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APPLICATION OF VARIATIONAL CONCEPTS TO SIZE EFFECTS IN ELASTIC HETEROGENEOUS BODIES

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HETEROGENEOUS materials that are not statistically uniform are considered. The basic assumptions and conditions underlying the concept of effective properties are reviewed. A general procedure for the experimental or numerical evaluation of the overall properties of the material when the concept of a representative volume does not apply or cannot be used is proposed and then applied to the elastic case as a first example. Three kinds of results are obtained: relationships between experimental results obtained on a large specimen and on an appropriate set of smaller ones, hierarchies between families of specimens of various sizes, absolute bounds for the two limiting cases where the size goes to zero, or to the representative volume when it exists. Possible applications and extensions are mentioned.

NOMENCLATURE

D_0	domain occupied in space by a material body
∂D_0	boundary of D_0
Γ	internal interface of a heterogeneous body
$\Gamma \to \Gamma^+$	in one medium, with outward normal n^+ : $a = a^+$ on Γ^+
$\Gamma \to \Gamma^-$	in one medium, with outward normal n^- : $a = a^-$ on Γ^-
$[a]^+_- = a^+ - a^-$	jump bracket of a
ρ	mass density
x	position of a material point at time t
v	velocity of a material point
σ	stress tensor
ε	strain tensor
d	strain-rate tensor
u	internal energy per unit mass
s	entropy per unit mass
q	heat current vector
T	absolute temperature
g	gradient of the reciprocal temperature
$f = \rho b$	external force density per unit volume
r	non-mechanical external energy supply per unit volume
b	external force density per unit mass
ξ	displacement vector

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P	traction vector density on a surface
$E[a] \Leftrightarrow \bar{a}$	mathematical expectation or ensemble average or stochastic average of the variable a
$\langle a \rangle$	spatial average of the variable a on a domain D
$\dot{a}$	material time derivative
a'	local fluctuation $a - \langle a \rangle$ of a around its volume average
$\times$	tensor product (dyadic)
$\cdot$	once contracted tensor product
$:$	twice contracted tensor product
a^T	transpose of the tensor a
sym a	symmetric part $\frac{1}{2}(a + a^T)$ of the tensor a
$F_{\tilde{\varepsilon}}, F_{\tilde{\sigma}}$	potential energy and complementary energy functionals, respectively

1. Introduction

Since the pioneering works by Hill (1952, 1963), Kröner (1953, 1958), Paul (1960), Hashin and Shtrikman (1963), Beran (1968) and Walpole (1966), the theory of random heterogeneous materials has experienced a great deal of development that results nowadays in the existence of a large set of sophisticated and thoroughly elaborated theories. For instance, when confining oneself to the techniques providing bounds for the overall properties, one may quote the systematic theory of Kröner (1972, 1977, 1981, 1986), and the recent developments of the method of stress polarizations by Willis (1981a, b, 1986) and Willis and Talbot (1989).

The main objective of these theories is to evaluate the so-called "effective properties" of the heterogeneous material from the knowledge upon its constitution which is available. These effective properties are expressed by material parameters involved in the relationship between appropriate averages of the field variables, say, to be concrete, of the stress tensor and of the strain tensor for the case of the elastic properties of solids. These relationships are then used as local constitutive equations in structural calculations, as if the material were a homogeneous one.

As extensively detailed in the classical literature on the subject, and, in particular, in most of the papers quoted above, the validity of this procedure is subordinated to the satisfaction of several conditions. Some of these conditions relate to the constitution of the material. Others are related to the loadings and other features of the whole particular problem under consideration, see for instance Kröner (1972, 1986).

Another condition is that each realization of the heterogeneous material into a given body must be statistically uniform. This means first that there must exist a "representative volume", i.e. a volume element large enough in comparison with the size of the coarser heterogeneities, and small enough in comparison with the structural element to be designed or checked for engineering purposes, the properties of which, defined as relationships between volume averages of the physical variables, can be taken as the overall properties of the material. Second, the relationship between the averaged variables must be independent of the position of this volume element. Third, all the statistical information about the material is supposed to be available in one single realization of the representative volume, this being called the ergodicity assumption.

When all these conditions are satisfied, it is possible, through more or less complicated calculations depending on the requested degree of accuracy and on the amount of available information about the internal constitution of the material, to make more or less accurate predictions about the range in which will lie the effective properties of the material. This is for the theoretical side of the problem.

From the experimental viewpoint, it is generally admitted that the concept of effective properties gives very good results for heterogeneous materials for which the microstructure is very fine by comparison with the dimensions of the engineering structural elements. For instance, for metals used in construction, the polycrystalline microstructure can be seen only through the use of a microscope. For this reason, the experimentally determined effective properties are generally applied with confidence to structural calculations.

But there are many fields in engineering for which the situation is completely different. For instance in structures made of cement concrete, the size of the coarser heterogeneities is frequently of the same order of magnitude as the smallest size, for instance the thickness, of the structural element: typically a few centimeters. Even for rather large specimens of ordinary concrete (16 cm in diameter, about 20 cm edge), the number of individual coarse grains along a diameter can be comprised of between 5 and 10 only. This results in a large amount of scatter in the experimental results showing that the ergodicity assumption generally made in the theories mentioned above is far from being adequate to these situations.

Many important examples do exist for which the other assumptions lying at the base of the theory of heterogeneous media are also inadequate. For instance, the larger grains in a dam concrete may have dimensions larger than 10 cm. If one would like to reach the limited objective of maintaining the same ratio between the grain and specimen sizes as in ordinary concrete, one would have to perform tests on specimens several square-meters in cross-section. This would require a testing machine with a loading capacity of several hundred thousands of kiloNewtons. There is no possibility to have available a testing machine of this capacity.

In fact, for this case, the tests are performed on ordinary specimens, or on specimens with dimensions a little bit larger. For instance, the loading capacity of the machines that exist in laboratories can lead in this case to perform the tests on cubic specimens with 30 cm edges at most.

Many other examples exist in various fields of civil engineering and in various industries. It is the case for instance for wood structural elements, mainly because wood is a curved laminated material with imperfect conical orthotropy and localized defects. It is also the case in road materials, soil mechanics, rock mechanics, geotechnics, geotectonics and so on.

Thus the question arises: how is it possible to provide scientific bases to the use of the effective properties concept in the safety and service design of such structures?

Of course one possible attitude could be to state that such kinds of problems of engineering fields have not to be dignified with a scientific approach, and that it is enough to leave them in their traditional empirical status.

Another possible attitude, in fact our preferred one, would be to examine, in the light of the available theoretical results, if there might be some way to provide some help and guide-lines to the people confronted daily with this kind of problem. For instance, this could be done by providing them with a better way of understanding, and a better way of interpreting and using, the effective properties concept for their case, and the results of the tests through which they determine, or try to determine, these effective properties.

In the present paper, as a first step along the way to such an objective, we especially consider this problem in reviewing the basic assumption used in the classical theory of heterogeneous media, and seeing what can be kept from the theory, or from the corresponding methods, when relaxing some of the assumptions that are not adequate to the kind of problems mentioned above.

Then we propose a general procedure that, as a starting point, we apply to the case of linear elasticity. From this procedure, we obtain for this case three kinds of results:

(i) relationships between results obtained on a large specimen and on an appropriate set of smaller ones;
(ii) hierarchies between families of specimens (or pieces) of various sizes;
(iii) absolute bounds for the two limiting cases where the size goes to zero, or to the representative volume when it exists.

Possibilities of extensions to other kinds of overall properties are examined in the conclusion.

The main notations used are given in the Nomenclature.

2. The Classical Concepts of Effective Modulus and Compliance Tensors: Mechanical Definitions

In order to see what can be changed and what must be kept from the classical situations, we first carefully restate here and in the following section the basic concepts relating to the effective modulus and compliance tensors.

Since we address here the point of making use of the effective properties in design calculations through the information provided by the experiment, we need only the most elementary aspects of the theory of heterogeneous media.

When the material is statistically uniform and linearly elastic, one defines classically the effective modulus tensor C^{eff} through the relationship between the averaged stress tensor $\langle\sigma\rangle$ and the averaged strain tensor $\langle\varepsilon\rangle$:

$$\langle\sigma\rangle = C^{\text{eff}} : \langle\varepsilon\rangle, \tag{2.1}$$

where $a:b$ denotes the twice contracted tensor product of a by b, and $\langle a\rangle$ the volume average of the variable a.

In (2.1), the averagings are performed on specimens with size equal to or larger than the representative volume. In addition, the specimens are supposed to be in loading conditions that would give homogeneous fields if the material were homogeneous. Then, by definition, the effective modulus tensor C^{eff} is supposed to be independent of the boundary conditions. For this reason, the consideration of particular boundary conditions leading to simple calculations is supposed to be sufficient for the determination of C^{eff}.

Conversely, stating (2.1) in the reverse direction, the effective compliance S^{eff} is defined through

$$\langle \varepsilon \rangle = S^{\text{eff}} : \langle \sigma \rangle. \tag{2.2}$$

Comparison with (2.1) yields

$$S^{\text{eff}} = (C^{\text{eff}})^{-1}, \tag{2.3}$$

meaning

$$S^{\text{eff}} : C^{\text{eff}} = C^{\text{eff}} : S^{\text{eff}} = I^4, \tag{2.4}$$

where I^4 is the fourth-rank unity tensor, with component I^4_{ijkl} defined by

$$I^4_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2.5}$$

with δ the unit tensor of 2nd rank (Kronecker symbol).

Giving to σ inside the brackets of (2.1) its local value

$$\sigma = C : \varepsilon \tag{2.6}$$

yields

$$C^{\text{eff}} : \langle \varepsilon \rangle = \langle C : \varepsilon \rangle. \tag{2.7}$$

Equation (2.1) or (2.7) can be considered as the mechanical definition of the effective modulus C^{eff}.

The RHS of (2.7) can be written in the form

$$\langle C : \varepsilon \rangle = \langle C \rangle : \langle \varepsilon \rangle + \langle C' : \varepsilon' \rangle, \tag{2.8}$$

where a' defines the fluctuation $a - \langle a \rangle$ of the variable a around its average $\langle a \rangle$.

From (2.8) it is seen that, except in very particular cases, C^{eff} differs from the average $\langle C \rangle$ of the moduli.

In fact, a celebrated result of HILL (1952, 1963), that was the starting point for a wide branch of the theory of heterogeneous media, was that $\langle C \rangle$ is an upper bound for C^{eff}, whatever the internal constitution of the elastic heterogeneous material may be. In fact, $\langle C \rangle$ is the approximate solution proposed by VOIGT as early as (1887). For this reason, it is often named the Voigt upper bound or, preferably, the Hill–Voigt upper bound for the modulus tensor.

