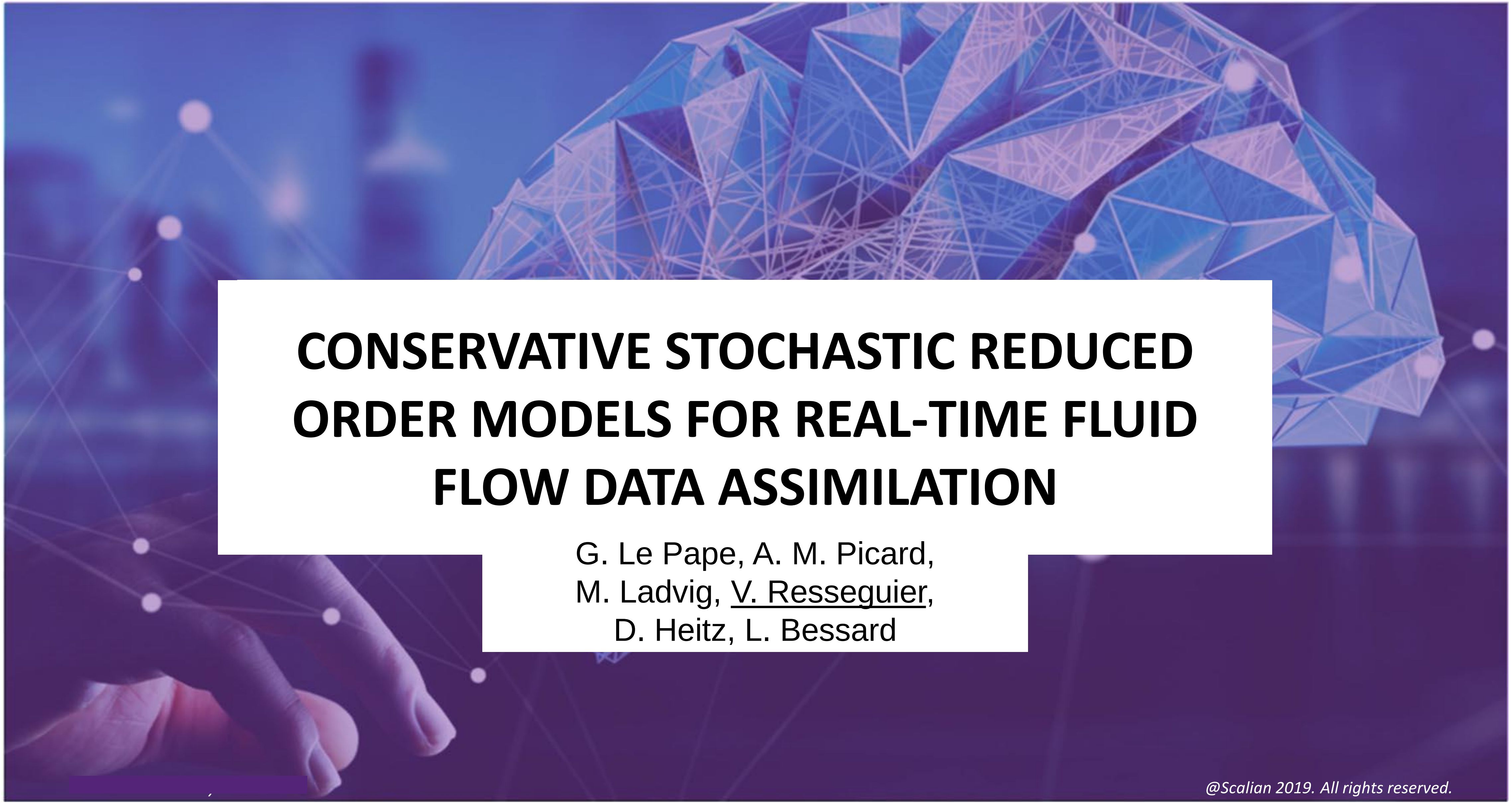

CONSERVATIVE STOCHASTIC REDUCED ORDER MODELS FOR REAL-TIME FLUID FLOW DATA ASSIMILATION

G. Le Pape, A. M. Picard,
M. Ladvig, V. Resseguier,
D. Heitz, L. Bessard

CONTENT

PART I

CONTEXT :

OBSERVER FOR WIND TURBINE APPLICATIONS

CONTEXT

Observer for wind turbine application

Application: Real-time estimation and prediction of 3D fluid flow
using strongly-limited computational resources & few sensors

Wind
turbine
blade

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Application: Real-time estimation and prediction of 3D fluid flow using strongly-limited computational resources & few sensors

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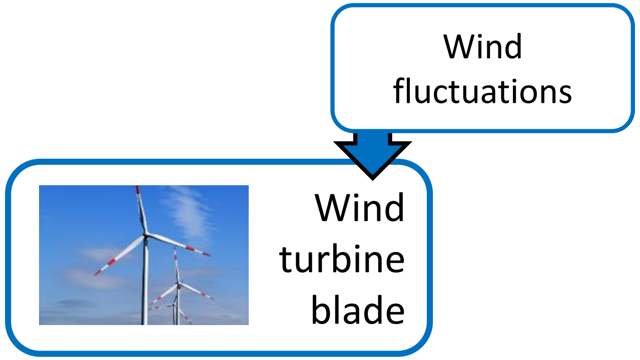

Wind fluctuations

Wind turbine blade

CONTEXT

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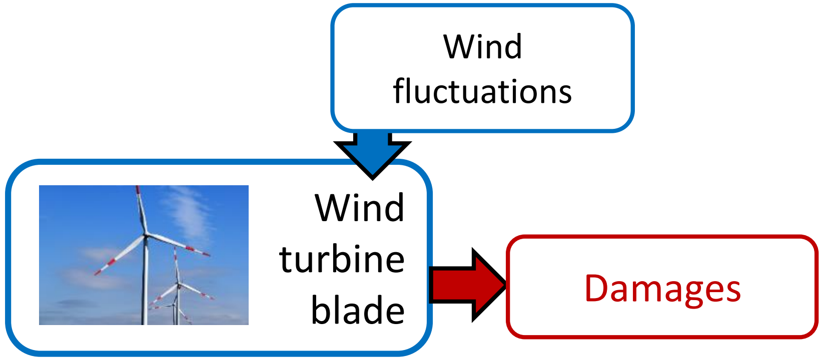


CONTEXT

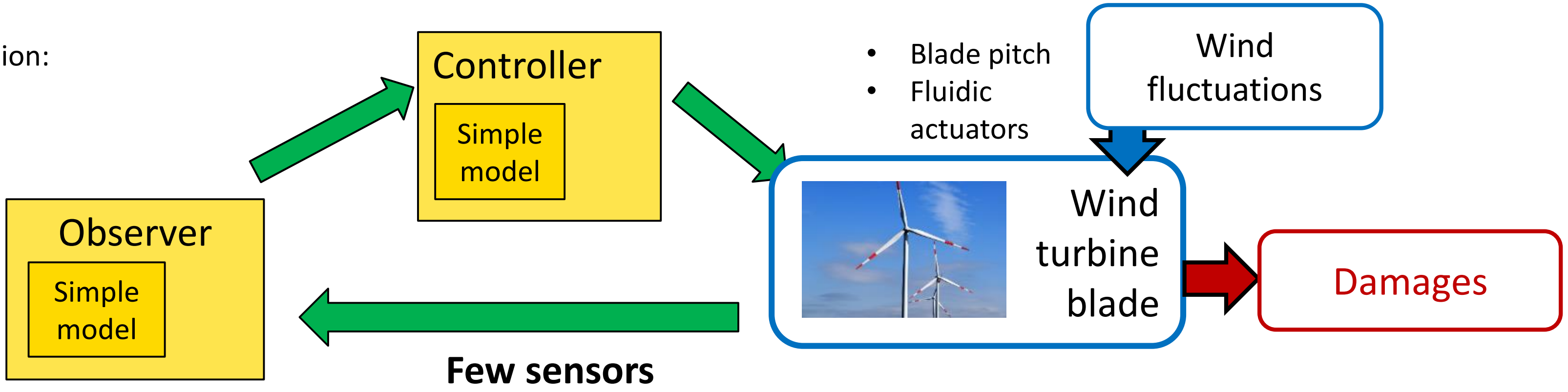
Observer for wind turbine application

Application: Real-time estimation and prediction of 3D fluid flow using strongly-limited computational resources & few sensors

---

Estimation and prediction:

- Air flow
- Lift, drag, inflow
- …



Controller

Simple model

- Blade pitch
- Fluidic actuators

Wind fluctuations

Observer

Simple model

Wind turbine blade

Damages

Few sensors

CONTEXT

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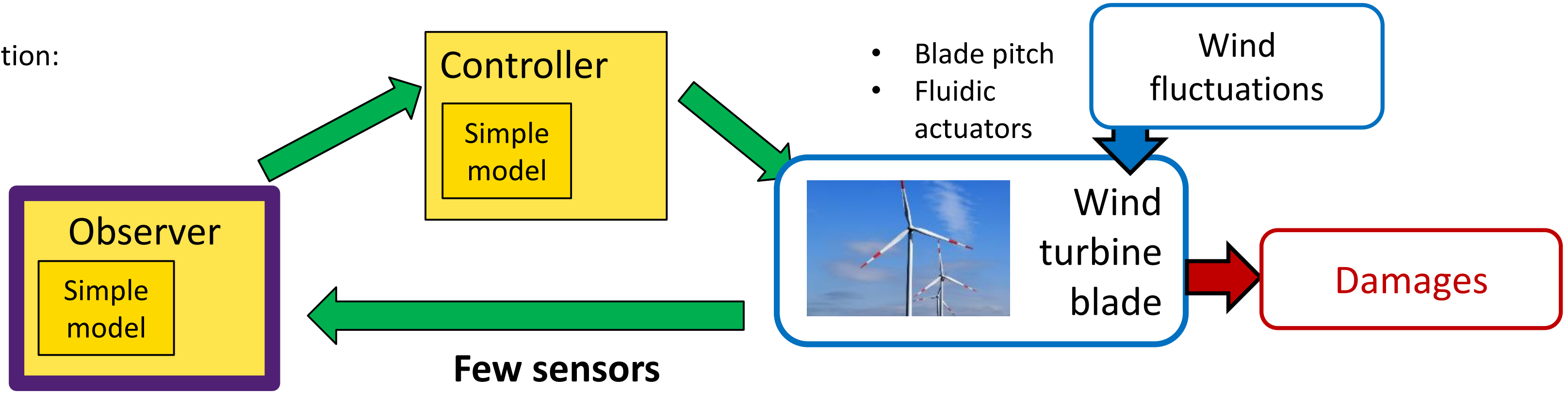

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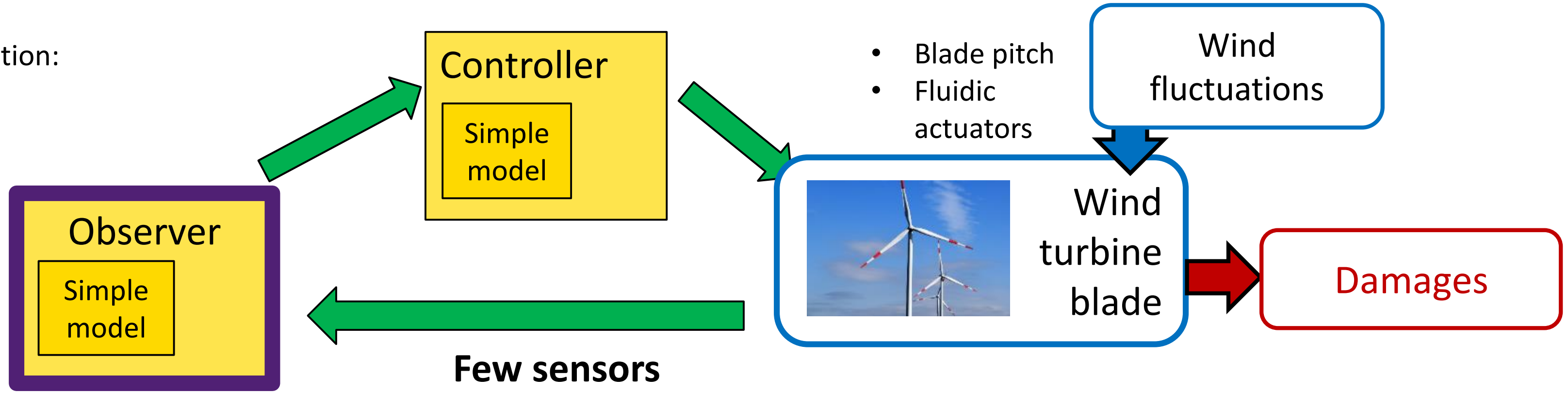

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Which simple model? How to combine model & measurements?

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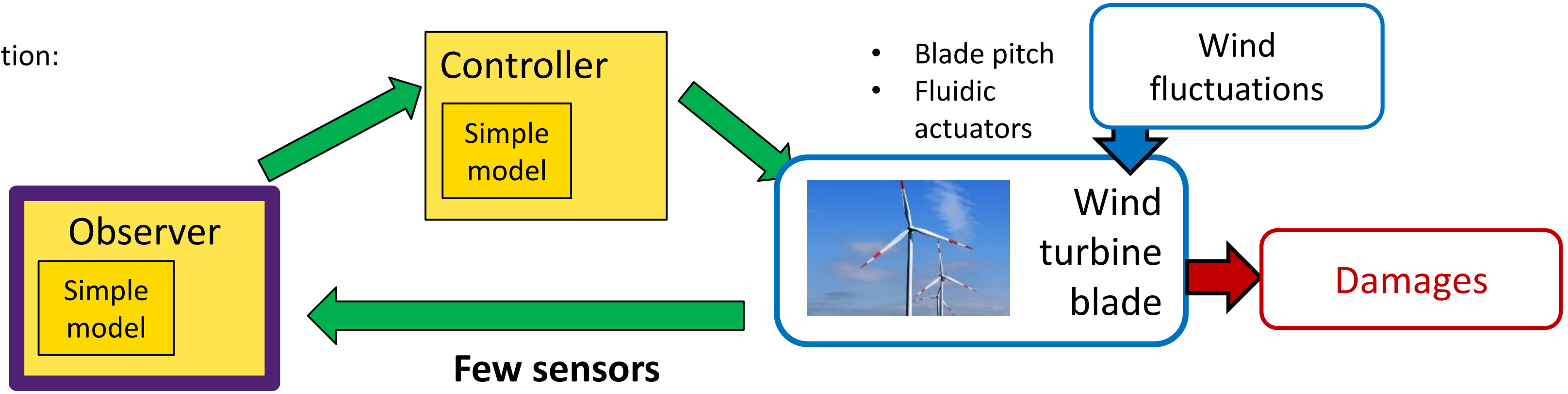


Which simple model? How to combine model & measurements?

Scientific problem :

Simulation & data assimilation under severe dimensional reduction

typically, $10^7 \rightarrow O(10)$ degrees of freedom

PART II

STATE OF THE ART

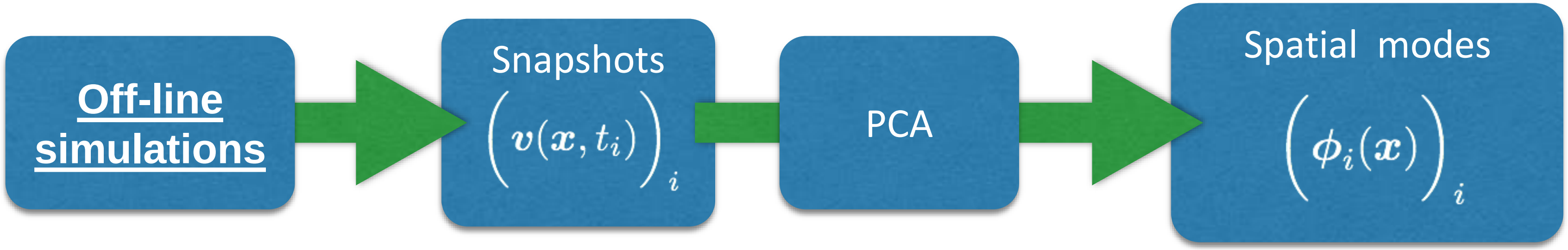
a. Intrusive reduced order model (ROM)

b. Data assimilation

INTRUSIVE REDUCED ORDER MODEL

Combine physical models and learning approches

- Principal Component Analysis (PCA) on a *dataset* to reduce the dimensionality:



Off-line simulations

Snapshots $\left(\boldsymbol{v}(\boldsymbol{x}, t_i)\right)_i$

PCA

Spatial modes $\left(\phi_i(\boldsymbol{x})\right)_i$

- Approximation:

$$v(x,t) \approx \sum_{i=0}^{n} b_i(t)\ \phi_i(x)$$

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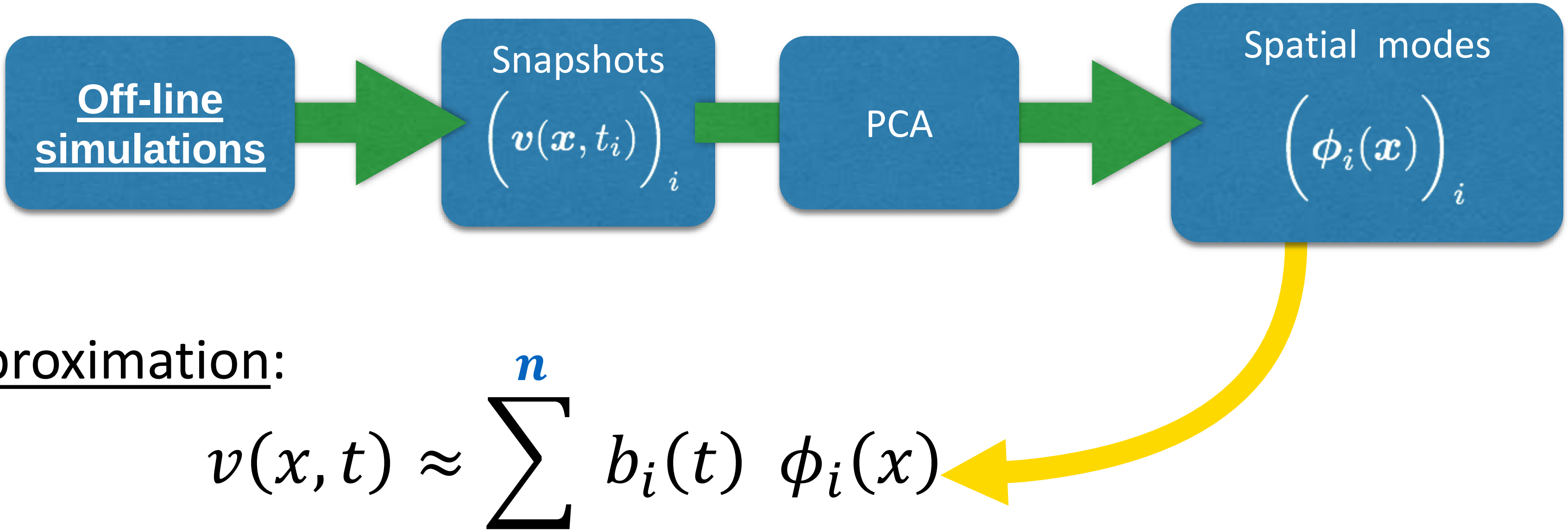

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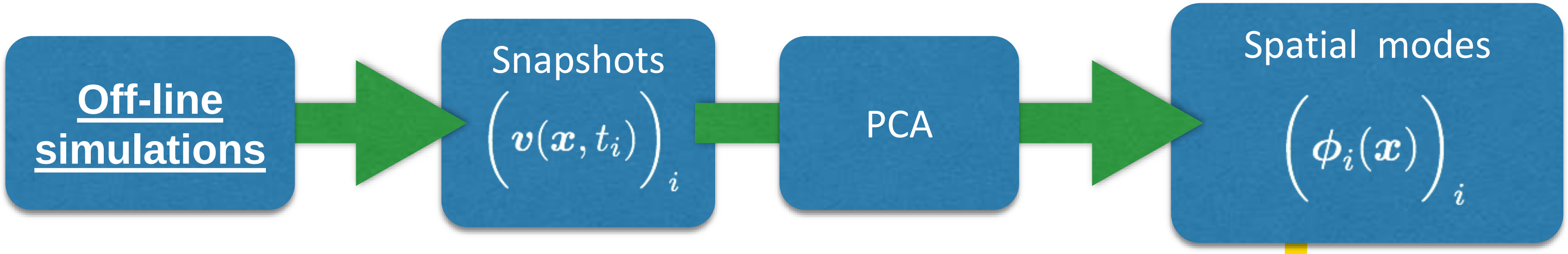

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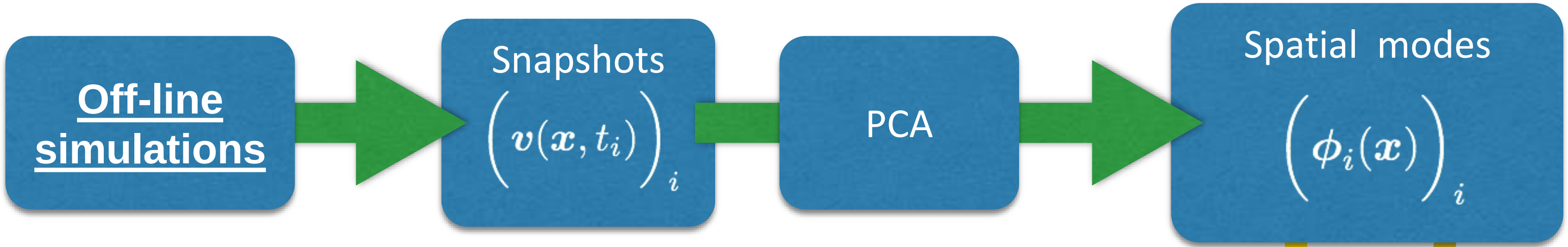

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$\int_\Omega dx\, \phi_i(x) \cdot$ (*Physical equation* (e.g. Navier-Stokes))

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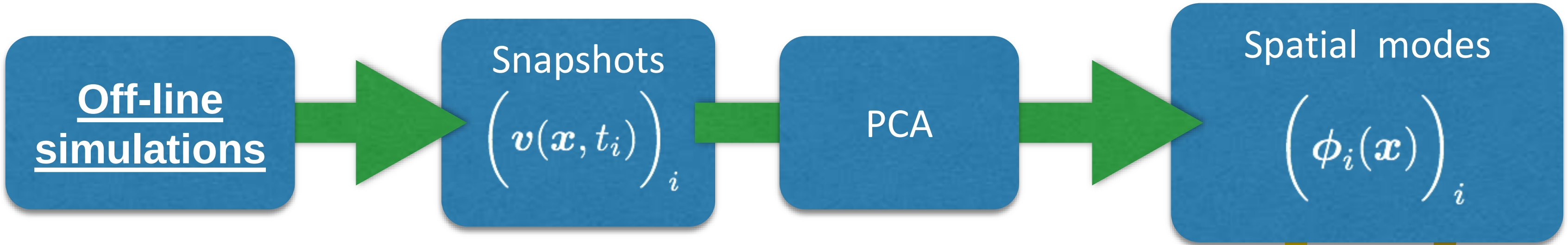

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→ **ROM for very fast simulation of temporal modes**

