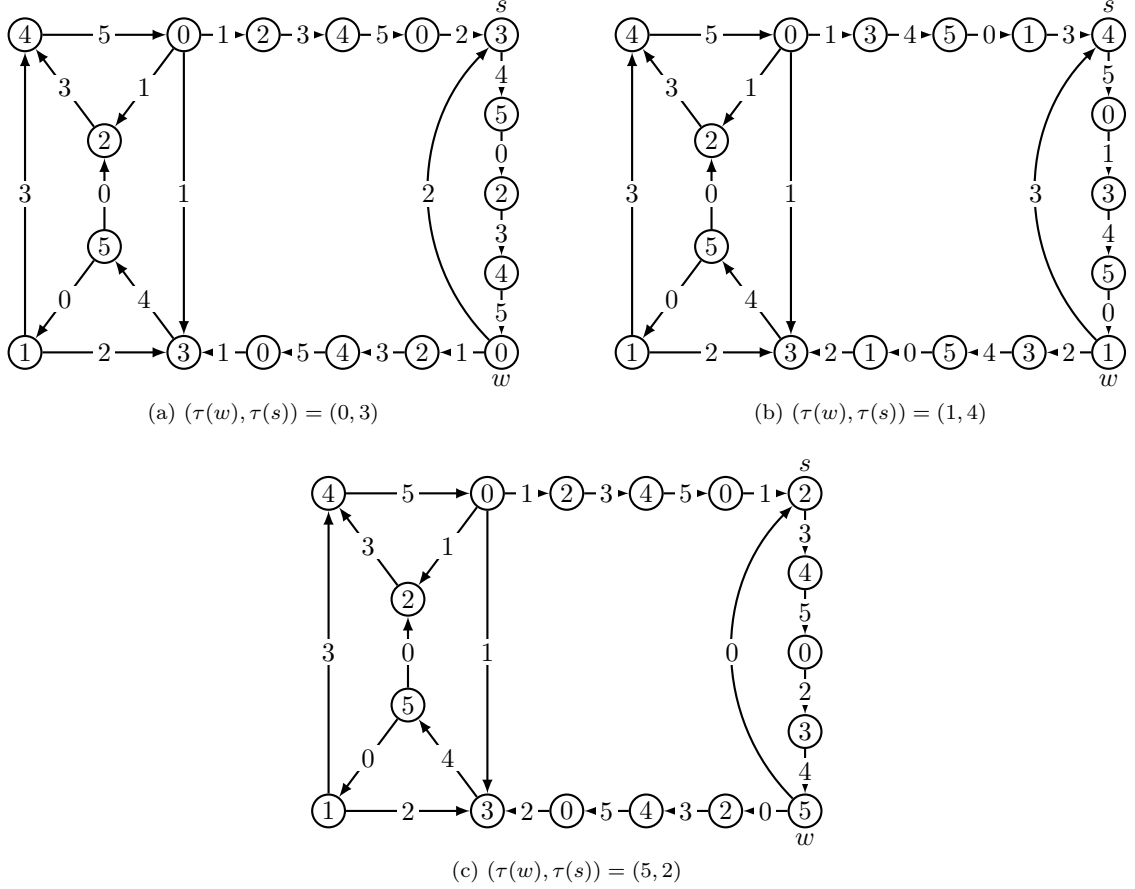
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(a) $(\tau(w), \tau(s)) = (0, 3)$

(b) $(\tau(w), \tau(s)) = (1, 4)$

(c) $(\tau(w), \tau(s)) = (5, 2)$

Figure 7: $\overrightarrow{T}$-total-colorings τ of $\overrightarrow{V}$, realizing any given good pair as $(\tau(w), \tau(s))$.

- If $\tau(s) = 4$, then, because ws is an arc, we have $\tau(w) \in \{1, 2\}$. By Lemma 5.5 applied to the directed 4-path linking w to d, we get that $\tau(w) = 1$.

Thus, $(\tau(w), \tau(s))$ must be a good pair, as claimed. For the last part, we provide in Figure 7 some colorings satisfying the requested properties. $\qquad\square$

We finish off by pointing out useful properties of $\overrightarrow{T}$-total-colorings in oriented graphs containing directed paths of certain length.

