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François Digne, Jean Michel

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FRANÇOIS DIGNE AND JEAN MICHEL

Abstract. We expound the properties of ribbons in a setting which is general enough to encompass spherical Artin monoids and dual braid monoids of well-generated complex reflection groups. We generalize to our setting results on parabolic subgroups of spherical Artin group of Godelle [4], Cumplido [2] and others.

This short note expounds the properties of ribbons in a setting (Assumptions 3 and 24) which is general enough to encompass spherical Artin monoids and dual braid monoids of well-generated complex reflection groups. We show in particular that this approach can recover, and generalize to our setting, results on parabolic subgroups of spherical Artin group of Godelle [4], Cumplido [2] and others. We have strived for the text to be self-contained apart from basic results on Garside theory for which we refer to [5]. The assumptions under which we study ribbons are slightly stronger than those in [5, VIII §1.4] but allow us to develop them much further.

Let M be Garside monoid; that is, M is Noetherian, left- and right-cancellative, any pair of elements have left- and right-gcd's and lcm's, and there exists a Garside element Δ, an element whose left- and right-divisors coincide and generate M (see [5, I 2.1]).

We recall that a factor is any left-divisor of a right-divisor (or equivalently a right-divisor of a left-divisor) and that Noetherian means there is no infinite sequence where each term is a proper factor of the previous one.

Definition 1. *A standard parabolic submonoid P of M is a submonoid closed by factors such that any element of M has a maximal left-divisor (the left P-head) in P and a maximal right-divisor (the right P-head) in P.*

This is the particular case, for a monoid, of Definition [5, 1.30 VII] which is in the context of a Garside category.

We denote by $H_P(g)$ the left P-head of $g \in M$ and we define $T_P(g)$ (the "P-tail") by $g = H_P(g)T_P(g)$. We say that g is P-reduced if $H_P(g) = 1$ or equivalently $g = T_P(g)$.

Any intersection of standard parabolic submonoids is a standard parabolic submonoid [5, VII 1.35]. It is clear that any standard parabolic submonoid of a standard parabolic submonoid is a standard parabolic submonoid.

Since M is Noetherian, it is generated by the set S of its atoms (elements which have no proper nontrivial divisors) and a standard parabolic submonoid being closed by factors is also generated by its atoms.

Lemma 2. *A standard parabolic submonoid P has a Garside element Δ_P given by $\Delta_P = H_P(\Delta)$.*

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Proof. First P is generated by the left-divisors of Δ_P since it is generated by its atoms which are left-divisors of Δ hence of Δ_P. Now the right- and left- divisors of Δ are the same, thus $\Delta \succcurlyeq \Delta_P$. Hence $H_P^{\succcurlyeq}(\Delta) \succcurlyeq \Delta_P$, if $H_P^{\succcurlyeq}(g)$ denotes the right P-head of g. Exchanging right and left, that is H_P and $H_P^{\succcurlyeq}$, we get symmetrically $H_P^{\succcurlyeq}(\Delta) \preccurlyeq \Delta_P$. Hence Δ_P is a factor of itself. It cannot be a strict factor by Noetherianity, thus $\Delta_P = H_P^{\succcurlyeq}(\Delta)$. The left-divisors of Δ_P are left, thus right-divisors of Δ, and since they are in P they are right-divisors of $H_P^{\succcurlyeq}(\Delta) = \Delta_P$. Conversely right-divisors of Δ_P are right, thus left-divisors of Δ and being in P are left-divisors of Δ_P. □

Assumption 3. *We assume that for any subset $S_1 \subset S$, the right-lcm of S_1 is the Garside element given by Lemma 2 of the smallest standard parabolic submonoid containing S_1.*

In the following we assume that Assumption 3 holds. It clearly holds in a spherical Artin monoid.

Proposition 4. *The dual braid monoid M of a well generated complex reflection group W satisfies Assumption 3.*

Proof. From the fact that any decomposition of Δ is obtained from one of them by the Hurwitz action (see [1, Proposition 7.6 and Proposition 8.5]), it follows that the same holds for any simple. It follows from that that any factor of a simple w is a left-divisor of w. It follows that any simple w is a common right multiple of the atoms which left-divide it. If the right-lcm of these atoms was a strict divisor of w, there would be an atom s whose square is a factor of w, thus of Δ. This does not exist, see [6, Property (iii) of Proposition 2.1 (M2)].

