Internal layer intersecting the boundary of a domain in a singular advection-diffusion equation

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Abstract

We perform an asymptotic analysis with respect to the parameter $\varepsilon > 0$ of the solution of the scalar advection-diffusion equation $y_t^\varepsilon + M(x,t)y_x^\varepsilon - \varepsilon y_{xx}^\varepsilon = 0$, $(x,t) \in (0,1)\times(0,T)$, supplemented with Dirichlet boundary conditions. For small values of ε, the solution y^ε exhibits a boundary layer of size $\mathcal{O}(\varepsilon)$ in the neighborhood of $x = 1$ (assuming $M > 0$) and an internal layer of size $\mathcal{O}(\varepsilon^{1/2})$ in the neighborhood of the characteristic starting from the point $(0,0)$. Assuming that these layers interact each other after a finite time $T > 0$ and using the method of matched asymptotic expansions, we construct an explicit approximation P^ε satisfying $\|y^\varepsilon - P^\varepsilon\|_{L^\infty(0,T;L^2(0,1))} = \mathcal{O}(\varepsilon^{1/2})$. We emphasize the additional difficulties with respect to the case M constant considered recently by the authors.

Key words: Asymptotic analysis, Singular perturbation, Internal and boundary layers, Sobolev estimates.
AMS subject classification (2020): 35C20, 35K67.

1 Introduction. Problem statement

Let $T > 0$ and $Q_T := (0,1)\times(0,T)$. This work is concerned with the scalar advection-diffusion equation

$$\begin{cases} y_t^\varepsilon(x,t) + M(x,t)\, y_x^\varepsilon(x,t) - \varepsilon y_{xx}^\varepsilon(x,t) = 0, & (x,t) \in Q_T, \\ y^\varepsilon(0,t) = v(t), \quad y^\varepsilon(1,t) = 0, & t \in (0,T), \\ y^\varepsilon(x,0) = y_0(x), & x \in (0,1), \end{cases} \tag{1}$$

where $\varepsilon \in (0,1)$ is the diffusion coefficient and $M(x,t) > 0$ is the transport velocity. For any initial data $y_0 \in H^{-1}(0,1)$ and Dirichlet condition $v \in L^2(0,T)$, there exists a unique solution $y^\varepsilon \in L^2(Q_T) \cap \mathcal{C}([0,T]; H^{-1}(0,1))$.

This apparently simple partial differential equation appears in many situation as it is a prototype of models where the diffusion coefficient is small compared to the others. As explained in [8], this model can notably be seen as an embedded system of the Navier-Stokes system with non-characteristic boundary condition and viscosity coefficient equals to ε. It also can be seen as a regularization of a transport equation and for this reasons well employed in numerical analysis (see notably [10, 25] and the references therein) when one wants to obtain robust numerical approximation uniformly with respect to ε small. Equation (1) appears in models of miscible displacement of compressible fluids in porous media, with small molecular diffusion and dispersion coefficients, see [6]. Last, it also appears in the context of exact boundary controllability when one wants to steer to zero the solution y^ε with a uniform control v: we mention the seminal works [9, 14] and the recent paper [18].

We are interested in this work with a precise asymptotic description of the solution y^ε when ε is small. This problem has been the subject of several studies in the last decades in the case for which the transport

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velocity is constant and the equation is defined over $\mathbb{R}_+ \times (0,T)$. We refer to [23, 22]. The case of a transport velocity depending only on the time variable has been formally discussed in [19] and deeper analyzed in [24].

For bounded domains with respect to the space variable, the asymptotic analysis is quite involved as several singular layers may appear in Q_T and interact each other. The constant transport velocity case has been analyzed in [3]. More precisely, if we denote the characteristic $t \mapsto X(t;x,s)$ through $(x,s) \in \overline{Q_T}$ as the solution in $\overline{Q_T}$ of

$$\frac{dX}{dt} = M(X,t), \quad X(s;x,s) = x, \tag{2}$$

then the violation of the compatibility conditions between y_0 and v at the point $(0,0)$ for which $y_0(0) \neq v(0)$ induces a thin inner region (called internal layer) of size $\mathcal{O}(\varepsilon^{1/2})$ in the vicinity of the characteristic $\{(x,t) \in Q_T, x - X(t;0,0) = 0\}$ where the solution y^ε exhibits rapid variations. Thus, if this characteristic gets arbitrarily close to the line $x = 1$ in a finite time $T_1 > 0$ unique solution of the equation

$$X(T_1;0,0) = 1, \tag{3}$$

then the internal layer interacts with the usual boundary layer of size $\mathcal{O}(\varepsilon)$ living along $x = 1$ and induced by the Dirichlet condition. Figure 1 provides a geometric description of this phenomenon in two cases: the case in Figure 1-Left for which the function M is constant, i.e. $M(x,t) = M > 0$ leading to a linear characteristic of equation $x - Mt = 0$. The internal layer lives in the red zone $\{(x,t) \in Q_T; |x - Mt| \leq \varepsilon^{1/2}\}$ and intersects the boundary layer occupying the blue zone $\{(x,t) \in Q_T, |x - 1| \leq \varepsilon\}$ in a small neighborhood of the point $(1, 1/M)$ such that $T_1 = 1/M$. This case has been extensively studied in [3] (see also [1, 2]). Figure 1-Right corresponds to more general situation for which the function M is not constant. The characteristic starting from the point $(0,0)$ has the equation $x - X(t,0,0) = 0$.

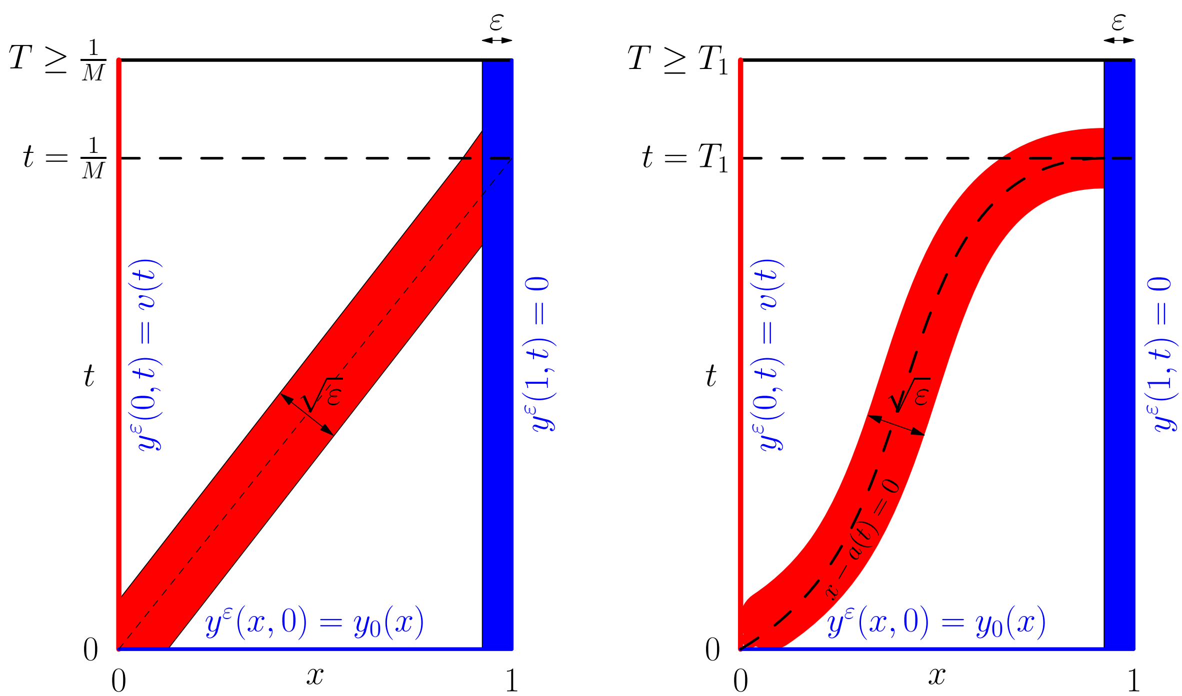

Figure 1: Internal (red) and boundary (blue) layer zones for y^ε for a constant velocity $M(x,t) = M > 0$ (left) and non constant velocity $M(x,t) > 0$ (right).

In this paper, we extend the analysis given in [3] devoted to a constant velocity $M(x,t) = M > 0$ and perform an asymptotic analysis of the solution y^ε of (1). Precisely, our main result reads as follows.

THEOREM 1.1. *Assume $y_0 \in \mathcal{C}^2([0,1])$, $v \in \mathcal{C}^2([0,T])$ and $M \in \mathcal{C}^2(\overline{Q_T}, \mathbb{R}_+^\star)$. For $\varepsilon > 0$ small enough, let y^ε be the solution of (1). Then, one may construct a function P^ε satisfying*

$$\|y^\varepsilon(\cdot,t) - P^\varepsilon(\cdot,t)\|_{L^2(0,1)} \leq c\,\varepsilon^{1/2}, \quad \forall t \in [0,T],$$

for some positive constant $c >$ independent of ε.

Compared to [3] where a rate of order $\mathcal{O}(\varepsilon^{3/2})$ is obtained for the constant case, we thus get a rate of order $\mathcal{O}(\varepsilon^{1/2})$. As we shall see, this is due to the fact that the non constant case under consideration here involves less explicit computations and developments leading to a more tedious analysis. The outline is as follows. In Section 2, we set up the matched asymptotic method by considering two terms in the expansion. In particular, we employ fundamental solutions for a parabolic equation with unbounded coefficients to construct an approximation of the solution in the internal layers of size $\varepsilon^{1/2}$ (we refer to Lemma 2.2). The method leads to a so-called composite approximations P^ε given by (32). Then, in section 3 by considering the system satisfied by $y^\varepsilon - P^\varepsilon$ and using a Gronwall-Bellman type inequality (see Lemma 3.4), we prove an *a priori* estimate for $y^\varepsilon - P^\varepsilon$ leading to Theorem 1.1. Some computations are collected in the appendice. In Section 4, we consider some explicit simple examples of transport velocity M and evaluate numerically the norm $\|y^\varepsilon - P^\varepsilon\|_{L^\infty(0,T;L^2(0,1))}$ with respect to ε in agreement with our theoretical estimate. We conclude with some remarks in Section 5.

2 Matched asymptotic expansions and approximate solutions

In order to construct an asymptotic approximation of the solution y^ε of (1), we use the method of matched asymptotic expansions ([17, 20, 26, 11, 15]). We also refer to [13] for a recent exposition. The solution y^ε exhibits two inner regions: an internal layer located along the characteristic $\{(x,t) \in Q_T, x - X(t;0,0) = 0\}$ and a boundary layer living along $x = 1$. The internal layer is of size $\mathcal{O}(\varepsilon^{1/2})$ while the boundary layer is of size $\mathcal{O}(\varepsilon)$. The outer region is the subset of $(0,1)$ consisting of the points far from the internal and boundary layers, it is of $\mathcal{O}(1)$ size. The occurrence of these three distinct regions require to introduce three distinct asymptotic expansions. The first one, the so-called outer expansion, lives far away from the inner regions and is given by

$$\sum_{k=0}^{m} \varepsilon^k y^k(x,t), \quad (x,t) \in Q_T, \quad x - X(t;0,0) \neq 0, \quad x < 1,$$

for some $m \in \mathbb{N}^\star$. A second one, the so-called first inner expansion, living in the neighborhood of $\{(x,t) \in Q_T, x - X(t;0,0) = 0\}$ is given by

$$\sum_{k=0}^{m} \varepsilon^{k/2} W^{k/2}(w,t), \quad w := \frac{x - X(t;0,0)}{\varepsilon^{1/2}} \in \left(-\frac{X(t;0,0)}{\varepsilon^{1/2}}, \frac{1 - X(t;0,0)}{\varepsilon^{1/2}}\right), \quad t \in (0,T).$$

Last, a third one, the so-called second inner expansion, living along $x = 1$, is given by

$$\sum_{k=0}^{m} \varepsilon^{k/2} Y^{k/2}(z,\tau,t), \quad z := \frac{1-x}{\varepsilon} \in (0, \varepsilon^{-1}), \quad \tau := \frac{T_1 - t}{\varepsilon^{1/2}},$$

where T_1 is the unique solution of (3). In particular, these expansions make appear several variables, at different scales, namely, x, t, $z = \varepsilon^{-1}(1-x)$, $w = \varepsilon^{-1/2}(x - X(t;0,0))$ and $\tau = \varepsilon^{-1/2}(T_1 - t)$.

