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Abstract

This paper deals with the estimation of the tail index of a conditional heavy-tailed distribution of a spatial process. We are particularly interested in the estimation of conditional spatial rare events when the process is $\beta-$mixing. Given a conditional stationary real-valued multidimensional spatial process $\left\{Y_{x_{\mathbf{i}}}, \mathbf{i} \in \mathbb{Z}^N\right\}$, we investigate its conditional heavy-tail index estimation and the corresponding conditional quantile. Asymptotic properties of the corresponding estimators are established under mild *mixing* conditions. The particularity of the tail proposed estimator is based on the spatial nature of the sample and its unbiased and reduced variance properties compared to well known conditional tail index estimators. A numerical study on synthetic and real data sets is conducted to assess the finite-sample behaviour of the proposed estimators.

keywords: Asymptotic normality; β-mixing; conditional extreme value index; bias correction; spatial dependence; functional estimation.

1 INTRODUCTION

Extreme value theory knows a growing dynamic in recent years motivated by the large number of applications in various and varied fields. The literature on statistical inference in extreme value theory, developing sophistic statistical tools for modeling extreme events towards several direction is then very extensive. We refer to Daouia et al. [2019, 2018], Basrak and Tafro [2014], Ledford and Tawn [1996, 1997], Beirlant and Vandewalle [2002], Heffernan and Tawn [2004], Draisma et al. [2004], Peng [2010, 1999] among many others.

Speaking of the estimation of the heavy distribution tail index, we refer to existing work, particularly that of Bobbia et al. [2019, 2021], Daouia et al. [2020], Bassene [2016], Ndao et al. [2014], Resnick et al. [1998], Hsing [1991], Hill [1975] to name but a few. The most tail index estimator is that of Hill [1975] for times series under the independent hypothesis while Resnick et al. [1998] discuss the consistency of Hill's estimator when it is applied to certain classes of heavy-tailed stationary and dependent processes. Other authors looks for Hill [1975] estimator properties and its applications in finance (risk measure), medicine... when Ndao et al. [2014] deal with censure data and Bassene [2016] extends these works in spatial case.

Almost all the existing statistical tools for estimating the heavy distribution tail index are developed for independent or time-dependent data despite the numerous situations where data are of spatial dependency nature. In fact, in many fields, data are now collected with geographical positions such as oceanography, epidemiology, forestry survey, economy and many others. The study of these kinds of data or any characteristic of such data cannot be done without taking into account their respective geographical positions and eventually spatial dependency. Spatial analysis is a general term to describe a technique that uses the spatial information in order to better handle the dependency of the observed data in an inference.

For modelling extreme spatial processes, the reader may refer to Tchazino et al. [2021], Bopp et al. [2021], Sharkey and Winter [2019], Opitz [2016], Bassene [2016], Basrak and Tafro [2014], Thibaud et al. [2013], Davison et al. [2012], Blanchet and Davison [2011], Turkman et al. [2010] among others. In particular, for tail index estimation, Basrak and Tafro [2014] considered the extremely behaviour of moving averages and moving maxima on a regular two dimension discrete grid while Bassene [2016] extended the previous works to a more general context under strongly conditions. Recently Tchazino et al. [2021] extend Bassene [2016] il the functional context under β-mixing condition.

Tail index estimation is important in many extreme value theory problems in particular when estimating extreme quantiles (see Bolancé and Guillen [2021], Velthoen et al. [2021], Chavez-Demoulin and Guillou [2018], Bassene [2016], Goegebeur et al. [2014] among others). Goegebeur et al. [2014] and Bassene [2016] proposed Weissman extreme quantile estimators for $\beta-$mixing non-spatial process (resp. $\alpha-$mixing spatial process) from tail estimation. Velthoen et al. [2021] proposed recently a gradient boosting procedure to estimate a conditional generalized Pareto quantiles while Bolancé and Guillen [2021] introduced a new method to estimate longevity risk based on the kernel estimation of extreme quantiles.

It is quite natural to note that the advent of a phenomena would not have been considered as independent of any other plagues but as the result of several related plagues.

Thus it would be convenient to associate a covariate (which can be a set of variables) with the process describing the phenomenon being studied. Speaking of the conditional estimation of the extreme distribution tail index, we refer to Bassene [2016], Ndao et al. [2014], Gardes and Stupfler [2014], Gardes et al. [2012], Davison et al. [2012], Gardes and Girard [2010, 2008], Resnick et al. [1998] to name but a few. Bassene [2016] proposed a non-parametric conditional tail index estimation and Weissman type estimator of extreme quantile under α-mixing condition while Ndao et al. [2014] early did the same but under random censoring. Gardes and Stupfler [2014] propose the estimation of the tail index in the presence of a finite-dimensional random covariate inspired by Gardes and Girard [2010] where the covariate is recorded simultaneously with the quantity of interest while early Gardes and Girard [2008] based their approach on a weighted sum of the log-spacing between some selected observations. Davison et al. [2012] introduced a latent variable modeling which allows a better fit to marginal distribution.

Let $(Y_i)_{i\in\mathbb{Z}}$ be a real and measurable process, where Y_i has the same distribution as Y defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and (x_i) a deterministic process observed to the point i ($x_i \in \mathbb{R}^d$, $d \in \mathbb{N}^*$). We provide $\mathbb{R}$ with the metric $d(\cdot,\cdot)$. We assume that the condition of the regular variation of the probability tail from Y conditionally to x is given by:

$$\forall y > 0, \qquad \mathbb{P}(Y > y, x) = y^{-\frac{1}{\gamma(x)}} L(y, x), \tag{1}$$

where $\gamma(\cdot)$ is unknown positive function of covariate x and $L(\cdot, x)$ is a slowly varying function at infinity.

The function $\gamma(\cdot)$ is referred to as conditional tail index function or conditional extreme value index function.

In a series of observations, since we are interested in extreme or unusual values, it is essential to find a method of identifying and collecting such values conditionally at x. In this context where we are interested in the Y_i process related to the information of the x process that we set, two methods are proposed in the literature; methods by moving window approach proposed by Gardes and Girard [2008] and the one of Goegebeur et al. [2014]. There is in the literature the conditional version of the estimator of Hill [1975] using either of the two extreme stock selection methods.

Another aspect of this estimator of the extreme distribution tail index that attracts the attention of many researchers is the functional one. To our knowledge, there are at least three works going in this direction; Tchazino et al. [2021], Chavez-Demoulin and Guillou [2018], Goegebeur and Guillou [2013].

Chavez-Demoulin and Guillou [2018] considered a β-mixing time series $(Y_i)_{i\in\mathbb{N}}$ and built an estimator of tail index γ:

$$\hat{\gamma}_k(K) = T_K(Q_n) = \int_0^1 \log\left(\frac{Q_n(t)}{Q_n(1)}\right) d(tK(t)), \tag{2}$$

where $Q_n(t) = Y_{n-\lfloor kt\rfloor,n}$ $0 < t < n/k$ is the quantile and K a function with support in $(0, 1)$.

In this paper, we are particularly interested in the conditional estimation of the tail index and extreme quantile for a process (Y_i) associated to a deterministic process (x_i). Indeed, we wish to extend the estimator proposed by Chavez-Demoulin and Guillou [2018] for a mixing time series process with the conditional framework.

This paper is organized as follows. Section 2 presents the estimator of the conditional tail index and a bias correction method and it's asymptotic properties while Section 3 deals with conditional extreme quantile estimator and it's asymptotic properties. In order to study the finite sample performance of our estimators,we also propose finite sample properties of the estimates with simulated study and a real data application in Section 4 and finally the proofs of the main results are presented in Section 5. All the figures and tables are in Section 6

2 Functional estimation of the conditional extreme value index

2.1 Methodology

Let $\mathbb{Z}^N$; $N \geq 1$ be an Euclidean space N-dimensional of the point indices and $\left\{Y_{\mathbf{i}}; \mathbf{i} \in \mathbb{Z}^N\right\}$ a real and measurable spatial process where $Y_{\mathbf{i}}$ has the same distribution as Y defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $(x_{\mathbf{i}})$ be a deterministic process observed to the point $\mathbf{i}$ ($x_{\mathbf{i}} \in \mathbb{R}^p$, $p \in \mathbb{N}^*$). d will denote the euclidean distance on $\mathbb{R}^p$. For reasons of simplicity, we note the couple $(Y_{\mathbf{i}}, x_{\mathbf{i}})$ as $Y_{x_{\mathbf{i}}}$. We assume that the condition of the regular variation of the Y's tail probability given x stated in equation (1) is satisfied; i. e. Y given x belongs to the Fréchet attraction domain of index $\alpha(x) = 1/\gamma(x)$; where $\gamma(\cdot)$ is unknown positive function of covariate x and $L(\cdot, x)$ is a slowly varying function at infinity that is:

$$\lim_{y \to \infty} \frac{L(ty, x)}{L(y, x)} = 1, \ \ \forall \ t > 0. \tag{3}$$

In the following, we are interested in the non-parametric point-wise estimation of this positive function of covariate x, $\gamma(\cdot)$ for spatial data.
For that purpose, let $\mathbf{i} = (i_1, \ldots, i_N) \in \mathbb{Z}^N$ be a site and consider the notation of the rectangular domain (see Bassene [2016]):

$$\mathtt{I}_{\mathbf{n}} = \{\mathbf{i} = (i_1, \ldots, i_N); 1 \leq i_k \leq n_k; k = 1, \ldots, N\}; \tag{4}$$

with lexicographical order; in the sense that $\mathbf{i} = (i_1, \ldots, i_N) \leq \mathbf{j} = (j_1, \ldots, j_N) \Leftrightarrow$ $i_1 \leq j_1$ or $i_1 = j_1$ and $i_2 \leq j_2$ or$, \ldots, i_k = j_k$ and $i_N \leq j_N,\ k = 1, \ldots, N-1$. Consider a sample $(Y_{\mathbf{i}}; x_{\mathbf{i}})_{\mathbf{i} \in \mathtt{I}_{\mathbf{n}}}$ of conditional dependents variables verifying the relationship (1). In the triangular ordering, the observations become $(Y_i, x_i)_{1 \leq i \leq \hat{\mathbf{n}}}$ where each index $i = 1, \ldots, \hat{\mathbf{n}} = n_1 \times n_2 \times ... \times n_N$, is identified by a site $\mathbf{i}$ in the region $\mathtt{I}_{\mathbf{n}}$ (see Robinson [2011]).
Let $\mathtt{I}_n = \{i; 1 \leq i \leq \hat{\mathbf{n}}\}$ and for simplicity set $\hat{\mathbf{n}} = n$. In the following all limits are considered for $n \to \infty$.

Let us recall the moving window approach proposed by Gardes and Girard [2008] that we use in the following for filtering process (Y_{x_i}). For $r > 0$, let

$$\mathrm{B}(x, r) = \left\{w \in \mathbb{R}^d,\ \ d(x, w) \leq r\right\} \tag{5}$$

be the ball with center x and radius r and let $r_{n,x}$ be a sequence of positive real numbers tending to 0 when $n \to \infty$. Since the estimator of $\gamma(x)$ for given x is based on the variables Y_i for which the associated covariates x_i belongs to the ball $\mathrm{B}(x, r_{n,x})$, the proportion of points in this ball is therefore given by

$$\phi(r_{n,x}) = \frac{1}{n} \sum_{i \in \mathrm{I}_n} \mathbf{1}_{\{x_i \in \mathrm{B}(x, r_{n,x})\}}.$$

From there, the number of observations Y_i for which the associated $x_i \in \mathrm{B}(x, r_{n,x})$, is given by $m_{n,x} = n\phi(r_{n,x})$. Our estimator is built according to the following steps:

- Data filtering: Let $\{Z_i(x) = Y_i, \ \ x_i \in \mathrm{B}(x, r_{n,x})\}$ be the set of Y_i's for which the associated covariates x_i belongs to the ball $\mathrm{B}(x, r_{n,x})$;

- Data ordering and construct the conditional quantile function $Q_{m_{n,x}}(t, x)$. The process $\{Z_i(x), \ \ i \in \mathrm{I}_n\}$ can be ordered in the following order : $Z_{1,m_{n,x}}(x) \leq Z_{2,m_{n,x}}(x) \leq \cdots \leq Z_{m_{n,x},m_{n,x}}(x)$. Let $(k_{m_{n,x}})$ be an intermediate sequence of integers such that $k_{m_{n,x}} \leq m_{n,x}$. We will assume that

$$k_x = k_{m_{n,x}} \to \infty; \ \ k_{m_{n,x}} = \mathrm{o}(m_{n,x}) \ \text{ as } \ n \to \infty. \tag{6}$$

  We set $Q_{m_{n,x}}(t, x) := Z_{m_{n,x} - \lfloor k_x t \rfloor, m_{n,x}}(x)$ where $0 < t < \frac{m_{n,x}}{k_x}$ the conditional quantile function measurable through (Y_i) for a given x_i in the ball $\mathrm{B}(x, r_{n,x})$. It is obvious that for all $0 < t \leq 1$, $Q_{m_{n,x}}(t, x) \geq Q_{m_{n,x}}(1, x)$;

- Estimation strategy: as in Chavez-Demoulin and Guillou [2018] and in Tchazino et al. [2021], for given x, consider $z(\cdot, x) : [0, 1] \longrightarrow \mathbb{R}$ a measurable function, and consider the function:

$$T_K(z) = \begin{cases} \displaystyle\int_0^1 \log\left(\frac{z(t,x)}{z(1,x)}\right) d(tK(t)), & \text{if the right-hand side is defined and finite,} \\ \\ 0 \ \ \text{otherwise.} \end{cases}$$

  Thus a class of estimators of $\gamma(x)$ from the model (1) is given by:

$$\hat{\gamma}_{k_x}(K, x) = T_K(Q_{m_{n,x}}) = \int_0^1 \log\left(\frac{Q_{m_{n,x}}(t,x)}{Q_{m_{n,x}}(1,x)}\right) d(tK(t)), \tag{7}$$

  where K is a function with support in $(0, 1)$.

Remark 2.1 *Under the differentiability conditions on the function K, the estimator in Bassene [2016] (conditional spatial version of Hill [1975]) is a special case of this functional conditional estimator for $K \equiv 1$).*

2.2 Asymptotic properties

To establish the asymptotic properties of our class of estimators, we need some assumptions and conditions.

Condition C_K: Let K be a function such that $0 < \int_0^1 K(t)dt < \infty$. Suppose that K is continuously differentiable on $(0,1)$ and that there exist $M > 0$ and $\tau \in [0, 1/2)$ such that $|K(t)| \leq Mt^{-\tau}$.

Condition C_M (mixing condition): Let $\sigma_Y(T) = \sigma(\{Y_{\mathbf{i}}, \mathbf{i} \in T\})$ denote the σ-field generated by $\{Y_{\mathbf{i}}, \mathbf{i} \in T\}$ for $T \subset \mathbb{Z}^N$. For any subsets T_1 and T_2 of $\mathbb{Z}^N$, the β-mixing coefficient between $\sigma_Y(T_1)$ and $\sigma_Y(T_2)$ is defined by

$$\tilde{\beta}(T_1, T_2) = \sup \frac{1}{2} \sum_{j=1}^{J} \sum_{s=1}^{S} |P(A_j \cap B_s) - P(A_j)P(B_s)|, \tag{8}$$

where the supremum is taken over all partitions $\{A_j\}_{j=1}^J \subset \sigma_Y(T_1)$ and $\{B_s\}_{s=1}^S \subset \sigma_Y(T_2)$ of $\mathbb{Z}^N$. Let $\mathcal{R}(b)$ denote the collection of all finite disjoint unions of cubes in $\mathbb{Z}^N$ with total volume not exceeding b. Then, let

$$\beta(a, b) = \sup \left\{ \tilde{\beta}(T_1, T_2); d(T_1, T_2) \geq a; T_1, T_2 \in \mathcal{R}(b) \right\}, \quad a, b > 0, \tag{9}$$

where $d(T_1, T_2) = \inf \{\|x - y\|; x \in T_1, y \in T_2\}$. We assume that there exist a nonincreasing function β_1 with $\lim_{a \to \infty} \beta_1(a) = 0$ and a nondecreasing function g (that may be unbounded) such that

$$\beta(a, b) \leq \beta_1(a) g(b); \quad a, b > 0. \tag{10}$$

Condition C_A (second order condition):
There is a constant $\rho(x) < 0$ and a rate function $\mathcal{A}(\cdot, x)$ with index $\rho(x)$; verifying $\mathcal{A}(y, x) \to 0$ when $y \to \infty$ for all $x \in \mathbb{R}^d$, such that for $t > 1$,

$$\lim_{y \to \infty} \frac{\log(\mathrm{U}(ty, x)/\mathrm{U}(y, x)) - \gamma(x)\log(t)}{\mathcal{A}(y, x)} = \log(t) \frac{t^{\rho(x)} - 1}{\rho(x)}$$

where U is the quantile function with regular variation defined by $\mathrm{U}(\cdot, x) = (1/(1 - F(\cdot, x)))^{\leftarrow}$ ($\leftarrow$ refers to the generalized inverse continuous on the left).
Note that this condition is a consequence of the Theorem B.3.1 (De Haan et al. [2006], De Haan and Stadtmüller [1996]) used in the literature as a second-order condition (see Ndao et al. [2014] and Bassene [2016]).

