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About the Power Ratios Relevant to a Passive Linear Time-Invariant 2-Port

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ABSTRACT We investigate a passive linear time-invariant 2-port, having a port coupled to a generator and a port coupled to a load, in the harmonic steady state. Two configurations are considered, in which the port at which the generator is connected and the port at which the load is connected are exchanged. We investigate 8 power ratios for each configuration. Under different assumptions, we establish several reciprocal relations between the 16 power ratios, some of which are known and some of which are new. Our results are used to discuss and generalize the Friis transmission formula.

INDEX TERMS Operating power gain, transducer power gain, available power gain, power transfer ratio, unnamed power gain, insertion power gain, passive circuits, linear circuits, reciprocity, circuit theory.

I. INTRODUCTION

In this article, a device under study (DUS) is a linear time-invariant (LTI) and passive 2-port operating in the harmonic steady state, at a given frequency. It is used in two configurations, which are shown in Fig. 1. In configuration A (CA), its port 1 is connected to an LTI generator of internal impedance Z_{S1} and its port 2 is connected to an LTI load of impedance Z_{S2}. In configuration B (CB) its port 1 is connected to an LTI load of impedance Z_{S1} and its port 2 is connected to an LTI generator of internal impedance Z_{S2}.

The average power available from a port, also referred to as "available power", is defined as the greatest average power that can be drawn from this port by an arbitrary LTI and passive load [1, Sec. 3-8]. Ignoring noise power contributions, we consider 10 average powers:

- P_{AAVG1} is the average power available from the generator connected to port 1 in CA;
- P_{ARP1} is the average power received by port 1, in CA;
- P_{AAVP2} is the average power available from port 2, in CA;
- P_{ADP2} is the average power delivered by port 2, in CA;
- P_{AW} is the average power which would be received by the load connected at port 2 in CA, if the DUS was not present and this load was directly connected to the generator connected at port 1 in CA;
- P_{BAVG2} is the average power available from the generator connected to port 2, in CB;
- P_{BRP2} is the average power received by port 2, in CB;

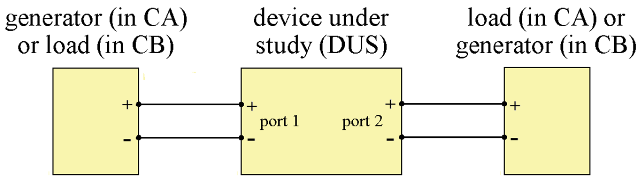

FIGURE 1. The two configurations, CA and CB.

- P_{BAVP1} is the average power available from port 1 in CB;
- P_{BDP1} is the average power delivered by port 1, in CB; and
- P_{BW} is the average power which would be received by the load connected at port 1 in CB, if the DUS was not present and this load was directly connected to the generator connected at port 2 in CB.

Our assumptions and the computation of the 10 average powers defined above, from the open-circuit voltages or the short-circuit currents of the generators, are covered in Section II. A new theorem on power products is stated and proven in Section III. It provides equalities between some products of average powers. In Section IV, we study 8 power ratios for each configuration, use this theorem to obtain reciprocal relations between the 16 power ratios, and discuss the novelty of these relations. Said power ratios include operating power gains, transducer power gains, available power gains, power transfer ratios, insertion power gains, and "unnamed power gains" defined in Section IV.

In Section V, we derive relations applicable to a lossless DUS, among which several new results. Examples are provided in Section VI. In Section VII, our results are used to discuss and generalize the Friis transmission formula [2], considered as a result on an unnamed power gain.

II. ASSUMPTIONS AND POWER COMPUTATIONS

A. ASSUMPTIONS, BASIC RESULTS AND NOTATIONS

As said above, we assume that the DUS is LTI and passive, and that the generators and the loads are LTI, so that Z_{S1} and Z_{S2} exist. Using $\mathrm{Re}(z)$ to denote the real part of a complex number z, we assume that $\mathrm{Re}(Z_{S1}) > 0$ and $\mathrm{Re}(Z_{S2}) > 0$. This ensures that the loads are passive and that P_{AAVG1} and P_{BAVG2} are defined. The DUS being passive, it follows that

$$0 \leqslant P_{ADP2} \leqslant P_{ARP1} \leqslant P_{AAVG1}\,; \tag{1}$$

P_{AAVP2} is defined and satisfies

$$0 \leqslant P_{ADP2} \leqslant P_{AAVP2} \leqslant P_{AAVG1}\,; \tag{2}$$

$$0 \leqslant P_{AW} \leqslant P_{AAVG1}\,; \tag{3}$$

$$0 \leqslant P_{BDP1} \leqslant P_{BRP2} \leqslant P_{BAVG2}\,; \tag{4}$$

P_{BAVP1} is defined and satisfies

$$0 \leqslant P_{BDP1} \leqslant P_{BAVP1} \leqslant P_{BAVG2}\,; \tag{5}$$

and

$$0 \leqslant P_{BW} \leqslant P_{BAVG2}\,. \tag{6}$$

It follows from $\mathrm{Re}(Z_{S1}) > 0$ that $Y_{S1} = 1/Z_{S1}$ exists and $\mathrm{Re}(Y_{S1}) > 0$. It follows from $\mathrm{Re}(Z_{S2}) > 0$ that $Y_{S2} = 1/Z_{S2}$ exists and $\mathrm{Re}(Y_{S2}) > 0$. Also, instead of assuming that Z_{S1} and Z_{S2} exist and have positive real parts, we could equivalently have assumed that Y_{S1} and Y_{S2} exist and have positive real parts.

We use V_{O1} and I_{S1} to denote the rms open-circuit voltage and the rms short-circuit current, respectively, of the generator connected to port 1 in CA. We use V_{O2} and I_{S2} to denote the rms open-circuit voltage and the rms short-circuit current, respectively, of the generator connected to port 2 in CB. We use V_1 and I_1 to denote the rms voltage across port 1 and the rms current flowing into port 1, respectively. We use V_2 and I_2 to denote the rms voltage across port 2 and the rms current flowing into port 2, respectively.

B. AUGMENTED MULTIPORTS

To avoid unnecessary assumptions, we will use the theory of parallel-augmented multiports and series-augmented multiports presented in Section II of [3]-[4] and also in Section II of [5].

Following Section II of [5], we introduce a parallel-augmented multiport composed of the DUS (as original multiport), a load of impedance Z_{S1} connected in parallel with port 1, and a load of impedance Z_{S2} connected in parallel with port 2. The admittance matrix of the added multiport is

$$\mathbf{Y}_{ADD} = \begin{pmatrix} Y_{S1} & 0 \\ 0 & Y_{S2} \end{pmatrix}. \tag{7}$$

Since $\mathrm{Re}(Y_{S1}) > 0$ and $\mathrm{Re}(Y_{S2}) > 0$, the hermitian part of $\mathbf{Y}_{ADD}$ is positive definite. By Theorem 1 and Corollary 1 of [5], the parallel-augmented multiport has an impedance matrix

$$\mathbf{Z}_{PAM} = \begin{pmatrix} Z_{PAM11} & Z_{PAM12} \\ Z_{PAM21} & Z_{PAM22} \end{pmatrix}, \tag{8}$$

and, in the special case where the DUS has an admittance matrix $\mathbf{Y}$, then: $\mathbf{Z}_{PAM}$ is invertible;

$$\mathbf{Z}_{PAM}^{-1} = \mathbf{Y} + \mathbf{Y}_{ADD}\,; \tag{9}$$

and $\mathbf{Z}_{PAM}$ is symmetric if and only if $\mathbf{Y}$ is symmetric.