Similarly, from (2.2) one has for the compliance

$$S^{\text{eff}} : \langle \sigma \rangle = \langle S : \sigma \rangle = \langle S \rangle : \langle \sigma \rangle + \langle S' : \sigma' \rangle \tag{2.9}$$

showing that, in general, S^{eff} differs from the average $\langle S \rangle$ of the compliance tensor. A second celebrated result of HILL (1952, 1963), is that $\langle S \rangle$ stands as an upper bound for the effective compliance S^{eff} whatever the internal constitution of the material may be.

Passing to the reciprocals yields that $\langle S \rangle^{-1}$ is a lower bound for C^{eff}. In fact, $\langle S \rangle^{-1}$ is often named the REUSS (1929) lower bound, or preferably the Hill–Reuss lower bound, for the modulus tensor.

But defining bounds means that one is able to state order relationships between two fourth-rank tensors A and B. This implies that some quadratic form with kernel the difference between these two tensors can be found to be of definite sign. For instance, we shall state that $B < A$ if

$$(A-B):a:a > 0 \quad \forall\, a \neq 0. \tag{2.10}$$

For elastic materials this can be stated only through energetic considerations. Thus one needs energetic counterparts to the mechanical definitions, (2.1) and (2.2), of the effective modulus and compliance tensors. These energetic counterparts are essential to the derivation of all kinds of bounds of the classical theory. We will see later that they are also essential for the derivation of the new results we are presenting in this paper.

3. The Energetic Definition of the Effective Properties and the Hill Condition

For a homogeneous medium, energetic considerations are, as is well known, of prime importance. For instance, in the elastic case, they provide us with symmetry properties of the isothermal modulus and compliance tensors, resulting from the existence of an elastic potential, the free energy, or the free enthalpy, densities. They provide also properties of positive definiteness that are very useful to provide relationships between components of these tensors, uniqueness of boundary value problems, and evaluation of approximate solutions.

Thus, the possibility of extending these concepts to the effective properties is an essential step to be performed if one is wanting, as is the case for the engineer, to ignore completely the heterogeneous character of the material.

It is well known that, for a homogeneous material, the free energy density is defined by

$$\varphi = \tfrac{1}{2}C:\varepsilon:\varepsilon \tag{3.1}$$

or, equivalently here, by the local Clapeyron equation

$$\varphi = \tfrac{1}{2}\sigma:\varepsilon. \tag{3.2}$$

Since the design engineer knows only averages and effective properties, he will make use of an averaged form of (3.2), involving only, in the form corresponding to (3.1), the averages of the free energy density φ and of the strain ε:

$$\langle\varphi\rangle = \tfrac{1}{2}\langle\sigma:\varepsilon\rangle \tag{3.3}$$

and

$$\langle\varphi\rangle = \tfrac{1}{2}C_e^{\mathrm{eff}}:\langle\varepsilon\rangle:\langle\varepsilon\rangle, \tag{3.4}$$

where C_e^{eff} stands, by analogy with (3.1) and (3.2), for an energetic definition of the effective modulus.

But (3.3) can be written

$$\langle\varphi\rangle = \tfrac{1}{2}\langle\sigma\rangle:\langle\varepsilon\rangle+\tfrac{1}{2}\langle\sigma':\varepsilon'\rangle = \tfrac{1}{2}C^{\mathrm{eff}}:\langle\varepsilon\rangle:\langle\varepsilon\rangle+\tfrac{1}{2}\langle\sigma':\varepsilon'\rangle, \tag{3.5}$$

where C^{eff} is the effective modulus tensor given by the mechanical definition, (2.1).

Substracting (3.4) from (3.6) yields

$$(C^{\text{eff}} - C_{\text{e}}^{\text{eff}}) : \langle \varepsilon \rangle : \langle \varepsilon \rangle = \langle \sigma' : \varepsilon' \rangle, \tag{3.6}$$

showing that the energetically defined effective modulus $C_{\text{e}}^{\text{eff}}$ is compatible with its mechanical counterpart C^{eff} iff

$$\langle \sigma' : \varepsilon' \rangle = 0 \tag{3.7}$$

or equivalently, from (3.3) and (3.6)

$$\langle \sigma : \varepsilon \rangle = \langle \sigma \rangle : \langle \varepsilon \rangle, \tag{3.8}$$

meaning that the average of the product equals the product of the averages.

This relation was for the first time derived by HILL (1963), and for this reason named the Hill condition by KRÖNER (1972).

Together with the concepts of statistical uniformity and of representative volume, the Hill condition is an essential ingredient of the theory of heterogeneous media in its various developments.

HUET (1981, 1982) derived universal conditions for the use of the concept of effective properties that are the generalization of the Hill condition to any kind of thermomechanical problem, irrespective of what the properties of the material and of its constituents might be. They have been obtained through the consideration of local ensemble averages, like in the work of BERAN (1968) for instance, applied to the universal set of balance equations of thermomechanics: conservation of mass, momentum, energy and balance of entropy.

The set of corresponding conditions is given in Tables 1 and 2 in two forms. The first form corresponds to necessary and sufficient conditions (Table 1), the second form to sufficient conditions only, easier to check in concrete situations (Table 2).

TABLE 1. *Assimilation conditions to a general effective continuum*

$\operatorname{div} \overline{\rho' v'} = 0$	(T1.1)
$\bar{\rho}\, \overline{v' \cdot \operatorname{grad} v'} + \overline{\rho' \dot{v}'} = \overline{\rho' b'}$	(T1.2)
$\bar{\rho}\, \overline{v' \cdot \operatorname{grad} u'} + \overline{\rho' \dot{u}'} = \overline{\sigma' : d'}$	(T1.3)
$\bar{\rho}\, \overline{v' \cdot \operatorname{grad} s'} + \overline{\rho' \dot{s}'} = \overline{(1/T)' r'} - \operatorname{div} [\overline{(1/T)' q'}]$	(T1.4)

TABLE 2. *Whole set of sufficient assimilation conditions to a general effective continuum*

$\overline{\rho' v'} = \overline{\rho' \dot{v}'} = \overline{\rho' b'} = \overline{\rho' \dot{u}'} = \overline{\rho' \dot{s}'} = 0$	(T2.1)
$\overline{v' \cdot \operatorname{grad} v'} = \overline{v' \cdot \operatorname{grad} u'} = \overline{v' \cdot \operatorname{grad} s'} = 0$	(T2.2)
$\overline{\sigma' : d'} = \overline{(1/T)' r'} = \overline{(1/T)' q'} = 0$	(T2.3)

In order to give sense to the concept of effective properties, it is necessary and sufficient that all the relationships of Table 1 be fulfilled by the characteristics of the problem and sufficient that all the relationships of Table 2 be fulfilled. Once this is granted, the corresponding information can be used in subsequent derivations.

It is easier to make use of these relationships in the reverse direction: as soon as one of these relationships of Table 1 is violated, it makes no sense to make use of the concepts of equivalent homogeneous medium and of associated effective properties, and the problem must be handled in another way. In some cases, the consideration of various phases, for each of which a complete set of coupled balance equations has to be written, will be sufficient to handle the problem. This yields to the so-called multiphasic flow theory, see for instance Ishii (1975).

In the case of isothermal elastic solids under quasi-static loading with negligible volume forces, the whole set of universal conditions reduces, through time integration and use of the ergodicity assumption, to the classical Hill condition only.

But it is important to be aware of the general conditions, since they are the key for obtaining generalizations to the first results we are presenting in this paper. A further presentation of these conditions and their consequences is presented in Huet (1988).

4. General Relationships Between Volume Averages and Boundary Conditions and the Case of Uniform Boundary Conditions

We denote by x the material point coordinates vector in a given fixed frame of reference.

By applying the Gauss theorem, in the form of the gradient theorem, to the strain tensor field $\varepsilon(x)$, inside a domain D with volume V, external boundary ∂D, and interface Γ between heterogeneities, one has, see for instance Huet (1981, 1984), for the volume average of the strain, the relationship:

$$\langle \varepsilon \rangle = \frac{1}{V}\int_D \text{sym grad}\,\xi\,\mathrm{d}V = \frac{1}{V}\int_{\partial D} \text{sym}\,(\xi \times n)\,\mathrm{d}\Sigma + \frac{1}{V}\int_\Gamma \text{sym}\,[\xi]^+_- \times n^+\,\mathrm{d}\Gamma, \tag{4.1}$$

where ξ is the displacement vector, "grad" the gradient operator, "sym" the operator giving the symmetrical part of a tensor (one half the sum of a tensor and its transpose), $\times$ the tensor product, dyadic), $\mathrm{d}\Sigma$ the area element of ∂D, and $\mathrm{d}\Gamma$ the area element of Γ. The symbol $[a]^+_-$ denotes the discontinuity bracket of the variable a defined by

$$[a]^+_- = a^+ - a^-, \tag{4.2}$$

where a^+ is the value taken by a at the point M on Γ considered as belonging to the material with unit normal n^+, and a^- the values taken by a at the same point considered as belonging to the other, adjacent, material.

One sees that, when the displacement ξ is and remains continuous on Γ (perfect interface, with no slips nor cracks), the strain average is completely determined by the knowledge of ξ at every point of the external boundary, giving

$$\langle \varepsilon \rangle = \frac{1}{V} \int_{\partial D} \text{sym}\,(\xi \times n)\, \mathrm{d}\Sigma. \tag{4.3}$$

By applying the same theorem, in the form of the divergence theorem applied to the product $(x \times \sigma)$, and by taking account of the conservation equation for the momentum, one has also, for the volume average of the stress, the relationship

$$\langle \sigma \rangle = \frac{1}{V} \int_{\partial D} \text{sym}\,(x \times P)\, \mathrm{d}\Sigma + \frac{1}{V} \int_{D} x \times (f - \rho \dot{v})\, \mathrm{d}V$$
$$+ \frac{1}{V} \int_{\Gamma} \text{sym}\,(x \times [\sigma \cdot n^{+}]_{-}^{+})\, \mathrm{d}\Gamma, \tag{4.4}$$

where P is the (prescribed or not) boundary traction vector defined by

$$P = \sigma \cdot n \tag{4.5}$$

and f the local external force density ρb per unit volume, b being the one per unit mass.

For impervious or perfect interfaces, the continuity of the stress vector $\sigma \cdot n^{+}$ across the interface Γ is granted. Thus, for negligible volume forces and accelerations, the volume average of the stress tensor is completely determined by the knowledge of the traction vector P at every point of the external boundary, through

$$\langle \sigma \rangle = \frac{1}{V} \int_{\partial D} \text{sym}\,(x \times P)\, \mathrm{d}\Sigma. \tag{4.6}$$

This is particularly adapted to the problem of experimental testing in the laboratory, where, in general, only the boundary conditions can be imposed or observed.

