Observer design for n + m linear hyperbolic ODE-PDE-ODE systems

Jean Auriol, Federico Bribiesca Argomedo

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Jean Auriol[1], Federico Bribiesca Argomedo[2]

***Abstract*— In this paper, an observer design is presented for a system of $n+m$ hetero-directional transport partial differential equations (PDEs) coupled on both boundaries of the domain to ordinary differential equations (ODEs). This class of systems can represent, for instance, actuator and load dynamics at the boundaries of a hyperbolic system. The results in this paper provide a constructive way to reconstruct the state of the ODEs and the PDEs using only available measurements on one of the two ODEs. This observer design completes existing full-state feedback designs for this class of systems and enables, together with previous results, the construction of output-feedback stabilizers. As a complementary result, this paper includes a simple (constructive) method to obtain a stable left-inverse of a specific transfer matrix required in the observer design, which can also be applied (after a transposition) to the computation of the analogous stable right-inverses required for the control design.**

I. Introduction

Many recent publications have focused on the control of interconnections involving hyperbolic PDEs (particularly ODE-PDE-ODE systems where the ODEs correspond to actuator and load dynamics). Indeed, this class of systems can model, e.g., the propagation of torsional waves in drilling systems [2] or deepwater construction vessels [27] (where the top of a cable is attached to a crane on a vessel at the ocean surface and the bottom attached to equipment to be installed at the seafloor).

Interconnections of hyperbolic PDEs and ODEs are not a new problem. Most constructive control results in the literature for these systems are based on the backstepping approach. Indeed, in the seminal paper [22], a re-interpretation of the classical Finite Spectrum Assignment [24] was proposed, modeling ODEs with input delays as PDE-ODE interconnections. Subsequently, this result has enabled the design of observers, controllers, or parameter estimation methods for a plethora of interconnected systems: systems with varying delays [8], [11], cascades of PDEs [4]. Of particular note are results concerning cascaded interconnections of hyperbolic PDE-ODE systems, such as [1], [19], [32]. More recently, constructive results on non-cascaded PDE-ODE systems have been proposed. In particular, a stabilizing state-feedback control law has been proposed in [15], [30]. Recently, a backstepping approach has been proposed in [13] for the output regulation for coupled linear wave–ODE systems. Some results also concern other types of PDE-ODE interconnections, such as parabolic PDEs, e.g. [28], [5], [12], however this class of systems is beyond the scope of this article.

Regarding ODE-hyperbolic PDE-ODE systems with full interconnections (not limited to cascades), a stabilizing observer-controller robust to delays has been proposed in the case of a scalar proximal ODE in [16]. In [14], an output-feedback controller is designed based on assumptions that guarantee the existence of a Byrnes–Isidori normal form for the proximal ODE, as well as a relative degree one condition. These restrictions are partially avoided in [9] for the case of a scalar PDE system as the proximal ODE is simply assumed to be minimum phase for the output that affects the PDE. In [10], a strictly-proper control law was proposed using less restrictive assumptions on the structure of the ODE components. This approach was later extended in [29] to encompass a state-observer (thus allowing the design of an output-feedback controller). However, the PDE subsystem is still assumed to be scalar in this work. Some recent results have also been obtained for interconnected PDE systems with non-linear ODEs [21].

This paper deals with observer design for systems of $n+m$ linear first-order hyperbolic Partial Differential Equations coupled with Ordinary Differential Equations at both boundaries of a one-dimensional spatial domain. The available measurements are obtained from one of the ODEs. The proposed approach extends the methodology introduced in [10] to $n+m$ transport equations (instead of $1+1$), and combines the backstepping methodology with time-delay approaches. The assumptions made on the structure of the ODE components do not require the system to be of relative degree 1, nor to be in a specific canonical form. As such, existing results in the literature do not cover this class of systems. Our approach relies on the backstepping methodology. Using an invertible integral transformation, we map the system into a simpler target system. Then, we design a state observer for this target system using a time-delay representation and an analysis in the Laplace domain. The proposed design relies on the stable dynamic left-inversion of the (measured) ODE dynamics. We present a simple constructive procedure to find such a stable left-inverse. This procedure requires the same condition on the transfer function (full-column rank for all $s \in \mathbb{C}^+$) than presented in Theorem 2 of [25], yet operates directly on the transfer function instead of requiring switching back and forth between a frequency-domain and time-domain representations. In practice, a state-space realisation of this inverse system will only be found after adding an adequate low-pass filter, as in the case of the control (e.g. [10]). Such a filtering procedure also guarantees the existence of robustness margins.

The paper is organized as follows: in Section II, we present the problem under consideration, as well as the hypotheses required for the observer design. In Section III, we design the state-observer using the backstepping methodology. In Section IV we present a simple constructive procedure to construct the a stable left-inverse for the ODE dynamics. Some concluding remarks are given in Section V.

Notations We denote the state space $\chi = \mathbb{R}^p \times L^2([0,1];\mathbb{R})^{n+m} \times \mathbb{R}^q$, where p,n,m,q are positive integers. For $(X_0,u,v,X_1) \in$

[1]Jean Auriol is with Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France. jean.auriol@l2s.centralesupelec.fr, [2]Federico Bribiesca Argomedo is with Université de Lyon, INSA Lyon, CNRS Ampère, F-69621, Villeurbanne, France

χ, we introduce the corresponding $\chi-$norm $||(X_0,u,v,X_1)||^2_\chi = ||X_0||^2_{\mathbb{R}^p} + ||u||^2_{L^2} + ||v||^2_{L^2} + ||X_1||^2_{\mathbb{R}^q}$. The variable s denotes the Laplace variable. To simplify the notations, the Laplace transform of a function $f(t)$ (provided it is well-defined) will be denoted $f(s)$. The space $\mathbb{C}^+$ corresponds to the complex right half plane: $\mathbb{C}^+ = \{s \in \mathbb{C},\ \mathrm{Re}(s) \geq 0\}$. The notation I_r stands for the $r \times r$ identity matrix (if the dimensions are not ambiguous, the subindex will be omitted).