Thus any simple w is the left-lcm of the set S_w of atoms which left divide it. The submonoid P_w generated by these atoms is standard parabolic since it is clearly stable by factor, and every element has a P-head : to see this it is sufficient to check it for a simple v by [5, VII, 1.25], in which case the left-gcd of v and w is the head.

Finally, for any set S_1 of atoms, the right-lcm w of this set defines thus a parabolic P_w which is clearly the minimal parabolic subgroup containing S_1. □

If $b, g \in M$ and $g \preccurlyeq bg$ we say that the conjugate b^g, equal to $g^{-1}bg$, is defined. For a set $P \subset M$ we say that P^g is defined if p^g is defined for any $p \in P$.

Definition 5. *For P a standard parabolic submonoid we call P-ribbon a P-reduced element $g \in M$ such that P^g is defined. We denote such a ribbon by $P \xrightarrow{g}$.*

Let P be a standard parabolic submonoid with Garside element Δ_P and s be an atom of M which is not in P; we define $P^{(s)}$ as the smallest standard parabolic submonoid containing P and s. Since $\Delta_{P^{(s)}}$ is left-divisible by Δ_P the following makes sense:

Definition 6. *For P a standard parabolic submonoid and $s \in S, s \notin P$, we define $v_{s,P}$ by $\Delta_{P^{(s)}} = \Delta_P v_{s,P}$.*

Recall that conjugation by Δ is an automorphism of M (see for example [5, V 2.17]).

Lemma 7. *For P a standard parabolic submonoid and for $s \in S$, $s \notin P$ the element $v_{s,P}$ is a P-ribbon.*

Proof. We have $\Delta_P = H_P(\Delta_{P^{(s)}})$ since any divisor in P of $\Delta_{P^{(s)}}$ divides Δ hence divides Δ_P. Hence $v_{s,P}$ is P-reduced. And $P^{v_{s,P}}$ is defined since the conjugation by $v_{s,P}$ is the composition of the inverse of the conjugation by Δ_P, which is an automorphism of P, and the conjugation by $\Delta_{P^{(s)}}$, which is an automorphism of $P^{(s)}$. □

Lemma 8. *If g is a P-ribbon and $s \in S$ left-divides g then $v_{s,P} \preccurlyeq g$.*

Proof. We have $\Delta_P g = g\Delta_P^g$. Thus $s \preccurlyeq \Delta_P g$, thus $\Delta_{P^{(s)}} \preccurlyeq \Delta_P g$ by Assumption 3, thus $v_{s,P} = \Delta_P^{-1}\Delta_{P^{(s)}} \preccurlyeq g$. □

Note that the converse of this lemma, that is $s \preccurlyeq v_{s,P}$, holds in an Artin monoid but not necessarily in a dual braid monoid, see Example 13.

Proposition 9. *Let P be a standard parabolic submonoid; then*

- *If $P \xrightarrow{g}$ is a ribbon then P^g is a standard parabolic submonoid, and $x \mapsto x^g$ is a monoid isomorphism $P \xrightarrow{\sim} P^g$.*
- *A P-ribbon is a product of elements $v_{s,P'}$ where P' are standard parabolic submonoids conjugate to P.*
- *If $P \xrightarrow{g}$ is a ribbon then the conjugate by g of the atoms of P are the atoms of P^g.*
- *If $P \xrightarrow{g}$ is a ribbon then $\Delta_P^g = \Delta_{P^g}$.*

Proof. Let s be an atom dividing g, then $v_{s,P}$ conjugates P on a standard parabolic submonoid since the conjugation by $\Delta_{P^{(s)}}$ is an automorphism of $P^{(s)}$, hence conjugates P on a standard parabolic submonoid of $P^{(s)}$ thus of M, and the conjugation by Δ_P conjugates P on itself. If $g_1 = v_{s,P}^{-1}g$, it follows that $P^{v_{s,P}} \xrightarrow{g_1}$ is a ribbon (g_1 is $P^{v_{s,P}}$-reduced since for an atom s if $s^{v_{s,P}} \preccurlyeq g_1$ then $s \preccurlyeq g$). By Noetherian induction we get the first two items.