For (twice) the strain energy, one has in a similar fashion

$$\langle \sigma : \varepsilon \rangle = \frac{1}{V} \int_{\partial D} (\sigma \cdot n) \cdot \xi\, \mathrm{d}\Sigma$$
$$+ \frac{1}{V} \int_{D} (f - \rho \dot{v}) \cdot \xi\, \mathrm{d}V$$
$$+ \frac{1}{V} \int_{\Gamma} [(\sigma \cdot n^{+}) \cdot \xi]_{-}^{+}\, \mathrm{d}\Gamma \tag{4.7}$$

which, when the two simplifying conditions above are fulfilled, reduces to

$$\langle \sigma : \varepsilon \rangle = \frac{1}{V} \int_{\partial D} (\sigma \cdot n) \cdot \xi\, \mathrm{d}\Sigma, \tag{4.8}$$

which is nothing else, for this particular case, than the principle of virtual work applied to the real solution fields.

As is well known in the classical literature, see for instance HILL (1963), there exist two kinds of particular boundary conditions that lead to drastic simplifications in the calculations and for which the Hill condition is satisfied. These are the uniform boundary conditions in the two, kinematic and static, following senses.

The first one corresponds to prescribed displacements ξ^{d} applied to the whole boundary ∂D of the body D, and of the form

$$\xi^{\mathrm{d}} = \varepsilon_0 \cdot x \quad \forall M(x) \quad \text{on} \quad \partial D, \tag{4.9}$$

where ε_0 is a given symmetric nondimensional tensor of the second rank with value in the order of magnitude encountered for the small strains of solids. For the sake of brevity, we name this case the case of *kinematic uniform boundary conditions of the kind* ε_0, or, in abridged notation, the case of ε_0-KUBC.

The second case corresponds to a prescribed traction vector surface density P^{d}, applied to the whole boundary ∂D of the body D and of the form

$$P^{\mathrm{d}} = \sigma_0 \cdot n \quad \forall M(x) \quad \text{on} \quad \partial D, \tag{4.10}$$

where σ_0 is a given symmetric tensor with dimension of stress and with values taken in the order of magnitude compatible with the elastic domain of the material. We name this case the case of *static uniform boundary conditions of the kind* σ_0 or, in abridged notations, the case of σ_0-SUBC.

As can be verified easily from replacing, in (4.3), ξ by its given value, (4.9), and, in (4.6), P by its given value (4.10), through application of the Gauss theorem in the reverse way, one gains the two well-known results that

$$\langle \varepsilon \rangle = \varepsilon_0 ; \quad \varepsilon_0\text{-KUBC} \tag{4.11}$$

and

$$\langle \sigma \rangle = \sigma_0 ; \quad \sigma_0\text{-KUBC.} \tag{4.12}$$

Moreover, in the same fashion, one sees from (4.8) that:

$$\langle \sigma : \varepsilon \rangle = \langle \sigma \rangle : \varepsilon_0 ; \quad \varepsilon_0\text{-KUBC} \tag{4.13}$$

and

$$\langle \sigma : \varepsilon \rangle = \sigma_0 : \langle \varepsilon \rangle ; \quad \sigma_0\text{-SUBC.} \tag{4.14}$$

Thus it turns out, by comparing (4.11) and (4.13) on one hand, (4.12) and (4.14) on the other hand, that, as extensively reported in the literature, for these two kinds of uniform boundary conditions, the Hill condition, (3.9), is trivially fulfilled.

This very important, although classical, result, is used extensively in the following parts of this paper.

Note that this fulfilment of the Hill condition, and the properties expressed by (4.11)–(4.14) are obtained without any statistical assumption like statistical uniformity nor ergodicity.

Since these statistical assumptions fail in the cases we are considering here, this particular feature will be specially useful for our purpose. In fact, this will allow us to make extensive use of the information that remains, for these boundary conditions, after having relaxed these statistical assumptions, and which is the information contained in (4.11) and (4.14).

Since, as is well known, these particular kinds of boundary conditions give homogeneous stress and strain fields in an homogeneous solid body, this information is specially adapted to the analysis of experimental testing in the laboratory, for which looking for homogeneous fields is most often taken as a goal. In fact, it is the case for which the inversion problem, a nontrivial one especially for nonlinear behaviors, that leads to the calculation of the parameters defining the properties of the material from the results of experiment, can be solved with the least difficulty.

5. Looking for Relationships Between Averages in the General Case: The Concept of Apparent Overall Properties

For any given boundary value problem with uniform boundary conditions of the kinds defined above, we can always, at least in principle, find the solution and calculate from it the average strain $\langle \varepsilon \rangle$ and the average strain $\langle \sigma \rangle$. For a linear problem it makes sense to relate $\langle \sigma \rangle$ to $\langle \varepsilon \rangle$ through a linear operator.

For instance, in ε_0-KUBC, this means that, when ε_0 is modified, $\langle \sigma \rangle$ is modified accordingly, but, because of the linearity of the problem, the relationship between $\langle \sigma \rangle$ and ε_0 remains invariant. This can be more precisely justified through the use of the modified Green functions, in the sense of Kröner (1972, 1986), of the given boundary value problem.

Thus, one can state that, when no physical nor chemical dimensional changes are involved, this relationship is of the form

$$\langle \sigma \rangle_\alpha = C^{\text{app}}_{\varepsilon\alpha} : \langle \varepsilon \rangle_\alpha = C^{\text{app}}_{\varepsilon\alpha} : \varepsilon_0 ; \quad \varepsilon_0\text{-KUBC} \tag{5.1}$$

for each body, or specimen, D_α, belonging to a given set $\{D_\beta | \beta = 0 \text{ to } n\}$ of bodies. In (5.1), $\langle a \rangle_\alpha$ denotes the volume average over D_α.

Equation (5.1) defines what we call the *kinematic apparent modulus tensor* $C^{\text{app}}_{\varepsilon\alpha}$ of the body D_α in kinematic uniform boundary conditions of the kind ε_0 or, briefly, in ε_0-KUBC.

Conversely, we can define the *kinematic apparent compliance tensor* $S^{\text{app}}_{\varepsilon\alpha}$ of the body D_α in ε_0-KUBC by

$$\varepsilon_0 = \langle \varepsilon \rangle = S^{\text{app}}_{\varepsilon\alpha} : \langle \sigma \rangle_\alpha ; \quad \varepsilon_0\text{-KUBC}. \tag{5.2}$$

From (5.1), we have

$$\varepsilon_0 = \langle \varepsilon \rangle_\alpha = (C^{\text{app}}_{\varepsilon\alpha})^{-1} : \langle \sigma \rangle_\alpha . \tag{5.3}$$

This yields

$$S^{\text{app}}_{\varepsilon\alpha} = (C^{\text{app}}_{\varepsilon\alpha})^{-1} . \tag{5.4}$$

Note that, when the statistical uniformity is realized, and when D_α has dimensions equal to or larger than the ones of the representative volume, these kinematic apparent compliance and modulus coincide with the effective ones.

But in other cases, they differ. In particular, since the linear relationship of the form (5.1) depends on the problem, i.e. upon the boundary conditions, the case of the static uniform boundary conditions in the sense defined in the preceding sections needs the introduction of another relationship between $\langle\sigma\rangle$ and $\langle\varepsilon\rangle$.

In σ_0-SUBC, we shall define first the *static apparent compliance tensor* $S^{\text{app}}_{\sigma\alpha}$ of the body D_α through

$$\langle\varepsilon\rangle_\alpha = S^{\text{app}}_{\sigma\alpha} : \langle\sigma\rangle_\alpha = S^{\text{app}}_{\sigma\alpha} : \sigma_0 ; \quad \sigma_0\text{-SUBC}. \tag{5.5}$$

Then we define the *static apparent modulus tensor* $C^{\text{app}}_{\sigma\alpha}$ of D_α by taking the reverse, leading to

$$\sigma_0 = \langle\sigma\rangle_\alpha = C^{\text{app}}_{\sigma\alpha} : \langle\varepsilon\rangle_\alpha \tag{5.6}$$

and thus

$$C^{\text{app}}_{\sigma\alpha} = (S^{\text{app}}_{\sigma\alpha})^{-1}. \tag{5.7}$$

Since we are in uniform boundary conditions, (4.13) and (4.14) of the preceding section apply. This means that the Hill condition is satisfied, and that $C^{\text{app}}_{\varepsilon\alpha}$, $C^{\text{app}}_{\sigma\alpha}$, $S^{\text{app}}_{\varepsilon\alpha}$, $S^{\text{app}}_{\sigma\alpha}$ thus defined are all compatible with their corresponding energetic definitions, meaning

$$\begin{aligned}\langle\sigma : \varepsilon\rangle_\alpha &= C^{\text{app}}_{\varepsilon\alpha} : \langle\varepsilon\rangle_\alpha : \langle\varepsilon\rangle_\alpha \\ &= C^{\text{app}}_{\varepsilon\alpha} : \varepsilon_0 : \varepsilon_0 \\ &= S^{\text{app}}_{\varepsilon\alpha} : \langle\sigma\rangle_\alpha : \langle\sigma\rangle_\alpha \\ &= \langle C : \varepsilon : \varepsilon\rangle_\alpha \\ &= \langle S : \sigma : \sigma\rangle_\alpha ; \quad \varepsilon_0\text{-KUBC},\end{aligned} \tag{5.8}$$

$$\begin{aligned}\langle\sigma : \varepsilon\rangle_\alpha &= S^{\text{app}}_{\sigma\alpha} : \langle\sigma\rangle_\alpha : \langle\sigma\rangle_\alpha \\ &= S^{\text{app}}_{\sigma\alpha} : \sigma_0 : \sigma_0 \\ &= C^{\text{app}}_{\sigma\alpha} : \langle\varepsilon\rangle_\alpha : \langle\varepsilon\rangle_\alpha \\ &= \langle S : \sigma : \sigma\rangle_\alpha \\ &= \langle C : \varepsilon : \varepsilon\rangle_\alpha ; \quad \sigma_0\text{-SUBC}.\end{aligned} \tag{5.9}$$

In the following sections, we turn to the evaluation problem.

6. Evaluating Effective Properties Through Small Specimen Experiments: Statistical Apparent Properties

In the Introduction we have seen that performing tests on a representative volume is not always possible. For instance, think again to the dam concrete, for which the representative volume can be a cube with an edge 2 m long. Nevertheless, since the dam can be itself 500 m long, 200 m high and 30 m thick, the huge representative volume considered above is significant as a local volume element for the design calculations of the dam. But, as we have seen, no testing machine exists capable of providing us with the effective properties of the dam concrete through tests performed on this representative volume.

When dealing with theoretical predictions of the effective properties of the material from the knowledge available upon its internal constitution, it has been found useful, as illustrated by the literature quoted in the Introduction, to satisfy oneself with approximate evaluations. Among these, the derivation of bounds turned out to be especially powerful and useful tools. This is because the exact theoretical calculations of the effective properties were simply impossible.