II. Problem statement

A. Presentation of the system

In this paper, we consider a $n+m$ linear hetero-directional hyperbolic system coupled through its boundaries with linear ODEs:

$$\begin{aligned} \dot{X}_0(t) &= A_0X_0(t) + E_0v(t,0), & (1)\\ u(t,0) &= C_0X_0(t) + Qv(t,0), & (2)\\ u_t + \Lambda^+ u_x &= \Sigma^{++}(x)u(t,x) + \Sigma^{+-}(x)v(t,x), & (3)\\ v_t - \Lambda^- v_x &= \Sigma^{-+}(x)u(t,x) + \Sigma^{--}(x)v(t,x), & (4)\\ v(t,1) &= Ru(t,1) + C_1X_1(t), & (5)\\ \dot{X}_1(t) &= A_1X_1(t) + E_1u(t,1), & (6) \end{aligned}$$

defined for a.e. $(t,x) \in [0,+\infty] \times [0,1]$. The state of the system is $(X_0(t), u(t,\cdot), v(t,\cdot), X_1(t)) \in \chi$. The initial condition is taken as $((X_0)_0, u_0, v_0, (X_1)_0) \in \chi$ and we consider weak solutions to (1)-(6) [7]. The system is well-posed in the sense of [7, Theorem A.6, page 254]. The matrices Λ^+ and Λ^- are diagonal and represent the transport velocities. We have $\Lambda^+ = \text{diag }(\lambda_i)$ and $\Lambda^- = \text{diag }(\mu_i)$ and we assume that their coefficients satisfy $-\mu_m < \cdots < -\mu_1 < 0 < \lambda_1 < \cdots < \lambda_n$. The spatially-varying matrices $\Sigma^{\cdot\cdot}$ are continuous (each coefficient of the matrix is a continuous function). With no loss of generality, we assume that the matrices Σ^{++} and Σ^{--} have zero diagonal elements [14]. The different coupling matrices satisfy $A_0 \in \mathbb{R}^{p\times p}$, $E_0 \in \mathbb{R}^{p\times m}$, $C_0 \in \mathbb{R}^{n\times p}$, $A_1 \in \mathbb{R}^{q\times q}$, $E_1 \in \mathbb{R}^{q\times n}$, $C_1 \in \mathbb{R}^{m\times q}$, $R \in \mathbb{R}^{m\times n}$, $Q \in \mathbb{R}^{n\times m}$. As we consider an observer design problem in this paper, and since the system under consideration is linear, we do not consider any control input in the system. We measure a part of one of the ODE state (as it is the case for drilling applications [2]), i.e. the measurement $Y(t)$ is defined by

$$Y(t) \doteq CX_1(t), \qquad (7)$$

where $C \in \mathbb{R}^{d\times q}$, $d \geq n$. Having $d \geq n$ means that the number of measurements is greater than the dimension of the PDE state u. Without this assumption, it may not be possible to reconstruct the value of the function $u(t,1)$. As no result currently exist for under-measured hyperbolic PDE systems, it is a reasonable assumption.

B. General objectives

The objective of this paper is to design a state observer for the system (1)-(6) based on the available measurement $Y(t)$ given by equation (7). Even if the velocities are considered constant here, the proposed methodology could be extended to the case of spatially-varying transport velocities (following the approach given in [20]). Note that, due to the symetry of the system, measuring the X_0 state would not change the nature of the problem.

C. Assumptions

Similarly to what has been done in [10], we make the following assumptions

Assumption 1: The pairs (A_1,C), (A_0,C_0) are detectable (i.e. there exist $L_0 \in \mathbb{R}^{p\times n}$ and $L_1 \in \mathbb{R}^{q\times d}$ such that $\tilde{A}_1 \doteq A_1 + L_1C$ and $\tilde{A}_0 \doteq A_0 + L_0C_0$ are Hurwitz).

Assumption 1 is a classical requirement found in most of the papers dealing with ODE-PDE-ODE systems [17], [14]. It is not overly conservative since without the detectability of (A_0,C_0) it becomes impossible to reconstruct the X_0-state independently of the PDE or interconnection structure. The detectability condition on (A_1,C) allows us to for a simpler observer design (no modes of X_1 are reconstructed indirectly through the PDE). It results in a set of conditions that can be easily tested.

Assumption 2: For all $s \in \mathbb{C}^+$, the matrices (A_1,E_1,C) satisfy

$$\text{rank }(\begin{pmatrix} sI - A_1 & E_1 \\ C & 0 \end{pmatrix}) = q+n. \qquad (8)$$

This last assumption (that requires $d \geq n$) serves multiples purposes. First, it implies that C and E_1 are not identically zero. It is necessary to observe the PDE and the X_1 states from the available measurements. Under Assumption 2, the function

$$P_1(s) \doteq C(sI - \tilde{A}_1)^{-1}E_1 \qquad (9)$$

does not have any zeros in $\mathbb{C}^+$ that is common to all its components. Thus, the function $P_1(s)$ admits a left inverse (denoted P_1^-) whose entries have no unstable poles (such a left inverse is not proper) [25]. A possible choice is given by the Moore-Penrose left inverse $P_1^-(s) = (P_1^T(s)P_1(s))^{-1}P_1^T(s)$ (which should be verified to be stable *a posteriori*). If this is not stable, then a more involved stable inversion procedure is needed, an example of which is given in Section IV. This assumption is directly used in the constructive design of a observer and can be tested in a simple way.

