Periodic solutions of approximated normal forms and simplification of normal transform by using Gröbner basis

Jean Flosi, Claude-Henri Lamarque, Alireza Ture Savadkoohi

Periodic solutions of approximated normal forms and simplification of normal transform by using Gröbner basis

J. Flosi[a,*], C.-H. Lamarque[a], A. Ture Savadkoohi[a]

[a] *Univ Lyon, ENTPE, LTDS UMR CNRS 5513, Rue Maurice Audin - F-69518 Vaulx-en-Velin Cedex, France*

Abstract

The paper proposes a methodology based on the normal form perturbation method and Gröbner basis approach. Spectrum rearrangement allows to exhibit resonant terms associated with periodic behaviors of the system. Normal forms generate compatibility equations when one looks for periodic solutions and define amplitude-frequency conditions. Gröbner generators are defined from these equations and associated ideal is generated in order to simplify results of final normal form using generalized Euclidean division. Normal transform are simplified, and in some cases can be linearized with nonlinearity remaining hidden in amplitude-frequency conditions.

Keywords:
Normal Form, Gröbner basis, nonlinear, spectrum

1. Introduction

Normal form has been introduced by Poincaré [1]. It is based on a change of variables introduced as the identity perturbation. Then several normal forms have been introduced, such as normal form of Birkhoff [2] or normal form of Gustavson [3], and more detailed information on normal forms are provided in [4–6]. In this paper, we will consider classical normal form and normal transform. When using normal forms of nonlinear dynamical systems, physicists normally introduce mono-harmonic periodic solutions of the normal form [7]. In such a case, the obtained amplitude equation provides a relation between amplitudes of oscillations (amplitude of normal coordinates) and the different parameters. Indeed, since the amplitude equation contains resonant terms, it can be written directly with normal coordinates and not only with amplitudes of normal coordinates. This leads to the following index: if amplitude equations are seen as generators of an ideal in a rig of polynomials, is it possible to simplify the expression of initial coordinates as functions of normal coordinates? Or, is it possible to simplify the normal transform taking into account amplitude

*Corresponding author-jean.flosi@entpe.fr

equations? As it is already used in nonlinear mechanics [8], a way to answer this question is to consider Gröbner basis [9] associated to the set of generators (amplitude equations), and to calculate the rest of the generalized "Euclidean division" for each normal transform. This is the idea that we intend to explain and illustrate in this paper. The paper is organized as it follows: in Sec.2, one normal form methodology is presented and illustrated for two particular cases. Section 3 proposes an approach leading to forced or internally resonant systems. Finally, sec.4 presents a use of Gröbner basis simplifying normal form results based on generators defined for the resonant periodic system. Applications are given in sec.5 and pros and cons are discussed in conclusion.

2. Normal form calculation

2.1. General case for autonomous dynamical systems

The normal form is applied on a general dynamical system following the governing matrix equation:

$$\frac{dX}{dt} = A.X + F(X) \tag{1}$$

where A is a linear operator. The system has a finite dimension and A can therefore be defined as a $2N$ times $2N$ matrix.
X is the vector composed of the dynamics coordinates. The vector $F(X)$ has a maximal chosen degree $n \geq 2$ and can be decomposed in $F(X) = F_2(X) + ... + F_n(X)$.

This approach is based on the normal form. For this matter, we introduce:

$$\begin{cases} U \text{ the normal coordinates} \\ \phi = \phi_2 + ... + \phi_n \text{ the normal transformation} \\ R = R_2 + ... + R_n \text{ the resonant terms} \end{cases} \tag{2}$$

where ϕ_k and R_k are decomposed in the base of monomials of degree k in the $2N$ variables $u_1, ..., u_{2N}$ for each component of vector equation k.
We introduce the identity perturbation for X:

$$X = U + \phi_2(U) + ... + \phi_n(U) \tag{3}$$

Injecting normal expression of X in Eq.1, we obtain:

$$\begin{aligned} \frac{dX}{dt} &= A.X + F(X) = A.\left(U + \phi_2(U) + ... + \phi_n(U)\right) \\ &+F\left(U + \phi_2(U) + ... + \phi_n(U)\right) \end{aligned} \tag{4}$$

The expression of normal forms reads:

$$\frac{dU}{dt} = A.U + R_2 + ... + R_n \tag{5}$$

and we can write the time derivative of Eq.3 as:

$$\frac{dX}{dt} = (I_d + \partial\,\phi_2 + ... + \partial\,\phi_n)\,\frac{dU}{dt} \tag{6}$$

Finally, resolution of normal form reads:

$$\left\{\begin{array}{l} (I_d + \partial\,\phi_2 + ... + \partial\,\phi_n)\,(A.U + R_2 + ... + R_n) \\ = A.\,(U + \phi_2(U) + ... + \phi_n(U)) + F\,(U + \phi_2(U) + ... + \phi_n(U)) \\ \\ \dfrac{dU}{dt} = A.U + R_2 + ... + R_n \end{array}\right. \tag{7}$$

Using Taylor decomposition, the first equation of Eq.7 can be re-written as:

$$\begin{array}{l} (I_d + \partial\,\phi_2 + ... + \partial\,\phi_n)\,(A.U + R_2 + ... + R_n) \\ = A.U + A.\phi_2(U) + ... + A.\phi_n(U) \\ +G_2(U) + ... + G_n(U) + \text{h. o. t.} \\ \\ = A.U + \underbrace{A.\phi_2(U) + G_2(U)}_{d_2} + ... + \underbrace{A.\phi_n(U) + G_n(U)}_{d_n} + \text{h. o. t.} \end{array} \tag{8}$$

In this decomposition:

$$\left\{\begin{array}{l} G_2 \text{ depends on } F_2 \\ G_k \text{ depends on } F_2...F_k, \phi_2...\phi_k, 3 \leq k \leq n \end{array}\right. \tag{9}$$