According to Section II of [5], we can also introduce a series-augmented multiport composed of the DUS (as original multiport), a load of impedance Z_{S1} connected in series with port 1, and a load of impedance Z_{S2} connected in series with port 2. The impedance matrix of the added multiport is

$$\mathbf{Z}_{ADD} = \begin{pmatrix} Z_{S1} & 0 \\ 0 & Z_{S2} \end{pmatrix} = \mathbf{Y}_{ADD}^{-1}\,. \tag{10}$$

Since $\mathrm{Re}(Z_{S1}) > 0$ and $\mathrm{Re}(Z_{S2}) > 0$, the hermitian part of $\mathbf{Z}_{ADD}$ is positive definite.

By Theorem 2 and Corollary 2 of [5], the series-augmented multiport has an admittance matrix

$$\mathbf{Y}_{SAM} = \begin{pmatrix} Y_{SAM11} & Y_{SAM12} \\ Y_{SAM21} & Y_{SAM22} \end{pmatrix}, \tag{11}$$

and, in the special case where the DUS has an impedance matrix $\mathbf{Z}$, then: $\mathbf{Y}_{SAM}$ is invertible;

$$\mathbf{Y}_{SAM}^{-1} = \mathbf{Z} + \mathbf{Z}_{ADD}\,; \tag{12}$$

and $\mathbf{Y}_{SAM}$ is symmetric if and only if $\mathbf{Z}$ is symmetric.

C. FORMULAS USING THE OPEN-CIRCUIT VOLTAGES

We want to compute the above-defined average powers, using the open-circuit voltages of the generators to define the excitations, and $\mathbf{Y}_{SAM}$ to define the DUS. By inspection, we find

$$P_{AAVG1} = \frac{|V_{O1}|^2}{4\mathrm{Re}(Z_{S1})}, \tag{13}$$

$$P_{ARP1} = \left[\mathrm{Re}(Y_{SAM11}) - |Y_{SAM11}|^2 \mathrm{Re}(Z_{S1})\right] |V_{O1}|^2, \tag{14}$$

$$P_{AAVP2} = \begin{cases} 0\ \mathrm{W} & \text{if } C_1 \text{ is true} \\ \dfrac{\left|\dfrac{Y_{SAM21}}{Y_{SAM22}}\right|^2 |V_{O1}|^2}{4\mathrm{Re}\left(\dfrac{1}{Y_{SAM22}} - Z_{S2}\right)} & \text{else} \end{cases}, \tag{15}$$

$$P_{ADP2} = \mathrm{Re}(Z_{S2})\, |Y_{SAM21} V_{O1}|^2, \tag{16}$$

$$P_{AW} = \frac{\mathrm{Re}(Z_{S2})}{|Z_{S1} + Z_{S2}|^2} |V_{O1}|^2, \tag{17}$$

$$P_{BAVG2} = \frac{|V_{O2}|^2}{4\mathrm{Re}(Z_{S2})}, \tag{18}$$

$$P_{BRP2} = \left[\mathrm{Re}(Y_{SAM22}) - |Y_{SAM22}|^2 \mathrm{Re}(Z_{S2})\right] \, |V_{O2}|^2 \,, \quad (19)$$

$$P_{BAVP1} = \begin{cases} 0 \text{ W} & \text{if } C_2 \text{ is true} \\ \dfrac{\left|\dfrac{Y_{SAM12}}{Y_{SAM11}}\right|^2 |V_{O2}|^2}{4\mathrm{Re}\left(\dfrac{1}{Y_{SAM11}} - Z_{S1}\right)} & \text{else} \end{cases} , \quad (20)$$

$$P_{BDP1} = \mathrm{Re}(Z_{S1}) \, |Y_{SAM12} \, V_{O2}|^2 \,, \quad (21)$$

and

$$P_{BW} = \frac{\mathrm{Re}(Z_{S1})}{|Z_{S1} + Z_{S2}|^2} \, |V_{O2}|^2 \,, \quad (22)$$

where the propositions C_1 and C_2 are

$$C_1 \Leftrightarrow \left(\mathrm{Re}(Y_{SAM22}) = |Y_{SAM22}|^2 \mathrm{Re}(Z_{S2})\right) \,, \quad (23)$$

and

$$C_2 \Leftrightarrow \left(\mathrm{Re}(Y_{SAM11}) = |Y_{SAM11}|^2 \mathrm{Re}(Z_{S1})\right) \,. \quad (24)$$

To obtain (13), (15), (18) and (20), we have used the classic maximum power transfer theorem for a single-port generator [6, Sec 7.4], [7, Sec. 11.1]. The propositions C_1 and C_2 deserve additional explanations. If $Y_{SAM22} = 0$ S, then C_1 is true, and (15) says that $P_{AAVP2} = 0$ W because we must have $Y_{SAM21} = 0$ S since, if this was not the case, port 2 of the series-augmented multiport and port 2 of the DUS would, for $V_{O1} \neq 0$ V, behave like current sources of nonzero current, so that we would obtain an infinite P_{AAVP2}, which is incompatible with (13) and $P_{AAVP2} \leqslant P_{AAVG1}$. If $Y_{SAM22} \neq 0$ S and the real part in (15) is zero, then C_1 is true, and (15) says that $P_{AAVP2} = 0$ W because we must have $Y_{SAM21} = 0$ S since, if this was not the case, port 2 of the DUS would, for $V_{O1} \neq 0$ V, behave like a generator of nonzero open-circuit voltage having a zero resistance, so that we would obtain an infinite P_{AAVP2}, which is incompatible with (13) and $P_{AAVP2} \leqslant P_{AAVG1}$. If $Y_{SAM22} \neq 0$ S and the real part in (15) is not zero, then C_1 is false, and the second line of (15) can be computed. This explains the proposition C_1. We note that, if the proposition C_1 is true, then $Y_{SAM21} = 0$ S. The explanation for the proposition C_2 is similar, and we find that, if the proposition C_2 is true, then $Y_{SAM12} = 0$ S.