Assumption 3: The system defined for all $i \in [1,n]$ by

$$z(t) = \sum_{k=1}^{m}\sum_{\ell=1}^{n} Q_{ik}R_{k\ell}z(t - \frac{1}{\mu_k} - \frac{1}{\lambda_\ell}), \qquad (10)$$

is exponentially stable.

Assumption 3 constitutes a reasonable assumption since it prevents system (1)-(6) from having an asymptotic chain of eigenvalues with non-negative real parts [18], [6]. It has been shown in [23] that having an open-loop transfer function with an infinite number of poles on the closed right half-plane implies no (delay-)robustness margins in closed-loop (i.e., the introduction of any arbitrarily small delay in the actuation will destabilize the closed-loop system). Therefore, Assumption 3 is slightly stronger than a necessary condition for delay-robust stabilization and guarantees exponential stability of system system (1)-(6). If the delays are rationally independent[1], Assumption 3 is equivalent to the following condition [18]: $\sup_{\theta_{k\ell}\in[0,2\pi]^{n\times m}} \text{Sp } (\sum_{k=1}^m \sum_{\ell=1}^n Q_{ik}R_{k\ell}\exp(j\theta_{k\ell})) < 1$, where Sp denotes the spectral radius and j is the imaginary unit. This condition is simplified if the delays are rationally dependent. Furthermore, since the spectral radius of a matrix

[1]Extending the variable z, it is always possible to rewrite the system in a situation where the delays are rationally independent [18, Chapter 9].

is upper-bounded by any norm of the matrix, easy to compute sufficient conditions for this spectral radius condition to hold can be derived using different norms of the matrices involved at the cost of increased conservatism.

III. Observer design

The objective of this section is to design a state-observer for the system (1)-(6) using the available measurement (7). In what follows, we denote with a $\hat{\cdot}$ superscript the estimated states and with a $\tilde{\cdot}$ superscript the error state, i.e. the difference between the real state and the observer state. The objective is that the estimated state converges to the real state or equivalently that the error state, converges to 0, in the sense of the χ-norm. To simplify the computations and the design of the observer, we will work with a target system obtained from (1)-(6) using a backstepping transformation, for which the in-domain coupling terms $\Sigma^{\cdot\cdot}$ have been moved to the boundary. Then, we design a state-observer that is a copy of this dynamics with some output injection operators.

A. Backstepping transformation

Inspired by [20], we use an integral transformation to move the local coupling terms $\Sigma^{\cdot\cdot}$ to the boundary (in the form of integral terms). Consider the Volterra transformation $\mathcal{T}$, similar to the one introduced in [20], [6]

$$X_0(t) = \xi(t) - \int_0^1 L_1(y)\alpha(y) + L_2(y)\beta(y)dy, \quad (11)$$

$$u(t,x) = \alpha(t,x) - \int_x^1 L^{\alpha\alpha}(x,y)\alpha(y)dy - \int_x^1 L^{\alpha\beta}(x,y)\beta(y)dy + \gamma_\alpha(x)X_1(t), \quad (12)$$

$$v(t,x) = \beta(t,x) - \int_x^1 L^{\beta\alpha}(x,y)\alpha(y)dy, - \int_x^1 L^{\beta\beta}(x,y)\beta(y)dy + \gamma_\beta(x)X_1(t), \quad (13)$$

$$X_1(t) = X_1(t), \quad (14)$$

where the kernels are bounded functions defined either on $\mathcal{T}_u = \{(x,y) \in [0,1]^2,\ x \leq y\}$, or $[0,1]$. This transformation rewrites $(X_0, u, v, X_1) = \mathcal{T}(\xi, \alpha, \beta, X_1)$. Denoting $\Lambda = \text{diag}(\Lambda^+, -\Lambda-)$, $\Sigma = \begin{pmatrix} \Sigma^{++} & \Sigma^{+-} \\ \Sigma^{-+} & \Sigma^{--} \end{pmatrix}$ and $L = \begin{pmatrix} L^{\alpha\alpha} & L^{\alpha\beta} \\ L^{\beta\alpha} & L^{\beta\beta} \end{pmatrix}$, $\gamma = (\gamma_\alpha, \gamma_\beta)$, we obtain

$$\Lambda L_x + L_y \Lambda = \Sigma(x) L, \quad \Lambda \gamma_x(x) = \Sigma(x)\gamma - \gamma^\top A_1 \quad (15)$$

$$(L_1(x))_x \Lambda^+ = A_0 L_1(x) + E_0 L^{\beta\alpha}(0,x) - L_0(L^{\alpha\alpha}(0,x) - QL^{\beta\alpha}(0,x) - C_0 L_1(x)), \quad (16)$$

$$(L_2(x))_x \Lambda^- = -A_0 L_2(x) - E_0 L^{\beta\beta}(0,x) + L_0(L^{\alpha\beta}(0,x) - QL^{\beta\beta}(0,x) - C_0 L_2(x)), \quad (17)$$