The third item is an immediate consequence of the first one, and the fourth also by Assumption 3. □

In view of the third item above, the definition of a P-ribbon could instead of asking that P^g is defined just ask that the conjugate of any atom of P by g is defined.

Lemma 10. *If P is a standard parabolic submonoid, if s is an atom of P and $P \xrightarrow{g}$ is a ribbon, then the right-lcm of s and g is sg.*

Proof. Since g is a P-ribbon it conjugates s into some atom s'. The right-lcm of s and g left-divides $sg = gs'$ and is a strict right-multiple of g (since g is P-reduced), hence is equal to gs'. □

Lemma 11. *Let $P \xrightarrow{g}$ be a ribbon; for $x \in P$, $y \in M$, it is equivalent that $g \preccurlyeq xy$ or that $g \preccurlyeq y$.*

Proof. If $y = gy'$, then $xy = gx^g y'$ so that $g \preccurlyeq xy$. To prove the converse, by Noetherian induction on x, it is sufficient to prove that if an atom $s \in P$ left-divides gh for some h, then $s^g \preccurlyeq h$. By Lemma 10, the right-lcm of s and g is $sg = gs^g$. Thus $s \preccurlyeq gh$ is equivalent to $gs^g \preccurlyeq gh$ which is finally equivalent to $s^g \preccurlyeq h$. □

Lemma 12. *Let $P \xrightarrow{g}$ be a ribbon and let $h \in M$. Then $T_P(gh) = gT_{P^g}(h)$ and $H_P(gh)^g = H_{P^g}(h)$. In particular if g is a P-ribbon and h is a P^g-ribbon, then gh is a P-ribbon.*

Proof. Let s be an atom of P and set $s' := s^g$. Both formulae clearly follow if we show that it is equivalent that $s \preccurlyeq gh$ or that $s' \preccurlyeq h$. This is true by the proof of Lemma 11. □

It follows from Lemma 12 and the first item of Proposition 9 that the ribbons form a category. The second item of Proposition 9 shows that the atoms of this category are the $P \xrightarrow{v_{s,P}}$ which are not strict multiple of some other one. This is always the case in a spherical Artin monoid but not in a dual braid monoid as the following example shows.

Example 13. We consider the Coxeter Group of type A_4 identified with the symmetric group on 5 letters. We number the transpositions $s_1, \ldots, s_{10}$ in the order $(1,2), (2,3), (3,4), (4,5), (1,3), (2,4), (3,5), (1,4), (2,5), (1,5)$. We choose as Coxeter element the product $s_1 \cdots s_4$. The atoms of the corresponding dual monoid are in one to one correspondence with the transpositions. We denote the atoms again by $s_1, \ldots s_{10}$. Let P be the standard parabolic subgroup generated by the atom s_5; then right-lcm$(s_5, s_6) = s_5 s_1 s_3$, thus $v(s_6, P) = s_1 s_3$ and right-lcm$(s_5, s_1) = s_5 s_1$ so that $v(s_1, P) = s_1$.

Note that s_6 does not divide $v(s_6, P) = s_1 s_3$.

Lemma 14. *If $b, g, g' \in M$ and b^g and $b^{g'}$ are defined, then so are $b^{\text{left-gcd}(g,g')}$ and $b^{\text{right-lcm}(g,g')}$.*

Proof. The conditions for b^g and $b^{g'}$ to be defined are $g \preccurlyeq bg$ and $g' \preccurlyeq bg'$. It follows easily that left-gcd$(g, g') \preccurlyeq b\,$left-gcd(g, g') and right-lcm$(g, g') \preccurlyeq b\,$right-lcm(g, g'). □

For $g \in M$ we denote by $H(g)$ the first term of its Garside normal form, equal to left-gcd(g, Δ) and define $T(g)$ by $g = H(g)T(g)$.

Proposition 15. *If $P \xrightarrow{g}$ is a ribbon, so are all the terms of its Garside normal form.*

Proof. It is sufficient to prove that $P \xrightarrow{H(g)}$ and $P^{H(g)} \xrightarrow{T(g)}$ are ribbons.