D. FORMULAS USING THE SHORT-CIRCUIT CURRENTS

We now wish to compute the above-defined average powers, using the short-circuit currents of the generators to define the excitations, and $\mathbf{Z}_{PAM}$ to define the DUS. By inspection, we find

$$P_{AAVG1} = \frac{|I_{S1}|^2}{4\mathrm{Re}(Y_{S1})} \,, \quad (25)$$

$$P_{ARP1} = \left[\mathrm{Re}(Z_{PAM11}) - |Z_{PAM11}|^2 \mathrm{Re}(Y_{S1})\right] \, |I_{S1}|^2 \,, \quad (26)$$

$$P_{AAVP2} = \begin{cases} 0 \text{ W} & \text{if } C_3 \text{ is true} \\ \dfrac{\left|\dfrac{Z_{PAM21}}{Z_{PAM22}}\right|^2 |I_{S1}|^2}{4\mathrm{Re}\left(\dfrac{1}{Z_{PAM22}} - Y_{S2}\right)} & \text{else} \end{cases} , \quad (27)$$

$$P_{ADP2} = \mathrm{Re}(Y_{S2}) \, |Z_{PAM21} \, I_{S1}|^2 \,, \quad (28)$$

$$P_{AW} = \frac{\mathrm{Re}(Y_{S2})}{|Y_{S1} + Y_{S2}|^2} \, |I_{S1}|^2 \,, \quad (29)$$

$$P_{BAVG2} = \frac{|I_{S2}|^2}{4\mathrm{Re}(Y_{S2})} \,, \quad (30)$$

$$P_{BRP2} = \left[\mathrm{Re}(Z_{PAM22}) - |Z_{PAM22}|^2 \mathrm{Re}(Y_{S2})\right] \, |I_{S2}|^2 \,, \quad (31)$$

$$P_{BAVP1} = \begin{cases} 0 \text{ W} & \text{if } C_4 \text{ is true} \\ \dfrac{\left|\dfrac{Z_{PAM12}}{Z_{PAM11}}\right|^2 |I_{S2}|^2}{4\mathrm{Re}\left(\dfrac{1}{Z_{PAM11}} - Y_{S1}\right)} & \text{else} \end{cases} , \quad (32)$$

$$P_{BDP1} = \mathrm{Re}(Y_{S1}) \, |Z_{PAM12} \, I_{S2}|^2 \,, \quad (33)$$

and

$$P_{BW} = \frac{\mathrm{Re}(Y_{S1})}{|Y_{S1} + Y_{S2}|^2} \, |I_{S2}|^2 \,, \quad (34)$$

where the propositions C_3 and C_4 are

$$C_3 \Leftrightarrow \left(\mathrm{Re}(Z_{PAM22}) = |Z_{PAM22}|^2 \mathrm{Re}(Y_{S2})\right) \,, \quad (35)$$

and

$$C_4 \Leftrightarrow \left(\mathrm{Re}(Z_{PAM11}) = |Z_{PAM11}|^2 \mathrm{Re}(Y_{S1})\right) \,. \quad (36)$$

To obtain (25), (27), (30) and (32), we have used an equivalent form of the maximum power transfer theorem for a single-port generator. The propositions C_3 and C_4 deserve some explanations. If $Z_{PAM22} = 0\ \Omega$, then C_3 is true, and (27) says that $P_{AAVP2} = 0$ W because we must have $Z_{PAM21} = 0\ \Omega$ since, if this was not the case, port 2 of the parallel-augmented multiport and port 2 of the DUS would, for $I_{S1} \neq 0$ A, behave like voltage sources of nonzero voltage, so that we would obtain an infinite P_{AAVP2}, which is incompatible with (25) and $P_{AAVP2} \leqslant P_{AAVG1}$. If $Z_{PAM22} \neq 0\ \Omega$ and the real part in (27) is zero, then C_3 is true, and (27) says that $P_{AAVP2} = 0$ W because we must have $Z_{PAM21} = 0\ \Omega$ since, if this was not the case, port 2 of the DUS would, for $I_{S1} \neq 0$ A, behave like a generator of nonzero short-circuit current having a zero conductance, so that we would obtain an infinite P_{AAVP2}, which is incompatible with (25) and $P_{AAVP2} \leqslant P_{AAVG1}$. If $Z_{PAM22} \neq 0\ \Omega$ and the real part in (27) is not zero, then C_3 is false, and the second line of (27) can be computed. This explains the proposition C_3. The explanation for the proposition C_4 is similar.

III. THE THEOREM ON POWER PRODUCTS

In what follows, when we write that the DUS is a reciprocal device, we refer to the definitions of reciprocal networks provided in [6, Ch. 16] or [8, Ch. 1], which are not limited to lumped networks.

Theorem on power products. Ignoring noise power contributions, we have

$$P_{ADP2}P_{BAVG2} = P_{AAVP2}P_{BRP2}\,, \tag{37}$$

$$P_{BDP1}P_{AAVG1} = P_{BAVP1}P_{ARP1}\,, \tag{38}$$

and

$$P_{BW}P_{AAVG1} = P_{BAVG2}P_{AW}\,. \tag{39}$$

Moreover, if the DUS is reciprocal and ignoring noise power contributions, we have

$$\begin{aligned} P_{ADP2}P_{BAVG2} &= P_{AAVP2}P_{BRP2} \\ &= P_{BDP1}P_{AAVG1} = P_{BAVP1}P_{ARP1}\,, \end{aligned} \tag{40}$$

and

$$P_{BDP1}P_{AW} = P_{ADP2}P_{BW}\,. \tag{41}$$

Proof: Using (16) and (18), we get

$$P_{ADP2}P_{BAVG2} = \frac{1}{4}|Y_{SAM21}V_{O1}V_{O2}|^2\,. \tag{42}$$

Using (15) and (19), we get

$$P_{AAVP2}P_{BRP2} = \frac{1}{4}|Y_{SAM21}V_{O1}V_{O2}|^2\,, \tag{43}$$

since, as explained in Section II.C, if the proposition C_1 is true, we must have $Y_{SAM21} = 0$ S. A comparison of (42) and (43) leads us to (37).

Using (13) and (21), we get

$$P_{BDP1}P_{AAVG1} = \frac{1}{4}|Y_{SAM12}V_{O1}V_{O2}|^2\,. \tag{44}$$

Using (14) and (20), we get

$$P_{BAVP1}P_{ARP1} = \frac{1}{4}|Y_{SAM12}V_{O1}V_{O2}|^2\,, \tag{45}$$

since, as explained in Section II.C, if the proposition C_2 is true, we must have $Y_{SAM12} = 0$ S. A comparison of (44) and (45) leads us to (38).

Using (13) and (22), we get

$$P_{BW}P_{AAVG1} = \frac{1}{4|Z_{S1}+Z_{S2}|^2}|V_{O1}V_{O2}|^2\,. \tag{46}$$

Using (17) and (18), we get

$$P_{BAVG2}P_{AW} = \frac{1}{4|Z_{S1}+Z_{S2}|^2}|V_{O1}V_{O2}|^2\,, \tag{47}$$

so that a comparison of (46) and (47) leads us to (39).

Using (17) and (21), we get

$$P_{BDP1}P_{AW} = \frac{\mathrm{Re}(Z_{S1})\mathrm{Re}(Z_{S2})}{|Z_{S1}+Z_{S2}|^2}|Y_{SAM12}\,V_{O1}V_{O2}|^2\,. \tag{48}$$

Using (16) and (22), we get

$$P_{ADP2}P_{BW} = \frac{\mathrm{Re}(Z_{S1})\mathrm{Re}(Z_{S2})}{|Z_{S1}+Z_{S2}|^2}|Y_{SAM21}\,V_{O1}V_{O2}|^2\,. \tag{49}$$

If we now assume that the DUS is reciprocal, it follows from Theorem 2 of [5] that $\mathbf{Y}_{SAM}$ is symmetric, so that: a comparison of (42), (43), (44) and (45) leads us to (40); and a comparison of (48) and (49) leads us to (41). □

In this proof, we have used the series-augmented multiport and the results of Section II.C, but we could have alternatively, and just as easily, used the parallel-augmented multiport and the results of Section II.D.