with the boundary conditions

$$\Lambda L(x,x) - L(x,x)\Lambda = \Sigma(x), \quad (18)$$

$$L_1(0) = L_0(\Lambda^+)^{-1},\ L_2(0)\Lambda^- = L_1(0)\Lambda^+ Q + E_0, \quad (19)$$

and $\gamma_\alpha(1) = 0,\ \gamma_\beta(1) = C_1$. Finally, we define $L_{ij}^{\alpha\alpha}(0,y)$ for $i \leq j$ by

$$L^{\alpha\alpha}(0,y) = QL^{\beta\alpha}(0,y) + C_0 L_1(y). \quad (20)$$

To this set of equations, we add arbitrary values for $L_{ij}^{\alpha\alpha}(x,1)$ (when $i > j$) and $L_{ij}^{\beta\beta}(x,1)$ (when $i > j$) and $L_{ij}^{\beta\beta}(0,y)$ (when $i \leq j$). Reinterpreting the ODEs in (16)-(17) as PDEs evolving in the triangular domain $\mathcal{T}_u$ with horizontal characteristic lines (since there is only an evolution along the x axis), it is possible to adjust the results from [15, Theorem 3.2] to guarantee that the set of PDEs and ODEs (15)-(20) has a unique solution which is piecewise continuous.The boundedness of transformation (12)-(14) is a direct consequence of the structure of the transform (identities, integral operator and matrices) and the regularity of the different kernels. Its invertibility is a consequence of the structure of the transformation which is block triangular with the blocks on the diagonal being either identities (for the ODEs) or invertible Volterra operators (for the PDEs).The invertible backstepping transformation (12)-(14) maps the original system (1)-(6) to the following target system

$$\dot{\xi}(t) = \tilde{A}_0 \xi(t) + G_3 \alpha(t,1) + G_4 X_1(t), \quad (21)$$

$$\alpha(t,0) = Q\beta(t,0) + C_0\xi(t) + (Q\gamma_\beta(0) - \gamma_\alpha(0))X_1(t) + \int_0^1 F^\alpha(y)\alpha(t,y) + F^\beta(y)\beta(t,y)dy, \quad (22)$$

$$\alpha_t(t,x) + \Lambda^+ \alpha_x(t,x) = G_1(x)\alpha(t,1), \quad (23)$$

$$\beta_t(t,x) - \Lambda^- \beta_x(t,x) = G_2(x)\alpha(t,1), \quad (24)$$

$$\beta(t,1) = R\alpha(t,1),\ \dot{X}_1(t) = A_1 X_1(t) + E_1 \alpha(t,1). \quad (25)$$

The functions G_1 and G_2 satisfy

$$G_1(x) = \int_x^1 L^{\alpha\alpha}(x,y)G_1(y) + L^{\alpha\beta}(x,y)G_2(y)dy - L^{\alpha\alpha}(x,1)\Lambda^+ + L^{\alpha\beta}(x,1)\Lambda^- R - \gamma_\alpha(x)E_1, \quad (26)$$

$$G_2(x) = \int_x^1 L^{\beta\alpha}(x,y)G_1(y) + L^{\beta\beta}(x,y)G_2(y)dy - L^{\beta\alpha}(x,1)\Lambda^+ + L^{\beta\beta}(x,1)\Lambda^- R - \gamma_\beta(x)E_1. \quad (27)$$

The set of equations (26)-(27) has a unique solution (Volterra equations of the second kind [31]). The matrices G_3 and G_4 are defined by $G_3(x) = L_2(x,1)\Lambda^- R - L_1(x,1)\Lambda^+ + \int_0^1 (L_1(x)G_1(x) + L_2(x)G_2(x))dx$, and $G_4(x) = E_0\gamma_\beta(0) + L_0(Q\gamma_\beta(0) - \gamma_\alpha(0))$. Finally, the matrix F^β the matrix F^α are defined by $F^\alpha(y) = L^{\alpha\alpha}(0,y) - QL^{\beta\alpha}(0,y) - C_0 L_1(y)$, and $F^\beta(y) = L^{\alpha\beta}(0,y) - QL^{\beta\beta}(0,y) - C_0 L_2(y)$. Note that F^α is **strictly lower triangular** due to equation (20). The measurement $Y(t)$ remains unchanged.

B. Observer equations

We can now design an observer for the target system (21)-(25). The observer state $(\hat{\xi}, \hat{\alpha}, \hat{\beta}, \hat{X}_1)$ is the solution of a set of equations that is a copy of the original dynamics to which we add dynamical output injection gains. We denote $\tilde{Y}(t) = Y(t) - C\hat{X}_1(t)$, the difference between the real output and the observer output. The observer equations read as

$$\dot{\hat{\xi}}(t) = \tilde{A}_0 \hat{\xi}(t) + G_3 \hat{\alpha}(t,1) + G_4 \hat{X}_1(t) - \mathcal{O}_0(\tilde{Y}), \quad (28)$$

$$\hat{\alpha}(t,0) = Q\hat{\beta}(t,0) + C_0\hat{\xi}(t) + (Q\gamma_\beta(0) - \gamma_\alpha(0))\hat{X}_1(t) + \int_0^1 F^\alpha(y)\hat{\alpha}(t,y) + F^\beta(y)\hat{\beta}(t,y)dy - \mathcal{O}_1(\tilde{Y}), \quad (29)$$

$$\hat{\alpha}_t(t,x) + \Lambda^+ \hat{\alpha}_x(t,x) = G_1(x)\hat{\alpha}(t,1) - \mathcal{O}_\alpha(x, \tilde{Y}), \quad (30)$$