We will construct outer and inner expansions which will be valid in the so-called outer and inner regions, respectively. There are intermediate regions between the outer region and the inner regions, with size $\mathcal{O}(\varepsilon^\gamma)$, $\gamma \in (0,1)$. To construct an approximate solution we require that inner and outer expansions coincide in each intermediate region, then some conditions must be satisfied in that region by the corresponding inner and outer expansions. These conditions are the so-called matching asymptotic conditions.

The strategy is as follows. We first identify the functions y^k, $k = 0, \dots, m$, in the outer region. Then, we identify the functions $W^{k/2}$, $k = 0, \dots, m$, of the first inner expansion satisfying the matching conditions (with the y^k). This allows to define an expansion in the form $p^\varepsilon = \sum_{k=0}^m \varepsilon^{k/2} p^{k/2}$, valid far away from $x = 1$, as a linear combination of the functions y^k and $W^{k/2}$. Then, we identify the functions Y^k, $k = 0, \dots, m$, of the second inner expansion satisfying the matching conditions (with the $p^{k/2}$). Eventually, we define an expansion P^ε, valid in the whole domain Q_T, as a linear combination of the functions $p^{k/2}$ and $Y^{k/2}$, and supposed to be an approximation of y^ε. In this work, for simplicity we restrict ourselves to the case $m = 1$ as it will allow to get an approximation with rate $\varepsilon^{1/2}$.

2.1 A property of the characteristics

We assume that M is a positive and smooth function in $\overline{Q_T}$. Let $t \mapsto X(t; x, s)$ be the characteristic through $(x, s) \in \overline{Q_T}$ defined as the solution in $\overline{Q_T}$ of equation (2). Using a classical result of differentiation with respect to a parameter of solutions of a differential equation we get that X is a smooth function of the variables t, x, and s. In particular,

$$\partial_t X_x = M_x(X, t) X_x, \quad X_x(s; x, s) = 1,$$

i.e. X_x satisfies the integral equation

$$X_x(t; x, s) = 1 + \int_s^t M_x(X(\sigma; x, s), \sigma) X_x(\sigma; x, s)\, d\sigma.$$

We set

$$\boxed{a(t) := X(t; 0, 0), \quad t \geq 0.}$$

Let $\gamma_0(x, t)$ denote the unique solution of the equation

$$X(\gamma_0(x, t); x, t) = 0, \;\text{ for } x < a(t). \tag{4}$$

Differentiating equality (4) with respect to x gives

$$X_t(\gamma_0(x, t); x, t)\gamma_{0x}(x, t) + X_x(\gamma_0(x, t); x, t) = 0 \;\text{ for all } (x, t) \text{ with } x < a(t),$$

hence

$$\gamma_{0x}(x, t) = -\frac{X_x(\gamma_0(x, t); x, t)}{M(0, \gamma_0(x, t))}, \quad \text{for } x < a(t), \tag{5}$$

since $X_t(\gamma_0(x, t); x, t) = M(X(\gamma_0(x, t); x, t), \gamma_0(x, t)) = M(0, \gamma_0(x, t))$.

2.2 Outer expansion

Putting $y^0(x, t) + \varepsilon y^1(x, t)$ into equation $(1)_1$, the identification of the powers of ε yields

$$\begin{aligned} \varepsilon^0: &\quad y_t^0 + M y_x^0 = 0, \\ \varepsilon: &\quad y_t^1 + M y_x^1 = y_{xx}^0. \end{aligned} \tag{6}$$

Taking the initial and boundary conditions into account we define y^0 and y^1 as functions satisfying the transport equations, respectively,

$$\begin{cases} y_t^0(x,t) + M(x,t) y_x^0(x,t) = 0, & (x,t) \in Q_T, \\ y^0(0,t) = v(t), & t \in (0,T), \\ y^0(x,0) = y_0(x), & x \in (0,1), \end{cases} \qquad \begin{cases} y_t^1(x,t) + M(x,t) y_x^1(x,t) = y_{xx}^0(x,t), & (x,t) \in Q_T, \\ y^1(0,t) = 0, & t \in (0,T), \\ y^1(x,0) = 0, & x \in (0,1). \end{cases}$$

We find explicit representations of solutions of these equations by using the method of characteristics. We get

$$\boxed{y^0(x,t) = \begin{cases} y_0(X(0;x,t), & x > a(t), \\ v\left(\gamma_0(x,t)\right), & x < a(t), \end{cases}} \tag{7}$$

and

$$y^1(x,t) = \begin{cases} \displaystyle\int_0^t y^0_{xx}(X(s;x,t),s)ds, & x > a(t), \\ \displaystyle\int_{\gamma_0(x,t)}^t y^0_{xx}(X(s;x,t),s)ds, & x < a(t). \end{cases}$$

2.3 Inner expansion along the characteristic $x - a(t) = 0$

We consider the change of variable $w = \frac{x-a(t)}{\varepsilon^{1/2}}$ and function $W^\varepsilon(w,t) = y^\varepsilon(x,t)$. W^ε satisfies the equation

$$W^\varepsilon_t(w,t) + \frac{M(\varepsilon^{1/2}w + a(t),t) - M(a(t),t)}{\varepsilon^{1/2}} W^\varepsilon_w(w,t) - W^\varepsilon_{ww}(w,t) = 0. \tag{8}$$

Using the Taylor expansion

$$M\left(\varepsilon^{1/2}w + a(t),t\right) - M(a(t),t) = \varepsilon^{1/2} w M_x(a(t),t) + \frac{\varepsilon w^2}{2} M_{xx}(a(t),t) + \mathcal{O}(\varepsilon^{3/2}),$$

then, putting $W^0(w,t) + \varepsilon W^{1/2}(w,t)$ into equation (8), the identification of the powers of ε yields

$$\varepsilon^0: \quad W^0_t(w,t) + M_x(a(t),t) w\, W^0_w(w,t) - W^0_{ww}(w,t) = 0,$$

$$\varepsilon^{1/2}: \quad W^{1/2}_t(w,t) + M_x(a(t),t) w\, W^{1/2}_w(w,t) - W^{1/2}_{ww}(w,t) = -\frac{w^2}{2} M_{xx}(a(t),t) W^0_w(w,t).$$

Obviously, the main difference with respect to the case M constant considered in [3] is the occurence of some aditionnal unbounded terms with respect to the variable w.

To get the asymptotic matching conditions we write that, for any fixed t and large w,

$$W^0(w,t) + \varepsilon^{1/2} W^{1/2}(w,t) = y^0(x,t) + \varepsilon y^1(x,t) + \mathcal{O}(\varepsilon^2).$$

Rewriting the right-hand side of the above equality in terms of w, t, and using Taylor expansions we have

$$\begin{aligned} W^0(w,t) + \varepsilon^{1/2} W^{1/2}(w,t) + \varepsilon W^1(w,t) &= y^0\left(\varepsilon^{1/2}w + a(t),t\right) + \varepsilon y^1(\varepsilon^{1/2}w + a(t),t) + \mathcal{O}(\varepsilon^2) \\ &= y^0(a(t),t) + \varepsilon^{1/2} w y^0_x(a(t),t) + \frac{\varepsilon w^2}{2} y^0_{xx}(a(t),t) + \mathcal{O}(\varepsilon^{3/2}). \end{aligned}$$

Therefore, the matching conditions read

$$W^0(w,t) \sim y^0((a(t))^\pm,t), \quad W^{1/2}(w,t) \sim y^0_x((a(t))^\pm,t)w, \quad \text{as } w \to \pm\infty. \tag{9}$$

Consequently, we obtain that the function W^0 must satisfy

$$\begin{cases} W^0_t(w,t) + M_x(a(t),t) w\, W^0_w(w,t) - W^0_{ww}(w,t) = 0, & (w,t) \in \mathbb{R} \times (0,T), \\ \displaystyle\lim_{w \to \pm\infty} W^0(w,t) = \lim_{x \to (a(t))^\pm} y^0(x,t), & t \in (0,T). \end{cases} \tag{10}$$

In view of (7), we have $\lim_{x\to(a(t))^+} y^0(x,t) = y_0(0)$ and $\lim_{x\to(a(t))^-} y^0(x,t) = v(0)$. Remark that these limits do not depend on t. Similarly, the function $W^{1/2}$ must satisfy

$$\begin{cases} W^{1/2}_t(w,t) + M_x(a(t),t) w\, W^{1/2}_w(w,t) - W^{1/2}_{ww}(w,t) = -\dfrac{w^2}{2} M_{xx}(a(t),t) W^0_w(w,t), & (w,t) \in \mathbb{R} \times (0,T), \\ \displaystyle\lim_{w\to\pm\infty} \left(W^{1/2}(w,t) - y^0_x(a(t)^\pm,t)\, w \right) = 0, & t \in (0,T). \end{cases} \tag{11}$$

In view of (7), $y^0_x(a(t)^+,t) = (y_0)^{(1)}(0)\, X_x(0;a(t),t)$ and $y^0_x(a(t)^-,t) = v^{(1)}(0)\, \gamma_{0x}(a(t)^-,t)$ where γ_{0x} is given by (5). Remark that these limits do depend on the variable t.

2.4 Representation of the functions W^0 and $W^{1/2}$

We express the functions W^0 and $W^{1/2}$ in term of the fundamental solution of the heat equation.

2.4.1 Fundamental solutions

Let us consider the differential operator

$$\mathcal{L}U(w,t) := U_t(w,t) + M_x(a(t),t)w\, U_w(w,t) - U_{ww}(w,t) \quad \text{in } \mathbb{R}\times(0,T). \tag{12}$$

Equation $\mathcal{L}U = 0$ is a parabolic equation with an unbounded coefficient. Linear second order parabolic equations with smooth but unbounded coefficients are studied in [5] (see also [16, 12, 7]). In [5], the authors consider a second order linear differential operator in the form

$$\widetilde{\mathcal{L}}u(w,t) = u_t(w,t) - c_{ij}(w,t)u_{w_iw_j}(w,t) - c_i(w,t)u_{w_i}(w,t) - c(w,t)u(w,t), \quad (w,t)\in\mathbb{R}^n\times(0,T),$$

and prove the existence of a so-called *fundamental solution* for $\widetilde{\mathcal{L}}u = 0$. Our operator $\mathcal{L}$ is obviously of the form of the operator $\widetilde{\mathcal{L}}$ with $n = 1$, $c \equiv 0$, $c_{11} \equiv 0$ and $c_1(w,t) = M_x(a(t),t)w$.

Definition 2.1. *A function $\widetilde{K}(w,t;\xi,s)$ defined for $w,\xi\in\mathbb{R}^n$ and $0\le s<t\le T$ is said to be a fundamental solution of $\widetilde{\mathcal{L}}u = 0$ if it has the following properties:*

- *considered as a function of (w,t) for each fixed $(\xi,s)\in\mathbb{R}^n\times[0,T]$, the derivatives of $\widetilde{K}$ which appear in $\widetilde{\mathcal{L}}$ exist and are continuous;*
- *$\widetilde{\mathcal{L}}\widetilde{K} = 0$ in $\mathbb{R}^n\times(s,T]$;*
- *If $g = g(w)$ is a continuous function with compact support in $\mathbb{R}^n$ then*

$$\lim_{(w,t)\to(w_0,s^+)} \int_{\mathbb{R}^n} \widetilde{K}(w,t;\xi,s)g(\xi)\,d\xi = g(w_0).$$

We shall employ the following lemma.

Lemma 2.1. *Let $f = f(w,t)$ be a given function in $\mathbb{R}^n\times(0,T)$. Assume that f is Hölder continuous on every compact subset of $\mathbb{R}^n\times(0,T)$. Then*

$$u(w,t) = \int_{\mathbb{R}^n} \widetilde{K}(w,t;\xi,0)g(\xi)\,d\xi + \int_0^t\int_{\mathbb{R}^n} \widetilde{K}(w,t;\xi,\tau)f(\xi,\tau)\,d\xi\,d\tau$$

is a solution of the Cauchy problem

$$\widetilde{\mathcal{L}}u(w,t) = f(w,t) \ \text{ in } \mathbb{R}^n\times(0,T), \quad u(w,0) = g(w) \ \text{ in } \mathbb{R}^n.$$

In the particular case of the operator $\mathcal{L}$, the solution of the corresponding Cauchy problem has an explicit representation in term of the fundamental solution of the heat equation $H_s - H_{vv} = 0$ in $\mathbb{R}\times(0,T)$ defined as follows

$$H(v,s;\xi,\tau) = H_0(v-\xi;s-\tau), \quad v,\xi\in\mathbb{R},\ 0\le\tau<s\le T, \tag{13}$$

with

$$H_0(v,s) := \frac{1}{\sqrt{4\pi s}}e^{-\frac{v^2}{4s}}, \quad v\in\mathbb{R}, s\in(0,T). \tag{14}$$

Precisely, we have the following result.