The normal form in Eq.8 is treated degree per degree as:

$$\left\{\begin{array}{l} d_1\,:\, A.U = A.U \\ d_2\,:\, \partial\phi_2 A.U - A.\phi_2(U) = G_2(U) - R_2 = F_2 - R_2 \\ \vdots \\ d_k\,:\, \partial\phi_k A.U - A.\phi_k(U) = G_k(U) - R_k \end{array}\right. \tag{10}$$

These equations can be set as a system of linear matrix using lexicographic order on monomials and using vector basis of $\mathbb{R}_{2N}$ (see [5]). It can be solved using Fredhom alternative for the linear system governed by matrix $\mathcal{L}$. In that case, resonant and normal terms are defined by:

$$\left\{\begin{array}{l} R_k = Proj_{Ker(\mathcal{L})}(G_k) \\ \phi_k\,:\, \text{solution of the linear system on Range}(\mathcal{L}\ ) \end{array}\right. \tag{11}$$

$Proj_{Ker(\mathcal{L})}(G_k)$ is the projection of matrix G_k on vector of the kernel space of matrix $\mathcal{L}$ and Range($\mathcal{L}$) is the column space of G_k. According to the form of the matrix A, we will distinguish two cases.

2.2. Matrix A being diagonal

In this section, we consider a diagonalized system. We set $A = D$:

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_{2N} \end{bmatrix} \tag{12}$$

The matrix $\mathcal{L}$ (associated to homological equation of Iooss-Adelmayer [5]) is now diagonal on the vectorial base composed by $2N$ vectors:

$$\left(\begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}, ..., \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \right) \tag{13}$$

In order to manipulate $\mathcal{L}$, we introduce $p_1 ... p_{2N}$ defining the projection $u_1^{p_1} ... u_{2N}^{p_{2N}}$ associated to the k^{th} vector (1 on the k^{th} position of the vector).
The k^{th} diagonal term of $\mathcal{L}$ has for expression:

$$\mathcal{L}_{kk} = \underbrace{[p_1, ..., p_{2N}]}_{P} . \underbrace{[\lambda_1, ..., \lambda_{2N}]^T}_{\Lambda} - \lambda_k \tag{14}$$

Therefore, we can distinguish two situations :

$$\begin{cases} \langle P \mid \Lambda \rangle - \lambda_k = 0 : & \text{this case provides resonant terms} \\ \langle P \mid \Lambda \rangle - \lambda_k \neq 0 : & \text{this case is non-resonant: the corresponding monomial} \\ & \text{can be eliminated} \end{cases} \tag{15}$$

<u>*Example 1:*</u> Let us consider a single degree of freedom (dof) nonlinear system defined as:

$$\frac{d^2x}{dt^2} + x + cx^3 = 0 \tag{16}$$

The matrix form of this system is:

$$X = \begin{bmatrix} \dot{x} \\ x \end{bmatrix}, \qquad \dot{X} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X + \begin{bmatrix} cx^3 \\ 0 \end{bmatrix} \tag{17}$$

This system can be diagonalized using:

$$\begin{cases} X = PY = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} Y \\ \\ \dot{Y} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} Y + \begin{bmatrix} \frac{i}{2} c(y_1 + y_2)^3 \\ -\frac{i}{2} c(y_1 + y_2)^3 \end{bmatrix} \end{cases} \tag{18}$$

Normal coordinates U are introduced. Third order terms are decomposed according to four monomials u_1^3, $u_1^2 u_2$, $u_1 u_2^2$ and u_2^3. Solving normal form equation at third order, matrix $\mathcal{L}$ is defined:

$$\underbrace{\begin{bmatrix} -2i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2i \end{bmatrix}}_{\mathcal{L}} \begin{bmatrix} \langle \phi_{31} \mid u_1^3 \rangle \\ \langle \phi_{31} \mid u_1^2 u_2 \rangle \\ \langle \phi_{31} \mid u_1 u_2^2 \rangle \\ \langle \phi_{31} \mid u_2^3 \rangle \\ \langle \phi_{32} \mid u_1^3 \rangle \\ \langle \phi_{32} \mid u_1^2 u_2 \rangle \\ \langle \phi_{32} \mid u_1 u_2^2 \rangle \\ \langle \phi_{32} \mid u_2^3 \rangle \end{bmatrix} = \begin{bmatrix} ic/2 \\ 3ic/2 \\ 3ic/2 \\ ic/2 \\ -ic/2 \\ -3ic/2 \\ -3ic/2 \\ -ic/2 \end{bmatrix} \tag{19}$$

The lines number 3 and number 6 are composing the kernel of $\mathcal{L}$ and are inducing resonant terms; corresponding ϕ_3 components are set to 0. All other lines are able to determine ϕ_3 components. Here, $\phi_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The calculation has been done up to the degree 3. Finally:

$$\begin{cases} X = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} Y \\ Y = U + \Phi_3(U) = \begin{bmatrix} u_1 - \frac{c}{4}u_1^3 - \frac{3c}{2}u_1^2 u_2 + \frac{c}{2}u_2^3 \\ \\ u_2 + \frac{c}{2}u_1^3 - \frac{3c}{2}u_1 u_2^2 - \frac{c}{4}u_2^3 \end{bmatrix} \end{cases} \tag{20}$$

2.3. Matrix A being a Jordan matrix

When the system can't be turned as a diagonal system, the Jordan elimination can be applied, which permits to obtain A as a block diagonal matrix:

$$A = \begin{bmatrix} J_{n_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_{n_p} \end{bmatrix} \text{ with } \sum_{j=1}^{p} \dim(J_{nj}) = 2N \tag{21}$$

$\dim(J_{nj})$ stands for the dimension of the matrix J_{nj}.
Each block J_{n_j} is a Jordan block or a diagonal matrix. For each Jordan block, the matrix $\mathcal{L}$ is defined by the same way as in the previous section. Therefore, if the block is diagonal, the resonant terms and normal transformation are defined the same way. If $\mathcal{L}$ is not diagonal due to the couplings between equations, Murdock [4] has developed a way to manage a simple expression of $\mathcal{L}$ using Gröbner basis. This use of Gröbner basis is different from what we are doing here. Otherwise, the resolution can be proceed terms by terms and favoring minimal resonant terms.