$$\hat{\beta}_t(t,x) - \Lambda^- \hat{\beta}_x(t,x) = G_2(x)\hat{\alpha}(t,1) - \mathcal{O}_\beta(x,\tilde{Y}), \quad (31)$$
$$\hat{\beta}(t,1) = R\hat{\alpha}(t,1),$$
$$\dot{\hat{X}}_1(t) = A_1\hat{X}_1(t) + E_1\hat{\alpha}(t,1) - L_1 C\tilde{Y}, \quad (32)$$

with any (arbitrary) initial conditions in χ. The **stable** operators $\mathcal{O}_i$ still have to be defined. Substracting the observer dynamics to the real one, we obtain the error system

$$\dot{\tilde{\xi}}(t) = \tilde{A}_0\tilde{\xi}(t) + G_3\tilde{\alpha}(t,1) + G_4\tilde{X}_1(t) + \mathcal{O}_0(\tilde{Y}), \quad (33)$$
$$\tilde{\alpha}(t,0) = C_0\tilde{\xi}(t) + Q\tilde{\beta}(t,0) + (Q\gamma_\beta(0) - \gamma_\alpha(0))\tilde{X}_1(t)$$
$$+ \int_0^1 F^\alpha(y)\tilde{\alpha}(t,y) + F^\beta(y)\tilde{\beta}(t,y)dy + \mathcal{O}_1(\tilde{Y}), \quad (34)$$
$$\tilde{\alpha}_t(t,x) + \Lambda^+\tilde{\alpha}_x(t,x) = G_1(x)\tilde{\alpha}(t,1) + \mathcal{O}_\alpha(x,\tilde{Y}) \quad (35)$$
$$\tilde{\beta}_t(t,x) - \Lambda^-\tilde{\beta}_x(t,x) = G_2(x)\tilde{\alpha}(t,1) + \mathcal{O}_\beta(x,\tilde{Y}) \quad (36)$$
$$\tilde{\beta}(t,1) = R\tilde{\alpha}(t,1), \quad (37)$$
$$\dot{\tilde{X}}_1(t) = \tilde{A}_1\tilde{X}_1(t) + E_1\tilde{\alpha}(t,1). \quad (38)$$

The objective is now to tune the different operators $\mathcal{O}_i$ such that the error system exponentially converges to zero. To do so, it is sufficient to show the convergence of $\tilde{\xi}$, $\tilde{\alpha}(t,1)$ and $\tilde{X}_1$ to zero. More precisely, we have the following lemma

Lemma 1: If $\tilde{\xi}(t)$, $\tilde{\alpha}(t,1)$ and $\tilde{X}_1(t)$ exponentially converge to zero, then the state $(\tilde{\xi},\tilde{\alpha},\tilde{\beta},\tilde{X}_1)$ converges to zero in the sense of the χ-norm. This implies the convergence of the observer state to the real state.

Proof: Due to the stability of the observer operators and using the transport structure of (35) and (36), the exponential convergence of $\tilde{X}_1$ and $\tilde{\alpha}(t,1)$ to zero imply the exponential convergence of the states $\tilde{\alpha}(t,x)$ and $\tilde{\beta}(t,x)$. ■

C. Design of the operators $\mathcal{O}_i$

We now want to define the operators $\mathcal{O}_i$ such that $\tilde{\xi}$, $\tilde{\alpha}(t,1)$ and $\tilde{X}_1$ exponentially converge to zero. The analysis will be done in the Laplace domain. The Laplace transform [2] of equation (38) yields

$$(sI - \tilde{A}_1)\tilde{X}_1(s) = E_1\tilde{\alpha}(s,1). \quad (39)$$

Due to Assumption 1, the matrix $(sI - \tilde{A}_1)$ is invertible on $\mathbb{C}^+$. This implies $\tilde{Y}(s) = C(sI - \tilde{A}_1)^{-1}E_1\tilde{\alpha}(s,1)$. Thus, we obtain $\tilde{\alpha}(s,1) = P_1^-(s)\tilde{Y}(s)$, where P_1^- is a left-inverse of P_1 in (9). This in turns implies $\tilde{X}_1(s) = (sI - \tilde{A}_1)^{-1}E_1P_1^-(s)\tilde{Y}(s)$. This means that the terms that are functions $\tilde{X}_1$ and $\tilde{\alpha}(s,1)$ that appear in the error system can directly be compensated using the observer gains. In particular, we can define $\mathcal{O}_0$ as

$$\mathcal{O}_0(\tilde{Y}(s)) = -(G_3P_1^-(s) + G_4(sI - \tilde{A}_1)^{-1}E_1P_1^-(s))\tilde{Y}(s) \quad (40)$$

so that equation (33) can be rewritten as $(sI - \tilde{A}_0)\tilde{\xi}(s) = 0$, which implies the exponential convergence of ξ to zero due to Assumption 1. Similarly, to get rid of the terms G_1 and G_2, we define the operators $\mathcal{O}_\alpha(x,\tilde{Y})$ and $\mathcal{O}_\beta(x,\tilde{Y})$ by

$$\mathcal{O}_\alpha(x,\tilde{Y}) = -G_1(x)P_1^-(s)\tilde{Y}(s) \quad (41)$$
$$\mathcal{O}_\beta(x,\tilde{Y}) = -G_2(x)P_1^-(s)\tilde{Y}(s), \quad (42)$$