Condition C_R: (regularity)
There is $\epsilon > 0$, a function $r : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.
Set $\mathcal{I}(\mathbf{j}) = \{\mathbf{i}; j_k(p+q) + 1 \leq i_k \leq j_k(p+q) + p; k = 1, \ldots, N\}$ a collections of disjoints sites of size p^N ($p^N = o(\hat{m}_{\mathbf{n},x})$, $p = p_{m_{\mathbf{n},x}} \to \infty$, $q = q_{m_{\mathbf{n},x}} \to \infty$, $q/p \to 0$; $\hat{m}_{\mathbf{n},x} = m_{n_1,x} \times \cdots \times m_{n_N,x}$) and separate at list by q. Set $(\hat{k}_{m_{\mathbf{n},x}})$ a sequence of integers such that $\hat{k}_{m_{\mathbf{n},x}} \leq \hat{m}_{\mathbf{n},x}$.

(a) $\frac{\beta(q,p^N)}{p^N} \hat{m}_{\mathbf{n},x} + p^N \frac{\log^2 \hat{k}_{m_{\mathbf{n},x}}}{\sqrt{\hat{k}_{m_{\mathbf{n},x}}}} \to 0$;

(b) $\frac{\hat{m}_{\mathbf{n},x}}{p^N \hat{k}_{m_{\mathbf{n},x}}} \mathrm{Cov}\left(\sum_{\substack{\mathbf{i}\in\mathcal{I}(\mathbf{j})\\ \mathbf{j}\in \mathtt{I}_{m_{\mathbf{n},x}}}} 1_{\{Z_{\mathbf{i}}(x)>F^{\leftarrow}(1-\hat{k}_{m_{\mathbf{n},x}} z/\hat{m}_{\mathbf{n},x})\}}, \sum_{\substack{\mathbf{i}\in\mathcal{I}(\mathbf{j})\\ \mathbf{j}\in \mathtt{I}_{m_{\mathbf{n},x}}}} 1_{\{Z_{\mathbf{i}}(x)>F^{\leftarrow}(1-\hat{k}_{m_{\mathbf{n},x}} y/\hat{m}_{\mathbf{n},x})\}} \right) \to$ $r(z,y)$,

$\forall\, 0 \le z, y \le 1+\epsilon$ and

$$\mathtt{I}_{m_{\mathbf{n},x}} = \{\mathbf{i} = (i_1, \ldots, i_N); 1 \le i_k \le m_{n_k,x}; k = 1, \ldots, N\}; \tag{11}$$

(c) there exists a constant C such that $\forall\ 0 \le z < y \le 1+\epsilon$:

$$\frac{\hat{m}_{\mathbf{n},x}}{p^N \hat{k}_{m_{\mathbf{n},x}}} E\left[\left(\sum_{\substack{\mathbf{i}\in\mathcal{I}(\mathbf{j})\\ \mathbf{j}\in \mathtt{I}_{m_{\mathbf{n},x}}}} 1_{\{F^{\leftarrow}(1-\hat{k}_{\mathbf{n}} y/\hat{m}_{\mathbf{n},x}) < Z_{\mathbf{i}}(x) \le F^{\leftarrow}(1-\hat{k}_{m_{\mathbf{n},x}} z/\hat{m}_{\mathbf{n},x})\}} \right)^4\right] \le C(y-z).$$

Remark 2.2 *The hypotheses C_A is classical in extreme value theory (see Gardes and Girard [2008], Ndao et al. [2014]) see also Drees [2000], Drees et al. [2003], de Haan et al. [2016] but without conditional data. The $\beta-$mixing random fields condition C_M is given in Tchazino et al. [2021] Kurisu et al. [2021], Bradley [1993], Dedecker et al. [2007]) in the non conditional case while condition C_R is an extension to conditional spatial context of the one-dimension regular condition given in Drees [2000]). Condition C_K is technical to achieve our goals. We need the additional assumptions (like C_2 and C_3 in Drees et al. [2003]) if one wants to obtain the regularity condition C_R. Particularly the condition $C_R - (a)$ is verified if (10) hold. The condition C_K follow the same line as in Tchazino et al. [2021].*

The following Theorem 2.2 is the conditional version of Theorem 2.1 in Tchazino et al. [2021] and the conditional spatial version of the Theorem 1 in Chavez-Demoulin and Guillou [2018]. The particularity lies in the spatial and conditional aspect of the considered process.

Definition 2.1 *(Zieliński [1998]).*
We define $Z_{m_{n,x}-\lfloor tk_x \rfloor, m_{n,x}}$ to be an $\mathbb{F}$-uniformly strongly consistent (or simply uniformly strongly consistent if the statistical model is fixed) estimator of the q^{th} quantile if

$$(\forall \epsilon > 0)(\forall \eta > 0)(\exists N)(\forall m \ge N),\ P_F\left\{ \sup_{h_{n,x}\ge m} |Z_{h_{n,x}-\lfloor tk_x \rfloor, h_{n,x}} - z_q(F)| > \epsilon \right\} < \eta, \tag{12}$$

where F is the distribution function of the process.

Theorem 2.1 *Let $\left\{Y_{x_i}; \boldsymbol{i} \in \mathbb{Z}^N\right\}$ be a conditional process, where $\left\{Y_{\boldsymbol{i}}; \boldsymbol{i} \in \mathbb{Z}^N\right\}$ is a stationary and spatial process. Assume that the condition of the regular variation of the Y's tail probability given x stated in equation (1) is satisfied. Let assume that the conditions C_M, C_R, C_A and C_K are satisfied. Assume also that the sample quantile $Q_{m_{n,x}}(t,x)$ is an $\mathbb{F}$-uniformly strongly consistent estimator of the q^{th} quantile $z_q = Q(q,t,x)$ as $m_{n,x} \to \infty$ and $tk_x/m_{n,x} \to q$. We have:*

$$\hat{\gamma}_{k_x}(K,x) \longrightarrow \gamma(x)\ \ a.s \text{ as } m_{n,x} \to \infty. \tag{13}$$

Theorem 2.2 *Let* $\left\{Y_{x_i}; \boldsymbol{i} \in \mathbb{Z}^N\right\}$ *be a conditional process, where* $\left\{Y_{\boldsymbol{i}}; \boldsymbol{i} \in \mathbb{Z}^N\right\}$ *is a stationary and spatial process. Suppose that the distribution function of* Y *given* x *is continuous. Let assume that the conditions* C_M*,* C_R*,* C_A *and* C_K *are satisfied. Let* $(k_{m_{n,x}})$ *be an intermediate sequence verifying (6) such that* $\sqrt{k_x}\mathcal{A}(b(m_{n,x}/k_x, x), x) \to \lambda(x) < \infty$, $n \to \infty$*. We have:*

$$\sqrt{k_x}\left[\hat{\gamma}_{k_x}(K,x) - \gamma(x) - \mathcal{A}(b(m_{n,x}/k_x,x),x)\frac{1}{\rho(x)}\right.$$
$$\left.\times \int_0^1 \left(t^{-\rho(x)} - 1 - \rho(x)t^{-\rho(x)}\log(t)\right) K(t)dt\right]$$
$$\xrightarrow{d} \gamma(x)\int_0^1 \left[t^{-1}W(t) - W(1)\right] d(tK(t)); \tag{14}$$

where $(W(t))_{t\in[0,1]}$ is a Gaussian centered process and covariance function r defined in C_R and

$$b(t,x) = F^{\leftarrow}(1-t^{-1},x) \qquad t>1, \tag{15}$$

$$F^{\leftarrow}(y,x) = \inf\{t, F(t,x) \geq y\}, \quad 0 \leq y \leq 1, \quad \text{for a given } x;$$

F being a distribution function and therefore subject to regular variation:

$$\bar{F}(b(t,x),x) \sim t^{-1}, \qquad \bar{F} := 1 - F. \tag{16}$$

Corollary 2.1 *Under the conditions of Theorem 2.2, we have:*

$$\sqrt{k_x}\left(\hat{\gamma}_{k_x}(K,x) - \gamma(x)\right) \xrightarrow{d} \mathcal{N}\left(\lambda(x)\mathcal{AB}(K,x),\ \mathcal{AV}(K,x)\right); \tag{17}$$

where $\mathcal{AB}(K,x) = \dfrac{1}{\rho(x)}\displaystyle\int_0^1 \left(t^{-\rho(x)} - 1 - \rho(x)t^{-\rho(x)}\log(t)\right) K(t)dt$

and $\mathcal{AV}(K,x) = \gamma(x)^2 \displaystyle\int_0^1\int_0^1 \left[\frac{r(t,s)}{ts} - \frac{r(t,1)}{t} - \frac{r(1,s)}{s} + r(1,1)\right] d(tK(t))d(sK(s))$.

Remark 2.3 *If we assume that the* $\left\{Y_{x_i}; \boldsymbol{i} \in \mathbb{Z}^N\right\}$ *are i.i.d and* $K(t)$ *=1 for every* $t \in (0,1)$*, we obtain the asymptotic normality of the same form as that of the estimator of Hill [1975].*

Since this estimator is biased, the bias reduction methods will be discussed below.

2.3 Bias correction

This section follows the same lines as in Tchazino et al. [2021]
We propose a bias reduction method, useful for low values of $|\rho|$ and for the corresponding quantile estimator. Knowing that the bias of the considered class of estimators depends on the function K, we will choose an optimal one, i.e. making both the bias almost zero and minimizing the variance. The particularity of our approach compared

to those of Chavez-Demoulin and Guillou [2018] and that of Goegebeur and Guillou [2013] lies in the consideration of a more wide class of functions K.
To do this let's consider two functions K_1 and K_2 verifying C_K and for a given x set:

$$K_{(\alpha(x),\beta(x))}(t) = \alpha(x)K_1(t) + \beta(x)K_2(t); \;\; \alpha(x), \;\; \beta(x) \;\; \in \;\; \mathbb{R}^* \tag{18}$$

such that $\frac{1}{\alpha(x)} + \frac{1}{\beta(x)} = -1$ (this condition imposed on $\alpha(x)$ and $\beta(x)$ is just technical for the results we want) and $\int_0^1 K_{(\alpha(x),\beta(x))}(t)dt > 0$. Let us now evaluate the bias of the estimator $\hat{\gamma}_{k_x}(K_{(\alpha(x),\beta(x))}, x)$.
We got

$$\frac{\lambda(x)}{\sqrt{k_x}}\mathcal{AB}\left(K_{(\alpha(x),\beta(x))}, x\right) = \frac{\lambda(x)}{\sqrt{k_x}}\left(\alpha(x)\mathcal{AB}(K_1, x) + \beta(x)\mathcal{AB}(K_2, x)\right).$$

Since we are dealing with bias reduction, let us find the values of $\alpha(x)$ and $\beta(x)$ for which the bias is close to 0. Then we obtain the system of equations:

$$\begin{cases} \alpha(x)\mathcal{AB}(K_1, x) + \beta(x)\mathcal{AB}(K_2, x) = 0 \\ \\ \frac{1}{\alpha(x)} + \frac{1}{\beta(x)} = -1. \end{cases}$$

Then we have $S = \left(\alpha(x) = \frac{\mathcal{AB}(K_2,x)-\mathcal{AB}(K_1,x)}{\mathcal{AB}(K_1,x)}, \; \beta(x) = \frac{\mathcal{AB}(K_1,x)-\mathcal{AB}(K_2,x)}{\mathcal{AB}(K_2,x)}\right) \;\in\; \mathbb{R}^2$ is the whole solution.
The resulting K_S leads to an asymptotically unbiased estimator.

Corollary 2.2 *Under the assumptions of the Theorem* 2.2, *and assuming that K_1 and K_2 satisfy the condition C_K, we have*

$$\sqrt{k_x}\left(\hat{\gamma}_{k_x}(K_S, x) - \gamma(x)\right) \xrightarrow{d} \mathcal{N}\left(0, \; \mathcal{AV}(K_S, x)\right). \tag{19}$$

Let $\mathcal{C}_{\gamma(x)} = \{\hat{\gamma}_{k_x}(K_S, x), K_1, \;\; K_2 \;\text{ verifying }\; C_K\}$ be the class of asymptotically unbiased estimators of $\gamma(x)$. The next part of the work consists in constructing the estimator with minimum variance.
In the case of the i.i.d variables, Goegebeur and Guillou [2013] (Theorem 2 and Corollary 4) establish that the pair function (K_1, K_2) verifying the condition C_K for which the variance is minimal is given by: $\left(1, \;\; (1-\rho(x))t^{-\rho(x)}\right)$. Although we have not established the optimal function K of minimum variance in our context, we propose to take the same couple $\left(1, \;\; (1-\rho(x))t^{-\rho(x)}\right)$ of minimum variance in the case of i.i.d.. Thus, the function K_{S^*} from the couple $\left(1, \;\; (1-\rho(x))t^{-\rho(x)}\right)$ of the form (18) is given by

$$K_{S^*}(t) = \frac{\rho(x)^3 - 4\rho(x)^2 + 2\rho(x)}{(1-2\rho(x))^2(2-\rho(x))} - \frac{\rho(x)^3 - 4\rho(x)^2 + 2\rho(x)}{(1-\rho(x))(2-3\rho(x))}t^{-\rho(x)}. \tag{20}$$

$\hat{\gamma}_{k_x}(K_{S^*}, x) \;\in\; \mathcal{C}_{\gamma(x)}$ and is of minimal variance.

Remark 2.4 *If we set* $a(x) = \dfrac{\rho(x)^3 - 4\rho(x)^2 + 2\rho(x)}{(1-2\rho(x))^2(2-\rho(x))}$ *and* $b(x) = \dfrac{\rho(x)^3 - 4\rho(x)^2 + 2\rho(x)}{(1-\rho(x))(2-3\rho(x))}$ *and under the assumption of independence, the variance of our estimator is given by:*

$$\mathcal{AV}(K_{S^*}, x) = \gamma(x)^2 \left((a(x) - b(x))^2 - \frac{2\rho(x)}{1-\rho(x)} a(x)b(x) \right)$$

This variance decreases and leads to 0 when $\rho(x) \to 0$ and explodes when $\rho(x) \to -\infty$; which is contrary to the existing estimators in the literature. In addition to this difference in the direction of variation, our estimator allows us to make the variance as small as desired. Thus, we propose to consider the estimator that achieves the trade-off between these two types (in terms of direction of variation) of variance in such a way as to minimize the variance depending on the values of $\rho(x)$. Such an estimator is one whose function K is given by:

$$K_{S\Delta} = \begin{cases} K_{\Delta^*} & \text{if } \rho(x) \leq \bar{\rho}(x) \\ \\ K_{S^*} & \text{if } \rho(x) \geq \bar{\rho}(x) \end{cases} \tag{21}$$

where $\bar{\rho}(x)$ is the unique solution to the equation: $\mathcal{AV}(K_{S^*}, x) = \mathcal{AV}(K_{\Delta^*}, x)$ and $K_{\Delta^*}(t) = \left(\dfrac{1-\rho(x)}{\rho(x)}\right)^2 - \dfrac{(1-\rho(x))(1-2\rho(x))}{\rho(x)^2} t^{-\rho(x)}$.