IV. AN INVESTIGATION OF 16 POWER RATIOS

A. INITIAL ASSUMPTIONS AND OBSERVATIONS

In what follows, we assume $V_{O1} \neq 0$ V and $V_{O2} \neq 0$ V, or equivalently $I_{S1} \neq 0$ A and $I_{S2} \neq 0$ A. This ensures that $P_{AAVG1} \neq 0$ W and $P_{BAVG2} \neq 0$ W.

Section II.C and Section II.D allow us to observe that:

- we have $P_{ARP1} \neq 0$ W if and only if C_2 is false (or equivalently if and only if C_4 is false);
- we have $P_{AAVP2} \neq 0$ W if and only if C_1 is false and $Y_{SAM21} \neq 0$ S (or equivalently if and only if C_3 is false and $Z_{PAM21} \neq 0\ \Omega$);
- we have $P_{BRP2} \neq 0$ W if and only if C_1 is false (or equivalently if and only if C_3 is false); and
- we have $P_{BAVP1} \neq 0$ W if and only if C_2 is false and $Y_{SAM12} \neq 0$ S (or equivalently if and only if C_4 is false and $Z_{PAM12} \neq 0\ \Omega$).

In CA, if $P_{ARP1} \neq 0$ W, we see that: a nonzero I_1 flows into port 1 and a nonzero V_1 exists across port 1; the generator connected to port 1 sees a nonzero impedance Z_{APP1}, which satisfies $V_1 = Z_{APP1}I_1$ and has a positive real part; Z_{APP1} is only determined by the DUS and the load connected to port 2; and, since C_2 is false, $Y_{SAM11} \neq 0$ S so that

$$Z_{APP1} = \frac{1}{Y_{SAM11}} - Z_{S1}\,. \tag{50}$$

In CB, if $P_{BRP2} \neq 0$ W, we see that: a nonzero I_2 flows into port 2 and a nonzero V_2 exists across port 2; the generator connected to port 2 sees a nonzero impedance Z_{BPP2}, which satisfies $V_2 = Z_{BPP2}I_2$ and has a positive real part, Z_{BPP2} is only determined by the DUS and the load connected to port 1; and, since C_1 is false, $Y_{SAM22} \neq 0$ S so that

$$Z_{BPP2} = \frac{1}{Y_{SAM22}} - Z_{S2}\,. \tag{51}$$

B. POWER TRANSFER RATIOS

We define the power transfer ratio in CA at port 1 of the DUS, given by

$$t_{A1} = \frac{P_{ARP1}}{P_{AAVG1}}\,, \tag{52}$$

and the power transfer ratio in CA without the DUS, given by

$$t_{AW} = \frac{P_{AW}}{P_{AAVG1}}\,, \tag{53}$$

which by (1) and (3) satisfy $0 \leqslant t_{A1} \leqslant 1$ and $0 \leqslant t_{AW} \leqslant 1$.

If $P_{AAVP2} \neq 0$ W, we define the power transfer ratio in CA at port 2 of the DUS, given by

$$t_{A2} = \frac{P_{ADP2}}{P_{AAVP2}}, \quad (54)$$

which by (2) is such that $0 \leqslant t_{A2} \leqslant 1$.

We define the power transfer ratio in CB at port 2 of the DUS, given by

$$t_{B2} = \frac{P_{BRP2}}{P_{BAVG2}}, \quad (55)$$

and the power transfer ratio in CB without the DUS, given by

$$t_{BW} = \frac{P_{BW}}{P_{BAVG2}}, \quad (56)$$

which by (4) and (6) satisfy $0 \leqslant t_{B2} \leqslant 1$ and $0 \leqslant t_{BW} \leqslant 1$.

If $P_{BAVP1} \neq 0$ W, we define the power transfer ratio in CB at port 1 of the DUS, given by

$$t_{B1} = \frac{P_{BDP1}}{P_{BAVP1}}, \quad (57)$$

which by (5) is such that $0 \leqslant t_{B1} \leqslant 1$.

The results of Section II.C can be used to easily show that the 6 power transfer ratios defined above neither depend on V_{O1} nor on V_{O2}.

Any one of the 6 power transfer ratios defined above is equal to one if and only if the condition of the maximum power transfer theorem is satisfied [6, Sec. 7.4], [7, Sec. 11.1]. For instance, using $\bar{z}$ to denote the complex conjugate of a complex number z, $t_{A1} = 1$ if and only if $P_{ARP1} \neq 0$ W and $Z_{S1} = \overline{Z_{APP1}}$. For instance, $t_{B2} = 1$ if and only if $P_{BRP2} \neq 0$ W and $Z_{S2} = \overline{Z_{APP2}}$. For instance, $t_{AW} = t_{BW} = 1$ if and only if $Z_{S1} = \overline{Z_{S2}}$.

It follows from (37)-(39) of the theorem on power products that:

$$(P_{BAVP1} \neq 0 \text{ W}) \Longrightarrow (t_{A1} = t_{B1}); \quad (58)$$

$$(P_{AAVP2} \neq 0 \text{ W}) \Longrightarrow (t_{A2} = t_{B2}); \quad (59)$$

and

$$t_{AW} = t_{BW}. \quad (60)$$

The reciprocal relations (58)-(60) are based on (37)-(39), and therefore valid whether the DUS is reciprocal or not. These reciprocal relations are not new since, for instance, according to [9, Appendix A], they are closely related to results on "power transmission coefficients" established in [10, Sec. III] using power waves.

According to [9, Sec. V], the power match figure at port 1 is $F_{M1} = \sqrt{1 - t_{A1}}$, and the power match figure at port 2 is $F_{M2} = \sqrt{1 - t_{B2}}$.

C. TRANSDUCER POWER GAINS

We can define two transducer power gains [1, Sec. 21-18]: the transducer power gain in CA, given by

$$G_{AT} = \frac{P_{ADP2}}{P_{AAVG1}}, \quad (61)$$

and the transducer power gain in CB, given by

$$G_{BT} = \frac{P_{BDP1}}{P_{BAVG2}}. \quad (62)$$

It follows from (1) and (4) that they satisfy $0 \leqslant G_{AT} \leqslant 1$ and $0 \leqslant G_{BT} \leqslant 1$. The results of Section II.C can be used to easily show that G_{AT} and G_{BT} neither depend on V_{O1} nor on V_{O2}.

It follows from (40) of the theorem on power products that, if the DUS is reciprocal, then

$$G_{AT} = G_{BT}. \quad (63)$$

This reciprocal relation was stated and proven in [10], using power waves. A less general version had been established 35 years earlier, using the entries of the impedance matrix of the DUS, in the case where this matrix exists [11].

D. INSERTION POWER GAINS

We can define two insertion power gains [1, Sec. 21-18]: the insertion power gain in CA, given by

$$G_{AI} = \frac{P_{ADP2}}{P_{AW}}, \quad (64)$$

and the insertion power gain in CB, given by

$$G_{BI} = \frac{P_{BDP1}}{P_{BW}}. \quad (65)$$

G_{AI} and G_{BI} are nonnegative, but they need not be less than or equal to one. The results of Section II.C can be used to easily show that G_{AI} and G_{BI} neither depend on V_{O1} nor on V_{O2}.