For the first fact, we have $H_P(H(g)) = 1$ since $H_P(g) = 1$, and $P^{H(g)}$ is defined by Lemma 14, since $H(g) = $ left-gcd(g, Δ).

For the second fact, since $(P^{H(g)})^{T(g)} = P^g$ is defined, it is sufficient to show that $T(g)$ is $P^{H(g)}$-reduced. By Lemma 12 with $H(g), T(g)$ for g, h, we get that $H_{P^{H(g)}}(T(g)) = H_P(H(g)T(g))^{H(g)} = 1$. □

Proposition 16. *If $P \xrightarrow{g}$ and $P \xrightarrow{g'}$ are in the ribbon category, then also the map $P \xrightarrow{\text{left-gcd}(g,g')}$*

Proof. It is clear that $H_P(\text{left-gcd}(g, g')) = 1$, and $P^{\text{left-gcd}(g,g')}$ is defined by lemma 14. □

Proposition 17. *If $P \xrightarrow{g}$ and $P \xrightarrow{g'}$ are in the ribbon category, then also the map $P \xrightarrow{\text{right-lcm}(g,g')}$.*

Proof. It follows from Lemma 14 that $P^{\text{right-lcm}(g,g')}$ is defined. It remains to show that $k = \text{right-lcm}(g, g')$ is P-reduced. Note that Lemma 12 implies that if $P \xrightarrow{g}$ is in the ribbon category and $g \preccurlyeq k$ then $g \preccurlyeq T_P(k)$. It follows that g and g' left-divide $T_P(k)$, thus $k = \text{right-lcm}(g, g')$ left-divides $T_P(k)$ which proves that k is P-reduced. $\square$

Proposition 18. *For P a standard parabolic submonoid, let $g \in M$ be P-reduced. Then there is a unique maximal left-divisor of g which is a P-ribbon. If we denote by $R_P(g)$ this left-divisor, then $R_P(g)^{-1}g$ is $P^{R_P(g)}$-reduced and there is equivalence between:*

(i) $R_P(g) = 1$.
(ii) *Any atom which left-divides $\Delta_P g$ is in P.*

Proof. The existence of $R_P(g)$ is a consequence of the fact that the ribbon category is stable by right-lcms. The fact that $R_P(g)^{-1}g$ is $P^{R_P(g)}$-reduced is an immediate consequence of Lemma 12.

We finally prove the equivalence of (i) and (ii) by observing that $R_P(g) \neq 1$ is equivalent to the existence of an atom $s \notin P$ such that $v_{s,P} \preccurlyeq g$, which is in turn equivalent to $\Delta_{P(s)} \preccurlyeq \Delta_P g$ which is equivalent to s left-dividing $\Delta_P g$. $\square$

Proposition 19. *Let P be standard parabolic submonoid of M, and let $b \in P$ and $g \in M$ be such that g is P-reduced and b^g is defined. Assume further that $\Delta_P \preccurlyeq b^i$ for some integer i. Then g is a P-ribbon.*

Proof. The fact that b^g is defined implies that $(b^i)^g$ is defined, thus replacing b by b^i we may assume that $\Delta_P \preccurlyeq b$; let us thus write $b = \Delta_P v$.

The assumption that b^g is defined can be written $g \preccurlyeq \Delta_P vg$. If s is an atom left-dividing g it follows, using Assumption 3, that $\text{right-lcm}(\Delta_P, s) = \Delta_{P(s)} \preccurlyeq \Delta_P vg$, whence $v_{s,P} \preccurlyeq vg$. By Lemma 11 it follows that $v_{s,P} \preccurlyeq T_P(vg) = g$.

We conclude by Noetherian induction on g, since replacing simultaneously g by $v_{s,P}^{-1}g$, P by $P^{v_{s,P}}$ and b by $b^{v_{s,P}}$ all the assumptions remain. $\square$

We conjecture that the assumption in Proposition 19 that $\Delta_P \preccurlyeq b^i$ for some i can be replaced by the assumption that P is the smallest standard parabolic submonoid containing b. For spherical Artin monoids, this is a result of the beautiful paper [3]. If we could prove that conjecture, we could extend all results of [3] to our setting.