Let $\tilde{\rho}(x)$ be a value taken by $\rho(x)$ in (20) or a point (or canonical) estimator of $\rho(x)$. We have

$$K_{\tilde{S}^*}(t) = \tilde{a}(x) - \tilde{b}(x) t^{-\tilde{\rho}(x)} \tag{22}$$

where $\tilde{a}(x)$ and $\tilde{b}(x)$ are the respective values of $a(x)$ and $b(x)$ by replacing $\rho(x)$ by $\tilde{\rho}(x)$. The following corollary holds:

Corollary 2.3 *Under the assumptions of the Theorem* 2.2*, and assuming that* K_1 *and* K_2 *satisfy the condition* C_K*, we have:*

$$\sqrt{k_x}\left(\hat{\gamma}_{k_x}(K_{\tilde{S}^*}, x) - \gamma(x)\right) \xrightarrow{d}$$
$$\mathcal{N}\left(\lambda(x) \left[\tilde{a}(x) \frac{2-\rho(x)}{(1-\rho(x))^2} - \frac{\tilde{b}(x)}{1-\rho(x)} \frac{2-\rho(x)-2\tilde{\rho}(x)}{(1-\rho(x)-\tilde{\rho}(x))^2} \right], \mathcal{AV}(K_{\tilde{S}^*}, x) \right). \tag{23}$$

The bias cancels out when $\rho(x)$ and $\tilde{\rho}(x)$ coincide.

Admittedly, $\hat{\gamma}_{k_x}(K_{\tilde{S}^*}, x)$ is biased but it is of particular interest for the control (reduction) of the bias because a judicious choice of $\tilde{\rho}(x)$ (since $\rho(x)$ is a parameter that controls the speed of convergence) would allow us to reduce the bias considerably, contrary to that in Corollary 2.1. On the other hand since it is possible to cancel this bias (for $\tilde{\rho}(x) = \rho(x)$) if we replace $\rho(x)$ by one of it's consistent estimator in probability $\hat{\rho}_{k_{m_{n,x},\rho(x)}}(x)$ which is a function of an intermediate sequence $(k_{m_{n,x},\rho(x)})_{m_{n,x} \in \mathbb{N}}$, we get the following theorem:

Theorem 2.3 *Let* $\left\{ Y_{x_{\boldsymbol{i}}}; \boldsymbol{i} \in \mathbb{Z}^N \right\}$ *be a conditional process, where* $\left\{ Y_{\boldsymbol{i}}; \boldsymbol{i} \in \mathbb{Z}^N \right\}$ *is a stationary spatial process with a continuous distribution function and verifying conditions*

C_M, C_R and C_A. Let $\hat{\rho}_{k_{m_{n,x},\rho(x)}}(x)$ be an consistent estimator of $\rho(x)$, which is a function of an intermediate sequence $(k_{m_{n,x},\rho(x)})_{m_{n,x}\in\mathbb{N}}$; and $(k_{m_{n,x}})$ another intermediate sequence verifying (6) such that $\sqrt{k_x}\mathcal{A}(b(m_{n,x}/k_x, x), x) \longrightarrow \lambda(x) < \infty, \quad n \to \infty$. We have:

$$\sqrt{k_x}\left(\hat{\gamma}_{k_x}(K_{\hat{S}^*}, x) - \gamma(x)\right) \xrightarrow{d} \mathcal{N}\left(0,\ \mathcal{AV}(K_{S^*}, x)\right), \tag{24}$$

where $K_{\hat{S}^*}$ is of the form (20) by replacing $\rho(x)$ by $\hat{\rho}_{k_{m_{n,x},\rho(x)}}(x) = \hat{\rho}(x)$ for simplicity.

Note that $K_{\hat{S}^*}$ depends on $\hat{\rho}(x)$ and $m_{n,x}$ then we need an additional condition on the term $\sqrt{k_{m_{n,x},\rho(x)}}\mathcal{A}(b(n/k_{m_{n,x},\rho(x)}, x), x)$. Although the $K_{\hat{S}^*}$ is in the form (20), it cannot be written in the form (18) (where K_1 and K_2 are functions of $\hat{\rho}(x)$ and α and β of the functions of $\rho(x)$). Thus it is necessary to estimate $\rho(x)$.
Gomes et al. [2002] proposed a possible family of $\hat{\rho}(x)$ used in De Haan et al. [2006] and in Chavez-Demoulin and Guillou [2018] expressed as follows.

$$\hat{\rho}(x) := \frac{-4 + 6S^{(2)}_{k_{m_{n,x},\rho(x)}} + \sqrt{3S^{(2)}_{k_{m_{n,x},\rho(x)}} - 2}}{4S^{(2)}_{k_{m_{n,x},\rho(x)}} - 3} \text{ with } S^{(2)}_{k_{m_{n,x},\rho(x)}} \in \left(\frac{2}{3}, \frac{3}{4}\right), \tag{25}$$

where

$$S^{(2)}_{k_{m_{n,x},\rho(x)}} := \frac{3}{4}\frac{\left[M^{(4)}_{k_{m_{n,x},\rho(x)}} - 24\left(M^{(1)}_{k_{m_{n,x},\rho(x)}}\right)^4\right]\left[M^{(2)}_{\hat{k}_n} - 2\left(M^{(1)}_{k_{m_{n,x},\rho(x)}}\right)^2\right]}{\left[M^{(3)}_{k_{m_{n,x},\rho(x)}} - 6\left(M^{(1)}_{k_{m_{n,x},\rho(x)}}\right)^3\right]^2},$$

with

$$M^{(\alpha)}_{k_{m_{n,x},\rho(x)}} := \frac{1}{k_{m_{n,x},\rho(x)}}\sum_{i=1}^{k_{m_{n,x},\rho(x)}}\left(\log Z_{m_{n,x}-i+1,m_{n,x}} - \log Z_{m_{n,x}-k_{m_{n,x},\rho(x)},m_{n,x}}\right)^{\alpha},\ \alpha \in \mathbb{N}. \tag{26}$$

In this family of the estimators $\hat{\rho}(x)$ defined in (25) we have the following corollary.

Corollary 2.4 *Let $\left\{Y_{x_{\boldsymbol{i}}}; \boldsymbol{i} \in \mathbb{Z}^N\right\}$ be a conditional process, where $\left\{Y_{\boldsymbol{i}}; \boldsymbol{i} \in \mathbb{Z}^N\right\}$ is a stationary spatial process with a continuous distribution function and verify the conditions C_M, C_R and C_A. Suppose also that condition C_K is satisfied. Let $\hat{\rho}(x)$ be an estimator of $\rho(x)$, where the intermediate sequence $(k_{m_{n,x},\rho(x)})_{m_{n,x}\in\mathbb{N}}$ is such that $\sqrt{k_{n,\rho(x)}}\mathcal{A}(b(m_{n,x}/k_{m_{n,x},\rho(x)}, x), x) \longrightarrow \infty$. Let $(k_{m_{n,x}})$ be another intermediate sequence such as $\sqrt{k_x}\mathcal{A}(b(m_{n,x}/k_x, x), x) \longrightarrow \lambda(x) < \infty, \quad n \to \infty$. We have:*

$$\sqrt{k_x}\left(\hat{\gamma}_{k_x}(K_{\hat{S}^*}, x) - \gamma(x)\right) \xrightarrow{d} \mathcal{N}\left(0,\ \mathcal{AV}(K_{S^*}, x)\right). \tag{27}$$

This corollary is similar to the Corollary 4 in Chavez-Demoulin and Guillou [2018]; the difference lies in the consideration of the process; spatial and conditional in our case. Here the sequence $(k_{m_{n,x},\rho(x)})_{n\in\mathbb{Z}^N}$ is such that $\sqrt{k_{m_{n,x},\rho(x)}}\mathcal{A}(b(m_{n,x}/k_{m_{n,x},\rho(x)}, x), x) \longrightarrow \infty$ is needed to ensure the consistency of the estimator $\hat{\rho}_{k_{m_{n,x},\rho(x)}}(x)$ of $\rho(x)$.
In practice we consider that K is a kernel, but this does not prevent us from using $K_{S\Delta}$, construct in this article. The estimation of $\gamma(\cdot)$ is the only (necessary) step in the

estimation of the quantile. In our approach, not only we have considered a conditional and spatial process, but also our condition C_K differ from those of our predecessors, notably Chavez-Demoulin and Guillou [2018], de Haan et al. [2016], Goegebeur and Guillou [2013] which are restricted to the case where the function K is a kernel.

3 Estimation of the conditional extreme distribution quantile

One of the main purposes of extreme value theory is prediction of future extreme events, the tail index estimation proposed in the previous section is useful in this situation. We then are interested in inference of the extreme quantile

$$z_p(x) = U(1/p, x) \quad p \to 0. \tag{28}$$

Especially, our contribution consists in estimating the extreme quantile by applying the tail index estimator.
From the condition C_A we obtain

$$\frac{U(tz,x)}{U(t,x)} = z^{\gamma(x)} \exp\left\{\mathcal{A}(t,x)\log(z)\frac{z^{\rho(x)}-1}{\rho(x)}\right\} + \mathrm{o}(1).$$

By setting $tz = 1/p$ and $t = Y_{m_{n,x}-\lfloor k_x\lambda\rfloor, m_{n,x}}(x)$, $0 < \lambda < \frac{m_{n,x}}{k_x}$ where Y_i is a random variable from a standard Pareto distribution and since $Z_{m_{n,x}-\lfloor k_x\lambda\rfloor, m_{n,x}}(x) = U(Y_{m_{n,x}-\lfloor k_x\lambda\rfloor, m_{n,x}}(x), x)$, we get the approximation

$$\begin{aligned} z_p(x) &\simeq Z_{m_{n,x}-\lfloor k_x\lambda\rfloor,m_{n,x}}(x)\left(\frac{1}{pY_{m_{n,x}-\lfloor k_x\lambda\rfloor,m_{n,x}}(x)}\right)^{\gamma(x)} \exp\left\{\mathcal{A}(Y_{m_{n,x}-\lfloor k_x\lambda\rfloor,m_{n,x}}(x),x)\right.\\ &\quad \left.\times\log\left(\frac{1}{pY_{m_{n,x}-\lfloor k_x\lambda\rfloor,m_{n,x}}(x)}\right)\frac{\left(\frac{1}{pY_{m_{n,x}-\lfloor k_x\lambda\rfloor,m_{n,x}}(x)}\right)^{\rho(x)}-1}{\rho(x)}\right\}\\ &\simeq Z_{m_{n,x}-\lfloor k_x\lambda\rfloor,m_{n,x}}(x)\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\gamma(x)}\\ &\quad \times\exp\left\{\mathcal{A}\left(b(m_{n,x}/k_x,x),x\right)\log\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)\frac{\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\rho(x)}-1}{\rho(x)}\right\} \end{aligned} \tag{29}$$

where the last step follows from replacing $Y_{m_{n,x}-\lfloor k_x\lambda\rfloor,m_{n,x}}(x)$ by its expected value $b(m_{n,x}/k_x,x)$. This estimator is accessible only if you replace $\gamma(x)$ and $\mathcal{A}\left(b(m_{n,x}/k_x,x),x\right)$ by their estimators. It should be noted that the term $\mathcal{A}\left(b(m_{n,x}/k_x,x),x\right)$ is seen as the moderator or corrector of the quantile estimator since we find the Weissman-type estimator if $m_{n,x}$ is large enough that is, if $\mathcal{A}\left(b(m_{n,x}/k_x,x),x\right) \to 0$ (see Weissman [1978]).

Remark 3.1 *This extreme quantile estimator has a peculiarity over those existing in the literature. Indeed the presence of the multiplicative term* $\log(Z_{m_{n,x}-\lfloor k_x\lambda\rfloor,m_{n,x}}(x))$ *rather large, implies that the speed of convergence of the term* $\mathcal{A}\left(b(m_{n,x}/k_x,x),x\right)$ *around* 0 *is bigger.*

Chavez-Demoulin and Guillou [2018] have proposed an estimator of $\mathcal{A}\left(b(m_{n,x}/k_x,x),x\right)$ in non-conditional cases that we adapt in our context. $\mathcal{A}\left(b(m_{n,x}/k_x,x),x\right)$ can be estimated by :

$$-\frac{(1-\xi(x))(1-2\xi(x))}{\xi(x)^2}[\hat{\gamma}_{k_x}(K_1,x)-\hat{\gamma}_{k_x}(K_{2,\xi(x)},x)],$$

where $\xi(x)$ is a negative canonical or a consistent estimator of $\rho(x)$. Thus by replacing $\gamma(x)$ and $\mathcal{A}\left(b(m_{n,x}/k_x,x),x\right)$ by their estimators in the relationship (29) we obtain:

$$\begin{aligned}\hat{z}_{p,\xi(x)}(x) &= Z_{m_{n,x}-\lfloor k_x\lambda\rfloor,m_{n,x}}(x)\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\hat{\gamma}_{k_x}(K_{\hat{S}^*},x)}\exp\left\{-\frac{(1-\xi(x))(1-2\xi(x))}{\xi(x)^2}\right.\\ &\quad\left.\times[\hat{\gamma}_{k_x}(K_1,x)-\hat{\gamma}_{k_x}(K_{2,\xi(x)},x)]\log\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)\frac{\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\xi(x)}-1}{\xi(x)}\right\}.\end{aligned}$$

Convergences in distribution under appropriate assumptions has been established by the same authors that we adapt in our spatial and conditional context.

Theorem 3.1 *Let* $\left\{Y_{x_{\boldsymbol{i}}};\boldsymbol{i}\in\mathbb{Z}^N\right\}$ *be a conditional process, where* $\left\{Y_{\boldsymbol{i}};\boldsymbol{i}\in\mathbb{Z}^N\right\}$ *is a stationary spatial process with a continuous distribution function and verifying conditions* C_M*,* C_R *and* C_A*. Let* $\hat{\rho}_{k_{m_{n,x},\rho(x)}}(x)$ *be an estimator of* $\rho(x)$*, consistent in probability, which is a function of an intermediate sequence* $(k_{m_{n,x},\rho(x)})_{m_{n,x}\in\mathbb{N}}$*; set* $(k_{m_{n,x}})$ *an intermediate sequence such as*
$\sqrt{k_x}\mathcal{A}(b(m_{n,x}/k_x,x),x)\longrightarrow\lambda(x)<\infty,\quad m_{n,x}\to\infty$ *and suppose that* $p=p_{m_{n,x}}$ *such that* $\frac{1}{pb(m_{n,x}/k_x,x)}\to\infty$*,* $\frac{\log\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)}{\sqrt{k_x}}\to 0$ *and* $m_{n,x}^{-a}\log p\to 0$ *for any* $a>0$*, then we have:*

$$\frac{\sqrt{k_x}}{\log\frac{1}{pb(m_{n,x}/k_x,x)}}\left(\frac{\hat{z}_{p,\xi(x)}(x)}{z_p(x)}-1\right)\xrightarrow{d}\mathcal{N}\left(0,\ \mathcal{AV}(K_{S^*},x)\right),\tag{30}$$

where $\xi(x)$ *is a negative parameter* $\tilde{\rho}(x)$ *or an estimator consistent in probability* $\hat{\rho}(x)$ *such that* $|\hat{\rho}(x)-\rho(x)|=O_{\mathbb{P}}(m_{n,x}^{-\epsilon})$ *for* $\epsilon>0$*.*

4 Finite sample properties

In this section, we illustrate the finite-sample performance of the proposed estimators using simulated and real datasets.