3. Resonant spectrum rearrangement

The normal form process can be applied with the spectrum of A without rearrangement, but particular cases could be considered. As found in other normal form applications [10, 11], artificial perturbation is introduced in the system. One of interesting situations to be considered is when the linear part is a perturbation of a matrix. For example:

$$A = A(\varepsilon) = A(0) + \varepsilon\alpha \tag{22}$$

where $A(0)$ possesses eigenvalues $\lambda_1, ..., \lambda_{2N}$, ε is a scalar representing the perturbation parameter and α is a matrix.
This approach permits to introduce a spectrum leading to a situation with a bigger number of resonant terms, or to take into account external resonances between external excitation and internal resonance.
The definition of a small parameter ϵ depends on the system: for nonlinear oscillations of mechanical systems, ϵ can be set according to the damping of the system as:

$$\begin{cases} \text{Without damping}: \ddot{x} + x + cx^3 = 0 \implies A(0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ \text{With damping}: \ddot{x} + \epsilon\dot{x} + x + cx^3 = 0 \implies A(\epsilon) = \begin{bmatrix} -\epsilon & -1 \\ 1 & 0 \end{bmatrix} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad = A(0) + \epsilon \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \end{cases} \tag{23}$$

3.1. Methodology for forced systems

For the forced system, we introduce a new variable Z:

$$\dot{\tilde{X}} = A.\tilde{X} + F(X,t) = A.\tilde{X} + \tilde{F}(\tilde{X}, Z) \tag{24}$$

where $Z = Z(t)$ is the solution of a system of differential equation $\dot{Z} = B.Z + H(Z)$ verified by the forcing.
In particular, if time dependency occurs via external excitation or parametric nonlinearities which can be expressed as functions of $\sin(\omega t)$ only, we can use:

$$\left.\begin{matrix} z_1 = \sin(\omega t) \\ Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{matrix}\right\} \implies \dot{Z} = \begin{bmatrix} 0 & -\omega^2 \\ 1 & 0 \end{bmatrix} Z = B.Z \tag{25}$$

We can now define a new vector X as:

$$X = \begin{bmatrix} \tilde{X} \\ Z \end{bmatrix} \implies \dot{X} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} X + F(X) \tag{26}$$

The forcing system is now expressed as an autonomous system with coordinate X.

We can now work on the spectrum of $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

The idea is to exploit a $1:k$ resonance with external excitation. The normal form doesn't justify to consider higher order nonlinearities and as it will be explained, we will not consider higher resonances than $k = 3$.

Example 2: The methodology is presented through an example of the following basic system:

$$\ddot{x} + x = \sin(\omega t) \tag{27}$$

The procedure is the same for other types of excitation. We can define:

$$\begin{aligned} \tilde{X} &= \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \\ \dot{\tilde{X}} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tilde{X} + \begin{bmatrix} \sin(\omega t) \\ 0 \end{bmatrix} \end{aligned} \tag{28}$$

Using Eq.25, we now have the equation:

$$\frac{d}{dt} \begin{bmatrix} \tilde{X} \\ Z \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega^2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{X} \\ Z \end{bmatrix} \tag{29}$$

The spectrum of this equation is $\{-i, i, -i\omega, i\omega\}$. In order to investigate the $1:1$ resonance, we need to set $\omega = 1$.
In order to match to the physical system, we prefer to define $A = A_0 + O(\epsilon)$ with:

$$A = P \left(\underbrace{\begin{bmatrix} -i\omega & 0 & 0 & 0 \\ 0 & i\omega & 0 & 0 \\ 0 & 0 & -i\omega & 0 \\ 0 & 0 & 0 & i\omega \end{bmatrix} + i\varepsilon \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{D} \right) P^{-1} \tag{30}$$

We can then deduce:

$$
A_0 = P \begin{bmatrix} -i\omega & 0 & 0 & 0 \\ 0 & i\omega & 0 & 0 \\ 0 & 0 & -i\omega & 0 \\ 0 & 0 & 0 & i\omega \end{bmatrix} P^{-1} = \begin{bmatrix} 0 & -\omega & 0 & \dfrac{\omega}{1+\omega} \\ \omega & 0 & \dfrac{1}{1+\omega} & 0 \\ 0 & 0 & 0 & -\omega^2 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$
$$
A - A_0 = \begin{bmatrix} 0 & \omega - 1 & 0 & \dfrac{1}{1+\omega} \\ 1-\omega & 0 & -\dfrac{1}{1+\omega} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{31}
$$

We now need to define $\omega - 1$ as a function of ϵ. We can distinguish 3 cases of $\omega - 1$:
- as a low order polynomial;
- as a series and do a truncation in order to consider the firsts n orders $n \geq k$ for the normal forms;
- as a singular.

The definition of normal forms Eq.5 allows to only consider the firsts orders because higher order resonant terms have less effects on the behavior. Therefore, we only consider the firsts orders of series or polynomials. Moreover, resonant terms of high order harmonics $1 : k$ would need to consider high order polynomials or series in order to obtain a 0 diagonal terms in the construction of matrix $\mathcal{L}$ (see Eq.14). Therefore, we do not consider higher order resonance $1 : k$ superior to $k = 3$.