We now give a version of Proposition 19 where g is not P-reduced.

Proposition 20. *Let P be a standard parabolic submonoid of M and let $b \in P$, $g \in M$ be such that $b^g \in M$. Assume that for some $i > 0$ we have $\Delta_P \preccurlyeq b^i$. Then $T_P(g)$ is a P-ribbon.*

Proof. Let us prove first that $b^{H_P(g)} \in P$, that is $H_P(g) \preccurlyeq bH_P(g)$. Indeed, from $g \preccurlyeq bg$ we get $H_P(g) \preccurlyeq H_P(bg) = bH_P(g)$. Since $T_P(g)$ is P-reduced we can now apply Proposition 19 with b replaced by $b^{H_P(g)}$ and g replaced by $T_P(g)$ and we get the result. $\square$

Lemma 21. *Let P and Q be standard parabolic submonoids of M and $g \in M$, $k > 0$ be such that $(\Delta_P^k)^g \in Q$; then $T_P(g)$ is a P-ribbon and $P^{T_P(g)} \subset Q$. If in addition $(\Delta_P^k)^g = \Delta_Q^k$ then $P^{T_P(g)} = Q$.*

Proof. Proposition 20 applied with $b=\Delta_P^k$ shows that $T_P(g)$ is a P-ribbon. Replacing Δ_P^k by some power, we can assume that $k>l_\Delta(H_P(g))$, where l_Δ is the Garside length, that is the number of factors in a Garside normal form. We then have $\Delta_P\preccurlyeq(\Delta_P^k)^{H_P(g)}\in P$. Now any atom of P left-divides $(\Delta_P^k)^{H_P(g)}$ and is thus conjugate to an element of Q by $T_P(g)$, hence $P^{T_P(g)}\subset Q$.

If $(\Delta_P^k)^g=\Delta_Q^k$, let $Q'=P^{T_P(g)}$. We have $\Delta_Q^k=(\Delta_P^k)^g=((\Delta_P^k)^{H_P(g)})^{T_P(g)}\subset Q'$. Hence the divisors of Δ_Q in particular all the atoms of Q are in Q', so that $Q=Q'$. $\square$

Garside Groups

We now denote by G the group of fractions of the Garside monoid M (which exists since M is an Ore monoid, see [5, 2.32, V and 3.11, II]).

We call standard parabolic subgroup G_P the group of fractions of a standard parabolic submonoid P.

Definition 22. *We say that $p^{-1}q$ is a left reduced fraction for $b\in G$ if $b=p^{-1}q$ with $p,q\in M$ and the left-gcd of p and q is trivial.*

Symmetrically there are right reduced fractions pq^{-1}. By “reduced fraction” we will mean left reduced fraction.

The reduced fraction for an element is unique; more precisely if $p^{-1}q$ is reduced and $p^{-1}q=p'^{-1}q'$ there exists d such that $p'=dp$ and $q'=dq$ (see [5, 3.11, II]).

By [5, II, 3.18] G_P is a subgroup of G and $G_P\cap M=P$. If an element of G_P has $p^{-1}q$ as its reduced fraction in G, then $p,q\in P$ since there is a reduced fraction $a^{-1}b$ in P and $a\succcurlyeq p$ and $b\succcurlyeq q$. It follows that if P and Q are standard parabolic submonoids and $P\subsetneq Q$ then $G_P\subsetneq G_Q$.

Lemma 23. *Let $b\in G_P$ and $g\in M$ be such that $b^g\in M$. Then $b^{H_P(g)}\in P$.*

Proof. Let $b'=b^g$, and write $g=H_P(g)T_P(g)$. Let $b^{H_P(g)}=pq^{-1}$ where the right-hand side is a right reduced fraction in P, that is $p,q\in P$. From the equality of the two fractions $T_P(g)b'T_P(g)^{-1}=pq^{-1}$ we deduce $q\preccurlyeq T_P(g)$ which implies $q=1$ since $T_P(g)$ is P-reduced. $\square$

Assumption 24. *We assume that the automorphism induced by the conjugation by Δ is of finite order, or equivalently that some power of Δ is central.*

From now on we assume that Assumption 24 holds. It holds in spherical Artin monoids and in the dual braid monoids of well generated complex reflection groups.