It follows from (41) of the theorem on power products that, if the DUS is reciprocal, then

$$G_{AI} = G_{BI}. \quad (66)$$

This reciprocal relation is not new, since it is a special case of Theorem 6 of [5]. It seems likely that (66) was known before [5], but we have not found any evidence of this.

E. OPERATING POWER GAINS

The operating power gains is sometimes called "power gain" [12, Sec. 3.2], and it could also be called "efficiency" since we are considering a passive DUS.

If $P_{ARP1} \neq 0$ W, we can define the operating power gain in CA, given by

$$G_{AO} = \frac{P_{ADP2}}{P_{ARP1}} = \frac{G_{AT}}{t_{A1}}. \quad (67)$$

If $P_{BRP2} \neq 0$ W, we can define the operating power gain in CB, given by

$$G_{BO} = \frac{P_{BDP1}}{P_{BRP2}} = \frac{G_{BT}}{t_{B2}}. \quad (68)$$

It follows from (1) and (4) that we have: $0 \leqslant G_{AO} \leqslant 1$ if G_{AO} is defined; and $0 \leqslant G_{BO} \leqslant 1$ if G_{BO} is defined. The results of Section II.C can be used to easily show that G_{AO} and G_{BO} neither depend on V_{O1} nor on V_{O2}.

In CA, if $P_{ARP1} \neq 0$ W, for a specified DUS and a specified load connected to port 2, it follows from (50) that P_{ADP2} and P_{ARP1} are completely determined by I_1, so that any change in Z_{S1} can be compensated by a change in V_{O1} to obtain the same nonzero I_1 and the same nonzero V_1, hence the same P_{ADP2} and the same P_{ARP1}, hence the same G_{AO}, so that G_{AO} does not depend on Z_{S1}.

In CB, if $P_{BRP2} \neq 0$ W, for a specified DUS and a specified load connected to port 1, it follows from (51) that P_{BDP1} and P_{BRP2} are completely determined by I_2, so that any change in Z_{S2} can be compensated by a change in V_{O2} to obtain the same nonzero I_2 and the same nonzero V_2, hence the same P_{BDP1} and the same P_{BRP2}, hence the same G_{BO}, so that G_{BO} does not depend on Z_{S2}.

We have just shown that, for a specified DUS: if G_{AO} is defined, it may depend on Z_{S2} but not on Z_{S1}; and, if G_{BO} is defined, it may depend on Z_{S1} but not on Z_{S2}.

F. AVAILABLE POWER GAINS

We can define two available power gains [1, Sec. 21-18]: the available power gain in CA, given by

$$G_{AA} = \frac{P_{AAVP2}}{P_{AAVG1}}\,, \tag{69}$$

and the available power gain in CB, given by

$$G_{BA} = \frac{P_{BAVP1}}{P_{BAVG2}}\,. \tag{70}$$

It follows from (2) and (5) that they satisfy $0 \leqslant G_{AA} \leqslant 1$ and $0 \leqslant G_{BA} \leqslant 1$. The results of Section II.C can be used to easily show that G_{AA} and G_{BA} neither depend on V_{O1} nor on V_{O2}.

If $P_{AAVP2} \neq 0$ W, we have

$$G_{AA} = \frac{G_{AT}}{t_{A2}}\,. \tag{71}$$

If $P_{BAVP1} \neq 0$ W, we have

$$G_{BA} = \frac{G_{BT}}{t_{B1}}\,. \tag{72}$$

We observe that: in CA, P_{AAVG1} depends on the generator connected to port 1, but neither on the DUS nor on the load connected to port 2; in CA, P_{AAVP2} depends on the generator connected to port 1 and on the DUS, but not on the load connected to port 2; in CB, P_{BAVG2} depends on the generator connected to port 2, but neither on the DUS nor on the load connected to port 1; and in CB, P_{BAVP1} depends on the generator connected to port 2 and on the DUS, but not on the load connected to port 1.

It follows that, for a specified DUS: G_{AA} may depend on Z_{S1} but not on Z_{S2}; and G_{BA} may depend on Z_{S2} but not on Z_{S1}.

It follows from (40) of the theorem on power products that, if the DUS is reciprocal and G_{BO} is defined, then

$$G_{AA} = G_{BO}\,; \tag{73}$$

and, if the DUS is reciprocal and G_{AO} is defined, then

$$G_{AO} = G_{BA}\,. \tag{74}$$

As far as we know, these reciprocal relations are new.

G. UNNAMED POWER GAINS

We are now considering a power gain which does not seem to have been named, so that we call it “unnamed power gain”.

If $P_{ARP1} \neq 0$ W, we can define the unnamed power gain in CA, given by

$$G_{AU} = \frac{P_{AAVP2}}{P_{ARP1}} = \frac{G_{AA}}{t_{A1}} = \frac{G_{AO}}{t_{A2}}\,. \tag{75}$$

If $P_{BRP2} \neq 0$ W, we can define the unnamed power gain in CB, given by

$$G_{BU} = \frac{P_{BAVP1}}{P_{BRP2}} = \frac{G_{BA}}{t_{B2}} = \frac{G_{BO}}{t_{B1}}\,. \tag{76}$$

If they exist, G_{AU} and G_{BU} are nonnegative, but they need not be less than or equal to one. The results of Section II.C can be used to easily show that G_{AU} and G_{BU} neither depend on V_{O1} nor on V_{O2}.

It follows from (40) of the theorem on power products that, if the DUS is reciprocal, if G_{AU} is defined and if G_{BU} is defined, then

$$G_{AU} = G_{BU}\,. \tag{77}$$

This reciprocal relation was stated and proven in [10], using power waves.

V. SPECIAL CASE OF A LOSSLESS DUS

The DUS is lossless if and only if, for any Z_{S1} and Z_{S2} such that $\mathrm{Re}(Z_{S1}) > 0$ and $\mathrm{Re}(Z_{S2}) > 0$, we have

$$P_{ADP2} = P_{ARP1} \text{ and } P_{BDP1} = P_{BRP2}\,. \tag{78}$$

In this Section V, we now use the assumptions of Section IV.A, and we assume that the DUS is lossless. It follows from (67) and (78) that, if $P_{ARP1} \neq 0$ W,

$$G_{AO} = 1 \text{ and } G_{AT} = t_{A1}\,. \tag{79}$$

Also, it follows from (68) and (78) that, if $P_{BRP2} \neq 0$ W,

$$G_{BO} = 1 \text{ and } G_{BT} = t_{B2}\,. \tag{80}$$

Thus, it follows from (63), (79) and (80) that, if the lossless DUS is reciprocal, $P_{ARP1} \neq 0$ W and $P_{BRP2} \neq 0$ W, then

$$t_{A1} = t_{B2}\,. \tag{81}$$

If $P_{ARP1} \neq 0$ W, since Z_{APP1} is defined and determined only by the DUS and by Z_{S2}, we can assume that we have chosen the generator in such a way that $Z_{S1} = \overline{Z_{APP1}}$. In this case, $P_{ARP1} = P_{AAVG1}$, so that, by (78), we have $P_{ADP2} = P_{AAVG1}$. Thus, (2) leads us to $P_{AAVP2} = P_{AAVG1} > 0$. By the maximum power transfer theorem, Z_{APP2} is defined and nonzero, and $Z_{S2} = \overline{Z_{APP2}}$.