Lemma 2.2. *Let $\mathcal{L}$ be the parabolic operator defined by* (12) *and H be the fundamental solution of the heat equation defined by* (13). *Then:*

i) Equation $\mathcal{L}U = 0$ has a fundamental solution given by

$$K(w,t;\xi,\tau) = \frac{1}{\beta(\tau)} H\left(\frac{w}{\beta(t)}, B(t); \frac{\xi}{\beta(\tau)}, B(\tau)\right), \quad w,\xi \in \mathbb{R},\ 0 \leq \tau < t \leq T, \tag{15}$$

with

$$\alpha(t) := M_x(a(t),t), \quad \beta(t) := e^{\int_0^t \alpha(s)\,ds}, \quad B(t) := \int_0^t \frac{1}{\beta(s)^2}\,ds.$$

ii) Let $f = f(w,t)$ be a continuous function in $\mathbb{R}\times[0,T]$ and Hölder continuous with respect to w uniformly for $t \in [0,T]$. Let $g = g(w)$ be a continuous function with compact support in $\mathbb{R}$. Then,

$$U(w,t) = \int_{\mathbb{R}} K(w,t;\xi,0)g(\xi)\,d\xi + \int_0^t\int_{\mathbb{R}} K(w,t;\xi,\tau)f(\xi,\tau)\,d\xi d\tau \tag{16}$$

is a solution of the Cauchy problem

$$\mathcal{L}U(w,t) = f(w,t) \ \text{ in } \mathbb{R}\times(0,T), \quad U(w,0) = g(w) \ \text{ in } \mathbb{R}. \tag{17}$$

Proof. We have

$$\beta'(t) = \alpha(t)\beta(t), \quad B'(t) = \frac{1}{\beta(t)^2}, \quad \beta(0) = 1, \quad B(0) = 0.$$

Note also that B is a stricty increasing function so that $0 \leq \tau < t \leq T$ is equivalent to $0 \leq B(\tau) < B(t) \leq B(T)$. Consider the function K defined by (15). We have

$$K(w,t;\xi,\tau) = \frac{1}{\beta(\tau)\sqrt{4\pi(B(t)-B(\tau))}} e^{-\frac{\left(\frac{w}{\beta(t)}-\frac{\xi}{\beta(\tau)}\right)^2}{4(B(t)-B(\tau))}}, \quad 0 \leq \tau < t \leq T.$$

For ξ,τ fixed, $\xi \in \mathbb{R}, 0 \leq \tau < t \leq T$, we have by a direct calculation

$$\mathcal{L}K(w,t;\xi,\tau) = \frac{1}{\beta(\tau)\beta^2(t)}\left[H_{0s}\left(\frac{w}{\beta(t)}-\frac{\xi}{\beta(\tau)}, B(t)-B(\tau)\right) - H_{0vv}\left(\frac{w}{\beta(t)}-\frac{\xi}{\beta(\tau)}, B(t)-B(\tau)\right)\right].$$

Then, $\mathcal{L}K(w,t;\xi,\tau) = 0$ for any $w \in \mathbb{R}$ and $t \in (0,T)$. If $g = g(w)$ is a continuous function with compact support in $\mathbb{R}$ we have

$$\int_{\mathbb{R}} K(w,t;\xi,\tau)g(\xi)\,d\xi = \frac{1}{\beta(\tau)\sqrt{4\pi(B(t)-B(\tau))}}\int_{\mathbb{R}} e^{-\frac{\left(\frac{w}{\beta(t)}-\frac{\xi}{\beta(\tau)}\right)^2}{4(B(t)-B(\tau))}} g(\xi)\,d\xi.$$

Using the change of variable $\frac{\frac{\xi}{\beta(\tau)}-\frac{w}{\beta(t)}}{2\sqrt{(B(t)-B(\tau))}} = v$, it holds that

$$\int_{\mathbb{R}} K(w,t;\xi,\tau)g(\xi)\,d\xi = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-v^2} g\left(\frac{\beta(\tau)}{\beta(t)}w + 2\beta(\tau)\sqrt{B(t)-B(\tau)}\,v\right)\,dv,$$

and deduce that

$$\lim_{t\to\tau^+}\int_{\mathbb{R}} K(w,t;\xi,\tau)g(\xi)\,d\xi = g(w) \text{ for any } w \in \mathbb{R}.$$

According to Definition 2.1, it follows that the function K given by (15) is a fundamental solution of the equation $\mathcal{L}U = 0$. Moreover, using that $\beta(0) = 1$ and $B(0) = 0$, we have

$$\int_{\mathbb{R}} K(w,t;\xi,0)g(\xi)\,d\xi = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-v^2} g\left(\frac{w}{\beta(t)} + 2\sqrt{B(t)}\,v\right)\,dv,$$

implying that

$$\lim_{t\to 0^+}\int_{\mathbb{R}} K(w,t;\xi,0)g(\xi)\,d\xi = g(w).$$

We also have, for any ξ fixed, $\mathcal{L}K(w,t;\xi,0)=0$ from which we check that $\mathcal{L}\left(\int_{\mathbb{R}} K(w,t;\xi,0)g(\xi)\,d\xi\right)=0$.

Consider now the second term in the right-hand side of (16) which we denote $\tilde{U}(w,t)$. We have

$$\begin{aligned}\mathcal{L}(\tilde{U})(w,t) &= \lim_{\tau\to t^-}\int_{\mathbb{R}} K(w,t;\xi,\tau)f(\xi,\tau)\,d\xi + \int_0^t\int_{\mathbb{R}} \mathcal{L}(K)(w,t;\xi,\tau)f(\xi,\tau)\,d\xi d\tau \\ &= \lim_{\tau\to t^-}\int_{\mathbb{R}} K(w,t;\xi,\tau)f(\xi,\tau)\,d\xi.\end{aligned}$$

Using again the change of variable $\frac{\frac{\xi}{\beta(\tau)}-\frac{w}{\beta(t)}}{2\sqrt{(B(t)-B(\tau))}}=v$, it holds that

$$\lim_{\tau\to t^-}\int_{\mathbb{R}} K(w,t;\xi,\tau)f(\xi,\tau)\,d\xi = \frac{1}{\sqrt{\pi}}\lim_{\tau\to t^-}\int_{\mathbb{R}} e^{-v^2} f\left(\frac{\beta(\tau)}{\beta(t)}w + 2\beta(\tau)\sqrt{(B(t)-B(\tau))}\,v,\tau\right)\,dv = f(w,t).$$

We conclude that U is a solution of the Cauchy problem (17). □

We now introduce the functions $g_{k/2}=g_{k/2}(w,t)$, $k=0,1$, defined as functions on $\mathbb{R}\times(0,T)$ and satisfying

$$\lim_{w\to\pm\infty}\left(g_0(w,t)-y^0(a(t)^\pm,t)\right)=0,\quad \lim_{w\to\pm\infty}\left(g_{1/2}(w,t)-y^0_x(a(t)^\pm,t)\,w\right)=0,\quad t\in(0,T).$$

We also introduce the functions $f_{k/2}=f_{k/2}(w,t)$, $k=0,1$, defined on $\mathbb{R}\times(0,T)$ by

$$f_0(w,t):=0,\quad f_{1/2}(w,t):=-\frac{w^2}{2}M_{xx}(a(t),t)\,W^0_w(w,t). \tag{18}$$

2.4.2 Explicit expression of W^0

In order to apply the second part of Lemma 2.2, it remains to choose an initial condition g (for the boundary value problem satisfied by W^0) so that the corresponding solution W^0 given generically by (16) satisfies the asymptotic conditions as $w\to\pm\infty$ (see (10)), i.e. $\lim_{w\to\infty}W^0(w,t)=y_0(0)$ and $\lim_{w\to-\infty}W^0(w,t)=v(0)$. The simplest choice, discussed in [3], is given by $g=g_0$ with

$$g_0(w):=\begin{cases} y_0(0), & w\geq 0,\\ v(0), & w<0,\end{cases} \tag{19}$$

leading to the explicit expression:

$$\boxed{W^0(w,t)=\frac{y_0(0)-v(0)}{2}erf\left(\frac{w}{2\beta(t)\sqrt{B(t)}}\right)+\frac{y_0(0)+v(0)}{2}.} \tag{20}$$

Recall that erf is the error function defined by $erf(s)=\frac{2}{\sqrt{\pi}}\int_0^s e^{-v^2}\,dv$ for all $s\in\mathbb{R}$, and that the complementary error function is defined by $erfc(s)=1-erf(s)=\frac{2}{\sqrt{\pi}}\int_s^{+\infty}e^{-v^2}\,dv$. Using the asymptotic behavior of the error function, we check that W^0 satisfies the prescribed asymptotic behavior in (10). Remark also that $lim_{t\to 0^+}W^0(w,t)$ equals $y_0(0)$ if $w\geq 0$ and $v^0(0)$ if $w<0$.

2.4.3 Integral representation of $W^{1/2}$

The computation of $W^{1/2}$ solution of the boundary value problem (11) posed over $\mathbb{R}\times(0,T)$ is more delicate since, first there is a right side, and second, because the asymptotic limits as $w\to\infty$ depend on the time variable, while the function g in (17) not. This fact requires a special treatment.

Let us consider the function $g_{1/2}(w,t)=y_x^0(a(t)^{\pm},t)\,w$. We have, for $(w,t)\in(]-\infty,0[\cup]0,+\infty[)\times(0,T)$,

$$\begin{aligned}\mathcal{L}(g_{1/2})(w,t)&=w\left(y_{xt}^0(a(t)^{\pm},t)+a'(t)y_{xx}^0(a(t)^{\pm},t)\right)+M_x(a(t),t)wy_x^0(a(t)^{\pm},t)\\&=w\left(y_{xt}^0(a(t)^{\pm},t)+M(a(t),t)y_{xx}^0(a(t)^{\pm},t)\right)+M_x(a(t),t)wy_x^0(a(t)^{\pm},t).\end{aligned}$$

Differentiating equation $(6)_1$ with respect to x yields

$$y_{xt}^0(x,t)+M_x(x,t)y_x^0(x,t)+M(x,t)y_{xx}^0(x,t)=0,\quad (x,t)\in\Omega^+\cup\Omega^-,$$

with $\Omega^+:=\{(x,t)\in Q_T,x>a(t)\}$ and $\Omega^-:=\{(x,t)\in Q_T,x<a(t)\}$. Letting $x\to a(t)^{\pm}$, we get

$$y_{xt}^0(a(t)^{\pm},t)+M_x(a(t),t)y_x^0(a(t)^{\pm},t)+M(a(t),t)y_{xx}^0(a(t)^{\pm},t)=0.$$

Therefore, for $(w,t)\in(]-\infty,0[\cup]0,+\infty[)\times(0,T)$, we have

$$\mathcal{L}(g_{1/2})(w,t)=-wM_x(a(t),t)y_x^0(a(t)^{\pm},t)+wM_x(a(t),t)y_x^0(a(t)^{\pm},t)=0.$$

When $y_x^0(a(t)^{\pm},t)$ has a jump on $w=0$, the function $(g_{1/2})_{ww}$ becomes singular on $w=0$. This leads to regularize the function $g_{1/2}$ as follows. Let $\rho\in C^\infty(\mathbb{R})$ such that

$$\rho(w)=\begin{cases}1 & \text{for } |w|\geq 2,\\ 0 & \text{for } |w|\leq 1.\end{cases}$$

We define

$$\widetilde{g}_{1/2}(w,t):=\rho(w)\,g_{1/2}(w,t),\quad w\in\mathbb{R},t\in(0,T).\tag{21}$$

Clearly, $\widetilde{g}_{1/2}\in C^\infty(\mathbb{R})$, $\widetilde{g}_{1/2}(w,t)=g_{1/2}(w,t)$ for $|w|\geq 2$ and $\widetilde{g}_{1/2}(w,t)=0$ for $|w|\leq 1$. Define

$$g_{1/2}^\star(w,t)=\mathcal{L}(\widetilde{g}_{1/2})(w,t),\quad w\in\mathbb{R},t\in(0,T),\tag{22}$$

and then $W^{1/2}$ as follows

$$\boxed{W^{1/2}(w,t)=\widetilde{g}_{1/2}(w,t)+\int_0^t\int_{\mathbb{R}}K(w,t;\xi,\tau)\Big(f_{1/2}(\xi,\tau)-g_{1/2}^\star(\xi,\tau)\Big)\,d\xi d\tau,\quad w\in\mathbb{R},t\in(0,T).}\tag{23}$$

LEMMA 2.3. *The function $W^{1/2}$ defined by* (23) *satisfies* (11)*, namely $\mathcal{L}(W^{1/2})(w,t)=f_{1/2}(w,t)$ in $\mathbb{R}\times(0,T)$ and $\lim_{w\to\pm\infty}\big(W^{1/2}(w,t)-g_{1/2}(w,t)\big)=0$ for all t in $(0,T)$.*

Proof. According to Lemma 2.2 we have $\mathcal{L}(W^{1/2})(w,t)=f_{1/2}(w,t)$ in $\mathbb{R}\times(0,T)$.