3.2. Methodology for internally resonant systems

If the system is composed of different resonators, it's possible to exploit internal resonances by adding a link between different modal frequencies.
Let's consider a system with two resonators verifying $\omega_2 = 2\omega_1 + \epsilon\sigma$.

$$
\begin{cases} \ddot{x}_1 + \omega_1^2 x_1 + \cdots = 0 \\ \ddot{x}_2 + \omega_2^2 x_2 + \cdots = 0 \end{cases} \quad \text{with } \omega_2 = 2\omega_1 + \varepsilon\sigma \tag{32}
$$

The same explained procedure can be applied, defining for example:

$$
D = \begin{bmatrix} -i\omega_1 & 0 & 0 & 0 \\ 0 & i\omega_1 & 0 & 0 \\ 0 & 0 & -2i\omega_1 & 0 \\ 0 & 0 & 0 & 2i\omega_1 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sigma & 0 \\ 0 & 0 & 0 & \sigma \end{bmatrix} \tag{33}
$$

and permitting the $1 : 2$ resonance between the two resonators.

4. Simplification of normal transform using Gröbner basis

To describe the method, let us consider a general case Eq.1 with diagonalizable matrix:

$$\left\{\begin{array}{l} X = P.Y \\ \dot{Y} = P^{-1}.A.P + P^{-1}.F(P.Y) \text{ with } P^{-1}.A.P = D \end{array}\right. \tag{34}$$

Let us assume that $\lambda_{2j} = \overline{\lambda}_{2j-1}$, $j = 1, \ldots, N$ and $\Im(\lambda_{2j-1}) < 0$.
The resolution is led after defining:

$$\left\{\begin{array}{l} \text{Identity perturbation: } Y = U + \phi_2(U) + \phi_3(U) + \ldots \\ \\ \text{Normal forms: } \dfrac{dU}{dt} = D.U + R_2(U) + R_3(U) + \ldots \\ \\ \text{Normal transform: } X = P.Y = P.(U + \phi_2(U) + \phi_3(U) + \ldots) \end{array}\right. \tag{35}$$

The last equation is the normal transform expressing X as a function of normal coordinates U.

4.1. Introduction of periodic solutions

Based on previous assumptions, we can look for approximated periodic solutions $U(t)$ of the normal form. We obtain approximate periodic solutions $X(t)$ via the normal transform. Since $\Im(\lambda_{2j-1}) < 0$, $j = 1, \ldots, N$, if the principal resonance is the only one considered, we can look for:

$$U = \begin{bmatrix} u_1 \\ \vdots \\ u_{2N} \end{bmatrix} = \begin{bmatrix} U_1 e^{-i\Omega t} \\ \vdots \\ \bar{U}_N e^{i\Omega t} \end{bmatrix} \tag{36}$$

with $u_{2j-1} = U_j e^{-i\Omega t}$ and $u_{2j} = \overline{U}_j e^{i\Omega t}$

4.2. Simplification via Gröbner basis

Introducing expression of U in normal form, one obtains for the $(2j-1)^{th}$ normal form equation:

$$\begin{array}{l} i\Omega U_{2j-1} e^{-i\Omega t} + \lambda_{2j-1} U_{2j-1} e^{-i\Omega t} + R_{2,2j-1}(u_1, \ldots, u_{2N}) + ... \\ +R_{n,2j-1}(u_1, \ldots, u_{2N}) = 0 \end{array} \tag{37}$$

and its conjugated expressions for the $2j^{th}$ normal form equation. In fact, because of resonance conditions, it is clear that this compatibility equation (and its conjugated) can be written for $j = 1, \ldots, N$ as:

$$\left\{\begin{array}{l} 0 = i\Omega u_{2j-1} + \lambda_{2j-1} u_{2j-1} + R_{2,2j-1}(u_1, \ldots, u_{2N}) + ... \\ +R_{n,2j-1}(u_1, \ldots, u_{2N}) \\ \\ 0 = -i\Omega u_{2j} + \lambda_{2j} u_{2j} + R_{2,2j}(u_1, \ldots, u_{2N}) + ... \\ +R_{n,2j}(u_1, \ldots, u_{2N}) \end{array}\right. \tag{38}$$

These $2N$ compatibility equations can be written as:

$$\begin{cases} g_{2j-1}(u_1,\ldots,u_{2N}) = 0 \\ g_{2j}(u_1,\ldots,u_{2N}) = 0 \end{cases} \qquad \text{with } j = 1,\ldots,N \tag{39}$$

and they provide $2N$ functions, generating an ideal in the Ring of polynomials with $2N$ variables. The generators of these basis are defined from amplitude equations which are obtained by successively supposing that the normal form solutions are periodic and averaging with respect to time the normal form equations, a methodology that is similar to the one in [10, 11]. The polynomial ideal is now used to get the best Euclidean division decomposition of the physical coordinates:

$$X(U) = \begin{bmatrix} \vdots \\ x_j(U) \\ \vdots \end{bmatrix}, \qquad j = 1,\ldots,N \tag{40}$$

For each x_j, we can write:

$$x_j = \sum_{k=1}^{2N} q_k(U) g_k(U) + r_k(U) \tag{41}$$

where q_k (standing for quotient) are polynomials in $u_1,\ldots,u_{2N}$ and r_k (standing for rest) also. The use of Gröbner basis allows to understand the Euclidean decomposition according to each element of the ideal.
This process should reduce de degree of x_j and it should lead in the best case to a linear expression $x_j(U) = x_{l,j}(U)$
The simplified expression of approximated periodic solution X is then:

$$X(U) = \begin{bmatrix} \vdots \\ x_j(U) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ r_j(U) \\ \vdots \end{bmatrix} \tag{42}$$

This process also justifies the choice of using a designed spectrum in order to produce adapted terms and perform the most complete Gröbner ideal to simplify the expression of $X(U)$.
In the considered examples, we will choose $n = 3$.