Lemma 5.7. *Let* $\overrightarrow{P} = s \cdots t$ *be the directed 8-path, and* τ *be a* $\overrightarrow{T}$*-total-coloring of* $\overrightarrow{P}$*. Then the pair* $(\tau(s), \tau(t))$ *cannot be good.*

Proof. Towards a contradiction, assume $\tau(s)$ is weak, $\tau(t)$ is strong, and $\tau(s) = \tau(t) + 3$. Let us denote by x the middle vertex of $\overrightarrow{P}$ and note that x is at distance precisely 4 from both s and t in P. Applying Lemma 5.5 twice (from s to x and from x to t) thus gives that

$$\tau(x) \in \{\tau(s) + 2, \tau(s) + 3, \tau(s) + 4\} \cap \{\tau(t) - 2, \tau(t) - 3, \tau(t) - 4\}.$$

Now, since $\tau(s) = \tau(t) + 3$, this intersection is empty, implying that there is no valid color for x by τ, a contradiction. $\qquad\square$

Lemma 5.8. *Let* $\overrightarrow{H}$ *be an oriented graph, and* τ *be a* $\overrightarrow{T}$*-total-coloring of* $\overrightarrow{H}$*. Assume s and t are two vertices of* $\overrightarrow{H}$*, such that:*

1. $\tau(xs) = \tau(s) - 1$ *for every arc xs;*

2. $\tau(tx) = \tau(t) + 1$ for every arc tx;

3. $\tau(s)$ is weak, $\tau(t)$ is strong, and $\tau(s) \neq \tau(t) + 3$.

Then the oriented graph obtained from $\overrightarrow{H}$ by adding a directed 8-path from s to t admits a $\overrightarrow{T}$-total-coloring extending τ.

—4→⑤—0→①—2→③—4→⑤—0→②—3→④—5→⓪—1→②—3→④—5→
(a) $(c, c') = (5, 4)$

—4→⑤—0→①—2→③—4→⑤—0→①—2→③—4→⑤—0→①—2→③—4→
(b) $(c, c') = (5, 3)$

—5→⓪—2→③—4→⑤—0→②—3→④—5→⓪—2→③—4→⑤—0→②—3→
(c) $(c, c') = (0, 2)$

—5→⓪—1→②—3→④—5→⓪—1→②—3→④—5→⓪—1→②—3→④—5→
(d) $(c, c') = (0, 4)$

—0→①—3→④—5→⓪—2→③—4→⑤—0→②—3→④—5→⓪—1→②—3→
(e) $(c, c') = (1, 2)$

—0→①—3→④—5→⓪—2→③—4→⑤—0→②—3→④—5→⓪—2→③—4→
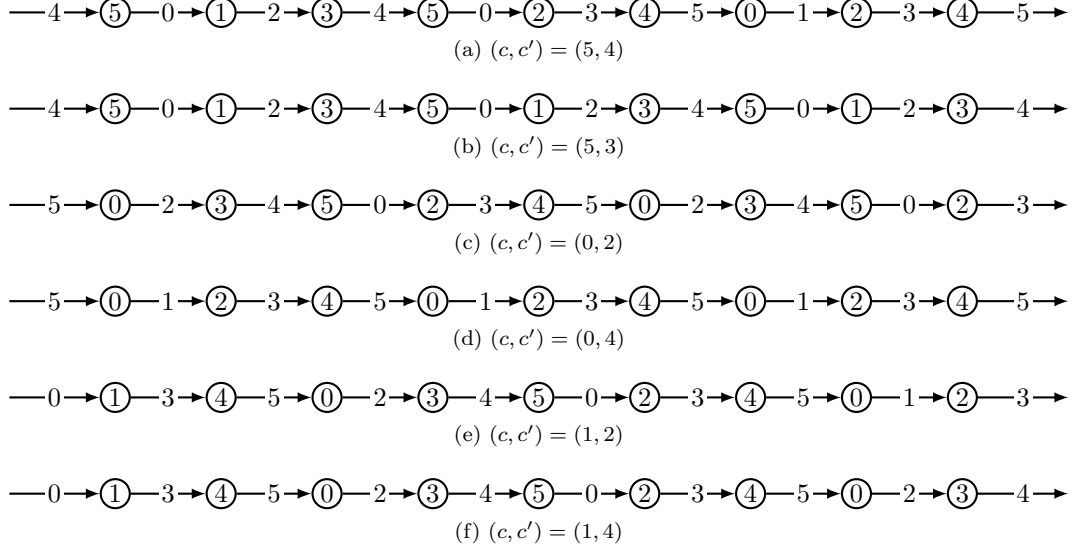(f) $(c, c') = (1, 4)$

Figure 8: $\overrightarrow{T}$-total-colorings τ of $\overrightarrow{P}$, showing that Lemma 5.8 is true.