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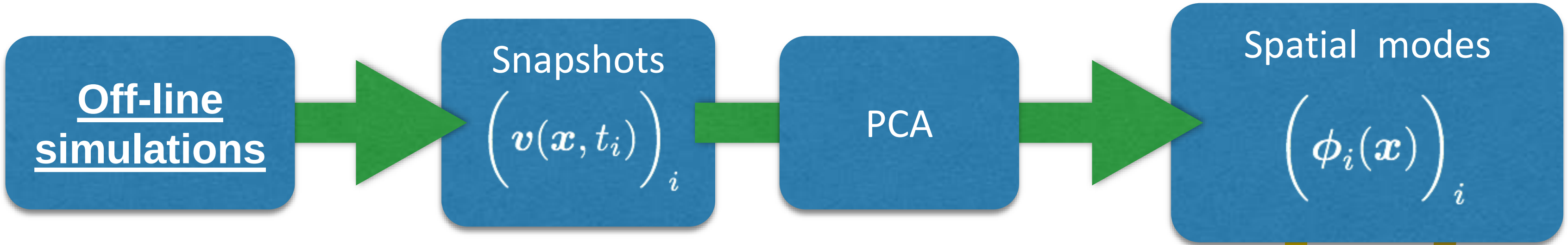

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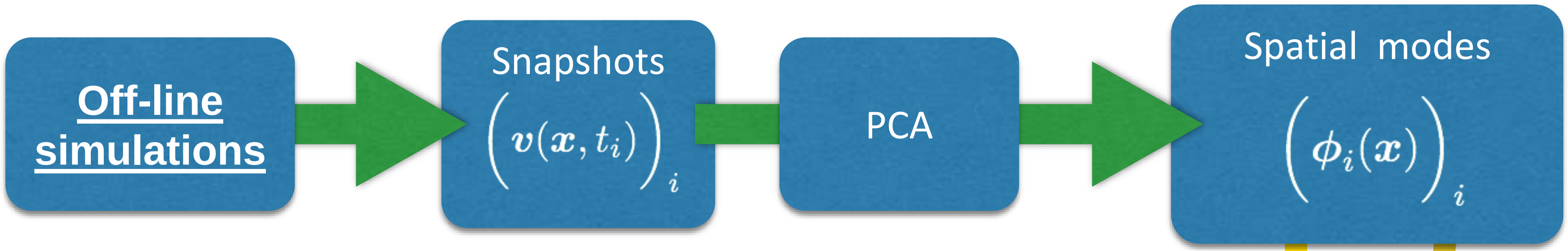

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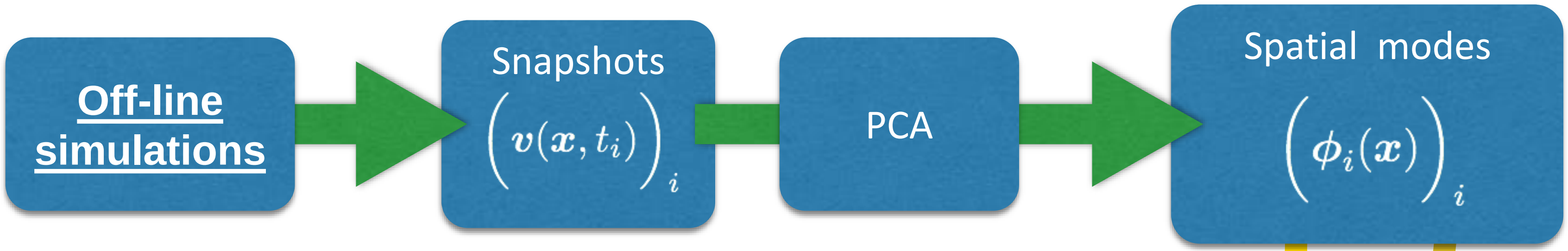

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Resolved modes

- Projection of the "*physics*" onto the spatial modes (POD-Galerkin)

$\int_\Omega dx\, \phi_i(x) \cdot$ (~~Physical equation~~ (e.g. Navier-Stokes))

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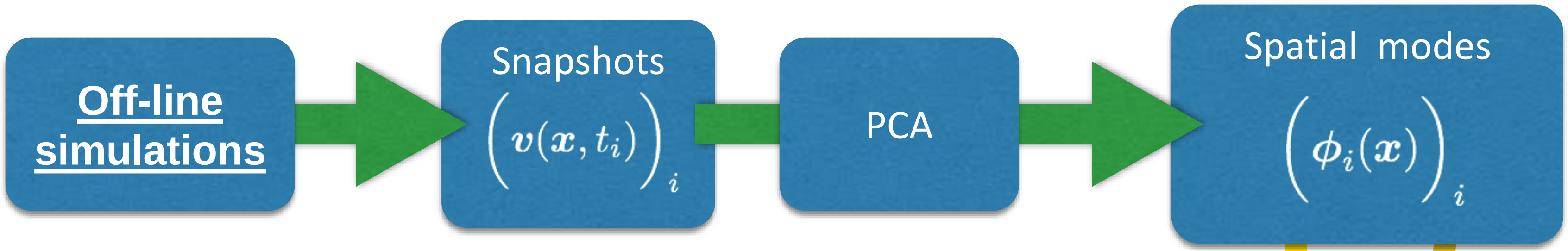

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SPOILER

$$\int_\Omega dx\, \phi_i(x) \cdot$$ (Randomized Navier-Stokes)

→ **ROM for very fast simulation of temporal modes**

DATA ASSIMILATION

= Coupling simulations and measurements y

Numerical Simulation (ROM)

→ erroneous

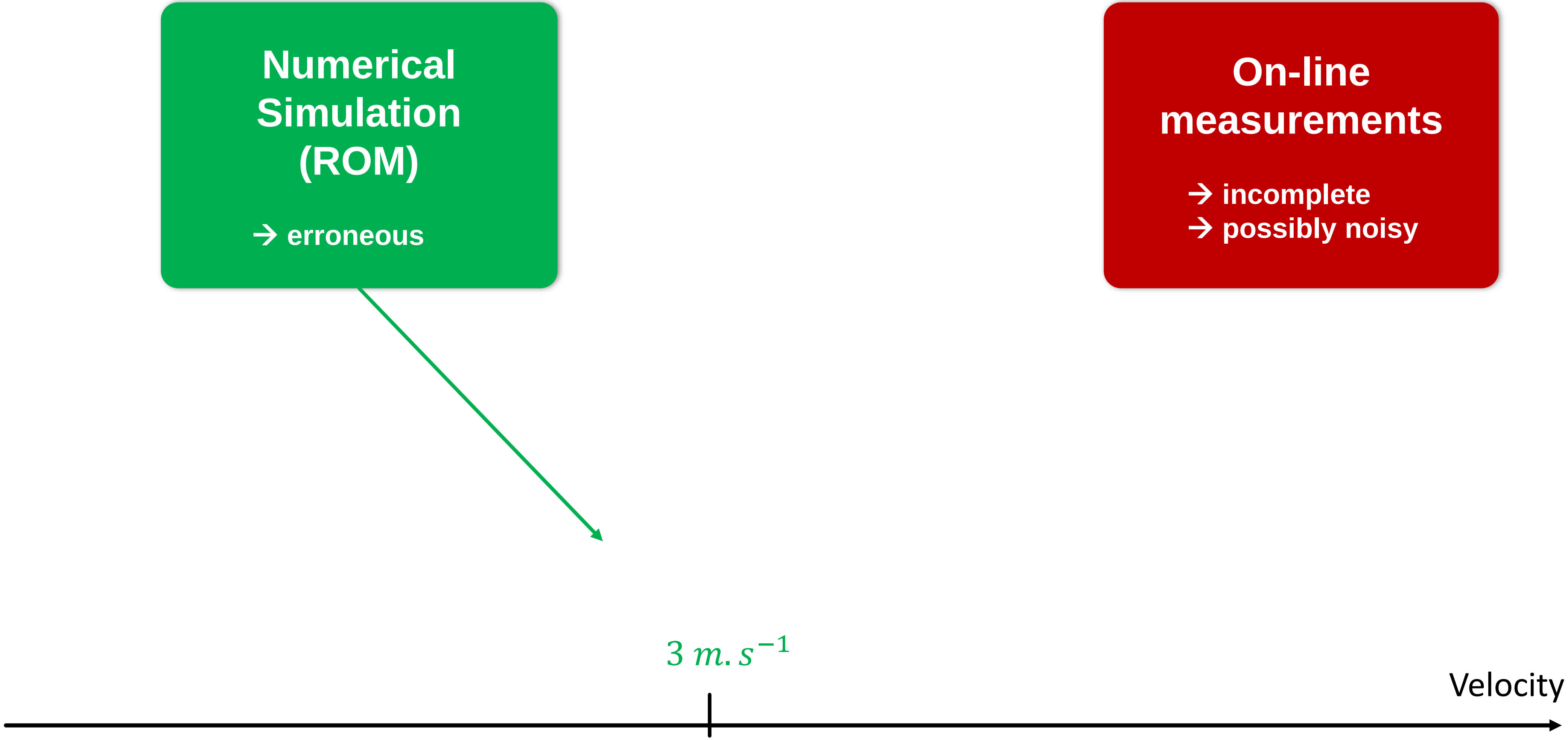
On-line measurements

→ incomplete
→ possibly noisy

Velocity

DATA ASSIMILATION

= Coupling simulations and measurements y



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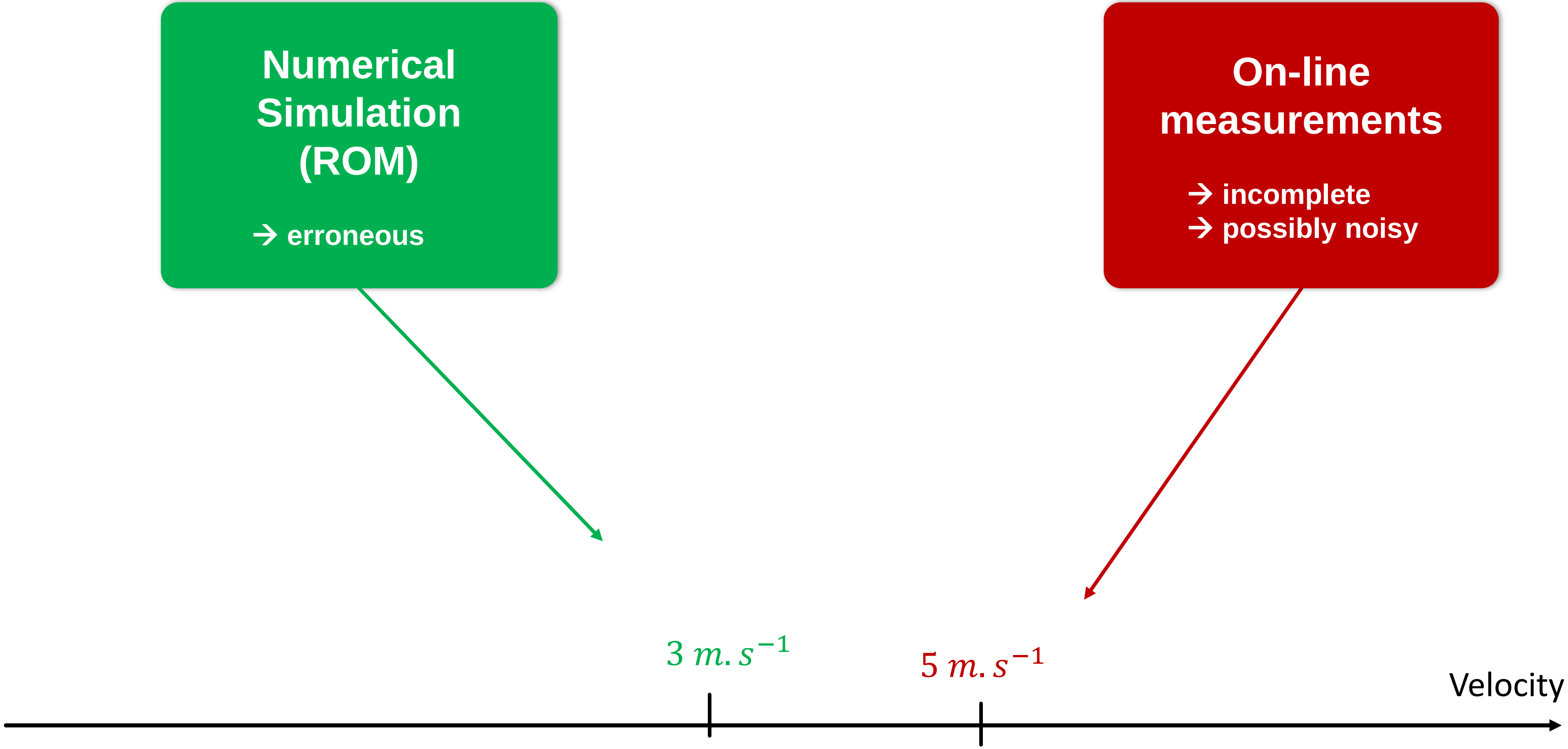

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$3\ m.s^{-1}$

$5\ m.s^{-1}$

Velocity

DATA ASSIMILATION

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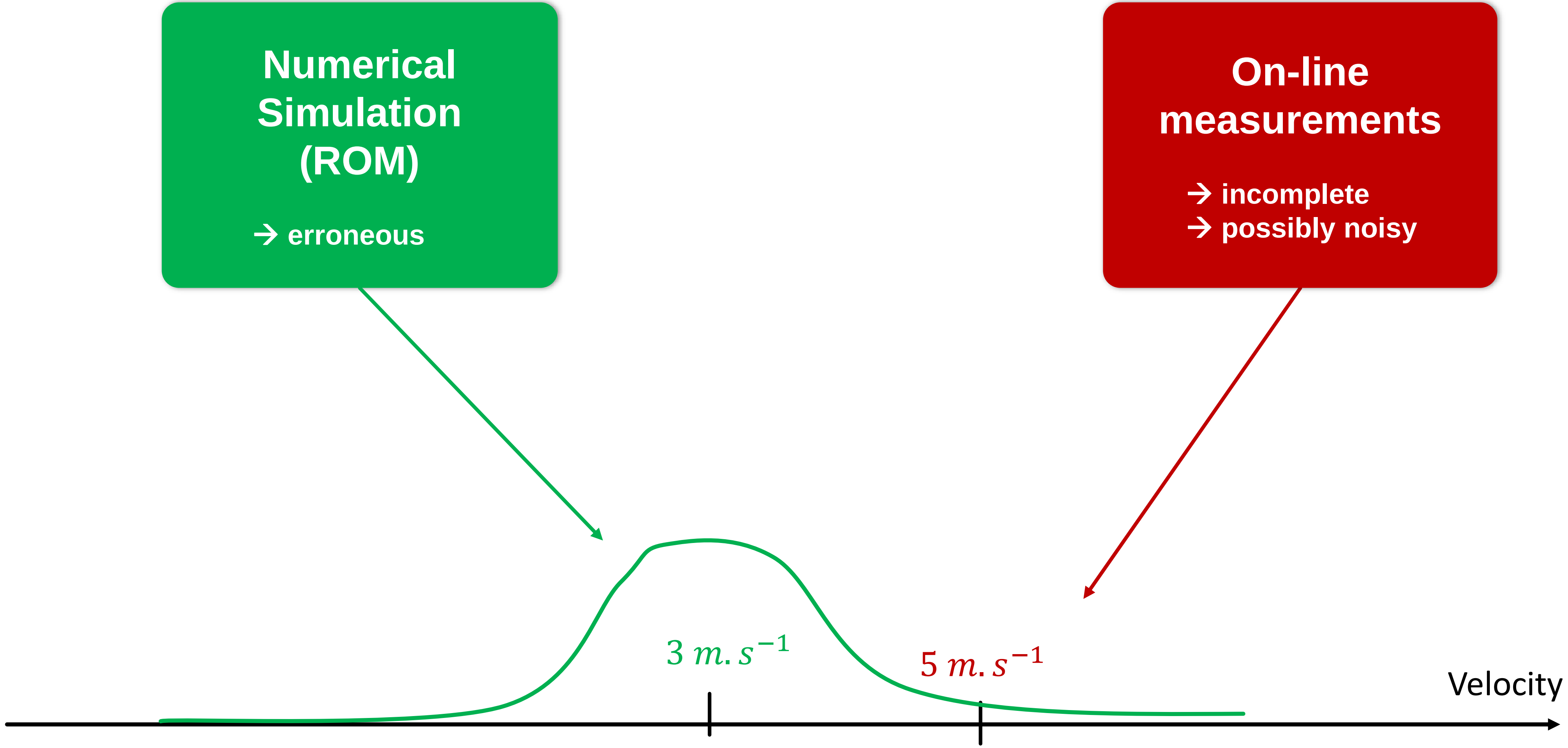

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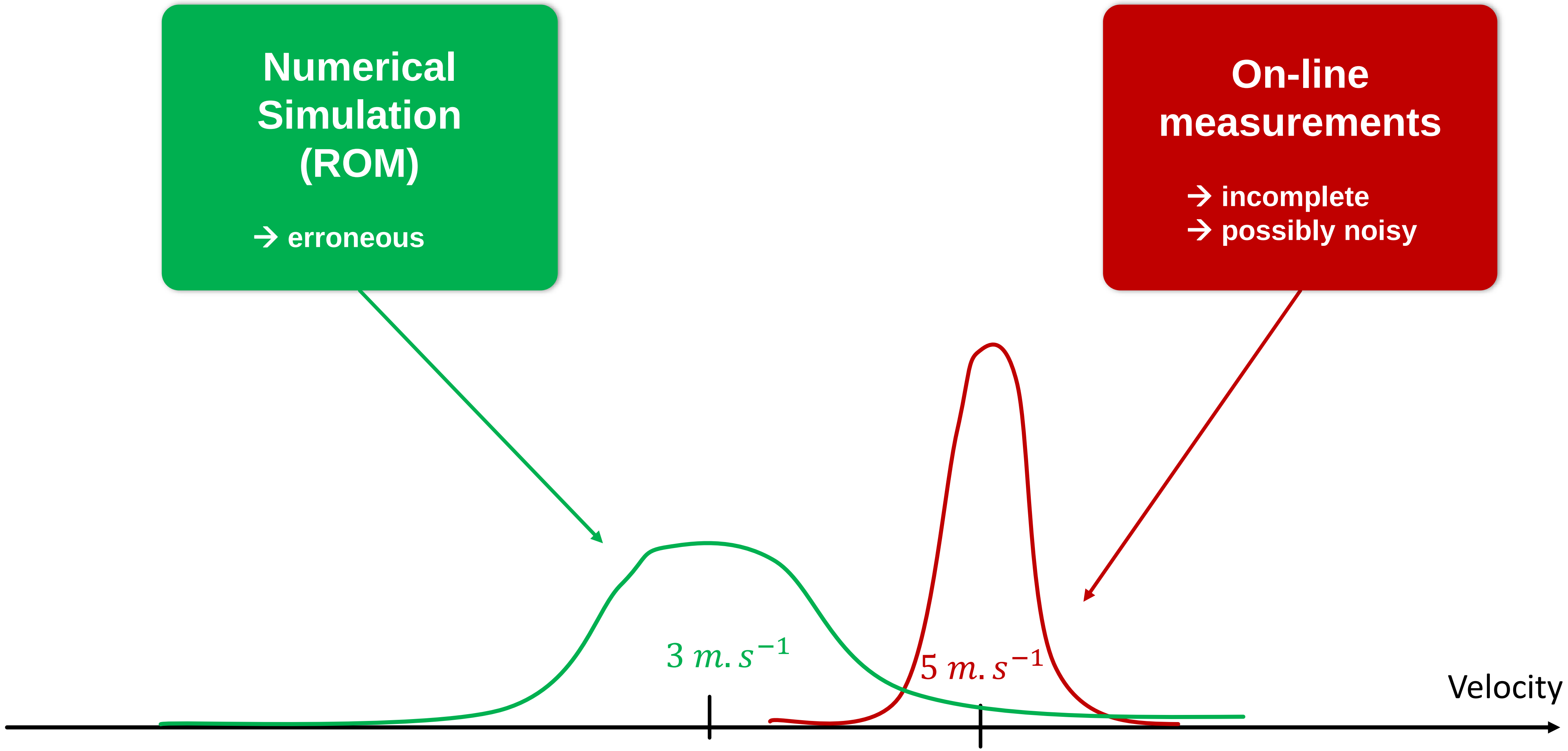

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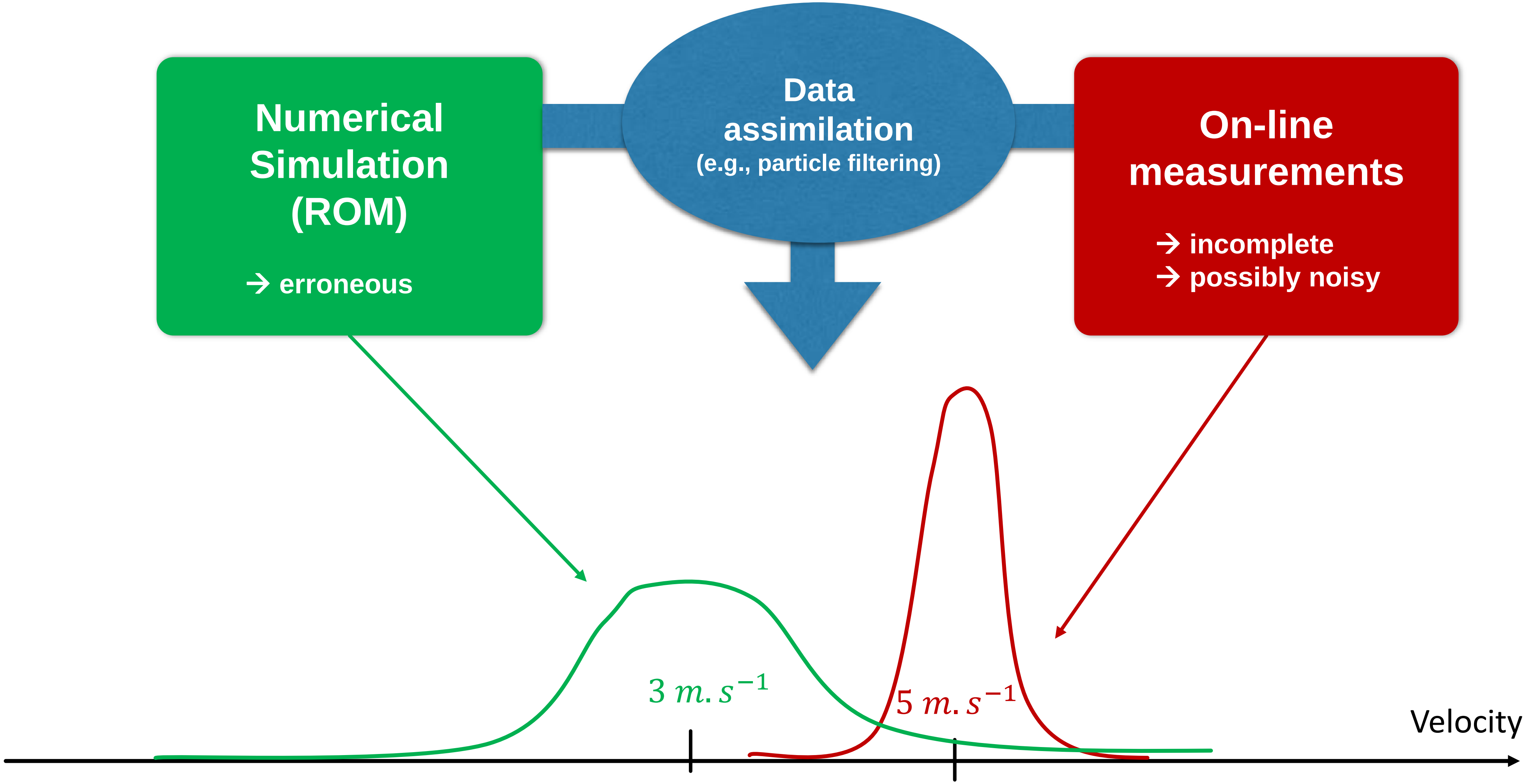