Proposition 25. *For two standard parabolic submonoids P and Q and $g\in G$ the three following properties are equivalent*

(i) $G_P^g\subset G_Q$
(ii) *There exists $k>0$ such that $(\Delta_P^k)^g\in G_Q$.*
(iii) *There exist $p\in G_P$ and a central power Δ^i of Δ such that $P\xrightarrow{pg\Delta^i}$ is a ribbon such that $P^{pg}\subset Q$.*

Proof. Clearly (iii) implies (i) and (i) implies (ii). We prove that (ii) implies (iii). Let $u=(\Delta_P^k)^g\in G_Q$. Multiplying g^{-1} by some central power Δ^i of Δ we get $h\in M$ such that $u^h=\Delta_P^k$. Then by the Lemma 23 we have $u^{H_Q(h)}\in Q$. We thus have $(\Delta_P^k)^{T_Q(h)^{-1}}\in Q$. Multiplying $T_Q(h)^{-1}$ by a central power Δ^j of Δ we get $m\in M$

such that $(\Delta_P^k)^m \in Q$. By Lemma 21 $P \xrightarrow{T_P(m)}$ is a ribbon. Now $h \in g^{-1}\Delta^i$, thus $T_Q(h) \in Qg^{-1}\Delta^i$, whence $m \in gG_Q\Delta^{j-i}$ and $T_P(m)$ is in $PgG_Q\Delta^{j-i} = G_Pg\Delta^{j-i}$, the last equality since g conjugates G_Q to G_P. □

Proposition 26. *Assume Assumption 24. For two standard parabolic submonoids P, Q we have equivalence between:*

- (i) *There is $g \in G$ such that $G_P^g \cap G_Q$ is of finite index in G_P^g and in G_Q.*
- (ii) *There is $g \in G$ such that $G_P^g = G_Q$.*
- (iii) *There exists an integer $k > 0$ and $g \in G$ such that $(\Delta_P^k)^g = \Delta_Q^k$.*
- (iv) *There exists a ribbon $P \xrightarrow{g}$ such that $P^g = Q$.*

Proof. It is clear that g satisfying (iv) satisfies (iii). If g satisfies (iii), we may assume that $g \in M$ up to multiplying g by some central power of Δ. Then Lemma 21 shows that $T_P(g)$ satisfies (iv). Thus (iv) and (iii) are equivalent.

It is clear that g satisfying (iv) satisfies (ii), and that g satisfying (ii) satisfies (i). It is thus enough to show that (i) implies (iv).

So we assume (i). Since $G_P^g \cap G_Q$ is of finite index in G_P, there is some positive power k such that $(\Delta_P^k)^g \in G_Q$. It follows by Proposition 25 that there exists $g' \in M$ of the form $g\Delta^i q$ with Δ^i a central power and $q \in G_Q$ such that $P \xrightarrow{g'}$ is a ribbon (and $P^{g'} \subset Q$).

The element g' satisfies clearly the same finite index assumptions as g. Thus there exists k such that $G_P^{g'} \ni \Delta_Q^k$. Let $u \in G_P$ be such that $u^{g'} = \Delta_Q^k$. By Lemma 23 we have $u^{H_P(g')} \in P$, but $H_P(g') = 1$ thus $P^{g'} \ni \Delta_Q^k$. This implies that $P^{g'} = Q$ since $P^{g'}$ is a standard parabolic submonoid, thus contains all divisors of its element Δ_Q^k, that is all atoms of Q. □

For a similar proposition for spherical Artin groups see [4, Théorème 0.1].

Corollary 27. *It is equivalent for standard parabolic submonoids P and Q that $g \in G$ conjugates G_P onto G_Q, or that it conjugates some central power Δ_P^k to Δ_Q^k. If $g \in M$ is P-reduced it is equivalent that it conjugates Δ_P to Δ_Q or that it conjugates P to Q.*

In particular if G_P is conjugate to G_Q and Δ_p^k is the smallest power central in P then Δ_Q^k is the smallest power central in Q.

Proof. We remark that in the proof (iii) ⇒ (iv) ⇒ (ii) of Proposition 26 the element obtained differs from g by a central power of Δ and an element of P, which gives that if g conjugates Δ_P^k to Δ_Q^k it conjugates P to Q.