Proof. By hypothesis, it is sufficient to consider the directed path $\overrightarrow{P} = x_1 \ldots x_{11}$ of length 10, and to show that, for every pair (c, c') that is not good, there exist $\overrightarrow{T}$-total-colorings τ of $\overrightarrow{P}$ such that $\tau(x_2) = c$, $\tau(x_{10}) = c'$, $\tau(x_1 x_2) = c - 1$ and $\tau(x_{10} x_{11}) = c' + 1$. We provide, in Figure 8, such $\overrightarrow{T}$-total-colorings for each of the six pairs of $\{0, 1, 5\} \times \{2, 3, 4\}$ to consider as (c, c'). □

We now conclude this section by showing the NP-hardness of OTC_6^*, which implies Theorem 5.1. To this end, we reduce the 3-COLORING problem, which is well known to be NP-hard.

The construction goes as follows. Let G be an instance of 3-COLORING. First, for every vertex v of G, we add, in $\overrightarrow{H}$, a copy $\overrightarrow{V_v}$ of the vertex gadget $\overrightarrow{V}$, for which we denote by w_v and s_v its vertices w and s. We then consider every edge uv of G, and join, in $\overrightarrow{H}$, the vertices w_u and s_v (resp. w_v and s_u) via a new directed 8-path (i.e., its internal vertices being new vertices) from w_u to s_v (resp. from w_v to s_u). Observe that $\overrightarrow{H}$ has neither sources nor sinks, due to the structure of $\overrightarrow{V}$, and that this oriented graph can be computed in polynomial time.

It remains to show that G is properly 3-colorable if and only if $\overrightarrow{H}$ has an oriented 6-total-coloring:

- Assume $\overrightarrow{H}$ admits an oriented 6-total-coloring τ. Since $\overrightarrow{H}$ contains copies of the vertex gadget $\overrightarrow{V}$, by Lemma 5.6, actually τ must be a $\overrightarrow{T}$-total-coloring. Also, for every $v \in V(G)$, the pair $(\tau(w_v), \tau(s_v))$ must be good. Recall that there are only three good pairs, namely $(0, 3)$, $(1, 4)$ and $(5, 2)$. Furthermore, for every $uv \in E(G)$, the vertices w_u and s_v of $\overrightarrow{H}$ are joined by a directed 8-path from w_u to s_v, and by Lemma 5.7 it must be that $\tau(w_u)$ is weak, $\tau(s_v)$ is strong, the pair $(\tau(w_u), \tau(s_v))$ is not good, and thus $\tau(w_u) \neq \tau(s_v) + 3$. This implies that the good pairs $(\tau(w_u), \tau(s_u))$ and $(\tau(w_v), \tau(s_v))$ are not the same. From this, we deduce that considering every vertex $v \in V(G)$ and assigning color $\tau(s_v)$ to v, we obtain a proper 3-coloring of G.

- Conversely, assume G admits a proper 3-coloring assigning colors $0, 1, 5$. Consider the $\overrightarrow{T}$-total-coloring τ' of $\overrightarrow{H}$ obtained as follows. For every vertex $v \in V(G)$, we set $\tau'(w_v) = \tau(v)$ and $\tau'(s_v) = \tau(v) + 3$. By Lemma 5.6, the resulting partial total-coloring τ' extends in an oriented way to all the $\overrightarrow{V_v}$'s. From the fact that τ is proper and that the resulting τ' can be assumed to satisfy all properties in Lemma 5.6, the resulting τ' actually verifies all conditions in Lemma 5.8, which extends in an oriented way also to the remaining elements of $\overrightarrow{H}$. As a result, we eventually obtain a $\overrightarrow{T}$-total-coloring of $\overrightarrow{H}$.

This concludes the proof of Theorem 5.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6. Oriented planar graphs

To make the results in this section more legible, we extend the oriented total chromatic number to graphs, by defining $\chi_o''(G)$ for a graph G as the maximum of $\chi_o''(\overrightarrow{G})$ over all orientations $\overrightarrow{G}$ of G. We also extend it to families of graphs, by defining $\chi_o''(\mathcal{F})$ for a family $\mathcal{F}$ of graphs as the maximum of $\chi_o''(G)$ over all members G of $\mathcal{F}$. Similarly, we extend all chromatic notions considered earlier to graphs and families of graphs.

A common class of graphs to consider when dealing with a new coloring problem is that of planar graphs. For $g \geqslant 3$, we define $\mathcal{P}_g$ as the family of planar graphs with girth at least g, i.e., in which shortest cycles have length at least g. Note that $\mathcal{P}_3$ is the class of all planar graphs.