Here we are confronted to the same kind of impossibility, but on the experimental side of the problem. Thus it is worthwhile to ask here if it could be possible to propose an experimental procedure that could provide us with evaluation of the effective properties of the material from a series of tests performed on specimens with dimensions smaller than the representative volume, for instance with dimensions of the kind currently used in the laboratories.

We shall see now that the answer is yes, at least up to the inaccuracy we are used to in experimental testing.

We shall state that, when one takes care of the kind of boundary conditions which are applied to the specimens, we can get bounds on the effective properties of the representative volume. Such care is not usual in ordinary laboratory testing because of the assumption of independence from boundary conditions which underlies the concept of effective properties. Thus, some adaptation of the usual experimental procedures will have to be done in the practical applications.

For this purpose, we propose the following general procedure:

> Take a representative volume in the form of a parallelepiped. Saw it into n parallelepipedic specimens with dimensions equal from one specimen to the others. Perform a series of tests in ε_0-KUBC on this set of specimens and take the stochastic average of the apparent moduli thus obtained. What is obtained is an upper bound for the effective modulus of the dam concrete or of any other material involving the same kind of problem.
>
> In addition, take another representative volume of the same material, with the same external shape and dimensions than the first one. Saw it into n parallelepipedic specimens of the same external shape and dimensions as for the first one. Perform a series of tests in σ_0-SUBC on this second set of specimens, and take the stochastic average of the apparent compliances thus obtained. The reciprocal of this average is a lower bound for the effective modulus of the material under consideration. Of course, if one can perform non destructive tests, one series of specimens would by sufficient, but this will rarely be the case.

The proof for the validity of this recipe will be given in Section 7 under the form of what we call the partition theorem. Additional results will then be presented.

But before going to the theorem mentioned above, let us still introduce further definitions.

Let us consider a set $\{D_\alpha | \alpha = 1 \text{ to } n\}$ of externally identical specimens. When performing a test on this set of specimens, one is more interested in the stochastic average of the obtained results than with the individual results. Let us see how this can be justified for the two kinds of uniform boundary conditions described in Section 2.

Let us denote by $\bar{a}$ the stochastic average of (5.1) on the set $\{D_\alpha\}$ for specimens all submitted to the same ε_0-KUBC. It follows that

$$\begin{aligned}\overline{\langle\sigma\rangle_\alpha} &= \overline{C^{\text{app}}_{\varepsilon\alpha} : \langle\varepsilon\rangle_\alpha} \\ &= \overline{C^{\text{app}}_{\varepsilon\alpha} : \varepsilon_0} \\ &= \overline{C^{\text{app}}_{\varepsilon\alpha}} : \varepsilon_0 \\ &= \overline{C^{\text{app}}_{\varepsilon\alpha}} : \overline{\langle\varepsilon\rangle_\alpha}; \quad \varepsilon_0\text{-KUBC}. \end{aligned} \tag{6.1}$$

This is because ε_0 here is a deterministic quantity, the same for each D_α. Thus (6.1) gives us a linear relationship between $\overline{\langle\sigma\rangle_\alpha}$ and $\overline{\langle\varepsilon\rangle_\alpha}$ which can be put into the form

$$\overline{\langle\sigma\rangle_\alpha} = C^{\text{app}}_{\varepsilon} : \overline{\langle\varepsilon\rangle_\alpha}; \quad \varepsilon_0\text{-KUBC} \tag{6.2}$$

with, from (6.1) again

$$C^{\text{app}}_{\varepsilon} = \overline{C^{\text{app}}_{\varepsilon\alpha}}. \tag{6.3}$$

We shall name $C^{\text{app}}_{\varepsilon}$ (without subscript α) defined by (6.2) the *statistical kinematic apparent modulus tensor* of the collection $\{D_\alpha | \alpha = 1 \text{ to } n\}$ of specimens in kinematic uniform boundary conditions.

Reversing (6.2), we shall write

$$\overline{\langle\varepsilon\rangle_\alpha} = S^{\text{app}}_{\varepsilon} : \overline{\langle\sigma\rangle_\alpha}; \quad \varepsilon_0\text{-KUBC} \tag{6.4}$$

and name $S^{\text{app}}_{\varepsilon}$ defined by (6.4) the *statistical kinematic apparent compliance tensor* of the collection $\{D_\alpha | \alpha = 1 \text{ to } n\}$ of specimens in kinematic uniform boundary conditions.

From (6.1) or (6.2) follows

$$S^{\text{app}}_{\varepsilon} = (\overline{C^{\text{app}}_{\varepsilon\alpha}})^{-1} = C^{\text{app}^{-1}}_{\varepsilon}; \quad \varepsilon_0\text{-KUBC}. \tag{6.5}$$

Let us insist on the fact that $S^{\text{app}}_{\varepsilon}$ *is not* the stochastic average of the individual kinematic apparent compliances $S^{\text{app}}_{\varepsilon\alpha}$, $\alpha = 1$ to n.

These definitions of the statistical kinematic apparent modulus and compliance tensors $C^{\text{app}}_{\varepsilon}$ and $S^{\text{app}}_{\varepsilon}$ are compatible with their corresponding energetic definitions since, from (5.8) one has, in ε_0-KUBC,

$$\begin{aligned}\overline{\langle\sigma : \varepsilon\rangle_\alpha} &= \overline{\langle C : \varepsilon : \varepsilon\rangle_\alpha} \\ &= \overline{C^{\text{app}}_{\varepsilon\alpha} : \langle\varepsilon\rangle_\alpha : \langle\varepsilon\rangle_\alpha} \\ &= \overline{C^{\text{app}}_{\varepsilon\alpha} : \varepsilon_0 : \varepsilon_0} \\ &= \overline{C^{\text{app}}_{\varepsilon\alpha}} : \varepsilon_0 : \varepsilon_0 \\ &= C^{\text{app}}_{\varepsilon} : \varepsilon_0 : \varepsilon_0 \\ &= C^{\text{app}}_{\varepsilon} : \overline{\langle\varepsilon\rangle_\alpha} : \overline{\langle\varepsilon\rangle_\alpha} \\ &= \overline{\langle\sigma\rangle_\alpha} : \overline{\langle\varepsilon\rangle_\alpha} \\ &= S^{\text{app}}_{\varepsilon} : \overline{\langle\sigma\rangle_\alpha} : \overline{\langle\sigma\rangle_\alpha}; \quad \varepsilon_0\text{-KUBC}. \end{aligned} \tag{6.6}$$

This would not have been the case with other definitions.

Similarly, let us take the stochastic average of (5.5) for the set $\{D_\alpha | \alpha = 1 \text{ to } n\}$ of specimens all submitted to the same σ_0-SUBC. There results

$$
\begin{aligned}
\overline{\langle \varepsilon_\alpha \rangle} &= \overline{S^{\mathrm{app}}_{\sigma\alpha} : \langle \sigma \rangle_\alpha} \\
&= \overline{S^{\mathrm{app}}_{\sigma\alpha} : \sigma_0} \\
&= \overline{S^{\mathrm{app}}_{\sigma\alpha}} : \sigma_0 \\
&= \overline{S^{\mathrm{app}}_{\sigma\alpha}} : \overline{\langle \sigma \rangle_\alpha}; \quad \sigma_0\text{-SUBC}.
\end{aligned}
\tag{6.7}
$$

This is because σ_0 here is a deterministic quantity, the same for each D_α. We can thus define the *static statistical apparent compliance tensor* S^{app}_σ of the collection $\{D_\alpha | \alpha = 1 \text{ to } n\}$ of specimens in static uniform boundary conditions by the relationship between the stochastic averages $\overline{\langle \varepsilon \rangle_\alpha}$ and $\overline{\langle \sigma \rangle_\alpha}$ of volume averages in (6.7):

$$
\overline{\langle \varepsilon \rangle_\alpha} = S^{\mathrm{app}}_\sigma : \overline{\langle \sigma \rangle_\alpha}
\tag{6.8}
$$

with

$$
S^{\mathrm{app}}_\sigma = \overline{S^{\mathrm{app}}_\sigma}.
\tag{6.9}
$$

Reversing (6.8) we shall write

$$
\overline{\langle \sigma \rangle_\alpha} = C^{\mathrm{app}}_\sigma : \overline{\langle \varepsilon \rangle_\alpha}; \quad \sigma_0\text{-SUBC}.
\tag{6.10}
$$

This defines the *static statistical apparent modulus tensor* C^{app}_σ, with value

$$
C^{\mathrm{app}}_\sigma = (\overline{S^{\mathrm{app}}_{\sigma\alpha}})^{-1} = S^{\mathrm{app}^{-1}}_\sigma; \quad \sigma_0\text{-SUBC}.
\tag{6.11}
$$

Let us insist, here too, on the fact that C^{app}_σ *is not* the stochastic average of the individual static apparent moduli $C^{\mathrm{app}}_{\sigma\alpha}$, $\alpha = 1$ to n.

Here again, these definitions of the statistical static apparent compliance and modulus tensors are compatible with their corresponding energetic definitions since, taking the stochastic average of (5.9), we get

$$
\begin{aligned}
\overline{\langle \sigma : \varepsilon \rangle_\alpha} &= \overline{\langle S : \sigma : \sigma \rangle_\alpha} \\
&= \overline{S^{\mathrm{app}}_{\sigma\alpha} : \langle \sigma \rangle_\alpha : \langle \sigma \rangle_\alpha} \\
&= \overline{S^{\mathrm{app}}_{\sigma\alpha} : \sigma_0 : \sigma_0} \\
&= \overline{S^{\mathrm{app}}_{\sigma\alpha}} : \sigma_0 : \sigma_0 \\
&= S^{\mathrm{app}}_\sigma : \sigma_0 : \sigma_0 \\
&= S^{\mathrm{app}}_\sigma : \overline{\langle \sigma \rangle_\alpha} : \overline{\langle \sigma \rangle_\alpha} \\
&= \overline{\langle \varepsilon \rangle_\alpha} : \overline{\langle \sigma \rangle_\alpha} \\
&= C^{\mathrm{app}}_\sigma : \overline{\langle \varepsilon \rangle_\alpha} : \overline{\langle \varepsilon \rangle_\alpha}; \quad \sigma_0\text{-KUBC}.
\end{aligned}
\tag{6.12}
$$

This would not have been the case with other definitions.

We turn now to the first result of this paper, the partition theorem.

7. Partition Theorem for a Representative Volume

In mathematical symbols, the *partition theorem* for a representative volume takes the following form, for any uniform partition of this representative volume:

$$C_\sigma^{\mathrm{app}} \leqslant C^{\mathrm{eff}} \leqslant C_\varepsilon^{\mathrm{app}}, \tag{7.1}$$

$$S_\varepsilon^{\mathrm{app}} \leqslant S^{\mathrm{eff}} \leqslant S_\sigma^{\mathrm{app}}, \tag{7.2}$$

where the apparent compliance and moduli are the statistical ones defined in Section 6.