[2] We omit the effect of the initial condition when taking the Laplace transform since it does not modify the stability analysis [18]

such that equations (35)-(36) rewrite as transport equations. Indeed, for $t > \frac{1}{\lambda_1} + \frac{1}{\mu_1}$, for every $1 \leq i \leq n$ and every $1 \leq j \leq m$ we now have for every $x \in [0,1]$

$$\tilde{\alpha}_i(t,x) = \tilde{\alpha}_i(t - \frac{x}{\lambda_i}, 0) \quad (43)$$
$$\tilde{\beta}_j(t,x) = \sum_{k=1}^n R_{jk}\tilde{\alpha}_k(t - \frac{1-x}{\mu_j}, 1). \quad (44)$$

In what follows, we consider that $t > \frac{1}{\lambda_1} + \frac{1}{\mu_1}$. The design of the operator $\mathcal{O}_1$ is more involved since this operator must compensate almost all the terms that appear in equation (34) (including the integral terms). To design this observer operator, we will omit the term $C_0\tilde{\xi}$ that appear in equation (34) since this term exponentially converges to zero. For all $1 \leq i \leq n$, we aim to obtain

$$\tilde{\alpha}_i(t,0) = \sum_{k=1}^m\sum_{\ell=1}^n Q_{ik}R_{k\ell}\tilde{\alpha}_\ell(t - \frac{1}{\mu_k} - \frac{1}{\lambda_\ell}, 0), \quad (45)$$

so that $\tilde{\alpha}(\cdot,0)$ will exponentially converge to zero in virtue of Assumption 3 (see [6] for details). Since the matrix F^α is strictly lower triangular, we will recursively compute the different components of $\mathcal{O}_1$. Consider the first line of equation (34) expressed in the Laplace domain. Using equations (43)-(44), we obtain

$$\tilde{\alpha}_1(s,0) = ((Q\gamma_\beta(0) - \gamma_\alpha(0))\tilde{X}_1)_1 + (\mathcal{O}_1(\tilde{Y}))_1$$
$$+ \sum_{k=1}^m\sum_{\ell=1}^n Q_{1k}R_{k\ell}\mathrm{e}^{-\frac{s}{\mu_k} - \frac{s}{\lambda_\ell}}\tilde{\alpha}_\ell(s,0)$$
$$+ \int_0^1 \sum_{k=1}^m\sum_{\ell=1}^n F_{1k}^\beta(\nu)R_{k\ell}\mathrm{e}^{-\frac{s(1-\nu)}{\mu_k}}\mathrm{e}^{-\frac{s}{\lambda_\ell}}\tilde{\alpha}_\ell(s,0)d\nu,$$

which gives $(\mathcal{O}_1(\tilde{Y}))_1 = -((Q\gamma_\beta(0) - \gamma_\alpha(0))(sI - \tilde{A}_1)^{-1} E_1P_1^-(s)\tilde{Y}(s))_1 - \sum_{k=1}^m\sum_{\ell=1}^n\int_0^1 F_{1k}^\beta(\nu)R_{k\ell}\mathrm{e}^{-\frac{s(1-\nu)}{\mu_k}} \mathrm{e}^{-\frac{s}{\lambda_\ell}}d\nu(P_1^-(s)\tilde{Y}(s))_\ell$. By induction, let us consider $i > 1$ and assume that for any $j < i$, we have managed to design the j^{th} component of our observer operator $\mathcal{O}_1$ such that equation (45) holds. Consequently, using the triangular structure or F^α, we obtain for any $y \in [0,1]$

$$(F^\alpha(y)\tilde{\alpha}(t,y))_i = \sum_{j=1}^{i-1} F_{ij}^\alpha(y)\tilde{\alpha}_j(t,y) \quad (46)$$
$$= \sum_{j=1}^{i-1} F_{ij}^\alpha(y)\sum_{k=1}^m\sum_{\ell=1}^n Q_{jk}R_{k\ell}\tilde{\alpha}_\ell(t - \frac{1}{\mu_k} - \frac{y}{\lambda_j}, 1),$$

where the last equality holds since $j < i$. Consequently, the i^{th} line of equation (34) now reads as

$$\tilde{\alpha}_i(s,0) = ((Q\gamma_\beta(0) - \gamma_\alpha(0))\tilde{X}_1)_i + (\mathcal{O}_1(\tilde{Y}))_i$$
$$+ \sum_{k=1}^m\sum_{\ell=1}^n Q_{ik}R_{k\ell}\mathrm{e}^{-\frac{s}{\mu_k} - \frac{s}{\lambda_\ell}}\tilde{\alpha}_\ell(s,0)$$
$$+ \int_0^1 \sum_{k=1}^m\sum_{\ell=1}^n F_{ik}^\beta(\nu)R_{k\ell}\mathrm{e}^{-\frac{s(1-\nu)}{\mu_k}}\tilde{\alpha}_\ell(s,1)d\nu$$
$$+ \int_0^1 \sum_{j=1}^i F_{ij}^\alpha(\nu)\sum_{k=1}^m\sum_{\ell=1}^n Q_{jk}R_{k\ell}\mathrm{e}^{-\frac{s\nu}{\lambda_j}}\mathrm{e}^{-\frac{s}{\mu_k}}\tilde{\alpha}_\ell(s,1)d\nu.$$