Note that results we have proved earlier provide first upper bounds on $\chi_o''(\mathcal{P}_3)$. Notably, we get $\chi_o''(\mathcal{P}_3) \leqslant 160$ from Theorem 4.3 and $\chi_o''(\mathcal{P}_3) \leqslant 38 + 2^{38}$ from Theorem 4.4, since oriented planar graphs have oriented chromatic number at most 80 and oriented chromatic index at most 38 (see [17]). Using that planar graphs have acyclic chromatic number at most 5 (see [1]), Theorem 4.6 also provides the following.

Theorem 6.1. $\chi_o''(\mathcal{P}_3) \leqslant 120$.

In the rest of this section, we deal with restricted oriented planar graphs. Namely, we deal with oriented planar graphs with large girth, and oriented outerplanar graphs.

6.1. Oriented planar graphs with large girth

A common trend when coloring planar graphs is that they require less and less colors when their girth increases, with some stabilization for large enough girths, similarly to what happens with standard coloring: $\chi(\mathcal{P}_g) = 3$ for $g \geqslant 4$ by Grötzsch's Theorem. In the oriented setting, it is known that $\chi_o(\mathcal{P}_g) = 5$ for every $g \geqslant 12$ and $\chi_o'(\mathcal{P}_g) \leqslant 4$ for every $g \geqslant 46$ (see [17]). For oriented total-coloring, this still holds, as shown by the following.

Theorem 6.2. For every $g \geqslant 26$, we have $\chi_o''(\mathcal{P}_g) = 7$.

We start by proving the upper bound in Theorem 6.2.

Lemma 6.3. For every $g \geqslant 26$, we have $\chi_o''(\mathcal{P}_g) \leqslant 7$.
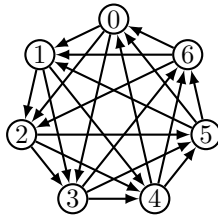

Figure 9: The tournament $\overrightarrow{T_7}$.

Proof. This is proved by showing that for every oriented planar graph $\overrightarrow{G}$ such that G has girth $g \geqslant 26$, its total digraph $T(\overrightarrow{G})$ admits a homomorphism to $\overrightarrow{T_7}$, the oriented graph depicted in Figure 9. Towards a contradiction, assume this is wrong, and let $\overrightarrow{M}$ be a minimal counterexample (in terms of order). That is, $M \in \mathcal{P}_g$ and the total digraph of $\overrightarrow{M}$ does not admit a homomorphism to $\overrightarrow{T_7}$. By minimality, M is connected. Also, we know that M, due to its girth, contains either a vertex v of degree 1, or a 6-path $x_0 \cdots x_6$ whose internal vertices $x_1, \ldots, x_5$ have degree 2 (see [5]). We exhibit in each case a contradiction with the minimality of $\overrightarrow{M}$.

- Assume M contains a degree-1 vertex v with unique neighbor u. Then, by minimality of $\overrightarrow{M}$, there must be a homomorphism $f : T(\overrightarrow{M} - v) \to \overrightarrow{T_7}$. Since $\overrightarrow{T_7}$ is vertex-transitive, without loss of generality we may assume that $f(u) = 0$.

  We extend f to $T(\overrightarrow{M})$ by setting $f(v) = 5$ and $f(e_{vu}) = 6$ if v is an in-neighbor of u, and $f(v) = 2$ and $f(e_{uv}) = 1$ otherwise, a contradiction.

- Assume M contains a 6-path $x_0 \cdots x_6$ whose internal vertices have degree 2. By minimality of $\overrightarrow{M}$, the oriented graph $\overrightarrow{M} - \{x_1, \ldots, x_8\}$ admits a $\overrightarrow{T_7}$-total-coloring f. Due to the vertex-transitivity and the arc-reversal symmetry of $\overrightarrow{T_7}$, without loss of generality we may assume that $f(x_0) = 0$ and x_1 is an out-neighbor of x_0. Moreover, assume that $f(x_6) = i \in V(\overrightarrow{T_7})$.

  Observe now that it is possible to extend f correctly to x_1 and $x_0 x_1$ by setting $f(x_0 x_1) = 1$ and $f(x_1) \in \{2, 3\}$. Indeed, every arc going to x_0 is colored with 5 or 6 (since the in-neighbor of x_0 incident to that arc is colored 4 or 5). Similarly, we can color the arc between x_5 and x_6 with $i \pm 1 \mod 7$ (depending on its orientation).