In the proof of (ii) ⇒ (iii) of Proposition 26 the element obtained is in $G_Pg\Delta^iG_Q$ for some central power Δ^i. If Δ_P^k is central this element has the same effect as g on it. This proves the reverse implication.

The second sentence results from Lemma 21. □

We call parabolic subgroups the conjugates of the standard parabolic subgroups. Note that the notion of parabolic subgroup of G depends on the Garside monoid M; in particular in a spherical Artin monoid the parabolic subgroups for the ordinary Garside structure are not the same as those for the dual Garside structure.

Definition 28. *If $K = G_P^g$ is a parabolic subgroup, we denote by z_K the element $(\Delta_P^k)^g$ where Δ_P^k is the smallest central power of Δ_P.*

The notation z_K makes sense thanks to the following proposition:

Proposition 29. *Let K, K' be parabolic subgroups of G.*

- *z_K depends only on K.*
- *It is equivalent that $g \in G$ conjugates K to K' or z_K to $z_{K'}$.*

Proof. For the first item, let $K = G_P^g$ and $K = G_{P'}^{g'}$ two ways of conjugating K to a standard parabolic subgroup. This defines two candidates for z_K which are $(\Delta_P^k)^g$ and $(\Delta_{P'}^k)^{g'}$. But these two elements are equal by Corollary 27 since $g'g^{-1}$ conjugates P' to P, thus $\Delta_{P'}^k$ to Δ_P^k.

For the second item, let $K = G_P^g$ and $K' = G_{P'}^{g'}$ be ways of conjugating K, K' to standard parabolic subgroups. Then $z_K = (\Delta_P^k)^g$ and $z_{K'} = (\Delta_{P'}^k)^{g'}$. Now $K^x = K'$ for some $x \in G$ is equivalent to $G_P^{gxg'^{-1}} = G_{P'}$ and we conclude again by Corollary 27. □

Proposition 30. *Let $K = G_P^b$ be a parabolic subgroup of G, where P is a standard parabolic submonoid and $b \in M$. Define b' by $H_P(b)R_P(T_P(b))b' = b$ and Q by $Q = P^{R_P(T_P(b))}$. Then $b'^{-1}\Delta_Q^k b'$ is the reduced fraction of z_K, where Δ_Q^k is the smallest central power of Δ_Q in Q.*

Proof. We first remark that by definition we have $(\Delta_P^k)^b = z_K$. We may clearly replace b in this equality by $T_P(b)$. Let $c = R_P(T_P(b))$; we have $(\Delta_P^k)^c = \Delta_Q^k$. We thus get $z_K = b'^{-1}\Delta_Q^k b'$. We claim this is a reduced fraction. Indeed by construction $R_Q(b') = 1$ thus by Proposition 18(ii) any atom left-dividing $\Delta_Q b'$ is in Q, thus the same is true for $\Delta_Q^k b'$ by induction, using that for $k > 1$ one has $H(\Delta_Q^k b') = H(\Delta_Q H(\Delta_Q^{k-1} b'))$. Since b' is Q-reduced, the fraction is reduced. □

Note that b' above is minimal such that ${}^{b'}K$ is standard, that is any $u \in M$ such that ${}^uK$ is standard is a left multiple of b', and G_Q is a "canonical" standard parabolic subgroup conjugate to K. For spherical Artin groups the proposition is [2, Theorem 3] and the element b' is called a minimal standardizer.

An immediate consequence of Proposition 30 if that it is equivalent that $z_K \in M$ or that K is standard.

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(F. Digne) Laboratoire Amiénois de Mathématique Fondamentale et Appliquée, CNRS UMR 7352, Université de Picardie-Jules Verne, 80039 Amiens Cedex France.

Email address: digne@u-picardie.fr

URL: www.lamfa.u-picardie.fr/digne

(J. Michel) Institut Mathématique de Jussieu – Paris rive gauche, CNRS UMR 7586, Université de Paris, Bâtiment Sophie Germain, 75013, Paris France.
Email address: `jean.michel@imj-prg.fr`
URL: `webusers.imj-prg.fr/~jean.michel`