To reach the desired target (as given by equation (45)) we choose

$$\begin{aligned}&(\mathcal{O}_1(\tilde{Y}))_i = -((Q\gamma_\beta(0) - \gamma_\alpha(0))(sI - \tilde{A}_1)^{-1}E_1 P_1^-(s)\\&\tilde{Y}(s))_i - \sum_{j=1}^{i}\sum_{k=1}^{m}\sum_{\ell=1}^{n}\int_0^1 F_{ij}^\alpha(\nu)Q_{jk}R_{k\ell}\mathrm{e}^{-\frac{s\nu}{\lambda_j}}\mathrm{e}^{-\frac{s}{\mu_k}}d\nu\\&(P_1^-(s)\tilde{Y}(s))_\ell - \sum_{k=1}^{m}\sum_{\ell=1}^{n}\int_0^1 F_{1k}^\beta(\nu)R_{k\ell}\mathrm{e}^{-\frac{s(1-\nu)}{\mu_k}}d\nu\\&\cdot(P_1^-(s)\tilde{Y}(s))_\ell.\end{aligned}\tag{47}$$

We can now write the following theorem

Theorem 1: Consider the operators $\mathcal{O}_0$, $\mathcal{O}_\alpha$, $\mathcal{O}_\beta$, $\mathcal{O}_1$, respectively defined by equations (40), (41), (42) and (47). Define the observer states $(\hat{X}_0, \hat{u}, \hat{v}, \hat{X}_1) = \mathcal{T}(\hat{\xi}, \hat{\alpha}, \hat{\beta}, \hat{X}_1)$, where the transformation $\mathcal{T}$ is defined by equation (12)-(14) and where $(\hat{\xi}, \hat{\alpha}, \hat{\beta}, \hat{X}_1)$ is the solution of the system (21)-(25). Then the state $(\hat{X}_0, \hat{u}, \hat{v}, \hat{X}_1)$ exponentially converges to (X_0, u, v, X_1) in the sense of the χ-norm.

Proof: With this choice of operators, we have already shown that $\tilde{X}_1$ and $\tilde{\xi}$ exponentially converges to zero. Moreover, the recursive design of $\mathcal{O}_1$ implies that for all $t > \frac{1}{\lambda_1} + \frac{1}{\mu_1}$, all $i \leq n$, $\tilde{\alpha}_i(t, 0)$ is solution of $\tilde{\alpha}_i(t, 0) = \sum_{k=1}^m \sum_{\ell=1}^n Q_{ik}R_{k\ell}\tilde{\alpha}_\ell(t - \frac{1}{\mu_k} - \frac{1}{\lambda_\ell}, 0) + \mathcal{O}(\tilde{\xi})$, where $\mathcal{O}$ is a linear bounded operator. Thus, $\tilde{\alpha}(t, 0)$ exponentially converges to zero (Assumption 3), which in turns imply the exponential convergence of $\tilde{\alpha}(t, 1)$ due to equation (43). Consequently, Lemma 1 implies that the state $(\tilde{\xi}_0, \tilde{\alpha}, \tilde{\beta}, \tilde{X}_1)$ exponentially converges to zero for the χ-norm. Using the invertibility and boundedness of the linear transformation $\mathcal{T}$, we can easily conclude the proof. ■

Using the linearity and invertibility of the different backstepping transformations, it is possible to express the observer system (28)-(32) in the original coordinates. This is omitted due to space restrictions.

Remark 1: In a similar way to what has been done in [26, Chapter 7] for the control and observer design in the $2 + 2$ case, it is possible (thanks to Assumption 3) to low-pass filter the measured output signal $Y(t)$ to use only strictly proper observer operators. Indeed, without filtering, the observer operators $\mathcal{O}_i$ may not be strictly proper (due to the use of the left inverses) and the observer system may consequently be sensitive to delays in the measurements.

Remark 2: The proposed observer could be combined with existing state-feedback laws [14], [17] to obtain output-feedback controllers. Note that we are currently extending the results of [10] to deal with the case of non-scalar hyperbolic systems using assumptions that are less restrictive than those existing in the literature.

IV. Stable left-inversion algorithm

In the construction of the observer, a stable left-inverse is required for the ODE system. In this section, we present a simple procedure to construct a stable left-inverse for $P_1(s)$. This procedure is given in order to give a complete, self-contained method in this paper and also to present an alternative, in many cases simpler, to the computation of a Hermite normal form of a matrix.

A. Preliminary Definitions

Recall that Assumption 2 guarantees that $P_1(s)$ is full-column rank for all $s \in \mathbb{C}^+$. Furthermore, taking $\det(sI_q - \tilde{A}_1)$ as a common denominator (of degree q, with all its roots in the complex open left half-plane) we can factor $P_1(s)$ as $P_1(s) \doteq \frac{1}{\det(sI_q - \tilde{A}_1)}P_1^{\text{num}}(s)$, where $P_1^{\text{num}}(s)$ has real polynomial entries of degree at most $q-1$ (it is a $d\times n$ matrix over a Principal Ideal Domain, which is in fact an Euclidean Domain, see e.g. [3]). The full-column rank property for $s \in \mathbb{C}^+$ also applies to $P_1^{\text{num}}(s)$.

Given a list of real polynomials $\mathcal{P} = (l_1(s), l_2(s), ...l_j(s))$, we will denote by $\gcd(l_1, l_2, ...l_j)$ the polynomial greatest common divisor of the elements of $\mathcal{P}$, and by $(a_1(s), a_2(s), ..., a_j(s)) \doteq \text{bezout}(l_1, l_2, ...l_j)$ a corresponding list of real polynomial coefficients such that $\sum_{k=1}^j a_k(s)l_k(s) = \gcd(l_1, l_2, ...l_j)$.