4.1 Simulation study

We would like to estimate the conditional extreme distribution tail index $\gamma(x)$ for a given x of the simulated log-Laplace process (see Tchazino et al. [2021]). To do this

we duplicate $N = 100$ processes of the said process and we consider a range of the highest values k ($k = 1, \cdots, 600$) to be considered to display the best estimator, i.e. the one of minimum error. We have compute these three family of estimators; that of Hill , Chavez-Demoulin and Guillou [2018] given by (2) and the ones built in this article given by (7). It will allow us to appreciate each estimator but also validate our estimator (21).
For this simulation study, we use the empirical form of the estimator (7) that is:

$$\hat{\gamma}_{k_x}(K, x) = \frac{1}{k_x}\sum_{i=1}^{k_x} \log\left(\frac{Z_{m_{n,x}-i+1,m_{n,x}}(x)}{Z_{m_{n,x}-k_x,m_{n,x}}(x)}\right) \times \left(K\left(\frac{i}{k_x}\right) + \frac{i}{k_x} \times K'\left(\frac{i}{k_x}\right)\right) \tag{31}$$

where K' is the first derivative of the function K and $Z_{m_{n,x}-i,m_{n,x}}(x)$ is the ordered dependent variable Y_{x_i} for which the non-random covariate x_i belongs to the ball $B(x_0, r)$. In this simulation study, the covariate x_i is assumed to be an realisation of the variable X belonging to the uniform $[0, 1]$ and we take $x \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$ and $r = 0.1$.
The following conditional tail-index function is similar to one in Daouia et al. [2011], see also Ndao et al. [2014].

$$\gamma(x) = (50/33) \times (.1 + \sin(\pi x)) \times (1.1 - .5 \times \exp(-64(x - .5)^2)) \tag{32}$$

The true quantile function (belongs to Frechet domain of attraction) is given by:

$$q(p, x) = (-ql(1 - p, x)\log(1 - p))^{-\gamma(x)} \tag{33}$$

where p is a probability and $ql(\cdot, x)$ is the conditional loglaplace quantile function.
A sample of true values of $\gamma(x)$ and $q(0.001, x)$ are given in Table 4.1:

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\gamma(x)$	0.682	1.145	1.461	1.331	1.000	1.331	1.461	1.145	0.682
$q(0.001, x)$	5.545	17.750	39.381	28.414	12.340	28.414	39.381	17.750	5.545

Table 4.1 *A sample of true values of* $\gamma(x)$ *and* $q(0,001, x)$ *for a given* x

We essentially verify the performance of our estimator (7) and that of Chavez-Demoulin and Guillou [2018] given by (2) for each given x according the values of $\rho(x)$.
For each $x \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$, we execute the following calculation program:

1. Compute estimators for each replication $i = 1, 2, \ldots, R$ and for $1 \leq k \leq m_x - 1$ noted $\hat{\gamma}_{i,k}(K, x)$.
2. Compute absolute error $\epsilon_{i,k} = |\hat{\gamma}_{i,k}(K, x) - \gamma(x)|$ for $i = 1, 2, \ldots, R$.
3. Recovering the number of extreme values $k_i = \underset{1\leq k\leq m_x-1}{\text{Arg}\min}(\epsilon_{i,k})$; the estimator of $\gamma(x)$ on the replication i is given by $\hat{\gamma}_{i,k_i}(K, x)$.
4. Compute the extreme quantile estimator for replication i; $q_i(p, x)$ is the one computed with $\hat{\gamma}_{i,k_i}(K, x)$.

5. The estimator of $\gamma(x)$ is then given by

$$\hat{\gamma}_{k_x}(K,x) = \frac{1}{R}\sum_{i=1}^{R}\hat{\gamma}_{i,k_i}(K,x),$$

where $k_x = \dfrac{1}{R}\sum_{i=1}^{R} k_i$ and that of $q(p,x)$ is given by:

$$\hat{q}(p,x) = \frac{1}{R}\sum_{i=1}^{R} q_i(p,x).$$

The results of our simulations allow us to conclude that our estimator (7) is also useful than (2) for low values of $\rho(x)$. Thus, based on these results, we recommend the use of the estimator (21) which realizes the compromise between the two estimators (7) and (2). The results of our simulations are reported in Table 6.1 for $\gamma(\cdot)$ and Table 6.2 for $q(0.001,\cdot)$.
Figures 6.1-6.8 and 6.9-6.16 show the boxplots of the N realisations of the estimator of $\gamma(x)$ and $q(0.001,x)$ respectively for every $x \in \{0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9\}$. As we have noted that the estimator of gamma is highly dependent on the covariate x, the quantile estimator is even more sensitive to it. Indeed, a bad estimator of $\gamma(x)$ has a negative effect on the predicted quantile making the estimation error more remarkable than that of $\gamma(x)$. In our simulation study, we can well notice that the performance of the predicted quantile deteriorates when the covariate is at the edge of the $(0,1)$ interval and acceptable in the center of this interval. Figures 6.9-6.16 illustrates this last remark well.
These results clearly show the influence of the parameter $\rho(x)$ and sample size on the estimators. We can also see that there is a relationship between the sample size n and the number k_x of extreme values to be considered.
Furthermore from these results it appears that the estimators of gamma and quantile capture quite well the shape of the conditional extreme value index function $x \longmapsto \gamma(x)$ and quantile function $x \longmapsto q(0.001,x)$ respectively.
Finally, one can clearly see from these simulation results that our estimators seem to have better performance compared to conditional Hill's and conditional Weissman's estimator. We can remark that our estimators quality deteriorates as the sample size decreases.

4.2 Application to real data

We illustrate here the behaviour of the proposed methodology on rainfall data from 1559 stations in the West Africa Region on first September 2019 (available on NASA-datalink). This data set contain the geographical position (longitude, latitude), rain, relative humidity, air pressure, temperature and wind speed.
The mesh was made over West Africa by varying the longitude in the interval $[-17, 3.5]$ and the latitude in $[4.5, 16]$.
The spatial aspect is materialized by longitude and latitude. The observed process has a minimum value of 0 and a maximum value of 65.94 with a mean of 6.94 and a median of 4.94. It can be seen that the mean is very close to the minimum observation and very

distant from the maximum observation. The median shows the extremal aspect of the observations quite large. Indeed 50% of the observations are concentrated in $[0, 4.94]$ and the rest scattered in $[4.94, 65.94]$; this aspect is visible on the Figure 6.25. We can thus think of the existence of a tail on the right, hence the interest of an extreme data study. The Figure 6.25. shows a grouping by similarity (color gradient) of the data on the geographical level. This makes us think of spatialized data or spatial dependence of data.
The Figure 6.26 6.25 illustrates two main aspects of the data: the spatial aspect materialized by the longitude and latitude and the extreme aspect is visualized by the color gradient which shows very few large observations (light blue, yellow and red). These figure also illustrates the spatial dependence marked by the grouping of data by similarity (size of observations) and according to geographical positions; we can notice the cluster of colors (materializing the size of observations) by location: dark blue (the most frequent observations), light blue which is found by geographical location, the largest observations in red and dark red (rare) surrounded by the observations more or less high (yellow). This data set is analysed in Tchazino et al. [2021]

We re-analyse the data taking into account some covariate. The selected covariate is air pressure (denoted by x in the following). Indeed, we performed a regression to identify the variable that best explains the variable of interest (rain); among the four explanatory variables (relative humidity, air pressure, temperature and wind speed). Air pressure best explains followed by relative humidity with a significance level above 99%.
We carry out, in Figure 6.28, visual checks of whether the heavy-tailed assumption makes sense for this sample of data (Y). The boxplot and histogram of the Y_i both give descriptive evidence that Y has a heavy right tail. To further confirm that the heavy-tailed framework is appropriate, we drew a quantile-quantile plot of the weighted log-spacings within the top of the data against the quantiles of the unit exponential distribution. Formally, let $Y_{1,n} \leq Y_{2,n} \leq \cdots \leq Y_{n,n}$ denote the order statistics of the sample $(Y_1, \cdots, Y_n)$. Let $Z_{i,n} = i \log(Y_{n-i+1,n}/Y_{n-i,n})$, $1 \leq i \leq n-1$, denote the weighted log-spacings computed from the consecutive top order statistics. It is known that, if Y is heavy-tailed with tail index $\gamma(\cdot)$ then, for low i, the $Z_{i,n}$ are approximately independent copies of an exponential random variable with mean (see e.g. Beirlant et al. [2006]). The Figure 6.28 therefore gives a quantile-quantile plot of the $Z_{i,n}$ for $1 \leq i \leq \lfloor n/5 \rfloor$ versus the exponential distribution. The relationship in this quantile-quantile plot is approximately linear, which constitutes further evidence that the heavy tail assumption on Y makes sense. The tail on the right is visible on Figure 6.28.
We can well notice that on average 1200 or 76.97% of the observations are close to the mean while very few observations (less than 6.41%) are very far from the majority.
Since the data shows heavy-tailed behavior, we estimate the conditional tail index $\gamma(x)$ and the quantile $q(0.001, x)$ of the conditional distribution Y (rain) given X (air pressure. We use the same estimation method described in Section 4.1 with respect to the empirical form of the estimator of $\gamma(\cdot)$ and the extreme quantile.
In the theory we have mentioned for the selection of conditional variables, the use of the ball centered on an x and of radius r that we give ourselves. Indeed we collect the realizations of the variable of interest for which the corresponding realizations

of the covariate fall into the ball. In practice the choice of the radius is a critical issue. We estimates the tail estimators and quantile for a given $x = 97.10$ (mean-standard deviation of air pression), $x = 98.91$ (mean of air pression) and $x = 100.72$ (mean+standard deviation of air pression). For each of these values we vary the radius $r \in \{1.5,\ sd(x) = 1.81,\ 1.96 \times \sqrt{sd(x)} = 2.65,\ 1.96 \times sd(x) = 3.55,\ 4\}$ and we retain the one achieving the smallest error calculated on $R = 100$ replications; each replication being a resampling of the starting sample.

Indeed for each replication $i = 1, \cdots, R$ we proceed as follows:

1. Compute the estimators of $\gamma(x)$ noted $\hat{\gamma}_{i,k}(K, x)$ for $1 \leq k \leq m_x - 1$.
2. we form several successive "blocks" of estimators of size $B = 15$.
3. we determine the $k-$value to be used (thereafter denoted by k_i) from the block with minimal standard deviation. Precisely, we take the middle value of the $k-$values in the block (see Goegebeur et al. [2014], Ndao et al. [2014])
4. Then the estimator of $\gamma(x)$ for replication i is $\hat{\gamma}_{i,k_i}(K, x)$

The rest of the algorithm follows the same lines as in the simulation part as the computation of the optimal k and the quantile $q(p, x)$.
The results of our studies are in Table 6.3 and the boxplots are in Figure 6.17-6.24.

Remark 4.1 *One can remark according these results, the central role of the function $\rho(\cdot)$ on the convergence of the estimators. For a given x, $\rho(x)$ does not give the same performance on the estimators.*

Conclusion

The estimation of conditional tail index proposed in this article generalizes that Tchazino et al. [2021] where no exogenous variable was considered. The asymptotic properties of the proposed conditional tail index estimators (biased and unbiased) have been established under mild conditions, in particular β-mixing condition compare to the α-mixing one in Bassene [2016]. The originality of the considered framework lies in the spatial nature of the dependent process studied but also on a wide class of tail index estimators, reducing the asymptotic bias and variance. We also proposed an asymptotically normal extreme quantile estimator. Future directions may include considering models with random exogenous variables or space-time processes, with a number of potential applications.

5 Proofs of the main results

Proofs of the main results

To establish the proofs of our results, we adopt Robinson [2011]'s notation of the spatial locations (for seek of simplicity). That is the process $\left\{Y_{\mathbf{i}}, \mathbf{i} \in \mathbb{Z}^N\right\}$ is written as $\{Y_i, 1 \leq i \leq n = n_1 \times n_2 \times \cdots \times n_N\}$ using for instance a triangular array notation and a lexicographic ordering. For this notation the mixing conditions C_M and C_R (regularity) are written as:

Condition C'_M (mixing condition): Let $(l_n)_{n\in\mathbb{N}^*}$ be a sequence of integers such that $1 \leq l_n \leq n$; set $\mathcal{B}_m^j = \sigma(Y_i, m \leq i \leq j)$ be σ-fields generated by the random variables $(Y_i)_i$ with $m \leq i \leq j$. The β-mixing condition is given by:

$$\beta(l_n) := \sup_{m\in\mathbb{N}^*} \mathbb{E}\left[\sup_{A\in\mathcal{B}_{l_n+m+1}^{+\infty}} |\mathbb{P}(A|\mathcal{B}_1^m) - \mathbb{P}(A)|\right] \underset{l_n\to\infty}{\longrightarrow} 0 \tag{34}$$

See Drees [2000] for a discussion on the β-mixing and examples.
Condition C'_R: (regularity)
There is $\epsilon > 0$, a function $r : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $(l_{m_{n,x}})$ defined above is such that $l_{m_{n,x}} = \mathrm{o}(m_{n,x}/k_x)$; and when $n \to \infty$. For simplicity set $l_n := l_{m_{n,x}}$.

(a') $\frac{\beta(l_n)}{l_n} m_{n,x} + l_n \frac{\log^2 k_x}{\sqrt{k_x}} \to 0$;

(b') $\frac{m_{n,x}}{l_n k_x} \mathrm{Cov}\left(\sum_{i=1}^{l_n} 1_{\{Z_i(x)>F^{\leftarrow}(1-k_x z/m_{n,x})\}}, \sum_{i=1}^{l_n} 1_{\{Z_i(x)>F^{\leftarrow}(1-k_x y/m_{n,x})\}}\right) \to r(z,y)$,

$\forall\, 0 \leq z, y \leq 1+\epsilon$;

(c') there exists a constant C such that $\forall\ 0 \leq z < y \leq 1+\epsilon$:

$$\frac{m_{n,x}}{l_n k_x} E\left[\left(\sum_{i=1}^{l_n} 1_{\{F^{\leftarrow}(1-k_x y/m_{n,x})<Z_i(x)\leq F^{\leftarrow}(1-k_x z/m_{n,x})\}}\right)^4\right] \leq C(y-z).$$

5.1 Proof of Theorem 2.1

To prove this theorem we need the following lemma and propositions.

Proposition 5.1 *Supose that* $\forall\ \epsilon > 0$, $\inf_F \min\left\{q - F\left(\frac{x_q-\epsilon}{2}\right), F\left(\frac{x_q+\epsilon}{2}\right) - q\right\} > 0$ *then sample quantile* $Z_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}}$ *such that* $tk_x/m_{n,x} \to q$ *as* $m_{n,x} \to \infty$ *is an* $\mathbb{F}$-*uniformly strongly consistent estimator of the* q^{th} *quantile* $z_q = z_q(F)$

Proof of Proposition 5.1.