In words, the primal result, (7.1), can be stated as follows:

> The effective modulus tensor of a statistically uniform material lies always between the static statistical apparent modulus tensor (lower bound) and the kinematic statistical apparent modulus tensor (upper bound) of any uniform partition of a representative volume into externally identical specimens submitted to static and kinematic uniform boundary conditions respectively. In other words, it lies always between the harmonic average of the individual apparent moduli obtained on the n specimens in σ_0-SUBC, and the arithmetic average of the individual apparent moduli obtained on these specimens in ε_0-KUBC.

And, for the dual result, (7.2):

> The effective compliance tensor of a statistically uniform material lies always between the kinematic statistical apparent compliance tensor (lower bound) and the static statistical apparent compliance tensor (upper bound) of any uniform partition of a representative volume into externally identical specimens submitted to kinematic and static uniform boundary conditions respectively. In other words, it lies between the harmonic average of the individual apparent compliances obtained on the n specimens in ε_0-KUBC and the harmonic average of the individual apparent compliances obtained on these specimens in σ_0-SUBC.

In these statements, special attention must be paid to the respective roles played by the two kinds of uniform boundary conditions considered above.

In the remainder of this section, we give the proof of this theorem.

Let us consider a single body that we call D_0. It may or may not be a representative volume.

Let us consider any uniform partition P of it into a set of smaller specimens $\{D_\alpha | \alpha = 1 \text{ to } n\}$.

Let us try to compare the individual apparent modulus $C_{\varepsilon 0}^{\mathrm{app}}$ defined for D_0 in Section 5, and the statistical apparent modulus $C_\varepsilon^{\mathrm{app}}$ defined, for the uniform partition P by the procedure with ε_0-KUBC explained in Section 7.

In ε_0-KUBC, D_0 exhibits a strain field $\varepsilon(x)$ which is the solution of the elasticity problem. This strain field is compatible with the displacement field

$$\xi^{\mathrm{d}}(x) = \varepsilon_0 \cdot x \quad \forall M(x) \in \partial D_0 \tag{7.3}$$

prescribed on the boundary ∂D_0 of D_0.

The same holds true for each D_α, with ∂D_0 replaced by ∂D_α.

But if we consider the specimens D_αs at their initial place in D_0, ∂D_0 is included in the union of the ∂D_αs, $\alpha = 1$ to $\boldsymbol{n}$:

$$\partial D_0 \subset \bigcup_\alpha \partial D_\alpha. \tag{7.4}$$

Thus the union on D_0 of the fields $\varepsilon_\alpha(x)$ that are obtained on each D_α when submitted separately at its boundary to the displacement field

$$\xi^{\mathrm{d}}(x) = \varepsilon_0 \cdot x \quad \forall M(x) \in \partial D_\alpha \tag{7.5}$$

is, as a whole, compatible with the displacement field, (7.3), prescribed for D_0.

This is because ε_α is, everywhere in D_α, the gradient of the displacement $\xi_\alpha(x)$ obtained from the boundary condition (7.5), and this for all D_αs. Further ε_α being a gradient, a translation on x is without influence on ε_α. Still further, for these specimens like D_β for which a part of ∂D_0 belongs to ∂D_β, the displacement prescribed on ∂D_β on that part of its boundary is the same as the displacement prescribed on that part of ∂D_0 belonging to ∂D_β. Last, because for the other parts of the boundaries of the D_βs, and for the D_αs that are without contact with ∂D_0 the displacements prescribed on each ∂D_α grants the continuity of the displacements across the interface between two adjacent specimens, despite the fact that they are mechanically separated (but not geometrically). This is because, at each point of the interface, in (7.5), ε_0 is the same and x is the same.

Thus, in these conditions, the union on D_0 of the fields, $\varepsilon_\alpha(x)$, $M(x) \in D_\alpha$, can be taken as an admissible field for D_0 when the latter is, as a whole, submitted to the ε_0-KUBC defined by (7.3).

Now, the classical minimum theorem of the potential energy for elasticity without volume forces states that, among the set $\{\tilde{\varepsilon}, \tilde{\xi}\}$ of all admissible solutions, the functional

$$F_{\tilde{\varepsilon}}(\tilde{\varepsilon}) = \frac{1}{2}\int_D C : \tilde{\varepsilon} : \tilde{\varepsilon}\, \mathrm{d}V - \int_{\partial D^\sigma} P^{\mathrm{d}} \cdot \tilde{\xi}\, \mathrm{d}\Sigma \tag{7.6}$$

is a minimum for the true solution (ε, ξ), where ∂D^σ stands for that part of ∂D on which the traction vector P^{d} is prescribed.

In general, we have, for any body

$$\partial D = \partial D^\sigma \cup \partial D^\varepsilon, \tag{7.7}$$

where ∂D^ε is that part of ∂D on which the displacements are prescribed.

Here, in ε_0-KUBC, the displacement is prescribed all over ∂D, and we have

$$\partial D^\sigma = \phi; \quad \partial D^\varepsilon = \partial D. \tag{7.8}$$

Thus, in ε_0-KUBC, $F_{\tilde{\varepsilon}}$ reduces to, for the body D_0,

$$F_{\tilde{\varepsilon}} = \frac{1}{2}\int_{D_0} C:\tilde{\varepsilon}:\tilde{\varepsilon}\,\mathrm{d}V = \frac{V_0}{2}\langle C:\tilde{\varepsilon}:\tilde{\varepsilon}\rangle. \tag{7.9}$$

In the same fashion, for the real solution on D_0, one has

$$F_{\varepsilon} = \frac{1}{2}\int_{D_0} C:\varepsilon:\varepsilon\,\mathrm{d}V = \frac{V_0}{2}\langle C:\varepsilon:\varepsilon\rangle. \tag{7.10}$$

Thus the fact that ε minimizes $F_{\tilde{\varepsilon}}$ can be expressed by

$$\langle C:\varepsilon:\varepsilon\rangle \leqslant \langle C:\tilde{\varepsilon}:\tilde{\varepsilon}\rangle, \tag{7.11}$$

the volume averages being taken on D_0.

But we have

$$\begin{aligned}\int_{D_0} C:\tilde{\varepsilon}:\tilde{\varepsilon}\,\mathrm{d}V &= \sum_{\alpha=1}^{n}\int_{D_\alpha} C:\tilde{\varepsilon}:\tilde{\varepsilon}\,\mathrm{d}V \\ &= \sum_{\alpha=1}^{n} V_\alpha\langle C:\varepsilon_\alpha:\varepsilon_\alpha\rangle_\alpha\end{aligned} \tag{7.12}$$

since the $\tilde{\varepsilon}$ have been taken as the real fields $\varepsilon_\alpha(x)$ on each D_α loaded separately.

From (5.8) of Section 5 it follows that

$$\langle C:\varepsilon_\alpha:\varepsilon_\alpha\rangle_\alpha = C^{\mathrm{app}}_{\varepsilon\alpha}:\varepsilon_0:\varepsilon_0. \tag{7.13}$$

Thus (7.12) becomes

$$\begin{aligned}V_0\langle C:\tilde{\varepsilon}:\tilde{\varepsilon}\rangle &= \sum_{\alpha=1}^{n} V_\alpha\langle C:\varepsilon_\alpha:\varepsilon_\alpha\rangle_\alpha \\ &= \sum_{\alpha=1}^{n} V_\alpha C^{\mathrm{app}}_{\varepsilon\alpha}:\varepsilon_0:\varepsilon_0\end{aligned} \tag{7.14}$$

or

$$\begin{aligned}\langle C:\tilde{\varepsilon}:\tilde{\varepsilon}\rangle &= \sum_{\alpha=1}^{n}\frac{V_\alpha}{V_0} C^{\mathrm{app}}_{\varepsilon\alpha}:\varepsilon_0:\varepsilon_0 \\ &= \left(\frac{1}{n}\sum_{\alpha=1}^{n} C^{\mathrm{app}}_{\varepsilon\alpha}\right):\varepsilon_0:\varepsilon_0 \\ &= \overline{C^{\mathrm{app}}_{\varepsilon\alpha}}:\varepsilon_0:\varepsilon_0 \\ &= C^{\mathrm{app}}_{\varepsilon}:\varepsilon_0:\varepsilon_0\end{aligned} \tag{7.15}$$

since, for n specimens D_β with equal volumes forming a uniform partition of D_0, we have:

$$V_\beta = \frac{1}{n}\sum_{\alpha=1}^{n} V_\alpha = \frac{1}{n}V_0;\quad \beta = 1 \quad \text{to} \quad n. \tag{7.16}$$

On the other hand, the LHS of the relation (7.11) can be written, from (5.1) with $\alpha = 0$

$$\langle C:\varepsilon:\varepsilon\rangle = C^{\text{app}}_{\varepsilon 0}:\varepsilon_0:\varepsilon_0. \tag{7.17}$$

Thus, the inequality (7.11) reads finally

$$C^{\text{app}}_{\varepsilon 0}:\varepsilon_0:\varepsilon_0 \leqslant C^{\text{app}}_{\varepsilon}:\varepsilon_0:\varepsilon_0 \tag{7.18}$$

or

$$C^{\text{app}}_{\varepsilon 0} \leqslant C^{\text{app}}_{\varepsilon} = \overline{C^{\text{app}}_{\varepsilon\alpha}}; \quad \alpha = 1 \quad \text{to} \quad n. \tag{7.19}$$

This proves the first part of the partition theorem in providing the upper bound.

To prove the second part, which gives the lower bound, we apply the same kind of procedure but in σ_0-SUBC, and we make use of the complementary energy minimum theorem.

For any body D, the latter states that for the real field $\sigma(x)$ solution of the problem, the functional

$$F_{\tilde{\sigma}}(\tilde{\sigma}) = \frac{1}{2}\int_{D_0} S:\tilde{\sigma}:\tilde{\sigma}\,\mathrm{d}V - \int_{\partial D_0^{\varepsilon}} \tilde{P}\cdot\xi^{\mathrm{d}}\,\mathrm{d}\Sigma \tag{7.20}$$

is a minimum among the values it takes for every stress field $\tilde{\sigma}(x)$ on D that is admissible. In the case considered here for σ_0-SUBC this means that

$$\operatorname{div}\tilde{\sigma} = 0 \quad \forall\, M(x) \in D, \tag{7.21}$$

$$\tilde{P} = \tilde{\sigma}\cdot n = P^{\mathrm{d}} \quad \forall\, M(x) \in \partial D^{\sigma}. \tag{7.22}$$

But here, in σ_0-SUBC, we have

$$\partial D^{\sigma} = \partial D; \quad \partial D^{\varepsilon} = \phi \tag{7.23}$$

and $F_{\tilde{\sigma}}$ reduces to

$$F_{\tilde{\sigma}} = \frac{1}{2}\int_{D} S:\tilde{\sigma}:\tilde{\sigma}\,\mathrm{d}V = \frac{V}{2}\langle S:\tilde{\sigma}:\tilde{\sigma}\rangle. \tag{7.24}$$

Now, considering the body D_0, submitted as a whole to σ_0-SUBC, one can take for $\tilde{\sigma}(x)$ the union $\{\sigma_\alpha(x) | \alpha = 1 \text{ to } n\}$ obtained separately on the specimens D_α, each being submitted itself to σ_0-SUBC.