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(e.g., particle filtering)

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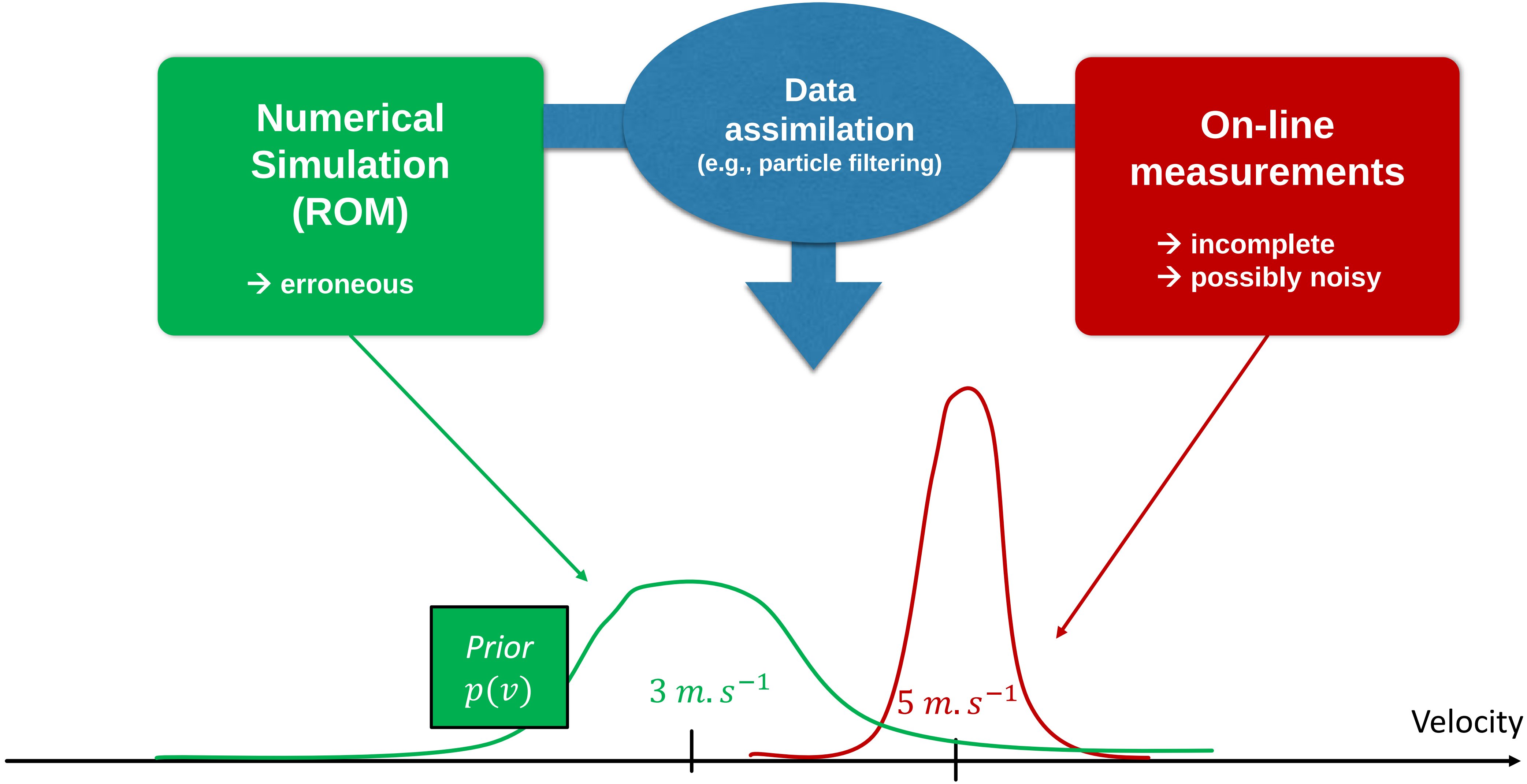

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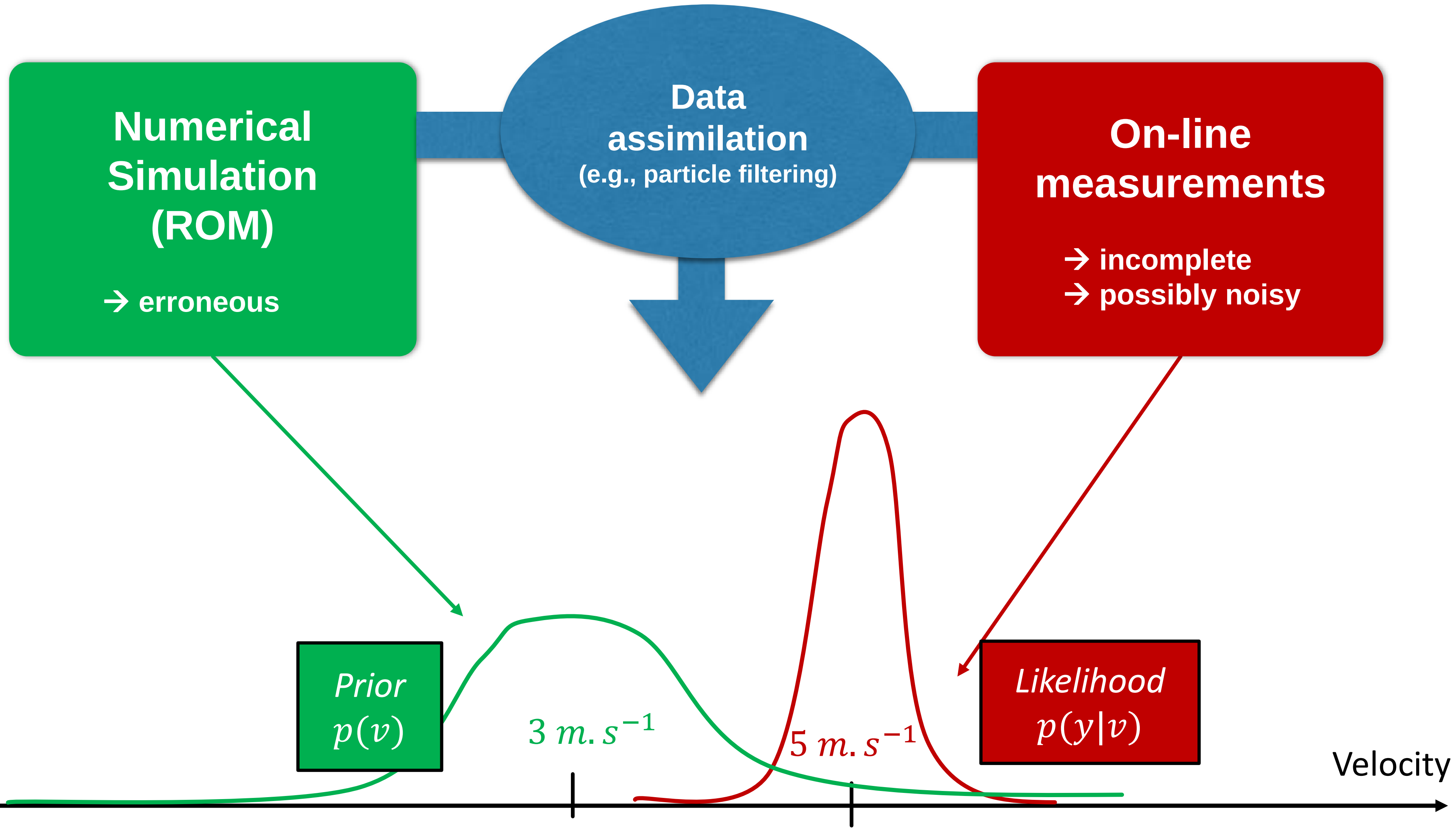

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Prior $p(v)$

$3\ m.s^{-1}$

$5\ m.s^{-1}$

Likelihood $p(y|v)$

DATA ASSIMILATION

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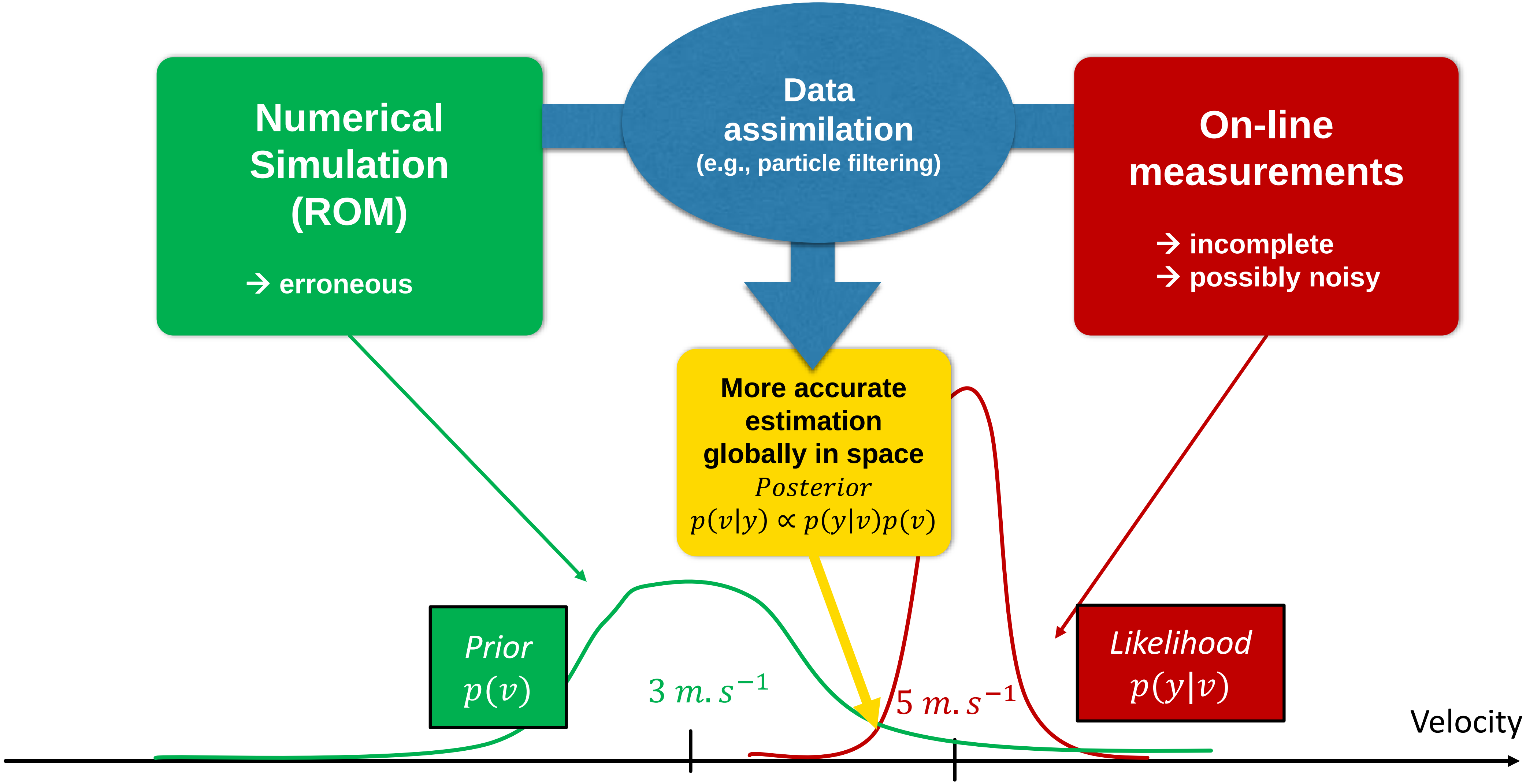

Numerical Simulation (ROM)

→ erroneous

Data assimilation

(e.g., particle filtering)

On-line measurements

→ incomplete
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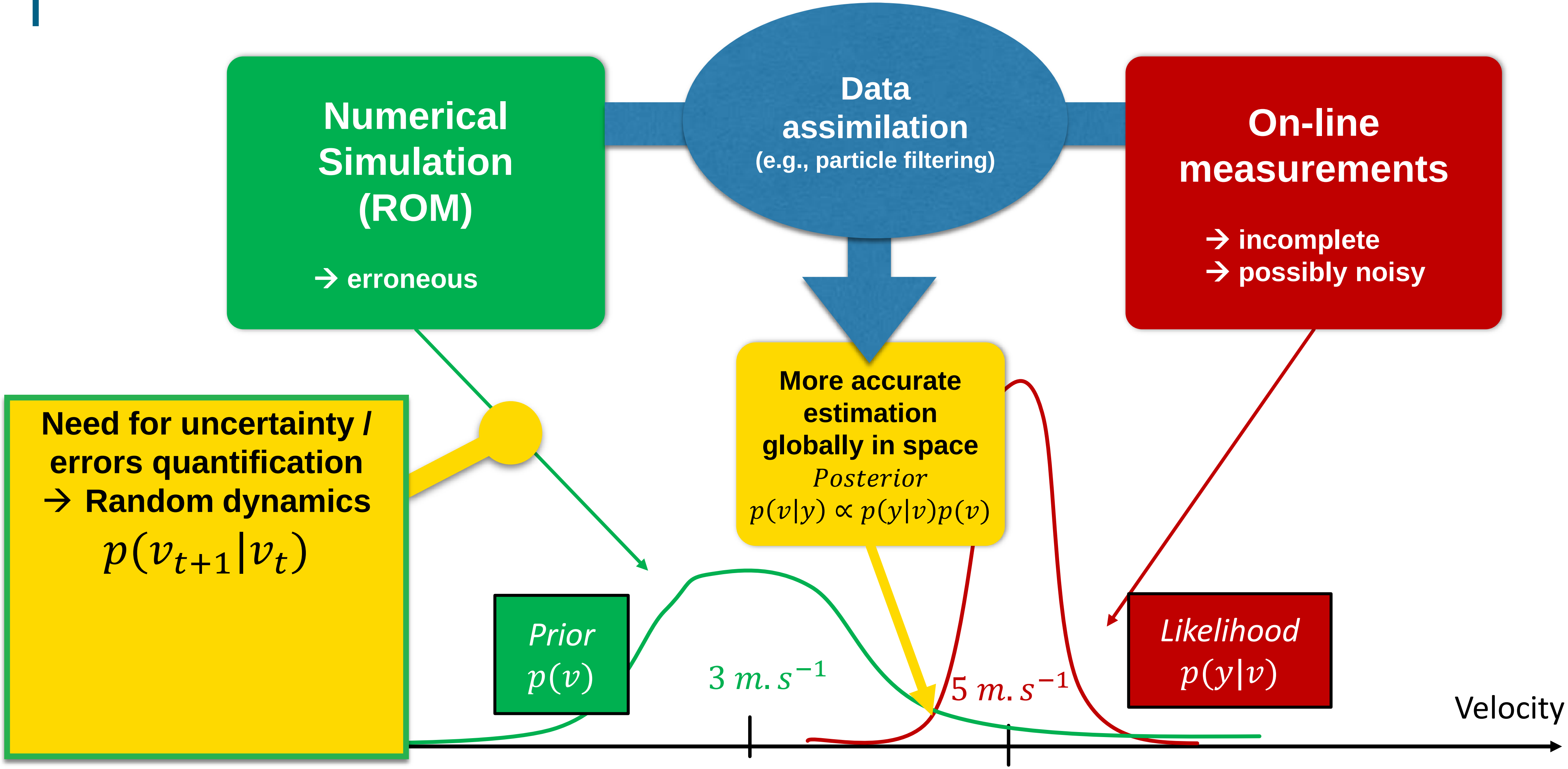
More accurate estimation globally in space

Posterior

$p(v|y) \propto p(y|v)p(v)$

DATA ASSIMILATION

= Coupling simulations and measurements y



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Data assimilation
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On-line measurements

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More accurate estimation globally in space
Posterior
$p(v|y) \propto p(y|v)p(v)$

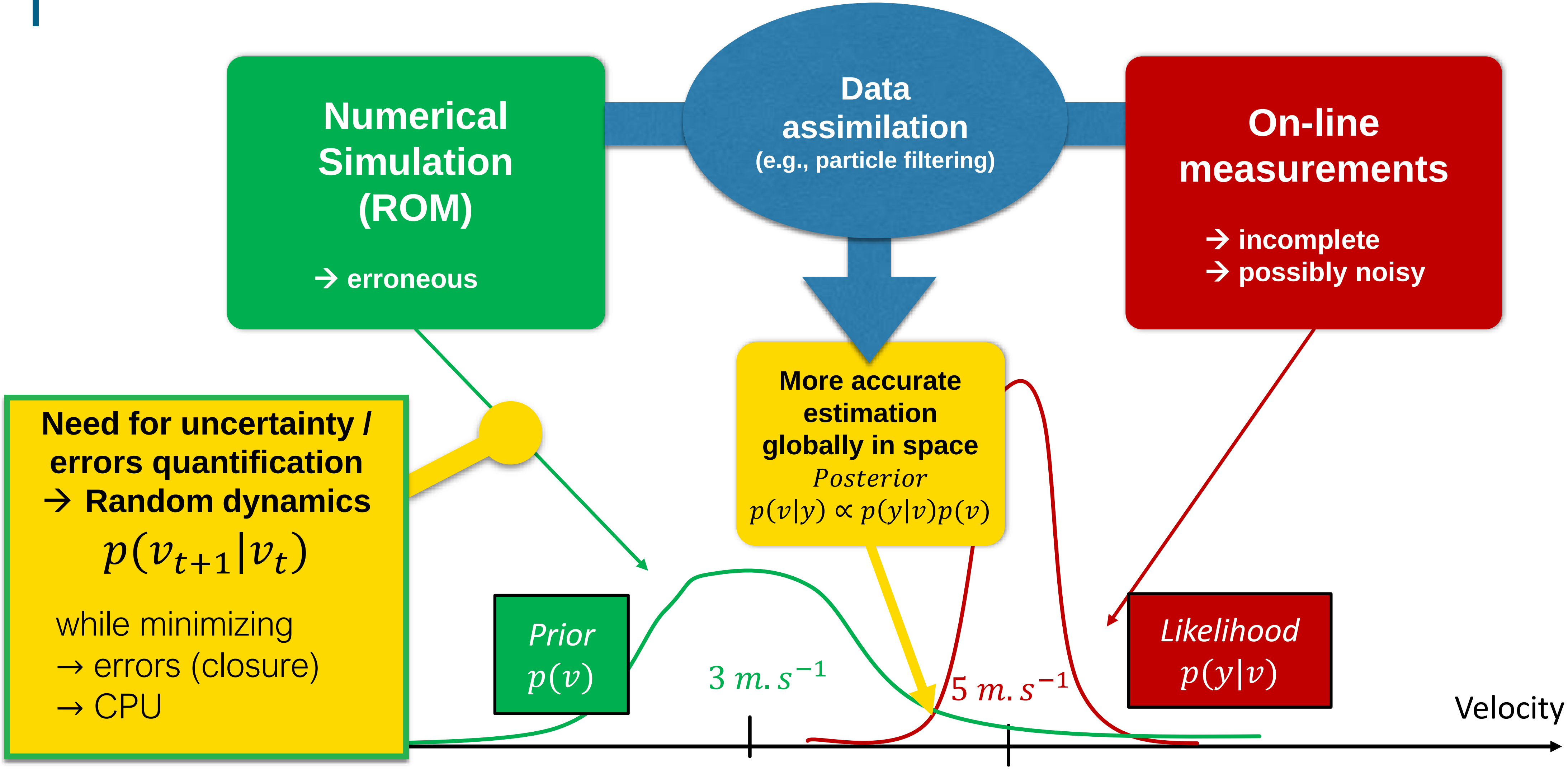
Need for uncertainty / errors quantification
→ Random dynamics
$p(v_{t+1}|v_t)$

Prior
$p(v)$

Likelihood
$p(y|v)$

DATA ASSIMILATION

= Coupling simulations and measurements y



Numerical Simulation (ROM)

→ **erroneous**

Data assimilation

(e.g., particle filtering)

On-line measurements

→ **incomplete**

→ **possibly noisy**

Need for uncertainty / errors quantification

→ **Random dynamics**

$$p(v_{t+1}|v_t)$$

while minimizing

→ errors (closure)

→ CPU

More accurate estimation globally in space

$Posterior$

$p(v|y) \propto p(y|v)p(v)$

PART III

REDUCED LOCATION UNCERTAINTY MODELS

a. Location uncertainty models (LUM)

b. Reduced LUM (Red LUM)

LOCATION UNCERTAINTY MODELS (LUM)

$$v = w + v'$$

Resolved fluid velocity:
$w = \sum_{i=0}^{n} b_i \phi_i$

Unresolved fluid velocity:
$v' = \sigma \dot{B}$

Assumed
(conditionally-)Gaussian
& white in time
(non-stationary in space)

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Randomized
Navier-Stokes model

- Good closure
- Good model error quantification **for data assimilation**

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Mikulevicius & Rozovskii, 2004
Flandoli, 2011

LUM

Memin, 2014
Resseguier et al. 2017 a, b, c, d
Cai et al. 2017
Chapron et al. 2018
Yang & Memin 2019
…

SALT

Holm, 2015
Holm and Tyranowski, 2016
Arnaudon et al. 2017

Crisan et al., 2017
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Cotter and al. 2017 Resseguier et al. 2020 a, b

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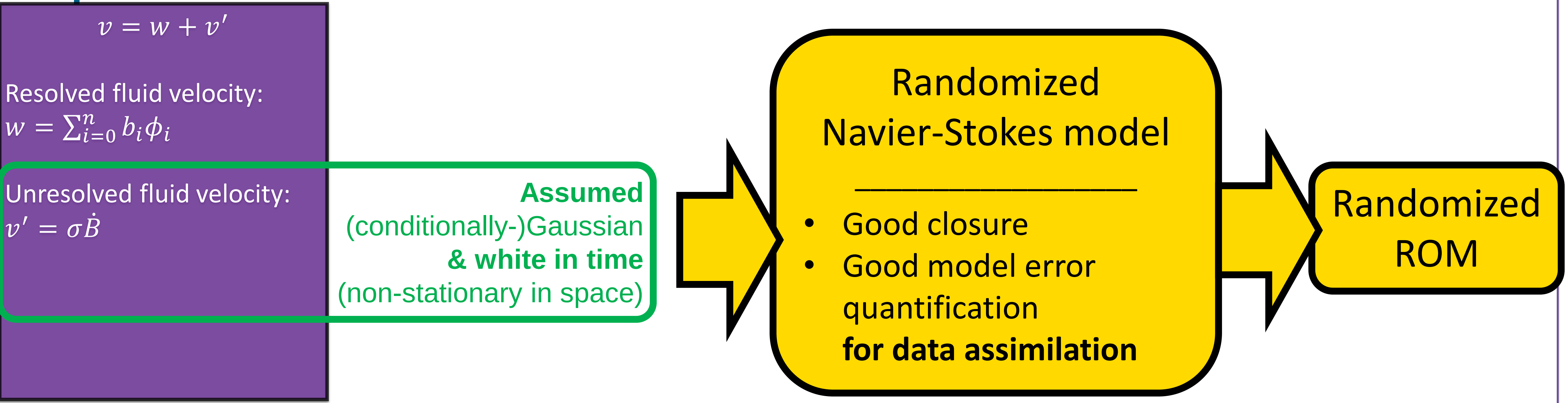

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ROM

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LOCATION UNCERTAINTY MODELS (LUM),

Randomized incompressible Navier-Stokes

$$v = w + v'$$

Resolved fluid velocity:
w

Unresolved fluid velocity:
$v' = \sigma \dot{B}$ (Gaussian, white wrt t)