It remains to check the asymptotic condition as $w\to\pm\infty$. For any $(w,t)\in\mathbb{R}\times(0,T)$, let us consider the integral

$$I_1(w,t):=\int_0^t\int_{\mathbb{R}}K(w,t;\xi,\tau)f_{1/2}(\xi,\tau)\,d\xi d\tau.$$

In view of (18) and (20), we obtain explicitly

$$f_{1/2}(w,t)=-\frac{w^2}{2}M_{xx}(a(t),t)\left(\frac{y_0(0)-v(0)}{2}\right)\frac{2}{\sqrt{\pi}}e^{-\frac{w^2}{4\beta(t)^2B(t)}}\left(\frac{1}{2\beta(t)\sqrt{B(t)}}\right),$$

then

$$I_1(w,t) = \frac{y_0(0)-v(0)}{2} \int_0^t\!\!\int_{\mathbb{R}} \frac{1}{\beta(\tau)\sqrt{4\pi(B(t)-B(\tau))}} e^{-\frac{\left(\frac{w}{\beta(t)}-\frac{\xi}{\beta(\tau)}\right)^2}{4(B(t)-B(\tau))}} \times$$
$$\times \left(-\frac{\xi^2}{2}\right) M_{xx}(a(\tau),\tau) \left(\frac{2}{\sqrt{\pi}}\right) e^{-\frac{\xi^2}{4\beta(\tau)^2 B(\tau)}} \left(\frac{1}{2\beta(\tau)\sqrt{B(\tau)}}\right) d\xi d\tau.$$

For the convergence of the integral $I_1(\omega,t)$ in the neighborhood of $\tau=0$, we consider the integral $I_1(\omega,t')$, with $0<t'<t$ fixed, and use the change of variable $\frac{\xi}{2\beta(\tau)\sqrt{B(\tau)}}=v$. This gives

$$I_1(w,t') := \frac{y_0(0)-v(0)}{2} \int_0^{t'}\!\!\int_{\mathbb{R}} \frac{1}{\beta(\tau)\sqrt{4\pi(B(t)-B(\tau))}} e^{-\frac{\left(\frac{w}{\beta(t)}-2\sqrt{B(\tau)}v\right)^2}{4(B(t)-B(\tau))}} \times$$
$$\times \left(-2\beta(\tau)^2 B(\tau)v^2\right) M_{xx}(a(\tau),\tau) \left(\frac{2}{\sqrt{\pi}}\right) e^{-v^2} dv d\tau.$$

The integrand is clearly uniformly bounded in w by $c(t',t)e^{-v^2}$ with a constant $c(t',t)$ depending only on t' and t. Moreover, for any fixed v and τ, the integrand tends to 0 as $|w|\to+\infty$. Using a result of limit passage on integrals depending on a parameter we get $\lim_{|w|\to+\infty} I_1(w,t')=0$.

For the convergence of the integral $I_1(w,t)$ in the neighborhood of $\tau=t$, we consider the integral

$$I_2(w,t) := \frac{y_0(0)-v(0)}{2} \int_{t'}^t\!\!\int_{\mathbb{R}} \frac{1}{\beta(\tau)\sqrt{4\pi(B(t)-B(\tau))}} e^{-\frac{\left(\frac{w}{\beta(t)}-\frac{\xi}{\beta(\tau)}\right)^2}{4(B(t)-B(\tau))}} \times$$
$$\times \left(-\frac{\xi^2}{2}\right) M_{xx}(a(\tau),\tau) \left(\frac{2}{\sqrt{\pi}}\right) e^{-\frac{\xi^2}{4\beta(\tau)^2 B(\tau)}} \left(\frac{1}{2\beta(\tau)\sqrt{B(\tau)}}\right) d\xi d\tau.$$

Using the change of variable $\frac{\frac{\xi}{\beta(\tau)}-\frac{w}{\beta(t)}}{2\sqrt{B(t)-B(\tau)}}=v$, we get

$$I_2(w,t) = \frac{y_0(0)-v(0)}{2}\frac{1}{\sqrt{\pi}} \int_{t'}^t\!\!\int_{\mathbb{R}} e^{-v^2} \left(-\frac{\left(\frac{\beta(\tau)}{\beta(t)}w+2\beta(\tau)\sqrt{B(t)-B(\tau)}\,v\right)^2}{2}\right) M_{xx}(a(\tau),\tau)\times$$
$$\times \left(\frac{2}{\sqrt{\pi}}\right) e^{-\frac{\left(\frac{\beta(\tau)}{\beta(t)}w+2\beta(\tau)\sqrt{B(t)-B(\tau)}\,v\right)^2}{4\beta(\tau)^2 B(\tau)}} \left(\frac{1}{2\beta(\tau)\sqrt{B(\tau)}}\right) dv d\tau$$

and conclude as before that $\lim_{|w|\to+\infty} I_2(w,t)=0$ and then that $\lim_{|w|\to+\infty} I_1(w,t)=0$.

Consider now the integral

$$\begin{aligned} I_3(w,t) : &= \int_0^t\!\!\int_{\mathbb{R}} K(w,t;\xi,\tau) g^\star_{1/2}(\xi,\tau)\, d\xi d\tau \\ &= \int_0^t\!\!\int_{\mathbb{R}} \frac{1}{\beta(\tau)\sqrt{4\pi(B(t)-B(\tau))}} e^{-\frac{\left(\frac{w}{\beta(t)}-\frac{\xi}{\beta(\tau)}\right)^2}{4(B(t)-B(\tau))}} g^\star_{1/2}(\xi,\tau)\, d\xi d\tau. \end{aligned}$$

Clearly, $g^\star_{1/2}$ is a smooth function on $\mathbb{R}\times(0,T)$ and satisfies $g^\star_{1/2}(w,t)=0$ for $|w|\geq 2$. The change of variable $\frac{\frac{\xi}{\beta(\tau)}-\frac{w}{\beta(t)}}{2\sqrt{(B(t)-B(\tau))}}=v$ gives

$$I_3(w,t) = \frac{1}{\sqrt{\pi}} \int_0^t\!\!\int_{\mathbb{R}} e^{-v^2} g^\star_{1/2}\left(\frac{\beta(\tau)}{\beta(t)}w+2\beta(\tau)\sqrt{B(t)-B(\tau)}\,v,\tau\right) dv d\tau.$$

The integrand is uniformly bounded in w by $c(t)e^{-v^2}$, where $c(t)$ depends only on t. Moreover, the integrand tends to 0 as $|w|\to+\infty$. Then $\lim_{|w|\to+\infty} I_3(w,t)=0$.

We then have

$$\lim_{|w|\to+\infty}\int_0^t\int_{\mathbb{R}} K(w,t;\xi,\tau)\Big(f_{1/2}(\xi,\tau)-g^\star_{1/2}(\xi,\tau)\Big)\,d\xi d\tau=0.$$

It results from (23) that, for each fixed t, $\lim_{w\to\pm\infty}\big(W^{1/2}(w,t)-\widetilde{g}_{1/2}(w,t)\big)=0$. Since $\lim_{w\to\pm\infty}\big(\widetilde{g}_{1/2}(w,t)-g_{1/2}(w,t)\big)=0$, we conclude that $\lim_{w\to\pm\infty}\big(W^{1/2}(w,t)-g_{1/2}(w,t)\big)=0$. □

2.5 Composite asymptotic approximation outside the boundary layer

We now define a composite approximation of the solution far from the $x=1$ obtained by adding, at each order, the inner and outer expansions and then by subtracting their common part.

At the first order, the common part of $y^0(x,t)$ and $W^0(w,t)$ (defined by (7) and (20) respectively) is equal to $y_0(0)$ for $x>a(t)$ and to $v(0)$ for $x<a(t)$. Thus, the first term of the composite approximation outside the boundary layer along $x=1$ is given by

$$\boxed{p^0(x,t)=y^0(x,t)+W^0(w,t)-y^0(a(t)^\pm,t).} \tag{24}$$

We check that the function p^0 is continuous along the characteristic as it satisfies $\lim_{x-a(t)\to 0^\pm}p^0(x,t)=W^0(0,t)$. The second term of the composite approximation is given by

$$\boxed{p^{1/2}(x,t)=W^{1/2}(w,t)-y^0_x(a(t)^\pm,t),} \tag{25}$$

where $W^{1/2}$ is defined by (23). Clearly, $p^{1/2}$ is also continuous along the characteristic. Then the following quantity is defined to be an asymptotic approximation of y^ε, outside the boundary layer along $x=1$,

$$p^\varepsilon(x,t):=p^0(x,t)+\varepsilon^{1/2}p^{1/2}(x,t),\quad (x,t)\in Q_T.$$

We easily verify the following property.

Proposition 2.1. *Assume that $v\in\mathcal{C}^1([0,T])$ and $y_0\in\mathcal{C}^1([0,1])$. Then p^ε belongs to $C^1([0,1]\times(0,T])$.*

2.6 Inner expansion along $x=1$

We consider the change of variables $z=\frac{1-x}{\varepsilon}$, $\tau=\frac{T_1-t}{\varepsilon^{1/2}}$, and function $Y^\varepsilon(z,\tau,t)=y^\varepsilon(x,t)$. Recall that T_1 defined by $a(T_1)=1$ is the time at which the characteristic $x-a(t)=0$ starting from the point $(0,0)$ intersects the right extremity of the domain. We assume that T_1 exists. Moreover, since $M>0$, T_1 is unique. The function Y^ε satisfies the equation

$$Y^\varepsilon_t(z,\tau,t)-\frac{1}{\varepsilon^{1/2}}Y^\varepsilon_\tau(z,\tau,t)-\frac{M(1-\varepsilon z,t)}{\varepsilon}Y^\varepsilon_z(z,\tau,t)-\frac{1}{\varepsilon}Y^\varepsilon_{zz}(z,\tau,t)=0. \tag{26}$$

Using the Taylor expansion

$$M(1-\varepsilon z,t)=M(1,t)-\varepsilon zM_x(1,t)+\frac{\varepsilon^2z^2}{2}M_{xx}(1,t)-\frac{\varepsilon^3z^3}{3!}M_{xxx}(1,t)+\cdots,$$

and putting $Y^0(z,\tau,t)+\varepsilon^{1/2}Y^{1/2}(z,\tau,t)$ into equation (26), the identification of the powers of ε yields

$$\begin{aligned}\varepsilon^{-1}:&\quad Y^0_{zz}(z,\tau,t)+M(1,t)Y^0_z(z,\tau,t)=0,\\ \varepsilon^{-1/2}:&\quad Y^{1/2}_{zz}(z,\tau,t)+M(1,t)Y^{1/2}_z(z,\tau,t)=-Y^0_\tau(z,\tau,t).\end{aligned}$$

It is important to note that the functions Y^k depend on three variables, namely z,t but also τ. As it is standard, the variable $z=(1-x)/\varepsilon$ is introduced to describe the boundary layer at $x=1^-$. Here, the variable $\tau=\frac{T_1-t}{\varepsilon^{1/2}}$ allows to take into account the interaction of the internal and boundary layers by making a zoom around the point $(1,T_1)$. If we do not introduce this variable τ, we see notably that $Y^{1/2}$ solves the same ordinary differential equation than Y^0, and the analysis leads to an error estimate (for the $L^\infty(L^2)$ norm) of order $\varepsilon^{1/4}$ only. Remark that the variable T_1, as a solution of $a(T_1) = 1$ is not explicit. Therefore, instead of $\tau = \frac{T_1 - t}{\varepsilon^{1/2}}$, we could have introduce $\tau = \frac{1-a(t)}{\varepsilon^{1/2}}$; however, we have observed that this leads to some incompatibilities with the matching condition.