Remark: If internal resonances $1 : k_j$ are considered, the vector must be adapted in order to consider these harmonics. We can define:

$$U = \begin{bmatrix} u_1 \\ \vdots \\ u_{2N} \end{bmatrix} = \begin{bmatrix} U_1 e^{-ik_1\Omega t} \\ \vdots \\ \bar{U}_N e^{ik_N\Omega t} \end{bmatrix} \tag{43}$$

with $u_{2j-1} = U_j e^{-ik_j\Omega t}$ and $u_{2j} = \overline{U}_j e^{ik_j\Omega t}$

5. Applications

5.1. Single dof forced system

The previous method will be applied on a simple single dof forced system. The normalized dynamical equation of the system is:

$$\ddot{x} + {w_1}^2 x + c x^3 = f \sin(wt) \tag{44}$$

As explained in the first section, forcing coordinates are integrated in principal coordinates using the following relation:

$$\begin{cases} \ddot{z} + w^2 z = 0 \\ z(0) = 0 \text{ and } \dot{z}(0) = w \end{cases} \tag{45}$$

We can now adopt the matrix form:

$$\underbrace{\frac{d}{dt}\begin{pmatrix} \dot{x} \\ x \\ \omega\cos(\omega t) \\ \sin(\omega t) \end{pmatrix}}_{\frac{dX}{dt}} = \underbrace{\begin{pmatrix} 0 & -\omega_1^2 & 0 & f \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega^2 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \dot{x} \\ x \\ \omega\cos(\omega t) \\ \sin(\omega t) \end{pmatrix}}_{X} + \underbrace{\begin{pmatrix} -c\,x^3 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{NL_3} \tag{46}$$

This system can take a diagonal form by using transfer matrix P defined as:

$$P = \begin{pmatrix} 0 & 0 & \frac{i}{2\omega} & \frac{1}{2} \\ 0 & 0 & -\frac{i}{2\omega} & \frac{1}{2} \\ \frac{i}{2\omega_1} & \frac{1}{2} & \frac{if}{2\omega_1(\omega^2-\omega_1^2)} & \frac{f}{2(\omega^2-\omega_1^2)} \\ -\frac{i}{2\omega_1} & \frac{1}{2} & -\frac{if}{2\omega_1(\omega^2-\omega_1^2)} & \frac{f}{2(\omega^2-\omega_1^2)} \end{pmatrix} \tag{47}$$

with diagonal linear operator D:

$$D = \begin{pmatrix} -i\omega & 0 & 0 & 0 \\ 0 & i\omega & 0 & 0 \\ 0 & 0 & -i\omega_1 & 0 \\ 0 & 0 & 0 & i\omega_1 \end{pmatrix} \tag{48}$$

ω is decomposed in order to allow the $1:1$ resonance as $\omega = \omega_1 + \varepsilon$, leading to:

$$D = \underbrace{\begin{pmatrix} -i\omega & 0 & 0 & 0 \\ 0 & i\omega & 0 & 0 \\ 0 & 0 & -i\omega & 0 \\ 0 & 0 & 0 & i\omega \end{pmatrix}}_{D_0} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i\varepsilon & 0 \\ 0 & 0 & 0 & -i\varepsilon \end{pmatrix}}_{D_1} \tag{49}$$

We introduce $Y = P^{-1}.X$ and $NL = P^{-1}.NL_3$. The system has now for equation:

$$\dot{Y} = D_0.Y + D_1.Y + NL_(Y) \tag{50}$$

We can now solve the system using the normal form at the third order because only cubic nonlinearity is present in the system. The coordinate matrix Y is decomposed in normal coordinates as:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = U + \Phi_2(U) + \Phi_3(U) \tag{51}$$

and compatibility equations are defined by:

$$\frac{d}{dt}U = D_0.U + R_2 + R_3 \tag{52}$$

The normal form equation presented in Eq.7 is applied to the system:

$$\begin{aligned}(I_d + \partial\,\Phi_2 + \partial\,\Phi_3)\left(\frac{d}{dt}U\right) &= D_0\left(\Phi_3(U) + \Phi_2(U) + U\right) \\ + \mathrm{NL}_2\left(\Phi_3(U) + \Phi_2(U) + U\right) &+ \mathrm{NL}_3\left(\Phi_3(U) + \Phi_2(U) + U\right)\end{aligned} \tag{53}$$

And because we only solve up to the third order, we can write:

$$\begin{aligned}(I_d + \partial\,\Phi_2 + \partial\,\Phi_3)\,(R_3 + R_2 + D_0.U) &= D_0\left(\Phi_3(U) + \Phi_2(U) + U\right) \\ + \mathrm{NL}_2\left(\Phi_2(U) + U\right) + \mathrm{NL}_3(U)\end{aligned} \tag{54}$$

The first order form of Eq.54 verifies:

$$D_0.U = D_0.U \tag{ordre 1}$$

The second order verifies:

$$R_2 + \partial\,\Phi_2 D_0.U = D_0.\phi_2(U) + D_1(\epsilon).U \tag{ordre 2}$$

and leads to:

$$R_2 = D_1.U \text{ and } \phi_2 = 0 \tag{55}$$

The third order verifies:

$$R_3 + \partial\,\Phi_3 D_0.U + \partial\,\Phi_2 R_2 = D_0.\phi_3(U) + D_1(\phi_2) + NL(U) \tag{ordre 3}$$

and according to second order identification in Eq.55, we have to solve:

$$R_3 + \partial\,\Phi_3 D_0.U = D_0.\phi_3(U) + NL(U) \tag{ordre 3}$$

Both ϕ_3 and R_3 are determined. X can now be deduced from this method in normal coordinates. In this case, we restrict the solution to only the first harmonic components of x. The solution in normal coordinates is the following highly nonlinear expression:

$$
\begin{aligned}
x &= u_3 + u_4 + \frac{3cf^3u_1^2u_2}{4\omega\omega_1(\omega-\omega_1)^3(\omega+\omega_1)^3} - \frac{3cf^2u_1^2u_4}{4\omega\omega_1(\omega-\omega_1)^2(\omega+\omega_1)^2} \\
&+ \frac{3cf^3u_1u_2^2}{4\omega\omega_1(\omega-\omega_1)^3(\omega+\omega_1)^3} - \frac{3cf^2u_1u_2u_3}{2\omega\omega_1(\omega-\omega_1)^2(\omega+\omega_1)^2} \\
&- \frac{3cf^2u_1u_2u_4}{2\omega\omega_1(\omega-\omega_1)^2(\omega+\omega_1)^2} - \frac{3cf^2u_2^2u_3}{4\omega\omega_1(\omega-\omega_1)^2(\omega+\omega_1)^2} \\
&+ \frac{3cfu_1u_3u_4}{2\omega\omega_1(\omega-\omega_1)(\omega+\omega_1)} + \frac{3cfu_1u_4^2}{4\omega\omega_1(\omega-\omega_1)(\omega+\omega_1)} + \frac{3cfu_2u_3^2}{4\omega\omega_1(\omega-\omega_1)(\omega+\omega_1)} \\
&+ \frac{3cfu_2u_3u_4}{2\omega\omega_1(\omega-\omega_1)(\omega+\omega_1)} - \frac{3cu_3^2u_4}{4\omega\omega_1} - \frac{3cu_3u_4^2}{4\omega\omega_1} - \frac{fu_1}{\omega^2-\omega_1^2} - \frac{fu_2}{\omega^2-\omega_1^2}
\end{aligned}
\tag{56}
$$

Compatibility equations deriving from Eq.52 are the two conjugated equations below:

$$
\begin{cases}
g_1(U) = -iu_3\varepsilon + \dfrac{3iu_3^2u_4}{2\omega_1} + \dfrac{3icf^3u_1^2u_2}{2\omega_1(-\omega+\omega_1)^3(\omega+\omega_1)} \\
+\dfrac{3icf^2u_1u_2u_3}{\omega_1(-\omega+\omega_1)^2(\omega+\omega_1)^2} + \dfrac{3icf^2u_1^2u_4}{2\omega_1(-\omega+\omega_1)^2(\omega+\omega_1)^2} \\
+\dfrac{3icfu_2u_3^2}{2\omega_1(-\omega+\omega_1)(\omega+\omega_1)} + \dfrac{3icfu_1u_3u_4}{\omega_1(-\omega+\omega_1)(\omega+\omega_1)} = 0 \\
g_2(U) = iu_4\varepsilon - \dfrac{3iu_3u_4^2}{2\omega_1} - \dfrac{3icf^3u_1u_2^2}{2\omega_1(-\omega+\omega_1)^3(\omega+\omega_1)} \\
-\dfrac{3icf^2u_1u_2u_4}{\omega_1(-\omega+\omega_1)^2(\omega+\omega_1)^2} - \dfrac{3icf^2u_2^2u_3}{2\omega_1(-\omega+\omega_1)^2(\omega+\omega_1)^2} \\
-\dfrac{3icfu_1u_4^2}{2\omega_1(-\omega+\omega_1)(\omega+\omega_1)} - \dfrac{3icfu_2u_3u_4}{\omega_1(-\omega+\omega_1)(\omega+\omega_1)} = 0
\end{cases}
\tag{57}
$$

The ideal is formed from these two equations and Euclidean division of $x(U)$ by Gröbner ideal components gives the final simplified and linear expression of x:

$$
x = r = x_l = u_3 + u_4 - \frac{f(u_1+u_2)}{(\omega-\omega_1)(\omega+\omega_1)} - \frac{\varepsilon(u_3+u_4)}{2\omega} \tag{58}
$$

This expression can't be used when $\omega = \omega_1$: in that case, the system can't be diagonalized and Jordan decomposition should be used.

5.2. An undamped two dof free system

The system is composed of 2 equal masses, linked together via a nonlinear function define as $c\alpha^3$ where α is relative displacements of the two masses. Both masses are grounded via linear springs k_1 and k_2. Therefore, the system takes the normalized matrix expression:

$$\dot{X} = \begin{pmatrix} 0 & -\omega_1^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega_2^2 \\ 0 & 0 & 1 & 0 \end{pmatrix} X + \begin{pmatrix} -c(x-y)^3 \\ 0 \\ c(x-y)^3 \\ 0 \end{pmatrix} \tag{59}$$

Using normal coordinates U and diagonalization of the system using matrix P, we can write:

$$\dot{U} = \begin{pmatrix} -i\omega_1 & 0 & 0 & 0 \\ 0 & i\omega_1 & 0 & 0 \\ 0 & 0 & -i\omega_2 & 0 \\ 0 & 0 & 0 & i\omega_2 \end{pmatrix} .U + \begin{pmatrix} -\dfrac{ic(u_1+u_2-u_3-u_4)^3}{2\omega_1} \\ \dfrac{ic(u_1+u_2-u_3-u_4)^3}{2\omega_1} \\ \dfrac{ic(u_1+u_2-u_3-u_4)^3}{2\omega_2} \\ -\dfrac{ic(u_1+u_2-u_3-u_4)^3}{2\omega_2} \end{pmatrix} \tag{60}$$