Fix $\epsilon > 0$ and let $\delta = \inf_F \min\left\{q - F\left(\frac{x_q-\epsilon}{2}\right), F\left(\frac{x_q+\epsilon}{2}\right) - q\right\}$. In the proof we shall use the following result of Hoeffding [1963]: if $\xi_1, \xi_2, \ldots, \xi_n$ are independent random variables such that, for some finite a and b, $P\{a < \xi_i < b\} = 1$, $i = 1, 2, \ldots, n$, then for $t > 0$,

$$P\left\{\frac{1}{n}\sum_{i=1}^{n}\xi_i - E\left(\frac{1}{n}\sum_{i=1}^{n}\xi_i\right) \geq t\right\} \leq \exp\left\{-2nt^2/(b-a)^2\right\}.$$

Take N such that $q - \frac{1}{2}\delta < tk_x/m_{n,x} < q + \frac{1}{2}\delta$ if $m_{n,x} \geq N$. Denote by F_n the empirical distribution function generated by the sample $Z_1, Z_2, \ldots, Z_{m_{n,x}}$ and by ξ_j the random variable equal to 1 if $2Z_j > z_q + \epsilon$, and equal to 0 otherwise. Set a sequence of independent r.v.'s $Z_1^*, Z_2^*, \ldots, Z_{m_{n,x}}^*$ independent of $Z_1, Z_2, \ldots, Z_{m_{n,x}}$ such that Z_i^* has the same distribution as Z_i and by ξ_j^* the random variable equal to 1 if $2Z_j^* > z_q+\epsilon$, and equal to 0 otherwise; and $0v_j$ an random variable such that $v_j = 1$ if $Z_j - Z_j^* > \frac{z_q+\epsilon}{2}$ and $v_j = -1$ otherwise
Then for $m_{n,x} \geq N$;

$$\begin{aligned}
P_F\left\{Z_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}} > z_q + \epsilon\right\} &= P_F\left\{Z_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}} - Z^*_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}}\right. \\
&\quad \left. + Z^*_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}} > z_q + \epsilon\right\} \\
&\leq P_F\left\{Z_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}} - Z^*_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}} > \frac{z_q+\epsilon}{2}\right\} \\
&\quad + P_F\left\{Z^*_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}} > \frac{z_q+\epsilon}{2}\right\} \\
&= A + B,
\end{aligned}$$

and

$$\begin{aligned}
P_F\left\{Z_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}} < z_q - \epsilon\right\} &= P_F\left\{Z_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}} - Z^*_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}}\right. \\
&\quad \left. + Z^*_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}} < z_q - \epsilon\right\} \\
&\leq P_F\left\{Z_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}} - Z^*_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}} < \frac{z_q-\epsilon}{2}\right\} \\
&\quad + P_F\left\{Z^*_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}} < \frac{z_q-\epsilon}{2}\right\} \\
&= A' + B'.
\end{aligned}$$

Hence for each $m_{n,x} \geq N$ and for each $F \in \mathbb{F}$

$$P_F\left\{|Z_{m_{n,x}-\lfloor tk_x\rfloor, m_{n,x}} - z_q| > \epsilon\right\} \leq A + A' + B + B'.$$

Firt we evaluate $B + B'$. Since Z_i^* are independent r.v.'s and $E\left(\xi_i^*\right) = 1 - F\left(\frac{z_q+\epsilon}{2}\right)$

we have:

$$\begin{aligned}
P_F\left\{Z^*_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} > \frac{z_q+\epsilon}{2}\right\} &= P_F\left\{\xi^*_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} \geq 1\right\}\\
&= P_F\left\{\frac{1}{m_{n,x}}\sum_{\lfloor tk_x\rfloor=1}^{m_{n,x}} \xi^*_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} \geq 1\right\}\\
&= P_F\left\{\frac{1}{m_{n,x}}\sum_{i=1}^{m_{n,x}}\xi_i^* - E\left\{\frac{1}{m_{n,x}}\sum_{i=1}^{m_{n,x}}\xi_i^*\right\} \geq 1\right.\\
&\quad \left. -E\left\{\frac{1}{m_{n,x}}\sum_{i=1}^{m_{n,x}}\xi_i^*\right\}\right\}\\
&= P_F\left\{\frac{1}{m_{n,x}}\sum_{i=1}^{m_{n,x}}\xi_i^* - E\left\{\frac{1}{m_{n,x}}\sum_{i=1}^{m_{n,x}}\xi_i^*\right\} \geq F\left(\frac{z_q+\epsilon}{2}\right)\right\}\\
&\leq P_F\left\{\frac{1}{m_{n,x}}\sum_{i=1}^{m_{n,x}}\xi_i^* - E\left\{\frac{1}{m_{n,x}}\sum_{i=1}^{m_{n,x}}\xi_i^*\right\} \geq \delta\right\}\\
&\leq P_F\left\{\frac{1}{m_{n,x}}\sum_{i=1}^{m_{n,x}}\xi_i^* - E\left\{\frac{1}{m_{n,x}}\sum_{i=1}^{m_{n,x}}\xi_i^*\right\} \geq \frac{\delta}{2}\right\}\\
&\leq \exp\left(-\frac{m_{n,x}\delta^2}{2}\right).
\end{aligned}$$

In the same way one gets:

$$P_F\left\{Z^*_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} < \frac{z_q-\epsilon}{2}\right\} \leq \exp\left(-\frac{m_{n,x}\delta^2}{2}\right).$$

Now we compute the term $A+A'$.
v_j as defined satisfy the condition C_M through Z_j verifying

$$\beta(n) \leq \exp\left(-cn^\tau\right) \text{ for any positive } n, \tag{35}$$

where c and τ are positives constants. Set c_1 and c_2 two positives constants depending only on c. Then by applying Bernstein type inequality given in Merlevède et al. [2009, 2011] one gets:

$$
\begin{aligned}
A + A' &= P_F\left\{Z_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} - Z^*_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} > \frac{z_q+\epsilon}{2}\right\} \\
&\quad + P_F\left\{Z_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} - Z^*_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} < \frac{z_q-\epsilon}{2}\right\} \\
&= P_F\left\{v_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} = 1\right\} + P_F\left\{v_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} = -1\right\} \\
&= P_F\left\{\sum_{\lfloor tk_x\rfloor=1}^{m_{n,x}} v_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} = m_{n,x}\right\} + P_F\left\{\sum_{\lfloor tk_x\rfloor=1}^{m_{n,x}} v_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} = -m_{n,x}\right\} \\
&= P_F\left\{\sum_{i=1}^{m_{n,x}} v_i = m_{n,x}\right\} + P_F\left\{\sum_{i=1}^{m_{n,x}} v_i = -m_{n,x}\right\} \\
&\leq P_F\left\{\left|\sum_{i=1}^{m_{n,x}} v_i\right| \geq m_{n,x}\right\} \\
&\leq \exp\left(-c_1 m_{n,x}^2/m_{n,x}\right) + \exp\left(-c_2 m_{n,x}/\left(\log(m_{n,x})\log(\log(m_{n,x}))\right)\right) \\
&\leq \exp\left(-c_1 m_{n,x}\right) + \exp\left(-c_2 m_{n,x}/\left(\log(m_{n,x})\log(\log(m_{n,x}))\right)\right).
\end{aligned}
$$

So we get

$$
\begin{aligned}
P_F\left\{|Z_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} - z_q| > \epsilon\right\} &\leq \exp\left(-c_2 m_{n,x}/\left(\log(m_{n,x})\log(\log(m_{n,x}))\right)\right) \\
&\quad + \exp\left(-c_1 m_{n,x}\right) + 2\exp\left(-\frac{m_{n,x}\delta^2}{2}\right).
\end{aligned}
$$

Each of the three terms $2\exp\left(-\frac{m_{n,x}\delta^2}{2}\right)$, $\exp\left(-c_2 m_{n,x}/\left(\log(m_{n,x})\log(\log(m_{n,x}))\right)\right)$ and $\exp\left(-c_1 m_{n,x}\right)$ tends to 0 as $m_{n,x}\to\infty$; hence $Z_{m_{n,x}-\lfloor tk_x\rfloor,m_{n,x}} \to z_q$ a.s. uniformly in $\mathbb{F}$.

Proof of Theorem 2.1
For all $\delta_1, \delta_2 > 0$ there exist $M > 0$ such that:

$$\begin{aligned}
|\hat{\gamma}_{k_x}(K,x) - \gamma(x)| &= \left| \int_0^1 \log\left(\frac{Q_{m_{n,x}}(t,x)}{Q_{m_{n,x}}(1,x)}\right) d(tK(t)) + \frac{\gamma(x)}{C}\int_0^1 \log(t)\, d(tK(t)) \right| \\
&\leq \int_0^1 |K(t) + tK'(t)| \left| \log\left(\left(\frac{Q_{m_{n,x}}(t,x)}{Q_{m_{n,x}}(1,x)}\right) \times (t)^{\frac{\gamma(x)}{C}}\right)\right| dt \\
&\leq \int_0^1 \left| \log\left(\left(\frac{Q_{m_{n,x}}(t,x)}{Q_{m_{n,x}}(1,x)}\right)^{|K(t)+tK'(t)|} \times (t)^{\frac{\gamma(x)}{C}|K(t)+tK'(t)|}\right)\right| dt \\
&\overset{K}{\leq} \int_0^1 \left| \log\left(\left(\frac{Q_{m_{n,x}}(t,x)}{Q_{m_{n,x}}(1,x)}\right)^{M} \times (t)^{\frac{\gamma(x)}{C}M}\right)\right| dt \\
&\leq \lim_{s\to 1}\int_0^s \left| \log\left(\left(\frac{Q_{m_{n,x}}(t,x)}{Q_{m_{n,x}}(s,x)}\right)^{M} \times (t)^{\frac{\gamma(x)}{C}M}\right)\right| dt \\
&\leq \lim_{s\to 1}\int_0^s \left| \log\left(\left(\frac{Q(t,x)}{Q(s,x)}\right)^{M} \times (s)^{\frac{\gamma(x)}{C}M}\right)\right| dt \\
&\overset{\text{Potter}}{\leq} \lim_{s\to 1}\int_0^s \left| \log\left((1+\delta_1)^M \left(\frac{t}{s}\right)^{M(\gamma(x)-\delta_2)} \times (s)^{\frac{\gamma(x)}{C}M}\right)\right| dt \\
&\overset{*}{\leq} \lim_{s\to 1}\int_0^s \left| \log\left((1+\delta_1)^M \times (s)^{\frac{\gamma(x)}{C}M}\right)\right| dt \\
&\leq \int_0^1 \left| \log(1+\delta_1)^M \right| dt \\
&\leq \left| \log(1+\delta_1)^M \right|,
\end{aligned}$$

where $C = \displaystyle\int_0^1 K(t)dt$ and since $Q_{m_{n,x}}(t,x) \to Q(t,x)$ $\mathbb{F}$-uniformly when $m_{n,x} \to \infty$ for $0 \leq t \leq 1$ and where $\overset{K}{\leq}$ is justified by Condition C_K, $\overset{\text{Potter}}{\leq}$ is due to Potter's lemma (see De Haan et al. [2006]) and $\overset{*}{\leq}$ is verify by choosing $\delta_2 \leq \gamma(x)$ for a given x.
One get the proof as $\delta_1 \to 0$
The others proofs of the theorems and corollary in Section 2 follows approximately the same lines as in Tchazino et al. [2021] so that are omitted.

5.2 Proof of Theorem 3.1

We need to show the asymptotic normality of $\dfrac{\sqrt{k_x}}{\log\frac{1}{pb(m_{n,x}/k_x,x)}}\log\dfrac{\hat{z}_{p,\xi}(x)}{z_p(x)}$. We have the following decomposition:

$$\begin{aligned}
&\frac{\sqrt{k_x}}{\log\frac{1}{pb(m_{n,x}/k_x,x)}}\log\frac{\hat{z}_{p,\xi}(x)}{z_p(x)}\\
=\;&\frac{\sqrt{k_x}}{\log\frac{1}{pb(m_{n,x}/k_x,x)}}\Bigg\{\log Z_{m_{n,x}-\lfloor k_x\lambda\rfloor,m_{n,x}}(x)+\hat{\gamma}_{k_x}(K_{\hat{S}^*},x)\log\frac{1}{pb(m_{n,x}/k_x,x)}-\log z_p(x)\\
&-\frac{(1-\xi(x))(1-2\xi(x))}{\xi(x)^2}[\hat{\gamma}_{k_x}(K_1,x)-\hat{\gamma}_{k_x}(K_{2,\xi(x)},x)]\log\frac{1}{pb(m_{n,x}/k_x,x)}\frac{\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\xi(x)}-1}{\xi(x)}\Bigg\}\\
=\;&\sqrt{k_x}\left(\hat{\gamma}_{k_x}(K_{\hat{S}^*},x)-\gamma(x)\right)+\frac{\sqrt{k_x}}{\log\frac{1}{pb(m_{n,x}/k_x,x)}}\log\frac{Q_{m_{n,x}}(\lambda,x)}{\mathrm{U}(b(m_{n,x}/k_x,x),x)}\\
&-\frac{\sqrt{k_x}}{\log\frac{1}{pb(m_{n,x}/k_x,x)}}\left\{\log\frac{\mathrm{U}(\frac{1}{p},x)}{\mathrm{U}(b(m_{n,x}/k_x,x),x)}-\gamma(x)\log\frac{1}{pb(m_{n,x}/k_x,x)}\right\}\\
&-\frac{(1-\xi(x))(1-2\xi(x))}{\xi(x)^2}\sqrt{k_x}[\hat{\gamma}_{k_x}(K_1,x)-\hat{\gamma}_{k_x}(K_{2,\xi},x)]\frac{\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\xi(x)}-1}{\xi(x)}\\
=\;&\sqrt{k_x}\left(\hat{\gamma}_{k_x}(K_{\hat{S}^*},x)-\gamma(x)\right)+\frac{\sqrt{k_x}}{\log\frac{1}{pb(m_{n,x}/k_x,x)}}\log\frac{Q_{m_{n,x}}(\lambda,x)}{\mathrm{U}(b(m_{n,x}/k_x,x),x)}\\
&-\frac{\sqrt{k_x}}{\log\frac{1}{pb(m_{n,x}/k_x,x)}}\tilde{\mathcal{A}}(b(m_{n,x}/k_x,x),x)\frac{\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\rho(x)}-1}{\rho(x)}\\
&-\frac{\sqrt{k_x}}{\log\frac{1}{pb(m_{n,x}/k_x,x)}}\tilde{\mathcal{A}}(b(m_{n,x}/k_x,x),x)\\
&\times\left\{\frac{\log\mathrm{U}(\frac{1}{p},x)-\log\mathrm{U}(b(m_{n,x}/k_x,x),x)-\gamma(x)\log\frac{1}{pb(m_{n,x}/k_x,x)}}{\tilde{\mathcal{A}}(b(m_{n,x}/k_x,x),x)}\right.\\
&\left.-\frac{\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\rho(x)}-1}{\rho(x)}\right\}\\
&-\frac{(1-\xi(x))(1-2\xi(x))}{\xi(x)^2}\sqrt{k_x}[\hat{\gamma}_{k_x}(K_1,x)-\hat{\gamma}_{k_x}(K_{2,\xi(x)},x)]\frac{\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\xi(x)}-1}{\xi(x)}\\
=\;&T_1+T_2-T_3-T_4-T_5
\end{aligned}$$

Now let's look at the 5 terms.
Theorem 2.3 guarantees the asymptotic normality of the term T_1

$$T_1\xrightarrow{d}\mathcal{N}\left(0,\ c^2\mathcal{AV}(K_{S^*},x)\right)$$

The corresponding Proposition 6.1 in Tchazino et al. [2021] guarantees that

$$T_2\xrightarrow{\mathbb{P}}0$$

Indeed

$$
\begin{aligned}
&\sup_{t\in[0,1]} t^{1/2+\epsilon}\left|\sqrt{k_x}\log\left(\frac{Q_{m_{n,x}}(t,x)}{\mathrm{U}(b(m_{n,x}/k_x,x),x)}\right)-\gamma(x)W(1)\right| \\
&\quad\le \sup_{t\in[0,1]} t^{\frac{1}{2}+\epsilon}\left|\sqrt{k_x}\left(\log\left(\frac{Q_{m_{n,x}}(t,x)}{\mathrm{U}\left(b(m_{n,x}/k_x,x),x\right)}\right)+\frac{\gamma(x)\log(t)}{\int_0^1 K(s)ds}\right)\right. \\
&\qquad\left. -\gamma(x)t^{-1}W(t)-\sqrt{k_x}\tilde{\mathcal{A}}\left(b(m_{n,x}/k_x,x),x\right)\frac{t^{-\rho(x)}-1}{\rho(x)}\right| \\
&\qquad+\sup_{t\in[0,1]} t^{\frac{1}{2}+\epsilon}\left|\sqrt{k_x}\left(\frac{\gamma(x)\log(t)}{\int_0^1 K(s)ds}\right)\right. \\
&\qquad\left. -\gamma(x)\left(t^{-1}W(t)-W(1)\right)-\sqrt{k_x}\tilde{\mathcal{A}}\left(b(m_{n,x}/k_x,x),x\right)\frac{t^{-\rho(x)}-1}{\rho(x)}\right| \\
&\quad= \mathrm{o}(1).
\end{aligned}
$$