This stress field $\tilde{\sigma}$ is admissible for D_0 in σ_0-SUBC since, first, the same traction vector $\sigma_0\cdot n$ is applied to ∂D_0 and to that part of the ∂D_αs that belong to ∂D_0 and since div $\tilde{\sigma}$ is zero inside each D_α. Moreover, the bracket $[\sigma\cdot n]^{+}_{-}$ over the intersection $\partial D_\alpha \cap \partial D_\beta$ of the boundaries of two adjacent specimens D_α and D_β is zero. This is because this bracket is equal to $[\sigma_0]^{+}_{-}\cdot n^{+}$ with σ_0 the same on ∂D_α and ∂D_β. This grants the continuity of div σ over D_0 when passing through $\partial D_\alpha \cap \partial D_\beta$, and completes the proof for the admissibility of the chosen field $\tilde{\sigma}$.

One thus shall have, on D_0,

$$\langle S:\sigma:\sigma\rangle \leqslant \langle S:\tilde{\sigma}:\tilde{\sigma}\rangle \tag{7.25}$$

with

$$V_0\langle S:\tilde{\sigma}:\tilde{\sigma}\rangle = \sum_{\alpha=1}^{n} V_\alpha \langle S:\sigma_\alpha:\sigma_\alpha\rangle_\alpha \tag{7.26}$$

or

$$\begin{aligned}\langle S:\tilde{\sigma}:\tilde{\sigma}\rangle &= \sum_{\alpha=1}^{n} \frac{V_\alpha}{V_0} S^{\text{app}}_{\sigma\alpha}:\sigma_0:\sigma_0 \\ &= \left(\frac{1}{n}\sum_{\alpha=1}^{n} S^{\text{app}}_{\sigma\alpha}\right):\sigma_0:\sigma_0 \\ &= \overline{S^{\text{app}}_{\sigma\alpha}}:\sigma_0:\sigma_0 \\ &= S^{\text{app}}_{\sigma}:\sigma_0:\sigma_0.\end{aligned} \tag{7.27}$$

On the other hand, the left-hand side of the inequality (7.25) reads

$$\langle S:\sigma:\sigma\rangle = S^{\text{app}}_{\sigma}:\sigma_0:\sigma_0. \tag{7.28}$$

Thus we have, for any given body on which a uniform partition can be performed:

$$S^{\text{app}}_{\sigma}:\sigma_0:\sigma_0 \leqslant S^{\text{app}}_{\sigma}:\sigma_0:\sigma_0 \tag{7.29}$$

or

$$S^{\text{app}}_{\sigma 0} \leqslant S^{\text{app}}_{\sigma} = \overline{S^{\text{app}}_{\sigma\alpha}}; \quad \alpha = 1 \quad \text{to} \quad n. \tag{7.30}$$

Going to the reciprocal, one gets:

$$C^{\text{app}}_{\sigma} = (S^{\text{app}}_{\sigma})^{-1} \leqslant (S^{\text{app}}_{\sigma 0})^{-1} = C^{\text{app}}_{\sigma 0}. \tag{7.31}$$

For the sake of completeness, we shall rederive in the next section the classical result that, in general, for one and the same domain D_0:

$$C^{\text{app}}_{\sigma 0} \leqslant C^{\text{app}}_{\varepsilon 0}. \tag{7.32}$$

But for D_0 here, taken as a specimen with the representative volume, we have by definition, because of the independence of the effective properties upon the boundary conditions:

$$C^{\text{app}}_{\sigma 0} = C^{\text{app}}_{\varepsilon 0} = C^{\text{eff}}. \tag{7.33}$$

Thus we have finally in this case

$$C^{\text{app}}_{\sigma} \leqslant C^{\text{eff}} \leqslant C^{\text{app}}_{\varepsilon} \tag{7.34}$$

and going to the reciprocals

$$S^{\text{app}}_{\varepsilon} \leqslant S^{\text{eff}} \leqslant S^{\text{app}}_{\sigma}. \tag{7.35}$$

This proves (7.1) and (7.2) and thus the partition theorem for a representative volume.

When the procedure used in Section 7 is applied to a parallelepipedic body D_0 with dimensions smaller than the representative volume, the derivation made in Section 7 still applies without change up to (7.31).

Thus we have here, for any partition of D_0:

$$C_{\varepsilon 0}^{\text{app}} \leqslant C_{\varepsilon}^{\text{app}}, \tag{8.1}$$

$$S_{\sigma 0}^{\text{app}} \leqslant S_{\sigma}^{\text{app}}, \tag{8.2}$$

$$C_{\sigma}^{\text{app}} \leqslant C_{\sigma 0}^{\text{app}}, \tag{8.3}$$

$$S_{\varepsilon}^{\text{app}} \leqslant S_{\varepsilon 0}^{\text{app}}, \tag{8.4}$$

where $C_{\varepsilon}^{\text{app}}$, S_{σ}^{app}, C_{σ}^{app}, $S_{\varepsilon}^{\text{app}}$ are statistical apparent modulus or compliance tensors (see Section 6).

Now let us show that, as announced by (7.32), we have, for any D_0

$$C_{\sigma 0}^{\text{app}} \leqslant C_{\varepsilon 0}^{\text{app}}. \tag{8.5}$$

This yields also

$$S_{\varepsilon 0}^{\text{app}} \leqslant S_{\sigma 0}^{\text{app}}. \tag{8.6}$$

To prove these classical relationships, we shall compare two problems for any given body D_0:

Problem 1 : D_0 submitted to ε_0-KUBC, with solution fields, $\{\xi_1, \varepsilon_1, \sigma_1\}$.

Problem 2: D_0 submitted to σ_0-SUBC, with solution fields, $\{\xi_2, \varepsilon_2, \sigma_2\}$.

Since volume forces and accelerations are supposed to be zero in both problems, the field $\{\sigma_2(x)\}$, which is divergence free everywhere in D_0, is an admissible stress field for Problem 1, since the latter involves no stress requirement on the boundary ∂D_0. Thus one has, for the corresponding virtual complementary energy F_σ^1 for Problem 1 :

$$-F_\sigma^1(\sigma_2) = -\frac{1}{2}\int_{D_0} S:\sigma_2:\sigma_2 \,\mathrm{d}V + \int_{\partial D_0} (\sigma_2)\cdot\xi_1^{\mathrm{d}} \,\mathrm{d}\Sigma \leqslant -F_\sigma^1(\sigma_1). \tag{8.7}$$

Since, in Problem 1, one has

$$\partial D_0^\sigma = \phi; \quad \partial D_0^\varepsilon = \partial D_0, \tag{8.8}$$

we have also

$$\begin{aligned} -F_\sigma^1(\sigma_1) &= -\frac{1}{2}\int_{D_0} S:\sigma_1:\sigma_1 \,\mathrm{d}V \\ &\quad + \int_{\partial D_0} (\sigma_1\cdot n)\cdot\xi_1^{\mathrm{d}} \,\mathrm{d}\Sigma \\ &= +\frac{1}{2}\int_{D_0} S:\sigma_1:\sigma_1 \,\mathrm{d}V, \end{aligned} \tag{8.9}$$

through the theorem of virtual work.

Thus we have

$$
\begin{aligned}
-F_{\sigma}^{1}(\sigma_1) &= \frac{V}{2}\langle \varepsilon_1 : \sigma_1 \rangle \\
&= \frac{V}{2}\langle C_1 : \varepsilon_1 : \varepsilon_1 \rangle \\
&= \frac{V}{2} C_{\varepsilon 0}^{\mathrm{app}} : \varepsilon_0 : \varepsilon_0 \\
&= F_{\varepsilon}^{1}(\varepsilon_1),
\end{aligned} \tag{8.10}
$$

where we find again the classical result of elasticity that, for the real solution fields, the complementary energy equals the opposite of the potential energy.

On the other hand, we have

$$
\begin{aligned}
\int_{\partial D_0} (\sigma_2 \cdot n) \cdot \xi_1^{\mathrm{d}} \, \mathrm{d}\Sigma &= \int_{\partial D_0} (\sigma_0 \cdot n) \cdot (\varepsilon_0 \cdot x) \, \mathrm{d}\Sigma \\
&= (\sigma_0 \cdot \varepsilon_0) : \int_{\partial D_0} n \times x \, \mathrm{d}\Sigma \\
&= (\sigma_0 \cdot \varepsilon_0) : \int_{D_0} \operatorname{grad} x \, \mathrm{d}\Sigma \\
&= V(\sigma_0 \cdot \varepsilon_0) : \delta \\
&= V\sigma_0 : \varepsilon_0 .
\end{aligned} \tag{8.11}
$$

Thus, the inequality (8.7) becomes

$$
\begin{aligned}
-\frac{1}{V} F_{\sigma}^{1}(\sigma_2) &= A \\
&= -\frac{1}{2} S_{\sigma 0}^{\mathrm{app}} : \sigma_0 : \sigma_0 + \sigma_0 : \varepsilon_0 \leqslant \frac{1}{2} C_{\varepsilon 0}^{\mathrm{app}} : \varepsilon_0 : \varepsilon_0 \;\; \forall \varepsilon_0, \sigma_0 .
\end{aligned} \tag{8.12}
$$

When ε_0 is maintained fixed and σ_0 varied, one has

$$
\frac{\partial A}{\partial \sigma_0} = -S_{\sigma 0}^{\mathrm{app}} : \sigma_0 + \varepsilon_0 , \tag{8.13}
$$

which is equal to zero for

$$
\sigma_0 = S_{\sigma 0}^{\mathrm{app}\;-1} : \varepsilon_0 = \sigma_0^{1} . \tag{8.14}
$$

One has also

$$\frac{\partial^2 A}{\partial \sigma_0^2} = -S_{\sigma 0}^{\mathrm{app}} < 0. \tag{8.15}$$

Thus, when ε_0 is fixed, A experiences a maximum for the value σ_0^{I} of σ_0 given by (8.14). Substituting into the expression for A the value σ_0^{I} of σ_0, it comes after reduction, for any value of ε_0:

$$\begin{aligned} A_{\max} = A(\sigma_0^{\mathrm{I}}) &= +\tfrac{1}{2} S_{\sigma 0}^{\mathrm{app}\;-1} : \varepsilon_0 : \varepsilon_0 \\ &= +\tfrac{1}{2} C_{\sigma 0}^{\mathrm{app}} : \varepsilon_0 : \varepsilon_0. \end{aligned} \tag{8.16}$$

Thus, the inequality (8.12) yields, for every value of σ_0 and ε_0, the inequality

$$\tfrac{1}{2} C_{\sigma 0}^{\mathrm{app}} : \varepsilon_0 : \varepsilon_0 \leqslant \tfrac{1}{2} C_{\varepsilon 0}^{\mathrm{app}} : \varepsilon_0 : \varepsilon_0, \tag{8.17}$$

equivalent to

$$C_{\sigma 0}^{\mathrm{app}} \leqslant C_{\varepsilon 0}^{\mathrm{app}} \quad \forall D_0. \tag{8.18}$$

This completes the proof of the recalled inequalities (7.32) or (8.5). This proof has been obtained here by elementary calculations. Other more straightforward but more abstract kinds of proof have been independently suggested by SUQUET (1986) in the framework of functional analysis, and given by WILLIS and TALBOT (1989) in the framework of general convex analysis applied to the non-linear case.