We impose $Y^{k/2}(0, \tau, t) = 0$ for $k = 0, 1$. To get the asymptotic matching conditions we write that, for any fixed τ, t and large z,

$$Y^0(z, \tau, t) + \varepsilon^{1/2} Y^{1/2}(z, \tau, t) = p^0(x, t) + \varepsilon^{1/2} p^{1/2}(x, t) + \mathcal{O}(\varepsilon^2).$$

In order to identify at each order the appropriate matching conditions, we need to rewrite the right-hand side of the above equality in terms of z and τ, t being fixed. We have the following equalities $x = 1 - \varepsilon z$, $w = \frac{x-a(t)}{\varepsilon^{1/2}} = -\varepsilon^{1/2} z + \frac{1-a(t)}{\varepsilon^{1/2}}$, $a(T_1) = 1$, $a'(t) = X_t(t; 0, 0) = M(a(t), t)$, $a'(T_1) = M(1, T_1)$. Writing $a(t) = a\left(T_1 - \varepsilon^{1/2}\tau\right)$, $M(1, T_1) = M\left(1, t + \varepsilon^{1/2}\tau\right)$, and using Taylor expansions we have

$$a(t) = a(T_1) - \varepsilon^{1/2}\tau a'(T_1) + \mathcal{O}(\varepsilon) = 1 - \varepsilon^{1/2}\tau M(1, T_1) + \mathcal{O}(\varepsilon), \quad M(1, T_1) = M(1, t) + \mathcal{O}(\varepsilon^{1/2}),$$

then

$$\frac{1 - a(t)}{\varepsilon^{1/2}} = \tau M(1, t) + \mathcal{O}(\varepsilon^{1/2}).$$

It results that

$$w = -\varepsilon^{1/2} z + \tau M(1, t) + \mathcal{O}(\varepsilon^{1/2}).$$

Writing $y^0(x, t) = y^0(1 - \varepsilon z, t)$, $W^0(w, t) = W^0\left(-\varepsilon^{1/2} z + \tau M(1, t) + \mathcal{O}(\varepsilon^{1/2}), t\right)$, and $W^{1/2}(w, t) = W^{1/2}\left(-\varepsilon^{1/2} z + \tau M(1, t) + \mathcal{O}(\varepsilon^{1/2}), t\right)$, and using again Taylor expansions we have

$$\begin{aligned} p^0(x, t) &= y^0(x, t) + W^0(w, t) - y^0(a(t)^\pm, t) \\ &= y^0(1, t) + W^0\left(\tau M(1, t), t\right) - y^0(a(t)^\pm, t) - \varepsilon^{1/2} z W^0_w\left(\tau M(1, t), t\right) + \mathcal{O}(\varepsilon^{1/2}), \end{aligned}$$

and

$$\begin{aligned} \varepsilon^{1/2} p^{1/2}(x, t) =& \varepsilon^{1/2}\left(W^{1/2}(w, t) - y^0_x(a(t)^\pm, t) w\right) \\ =& \varepsilon^{1/2}\left(W^{1/2}(\tau M(1, t), t) - \tau M(1, t)\, y^0_x(a(t)^\pm, t)\right) + \mathcal{O}(\varepsilon). \end{aligned}$$

We deduce that

$$p^0(x, t) + \varepsilon^{1/2} p^{1/2}(x, t) = C_0(\tau, t) + \varepsilon^{1/2} C_{1/2}(z, \tau, t) + \mathcal{O}(\varepsilon),$$

with

$$\begin{aligned} C_0(\tau, t) =& y^0(1, t) + W^0(\tau M(1, t), t) - y^0(a(t)^\pm, t), \\ C_{1/2}(z, \tau, t) =& W^{1/2}(\tau M(1, t), t) - \tau M(1, t) y^0_x(a(t)^\pm, t) - z W^0_w\left(\tau M(1, t), t\right). \end{aligned} \tag{27}$$

Therefore, the matching conditions read

$$Y^0(z, \tau, t) \sim C_0(\tau, t), \quad Y^{1/2}(z, \tau, t) \sim C_{1/2}(z, \tau, t), \quad \text{as } z \to \infty. \tag{28}$$

• We define Y^0 as a solution of

$$\begin{cases} Y^0_{zz}(z, \tau, t) + M(1, t) Y^0_z(z, \tau, t) = 0, & (z, \tau, t) \in \mathbb{R}^\star_+ \times \mathbb{R} \times (0, T), \\ Y^0(0, \tau, t) = 0, \quad \lim\limits_{z \to +\infty} Y^0(z, \tau, t) = C_0(\tau, t), & t \in (0, T). \end{cases}$$

The solution is

$$\boxed{Y^0(z, \tau, t) = C_0(\tau, t)\left(1 - e^{-M(1,t) z}\right), \quad (z, \tau, t) \in \mathbb{R}_+ \times \mathbb{R} \times [0, T].}$$

• We define $Y^{1/2}$ as a solution of

$$\begin{cases} Y^{1/2}_{zz}(z,\tau,t) + M(1,t)Y^{1/2}_{z}(z,\tau,t) = -Y^0_\tau(z,\tau,t), & (z,\tau,t) \in \mathbb{R}^\star_+ \times \mathbb{R} \times (0,T), \\ Y^{1/2}(0,\tau,t) = 0, \quad \lim_{z\to+\infty} \Big(Y^{1/2}(z,\tau,t) - C_{1/2}(z,\tau,t)\Big) = 0, & t \in (0,T). \end{cases}$$

Writing that $Y^0_\tau(z,\tau,t) = C_{0,\tau}(\tau,t)\left(1 - e^{-M(1,t)z}\right)$, we obtain, for $(z,\tau,t) \in \mathbb{R}^\star \times \mathbb{R} \times (0,T)$,

$$\boxed{Y^{1/2}(z,\tau,t) = C_{1/2}(z,\tau,t) + e^{-M(1,t)z}\Bigg(-C_{1/2}(0,\tau,t) - W^0_w(\tau M(1,t),t)z\Bigg), \quad (z,\tau,t) \in \mathbb{R}_+ \times \mathbb{R} \times [0,T].} \tag{29}$$

2.7 Asymptotic composite approximation in Q_T

We are now in position to define what is supposed to be an asymptotic approximation of the solution y^ε. We proceed as before by adding at each order the function $p^{k/2}$, approximation outside the boundary layer along $x = 1$, and the function $Y^{k/2}$, approximation in the boundary layer along $x = 1$, then subtracting their common part. At the first order, the composite approximation is given by

$$\boxed{P^0(x,t) = p^0(x,t) + Y^0(z,\tau,t) - C_0(\tau,t) = p^0(x,t) - C_0(\tau,t)e^{-M(1,t)z},} \tag{30}$$

i.e. explicitly

$$P^0(x,t) = y^0(x,t) + W^0(w,t) - y^0(a(t)^\pm,t) - \Bigg(y^0(1,t) + W^0(\tau M(1,t),t) - y^0(a(t)^\pm,t)\Bigg)e^{-M(1,t)z}.$$

Repeating the arguments at the next order, we define

$$\boxed{P^{1/2}(x,t) = p^{1/2}(x,t) + Y^{1/2}(z,\tau,t) - C_{1/2}(z,\tau,t),} \tag{31}$$

then we define an asymptotic composite approximation of y^ε in Q_T by

$$P^\varepsilon(x,t) := P^0(x,t) + \varepsilon^{1/2}P^{1/2}(x,t), \quad (x,t) \in Q_T. \tag{32}$$

3 Convergence of the sequence $(P^\varepsilon)_{(\varepsilon>0)}$ - Proof of Theorem 1.1

This section is devoted to the study of the convergence of the sequence $(P^\varepsilon)_{\varepsilon>0}$ stated in Theorem (1.1). In order to prove this theorem we need to establish a number of preliminary results. We define the error as follows:

$$z^\varepsilon(x,t) := P^\varepsilon(x,t) - y^\varepsilon(x,t) - \theta^\varepsilon(x,t), \quad (x,t) \in Q_T, \tag{33}$$

where θ^ε is the initial layer corrector defined as a solution of the equation

$$\begin{cases} \theta^\varepsilon_t(x,t) + M(x,t)\theta^\varepsilon_x - \varepsilon\theta^\varepsilon_{xx}(x,t) = 0, & (x,t) \in Q_T, \\ \theta^\varepsilon(0,t) = \theta^\varepsilon(1,t) = 0, & t \in (0,T), \\ \theta^\varepsilon(x,0) = P^\varepsilon(x,0) - y^\varepsilon(x,0), & x \in (0,1). \end{cases} \tag{34}$$

3.1 Preliminary results

From now on, in order to shorten some equations, we shall use the following notation:

$$L_\varepsilon y := y_t + M(x,t)y_x - \varepsilon y_{xx}. \tag{35}$$

3.1.1 Estimate of the initial layer corrector θ^ε

The following lemma gives an exponential decay property of the initial layer corrector.

Lemma 3.1. *Let θ^ε be the solution of problem (34) and $\gamma \in (0,1/2]$. There exists a constant c independent of ε such that*

$$\|\theta^\varepsilon(\cdot,t)\|_{L^2(0,1)} \le c\, e^{-\frac{M(1,0)\varepsilon^\gamma}{\varepsilon}} + c\,\varepsilon^{1/2} e^{-\frac{M_0^2}{2\varepsilon^\gamma}t}, \quad \forall t \in [0,T]. \tag{36}$$

Proof. i) We first check that the initial data $\theta^\varepsilon(\cdot,0)$ is given by

$$\theta^\varepsilon(x,0) = -y_0(1)e^{-M(1,0)z}, \quad \forall x \in (0,1], \tag{37}$$

showing that the initial condition gets concentrated in the neighborhood of $x=1$. Indeed, from (30)–(32) we have

$$\begin{aligned}
\theta^\varepsilon(x,0) &= \sum_{k=0}^{1} \varepsilon^{k/2} \lim_{t\to 0} P^{k/2}(x,t) - y_0(x) \\
&= \lim_{t\to 0} \left(p^0(x,t) - C_0(\tau,t)e^{-Mz}\right) - y_0(x) \\
&\quad + \lim_{t\to 0} \varepsilon^{1/2} \left(p^{1/2}(x,t) + Y^{1/2}(z,\tau,t) - C_{1/2}(z,\tau,t)\right), \quad x \in (0,1).
\end{aligned}$$

We have from (30)

$$\begin{aligned}
&\lim_{t\to 0} \left(p^0(x,t) - C_0(\tau,t)e^{-M(1,t)z}\right) - y_0(x) \\
&= \lim_{t\to 0} \left(W^0(w,t) - y_0(0) - C_0(\tau,t)e^{-M(1,t)z}\right) = -\lim_{t\to 0} C_0(\tau,t)e^{-M(1,t)z} \\
&= -\lim_{t\to 0} \left(y^0(1,t) + W^0(\tau M(1,t),t) - y^0(a(t))^{\pm},t)\right) e^{-M(1,t)z} = -y_0(1)e^{-M(1,0)z}.
\end{aligned}$$

Using the matching conditions of $W^{1/2}$ with y^0, we check that $\lim_{t\to 0} p^{1/2}(x,t) = 0$, for $x \in (0,1]$ and that

$$\begin{aligned}
&\lim_{t\to 0} \left(Y^{1/2}(z,\tau,t) - C_{1/2}(z,\tau,t)\right) \\
&= \lim_{t\to 0} \left(-W^{1/2}(\tau M(1,t),t) + \tau y_x^0(a(t)^{\pm},t) - zW_w^0(\tau M(1,t),t)\right) e^{-M(1,t)z} = 0, \quad \text{for } x>0.
\end{aligned}$$

It results that $P^{1/2}(x,0) = 0$ for all $x \in (0,1]$. These computations lead to (37).

ii-a) We now introduce a C^∞ cut-off function $\mathcal{X} : \mathbb{R} \to [0,1]$ such that $\mathcal{X}(s) = 0$ if $s \le 1$ and $\mathcal{X}(s) = 1$ if $s \ge 2$ and define, for $\gamma \in (0,1/2]$, the function $\mathcal{X}_\varepsilon : [0,1] \to [0,1]$ by $\mathcal{X}_\varepsilon(x) = \mathcal{X}\left(\frac{1-x}{\varepsilon^\gamma}\right)$. The solution θ^ε of the linear system (34) can be decomposed as $\theta^\varepsilon = \theta^{\varepsilon,1} + \theta^{\varepsilon,2}$ with

$$\begin{cases}
L_\varepsilon \theta^{\varepsilon,1} = 0, & (x,t) \in Q_T, \\
\theta^{\varepsilon,1}(0,t) = \theta^{\varepsilon,1}(1,t) = 0, & t \in (0,T), \\
\theta^{\varepsilon,1}(x,0) = \mathcal{X}_\varepsilon(x)\theta^\varepsilon(x,0), & x \in (0,1),
\end{cases} \tag{38}$$

$$\begin{cases}
L_\varepsilon \theta^{\varepsilon,2} = 0, & (x,t) \in Q_T, \\
\theta^{\varepsilon,2}(0,t) = \theta^{\varepsilon,2}(1,t) = 0, & t \in (0,T), \\
\theta^{\varepsilon,2}(x,0) = (1-\mathcal{X}_\varepsilon(x))\theta^\varepsilon(x,0), & x \in (0,1).
\end{cases} \tag{39}$$