  Now, one can show by induction that for every j, there are at least $j + 1$ colors that are possible for x_j, meaning that if we set $f(x_j)$ to be one of them, then we can extend f to the oriented path from x_0 to x_j. To obtain a homomorphism of $T(\overrightarrow{M})$ to $\overrightarrow{T_7}$, note that either $i - 2$ or $i - 3$, and either $i + 2$ or $i + 3$ have to be among the six possible colors for x_5. We may thus color x_5 regardless of the orientation of the arc $x_5 x_6$, a contradiction. $\square$

We now focus on proving the lower bound in Theorem 6.2. Note that Theorems 3.3 and 3.4 already provide that $\chi_o''(\mathcal{P}_g) \geqslant 6$ holds for every $g \geqslant 26$. Towards a contradiction, let us assume that $\chi_o''(\mathcal{P}_g) = 6$. Since $\mathcal{P}_g$ is closed under taking disjoint unions of graphs, there must exist a tournament $\overrightarrow{T_6}$ such that every orientation of a graph in $\mathcal{P}_g$ admits a $\overrightarrow{T_6}$-total-coloring. In what follows, we investigate what $\overrightarrow{T_6}$ should look like.

Lemma 6.4. $\overrightarrow{T_6}$ *contains exactly three vertices with out-degree* 3 *and in-degree* 2*, while the other three vertices have in-degree* 3 *and out-degree* 2*.*

Proof. It suffices to show that every vertex of $\overrightarrow{T_6}$ has out-degree and in-degree at least 2.

Recall that, whatever $g \geqslant 26$ is, there exists an oriented cycle $\overrightarrow{C}$ of length at least g satisfying $\chi_o''(\overrightarrow{C}) = 6$. Consider $\overrightarrow{C}^*$, the oriented graph obtained by attaching a new pending out-neighbor x^+ and a new pending in-neighbor x^- at every vertex x of $\overrightarrow{C}$. By our assumption, there exists a homomorphism $\tau : T(\overrightarrow{C}^*) \to \overrightarrow{T_6}$. Since $\chi_o''(\overrightarrow{C}) = 6$, each color must appear on some x in the unique cycle of C^*. But then the vertices x^+ and e_{xx^+} (resp. x^- and e_{x^-x}) must be mapped, in $\overrightarrow{T_6}$, to distinct out-neighbors (resp. in-neighbors) of $\tau(x)$, which concludes. $\square$

We now prove a key structural property of $\overrightarrow{T_6}$.

Lemma 6.5. $\overrightarrow{T_6}$ *does not contain a transitive oriented* 3*-cycle with source* x *and sink* y *such that the out-degree of* x *and the in-degree of* y *are both* 2 *in* $\overrightarrow{T_6}$*.*

Proof. Assume this is wrong, i.e., suppose $\overrightarrow{T_6}$ contains a transitive oriented 3-cycle with source x and sink y such that the out-degree of x and the in-degree of y are both 2 in $\overrightarrow{T_6}$.

Consider an orientation $\overrightarrow{A}$ of a $(2g+1)$-cycle $A = u_0 u_1 v_1 u_2 v_2 \ldots u_g v_g u_0$ where each v_i is a sink and u_0 is a source. Let $\tau : T(\overrightarrow{A}) \to \overrightarrow{T_6}$ be a homomorphism. If $\tau(u_1) = x$, then we must have

$\tau(v_i) = y$ and $\tau(u_i) = x$ for every $i \in \{1, \ldots, g\}$. This implies $\tau(u_0) = \tau(u_1)$ while u_0 and u_1 are adjacent. Thus, there cannot exist a homomorphism $\tau : T(\overrightarrow{A}) \to \overrightarrow{T_6}$ where $\tau(u_1) = x$.

Now consider an oriented cycle $\overrightarrow{C}$ of length at least g satisfying $\chi''_o(\overrightarrow{C}) = 6$. For every vertex u of $\overrightarrow{C}$, add a new copy $\overrightarrow{A_u}$ of $\overrightarrow{A}$ to the oriented graph, and identify u with the vertex u_1 of that new $\overrightarrow{A}$. Let us denote by $\overrightarrow{B}$ the resulting oriented graph (whose girth is still at least g).

Let $\tau : T(\overrightarrow{B}) \to \overrightarrow{T_6}$. Since $\chi''_o(C) = 6$, one of the vertices u of $\overrightarrow{C}$ must satisfy $\tau(u) = x$. Consider now the restriction of τ to $\overrightarrow{A_u}$. Since $\tau(u_1) = x$, then we must have $\tau(v_i) = y$ and $\tau(u_i) = x$ for every $i \in \{1, \ldots, g\}$. This implies $\tau(u_0) = \tau(u_1)$ while u_0 and u_1 are adjacent, a contradiction. $\qquad\square$

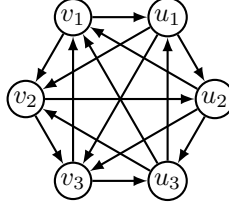We are now ready to show that there is only one candidate tournament as $\overrightarrow{T_6}$.