(assuming $\nabla \cdot w = 0$ and $\nabla \cdot v' = 0$)

Momentum conservation

$$\frac{Dw}{Dt} = F \text{ (Forces)}$$

Resseguier V. & al. (2017). Part I. Geophys. Astro. Fluid. hal-01391420

LOCATION UNCERTAINTY MODELS (LUM),

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$v = w + v'$

Resolved fluid velocity:
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(assuming $\nabla \cdot w = 0$ and $\nabla \cdot v' = 0$)

Variance tensor:
$a(x,x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt}$

From Ito-Wentzell formula (Kunita 1990) with Ito notations

$$\partial_t w + w^* \cdot \nabla w + \sigma \dot{B} \cdot \nabla w - \nabla \cdot \left(\frac{1}{2} a \nabla w \right) = F$$

Resseguier V. & al. (2017). Part I. Geophys. Astro. Fluid. hal-01391420

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From Ito-Wentzell formula (Kunita 1990) with Ito notations

$$\partial_t w + \overbrace{w^* \cdot \nabla w + \sigma \dot{B} \cdot \nabla w}^{\text{Advection}} \overbrace{- \nabla \cdot \left(\frac{1}{2} a \nabla w\right)}^{\text{“Turbulent” diffusion}} \overbrace{= F}^{\text{Forces}}$$

Resseguier V. & al. (2017). Part I. Geophys. Astro. Fluid. hal-01391420

LOCATION UNCERTAINTY MODELS (LUM),
Randomized Navier-Stokes

$$v = w + v'$$

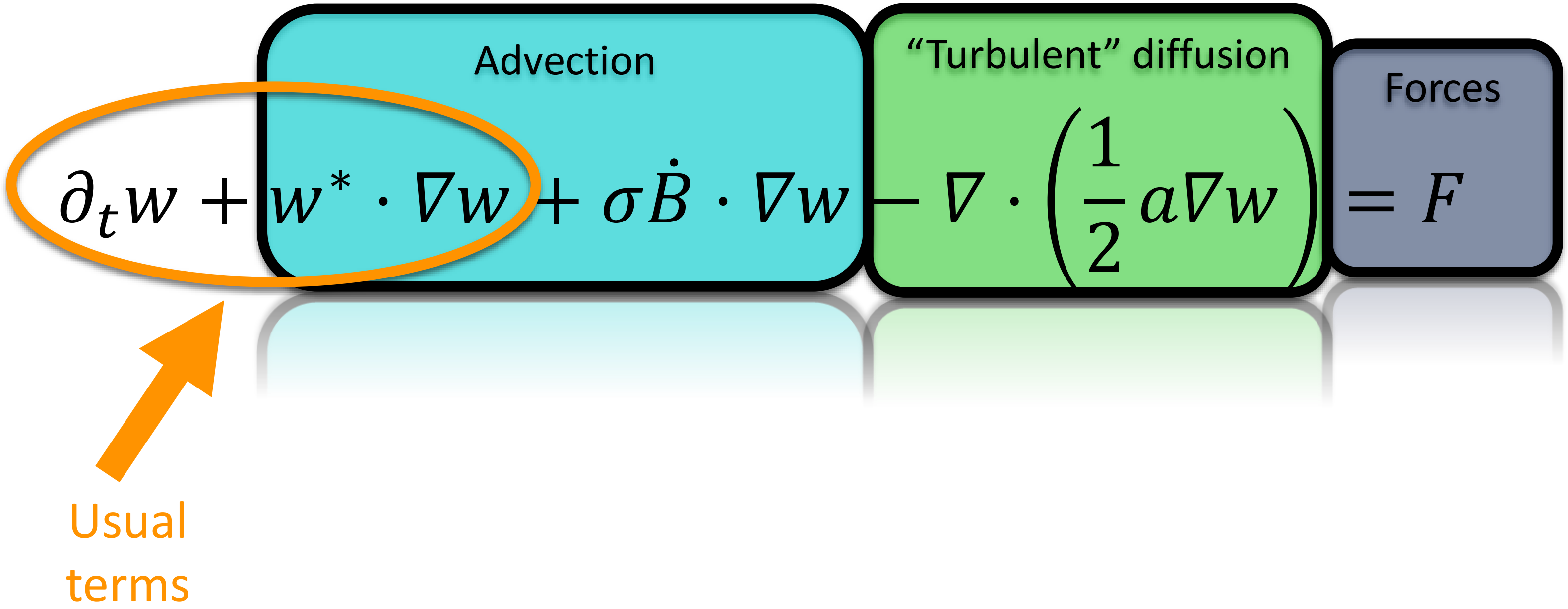
Resolved fluid velocity:
w

Unresolved fluid velocity:
$v' = \sigma\dot{B}$ (Gaussian, white wrt t)

(assuming $\nabla \cdot w = 0$ and $\nabla \cdot v' = 0$)

Variance tensor:
$a(x,x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt}$

From Ito-Wentzell formula (Kunita 1990) with Ito notations



Advection

"Turbulent" diffusion

Forces

$$\partial_t w + w^* \cdot \nabla w + \sigma\dot{B} \cdot \nabla w - \nabla \cdot \left(\frac{1}{2} a \nabla w\right) = F$$

Usual terms

Resseguier V. & al. (2017). Part I. Geophys. Astro. Fluid. hal-01391420

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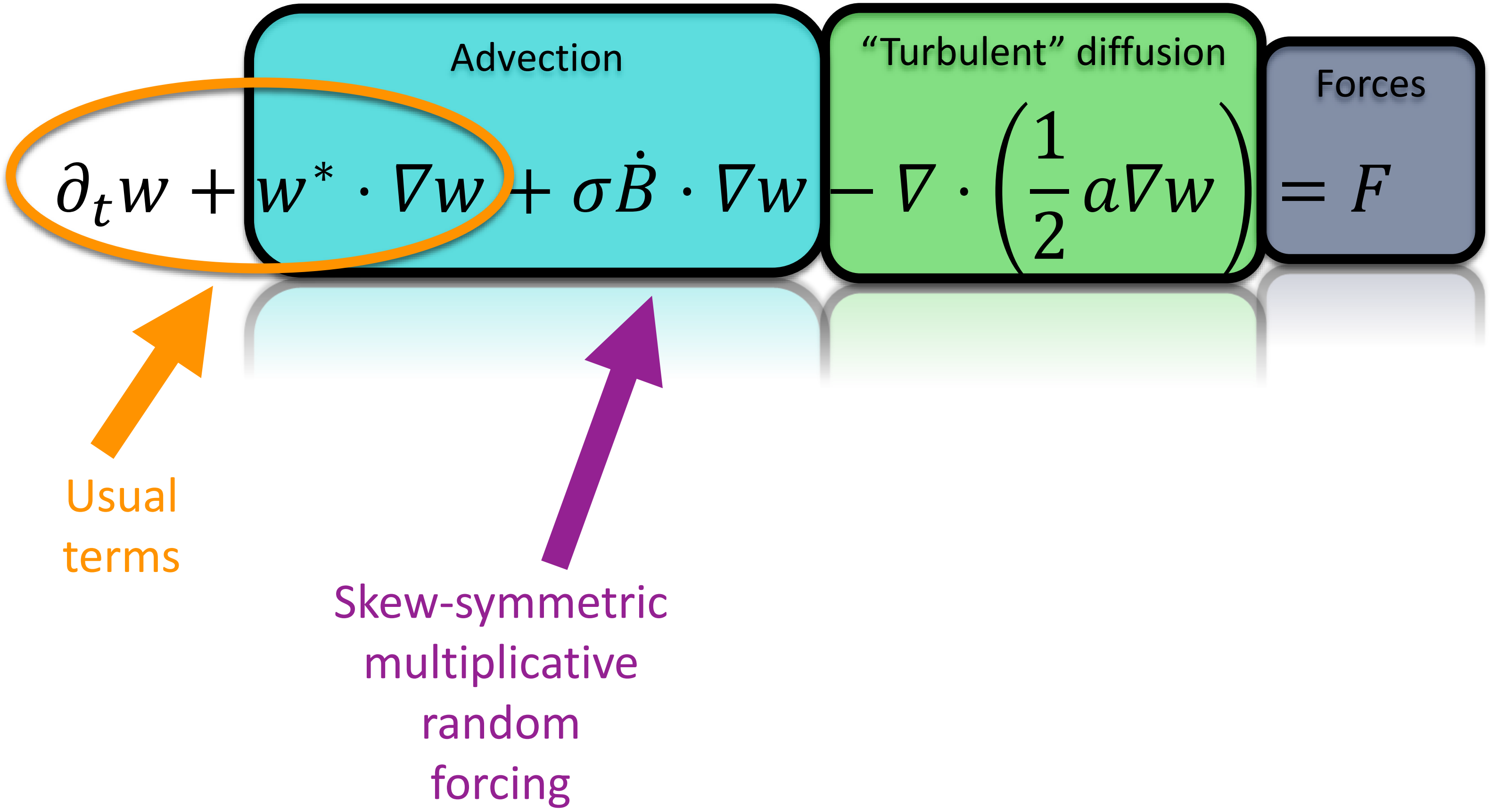

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$$\partial_t w + w^* \cdot \nabla w + \sigma \dot{B} \cdot \nabla w - \nabla \cdot \left(\frac{1}{2} a \nabla w \right) = F$$

Usual terms

Skew-symmetric multiplicative random forcing

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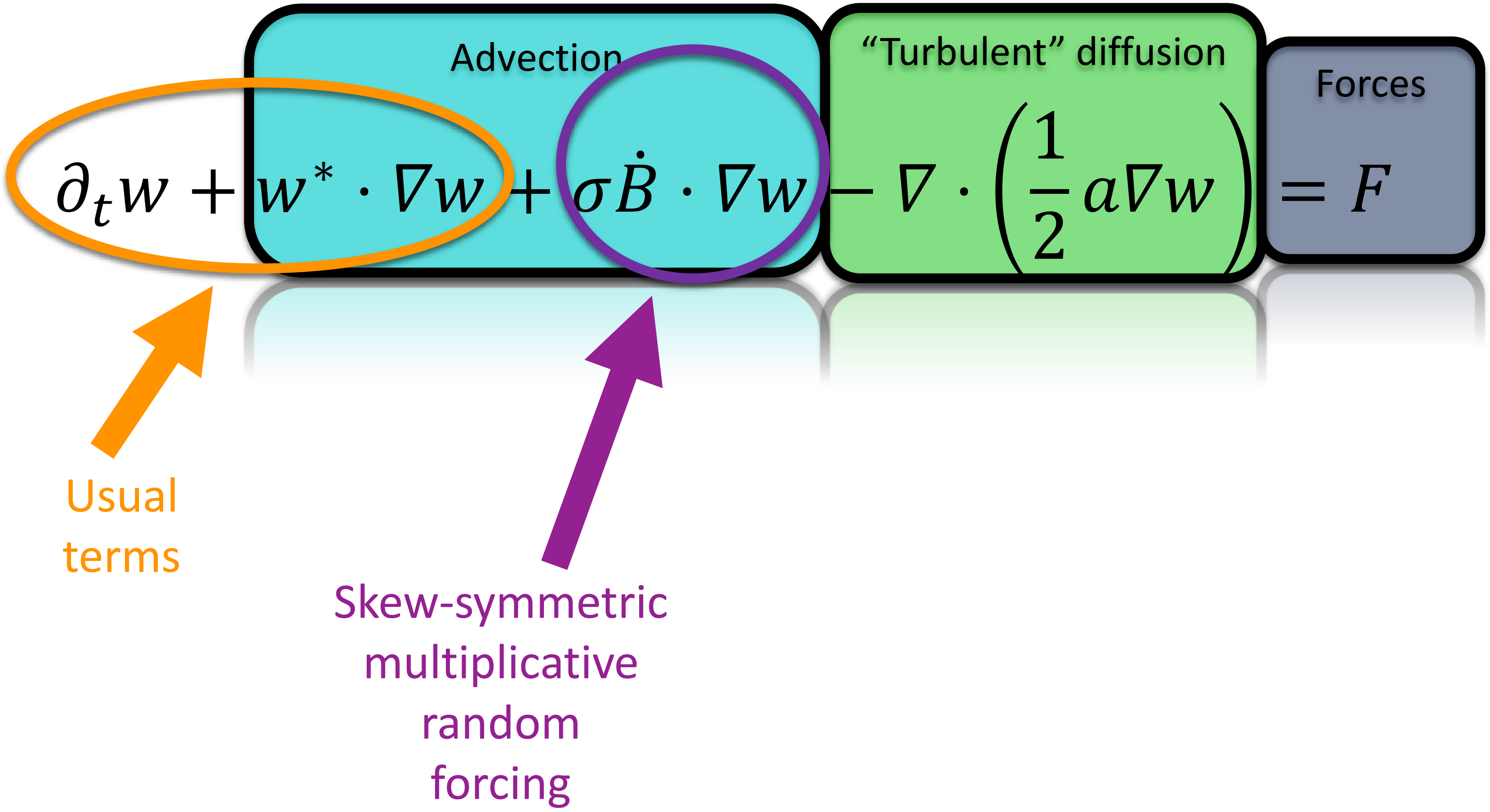

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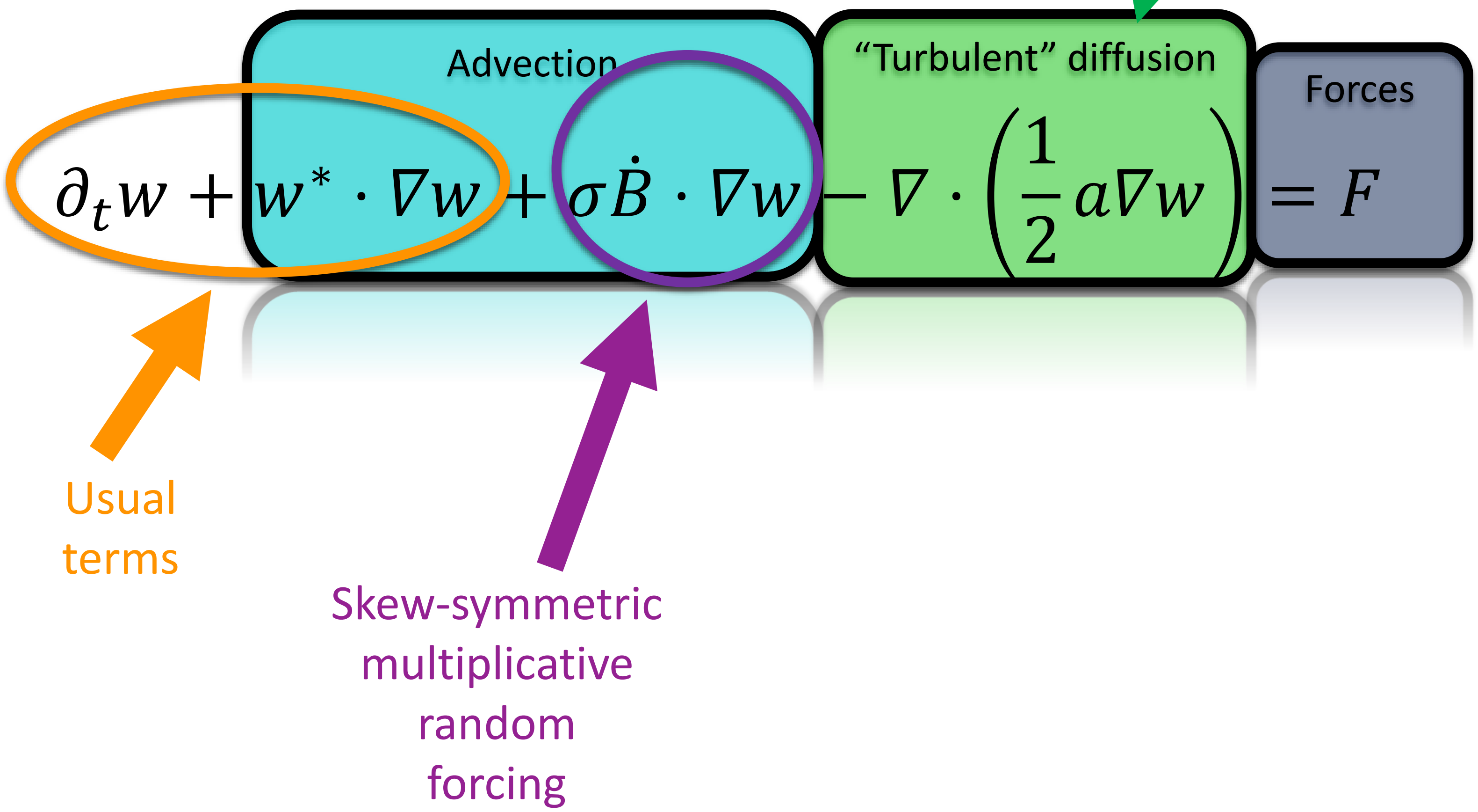
Symmetric negative

From Ito-Wentzell formula (Kunita 1990) with Ito notations



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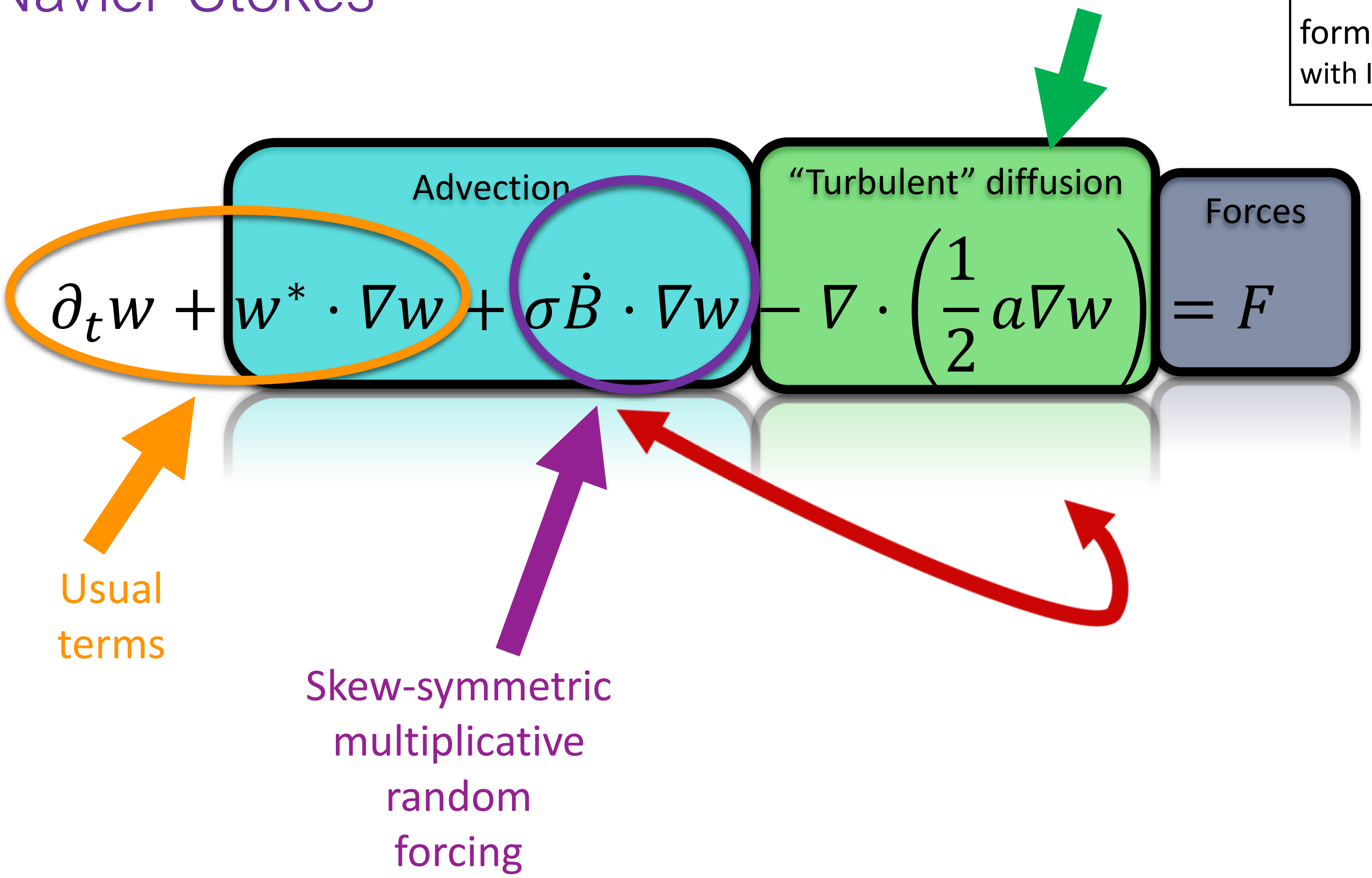

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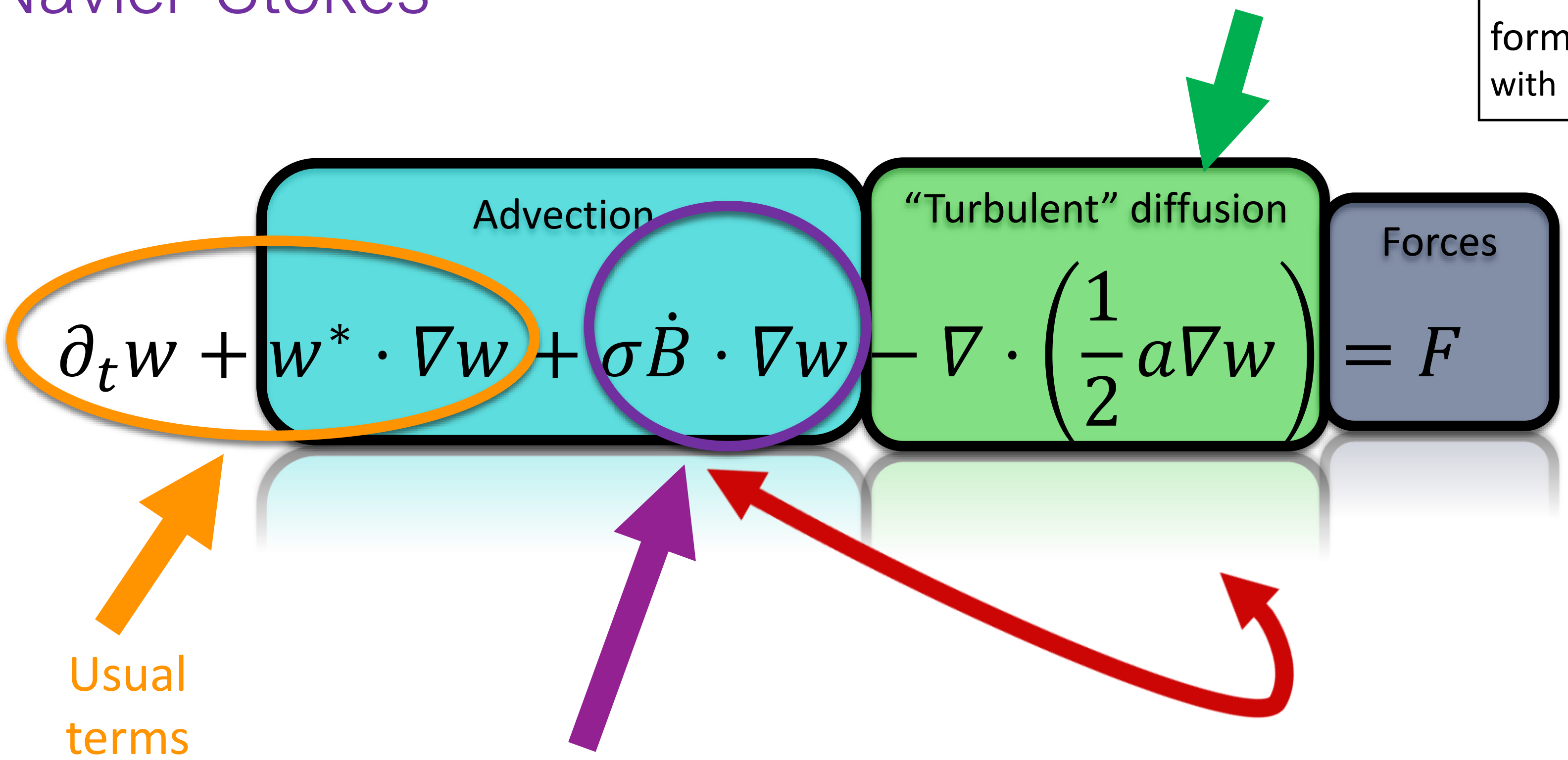