Let us no longer assume that $Z_{S1} = \overline{Z_{APP1}}$. We have just shown that, if $P_{ARP1} \neq 0$ W, then

$$(Z_{S1} = \overline{Z_{APP1}}) \Longrightarrow (P_{BRP2} \neq 0 \text{ W and } Z_{S2} = \overline{Z_{APP2}})\,. \tag{82}$$

Using CB, we can also prove that, if $P_{BRP2} \neq 0$ W, then

$$(Z_{S2} = \overline{Z_{APP2}}) \Longrightarrow (P_{ARP1} \neq 0 \text{ W and } Z_{S1} = \overline{Z_{APP1}})\,. \tag{83}$$

TABLE 1. Results for the first example, in the case $Z_{S1} = (320 + 39j)\,\Omega$ and $Z_{S2} = (51 + 87j)\,\Omega$.

Quantity	CA	CB
power transfer ratio at port 1 of the DUS	0.209313	0.209313
power transfer ratio at port 2 of the DUS	0.131104	0.131104
power transfer ratio without the DUS	0.425230	0.425230
transducer power gain	0.060841	0.095689
insertion power gain	0.143077	0.225028
operating power gain	0.290669	0.729867
available power gain	0.464063	0.457156
unnamed power gain	2.217077	3.486959

Let us assume that $P_{BRP2} \neq 0$ W. In CA, if we choose Z_{S2} such that $Z_{S2} = \overline{Z_{BPP2}}$, we get: $P_{ADP2} = P_{AAVP2}$; and $P_{ARP1} = P_{AAVG1}$ by (83). It follows from $P_{ADP2} = P_{ARP1}$ that $P_{AAVP2} = P_{AAVG1}$. It must be stressed that this result is independent of the value of Z_{S2}, because Z_{S2} has no effect on P_{AAVG1} and no effect on P_{AAVP2}. Thus, in CA, for any value of Z_{S2}, we have $P_{AAVP2} \neq 0$ W, $G_{AA} = 1$ and $t_{A1} = t_{A2}$ because $P_{ADP2} = P_{ARP1}$. If, instead of assuming $P_{BRP2} \neq 0$ W, we assume that $P_{ARP1} \neq 0$ W and consider CB, we likewise obtain $P_{BAVP1} \neq 0$ W, $G_{BA} = 1$ and $t_{B1} = t_{B2}$.

Consequently, we have just shown that:

$$(P_{BRP2} \neq 0 \text{ W}) \Longrightarrow (G_{AA} = 1)\,, \tag{84}$$

$$(P_{BRP2} \neq 0 \text{ W}) \Longrightarrow (P_{AAVP2} \neq 0 \text{ W} \text{ and } G_{AT} = t_{A1} = t_{A2} = t_{B2} = G_{BT})\,, \tag{85}$$

$$(P_{ARP1} \neq 0 \text{ W}) \Longrightarrow (G_{BA} = 1)\,, \tag{86}$$

and

$$(P_{ARP1} \neq 0 \text{ W}) \Longrightarrow (P_{BAVP1} \neq 0 \text{ W} \text{ and } G_{BT} = t_{B2} = t_{B1} = t_{A1} = G_{AT})\,, \tag{87}$$

where we have used (58) and (59) to obtain (85) and (87). Note that (85) and (87) are much stronger results than (81), because they apply to a non-reciprocal DUS, as well as a reciprocal one. It follows from (75), (76), and (84)-(87) that

$$(P_{ARP1} \neq 0 \text{ W and } P_{BRP2} \neq 0 \text{ W}) \Longrightarrow \left(G_{AU} = \frac{1}{t_{A1}} = \frac{1}{t_{B2}} = G_{BU} \right). \tag{88}$$

According to (61), (62), (85) and (87), if $P_{ARP1} \neq 0$ W or $P_{BRP2} \neq 0$ W, then $P_{ADP2} P_{BAVG2} = P_{BDP1} P_{AAVG1}$, so that according to (42) and (44), we have $|Y_{SAM12}| = |Y_{SAM21}|$, so that according to (48) and (49), we have $P_{BDP1} P_{AW} = P_{ADP2} P_{BW}$. Thus, using (64) and (65), we obtain

$$(P_{ARP1} \neq 0 \text{ W or } P_{BRP2} \neq 0 \text{ W}) \Longrightarrow (G_{AI} = G_{BI})\,. \tag{89}$$

The results (84)-(89) are new.

TABLE 2. Results for the first example, in the case $Z_{S1} = (22 - 72j)\,\Omega$ and $Z_{S2} = (51 + 87j)\,\Omega$.

Quantity	CA	CB
power transfer ratio at port 1 of the DUS	0.998007	0.998007
power transfer ratio at port 2 of the DUS	0.615677	0.615677
power transfer ratio without the DUS	0.808066	0.808066
transducer power gain	0.290089	0.456245
insertion power gain	0.358992	0.564613
operating power gain	0.290669	0.741045
available power gain	0.471171	0.457156
unnamed power gain	0.472112	0.742525

TABLE 3. Results for the first example, in the case $Z_{S1} = (320 + 39j)\,\Omega$ and $Z_{S2} = (74 - 225j)\,\Omega$.

Quantity	CA	CB
power transfer ratio at port 1 of the DUS	0.863586	0.863586
power transfer ratio at port 2 of the DUS	0.999862	0.999862
power transfer ratio without the DUS	0.498968	0.498968
transducer power gain	0.463999	0.729766
insertion power gain	0.929919	1.462551
operating power gain	0.537294	0.729867
available power gain	0.464063	0.845041
unnamed power gain	0.537368	0.845158

VI. SOME EXAMPLES

A. FIRST EXAMPLE

In a first example, we assume that the DUS has an impedance matrix, given by

$$\mathbf{Z} = \begin{pmatrix} 11 + 125j & 15 + 150j \\ -7 + 120j & 30 + 250j \end{pmatrix} \Omega\,. \tag{90}$$

Here, $\mathbf{Z}$ has a positive definite hermitian part and is not symmetric. Thus, the DUS is passive, not reciprocal, and not lossless. For each value of Z_{S1} and Z_{S2} considered below, the 16 power ratios defined in Section IV have been computed a first time using the formulas of Section II.C and $\mathbf{Y}_{SAM}$ given by (12), and a second time using the formulas of Section II.D and $\mathbf{Z}_{PAM}$ given by (9). Both methods give exactly the same values, shown in Table 1 to Table 3.

In Table 1, we assume that $Z_{S1} = (320 + 39j)\,\Omega$ and $Z_{S2} = (51 + 87j)\,\Omega$. In Table 2, we use $Z_{S1} = (22 - 72j)\,\Omega$ and Z_{S2} is the same as in Table 1. In Table 3, Z_{S1} is the same as in Table 1, and we use $Z_{S2} = (74 - 225j)\,\Omega$.