Under the hypothesis of the said theorem, $T_3\to 0$.
To prove that $T_4=\mathrm{o}(1)$, we need the following inequality:
Suppose the relationship R2 (from condition C_A). By applying the function $z\longmapsto \log\mathrm{U}(z)-\gamma\log(z)$ to Theorem B.2.18 in De Haan et al. [2006] we gets:
$\forall\ \epsilon,\ \delta>0\ \exists\ u_0=u_0(\epsilon,\delta)$ such that $\forall\ ux\ge u_0$;

$$
(***)\quad\left|\frac{\log(\mathrm{U}(uy,x)/\mathrm{U}(u,x))-\gamma(x)\log(y)}{\tilde{\mathcal{A}}(b(u,x),x)}-\log(y)\frac{y^{\rho(x)}-1}{\rho(x)}\right|\le \epsilon y^{\rho(x)}\max(y^{\delta},y^{-\delta}).
$$

Then this last inequality leads to:

$$
\begin{aligned}
|T_4| &\le \frac{\sqrt{k_x}}{\log\frac{1}{pb(m_{n,x}/k_x,x)}}|\tilde{\mathcal{A}}(b(m_{n,x}/k_x,x))| \\
&\quad\times\left|\frac{\log\ \mathrm{U}(\frac{1}{p})-\log\ \mathrm{U}(b(m_{n,x}/k_x,x))-\gamma(x)\log\frac{1}{pb(m_{n,x}/k_x,x)}}{\tilde{\mathcal{A}}(b(m_{n,x}/k_x,x),x)}\right. \\
&\quad\left.-\frac{\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\rho(x)}-1}{\rho(x)}\right| \\
&\le \frac{\sqrt{k_x}}{\log\frac{1}{pb(m_{n,x}/k_x,x)}}|\tilde{\mathcal{A}}(b(m_{n,x}/k_x,x),x)|\epsilon\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\rho(x)+\delta} \\
&= \mathrm{o}(1)
\end{aligned}
$$

for all $0<\delta<-\rho(x)$.
Note that the term T_5 is function of $\xi(x)$ which can be a canonical value or an estimator consisting of the probability of $\rho(x)$.

- $\xi(x)=\tilde{\rho}(x)$;
  we have :

$$
\begin{aligned}
\sqrt{k_x}[\hat{\gamma}_{k_x}(K_1,x)-\hat{\gamma}_{k_x}(K_{2,\xi},x)] &= \sqrt{k_x}[\hat{\gamma}_{k_x}(K_1,x)-\gamma(x)]-\sqrt{k_x}[\hat{\gamma}_{k_x}(K_{2,\xi},x)-\gamma(x)] \\
&= \mathrm{O}_{\mathbb{P}}(1)
\end{aligned}
$$

according to Corollary 2.1. This leads to $T_5 = \mathrm{o}_{\mathbb{P}}(1)$.

- $\xi(x) = \hat{\rho}(x)$;

$$\begin{aligned}
T_5 &= \frac{(1-\hat{\rho}(x))(1-2\hat{\rho}(x))}{\hat{\rho}^2(x)}\sqrt{k_x}[\hat{\gamma}_{k_x}(K_1,x) - \hat{\gamma}_{k_x}(K_{2,\hat{\rho}(x)},x)] \\
&\times \frac{\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\rho(x)} - 1}{\rho(x)} \\
&+ \frac{(1-\hat{\rho}(x))(1-2\hat{\rho}(x))}{\hat{\rho}^2(x)}\sqrt{k_x}[\hat{\gamma}_{k_x}(K_1,x) - \hat{\gamma}_{k_x}(K_{2,\hat{\rho}(x)},x)] \\
&\times \left(\frac{\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\hat{\rho}(x)} - 1}{\hat{\rho}(x)} - \frac{\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\rho(x)} - 1}{\rho(x)} \right),
\end{aligned}$$

now, based on Corollary 2.1 and Theorem 2.3

$$\begin{aligned}
\sqrt{k_x}[\hat{\gamma}_{k_x}(K_1,x) - \hat{\gamma}_{k_x}(K_{2,\hat{\rho}(x)},x)] &= \sqrt{k_x}[\hat{\gamma}_{k_x}(K_1,x) - \gamma(x)] - \sqrt{k_x}[\hat{\gamma}_{k_x}(K_{2,\rho(x)},x) - \gamma(x)] \\
&\quad -\sqrt{k_x}[\hat{\gamma}_{k_x}(K_{2,\hat{\rho}(x)},x) - \hat{\gamma}_{k_x}(K_{2,\rho(x)},x)] \\
&= \mathrm{O}_{\mathbb{P}}(1),
\end{aligned}$$

then T_5 gives

$$\begin{aligned}
T_5 &= \mathrm{o}_{\mathbb{P}}(1) + \mathrm{o}_{\mathbb{P}}(1) \left\{ \frac{\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\hat{\rho}(x)} - 1}{\hat{\rho}(x)} - \frac{\left(\frac{1}{pb(m_{n,x}/k_x,x)}\right)^{\rho(x)} - 1}{\rho(x)} \right\} \\
&= \mathrm{o}_{\mathbb{P}}(1) + \mathrm{o}_{\mathbb{P}}(1) \int_0^{\frac{1}{pb(m_{n,x}/k_x,x)}} s^{\rho(x)-1}(s^{\hat{\rho}(x)-\rho(x)} - 1)ds.
\end{aligned}$$

Inspired by Chavez-Demoulin and Guillou [2018] we get
$\int_0^{\frac{1}{pb(m_{n,x}/k_x,x)}} s^{\rho(x)-1}(s^{\hat{\rho}(x)-\rho(x)} - 1)ds = \mathrm{o}_{\mathbb{P}}(1)$.

Which leads to the conclusion that $T_5 = \mathrm{o}_{\mathbb{P}}(1)$ and thus we get proof of Theorem 3.1.

6 Appendix

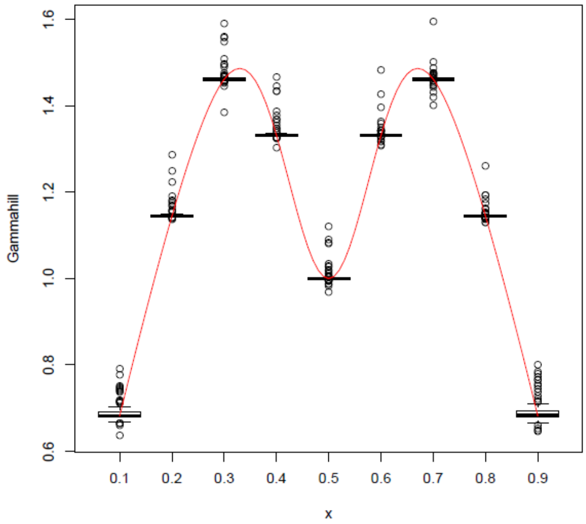

Figure 6.1 *The boxplot of* $\hat{\gamma}_H(x)$ *(Hill's estimator) and the true value of* $\gamma(x)$ *(red line) for* $\rho = -3.67$ *and* $n = 10000$.

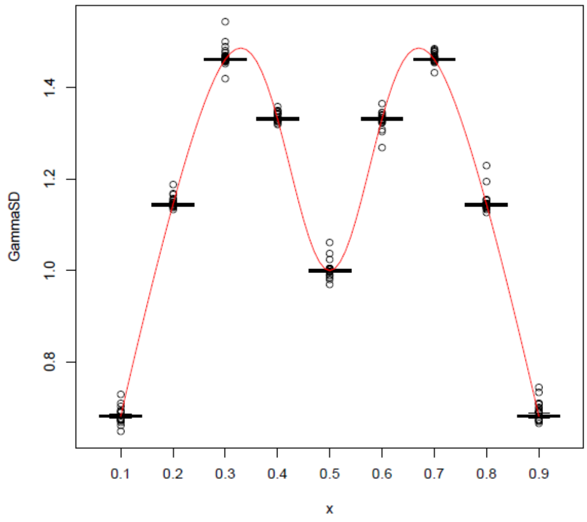

Figure 6.2 *The boxplot of* $\hat{\gamma}_{k_x}(K_{\Delta S}, x)$ *and the true value of* $\gamma(x)$ *(red line) for* $\rho = -3.67$ *and* $n = 10000$.

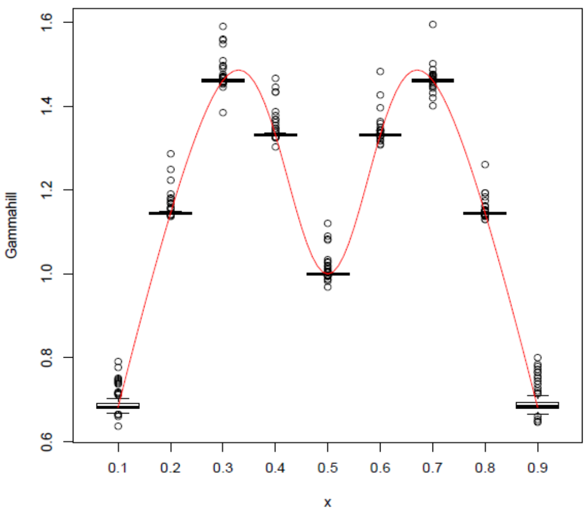

Figure 6.3 *The boxplot of* $\hat{\gamma}_H(x)$ *and the true value of* $\gamma(x)$ *(red line) for* $\rho = -10$ *and* $n = 10000$.

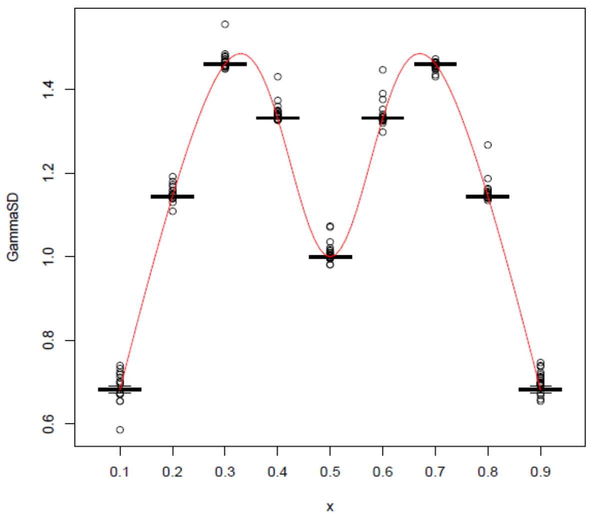

Figure 6.4 *The boxplot of* $\hat{\gamma}_{k_x}(K_{\Delta S}, x)$ *and the true value of* $\gamma(x)$ *(red line) for* $\rho = -10$ *and* $n = 10000$.

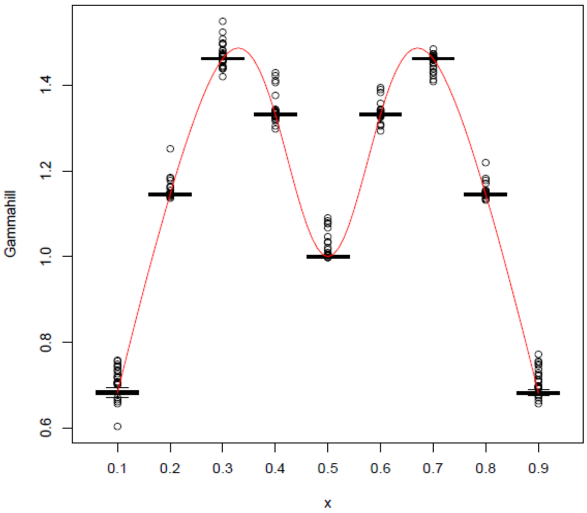

Figure 6.5 *The boxplot of $\hat{\gamma}_H(x)$ and the true value of $\gamma(x)$ (red line) for $\rho = -3.67$ and $n = 40000$.*

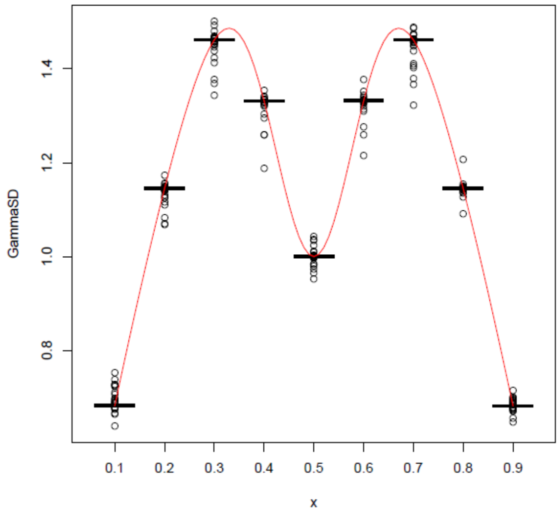

Figure 6.6 *The boxplot of $\hat{\gamma}_{k_x}(K_{\Delta S}, x)$ and the true value of $\gamma(x)$ (red line) for $\rho = -3.67$ and $n = 40000$.*

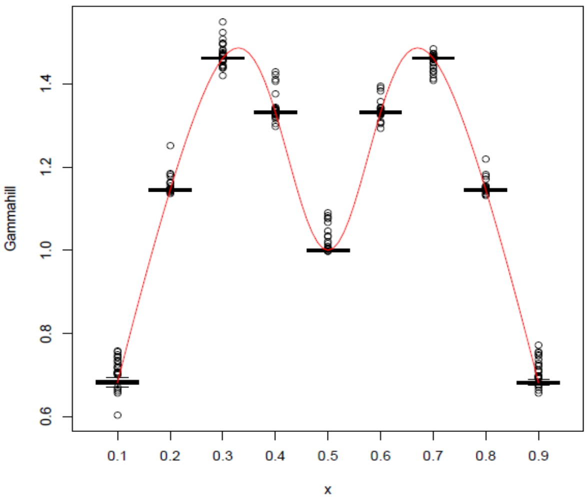

Figure 6.7 *The boxplot of $\hat{\gamma}_H(x)$ and the true value of $\gamma(x)$ (red line) for $\rho = -10$ and $n = 40000$.*

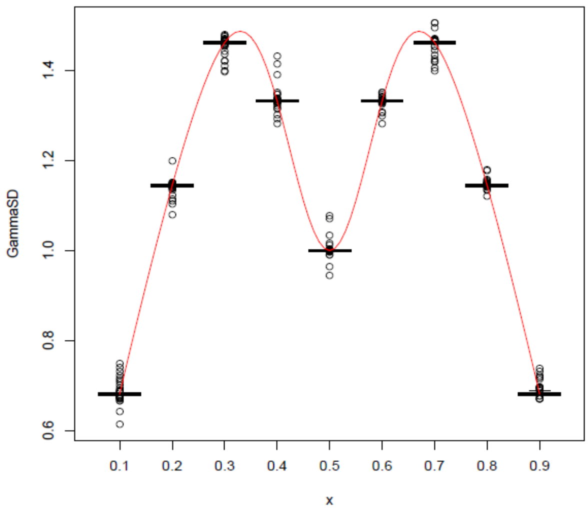

Figure 6.8 *The boxplot of $\hat{\gamma}_{k_x}(K_{\Delta S}, x)$ and the true value of $\gamma(x)$ (red line) for $\rho = -10$ and $n = 40000$.*