Once more, X is expressed using periodic normal coordinates and we only keep the first harmonic of the solution. We finally have:

$$\begin{aligned} x(U) = {} & u_1 + u_2 - \frac{3cu_1^2u_2}{4\omega_1^2} - \frac{3cu_1^2u_4}{2\omega_1(\omega_1-\omega_2)} - \frac{3cu_1u_2^2}{4\omega_1^2} + \frac{3cu_1u_2u_3}{\omega_1(\omega_1-\omega_2)} \\ & + \frac{3cu_1u_2u_3}{\omega_1(\omega_1+\omega_2)} + \frac{3cu_1u_2u_4}{\omega_1(\omega_1-\omega_2)} + \frac{3cu_1u_2u_4}{\omega_1(\omega_1+\omega_2)} - \frac{3cu_1u_3u_4}{2\omega_1^2} - \frac{3cu_1u_4^2}{4\omega_1(\omega_1-\omega_2)} \\ & - \frac{3cu_1u_4^2}{4\omega_1\omega_2} - \frac{3cu_2^2u_3}{2\omega_1(\omega_1-\omega_2)} - \frac{3cu_2u_3^2}{4\omega_1(\omega_1-\omega_2)} - \frac{3cu_2u_3^2}{4\omega_1\omega_2} - \frac{3cu_2u_3u_4}{2\omega_1^2} \\ & + \frac{3cu_3^2u_4}{2\omega_1(\omega_1-\omega_2)} + \frac{3cu_3^2u_4}{2\omega_1(\omega_1+\omega_2)} + \frac{3cu_3u_4^2}{2\omega_1(\omega_1-\omega_2)} + \frac{3cu_3u_4^2}{2\omega_1(\omega_1+\omega_2)} \end{aligned} \tag{61}$$

Introducing periodic normal coordinates in normal equations, the method leads to 4 compatibility equations conjugated two by two:

$$\begin{cases} g_1(U) = \dfrac{3icu_1^2u_2}{2\omega_1} + \dfrac{3icu_1u_3u_4}{\omega_1} - iu_1(\omega-\omega_1) = 0 \\ g_2(U) = -\dfrac{3icu_1u_2^2}{2\omega_1} - \dfrac{3icu_2u_3u_4}{\omega_1} + iu_2(\omega-\omega_1) = 0 \\ g_3(U) = \dfrac{3icu_1u_2u_3}{\omega_2} + \dfrac{3icu_3^2u_4}{2\omega_2} - iu_3(\omega-\omega_2) = 0 \\ g_4(U) = -\dfrac{3icu_1u_2u_4}{\omega_2} - \dfrac{3icu_3u_4^2}{2\omega_2} + iu_4(\omega-\omega_2) = 0 \end{cases} \tag{62}$$

The ideal is formed from these two equations and Euclidean division of x by Gröbner ideal components gives the final simplified expression:

$$\begin{aligned} x = {} & \frac{1}{4\omega_1\omega_2(\omega_1-\omega_2)(\omega_1+\omega_2)} \left(-3c(\omega_1+\omega_2)\left(u_1u_4(2u_1\omega_2+u_4\omega_1)\right.\right. \\ & \left. +2u_2^2u_3\omega_2 + u_2u_3^2\omega_1\right) + 2\omega_2\omega\left(\omega_1^2(-(u_1+u_2)) + \omega_2^2(u_1+u_2)\right. \\ & \left.\left. +4\omega_1\omega_2(u_3+u_4)\right) - 2\omega_1\omega_2^3(3u_1+3u_2+4(u_3+u_4)) + 6\omega_1^3\omega_2(u_1+u_2)\right) \end{aligned} \tag{63}$$

This process is expanded to all components of X and we can have the final expression $X = M_0(\omega).U + M_1(U).U$ with $M_{0j,k}$ and $M_{1j,k}$, $j,k \in [1,4]^2$ being given in Appendix.
Solving compatibility equations in the case where ω_1 and ω_2 being very different leads to set either $u_1, u_2 = 0$ or $u_3, u_4 = 0$ in order to satisfy the small amplitude condition of normal form. In both cases, $M_1 = 0$ and a final linear expression of X is obtained.

6. Conclusion

A methodology based on the normal forms has been proposed. We are managing the spectrum in the methodology which is possible by the introduction of a first order small parameter and of the forcing term in order to exploit resonances with the excitation and/or internal resonances of the system. Once the determination of normal transformation and resonant terms is achieved, the introduction of mono-harmonic periodic normal coordinates in the normal forms allows to obtain compatibility equations. These equations are amplitude-frequency relations that have to be verified to ensure periodic solutions. Gröbner generators have been defined based on these compatibility equations. Because of the previous resonant spectrum, the compatibility equations lead to the most complete and efficient Euclidean division of the normal form solutions by the ideal. This methodology aims to reduce the degree of final normal solutions of the system but they cannot be expressed linearly in the general case. In some cases (diagonal system without internal resonance for example), we can however obtain a linear expression. Nonlinearity would still remain hidden because of compatibility equations. Furthermore, system cannot be always expressed in a diagonal manner. Jordan decomposition can be used but it will induce more resonant terms. Solving the normal form is difficult due to number of degrees of freedom and coupling terms and the ideal will need more calculation costs to be obtained. Finally, Euclidean division will be less effective and more nonlinear terms will be remaining in the final expression. When increasing the number of couplings and degrees of freedom, computation can be simplified by considering numerical values of the multivariate polynomials involved in Gröbner generators and $X(U)$ expression. A perspective work could be the introduction of Gröbner basis in order to define new normal coordinates and to introduce less pairing, especially when A takes a Jordan form. Time dependent periodic functions could also be introduced in Gröbner basis as it is the case in [12] to improve the simplification of normal form solutions.

Acknowledgments

The authors would like to thank the following organizations for supporting this research: (i) The "Ministère de la transition écologique et solidaire" and (ii) LABEX CELYA (ANR-10-LABX-0060) of the "Université de Lyon" within the program "Investissement d'Avenir" (ANR-11-IDEX- 0007) operated by the French National Research Agency (ANR).