A comparison of Table 1 to Table 3 teaches that t_{A1}, t_{A2}, t_{AW}, t_{B1}, t_{B2}, t_{BW}, G_{AT}, G_{BT}, G_{AI}, G_{BI}, G_{AU} and G_{BU} depend on Z_{S1} and Z_{S2}. A comparison of Table 1 to Table 3 also teaches that: G_{BO} and G_{AA} depend on Z_{S1}; and G_{AO} and G_{BA} depend on Z_{S2}. The results shown in Table 1 to Table 3 are compatible with the fact that, as shown in Section IV.E and Section IV.F: G_{BO} and G_{AA} do not depend on Z_{S2}; and G_{AO} and G_{BA} do not depend on Z_{S1}.

The results shown in Table 1 to Table 3 are compatible with the fact that, as shown in Section IV.B, Section IV.C, Section IV.E and Section IV.F, t_{A1}, t_{A2}, t_{AW}, t_{B1}, t_{B2}, t_{BW}, G_{AT}, G_{BT}, G_{AO}, G_{BO}, G_{AA}, G_{BA} are less than or equal to one.

The results shown in Table 1 and Table 3 teach that G_{BI}, G_{AU} and G_{BU} can be greater than one, in line with the assertions found in Section IV.D and Section IV.G, according to which G_{AI}, G_{BI}, G_{AU} and G_{BU} need not be less than or equal to one.

We find that the computed values are compatible with (58)-(60). We also find that (63), (66), (73)-(74) and (77) need not be true in a case where the DUS is not reciprocal, and not lossless.

B. SECOND EXAMPLE

In a second example, we assume that $Z_{S1} = (32 + 39j)\,\Omega$ and $Z_{S2} = (51+87j)\,\Omega$, and that the DUS has an admittance matrix, given by

$$\mathbf{Y} = \begin{pmatrix} 5 + 28j & -3 - 22j \\ -3 - 22j & 2 + 14j \end{pmatrix} \text{mS}\,. \tag{91}$$

Here, $\mathbf{Y}$ has a positive definite hermitian part and is symmetric. Thus, the DUS is passive, reciprocal, and not lossless. The 16 power ratios defined in Section IV have been computed a first time using the formulas of Section II.C and $\mathbf{Y}_{SAM}$ given by (12), and a second time using the formulas of Section II.D and $\mathbf{Z}_{PAM}$ given by (9). Both methods give exactly the same values, shown in Table 4.

TABLE 4. Results for the second example.

Quantity	CA	CB
power transfer ratio at port 1 of the DUS	0.534733	0.534733
power transfer ratio at port 2 of the DUS	0.539872	0.539872
power transfer ratio without the DUS	0.286756	0.286756
transducer power gain	0.439568	0.439568
insertion power gain	1.532900	1.532900
operating power gain	0.822034	0.814209
available power gain	0.814209	0.822034
unnamed power gain	1.522646	1.522646

The results shown in Table 4 teach that G_{AI}, G_{BI}, G_{AU} and G_{BU} can be greater than one, in line with the assertions found in Section IV.D and Section IV.G.

We find that the values computed for this reciprocal DUS are compatible with (58)-(60), (63), (66), (73)-(74) and (77).

C. THIRD EXAMPLE

In a third example, the DUS is an ideal transformer, which has neither an impedance matrix nor an admittance matrix. However, we can directly compute $\mathbf{Z}_{PAM}$ and $\mathbf{Y}_{SAM}$, by inspection. Let n be the ratio of the number of turns in the secondary to the number of turns in the primary. We obtain:

$$\mathbf{Z}_{PAM} = \frac{1}{Y_{S1} + n^2 Y_{S2}} \begin{pmatrix} 1 & n \\ n & n^2 \end{pmatrix} \tag{92}$$

and

$$\mathbf{Y}_{SAM} = \frac{1}{n^2 Z_{S1} + Z_{S2}} \begin{pmatrix} n^2 & n \\ n & 1 \end{pmatrix}\,. \tag{93}$$

$\mathbf{Z}_{PAM}$ and $\mathbf{Y}_{SAM}$ are singular. For $Z_{S1} = (32 + 39j)\,\Omega$, $Z_{S2} = (111 - 120j)\,\Omega$ and $n = 2$, we get:

$$\mathbf{Z}_{PAM} \simeq \begin{pmatrix} 33.98 - 3.07j & 67.96 - 6.14j \\ 67.96 - 6.14j & 135.92 - 12.29j \end{pmatrix} \Omega \tag{94}$$

and

$$\mathbf{Y}_{SAM} \simeq \begin{pmatrix} 16.37 - 2.47j & 8.18 - 1.23j \\ 8.18 - 1.23j & 4.09 - 0.62j \end{pmatrix} \text{mS}\,. \tag{95}$$

Here, the DUS is passive, reciprocal, and lossless. The 16 power ratios defined in Section IV have been computed a first time using the formulas of Section II.C and $\mathbf{Y}_{SAM}$ given by (93), and a second time using the formulas of Section II.D and $\mathbf{Z}_{PAM}$ given by (92). Both methods give exactly the same values, shown in Table 5.

TABLE 5. Results for the third example.

Quantity	CA	CB
power transfer ratio at port 1 of the DUS	0.972867	0.972867
power transfer ratio at port 2 of the DUS	0.972867	0.972867
power transfer ratio without the DUS	0.526027	0.526027
transducer power gain	0.972867	0.972867
insertion power gain	1.849462	1.849462
operating power gain	1.000000	1.000000
available power gain	1.000000	1.000000
unnamed power gain	1.027889	1.027889

The results shown in Table 5 teach that G_{AI}, G_{BI}, G_{AU} and G_{BU} can be greater than one, in line with the assertions found in Section IV.D and Section IV.G.

We find that the values computed for this reciprocal DUS are compatible with (58)-(60), (63), (66), (73)-(74) and (77). We also find that the values computed for this lossless DUS are compatible with (79)-(80) and (84)-(89).

D. FOURTH EXAMPLE

In a fourth example, we assume that $Z_{S1} = (320 + 39j)\,\Omega$ and $Z_{S2} = (51+87j)\,\Omega$, and that the DUS has an impedance matrix, given by

$$\mathbf{Z} = \begin{pmatrix} 125j & 15 + 200j \\ -15 + 200j & 250j \end{pmatrix} \Omega\,. \tag{96}$$

Here, $\mathbf{Z}$ has a null hermitian part and is not symmetric. Thus, the DUS is passive, not reciprocal, and lossless. The 16 power ratios defined in Section IV have been computed a first time using the formulas of Section II.C and $\mathbf{Y}_{SAM}$ given by (12), and a second time using the formulas of Section II.D and $\mathbf{Z}_{PAM}$ given by (9). Both methods give exactly the same values, shown in Table 6.

TABLE 6. Results for the fourth example.

Quantity	CA	CB
power transfer ratio at port 1 of the DUS	0.194438	0.194438
power transfer ratio at port 2 of the DUS	0.194438	0.194438
power transfer ratio without the DUS	0.425230	0.425230
transducer power gain	0.194438	0.194438
insertion power gain	0.457254	0.457254
operating power gain	1.000000	1.000000
available power gain	1.000000	1.000000
unnamed power gain	5.143022	5.143022

We find that the values computed for this lossless DUS are compatible with (79)-(80) and (84)-(89).