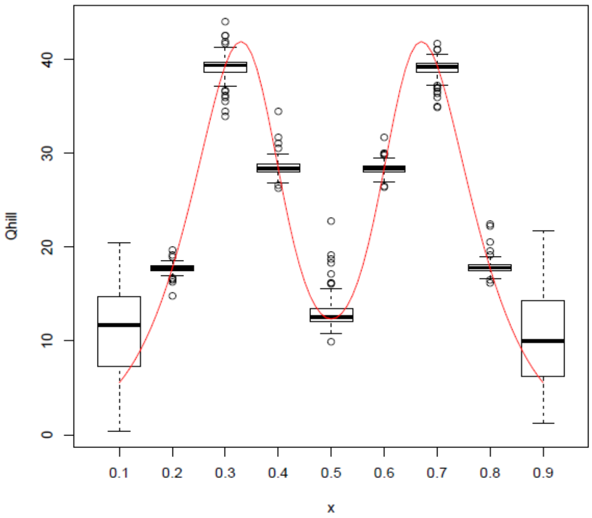

Figure 6.9 *The boxplot of $q(1/1000, x)$ Hill and the true value of $q(p, x)$ (red line) for $\rho = -3.67$ and $n = 10000$.*

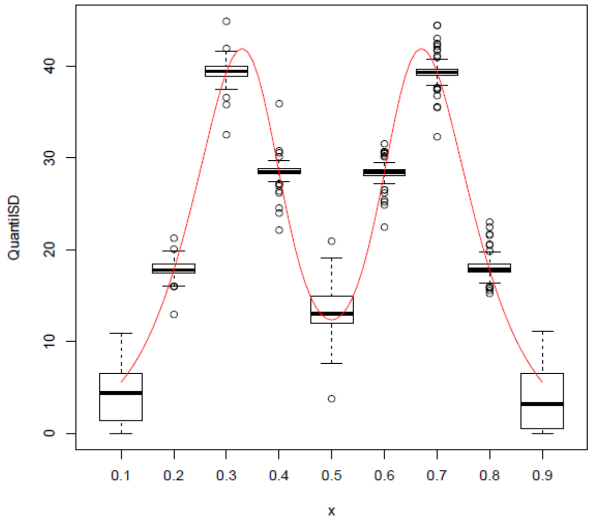

Figure 6.10 *The boxplot of $q_{\Delta S}(1/1000, x)$ and the true value of $q(p, x)$ (red line) for $\rho = -3.67$ and $n = 10000$.*

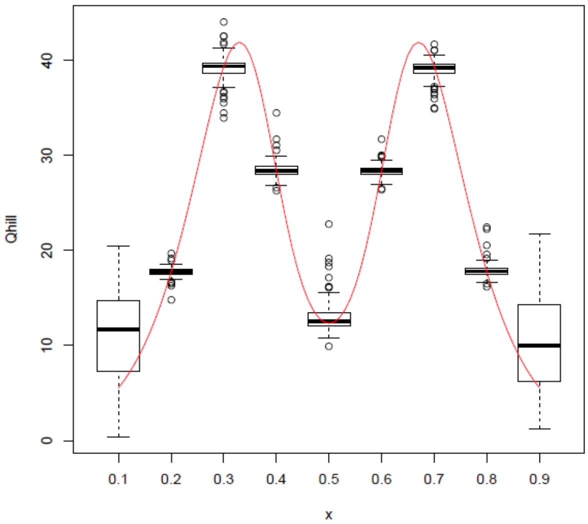

Figure 6.11 *The boxplot of $q(1/1000, x)$ Hill and the true value of $q(p, x)$ (red line) for $\rho = -10$ and $n = 10000$.*

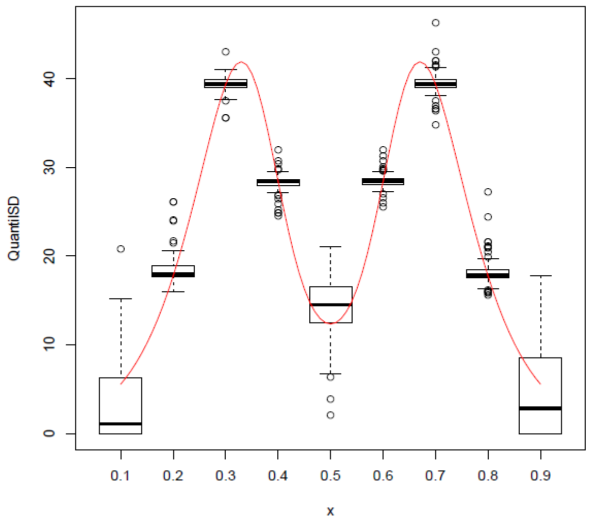

Figure 6.12 *The boxplot of $q_{\Delta S}(1/1000, x)$ and the true value of $q(p, x)$ (red line) for $\rho = -10$ and $n = 10000$.*

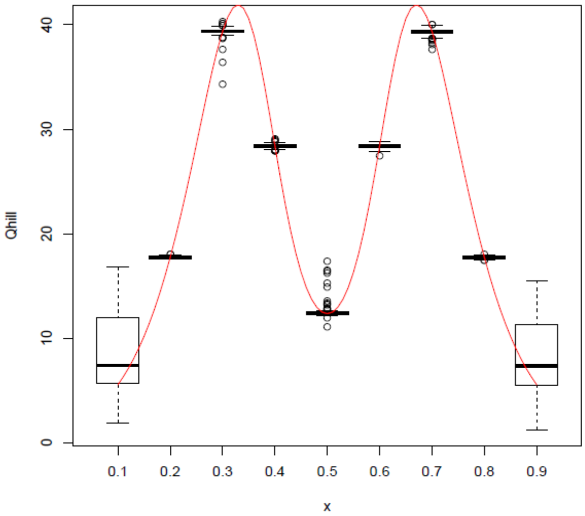

Figure 6.13 *The boxplot of $q(1/1000, x)$ Hill and the true value of $q(p, x)$ (red line) for $\rho = -3.67$ and $n = 40000$.*

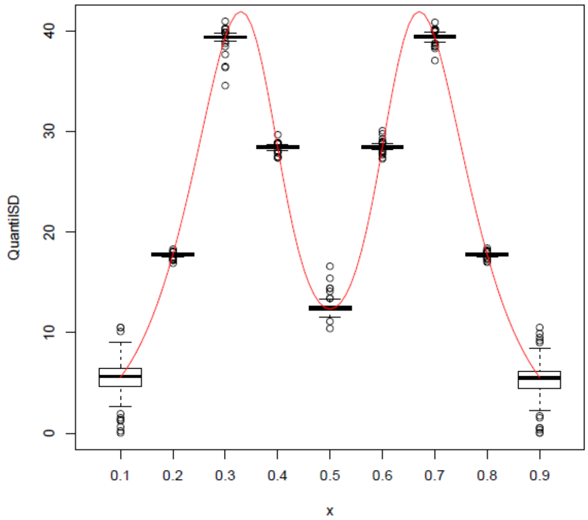

Figure 6.14 *The boxplot of $q_{\Delta S}(1/1000, x)$ and the true value of $q(p, x)$ (red line) for $\rho = -3.67$ and $n = 40000$.*

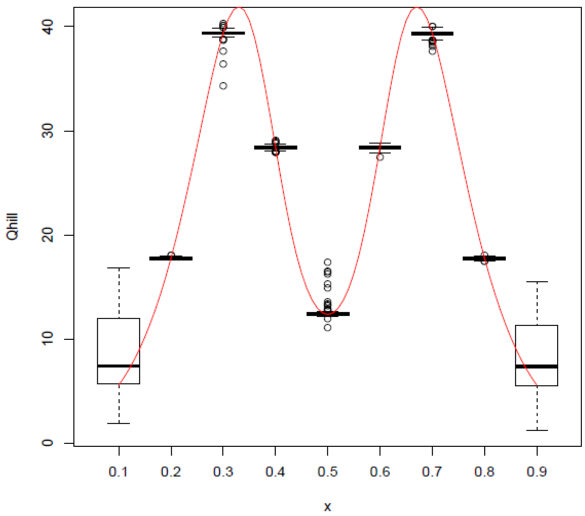

Figure 6.15 *The boxplot of $q(1/1000, x)$ (Hill) and the true value of $q(p, x)$ (red line) for $\rho = -10$ and $n = 40000$.*

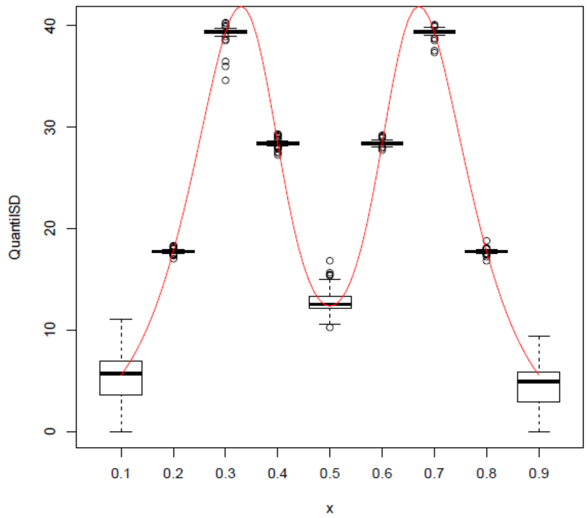

Figure 6.16 *The boxplot of $q_{\Delta S}(1/1000, x)$ and the true value of $q(p, x)$ (red line) for $\rho = -10$ and $n = 40000$.*

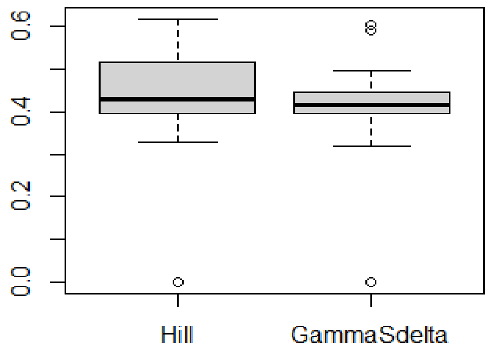

Figure 6.17 *The boxplot of* $\gamma(x = \bar{X})$ *for* $\rho = -3.67$.

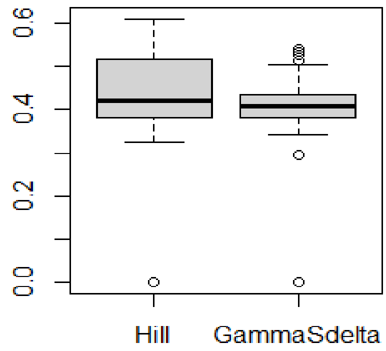

Figure 6.18 *The boxplot of* $\gamma(x = median(X))$ *for* $\rho = -3.67$.

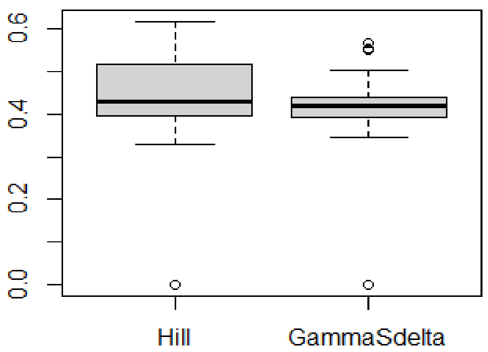

Figure 6.19 *The boxplot of* $\gamma(x = \bar{X})$ *for* $\rho = -5$.

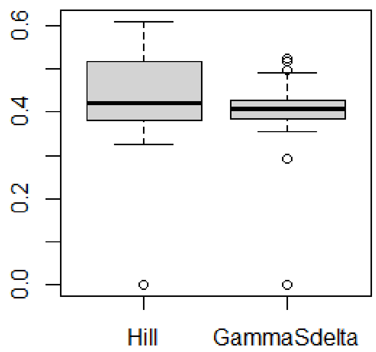

Figure 6.20 *The boxplot of* $\gamma(x = median(X))$ *for* $\rho = -5$.

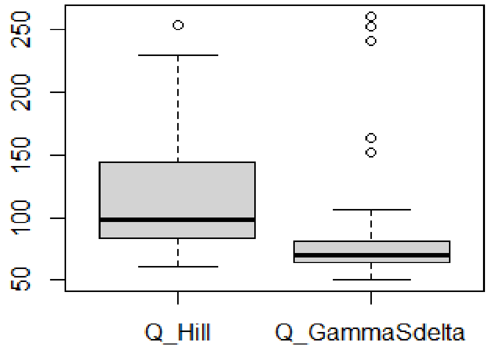

Figure 6.21 *The boxplot of* $q(1/1000, x = \bar{X})$ *for* $\rho = -3.67$.

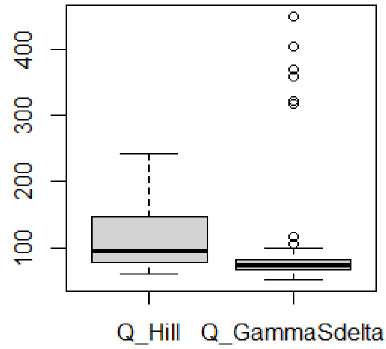

Figure 6.22 *The boxplot of* $q(1/1000, x = median(X))$ *for* $\rho = -3.67$.

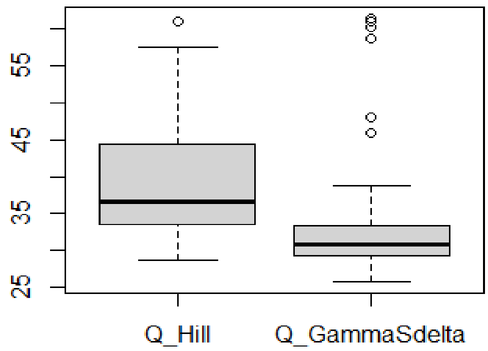

Figure 6.23 *The boxplot of* $q(1/100, x = \bar{X})$ *for* $\rho = -3.67$.

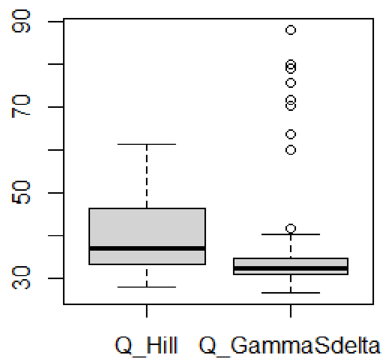

Figure 6.24 *The boxplot of* $q(1/100, x = median(X))$ *for* $\rho = -3.67$.