Appendix

$$M_{01,1} = -\frac{1}{2}i(\omega_1+\omega)$$
$$M_{01,2} = \frac{1}{2}i(\omega_1+\omega)$$
$$M_{01,3} = \frac{2i\omega_2^2(\omega_2-\omega)}{(\omega_1-\omega_2)(\omega_1+\omega_2)}$$
$$M_{01,4} = -\frac{2i\omega_2^2(\omega_2-\omega)}{(\omega_1-\omega_2)(\omega_1+\omega_2)}$$
$$M_{02,1} = \frac{3}{2}-\frac{\omega}{2\omega_1}$$
$$M_{02,2} = \frac{3}{2}-\frac{\omega}{2\omega_1}$$
$$M_{02,3} = \frac{2\omega_2(\omega-\omega_2)}{(\omega_1-\omega_2)(\omega_1+\omega_2)}$$
$$M_{02,4} = \frac{2\omega_2(\omega-\omega_2)}{(\omega_1-\omega_2)(\omega_1+\omega_2)}$$
$$M_{03,1} = -\frac{2i\omega_1^2(\omega_1-\omega)}{(\omega_1-\omega_2)(\omega_1+\omega_2)}$$
$$M_{03,2} = \frac{2i\omega_1^2(\omega_1-\omega)}{(\omega_1-\omega_2)(\omega_1+\omega_2)}$$
$$M_{03,3} = -\frac{1}{2}i(\omega_2+\omega)$$
$$M_{03,4} = \frac{1}{2}i(\omega_2+\omega)$$
$$M_{04,1} = \frac{2\omega_1(\omega_1-\omega)}{(\omega_1-\omega_2)(\omega_1+\omega_2)}$$
$$M_{04,2} = \frac{2\omega_1(\omega_1-\omega)}{(\omega_1-\omega_2)(\omega_1+\omega_2)}$$
$$M_{04,3} = \frac{3}{2}-\frac{\omega}{2\omega_2}$$
$$M_{04,4} = \frac{3}{2}-\frac{\omega}{2\omega_2}$$

$$M_{11,1} = \frac{3icu_1u_4}{2(\omega_1-\omega_2)}$$
$$M_{11,2} = -\frac{3icu_2u_3}{2(\omega_1-\omega_2)}$$
$$M_{11,3} = -\frac{3icu_2u_3(\omega_1-2\omega_2)}{4\omega_2(\omega_1-\omega_2)}$$
$$M_{11,4} = -\frac{3icu_1u_4(2\omega_2-\omega_1)}{4\omega_2(\omega_1-\omega_2)}$$
$$M_{12,1} = -\frac{3cu_1u_4}{2\omega_1(\omega_1-\omega_2)}$$
$$M_{12,2} = -\frac{3cu_2u_3}{2\omega_1(\omega_1-\omega_2)}$$
$$M_{12,3} = -\frac{3cu_2u_3}{4\omega_2(\omega_1-\omega_2)}$$
$$M_{12,4} = -\frac{3cu_1u_4}{4\omega_2(\omega_1-\omega_2)}$$
$$M_{13,1} = -\frac{3icu_1u_4(2\omega_1-\omega_2)}{4\omega_1(\omega_1-\omega_2)}$$
$$M_{13,2} = \frac{3icu_2u_3(2\omega_1-\omega_2)}{4\omega_1(\omega_1-\omega_2)}$$
$$M_{13,3} = -\frac{3icu_2u_3(\omega_1-2\omega_2)}{(\omega_1-3\omega_2)(\omega_1-\omega_2)}$$
$$M_{13,4} = \frac{3icu_1u_4(\omega_1-2\omega_2)}{(\omega_1-3\omega_2)(\omega_1-\omega_2)}$$
$$M_{14,1} = \frac{3cu_1u_4}{4\omega_1(\omega_1-\omega_2)}$$
$$M_{14,2} = \frac{3cu_2u_3}{4\omega_1(\omega_1-\omega_2)}$$
$$M_{14,3} = -\frac{3cu_2u_3}{(\omega_1-3\omega_2)(\omega_1-\omega_2)}$$
$$M_{14,4} = -\frac{3cu_1u_4}{(\omega_1-3\omega_2)(\omega_1-\omega_2)}$$

References

[1] H. Poincaré, 1889. Les Méthodes Nouvelles de la Mécanique Céleste. Paris: Gauthier-Villars.

[2] G.D. Birkhoff, 1927. Dynamical systems, American Mathematical Soc.

[3] F.G. Gustavson, 1966. On constructing formal integrals of a Hamiltonian system near an equilibrium point, The Astronomical Journal 71.

[4] J.A. Murdock, 2003. Normal Forms and Unfoldings for Local Dynamical Systems, Springer Monographs in Mathematics 978-1-4419-3013-2 2003 Springer-Verlag New York, Inc.

[5] G. Iooss and M. Adelmeyer, 1992. Topics in Bifurcation Theory and Applications, World Scientific.

[6] G.I. Barenblatt, G. Iooss, D.D. Joseph, 1983. Nonlinear Dynamics and Turbulence, Pitman.

[7] P.H. Coullet and E. A. Spiegel, 1983. Amplitude Equations for Systems with Competing Instabilities, SIAM J. Appl. Math.

[8] D. Armbruster, 1985. Bifurcation theory and computer algebra: An initial approach, Lecture Notes in Computer Science 204, Pages 126-137.

[9] B. Buchberger, 1965. Bruno Buchberger PhD thesis 1965 : An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal. Journal of Symbolic Computation 41.

[10] V.F. Zhuravlev, 1997. A new algorithm for Birkhoff normalization of Hamiltonian systems. Journal of Applied Mathematics and Mechanics. Volume 61, Issue 1, Pages 9-13.

[11] V.F. Zhuravlev, 2002. The normal form of perturbations of non-linear oscillatory systems. Journal of Applied Mathematics and Mechanics. Volume 66, Issue 6, Pages 881-887.

[12] V.F. Zhuravlev, 2002. Invariant normalization of non-autonomous Hamiltonian systems. Journal of Applied Mathematics and Mechanics. Volume 66, Issue 3, Pages 347-355.