In view of the definition of $\mathcal{X}_\varepsilon$, we see that $\theta^{\varepsilon,1}(x,0) = 0$ for all $x \ge 1-\varepsilon^\gamma$. Then, in view of (37), we check that there exists a constant $c_1 > 0$ independent of ε such that $|\theta^{\varepsilon,1}(x,0)| \le c_1 e^{-\frac{M(1,0)\varepsilon^\gamma}{\varepsilon}}$ for all $x \in (0,1)$. By a maximum principle, it follows that

$$|\theta^{\varepsilon,1}(x,t)| \le c_1 e^{-\frac{M(1,0)\varepsilon^\gamma}{\varepsilon}}, \quad \forall (x,t) \in Q_T. \tag{40}$$

ii-b) Concerning $\theta^{\varepsilon,2}(\cdot,0)$, we check that $\theta^{\varepsilon,2}(x,0)=0$ for all $x\le 1-2\varepsilon^\gamma$ and that $\|\theta^{\varepsilon,2}(\cdot,0)\|_{L^2(0,1)}\le c_2\varepsilon^{\frac12}$ for some constant $c_2>0$, independent of ε. We now estimate $\|\theta^{\varepsilon,2}(\cdot,t)\|_{L^2(0,1)}$. For that, for any $\alpha>0$, we check that the function $\rho^\varepsilon(x,t):=e^{\frac{-M_0\alpha x}{2\varepsilon}}\theta^{\varepsilon,2}(x,t)$ with $M_0:=\min_{(x,t)\in\overline{Q_T}}M(x,t)>0$ solves

$$\begin{cases}\rho^\varepsilon_t-\varepsilon\rho^\varepsilon_{xx}+(M-\alpha M_0)\,\rho^\varepsilon_x-\dfrac{M_0}{4\varepsilon}\left(\alpha^2M_0-2\alpha M\right)\rho^\varepsilon=0, & (x,t)\in Q_T,\\ \rho^\varepsilon(0,t)=\rho^\varepsilon(1,t)=0, & t\in(0,T),\\ \rho^\varepsilon(x,0)=e^{\frac{-M_0\alpha x}{2\varepsilon}}\theta^{\varepsilon,2}_0(x), & x\in(0,1),\end{cases}\tag{41}$$

and satisfies the estimates

$$\begin{aligned}\frac{d}{dt}\|\rho^\varepsilon(\cdot,t)\|^2_{L^2(0,1)}+2\varepsilon\|\rho^\varepsilon_x(\cdot,t)\|^2_{L^2(0,1)}&=\int_0^1\left(M_x+\frac{M_0}{2\varepsilon}(\alpha^2M_0-2\alpha M\right)(\rho^\varepsilon(\cdot,t))^2\,dx\\ &\le\int_0^1\left(C+\frac{M_0^2}{2\varepsilon}(\alpha^2-2\alpha\right)(\rho^\varepsilon(\cdot,t))^2\,dx\end{aligned}$$

with $C=\max_{(x,t)\in\overline{Q_T}}|M_x(x,t)|$. Gronwall inequality leads to $\|\rho^\varepsilon(\cdot,t)\|_{L^2(0,1)}\le\|\rho^\varepsilon(\cdot,0)\|_{L^2(0,1)}\,e^{\frac{C}{2}t}\,e^{\frac{M_0^2}{4\varepsilon}(\alpha^2-2\alpha)t}$, which is equivalent to

$$\|e^{-\frac{M_0\alpha x}{2\varepsilon}}\theta^{\varepsilon,2}(\cdot,t)\|_{L^2(0,1)}\le\|e^{-\frac{M_0\alpha x}{2\varepsilon}}\theta^{\varepsilon,2}(\cdot,0)\|_{L^2(0,1)}\,e^{\frac{C}{2}t}\,e^{\frac{M_0^2}{4\varepsilon}(\alpha^2-2\alpha)t}.$$

Consequently,

$$\begin{aligned}\|\theta^{\varepsilon,2}(\cdot,t)\|_{L^2(0,1)}&=\|e^{\frac{M_0\alpha x}{2\varepsilon}}e^{-\frac{M_0\alpha x}{2\varepsilon}}\theta^{\varepsilon,2}(\cdot,t)\|_{L^2(0,1)}\le\|e^{\frac{M_0\alpha x}{2\varepsilon}}\|_{L^\infty(0,1)}\|e^{-\frac{M_0\alpha x}{2\varepsilon}}\theta^{\varepsilon,2}(\cdot,t)\|_{L^2(0,1)}\\ &\le\|e^{\frac{M_0\alpha x}{2\varepsilon}}\|_{L^\infty(0,1)}\|e^{-\frac{M_0\alpha x}{2\varepsilon}}\theta^{\varepsilon,2}(\cdot,0)\|_{L^2(0,1)}\,e^{\frac{C}{2}t}\,e^{\frac{M_0^2}{4\varepsilon}(\alpha^2-2\alpha)t}\\ &\le\|e^{\frac{M_0\alpha x}{2\varepsilon}}\|_{L^\infty(0,1)}\|e^{-\frac{M_0\alpha x}{2\varepsilon}}\theta^{\varepsilon,2}(\cdot,0)\|_{L^2(1-2\varepsilon^\gamma,1)}\,e^{\frac{C}{2}t}\,e^{\frac{M_0^2}{4\varepsilon}(\alpha^2-2\alpha)t}\\ &\le e^{\frac{M_0\alpha}{2\varepsilon}}e^{\frac{-M_0\alpha(1-2\varepsilon^\gamma)}{2\varepsilon}}\|\theta_0^{\varepsilon,2}\|_{L^2(1-2\varepsilon^\gamma,1)}\,e^{\frac{C}{2}t}\,e^{\frac{M_0^2}{4\varepsilon}(\alpha^2-2\alpha)t}\\ &=\|\theta_0^{\varepsilon,2}\|_{L^2(1-2\varepsilon^\gamma,1)}\,e^{\frac{C}{2}t}\,e^{-\frac{M_0\alpha}{2\varepsilon}\left(-2\varepsilon^\gamma+\left(1-\frac{\alpha}{2}\right)M_0t\right)},\end{aligned}$$

using that (recall that $\alpha>0$) $\|e^{\frac{M_0\alpha x}{2\varepsilon}}\|_{L^\infty(0,1)}=e^{\frac{M_0\alpha}{2\varepsilon}}$ and $\|e^{-\frac{M_0\alpha x}{2\varepsilon}}\|_{L^\infty(1-2\varepsilon^\gamma,1)}=e^{\frac{-M_0\alpha(1-2\varepsilon^\gamma)}{2\varepsilon}}$. The value $\alpha=\varepsilon^{1-\gamma}$ then leads to

$$\|\theta^{\varepsilon,2}(\cdot,t)\|_{L^2(0,1)}\le\|\theta_0^{\varepsilon,2}\|_{L^2(1-2\varepsilon^\gamma,1)}\,e^{\frac{C}{2}t}\,e^{M_0}e^{\frac{M_0^2\varepsilon^{1-2\gamma}}{4}t}e^{-\frac{M_0^2}{2\varepsilon^\gamma}t}\le c_3\varepsilon^{1/2}e^{-\frac{M_0^2}{2\varepsilon^\gamma}t}\tag{42}$$

for some constant $c_3>0$, using that $\gamma\in(0,1/2]$. From inequalities (40) and (42) we deduce (36). The initial layer concentrated in the neighborhood of $x=1$ vanishes exponentially fast with respect to t since the velocity transport is strictly positive. □

3.1.2 Estimate of $\|L_\varepsilon P^\varepsilon\|_{L^1(0,t;L^2(0,1))}$

Recall that L_ε is the differential operator defined by (35). We have the following result proven in Appendice A.1.

LEMMA 3.2. *Assume hypotheses of Theorem 1.1. The function P^ε defined by (32) satisfies*

$$L_\varepsilon(P^\varepsilon) = \sum_{i=1}^{5} I_i^\varepsilon \text{ in } (0,1)\times(0,T),$$

with

$$\begin{aligned}
I_1^\varepsilon(x,t) =& -\varepsilon y^0_{xx}(x,t),\\
I_2^\varepsilon(x,t) =& \varepsilon^{1/2}\Big(M_{xx}(\kappa_1,t) - M_{xx}(a(t),t)\Big)\frac{w^2}{2}W^0_w(w,t) + \varepsilon M_{xx}(\kappa_1,t)\frac{w^2}{2}\left(W^{1/2}_w(w,t) - y^0_x(a(t)^\pm,t)\right),\\
I_3^\varepsilon(x,t) =& -y^0_t(1,t)e^{-M(1,t)z} + \Big(M_t(1,t) + M(1,t)M_x(\kappa_2,t)\Big)\Big(y^0(1,t) - y^0(a(t)^\pm,t)\Big)ze^{-M(1,t)z}\\
&+\Big(M_t(1,t) + M(1,t)M_x(\kappa_2,t)\Big)W^0(\tau M(1,t),t)ze^{-M(1,t)z}\\
&-\left(W^0_t(\tau M(1,t),t) + \left(\tau M_t(1,t)) - \frac{M(1,t)}{\varepsilon^{1/2}}\right)W^0_w(\tau M(1,t),t)\right)e^{-M(1,t)z},\\
I_4^\varepsilon(x,t) =& -\varepsilon^{1/2}\left(W^0_{wt}(\tau M(1,t),t) + \left(\tau M_t(1,t) - \frac{M(1,t)}{\varepsilon^{1/2}}\right)W^0_{ww}(\tau M(1,t),t)\right)ze^{-M(1,t)z}\\
&-\varepsilon^{1/2}M_x(\kappa_2,t)W^0_w(\tau M(1,t),t)ze^{-M(1,t)z}\\
&+\varepsilon^{1/2}\Big(M_t(1,t) + M(1,t)M_x(\kappa_2,t)\Big)W^0_w(\tau M(1,t),t)z^2e^{-M(1,t)z},\\
I_5^\varepsilon(x,t) =& \varepsilon^{1/2}\Big(M_t(1,t) + M(1,t)M_x(\kappa_2,t)\Big)C_{1/2}(0,\tau,t)ze^{-M(1,t)z}\\
&-\varepsilon^{1/2}\left(C_{1/2,t}(0,\tau,t) - \frac{C_{1/2,\tau}(0,\tau,t)}{\varepsilon^{1/2}}\right)e^{-M(1,t)z},
\end{aligned}$$

with $inf\{a(t), a(t)+\varepsilon^{1/2}w\} < \kappa_1 < sup\{a(t), a(t)+\varepsilon^{1/2}w\}$, $1-\varepsilon z < \kappa_2 < 1$.

For the previous expansion, we get the following estimate for the $L^1(0,T;L^2(0,1))$ norm. Details of the proof are given in Appendix A.3.

LEMMA 3.3. *Assume hypotheses of Theorem 1.1. Let P^ε be the function defined by (32). There exists a constant c independent of ε such that*

$$\|L_\varepsilon P^\varepsilon\|_{L^1(0,t;L^2(0,1))} \le c\,\varepsilon^{1/2}, \quad \forall t \in (0,T]. \tag{43}$$

3.1.3 Gronwall-Bellman type estimate

We now derive *a priori* estimates for the function z^ε. Preliminary, since z^ε is not vanishing at $x=0$ and $x=1$, we define

$$Z^\varepsilon(x,t) := z^\varepsilon(x,t) - z^\varepsilon(0,t)f^0_\varepsilon(x) - z^\varepsilon(1,t)f^1_\varepsilon(x), \quad (x,t)\in Q_T, \tag{44}$$

where $f^0_\varepsilon \in C^2([0,1],\mathbb{R}^+)$ and satisfies $f^0_\varepsilon(0)=1$ and $f^0_\varepsilon(1)=0$, and $f^1_\varepsilon(x) = f^0_\varepsilon(1-x)$. The function Z^ε solves the equation

$$\begin{cases} L_\varepsilon Z^\varepsilon = L_\varepsilon P^\varepsilon - L_\varepsilon(z^\varepsilon(0,t)f^0_\varepsilon(x)) - L_\varepsilon(z^\varepsilon(1,t)f^1_\varepsilon(x)), & (x,t)\in Q_T,\\ Z^\varepsilon(0,t) = Z^\varepsilon(1,t) = 0, & t\in(0,T),\\ Z^\varepsilon(x,0) = -z^\varepsilon(0,0)f^0_\varepsilon(x) - z^\varepsilon(1,0)f^1_\varepsilon(x), & x\in(0,1),\end{cases} \tag{45}$$

where $z^\varepsilon(0,0) := \lim_{t\to0^+} z^\varepsilon(0,t)$, $z^\varepsilon(1,0) := \lim_{t\to0^+} z^\varepsilon(1,t)$. Recall that L_ε is defined by (35).