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Usual terms

Skew-symmetric multiplicative random forcing

Balanced energy fluxes

From Ito-Wentzell formula (Kunita 1990) with Ito notations

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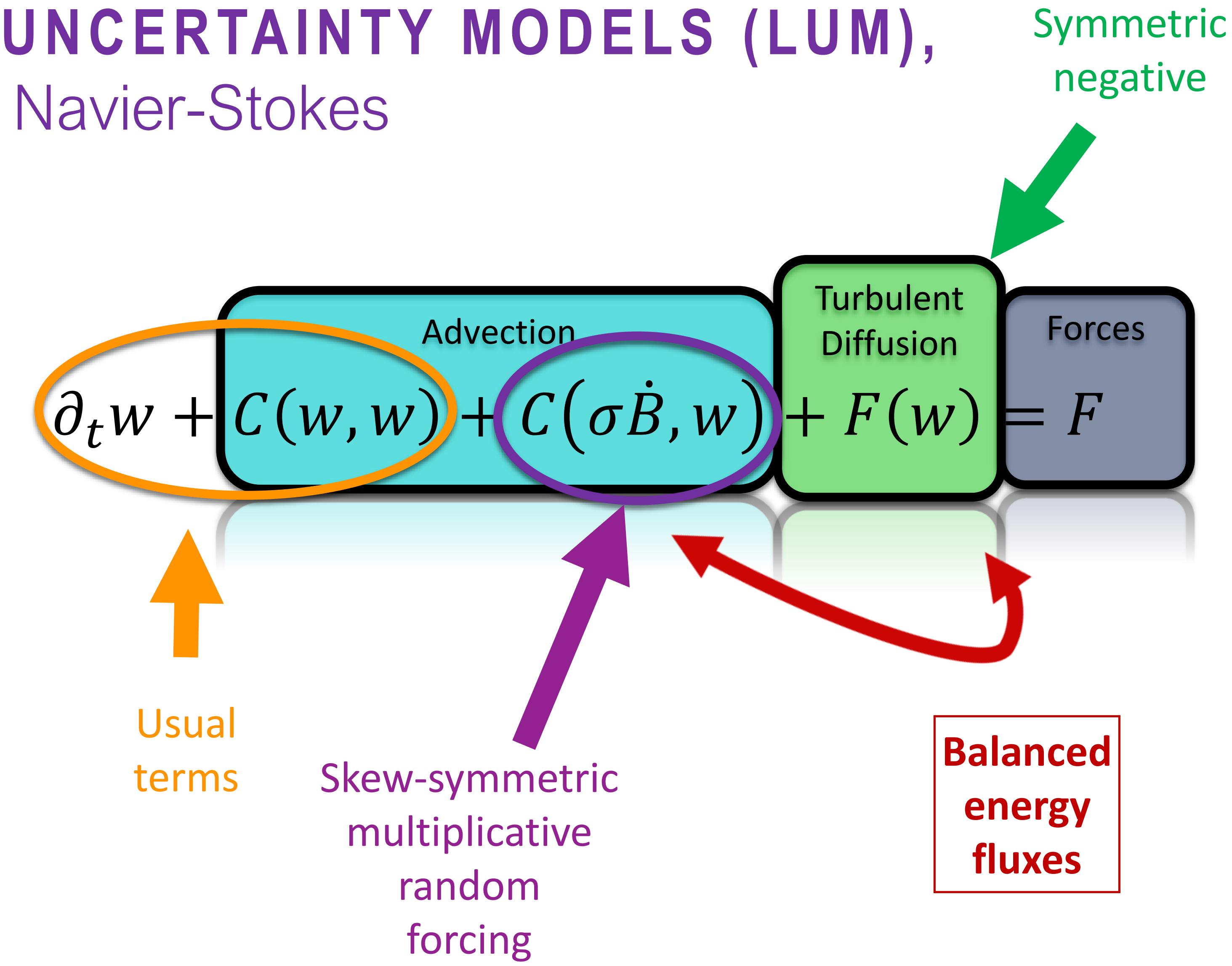

Symmetric negative

Advection

Turbulent Diffusion

Forces

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Resseguier V. & al. (2017). Part I. Geophys. Astro. Fluid. hal-01391420

REDUCED LUM (RED LUM)

POD-Galerkin gives SDEs for resolved modes

Full order : $M \sim 10^7$
Reduced order : $n \sim 10$

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Advection : 2^{nd} order polynomial

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Multiplicative
skew-symmetric noise

Advection : 2nd order polynomial

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Advection : 2nd order polynomial

Multiplicative
skew-symmetric noise

$K_{jq}[\xi] = -\int_\Omega \phi_j \cdot C(\xi, \phi_q)$

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Multiplicative skew-symmetric noise

→ Covariance to estimate

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New estimator

- Consistency proven $(\Delta t \to 0)$
- Numerically efficient
- Data-based & Physics-based

→ Robustness in extrapolation

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Advection : 2nd order polynomial

Multiplicative
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"Turbulent" diffusion

with $a(x) \approx \Delta t\ \overline{v' \, (v')^T}$

New estimator

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≈ globally balanced energy fluxes

New estimator

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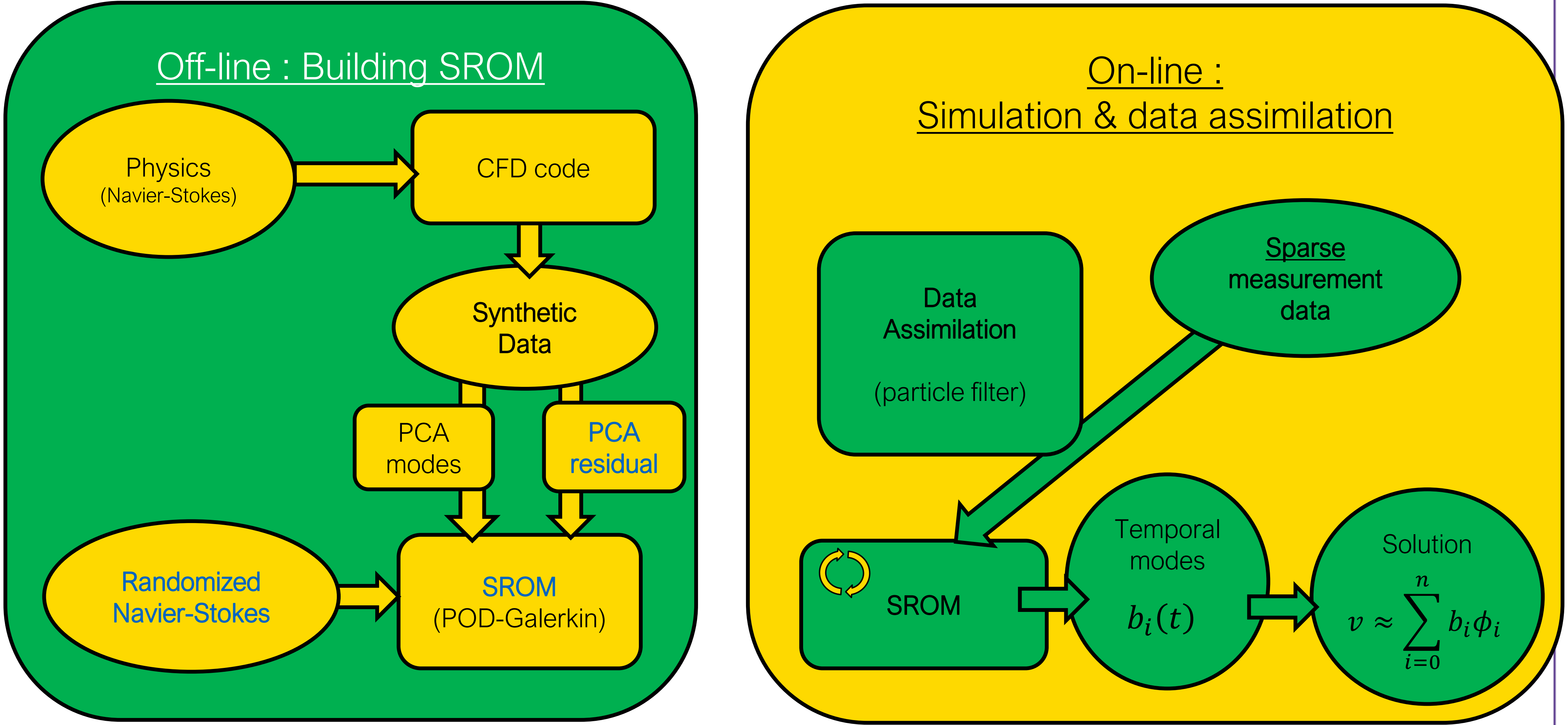
SUMMARY

Stochastic ROM + Data assimilation



Off-line : Building SROM

Physics
(Navier-Stokes)

CFD code

Synthetic
Data

PCA
modes

PCA
residual

Randomized
Navier-Stokes

SROM
(POD-Galerkin)

On-line :
Simulation & data assimilation

Data
Assimilation

(particle filter)

Sparse
measurement
data

SROM

Temporal
modes

$b_i(t)$

Solution

$v \approx \sum_{i=0}^{n} b_i \phi_i$

Resseguier et al. (2022). *J Comp.Phys* . hal-03445455

SUMMARY

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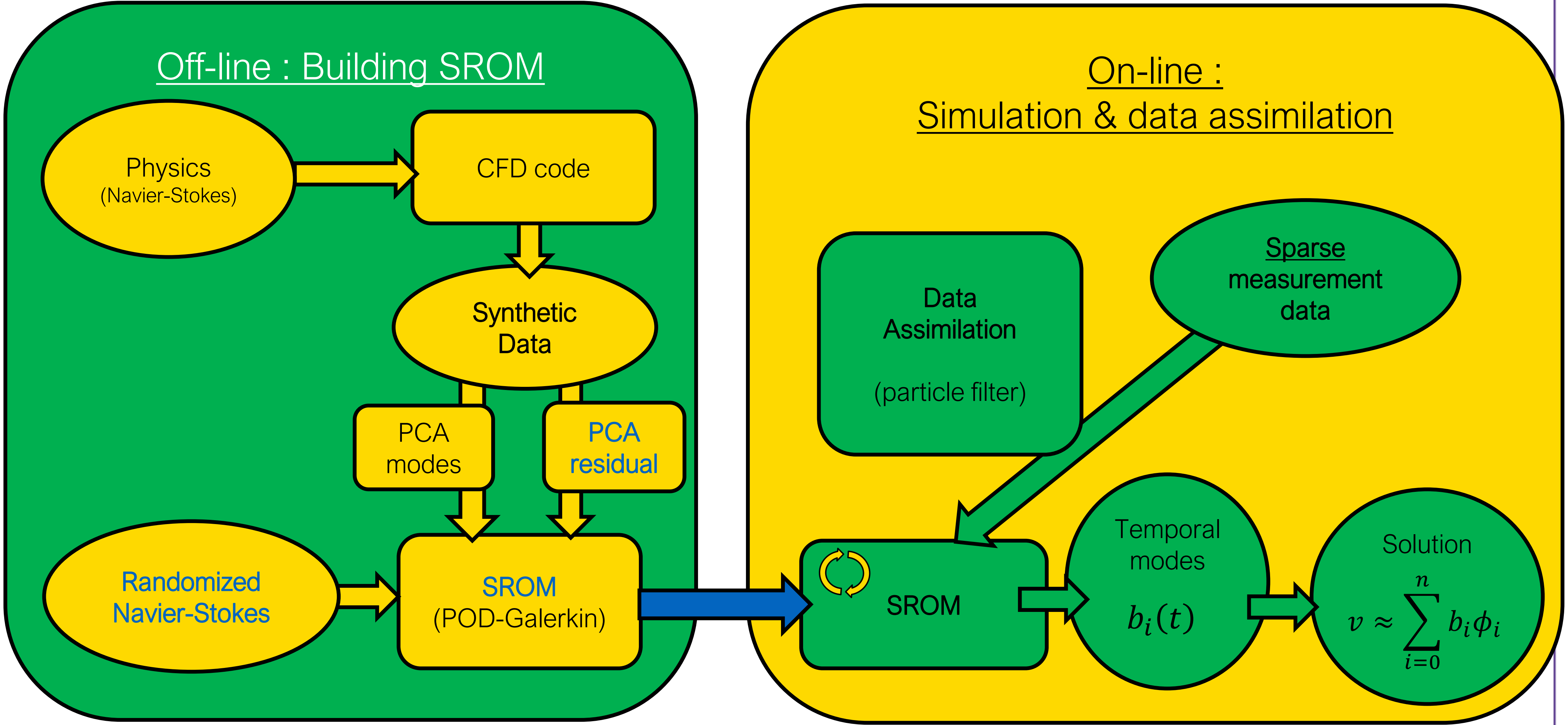

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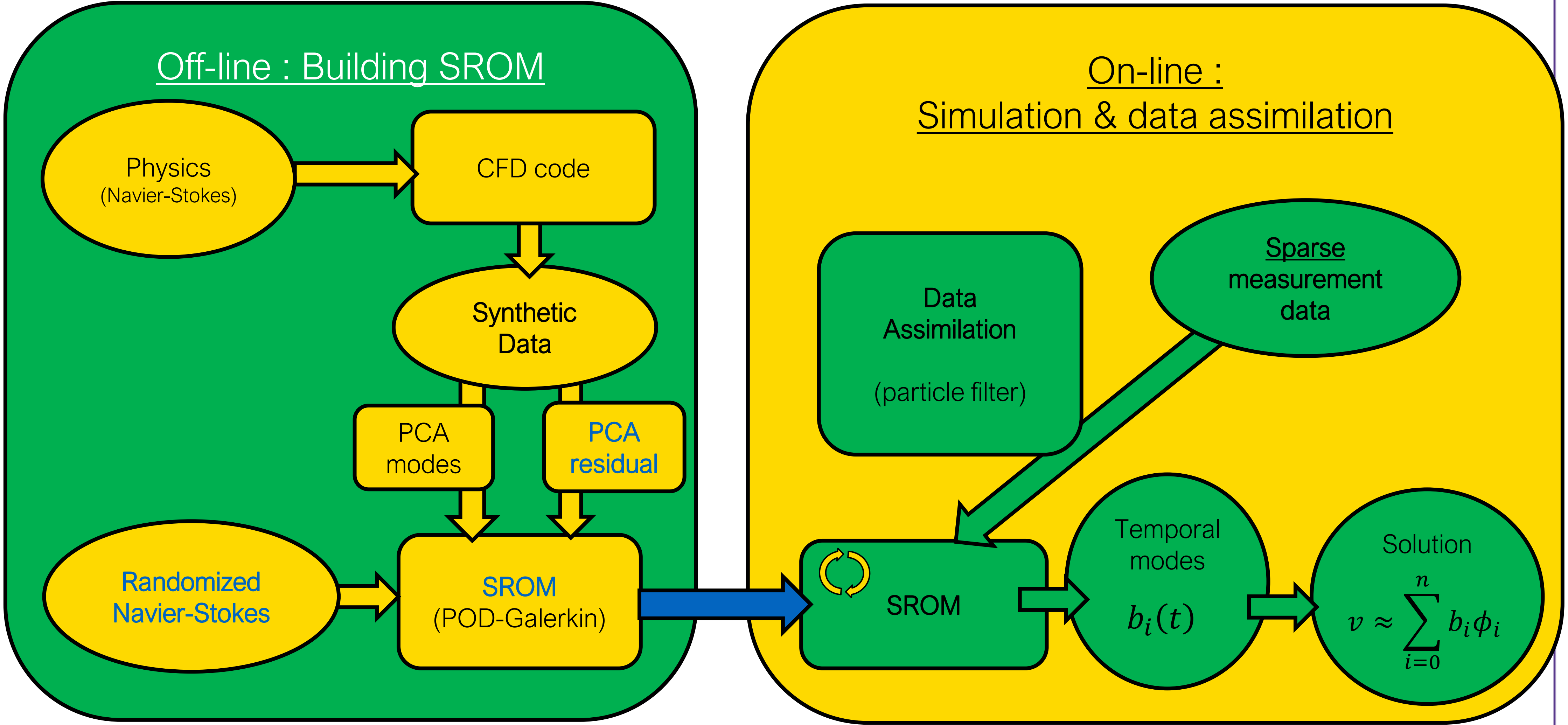

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PART IV

NUMERICAL RESULTS

a. Uncertainty quantification (Prior)

b. Data assimilation (Posterior)

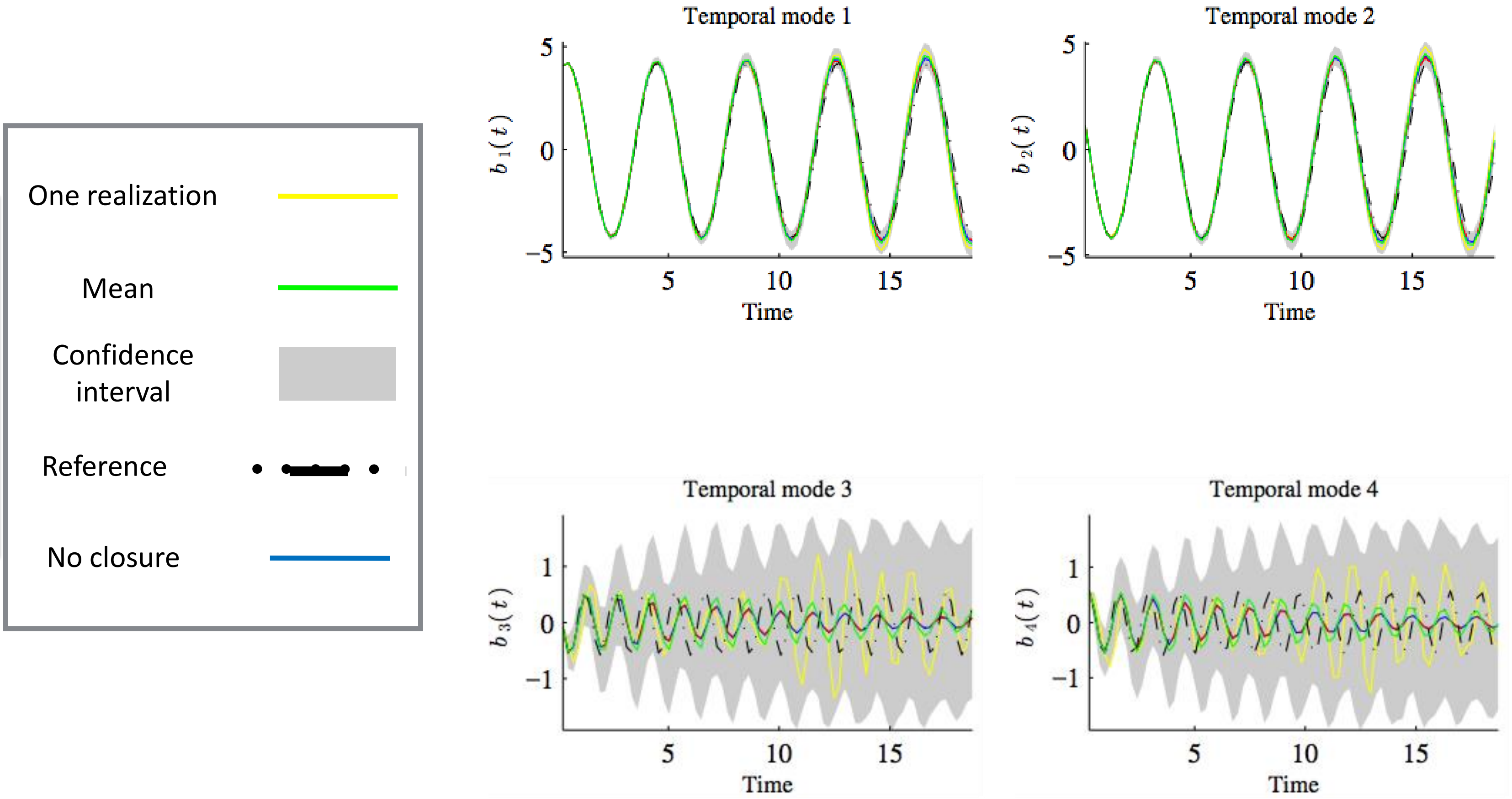
UNCERTAINTY QUANTIFICATION (PRIOR)

$n = 4$ resolved degrees of freedom
No data assimilation
Known initial conditions $b(t = 0)$

$$v = w + v'$$

Resolved fluid velocity:
$w = \sum_{i=0}^{n} b_i \phi_i$

Unresolved fluid velocity:
v'



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UNCERTAINTY QUANTIFICATION (PRIOR)

$n = 4$ resolved degrees of freedom
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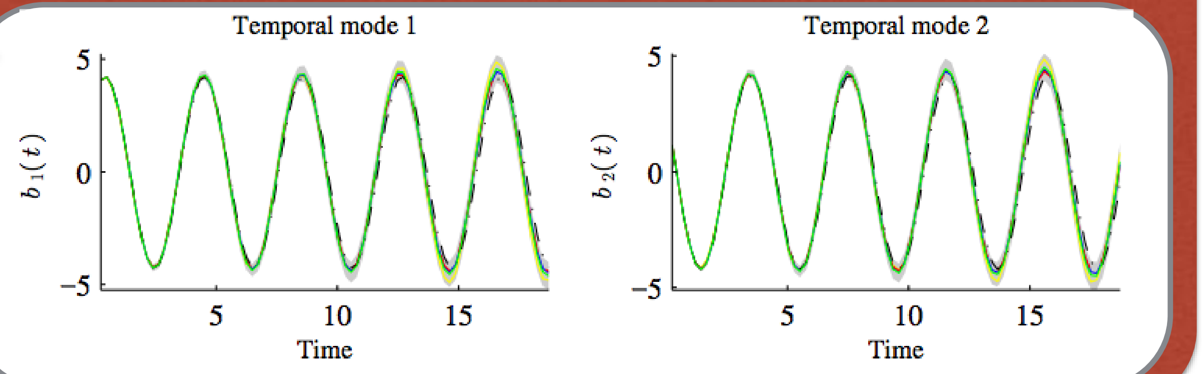
Known initial conditions $b(t = 0)$