Going to the reciprocals yields:

$$S_{\varepsilon 0}^{\mathrm{app}} \leqslant S_{\sigma 0}^{\mathrm{app}} \quad \forall D_0. \tag{8.19}$$

Now, we are able to order the four inequalities (8.1) to (8.4) in the following fashion:

$$C_{\sigma}^{\mathrm{app}} \leqslant C_{\sigma 0}^{\mathrm{app}} \leqslant C_{\varepsilon 0}^{\mathrm{app}} \leqslant C_{\varepsilon}^{\mathrm{app}} \quad \forall D_0 \tag{8.20}$$

$$S_{\varepsilon}^{\mathrm{app}} \leqslant S_{\varepsilon 0}^{\mathrm{app}} \leqslant S_{\sigma 0}^{\mathrm{app}} \leqslant S_{\sigma}^{\mathrm{app}} \quad \forall D_0 \tag{8.21}$$

for any uniform partition of any given body D_0, with or without the representative volume. When it is with the representative volume, $C_{\sigma 0}^{\mathrm{app}}$ and $C_{\varepsilon 0}^{\mathrm{app}}$ merge into C^{eff}, and the basic inequalities (7.32) and (7.33) of Section 7 are recovered.

9. HIERARCHIES BETWEEN ITERATED PARTITIONS AND LIMIT BOUNDINGS

Now, let us consider the case of a given uniform partition P_c of D_0 into n_c coarse specimens $\{D_{\alpha c} | \alpha_c = 1 \text{ to } n_c\}$.

Let us partition uniformly each $D_{\alpha c}$ into fine specimens $\{D_{\alpha cf} | \alpha_{cf} = 1 \text{ to } m\}$ in such a way that the $D_{\alpha cf}$s are of the same shape and dimensions for all $D_{\alpha c}$. We name this operation an *iterated partitioning* of D_0. The result is a partition P_f of D_0 into n_f fine specimens $\{D_{\alpha f} | \alpha_f = 1 \text{ to } n_f\}$ with:

$$n_{\mathrm{f}} = m n_{\mathrm{c}}. \tag{9.1}$$

We name this result an *iterated partition* of D_0 into fine specimens $D_{\alpha\mathrm{f}}$ through the coarse specimens $D_{\alpha\mathrm{c}}$ of a first partition.

For each specimen of this partition, the results of Section 8 apply. Let us denote with the subscript c the statistical apparent properties of the coarse uniform partition, and with the subscript f the ones of the fine uniform partition. One has thus, for the coarse partition of D_0:

$$C^{\mathrm{app}}_{\sigma\mathrm{c}} \leqslant C^{\mathrm{app}}_{\sigma 0} \leqslant C^{\mathrm{app}}_{\varepsilon 0} \leqslant C^{\mathrm{app}}_{\varepsilon\mathrm{c}} \quad \forall D_0 \tag{9.2}$$

and, for the fine partition of D_0:

$$C^{\mathrm{app}}_{\sigma\mathrm{f}} \leqslant C^{\mathrm{app}}_{\sigma 0} \leqslant C^{\mathrm{app}}_{\varepsilon 0} \leqslant C^{\mathrm{app}}_{\varepsilon\mathrm{f}} \quad \forall D_0. \tag{9.3}$$

Converse inequalities hold for the corresponding compliances.

Let us show now that the apparent moduli tensors involved in the inequalities (9.2) and (9.3) can be put into further ordering relationships, and the same for the compliances, with the results:

$$C^{\mathrm{app}}_{\sigma\mathrm{f}} \leqslant C^{\mathrm{app}}_{\sigma\mathrm{c}} \leqslant C^{\mathrm{app}}_{\sigma 0} \leqslant C^{\mathrm{app}}_{\varepsilon 0} \leqslant C^{\mathrm{app}}_{\varepsilon\mathrm{c}} \leqslant C^{\mathrm{app}}_{\varepsilon\mathrm{f}}, \tag{9.4}$$

$$S^{\mathrm{app}}_{\varepsilon\mathrm{f}} \leqslant S^{\mathrm{app}}_{\varepsilon\mathrm{c}} \leqslant S^{\mathrm{app}}_{\varepsilon 0} \leqslant S^{\mathrm{app}}_{\sigma 0} \leqslant S^{\mathrm{app}}_{\sigma\mathrm{c}} \leqslant S^{\mathrm{app}}_{\sigma\mathrm{f}}. \tag{9.5}$$

This is because one has also, for each specimen $D_{\alpha\mathrm{c}}$ of the coarse partition

$$C^{\mathrm{app}}_{\varepsilon\alpha_{\mathrm{c}}} \leqslant \overline{C^{\mathrm{app}}}^{(\alpha_{\mathrm{c}})} = \frac{1}{m} \sum_{\alpha_{\mathrm{f}} = q_{\alpha_{\mathrm{c}}}}^{q_{\alpha_{\mathrm{c}}}+m} C^{\mathrm{app}}_{\varepsilon\alpha_{\mathrm{f}}}, \tag{9.6}$$

where $\overline{(\,\cdot\,)}^{(\alpha_{\mathrm{c}})}$ denotes the stochastic average upon the m specimens $\{D_{\alpha\mathrm{f}} | \alpha_{\mathrm{f}} = q_{\alpha_{\mathrm{c}}}$ to $q_{\alpha_{\mathrm{c}}} + m\}$ of the fine partition constituting the coarse specimen $D_{\alpha\mathrm{c}}$.

Since the $C^{\mathrm{app}}_{\varepsilon\alpha_{\mathrm{f}}}$ and $C^{\mathrm{app}}_{\varepsilon\alpha_{\mathrm{c}}}$ are all positive semi-definite one has also, taking the stochastic average of (9.6) over the partition P_{c} of D_0:

$$\begin{aligned} C^{\mathrm{app}}_{\varepsilon\mathrm{c}} &= \frac{1}{n} \sum_{\alpha_{\mathrm{c}}=1}^{n_{\mathrm{c}}} C^{\mathrm{app}}_{\varepsilon\alpha_{\mathrm{c}}} \\ &\leqslant \frac{1}{n_{\mathrm{c}}} \frac{1}{m} \sum_{\alpha_{\mathrm{c}}=1}^{n_{\mathrm{c}}} \sum_{\alpha_{\mathrm{f}} = q_{\alpha_{\mathrm{c}}}}^{q_{\alpha_{\mathrm{c}}}+m} C^{\mathrm{app}}_{\varepsilon\alpha_{\mathrm{f}}} \\ &= \frac{1}{n_{\mathrm{f}}} \sum_{\alpha_{\mathrm{f}}=1}^{n_{\mathrm{f}}} C^{\mathrm{app}}_{\varepsilon\alpha_{\mathrm{f}}} \\ &= \overline{C^{\mathrm{app}}_{\varepsilon\alpha_{\mathrm{f}}}} \\ &= C^{\mathrm{app}}_{\varepsilon\mathrm{f}}, \end{aligned} \tag{9.7}$$

or

$$C^{\mathrm{app}}_{\varepsilon\mathrm{c}} \leqslant C^{\mathrm{app}}_{\varepsilon\mathrm{f}}. \tag{9.8}$$

Going to the reciprocals yields

$$S_{\varepsilon f}^{app} \leqslant S_{\varepsilon c}^{app}. \tag{9.9}$$

Similarly, one has for the compliances in σ_0-KUBC:

$$\begin{aligned} S_{\sigma c}^{app} &= \frac{1}{n_c} \sum_{\alpha_c = 1}^{n_c} C_{\varepsilon\alpha_c}^{app} \\ &\leqslant \frac{1}{n_c} \frac{1}{m} \sum_{\alpha_c = 1}^{n_c} \sum_{\alpha_f = q_{\alpha_c}}^{q_{\alpha_c}+m} S_{\sigma\alpha_f}^{app} \\ &= \frac{1}{n_f} \sum_{\alpha_f = 1}^{n_f} S_{\sigma\alpha_f}^{app} \\ &= \overline{S_{\sigma\alpha_f}^{app}} \\ &= S_{\sigma_f}^{app}, \end{aligned} \tag{9.10}$$

or

$$S_{\sigma c}^{app} \leqslant S_{\sigma f}^{app}. \tag{9.11}$$

Going to the reciprocals gives

$$C_{\sigma f}^{app} \leqslant C_{\sigma c}^{app}. \tag{9.12}$$

Inequalities (9.12), (9.8) together with inequalities (9.2) prove inequalities (9.4), while inequalities (9.9), (9.11) together with the compliances version of inequalities (9.2) prove inequalities (9.5).

Considering inequalities (9.4) and (9.5), we can say that a hierarchy is introduced between any two iterated partitions of the same body D_0 since the statistical apparent moduli or compliances of the coarse one are bounded on both sides by the statistical apparent moduli or compliances of the finer one.

In the limits of applicability of continuum mechanics, the above results apply whatever the sizes of the specimens may be.

Now let it be taken, as a fine partition, a partition for which the size of each specimen is smaller than the size of all the homogeneous subregions inside D_0. This is always possible for bodies that are of the non fractal type, or for which there exists a lower limit to fractality, in fact always the case for real materials.

In these conditions, all the specimens are homogeneous except the few ones containing one portion of the interface Γ. But Γ is a surface, and thus of measure zero in comparison with the volume. Thus, for sufficiently fine such partitions, the statistical weight of these specimens will become negligible.

Since C is uniform over each $D_{\alpha f}$, one has then, over P_f and in ε_0-KUBC:

$$\begin{aligned} \overline{\langle \sigma \rangle_{\alpha_f}} &= \overline{\langle C : \varepsilon \rangle_{\alpha_f}} \\ &= \overline{C : \langle \varepsilon \rangle_{\alpha_f}} \\ &= \overline{C : \varepsilon_0} \end{aligned}$$

$$= \bar{C} : \varepsilon_0 \tag{9.13}$$

and thus, for the particular fine partition P_f considered here:

$$C_{\varepsilon f}^{app} = \bar{C}. \tag{9.14}$$

This yields, from inequality (9.8),

$$C_{\varepsilon}^{app} \leqslant \bar{C} \tag{9.15}$$

for any partition of D_0 since any partition of D_0 into heterogeneous specimens is coarser than the particular P_f considered here.