In order to derive an estimate for Z^ε, we shall use the following Gronwall-Bellman type inequality (see [21]).

LEMMA 3.4. *Let $I = [t_0, T] \subset \mathbb{R}$, $k, b, p \in \mathcal{C}(I, \mathbb{R}^+)$. If $\zeta \in \mathcal{C}(I, \mathbb{R}^+)$ satisfies*

$$\zeta(t) \leq k(t) + \int_{t_0}^{t} b(s)\zeta(s)\, ds + \int_{t_0}^{t} p(s)\zeta^{\gamma}(s)\, ds, \quad t \in I,$$

with $0 \leq \gamma < 1$, then for $t \in I$,

$$\zeta(t) \leq \left[A^{1-\gamma}(t) + (1-\gamma) \int_{t_0}^{t} e^{(\gamma-1)\int_{t_0}^{t} b(\sigma)\, d\sigma} p(s)\, ds \right]^{\frac{1}{1-\gamma}} e^{\int_{t_0}^{t} b(s)\, ds},$$

where $A(t) = \max_{t_0 \leq s \leq t} k(s)$.

LEMMA 3.5. *Let Z^ε be the function defined by (44) and P^ε the function defined by (32). There is a constant c independent of ε such that, for each t in $[0, T]$,*

$$\begin{aligned} &\|Z^\varepsilon(\cdot,t)\|^2_{L^2(0,1)} + 2\varepsilon \int_0^t \|Z^\varepsilon_x(\cdot,s)\|^2_{L^2(0,1)}\, ds \\ &\leq \left[\left(|z^\varepsilon(0,0)|^2 \|f^0_\varepsilon\|^2_{L^2(0,1)} + |z^\varepsilon(1,0)|^2 \|f^1_\varepsilon\|^2_{L^2(0,1)} \right)^{1/2} \right. \\ &\left. + e^{\frac{Ct}{2}} \int_0^t \left(\int_0^1 \left(L_\varepsilon P^\varepsilon(x,s) - L_\varepsilon(f^0_\varepsilon(x) z^\varepsilon(0,s) - L_\varepsilon(z^\varepsilon(1,s) f^1_\varepsilon(x))) \right)^2 dx \right)^{1/2} ds \right]^2 e^{Ct}, \end{aligned} \tag{46}$$

with $C := \max_{(x,t)\in\overline{Q_T}} |M_x(x,t)|$.

Proof. Multiplying equation (45) by Z^ε and integrating over $(0,1) \times (0,t)$ gives

$$\begin{aligned} &\frac{1}{2}\|Z^\varepsilon(\cdot,t)\|^2_{L^2(0,1)} + \varepsilon \int_0^t \|Z^\varepsilon_x(\cdot,s)\|^2_{L^2(0,1)}\, ds \\ &= \frac{1}{2} \int_0^t\!\!\int_0^1 M_x(x,s) Z^\varepsilon(x,s)^2\, dxds + \frac{1}{2}|z^\varepsilon(0,0)|^2 \|f^0_\varepsilon\|^2_{L^2(0,1)} - \frac{1}{2}|z^\varepsilon(1,0)|^2 \|f^1_\varepsilon\|^2_{L^2(0,1)} \\ &\quad + \int_0^t\!\!\int_0^1 \left(L_\varepsilon P^\varepsilon(x,s) - L_\varepsilon(z^\varepsilon(0,s) f^0_\varepsilon(x)) - L_\varepsilon(z^\varepsilon(1,s) f^1_\varepsilon(x)) \right) Z^\varepsilon(x,s)\, dxds. \end{aligned}$$

Applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} &\|Z^\varepsilon(\cdot,t)\|^2_{L^2(0,1)} + 2\varepsilon \int_0^t \|Z^\varepsilon_x(\cdot,s)\|^2_{L^2(0,1)}\, ds \leq C \int_0^t\!\!\int_0^1 Z^\varepsilon(x,s)^2\, dxds \\ &+ |z^\varepsilon(0,0)|^2 \|f^0_\varepsilon\|^2_{L^2(0,1)} + |z^\varepsilon(1,0)|^2 \|f^1_\varepsilon\|^2_{L^2(0,1)} \\ &+ 2\int_0^t \left(\int_0^1 \left(L_\varepsilon P^\varepsilon(x,s) - L_\varepsilon(z^\varepsilon(0,s) f^0_\varepsilon(x)) - L_\varepsilon(z^\varepsilon(1,s) f^1_\varepsilon(x)) \right)^2 dx \right)^{1/2} \left(\int_0^1 Z^\varepsilon(x,s)^2\, dx \right)^{1/2} ds, \end{aligned}$$

with $C := \max_{(x,s)\in\overline{Q_T}} |M_x(x,s)|$. Applying Lemma 3.4 with

$$\begin{aligned} &t_0 = 0,\ b(s) = C,\ \gamma = \frac{1}{2}, \quad A = k = |z^\varepsilon(0,0)|^2 \|f^0_\varepsilon\|^2_{L^2(0,1)} + |z^\varepsilon(1,0)|^2 \|f^1_\varepsilon\|^2_{L^2(0,1)}, \\ &\zeta(t) = \|Z^\varepsilon(\cdot,t)\|^2_{L^2(0,1)} + 2\varepsilon \int_0^t \|Z^\varepsilon_x(\cdot,s)\|^2_{L^2(0,1)}\, ds, \\ &p(s) = 2\left(\int_0^1 \left(L_\varepsilon P^\varepsilon(x,s) - L_\varepsilon(z^\varepsilon(0,s) f^0_\varepsilon(x) - L_\varepsilon(z^\varepsilon(1,s) f^1_\varepsilon(x)) \right)^2 dx \right)^{1/2}, \end{aligned}$$

we obtain (46). □

3.1.4 Estimate of $\|z^\varepsilon(x_0,\cdot)\|_{L^1(0,s)}$, $\|z_t^\varepsilon(x_0,\cdot)\|_{L^1(0,s)}$ and $|z^\varepsilon(x_0,0)|$, $x_0=0,1$

We now estimate each term of the right-hand side of (46).

LEMMA 3.6. *There is a constant c independent of ε such that, $\forall s\in(0,T]$,*

$$\|z^\varepsilon(0,\cdot)\|_{L^1(0,s)}+\|z^\varepsilon(1,\cdot)\|_{L^1(0,s)}\le c\varepsilon, \tag{47}$$
$$\|z_t^\varepsilon(0,\cdot)\|_{L^1(0,s)}+\|z_t^\varepsilon(1,\cdot)\|_{L^1(0,s)}\le c, \tag{48}$$
$$|z^\varepsilon(x_0,0)|\le c, \tag{49}$$

with $z^\varepsilon(x_0,0):=\lim_{t\to0}z^\varepsilon(x_0,t),\ x_0=0,1$.

Proof. Let $s\in(0,T]$ be arbitrary, and $w_0(t)=-\frac{a(t)}{\varepsilon^{1/2}},\ w_1(t)=\frac{1-a(t)}{\varepsilon^{1/2}}$, $t\in[0,T]$. We have from (33) that

$$\begin{aligned}z^\varepsilon(0,t)&=W^0(w_0(t),t)-v(0)-\Big(y_0(X(0;1,t))+W^0(\tau M(1,t),t)-y_0(0)\Big)e^{-\frac{M(1,t)}{\varepsilon}}\\&\quad+\varepsilon^{1/2}\left(W^{1/2}(w_0(t),t)-y_x^0(a(t)^\pm,t)w_0(t)\right)-\varepsilon^{1/2}\Big(C_{1/2}(0,\tau,t)+\frac{1}{\varepsilon}W_w^0(\tau M(1,t),t)\Big)e^{-\frac{M(1,t)}{\varepsilon}}\\&=W^0(w_0(t),t)-v(0)+\varepsilon^{1/2}\left(W^{1/2}(w_0(t),t)-y_x^0(a(t)^\pm,t)w_0(t)\right)+\mathcal{O}(e^{-\frac{M(1,t)}{\varepsilon}}).\end{aligned}$$

We have

$$|W^0(w_0(t),t)-v(0)|=\frac{|y_0(0)-v(0)|}{2}\,erfc\Big(\frac{-w_0(t)}{2\beta(t)\sqrt{B(t)}}\Big)\le|y_0(0)-v(0)|e^{-\frac{a^2(t)}{4\varepsilon\beta^2(t)B(t)}},$$

then we deduce that

$$\left\|W^0(w_0(t),t)-v(0)\right\|_{L^1(0,s)}\le c\varepsilon.$$

By similar arguments to that used in the proof of Lemma 3.3 (point (e.2)), we show that $\|W^{1/2}(w_0(t),t)-y_x^0(a(t)^\pm,t)w_0(t)\|_{L^1(0,s)}\le c\varepsilon^{1/2}$ allowing to conclude that $\|z^\varepsilon(0,\cdot)\|_{L^1(0,s)}\le c\varepsilon$.

We also have

$$\begin{aligned}z^\varepsilon(1,t)&=W^0(w_1(t),t)-W^0(\tau M(1,t),t)+\varepsilon^{1/2}\left(W^{1/2}(w_1(t),t)-y_x^0(a(t)^\pm,t)w_1(t)\right)\\&\quad-\varepsilon^{1/2}\left(W^{1/2}(\tau M(1,t),t)-\tau M(1,t)y_x^0(a(t)^\pm,t)\right).\end{aligned}$$

Writing $W^0(w_1(t),t)-W^0(\tau M(1,t),t)=(W^0(w_1(t),t)-v(0))-(W^0(\tau M(1,t),t)-v(0)$, and arguing as above we get that

$$\left\|W^0(w_1(t),t)-v(0)\right\|_{L^1(0,s)}+\left\|(W^0(\tau M(1,t),t)-v(0)\right\|_{L^1(0,s)}\le c\varepsilon.$$

Similarly as above we have

$$\begin{aligned}&\|W^{1/2}(w_1(t),t)-y_x^0(a(t)^\pm,t)w_1(t)\|_{L^1(0,s)}\le c\varepsilon^{1/2},\\&\|W^{1/2}(\tau M(1,t)),t)-y_x^0(a(t)^\pm,t)\tau M(1,t)\|_{L^1(0,s)}\le c\varepsilon^{1/2},\end{aligned}$$

then conclude that $\|z^\varepsilon(1,\cdot)\|_{L^1(0,s)}\le c\varepsilon$. Inequality (47) follows.

Concerning the second inequality, we have

$$\begin{aligned}z_t^\varepsilon(1,t)&=W_t^0(w_1(t),t)+w_1'(t)W_w^0(w_1(t),t)\\&\quad-\left(W_t^0(\tau M(1,t),t)+\left(\tau M_t(1,t)-\frac{M(1,t)}{\varepsilon^{1/2}}\right)W_w^0(\tau M(1,t),t)\right)\\&\quad+\varepsilon^{1/2}W_t^{1/2}(w_1(t),t)+\varepsilon^{1/2}w_1'(t)\left(W_w^{1/2}(w_1(t),t)-y_x^0(a(t)^\pm,t)\right)+\varepsilon^{1/2}M_x(a(t),t)y_x^0(a(t)^\pm,t)w_1(t)\\&\quad-\varepsilon^{1/2}W_t^{1/2}(\tau M(1,t),t)-\varepsilon^{1/2}\left(\tau M_t(1,t)-\frac{M(1,t)}{\varepsilon^{1/2}}\right)\left(W_w^{1/2}(\tau M(1,t),t),t)-y_x^0(a(t)^\pm,t)\right)\\&\quad-\varepsilon^{1/2}M_x(a(t),t)\,y_x^0\big(a(t)^\pm,t\big)\,\tau M(1,t),t).\end{aligned}$$

We have the following estimates, see the proof of Lemma 3.3 (points (c.3), (e.3), and (e.4), respectively),

$$\left\| \left(W_t^0(\tau M(1,t),t) + \left(\tau M_t(1,t) - \frac{M(1,t)}{\varepsilon^{1/2}}\right) W_w^0(\tau M(1,t),t)\right)\right\|_{L^1(0,s)} \le c,$$
$$\|W_t^{1/2}(\tau M(1,t),t)\|_{L^1(0,s)} \le c\varepsilon^{1/2},$$
$$\left\|W_w^{1/2}(\tau M(1,t),t) - y_x^0(a(t)^\pm,t)\right\|_{L^1(0,s)} \le c\varepsilon^{1/2}.$$