Mainly from the mean $\bar{v} = \phi_0$

Energy

Linearly unstable modes



$v = w + v'$

Resolved fluid velocity:
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Unresolved fluid velocity:
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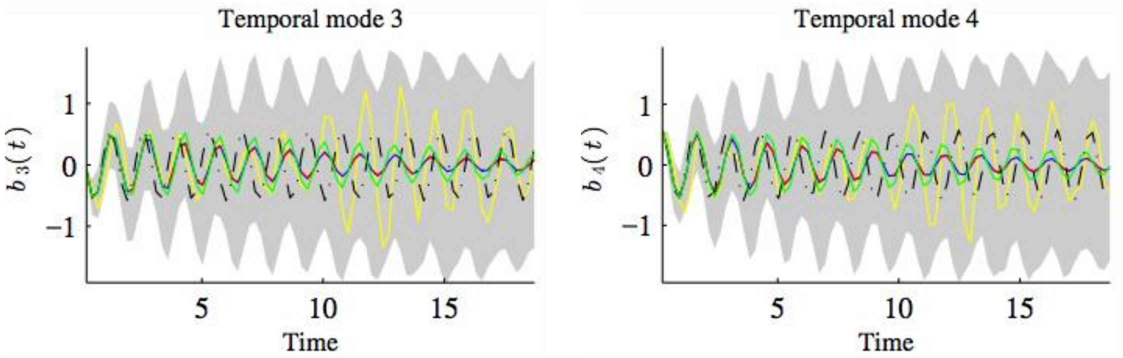
One realization	—
Mean	—
Confidence interval	▨
Reference	- - -
No closure	—



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$c(b(t), b(t))$

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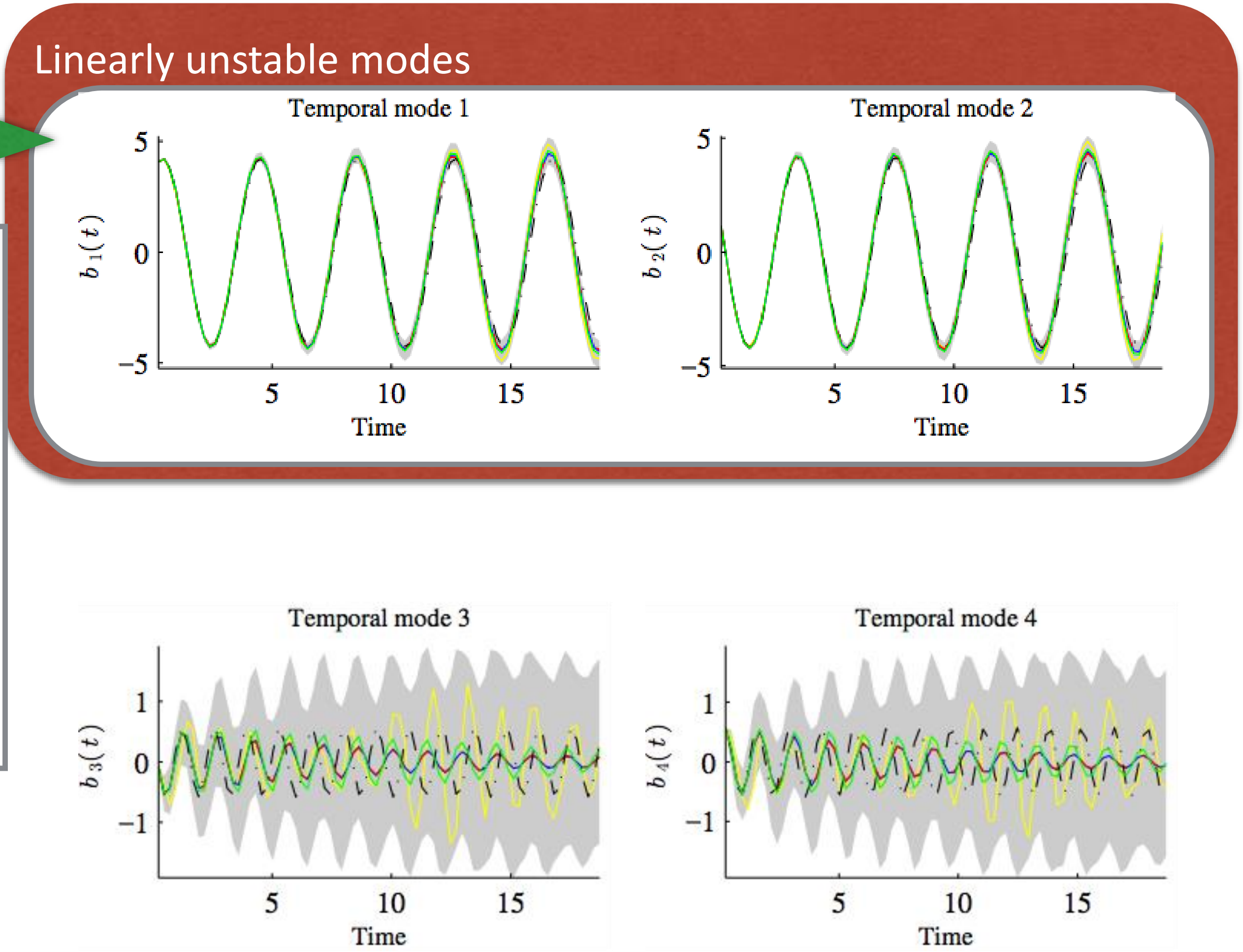
One realization	🟨
Mean	🟩
Confidence interval	▨
Reference	• ▬ •
No closure	🟦



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One realization	━
Mean	━
Confidence interval	▨
Reference	-·-
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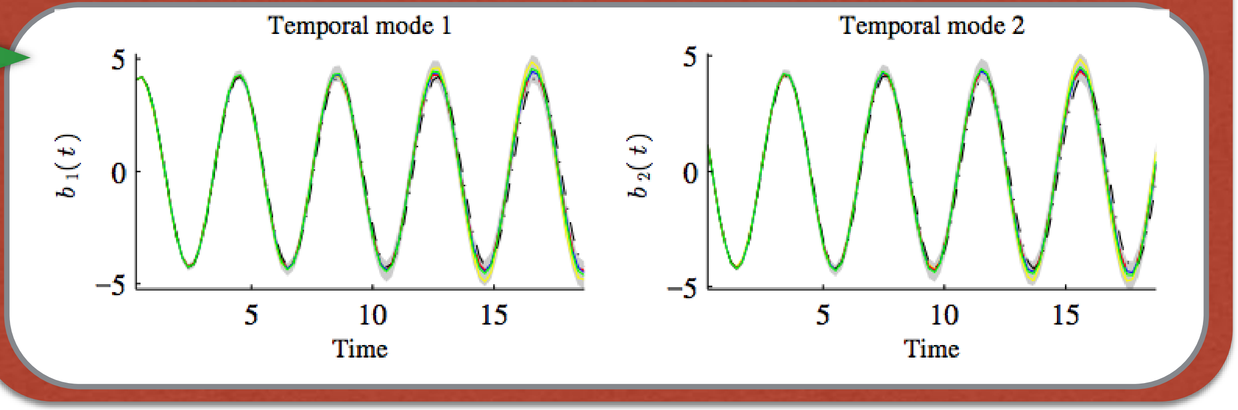
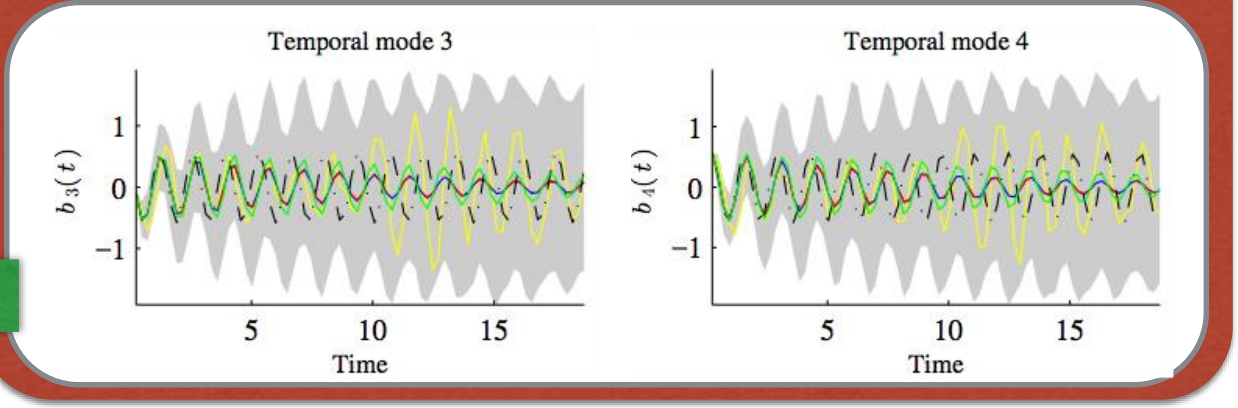


Linearly stable modes



Mainly to the mean $\bar{v} = \phi_0$

Energy

Resseguier et al. (2021). *SIAM-ASA J Uncertain* . hal- 03169957

UNCERTAINTY QUANTIFICATION (PRIOR)

$n = 4$ resolved degrees of freedom
No data assimilation
Known initial conditions $b(t = 0)$

Mainly from the mean $\bar{v} = \phi_0$

Energy

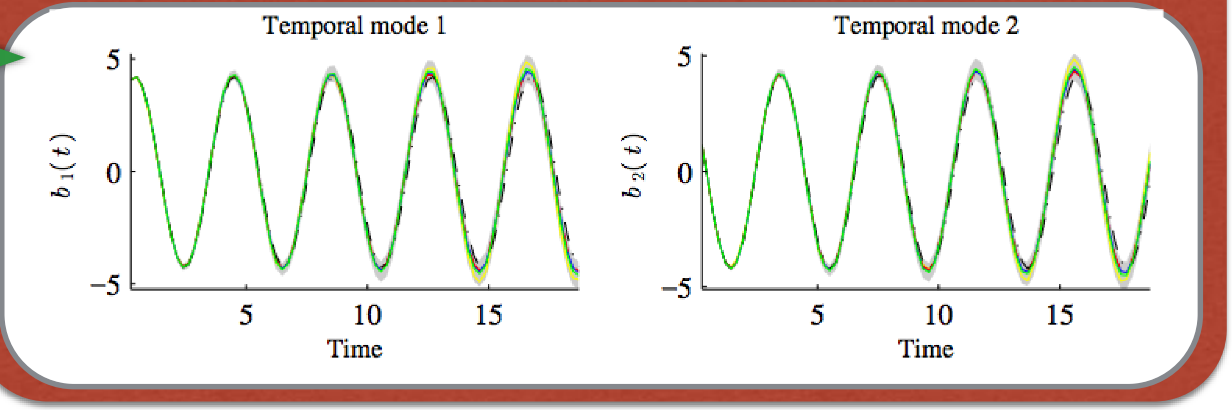
$c(b(t), b(t))$

$v = w + v'$

Resolved fluid velocity: $w = \sum_{i=0}^{n} b_i \phi_i$

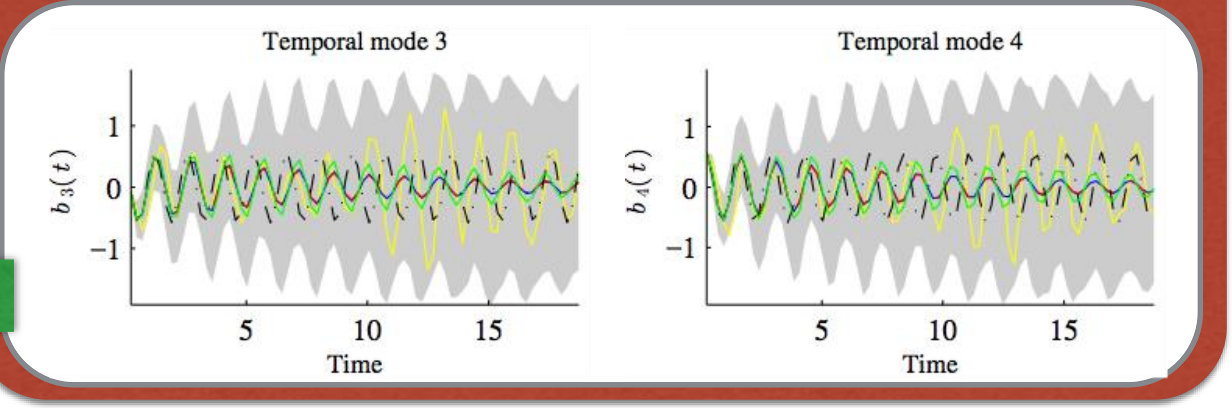
Unresolved fluid velocity: v'

Linearly unstable modes



One realization	🟨
Mean	🟩
Confidence interval	▨
Reference	• ▬ •
No closure	🟦

Linearly stable modes



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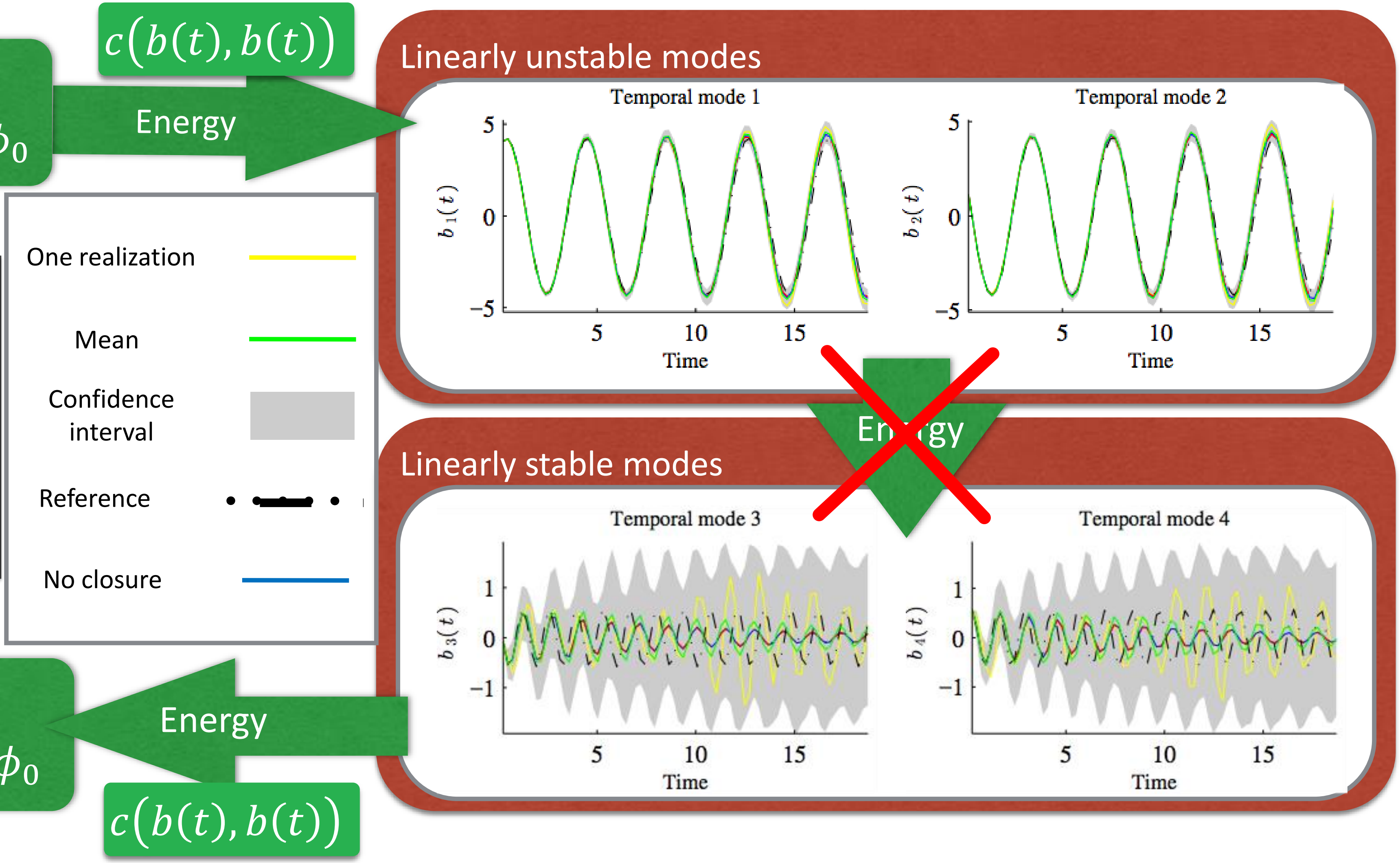

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Linearly unstable modes

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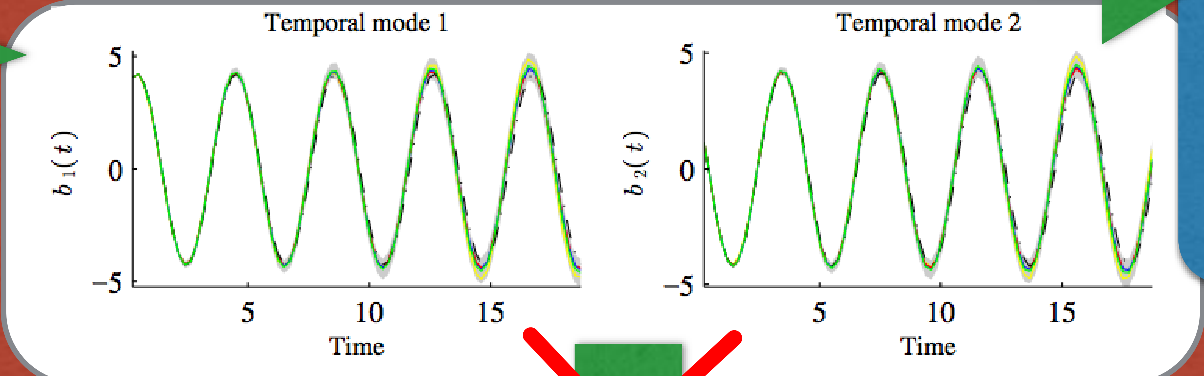
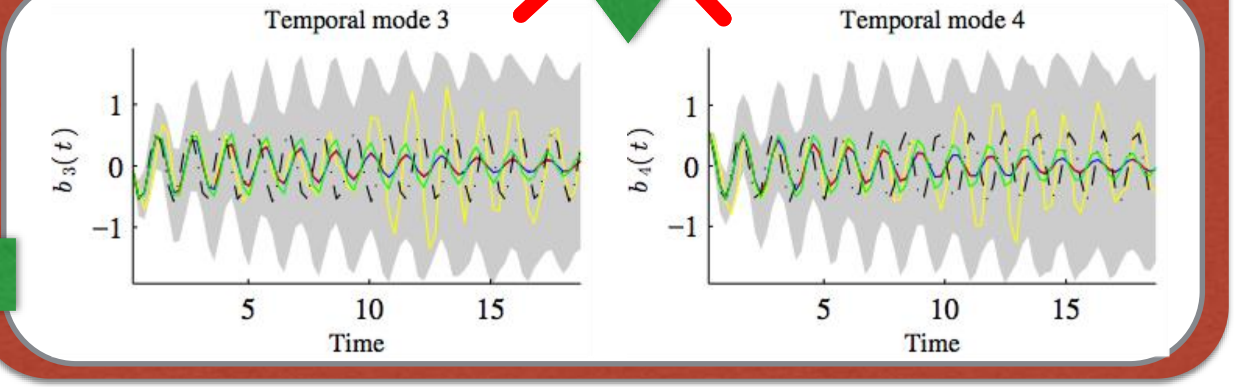


Stabilized by turbulent diffusion

$f\ b(t)$

Energy

Linearly stable modes



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Mean 🟩
Confidence interval ▨
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Mainly to the mean $\bar{v} = \phi_0$

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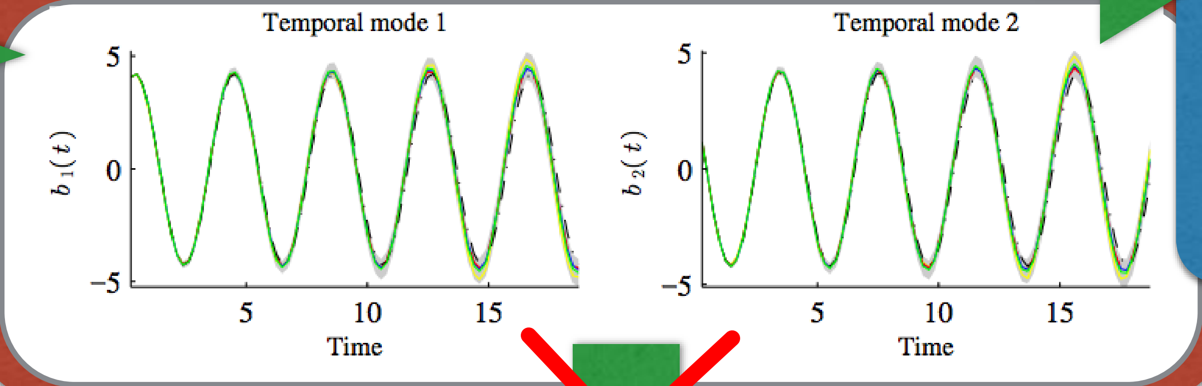

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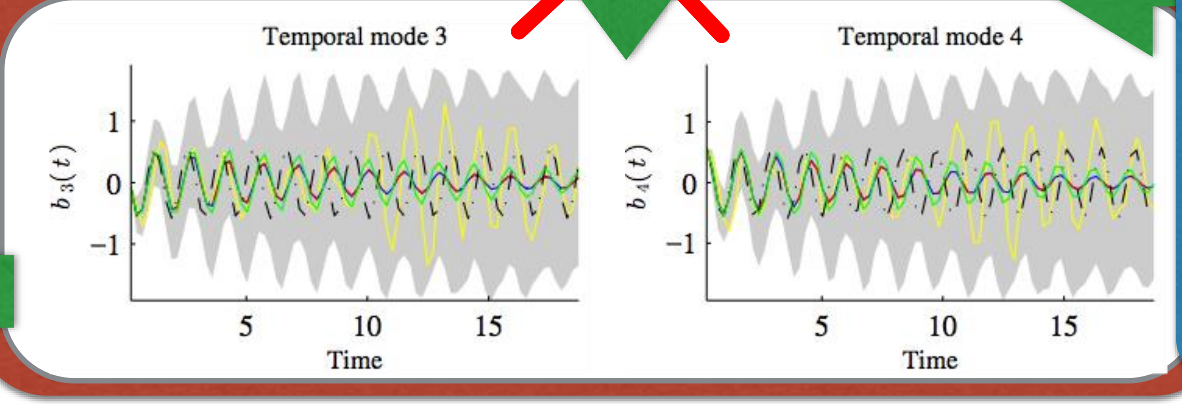