Figure 10: The tournament $\overrightarrow{T_6}$.

Lemma 6.6. $\overrightarrow{T_6}$ *is the tournament depicted in Figure 10.*

Proof. By Lemma 6.4, we know that $\overrightarrow{T_6}$ contains exactly three vertices u_1, u_2, u_3 having out-degree 3 and in-degree 2. Also, the other three vertices v_1, v_2, v_3 of $\overrightarrow{T_6}$ have in-degree 3 and out-degree 2.

For every $i \in \{1, 2, 3\}$, the two in-neighbors of u_i cannot both belong to $\{v_1, v_2, v_3\}$ due to Lemma 6.5, and, thus, $\{u_1, u_2, u_3\}$ must induce a directed 3-cycle, say $u_1 u_2 u_3 u_1$ by symmetry. Similarly $\{v_1, v_2, v_3\}$ must induce a directed 3-cycle, say $v_1 v_2 v_3 v_1$.

Now, note that the u_i's (resp. v_i's), due to their out-degrees and in-degrees, must have exactly one in-neighbor (resp. out-neighbor) among the v_i's (resp. u_i's). Without loss of generality, we have the arc $v_i u_i$ for every $i \in \{1, 2, 3\}$ and all the other arcs are directed from $\{u_1, u_2, u_3\}$ to $\{v_1, v_2, v_3\}$, which concludes. $\qquad\square$

We end this section by exhibiting an oriented graph $\overrightarrow{G}$ without $\overrightarrow{T_6}$-total-coloring and such that $G \in \mathcal{P}_g$. Consider the directed $(3g+1)$-cycle $\overrightarrow{C_{3g+1}} = x_0 \cdots x_{3g} x_0$. Note that C_{3g+1} is a planar graph with girth at least g. Since $\chi''_o(\mathcal{P}_g) = 6$, there must exist $\tau : T(\overrightarrow{C_{3g+1}}) \to \overrightarrow{T_6}$. Observe that for each arc xy, if $\tau(x) = u_i$ then $\tau(y) = u_{i+1}$ and if $\tau(x) = v_i$ then $\tau(y) = v_{i+1}$ (where indices are taken modulo 3). In particular, that means we have $\tau(x_0) = \tau(x_{3g})$, which is impossible since these vertices are adjacent. This concludes the proof of Theorem 6.2.

6.2. Oriented outerplanar graphs

We now turn our attention to the oriented total chromatic number of outerplanar graphs. Denoting by $\mathcal{O}$ the family of all outerplanar graphs, our main result in this section reads as follows:

Theorem 6.7. $10 \leqslant \chi''_o(\mathcal{O}) \leqslant 13$.

For comparison, let us mention, see e.g. [17], that $\chi_o(\mathcal{O}) = \chi'_o(\mathcal{O}) = 7$. As in the previous section, we split the proof of Theorem 6.7 into smaller proofs. We first prove the upper bound.

Lemma 6.8. $\chi''_o(\mathcal{O}) \leqslant 13$.

Proof. By [15], every oriented outerplanar graph admits a homomorphism to the Paley tournament $\overrightarrow{Pal_7}$ (depicted in Figure 11) on 7 vertices. The result now follows from Proposition 2.1 since $\overrightarrow{Pal_7}$ admits an oriented 13-total-coloring (see again Figure 11). $\qquad\square$

We now focus on proving the lower bound of Theorem 6.7.
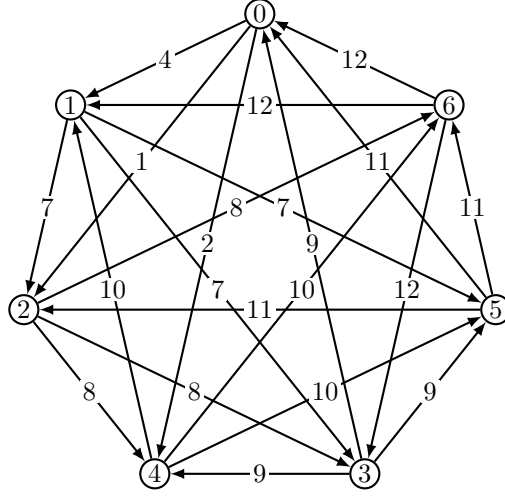
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Figure 11: The Paley tournament $\overrightarrow{Pal_7}$ on 7 vertices together with an oriented 13-total-coloring.