In order to construct a left-inverse for $P_1^{\text{num}}(s)$ we will first transform it into an upper-triangular form. One possible upper-triangular form is the Hermite normal form, see for instance Theorem 2.9, in [3, Ch. 5]. However, in practice we do not require the uniqueness provided by this normal form, and it might be simpler to find a different upper-triangular form with the right properties. We provide a simple method that allows for one such construction. We begin defining some matrices that we will use to operate on the rows of $P_1^{\text{num}}(s)$ to construct the desired upper-triangular form. The first matrix, a $d \times d$ upper-triangular matrix with *real polynomial* entries (of degree at most $q-1$) will allow us to replace the i-th row of a matrix by a combination of that row and the following ones and allow us to place a "pivot" element in the diagonal of the transformed matrix:

$$T_p^i[c_i, ..., c_d](s) \doteq \begin{bmatrix} I_{i-1} & 0_{i-1,d-i+1} \\ 0_{d-i+1,i-1} & U[c_i, ..., c_d](s) \end{bmatrix} \tag{48}$$

where $U[c_i, ..., c_d](s)$ is *any* $d-i+1 \times d-i+1$ polynomial matrix with full rank for $s \in \mathbb{C}^+$ and having as first row the polynomials $[c_i, ..., c_d]$. Note that a *particular* choice for this matrix would be the unimodular (invertible) matrix used to construct the Hermite normal form (see for instance the matrix U_1 in the inductive proof of Theorem 2.9 in [3, Ch. 5]), it is also worth mentioning that, unlike the construction of the Hermite normal form, we do not require the elements above the diagonal to belong to a set of residues modulo the element on the diagonal. We believe that for the application considered in this paper, this formulation simplifies the necessary computations since, in many cases, one can simply complete the first line with $d - i$ adequately chosen rows of the identity I_{d-i+1}, as long as one avoids rank deficiencies in $\mathbb{C}^+$ (trivial if at least one of the c_k polynomials has no roots in $\mathbb{C}^+$). A second, $d \times d$ lower-triangular matrix with with *real polynomial* entries, $T_l^i[p_{i+1}, ..., p_d](s)$, will allow us eliminate the elements under the previously constructed "pivot". It is constructed by replacing all the elements below the diagonal on the i'th column of a $d \times d$ identity matrix, by the column of polynomials $[p_{i+1}, ..., p_d]^T$.

B. Construction of a stable left-inverse

A stable left-inverse of $P_1(s)$ can then be found by a method similar to Gaussian elimination detailed in Algorithm 1. Let us remark that a completely analogous algorithm can be used to find a stable right-inverse in the control case (simply acting on the columns of the transfer matrix, instead of the rows, or transposing the system).

If the reader already knows $P(s)$, the Hermite normal form of $P_1^{\text{num}}(s)$ and associated unimodular matrix $T(s)$ such that

$P(s) = T(s)P_1^{\text{num}}(s)$, they can skip directly to step 11 of the algorithm. Remark that we do not require the classical condition of the Hermite normal form of having the elements above the diagonal (in the full column rank case) belonging to a complete set of residues modulo the elements of the diagonal (see [3, Ch. 5]), which also simplifies the procedure.

Algorithm 1

1: $P(s) \leftarrow P_1^{\text{num}}(s)$
2: $T(s) \leftarrow I_d$
3: **for** $i = [1, 2, ..., n]$ **do**
4: $\quad (c_i, ..., c_d) \leftarrow \text{bezout}(P_{i:d,i}(s))$
5: $\quad P(s) \leftarrow T_p^i[c_i, ..., c_d](s)P(s)$
6: $\quad T(s) \leftarrow T_p^i[c_i, ..., c_d](s)T(s)$
7: $\quad (p_{i+1}, ..., p_d) = -P_{i+1:d,i}(s)/P_{i,i}(s)$
8: $\quad P(s) \leftarrow T_l^i[p_{i+1}, ..., p_d](s)P(s)$
9: $\quad T(s) \leftarrow T_l^i[p_{i+1}, ..., p_d](s)T(s)$
10: **end for** $\triangleright$ At the end of this loop, we obtain an upper triangular polynomial matrix $P(s)$ with Hurwitz polynomials on the diagonal and zeros below the diagonal.
11: $P(s) \leftarrow [I_n \quad 0_{n,d-n}]\, P(s)$ $\triangleright$ We extract the first n rows of the matrix $P(s)$, which are full rank in $\mathbb{C}^+$.
12: $T(s) \leftarrow P^{-1}(s)\, [I_n \quad 0_{n,d-n}]\, T(s)$ $\triangleright$ P, at this step a square, triangular matrix with Hurwitz polynomial entries in the diagonal, has a trivial stable inverse, and $T(s)$ is therefore a stable, left inverse of $P_1^{\text{num}}(s)$
13: $P_1^-(s) \leftarrow \det(sI_q - \tilde{A}_1)T(s)$ $\triangleright$ $P_1^-(s)$ now contains a stable, left-inverse of $P_1(s)$

V. Concluding remarks

In this paper, we have designed a dynamic observer for a class of $n + m$ linear hyperbolic ODE-PDE-ODE systems. The proposed approach combines backstepping transformations and frequency-domain design methods. The resulting observer requires the computation of a stable left-inverse, for which we have proposed a *simple* constructive procedure that can be easier to construct than a Hermite normal form. This observer can be combined with existing state-feedback control laws to obtain output-feedback controllers. In future works we will consider networks of ODE and PDEs with a more complex structure (star-shaped networks).

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