Variability maintained by random energy transfert

$K(\sigma \dot{B})\ b(t)$

Mainly to the mean $\bar{v} = \phi_0$

Energy

$c(b(t), b(t))$

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$n = 8$ resolved degrees of freedom
No data assimilation
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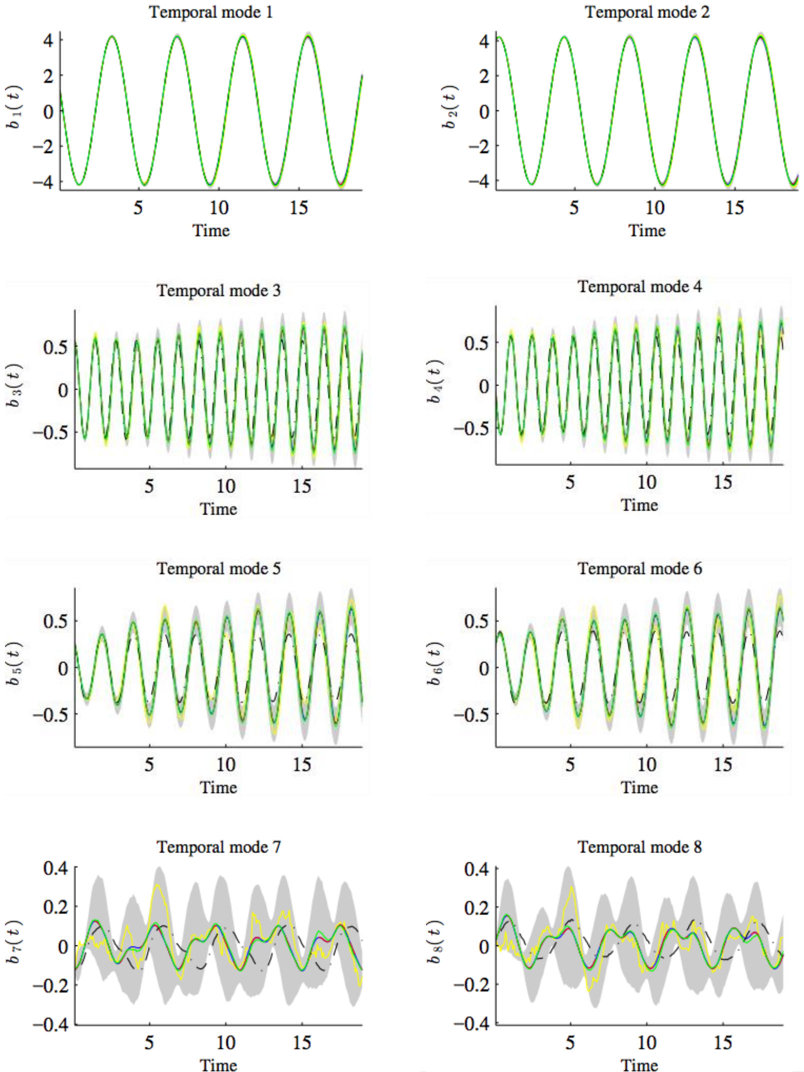

One realization	—
Mean	—
Confidence interval	▨
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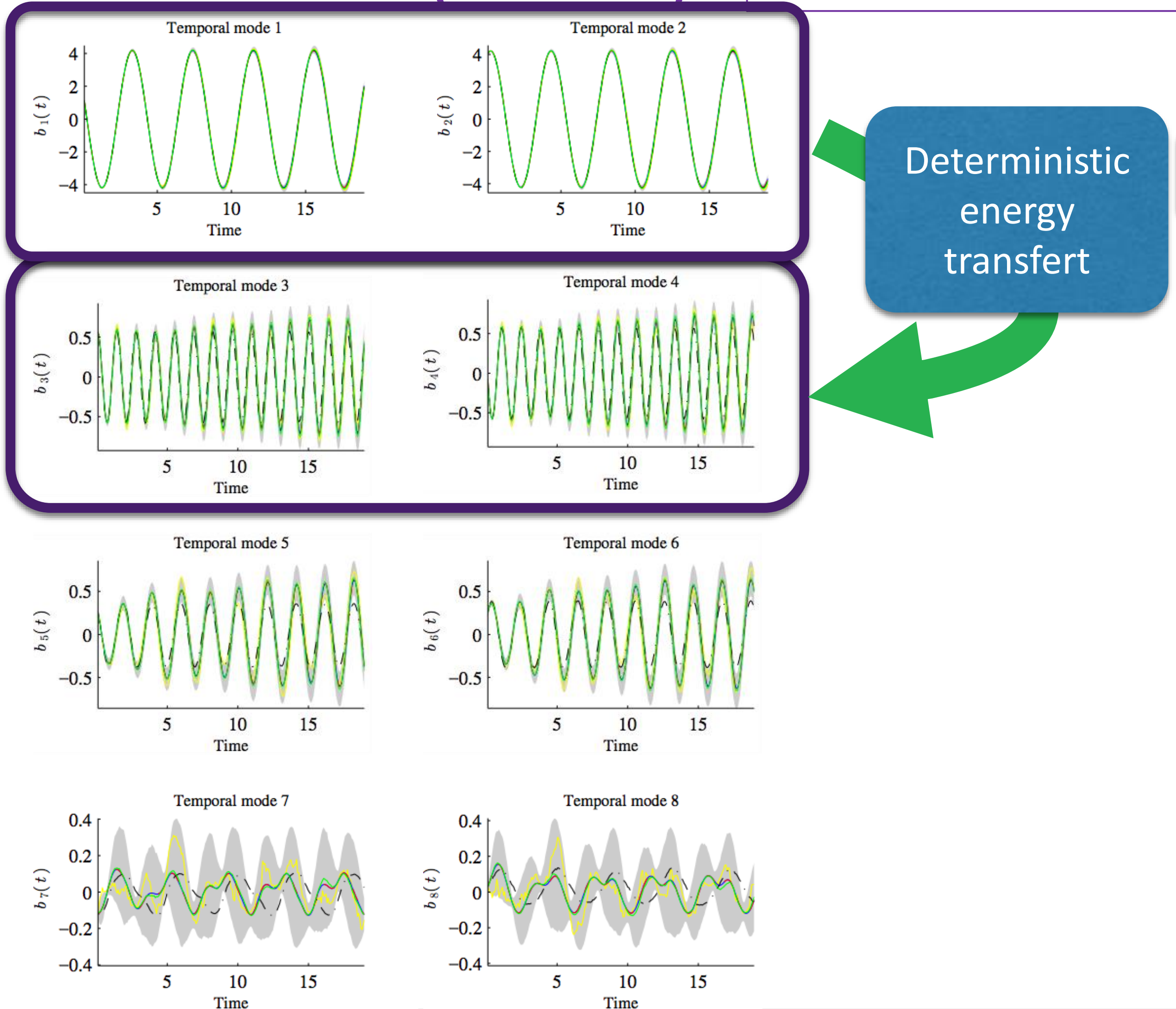

Deterministic energy transfert

$c\big(b(t), b(t)\big)$

One realization ━
Mean ━
Confidence interval ▨
Reference ●▬●
No closure ━

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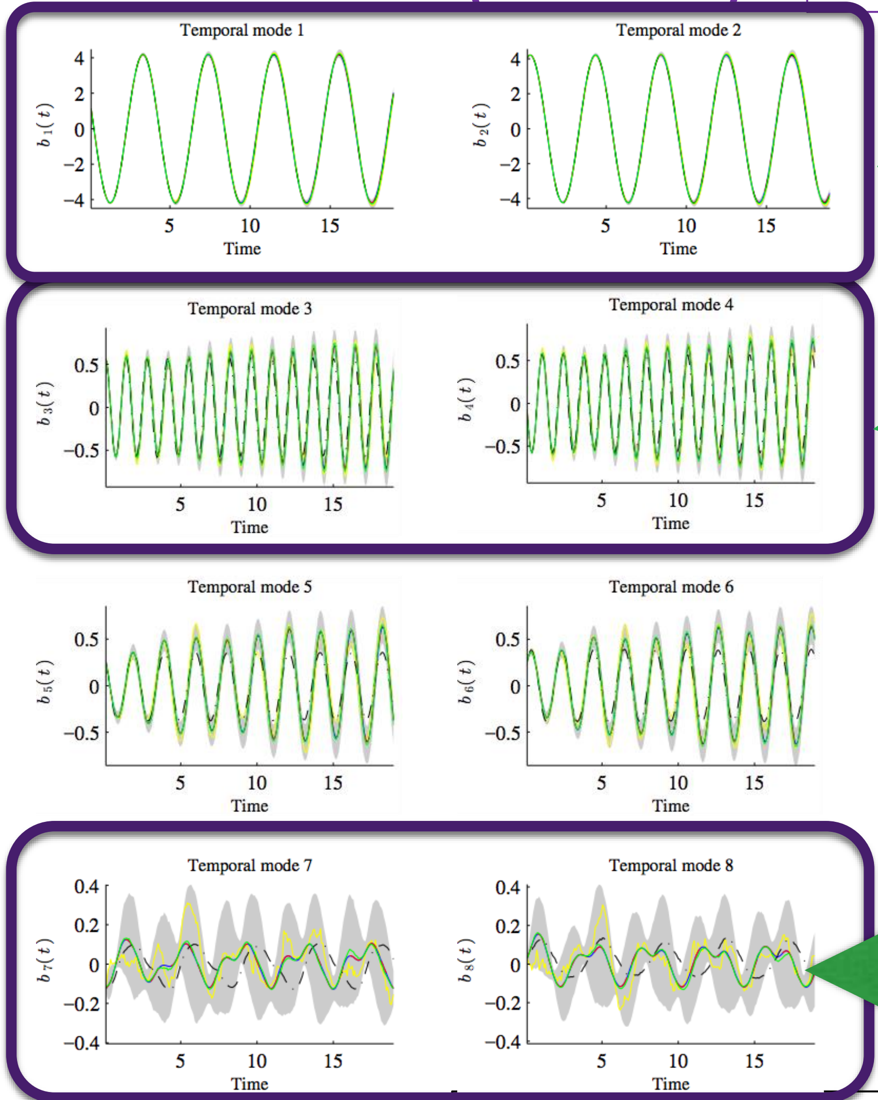


Deterministic energy transfert

$c(b(t), b(t))$

Random energy transfert

$K(\sigma \dot{B})\, b(t)$

Resseguier et al. (2021). *SIAM-ASA J Uncertain* . hal- 03169957

DATA ASSIMILATION (POSTERIOR)

On-line estimation of the solution

From 10^7 to 8 degrees of freedom

● Single measurement point (blurred & noisy velocity)

Reference

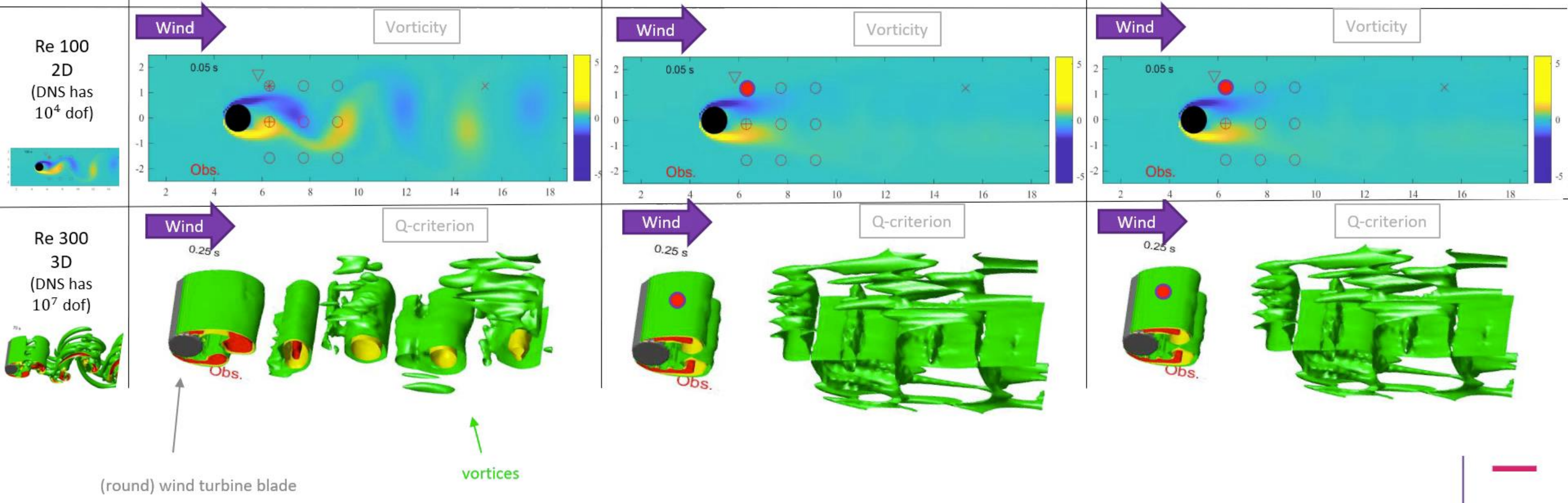
PCA-projection of the DNS
(Optimal from 8-dof linear decomposition)

Our method

POD-Galerkin with Navier-Stokes
under location uncertainty (LUM)

State-of-the-art

POD-Galerkin with Navier-Stokes + optimally
tuned eddy viscosity & additive noise



Resseguier et al. (2022). *J Comp.Phys* . hal-03445455

CONCLUSION

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- Unsteady CFD ROM with severe truncation ($\mathrm{O}(10^7) \rightarrow O(10)$ degrees of freedom)
  - Intrusive ROM (Combine data & physics)
  - **Conservative stochastic closure** (LUM)
    - Stabilization of the unstable modes
    - Maintain variability of stable modes
  - Efficient estimator for the conservative multiplicative noise
  - **Efficient generation of prior / Model error quantification**
- **Data assimilation** (Bayesian inverse problem) :
  to correct the fast simulation on-line by incomplete/noisy measurements
- First results at $Re = 100$ and 300 :
  - Quasi-optimal **unsteady 3D flow** estimation
  - **Robust far outside the training set**

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NEXT STEP :

- Increasing Reynolds (ROM of LES, DDES)
    - Hyperreduction
    - Error quantification of hyperreduction

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BONUS SLIDES

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REDUCED LUM (RED LUM)

POD-Galerkin gives SDEs for resolved modes

Full order : $M \sim 10^7$
Reduced order : $n \sim 10$

$$v = w + v'$$

Resolved fluid velocity:
$w(x,t) = \sum_{i=0}^{n} b_i(t)\phi_i(x)$

Unresolved fluid velocity:
$v' = \sigma\dot{B}$ (Gaussian, white wrt t)

Variance tensor:
$a(x,x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt}$

$$\int_\Omega \phi_i(x) \cdot (d_t w + C(w,w)dt + F(w)dt + C(\sigma dB_t, w) = dF)\, dx$$

➡

$$\frac{db(t)}{dt} = H\big(b(t)\big) + K\big(\sigma\dot{B}\big)\, b(t)$$

$$K_{jq}[\xi] = -\int_\Omega \phi_j \cdot C\big(\xi, \phi_q\big)$$

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2nd order polynomial

Coefficients given by :
- Randomized Navier-Stokes
- $(\phi_j)_j$
- $a(x) \approx \Delta t\,\overline{v'\,(v')^T}$

$$\overline{f} = \frac{1}{T}\int_0^T f$$

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Multiplicative skew-symmetric noise

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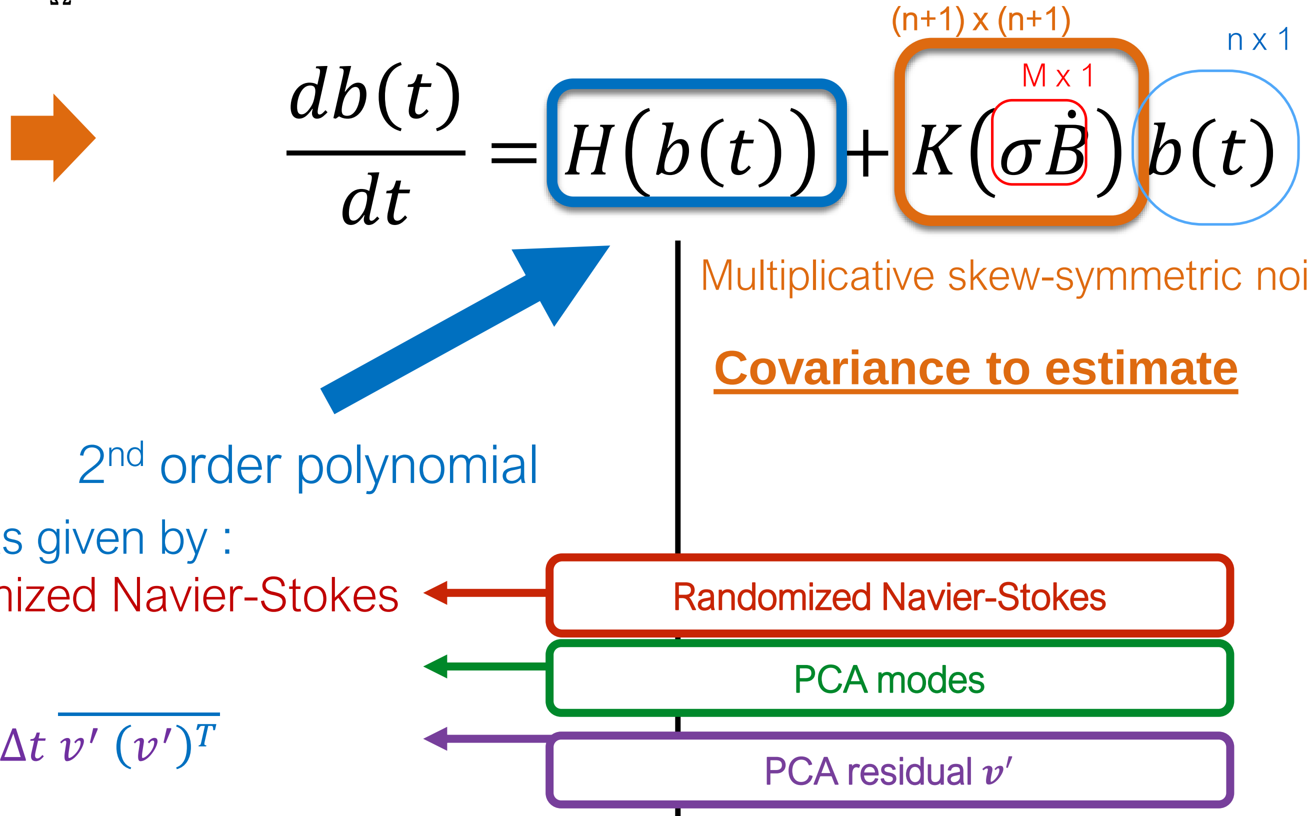

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Covariance to estimate

$$\mathbb{E}\left(K_{jq}(\sigma dB_t)\, K_{ip}(\sigma dB_t)\right)/dt \approx \Delta t\, K_{jq}\left[\overline{\frac{b_p}{\overline{b_p^2}}\, \frac{\Delta b_i}{\Delta t}\, v'}\right]$$

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$w(x,t) = \sum_{i=0}^{n} b_i(t)\phi_i(x)$

Unresolved fluid velocity:
$v' = \sigma\dot{B}$ (Gaussian, white wrt t)

Variance tensor:
$a(x,x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt}$

$$\int_\Omega \phi_i(x) \cdot (d_t w + C(w,w)dt + F(w)dt + C(\sigma dB_t, w) = dF)\, dx$$

$$\frac{db(t)}{dt} = \boxed{H\big(b(t)\big)} + \overset{(n+1)\times(n+1)}{\boxed{K\Big(\overset{M\times 1}{\boxed{\sigma\dot{B}}}\Big)}}\ \overset{n\times 1}{b(t)}$$

Multiplicative skew-symmetric noise

2nd order polynomial

Covariance to estimate

$$\mathbb{E}\Big(K_{jq}(\sigma dB_t)\, K_{ip}(\sigma dB_t)\Big)/dt \approx \Delta t\, K_{jq}\left[\overline{\frac{b_p}{\overline{b_p^2}}\,\frac{\Delta b_i}{\Delta t}\,v'}\right]$$

Coefficients given by :

- Randomized Navier-Stokes
- $(\phi_j)_j$
- $a(x) \approx \Delta t\, \overline{v'\,(v')^T}$

Randomized Navier-Stokes

PCA modes

PCA residual v'

$$\overline{f} = \frac{1}{T}\int_0^T f$$

$$K_{jq}[\xi] = -\int_\Omega \phi_j \cdot C(\xi, \phi_q)$$

Resseguier et al. (2021). *SIAM-ASA J Uncertain* . hal- 03169957

REDUCED LUM (RED LUM)

POD-Galerkin gives SDEs for resolved modes

Full order : $M \sim 10^7$
Reduced order : $n \sim 10$

$$v = w + v'$$

Resolved fluid velocity:
$w(x,t) = \sum_{i=0}^{n} b_i(t)\phi_i(x)$

Unresolved fluid velocity:
$v' = \sigma \dot{B}$ (Gaussian, white wrt t)

Variance tensor:
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2nd order polynomial

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Covariance to estimate

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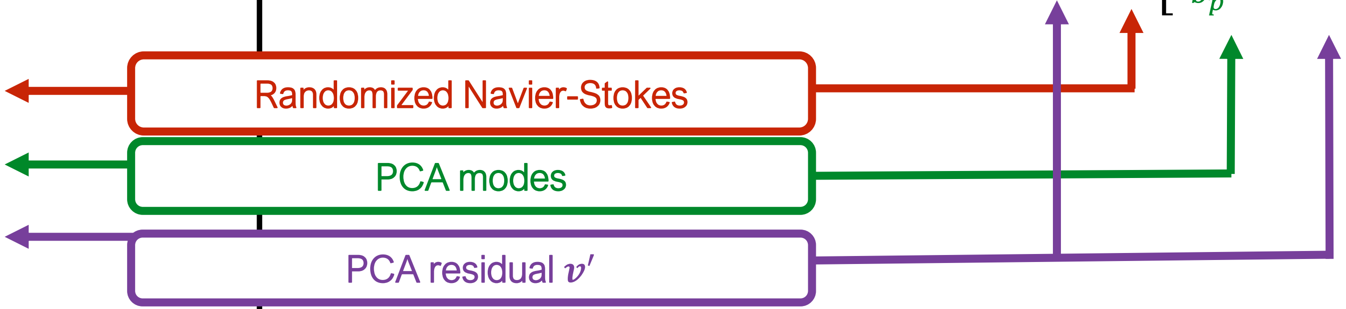

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$$\int_\Omega \phi_i(x)\cdot(d_t w + C(w,w)dt + F(w)dt + C(\sigma dB_t, w) = dF)\,dx$$

(n+1) x (n+1)

M x 1

n x 1

$$\frac{db(t)}{dt} = \boxed{H(b(t))} + \boxed{K(\boxed{\sigma\dot{B}})}\,b(t)$$

2nd order polynomial

Multiplicative skew-symmetric noise

Covariance to estimate

$$\mathbb{E}\left(K_{jq}(\sigma dB_t)\,K_{ip}(\sigma dB_t)\right)/dt \approx \Delta t\,K_{jq}\left[\overline{\frac{b_p}{\overline{b_p^2}}\,\frac{\Delta b_i}{\Delta t}\,v'}\right]$$

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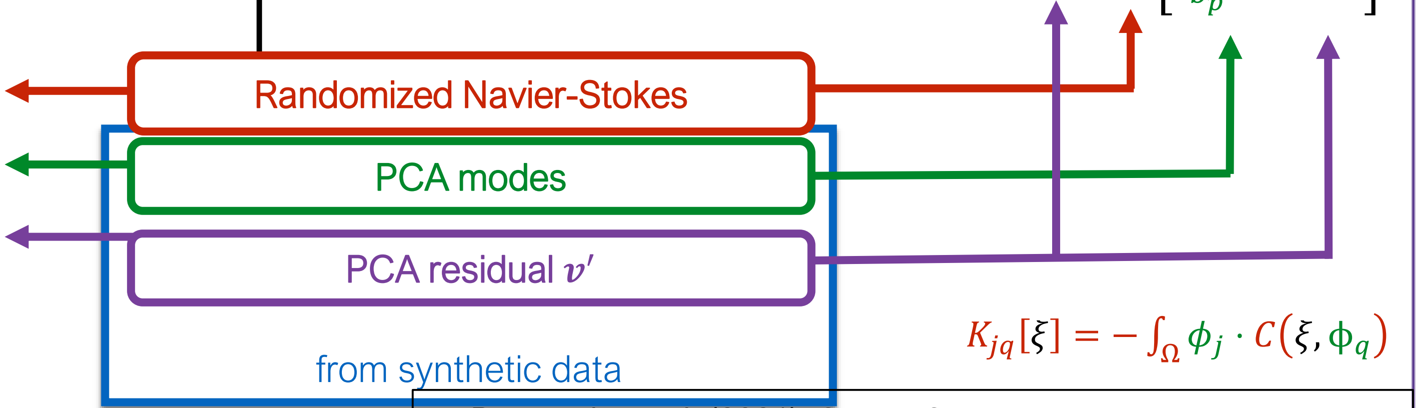