VII. ABOUT THE FRIIS TRANSMISSION FORMULA

The original Friis transmission formula is about “a radio circuit made up of a transmitting antenna and a receiving antenna in free space”, and it reads [2]:

$$\frac{P_{avr}}{P_t} = \frac{A_r A_t}{d^2 \lambda^2}\,, \tag{97}$$

where: P_{avr} is the power available at the port of the receiving antenna; P_t is the power fed into the transmitting antenna at its port; A_r is the effective area of the receiving antenna, in the direction of the transmitting antenna; A_t is the effective area of the transmitting antenna, in the direction of the receiving antenna; d is the distance between the antennas; and λ is the wavelength. The correct definition of the effective area of an antenna, to be used in (97), is: “the ratio of the available power at the terminals of a receiving antenna to the power flux density of a plane wave incident on the antenna from that direction, the wave being polarization matched to the antenna” [13].

As pointed out in [2], (97) is not based on any assumption regarding antenna efficiency or antenna losses. However, it assumes that the antennas are polarization matched, that d is sufficiently large (far field condition), and that the transmitting antenna is reciprocal. Here, “reciprocal antenna” means an antenna to which we could apply the Lorentz reciprocity theorem if it was used in free space [14, Sec. 13.1].

For polarization-matched antennas and sufficiently large values of d, another form of (97) is [14, Sec. 4.4.2], [15]:

$$\frac{P_{avr}}{P_t} = \frac{A_r G_t}{4\pi d^2}\,, \tag{98}$$

where G_t is the gain of the transmitting antenna, in the direction of the receiving antenna. The gain of an antenna in a given direction being defined as “the ratio of the radiation intensity in a given direction to the radiation intensity that would be produced if the power accepted by the antenna were isotropically radiated” [13], (98) directly follows from the definitions of A_r and G_t, and therefore does not require any assumption on the reciprocity of the transmitting antenna.

Another common form of (97) for polarization-matched antennas and sufficiently large values of d is [14, Sec. 4.4.2]:

$$\frac{P_{avr}}{P_t} = G_r G_t \left(\frac{\lambda}{4\pi d}\right)^2\,, \tag{99}$$

where G_r is the effective area of the receiving antenna, in the direction of the transmitting antenna. This formula applies only to a reciprocal receiving antenna.

From our perspective, the ratio P_{avr}/P_t is an unnamed power gain. Thus, (97)-(99) are suitable for computing this unnamed power gain, and they are valid for any LTI generator connected to the transmitting antenna, and any LTI load connected to the receiving antenna, provided $P_t \neq 0$ W. Some authors use “Friis transmission formula” to designate formulas which, instead of providing the value of the unnamed power gain, give the value of other power ratios, such as the ratio of the power delivered by the receiving antenna to P_t [16, Sec. 3-12], [17, Sec 5.3], [18, Sec. 2.17.1], [19]. We believe that this is regrettable.

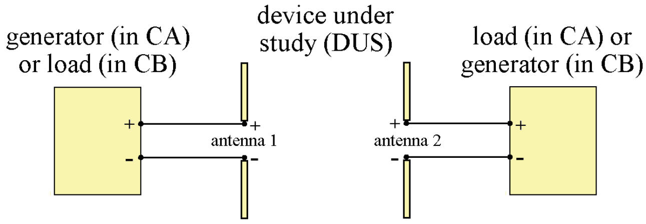

FIGURE 2. The configurations considered in Section VII, in which the DUS comprises antenna 1 and antenna 2.

In the case where both antennas are reciprocal, the original Friis transmission formula (97) conveys two teachings: how to compute the unnamed power gain P_{avr}/P_t; and that, if, without moving the antennas, their roles are reversed (i.e., the receiving antenna becomes the transmitting antenna and vice versa), then the unnamed power gain does not change. These teachings are also imparted by (99). The first teaching is specific to the configuration of two polarization-matched antennas in free space, at a sufficient distance from one another, at least one of them being reciprocal, as explained above. The second teaching can be generalized.

To this end, we now consider a DUS comprising two antennas and whatever lies around them, as shown in Fig. 2. We neither assume polarization-matched antennas, nor a large value of d, nor a free space environment. In CA, antenna 1 is used for emission and antenna 2 for reception. In CB, antenna 2 is used for emission and antenna 1 for reception. We assume, however, that both antennas are reciprocal and that the medium surrounding them is reciprocal [20, Sec. 13.06]. Thus, we can use theorem II of [21], known as the “Rayleigh-Carson reciprocity theorem” and corresponding to [20, eq. (13-40)], to assert that $\mathbf{Z}_{PAM}$ and $\mathbf{Y}_{SAM}$ are symmetric. The DUS being consequently reciprocal, (77) holds, that is: if $P_{ARP1} \neq 0$ W and $P_{BRP2} \neq 0$ W, then $G_{AU} = G_{BU}$.

This reciprocal relation generalizes said second teaching of the original Friis transmission formula (97). Other reciprocal relations obtained above for a reciprocal DUS can also be used, such as (63), (66) and (73)-(74). Note, however, that lossless antennas operating in a lossless medium do not lead to a lossless DUS, in the meaning of Section V. Note also that ionospheric propagation may involve a significant Faraday rotation, which makes the propagation medium non-reciprocal [17, Sec. 6.6], [20, Sec. 17.10], [21].

In CA, P_{AAVP2} depends on the generator connected to port 1 and on the DUS, but not on Z_{S2}. For sufficiently large values of d (far field condition), and if $P_{ARP1} \neq 0$ W, we can say that Z_{APP1} depends very little on Z_{S2}, so that P_{ARP1} depends very little on Z_{S2}. Thus, in this case, G_{AU} depends very little on Z_{S2}. Likewise, for sufficiently large values of d and if $P_{BRP2} \neq 0$ W, G_{BU} depends very little on Z_{S1}. If follows that, if the DUS is reciprocal, $P_{ARP1} \neq 0$ W, and $P_{BRP2} \neq 0$ W, then $G_{AU} = G_{BU}$ depends very little on Z_{S1} and very little on Z_{S2}. However, the dependence of $G_{AU} = G_{BU}$ on Z_{S1} and Z_{S2} exists, as shown in Section VI.A. Thus, (97)-(99) must be considered as approximations, because they ignore this dependence.

VIII. CONCLUSION

We have stated and proven a new theorem on power products, which can be used to directly obtain 8 reciprocal relations between 16 power ratios. Five of these reciprocal relations hold for a reciprocal DUS. The beauty of the theorem on power products lies in its simplicity and generality, whereas 5 of the reciprocal relations between power ratios necessitate assumptions ensuring nonzero denominators. We have also directly derived several results on power ratios, applicable to a lossless DUS that need not be reciprocal.

We used our results on the unnamed power gain to discuss and generalize the Friis transmission formula.

The formulas (63) and (66) were generalized to a DUS having more than two ports, in [5]. The formula (60) has been generalized to multiport generators and loads, in [9]. We plan to generalize other results of this article, to a DUS having more than two ports.

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