Lemma 6.9. *The oriented outerplanar graph $\overrightarrow{O}$ depicted in Figure 12 does not admit an oriented 9-total-coloring.*
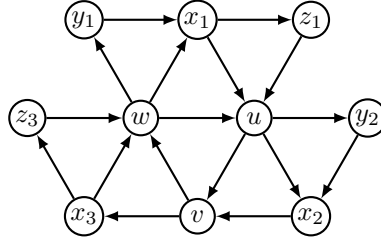

Figure 12: An oriented outerplanar graph with oriented total chromatic number 10.

Proof. Assume this is wrong, that is let $\tau : V(\overrightarrow{O}) \to \{1, \ldots, 9\}$ be an oriented 9-total-coloring of $\overrightarrow{O}$. By Lemma 2.2, recall that if two vertices of $T(\overrightarrow{O})$ are either adjacent or connected by a directed 2-path, then their corresponding vertices or arcs in $\overrightarrow{O}$ must be assigned distinct colors by τ. This implies that $x_1u, ux_2, x_2v, vx_3, x_3w, wx_1, u, v, w$ must get colored with 1 to 9 in this order (up to renaming colors).

From this, one can easily check that the only available colors for $x_1, x_2, x_3, uv, vw, wu$ are the ones listed in the following table:

	x_1	x_2	x_3	uv	vw	wu
τ	$3,4$	$5,6$	$1,2$	$2,3$	$4,5$	$6,1$

Moreover, either they are all colored with the first element of their list, or they are all colored with the second. In both cases, observe that $\tau(y_1x_1)$ cannot lie in $\{3, 4\}$. Therefore $\tau(y_1x_1) \in \{6, 8\}$. Similarly, we get the following constraints:

	x_3z_3	z_3w	wy_1	y_1x_1	x_1z_1	z_1u	uy_2	y_2x_2
τ	$5,7$	$5,8$	$6,7$	$6,8$	$1,8$	$1,9$	$2,8$	$2,9$

Now, if $\tau(y_1x_1) = 6$, then $\tau(wy_1) = 7$, hence $\tau(x_3z_3) = 5$, and finally $\tau(z_3w) = 8$. Therefore, $8 \in \{\tau(y_1x_1), \tau(z_3w)\}$. But, by symmetry, we also get that $8 \in \{\tau(x_1z_1), \tau(uy_2)\}$, a contradiction since y_1x_1, z_3w, x_1z_1 and z_3w form a relative oriented clique. Thus, $\overrightarrow{O}$ admits no oriented 9-total-coloring. $\square$

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Note that there is some gap between the two bounds in Theorem 6.7; we thus leave the following question open for now.

Question 6.10. *What is the exact value of $\chi''_o(\overrightarrow{\mathcal{O}})$?*

7. Conclusions

Our main goal in this work was to introduce a notion of total-coloring of oriented graphs that would stand as a generalization of proper total-colorings, fitting in a natural way into the area comprising oriented colorings and oriented arc-colorings. From this point of view, the way the three coloring notions relate, as described in Section 2, is indeed quite reminiscent of the connections between the corresponding three notions in undirected graphs.

Following the previous work on the oriented chromatic number and index yields a wealth of directions for studying oriented total-colorings. In this work, we chose to focus on presenting several kinds of results, about particular families of oriented graphs, connections with other graph parameters, and algorithmic aspects. In this concluding section, we would like to summarize the open questions disseminated throughout this work and present possible directions for further work on the topic.

- Regarding our results in Section 4, certainly the most interesting question is Question 4.1. Having $\chi''_o(\overrightarrow{G}) \leqslant \chi_o(\overrightarrow{G}) + \chi'_o(\overrightarrow{G})$ for every oriented graph $\overrightarrow{G}$ would definitely seem very natural, as it would be reminiscent of a similar relationship in the undirected setting, but we failed proving it because of situations such as the one depicted in Figure 4. As mentioned earlier, recall that this inequality, if true, would imply a certain number of things, such as a direct proof of Theorem 4.3.

  In connection with Question 4.1, we proved through Theorem 4.2 that the difference of $\chi_o(\overrightarrow{G}) + \chi'_o(\overrightarrow{G})$ and $\chi''_o(\overrightarrow{G})$ can be arbitrarily large for an oriented graph $\overrightarrow{G}$. One additional question could be whether the construction we came up with is a very peculiar one, or whether there exist other constructions that are as worse regarding the difference of these parameters.