In σ_0-KUBC, one has similarly, for such a fine uniform partition P_f of D_0, S being uniform on each $D_{\alpha f}$:

$$\begin{aligned} \overline{\langle \varepsilon \rangle_{\alpha_f}} &= \overline{\langle S : \sigma \rangle_{\alpha_f}} \\ &= \overline{S : \langle \sigma \rangle_{\alpha_f}} \\ &= \overline{S : \sigma_0} \\ &= \bar{S} : \sigma_0 \end{aligned} \tag{9.16}$$

and thus, for the particular P_f considered here

$$S_{\sigma f}^{app} = \bar{S}. \tag{9.17}$$

This yields similarly, from inequality (9.11),

$$S_{\sigma}^{app} \leqslant \bar{S} \tag{9.18}$$

for any partition of D_0.

Going to the reciprocals, inequalities (9.15) and (9.18) yield

$$(\bar{S})^{-1} = (\overline{C^{-1}})^{-1} \leqslant C_{\sigma}^{app} \tag{9.19}$$

and

$$(\bar{C})^{-1} = (\overline{S^{-1}})^{-1} \leqslant S_{\varepsilon}^{app} \tag{9.20}$$

for any partition of D_0.

The inequalities (9.4) and (9.5) can thus be completed by the bounds given by inequalities (9.15), (9.18), (9.19) and (9.20) that are absolute limits to the bounds that can be obtained when uniformly partitioning any given body D_0 into specimens of any size:

$$(\overline{C^{-1}})^{-1} \leqslant C_{\sigma f}^{app} \leqslant C_{\sigma c}^{app} \leqslant C_{\sigma 0}^{app} \leqslant C_{\varepsilon 0}^{app} \leqslant C_{\varepsilon c}^{app} \leqslant C_{\varepsilon f}^{app} \leqslant \bar{C}, \tag{9.21}$$

$$(\overline{S^{-1}})^{-1} \leqslant S_{\varepsilon f}^{app} \leqslant S_{\varepsilon c}^{app} \leqslant S_{\varepsilon 0}^{app} \leqslant S_{\sigma 0}^{app} \leqslant S_{\sigma c}^{app} \leqslant S_{\sigma f}^{app} \leqslant \bar{S}. \tag{9.22}$$

It is striking to recognize, in these absolute limit bounds, the classical (first order) Hill–Reuss and Hill–Voigt bounds mentioned in Section 2.

These results hold for any body D_0, taken or not from a material with statistical uniformity.

But, when the material is uniform, any specimen with dimensions larger than the representative volume will have one and the same value of C_{σ}^{app} and C_{ε}^{app}, both equal to the effective modulus C^{eff}. Taking such a specimen as the body D_0 allows one to write the inequalities (9.21) and (9.22) in the form:

$$(\overline{C^{-1}})^{-1} \leqslant C_{\sigma f}^{app} \leqslant C_{\sigma c}^{app} \leqslant C^{eff} \leqslant C_{\varepsilon c}^{app} \leqslant C_{\varepsilon f}^{app} \leqslant \bar{C}, \tag{9.23}$$

$$(\overline{S^{-1}})^{-1} \leqslant S_{\varepsilon f}^{app} \leqslant S_{\varepsilon c}^{app} \leqslant S^{eff} \leqslant S_{\sigma c}^{app} \leqslant S_{\sigma f}^{app} \leqslant \bar{S}. \tag{9.24}$$

This completes the set of results considered in the present paper.

10. Conclusion

One has seen that, when the material is statistically uniform, the effective modulus acts as a separator in the inequality between the static and kinematic apparent moduli of any partition of the representative volume into specimens of any size. And the same for the compliances.

The classical Hill–Voigt and Hill–Reuss bounds for the effective moduli are absolute limit bounds for the apparent moduli of any body, or for the apparent moduli of any uniform partition of this body.

Thus, when the kind of boundary conditions is known, say ε_0-KUBC, the corresponding apparent modulus is bounded on both sides by the effective modulus on one hand, and the Hill limit bound on the other hand.

When the effective modulus of the material is known, this can be useful in specific problems dealing with bodies taken from a material which is statistically uniform, but with dimensions smaller than the ones of the representative volume.

Conversely, the effective properties are bounded on both sides by the apparent properties, obtained in static and kinematic uniform boundary conditions respectively for any set of specimens forming a uniform partition of the representative volume. This will be especially useful for the cases where, as in the quoted example of the dam concrete, the representative volume is too large to be itself submitted to direct experimental testing.

When the body which is used does not have the representative volume, its apparent properties are also bounded on both sides by the results obtained on sets of smaller specimens forming a uniform partition of this body.

When many such bodies of the same size are produced by a factory, repeating this procedure for several bodies will give indications about the average technical quality and the scatter in quality of these bodies. This will be useful for the producer and also for the user of these products. This can be used for instance in comparing two kinds of such products obtained by two different processes, and to study the influence of modifications in the process.

Of course, by comparison with the testing methods presently used, some adaptation of the testing facilities and procedures will be necessary. Since the corresponding boundary conditions will be difficult to realize with precision, it will be interesting to see in what cases similar results could be obtained in various kinds of mixed-boundary conditions. This will be the subject of further research.

It will also be interesting to see, for instance through numerical simulation, how far from the effective properties will be the apparent properties obtained on specimens with sizes smaller than the representative volume, in terms of this size. This will lead to optimization in the testing procedure for a testing machine of given capacity.

The results obtained here are also valid for shapes other than the parallelepipedic one considered in this paper, and which is the most suitable for the testing when applying the proposed procedure. In numerical computations, use of other shapes of specimens and of bodies will be possible, provided the body will be decomposable into specimens with shapes capable of filling the space. It is well known that the number of such shapes is limited.

As indicated in Section 9, it is striking to see that the classical bounds of HILL (1952, 1963), that were at the root of all the following developments of the bounding methods in the theory of heterogeneous media, are also absolute bounds for the size effects encountered in bodies with sizes smaller than the representative volume, or made of materials that are not statistically homogeneous. Since the original derivation of these Hill bounds for the effective moduli were obtained by the successive considerations of strain and stress fields that were uniform all over the specimen, and not only, as here, on its boundary, it can be seen that our derivation provides a new independent proof of the classical Hill bounds for the statistically uniform materials. In addition, our derivation provides a way to get a better feeling of their real status, and of their distance to the effective modulus.

It can be shown easily that the HILL (1963) modification theorems, use of which was made in HUET *et al.* (1989) are still valid for the apparent properties considered here in this paper. Thus, zeroth order bounds can be obtained from this modification theorem, but they are already outside the range delimited by the first order Hill bounds.

It is expected that all these theoretical results will be useful by providing a rational status and improvements to the experimental procedures and ways of better interpretation for the very important problem of obtaining the overall properties of heterogeneous materials from experimental testing.

Of course, straightforward extensions will be obtained easily to all the other kinds of linear physical problems that can be expressed by systems of equation with structure similar to the one of the elasticity problem considered here. It is well known that it is the case for instance for problems dealing with properties like heat conductivity, dielectric permittivity, magnetic susceptibility and so on.

ACKNOWLEDGEMENTS

We are grateful to PROF. J. R. WILLIS for having provided us with the manuscript of his paper with DR D. R. S. TALBOT (1989) prior to publication.

REFERENCES

BERAN, M. J. 1968 *Statistical Continuum Theories*. Wiley, New York.

HASHIN, Z. and SHTRIKMAN, S. 1963 *J. Mech. Phys. Solids* **11**, 127.

HILL, R. 1952 *Proc. Phys. Soc.* (*Lond.*) **A65,** 349.

HILL, R. 1963 *J. Mech. Phys. Solids* **11,** 357.

HUET, C. 1981 In *Rheological Behavior and Structure of Materials* (edited by C. HUET and A. ZAOUI), p. 231. Presses ENPC, Paris.

HUET, C. 1982 *Mech. Res. Comm.* **9,** 165.

HUET, C. 1984 *Mech. Res. Comm.* **11,** 195.

HUET, C., NAVI, P. and ROELFSTRA, P. 1989 A new homogeneisation technique based on Hill modification theorem. In *Proc. Sixth Symp. Continuum Models and Discrete Systems*, Dijon (edited by G. MAUGIN). Longmans, London, to appear.

ISHII, M. 1975 *Thermo-fluid Dynamic Theory of Two-phase Flow*. Eyrolles, Paris.

KRÖNER, E. 1953 *Z. Phys.* **136,** 402.

KRÖNER, E. 1958 *Z. Phys.* **151,** 504.

KRÖNER, E. 1972 *Statistical Continuum Mechanics*. Springer, Berlin.

KRÖNER, E. 1977 *J. Mech. Phys. Solids* **25,** 137.

KRÖNER, E. 1981 In *Rheological Behaviour and Structure of Materials* (edited by C. HUET and A. ZAOUI), p. 15. Presses ENPC, Paris.

KRÖNER, E. 1986 In *Modelling Small Deformations of Polycrystals* (edited by J. ZARKA and J. GITTUS), p. 229. Elsevier, Amsterdam.

PAUL, B. 1960 *Trans. Am. Inst. Min. Metall. Pet. Engrs* **218,** 36.

REUSS, A. 1929 *Z. angew. Math. Mech.* **9,** 55.

SUQUET, P. M. 1986 In *Homogenization Techniques for Composite Media* (I. C. M. S. course, Udine, 1985), Lecture notes in Physics 272 (edited by E. SANCHEZ-PALENCIA and A. ZAOUI), p. 193. Springer-Verlag, Berlin.

VOIGT, W. 1887 *Abh. Kgl. Ges. Wiss., Math. Kl.* **34,** 1, 47.

WALPOLE, L. J. 1966 *J. Mech. Phys. Solids* **14,** 151 and 289.

WILLIS, J. R. 1981a In *4th Int. Symp. Continuum Models and Discrete Systems*, (CMDS 4) (edited by O. BRULIN and R. K. T. HSIEH), p. 471. North-Holland, The Hague.

WILLIS, J. R. 1981b In *Advances in Applied Mechanics* (edited by YIH CHIA-SHUN), Vol. 21, p. 1. Academic Press, New York.

WILLIS, J. R. 1986 In *Homogenization Techniques for Composite Media* (I. C. M. S. course, Udine, 1985). Lecture notes in Physics 272 (edited by E. SANCHEZ-PALENCIA and A. ZAOUI), p. 279. Springer-Verlag, Berlin.

WILLIS, J. R. and TALBOT, D. R. S. 1989 In *Proc. Sixth Symp. Continuum Models and Discrete Systems*, Dijon (edited by G. MAUGIN), Longmans, London, to appear.