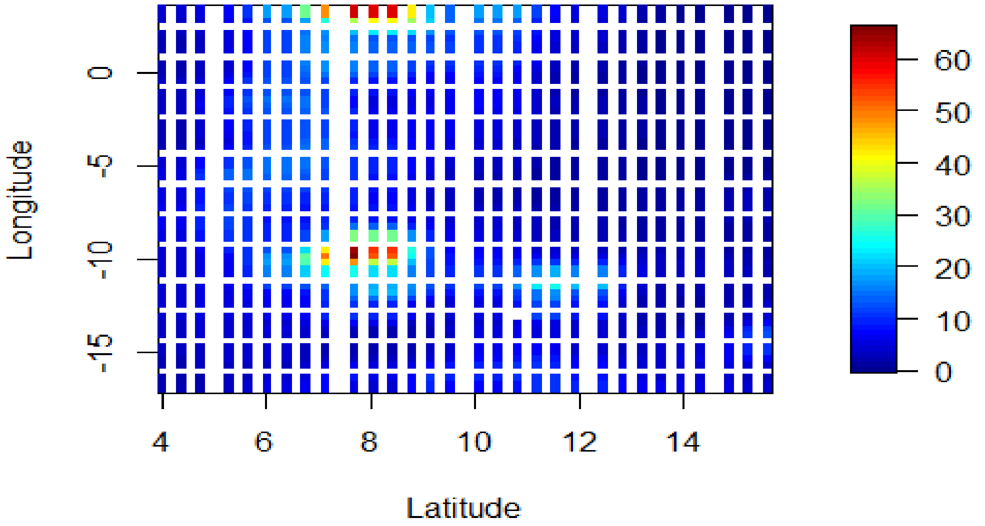

Figure 6.25 *Spatial representation of data.*

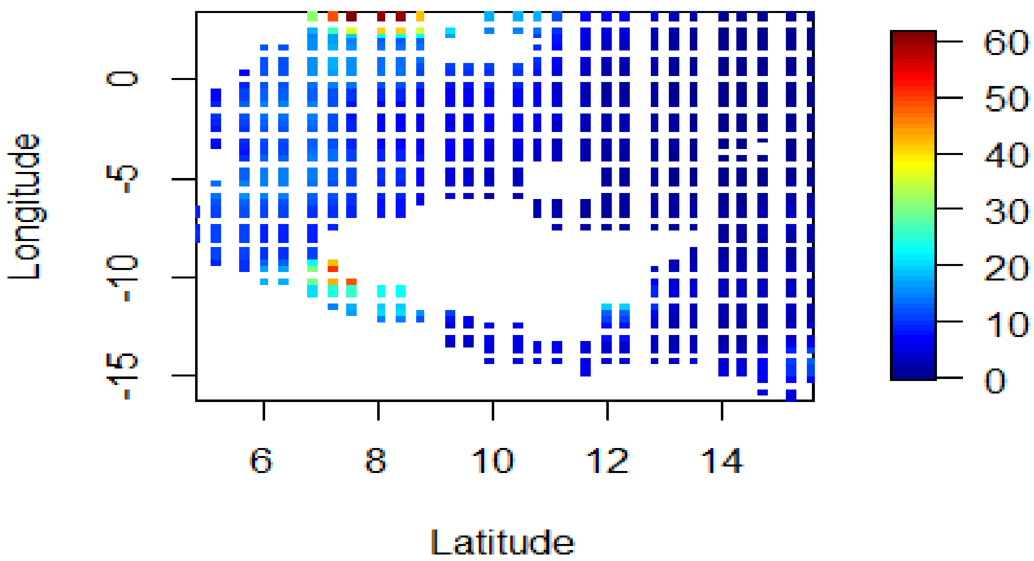

Figure 6.26 *Spatial representation of data with cavariate.*

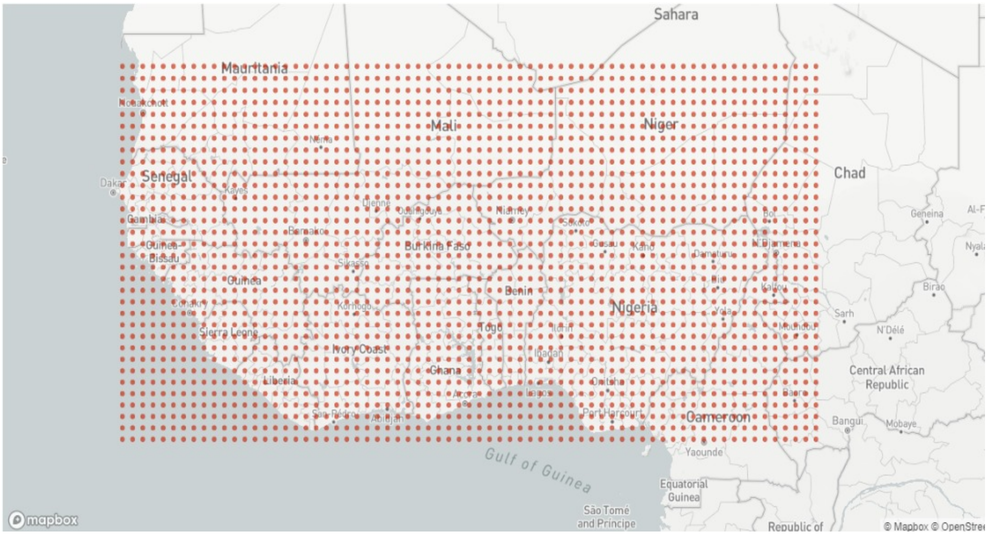

Figure 6.27 *Geographical representation of data.*

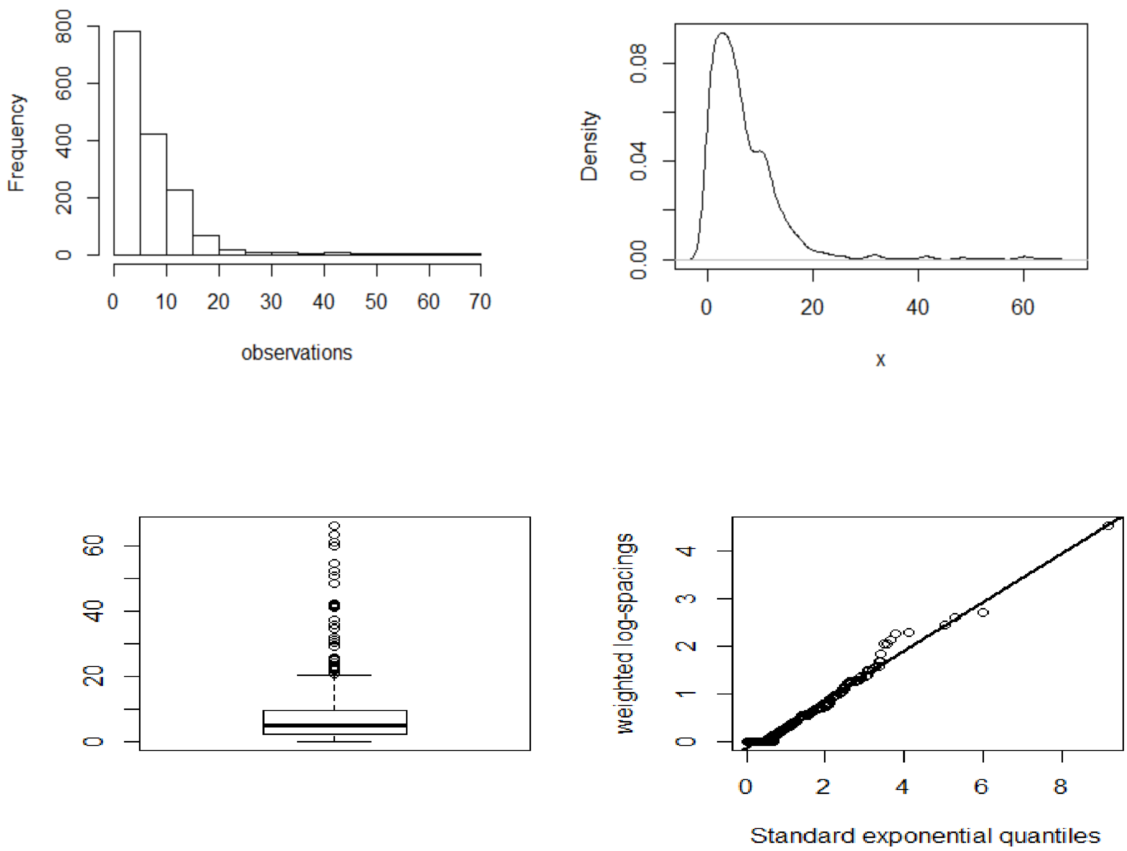

Figure 6.28 *Top row, left panel: histogram of the data. Top row, right panel: density. Bottom row, left panel: boxplot of the data. Bottom row, right panel: quantile-quantile plot of weighted log-spacings $Z_{i,n}$ for $1 \leq i \leq \lfloor n/5 \rfloor$ versus the standard exponential quantiles.*

	n=10 000		n=40 000	
value of x	$\gamma_{\mathrm{H}}(x)$	$\gamma_{K_{S\Delta}}(x)$	$\gamma_{\mathrm{H}}(x)$	$\gamma_{K_{S\Delta}}(x)$
		$\rho = -3.67$		
0.1	0.688 36 [0.0215] (0.0206)	0.685 72 [0.0144] (0.0140)	0.690 75 [0.0249] (0.0236)	0.682 129 [0.0078] (0.0079)
0.2	1.147 124 [0.0129] (0.0127)	1.142 221 [0.0141] (0.0138)	1.150 224 [0.0211] (0.0204)	1.146 291 [0.0061] (0.0060)
0.3	1.467 124 [0.0236] (0.0232)	1.458 198 [0.0196] (0.0194)	1.464 215 [0.0151] (0.0150)	1.463 213 [0.0110] (0.0109)
0.4	1.339 118 [0.0251] (0.0242)	1.333 200 [0.0186] (0.0183)	1.335 246 [0.0185] (0.0183)	1.329 250 [0.0049] (0.0048)
0.5	1.0057 102 [0.0204] (0.0200)	1.0006 169 [0.0101] (0.0102)	1.0043 209 [0.0190] (0.0182)	0.9999 267 [0.0086] (0.0087)
0.6	1.3357 139 [0.0200] (0.0197)	1.3297 228 [0.0160] (0.0159)	1.3330 233 [0.0119] (0.0119)	1.3311 269 [0.0084] (0.0085)
0.7	1.4597 141 [0.0168] (0.0169)	1.4570 226 [0.0218] (0.0214)	1.4625 247 [0.0105] (0.0169)	1.4626 277 [0.0057] (0.0056)
0.8	1.1475 123 [0.0149] (0.0147)	1.1456 218 [0.0102] (0.0103)	1.1463 219 [0.0096] (0.0095)	1.1445 277 [0.0086] (0.0087)
0.9	0.6931 45 [0.0298] (0.0277)	0.6843 82 [0.0106] (0.0103)	0.6883 82 [0.02030] (0.0192)	0.6821 120 [0.0071] (0.0071)

$\rho = -10$				
0.1	0.6899	0.6834	0.6881	0.6832
	36	56	75	92
	[0.0215]	[0.0150]	[0.0249]	[0.0152]
	(0.0206)	(0.0150)	(0.0236)	(0.0151)
0.2	1.147	1.1456	1.150	1.1431
	124	172	224	261
	[0.0129]	[0.0110]	[0.0211]	[0.0084]
	(0.0127)	(0.0109)	(0.0204)	(0.0083)
0.3	1.467	1.4593	1.464	1.4630
	124	198	215	213
	[0.0236]	[0.0129]	[0.0151]	[0.0104]
	(0.0232)	(0.0127)	(0.0150)	(0.0103)
0.4	1.339	1.3341	1.335	1.3331
	118	165	246	236
	[0.0251]	[0.0163]	[0.0185]	[0.0116]
	(0.0242)	(0.0164)	(0.0183)	(0.0114)
0.5	1.0057	1.0023	1.0043	1.0008
	102	129	209	249
	[0.0204]	[0.0131]	[0.0190]	[0.0117]
	(0.0200)	(0.0131)	(0.0182)	(0.0115)
0.6	1.3357	1.3337	1.3330	1.3311
	139	228	233	269
	[0.0200]	[0.0144]	[0.0119]	[0.0078]
	(0.0197)	(0.0143)	(0.0119)	(0.0079)
0.7	1.4597	1.4603	1.4625	1.4607
	141	188	247	279
	[0.0168]	[0.0136]	[0.0105]	[0.0052]
	(0.0169)	(0.0136)	(0.0169)	(0.0051)
0.8	1.1475	1.1467	1.1463	1.1454
	123	154	219	244
	[0.0149]	[0.0135]	[0.0096]	[0.0060]
	(0.0147)	(0.0134)	(0.0095)	(0.0059)
0.9	0.6931	0.6859	0.6883	0.6851
	45	56	82	91
	[0.0298]	[0.0143]	[0.02030]	[0.0115]
	(0.0277)	(0.0138)	(0.0192)	(0.0110)

** k: number of extremes values
[MSE]: Mean Square Error
(sd): standard deviation

Table 6.1 *Gamma estimators*

	n=10 000		n=40 000	
value of x	$\hat{q}_{\mathrm{Hill}}(x)$	$\hat{q}_{K_{S\Delta}}(x)$	$\hat{q}_{\mathrm{Hill}}(x)$	$\hat{q}_{K_{S\Delta}}(x)$
$\mathbb{P}(Y_i \geq q/x) = 10^{-3}$, $\rho = -3.67$				
0.1	10.893 [7.129] (4.739)	4.352 [3.212] (2.997)	8.691 [4.847] (3.706)	5.500 [1.892] (1.901)
0.2	17.738 [0.611] (0.614)	17.772 [0.562] (0.544)	17.743 [0.094] (0.094)	17.761 [0.019] (0.019)
0.3	39.261 [1.421] (1.422)	39.478 [1.357] (1.361)	39.306 [0.667] (0.666)	39.287 [0.732] (0.730)
0.4	28.561 [1.013] (1.008)	28.415 [1.010] (1.012)	28.399 [0.191] (0.192)	28.390 [0.297] (0.298)
0.5	13.056 [2.032] (1.912)	13.290 [2.077] (1.623)	12.626 [0.979] (0.941)	12.502 [0.769] (0.756)
0.6	28.387 [0.754] (0.757)	28.352 [1.014] (1.018)	28.385 [0.215] (0.214)	28.438 [0.134] (0.134)
0.7	39.036 [1.178] (1.132)	39.394 [1.515] (1.523)	39.299 [0.389] (0.382)	39.375 [0.242] (0.244)
0.8	17.894 [0.891] (0.883)	18.035 [1.271] (1.245)	17.740 [0.102] (0.103)	17.728 [0.175] (0.175)
0.9	10.371 [6.787] (4.796)	3.835 [3.818] (3.432)	8.430 [4.430] (3.379)	5.293 [1.972] (1.966)

$\mathbb{P}(Y_i \geq q/x) = 10^{-3}$, $\rho = -10$				
0.1	10.893	4.699	8.691	5.758
	[7.129]	[4.696]	[4.847]	[1.964]
	(4.739)	(4.643)	(3.706)	(1.962)
0.2	17.738	18.489	17.743	17.720
	[0.611]	[1.861]	[0.094]	[0.176]
	(0.614)	(1.717)	(0.094)	(0.175)
0.3	39.261	39.405	39.306	39.274
	[1.421]	[0.960]	[0.667]	[0.704]
	(1.422)	(0.964)	(0.666)	(0.699)
0.4	28.561	28.306	28.399	28.402
	[1.013]	[1.054]	[0.191]	[0.297]
	(1.008)	(1.054)	(0.192)	(0.298)
0.5	13.056	13.322	12.626	12.818
	[2.032]	[2.959]	[0.979]	[0.218]
	(1.912)	(2.805)	(0.941)	(0.260)
0.6	28.387	28.460	28.385	28.417
	[0.754]	[0.853]	[0.215]	[0.208]
	(0.757)	(0.856)	(0.214)	(0.209)
0.7	39.036	39.542	39.299	39.359
	[1.178]	[1.351]	[0.389]	[0.358]
	(1.132)	(1.348)	(0.382)	(0.360)
0.8	17.894	18.206	17.740	17.736
	[0.891]	[1.673]	[0.102]	[0.195]
	(0.883)	(1.618)	(0.103)	(0.196)
0.9	10.371	6.235	8.430	5.723
	[6.787]	[4.577]	[4.430]	[1.856]
	(4.796)	(4.548)	(3.379)	(1.857)

** [MSE]: Mean Square Error.
(sd): standard deviation

Table 6.2 *Quantiles predictions*

value of x	$\hat{\gamma}_{\text{Hill}}(x)$ $(k_{m_{n,x}})$	$\hat{\gamma}_{k_n}(K_{S\Delta}, x)$ $(k_{m_{n,x}})$	$q(1/1000, x)$		$q(1/100, x)$	
			Hill	$\hat{q}_{K_{S\Delta}}$	Hill	$\hat{q}_{K_{S\Delta}}$
$\rho = -3.67$						
100.72	0.5374 (366) [0.096]	0.3392 (208) [0.075]	178.85 [76.82]	100.33 [101.50]	48.02 [12.26]	33.24 [17.12]
98.91	0.4003 (360) [0.0583]	0.4097 (282) [0.0524]	88.34 [16.89]	72.96 [11.64]	34.73 [4.33]	31.60 [4.18]
97.10	0.4381 (323) [0.086]	0.4430 (269) [0.061]	133.20 [66.83]	99.35 [16.82]	45.73 [10.22]	39.88 [4.43]

[sd]: standard deviation

Table 6.3 *Gamma estimation and quantile prediction on real data.*

References

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