  Finally, regarding Section 4, recall that we still miss a proof that all bounds are tight in Theorems 4.4 and 4.6, recall Questions 4.5 and 4.9.

- Regarding our results in Section 5, one could wonder about restrictions of Theorem 5.1 in particular classes of oriented graphs. Interesting candidates could include classes of oriented graphs for which determining the oriented chromatic number or oriented chromatic index is hard, such as oriented graphs being bipartite, of bounded degree, acyclic, planar, etc. [17].

- Regarding our results in Section 6, improving the upper bound in Theorem 6.1 would definitely be an appealing direction. We did not discuss about lower bounds on $\chi''_o(\mathcal{P}_3)$, but, due to the fact that $\chi_o(\mathcal{P}_3) \geqslant 18$ (as proved by Marschall [10]), we have $\chi''_o(\mathcal{P}_3) \geqslant 18$ as well. Improving any of these two bounds on $\chi''_o(\mathcal{P}_3)$ would thus be interesting.

  In Theorem 6.2, we have determined $\chi''_o(\mathcal{P}_g)$ when $g \geqslant 26$. A natural question is about oriented graphs with smaller girth, i.e., for any $g \in \{3, \ldots, 25\}$. Again, lower bounds can be established from similar bounds known for the oriented chromatic number, or some of our results such as Theorem 3.4.

  Lastly, it would be interesting to refine Theorem 6.7, the two bounds being not that distant from each other. Recall that $\chi_o(\mathcal{O}) = \chi'_o(\mathcal{O}) = 7$, so the class of oriented outerplanar graphs is well understood for the oriented chromatic number and oriented chromatic index.

- In the case of graphs, a common way for bounds on the chromatic number, chromatic index, and total chromatic number to be expressed is by a function of the maximum degree. In particular, for a graph G with maximum degree Δ, we always have $\chi(G) \leqslant \Delta + 1$ by Brooks' Theorem, $\chi'(G) \in \{\Delta, \Delta + 1\}$ by Vizing's Theorem, and it is conjectured, through the Total Coloring Conjecture of Behzad and Vizing, that we always have $\chi''(G) \leqslant \Delta + 2$.

  In the case of oriented graphs $\overrightarrow{G}$, one can similarly wonder abound upper bounds on $\chi''_o(\overrightarrow{G})$ that are functions of some maximum degrees (maximum degree of G, maximum out-degree

and in-degree in $\overrightarrow{G}$, or combinations of these parameters). For instance, setting $\Delta = \Delta(G)$, it is known that we always have $\chi_o(\overrightarrow{G}) \leqslant 2\Delta^2 \cdot 2^\Delta$ (see [8]), from which we deduce that we always have $\chi''_o(\overrightarrow{G}) \leqslant 4\Delta^2 \cdot 2^\Delta$ by Theorem 4.3. One legitimate direction for further research on the topic could be to improve this last upper bound on the oriented total chromatic number.

- Still about the maximum degree Δ of the underlying graph, one could also wonder about situations where Δ is small. Let us denote by $\mathcal{D}_\Delta$ the class of graphs with maximum degree Δ. Through Theorems 3.3 and 3.4, note that we have fully determined the oriented total chromatic number when $\Delta = 2$, as we proved that $\chi''_o(\mathcal{D}_2) = 8$. For comparison, recall that $\chi_o(\mathcal{D}_2) = \chi'_o(\mathcal{D}_2) = 5$.

  In that line, the next natural step to make would be to consider the class $\mathcal{D}_3$ of subcubic graphs. We recall that, for now, it is known that $\chi_o(\mathcal{D}_3) \in \{7, 8\}$ and $\chi'_o(\mathcal{D}_3) \in \{6, 7\}$ (see [7, 17]). From our results in this work, from Theorem 3.4 we get that $\chi''_o(\mathcal{D}_3) \geqslant 8$, while the best upper bound our results provide, through Theorem 4.3, gives $\chi''_o(\mathcal{D}_3) \leqslant 16$. Thus, quite some efforts remain to be made towards determining $\chi''_o(\mathcal{D}_3)$.

- Through Theorem 6.2, we have considered oriented graphs with a very sparse structure. In that line, it would be interesting to study the oriented total chromatic number of sparse oriented graphs, which is a line of research that has already been quite investigated regarding the oriented chromatic number and the oriented chromatic index (see [17] for pointers). Particularly, one could consider orientations of graphs with bounded maximum average degree, and investigate how the discharging method, which is one of the main tools in this context, can be set to use.

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