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$$\frac{db(t)}{dt} = \boxed{H\big(b(t)\big)} + \overset{(n+1)\times(n+1)}{\boxed{K\Big(\overset{M\times 1}{\boxed{\sigma\dot{B}}}\Big)}}\,\overset{n\times 1}{b(t)}$$

2nd order polynomial

Multiplicative skew-symmetric noise

Covariance to estimate

$$\mathbb{E}\Big(K_{jq}(\sigma dB_t)\,K_{ip}(\sigma dB_t)\Big)/dt \approx \Delta t\, K_{jq}\left[\overline{\frac{b_p}{\overline{b_p^2}}\,\frac{\Delta b_i}{\Delta t}\,v'}\right]$$

New estimator

- Consistency proven ($\Delta t \to 0$)
- Numerically efficient
- Physically-based
  → Robustness in extrapolation

Coefficients given by :
- Randomized Navier-Stokes
- $(\phi_j)_j$
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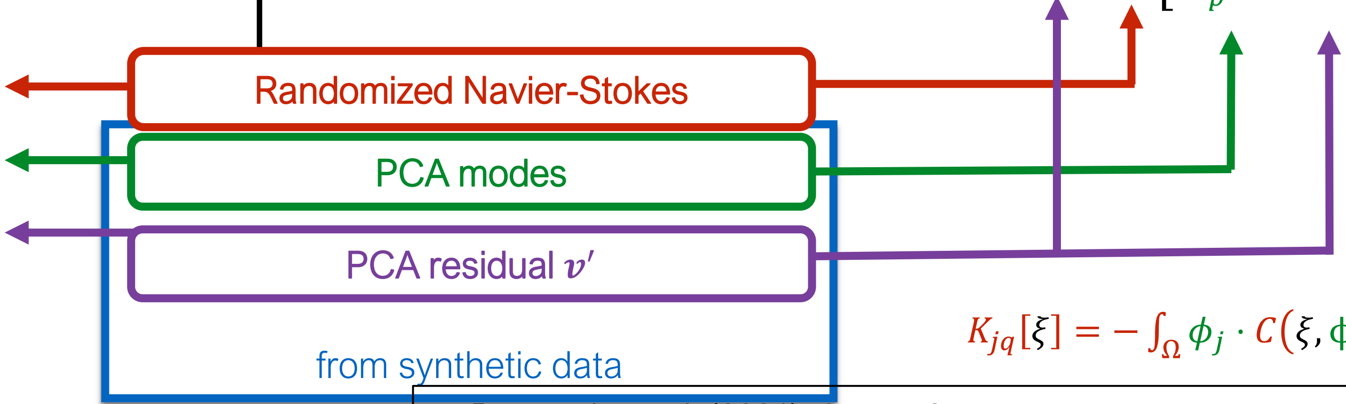

Randomized Navier-Stokes

PCA modes

PCA residual v'

from synthetic data

$K_{jq}[\xi] = -\int_\Omega \phi_j\cdot C(\xi,\phi_q)$

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Multiplicative noise covariance

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Number of time steps : $N \sim 10^4$

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Resolved fluid velocity:
$w(x,t) = \sum_{i=0}^{n} b_i(t)\phi_i(x)$

Unresolved fluid velocity:
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Randomized Navier-Stokes

PCA modes

PCA residual v'

from synthetic data

$$db(t) = H\big(b(t)\big)\,dt + \underset{\text{(n+1) x (n+1)}}{\boxed{K\big(\overset{\text{M x 1}}{\boxed{\sigma dB_t}}\big)}}\; b(t) \quad \text{with} \quad K_{jq}[\xi] = -\int_\Omega \phi_j \cdot C(\xi, \phi_q)$$

- **Curse of dimensionality**
  - Since σdB_t is white in time,
    $\Sigma_{jq,ip} = \mathbb{E}\Big(K_{jq}(\sigma dB_t)\; K_{ip}(\sigma dB_t)\Big)/dt \approx \Delta t\, \overline{K_{jq}(v')\, K_{ip}(v')}$
  - K is a matrix of integro-differential operators → cannot be evaluated on $v'(x,t)$ at every time t
  - Covariance of $\sigma dB_t \approx \Delta t^2\, \overline{\big(v'(x,t)\big)\big(v'(y,t)\big)^T}$: $M \times M \sim 10^{13}$ coefficients → intractable

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M x 1

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New estimator

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- **Consistency of our estimator** (convergence in probability for $\Delta t \to 0$, using stochastic calculus and continuity of K)

$$\Delta t\, K_{jq}\left[\overline{b_p \frac{\Delta b_i}{\Delta t}\, v'}\right] = \Delta t\, \overline{b_p \frac{\Delta b_i}{\Delta t} K_{jq}[v']} \approx \frac{1}{T}\int_0^T b_p\, d < b_i, K_{jq}(\sigma B) > = \frac{1}{T}\int_0^T b_p \sum_{\mathrm{r}=0}^{\mathrm{n}} b_r d < K_{ir}(\sigma B), K_{jq}(\sigma B) > = \sum_{\mathrm{r}=0}^{\mathrm{n}} \Sigma_{jq,ir}\, \overline{b_p b_r} = \Sigma_{jq,ip}\, \overline{b_p^2} \quad \text{(orthogonality from PCA)}$$

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PCA modes

PCA residual v'

from synthetic data

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- **Optimal time subsampling at Δt needed** to meet the white assumption

New estimator

- Consistency proven ($\Delta t \to 0$)
- Numerically efficient
- Physically-based
  → Robustness in extrapolation

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Randomized Navier-Stokes

PCA modes

PCA residual v'

from synthetic data

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- **Optimal time subsampling at Δt needed** to meet the white assumption
- **Additional reduction for efficient sampling** :
  diagonalization of Σ → $K(\sigma dB_t) \approx \alpha(d\beta_t)$ with a n-dimensional (instead of (n+1)²-dimensional) Brownian motion β

$$\overline{f} = \frac{1}{T}\int_0^T f$$

New estimator

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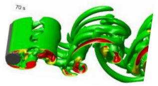
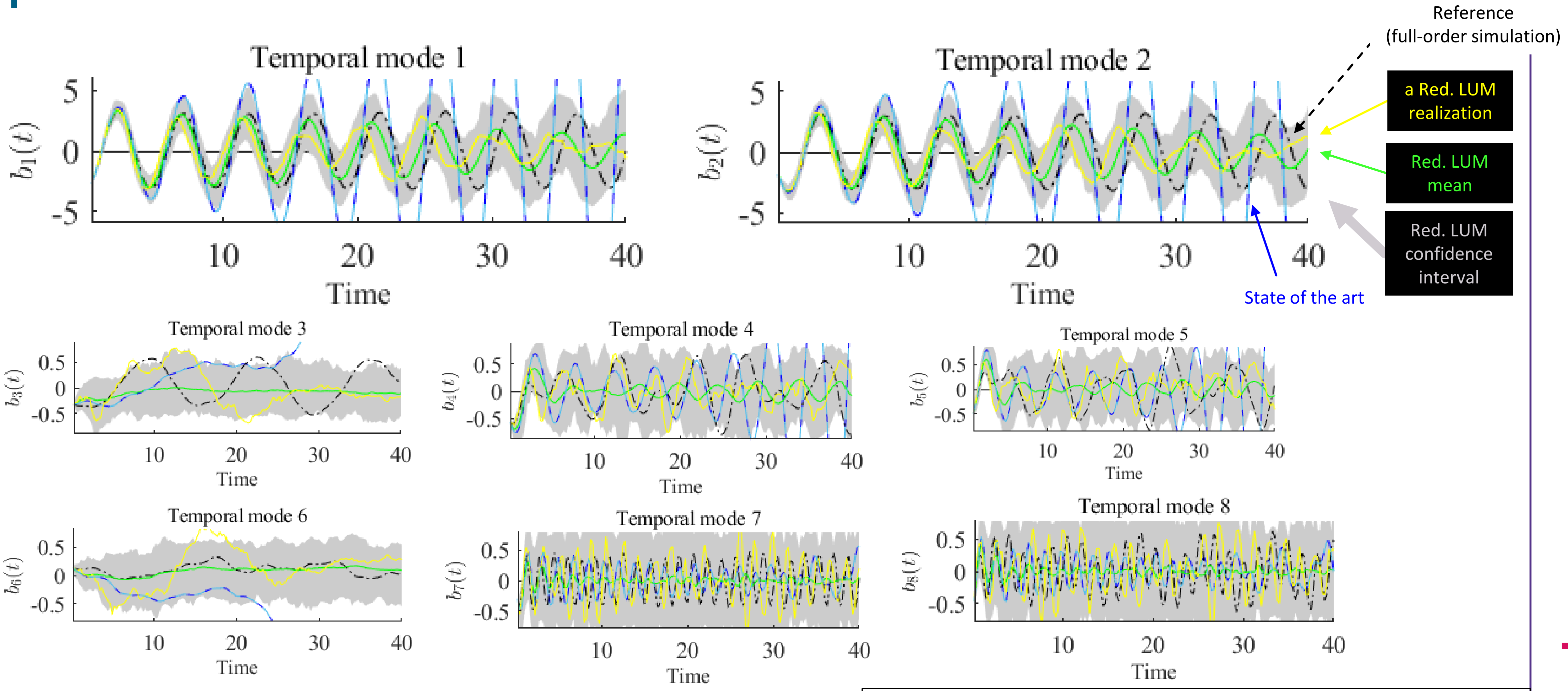
UNCERTAINTY QUANTIFICATION (PRIOR)

$b_i(t)$ VS reference



From 10^7 to 8 degrees of freedom
No data assimilation
Known initial conditions $b(t = 0)$



Reference
(full-order simulation)

a Red. LUM realization

Red. LUM mean

Red. LUM confidence interval

State of the art

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UNCERTAINTY QUANTIFICATION (PRIOR)

Error on the reduced solution w

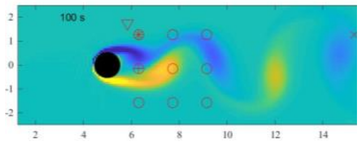
From 10^7 to 8 degrees of freedom
No data assimilation
Known initial conditions $b(t=0)$

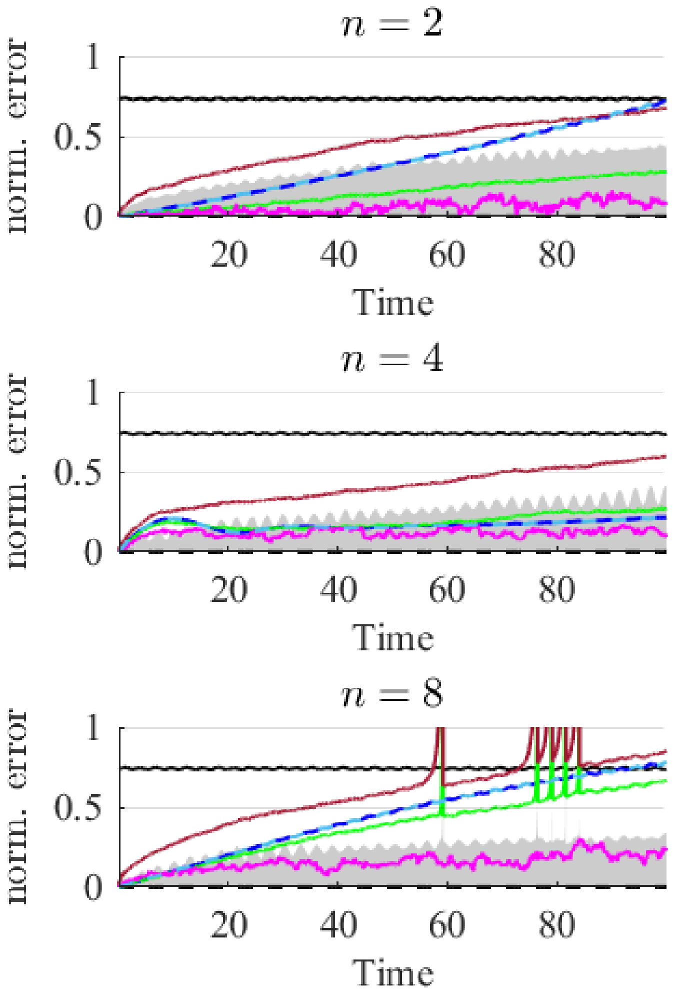
$$v = w + v'$$

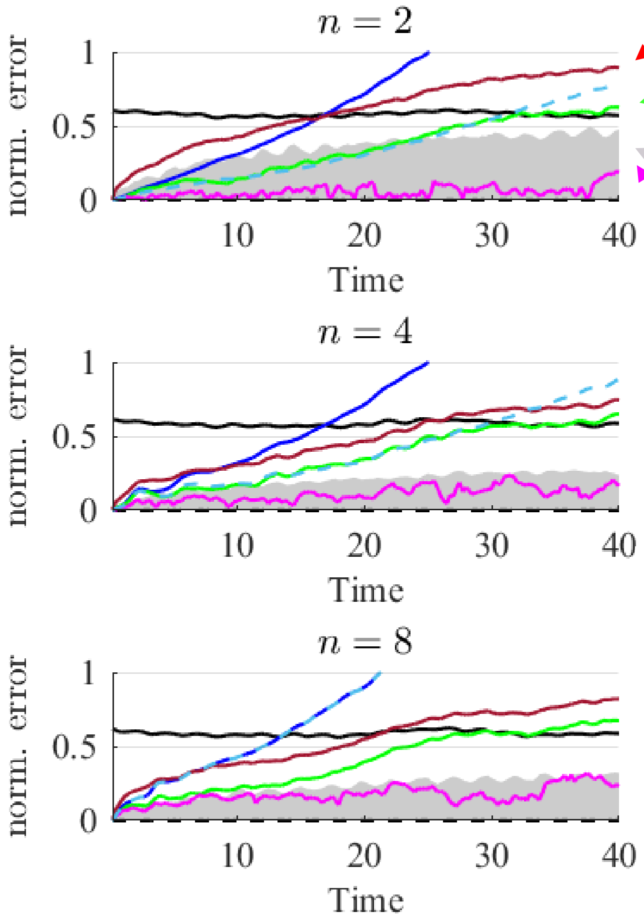
Resolved fluid velocity:
$w = \sum_{i=0}^{n} b_i \phi_i$

Unresolved fluid velocity:
v'

Reynolds number (Re) = 100 / 2D
(full-order simulation has 10^4 dof)





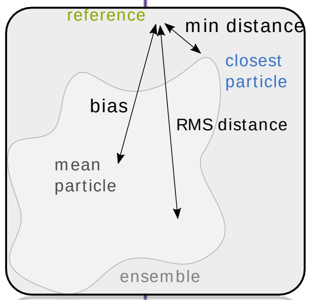


Red. LUM
RMSE

Red. LUM
bias

Red. LUM
std

Red. LUM
ensemble
minimal
distance
to the reference



Reynolds number (Re) = 300 3D
(full-order simulation has 10^7 dof)

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UNCERTAINTY QUANTIFICATION (PRIOR)

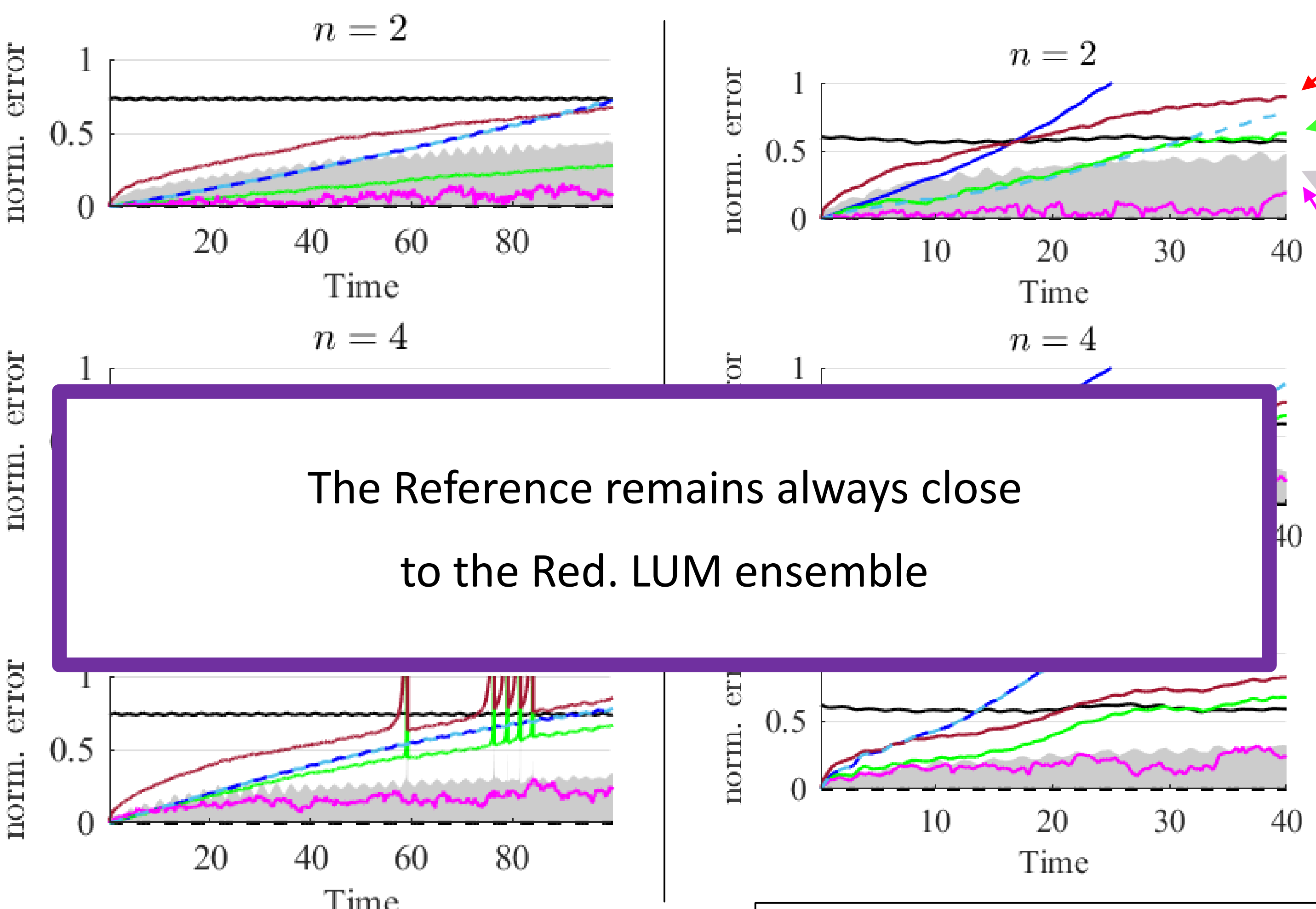
Error on the reduced solution w

From 10^7 to 8 degrees of freedom
No data assimilation
Known initial conditions $b(t=0)$

$v = w + v'$

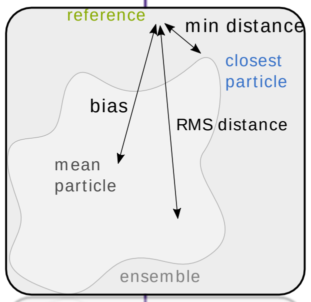
Resolved fluid velocity:
$w = \sum_{i=0}^{n} b_i \phi_i$

Unresolved fluid velocity:
v'

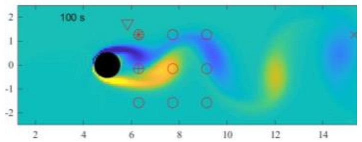


The Reference remains always close

to the Red. LUM ensemble



Reynolds number (Re) = 100 / 2D
(full-order simulation has 10^4 dof)



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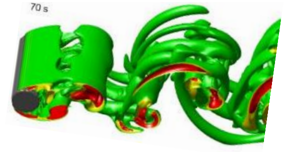

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DATA ASSIMILATION (POSTERIOR)

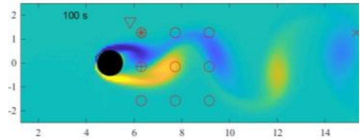
Error on the solution estimation

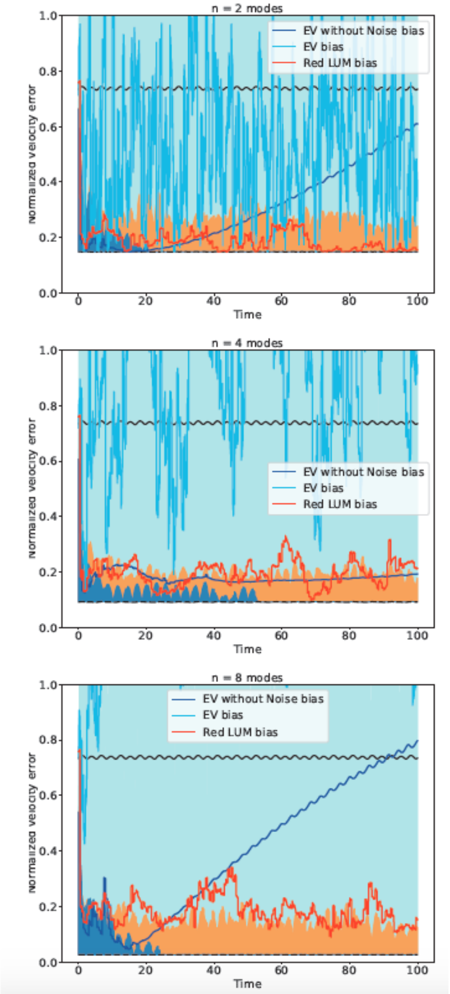
$$v = w + v'$$

Resolved fluid velocity:
$w = \sum_{i=0}^{n} b_i \phi_i$

Unresolved fluid velocity:
v'

Reynolds number (Re) = 100 / 2D
(full-order simulation has 10^4 dof)



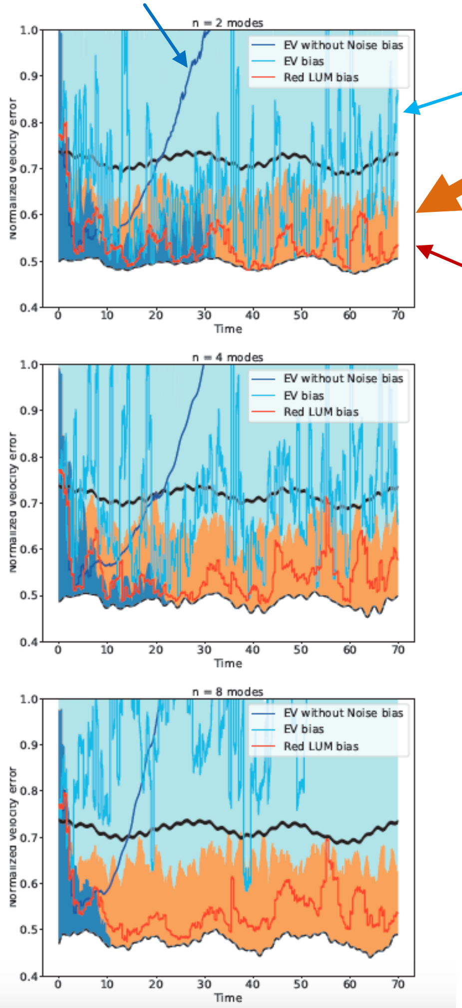


State of the art

State of the art

Red. LUM std

Red. LUM bias



Reynolds number (Re) = 300 3D
(full-order simulation has 10^7 dof)

Resseguier et al. (2022). *J Comp.Phys* . hal-03445455