Then clearly,

$$\left\|\varepsilon^{1/2}\left(\tau M_t(1,t) - \frac{M(1,t)}{\varepsilon^{1/2}}\right)\left(W_w^{1/2}(\tau M(1,t),t),t) - y_x^0(a(t)^\pm,t)\right)\right\|_{L^1(0,s)} \le c\varepsilon^{1/2},$$
$$\left\|\varepsilon^{1/2}M_x(a(t),t)y_x^0(a(t)^\pm,t)\tau M(1,t),t)\right\|_{L^1(0,s)} + \left\|\varepsilon^{1/2}M_x(a(t),t)y_x^0(a(t)^\pm,t)w_1(t)\right\|_{L^1(0,s)} \le c.$$

We have from (74)

$$W_t^0(w,t) + CW_w^0(w,t) = \frac{c^- - c^+}{2}\frac{1}{2\sqrt{\pi}}\frac{2w\alpha(t)\gamma(t) + w - 2C\gamma(t)}{\gamma(t)^{3/2}}e^{-\frac{w^2}{4\gamma(t)}},$$

then taking $w = w_1(t) = \frac{1-a(t)}{\varepsilon^{1/2}}$ and $C = w_1'(t) = -\frac{a'(t)}{\varepsilon^{1/2}}$ there holds that

$$\|W_t^0(w,t) + CW_w^0(w,t)\|_{L^1(0,t)} = C\mathcal{O}(\varepsilon^{1/2}) + \mathcal{O}(\epsilon) = \mathcal{O}(1),$$

then $\left\|W_t^0(w_1(t),t) + w_1'(t)W_w^0(w_1(t),t)\right\|_{L^1(0,s)} \le c$. We show, as above, that

$$\|\varepsilon^{1/2}w_1'(t)\left(W_w^{1/2}(w_1(t),t) - y_x^0(a(t)^\pm,t)\right)\|_{L^1(0,s)} \le c\varepsilon^{1/2}.$$

As in the proof of Lemma 3.3 (point (e.3)), we get $\left\|\varepsilon^{1/2}W_t^{1/2}(w_1(t),t)\right\|_{L^1(0,s)} \le c\varepsilon$. Thus $\|z_t(1,\cdot)\|_{L^1(0,s)} \le c$. Similarly $\|z_t(0,\cdot)\|_{L^1(0,s)} \le c$, so that (48) follows. Eventually, we easily check that (49) holds. This ends the proof of the lemma. $\square$

3.2 End of the proof of Theorem 1.1

We are now in position to finish the proof of Theorem 1.1. It remains to choose the function $f_\varepsilon^0 \in \mathcal{C}^2([0,1])$, satisfying $f_\varepsilon^0(0) = 1$ and $f_\varepsilon^0(1) = 0$ so as to minimize the terms in the right side of (46), asymptotically with respect to ε. We consider the functions

$$f_\varepsilon^0(x) = (1-x)e^{-\frac{x}{\varepsilon}}, \quad f_\varepsilon^1(x) = xe^{-\frac{1-x}{\varepsilon}}, \quad x \in [0,1],$$

so that $\|f_\varepsilon^i\|_{L^2(0,1)} \le c\,\varepsilon^{1/2}$ and $\|-\varepsilon f_\varepsilon^{i\prime\prime} + Mf_\varepsilon^{i\prime}\|_{L^2(0,1)} \le c\,\varepsilon^{-1/2}$, for $i = 0,1$. It follows that, for $x_0 = 0,1$, and for each $t \in (0,T]$,

$$\|f_\varepsilon\|_{L^2(0,1)}\left(\|z_t^\varepsilon(x_0,\cdot)\|_{L^1(0,t)} + |z^\varepsilon(x_0,x_0)|\right) \le c\varepsilon^{1/2},$$
$$\|-\varepsilon f_\varepsilon^{i\prime\prime} + Mf_\varepsilon^{i\prime}\|_{L^2(0,1)}\|z^\varepsilon(x_0,\cdot)\|_{L^1(0,t)} \le c\varepsilon^{1/2}.$$

Coming back to Lemma 3.5, using the two previous estimates and Lemma 3.3, there holds that

$$\|Z^\varepsilon(\cdot,t)\|_{L^2(0,1)} + \varepsilon^{1/2}\|Z_x^\varepsilon\|_{L^2((0,1)\times(0,t))} \le c\varepsilon^{1/2}, \quad \forall t \in [0,T].$$

Now, since $z^\varepsilon(x,t) = Z^\varepsilon(x,t) + f_\varepsilon^0(x)z^\varepsilon(0,t) + f_\varepsilon^1(x)z^\varepsilon(1,t)$, we deduce that

$$\begin{aligned}\|z^\varepsilon(\cdot,t)\|_{L^2(0,1)} &\le \|Z^\varepsilon(\cdot,t)\|_{L^2(0,1)} + |z^\varepsilon(0,t)|\|f_\varepsilon^0\|_{L^2(0,1)} + |z^\varepsilon(1,t)|\|f_\varepsilon^1\|_{L^2(0,1)} \\ &\le c\varepsilon^{1/2} + c\big(|z^\varepsilon(0,t)| + |z^\varepsilon(1,t)|\big)\varepsilon^{1/2}, \quad \forall t \in [0,T]. \end{aligned} \tag{50}$$

Writing $z^\varepsilon(x_0,t) = z^\varepsilon(x_0,0) + \int_0^t z_t^\varepsilon(x_0,s)\,ds$ and using (49), there holds that

$$|z^\varepsilon(x_0,t)| \le |z^\varepsilon(x_0,0)| + \|z_t^\varepsilon(x_0,\cdot)\|_{L^1(0,t)} \le c, \quad \forall t \in [0,T].$$

These estimate allow to deduce from (50) that $\|z^\varepsilon(\cdot,t)\|_{L^2(0,1)} \le c\varepsilon^{1/2}$ for all $t \in [0,T]$. Finally, since $P^\varepsilon - y^\varepsilon = z^\varepsilon - \theta^\varepsilon$, using Lemma 3.1, we derive the estimate

$$\|P^\varepsilon(\cdot,t) - y^\varepsilon(\cdot,t)\|_{L^2(0,1)} \le c\,\varepsilon^{1/2} + c\,e^{-\frac{M(1,0)\varepsilon^\gamma}{\varepsilon}} + c\,\varepsilon^{1/2} e^{-\frac{M_0^2}{2\varepsilon^\gamma}t} \le c\,\varepsilon^{1/2} + c\,\varepsilon^{1/2} e^{-\frac{M_0^2}{2\varepsilon^\gamma}t}, \quad \forall t \in [0,T].$$

This ends the proof of Theorem 1.1. □

4 Some explicit examples

We explicit in this section some expansion of P^ε for some particular case of the function $M = M(x,t)$ then numerically evaluate the $L^\infty(0,T;L^2(0,1))$ norm of the difference $y^\varepsilon - P^\varepsilon$.

4.1 The case $M(x,t) = 1 + x$, $y_0(x) \equiv y_0$ in $(0,1)$, $v(t) \equiv v$ in $(0,T)$

The characteristic starting from $(0,0)$ is given by $x - a(t) = 0$ where the function $a(t)$ solves

$$\begin{cases} a'(t) = M(a(t),t), & t > 0, \\ a(0) = 0, \end{cases}$$

and is given by $a(t) = e^t - 1$. T_0 such that $a(T_0) = 1$ is then given by $T_0 = \ln 2$. Then,

$$\alpha(t) := M_x(a(t),t) = 1, \quad \beta(t) := e^{\int_0^t \alpha(s)\,ds} = e^t, \quad B(t) := \int_0^t \frac{1}{\beta(s)^2}\,ds = \frac{1 - e^{-2t}}{2},$$

leading to $2\beta(t)\sqrt{B(t)} = \sqrt{2(e^{2t}-1)}$. We obtain that y^0 defined by (7) is given by

$$y^0(x,t) = \begin{cases} y_0, & x + 1 > e^t, \\ v, & x + 1 < e^t, \end{cases}$$

and

$$W^0(w,t) = \frac{y_0 - v}{2} erf\left(\frac{w}{\sqrt{2(e^{2t}-1)}}\right) + \frac{y_0 + v}{2}, \quad W^{1/2}(w,t) = 0, \quad w = \frac{x + 1 - e^t}{\varepsilon^{1/2}}.$$

p^0 defined by (24) is given by $p^0(x,t) = W^0(w,t)$ while simply $p^{1/2}(x,t) = 0$. Then, we get

$$C_0(\tau,t) = W^0(2\tau,t), \quad C_{1/2}(z,\tau,t) = -zW_w^0(2\tau,t), \quad z = \frac{1-x}{\varepsilon}, \quad \tau = \frac{\ln 2 - t}{\varepsilon^{1/2}},$$

implying

$$P^0(x,t) = W^0(w,t) - W^0(2\tau,t)e^{-2z}, \quad P^{1/2}(x,t) = -ze^{-2z}W_w^0(2\tau,t).$$

Consequently, our approximation is given by

$$\boxed{\begin{aligned} &P^\varepsilon(x,t) = W^0(w,t) - W^0(2\tau,t)e^{-2z} - \varepsilon^{1/2} z e^{-2z} W_w^0(2\tau,t), \quad (x,t) \in Q_T, \\ &w = \frac{x+1-e^t}{\varepsilon^{1/2}}, z = \frac{1-x}{\varepsilon}, \quad \tau = \frac{\ln 2 - t}{\varepsilon^{1/2}}. \end{aligned}} \tag{51}$$

Remark that $\partial_w erf(\frac{w}{g(t)}) = \frac{2}{\sqrt{\pi}} \frac{e^{-\frac{w^2}{g(t)}}}{g(t)}$ so that $W_w^0(w,t) = \frac{y_0 - v}{\sqrt{\pi}} \frac{e^{-\frac{w^2}{\sqrt{2(e^{2t}-1)}}}}{\sqrt{2(e^{2t}-1)}}$. As an illustration, Figure 2 depicts the function P^ε in Q_T for $\varepsilon = 10^{-1}, 10^{-2}, 5 \times 10^{-3}$ with $T = 0.8 > T_0$. In particular, observe that P^ε does not vanish at $x = 0$ and $x = 1$ but gets small as ε decreases. Table 1 provides some numerical values of the norm $\|y^\varepsilon - P^\varepsilon\|_{L^\infty(0,T;L^2(0,1))}$ for several values of the parameter ε. The unknown solution y^ε of (1) is obtained here using a numerical approximation. We obtain $\|y^\varepsilon - P^\varepsilon\|_{L^\infty(0,T;L^2(0,1))} = \mathcal{O}(\varepsilon^{0.61})$ in full agreement with our estimate in Theorem 1.1.

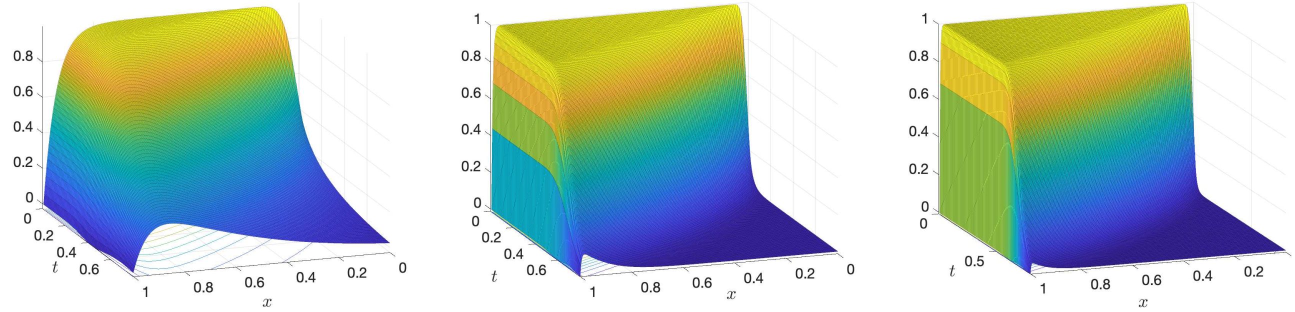

Figure 2: Function P^ε given by (51) in Q_T for $\varepsilon = 10^{-1}, 10^{-2}, 5\times 10^{-3}$.

ε	10^{-1}	5×10^{-2}	10^{-2}	5×10^{-3}	10^{-3}
	9.24×10^{-2}	6.14×10^{-2}	2.07×10^{-2}	1.4×10^{-2}	5.30×10^{-3}

Table 1: $\|y^\varepsilon - P^\varepsilon\|_{L^\infty(0,T;L^2(0,1))}$ w.r.t. ε for $T=0.8$, $y_0=1$ and $v=0$.

4.2 The case $M(x,t) = 1+t$, $y_0(x)\equiv y_0$ in $(0,1)$, $v(t)\equiv v$ in $(0,T)$

In this case, the function $a(t)$ is given by $a(t) = \frac{t(t+2)}{2}$. T_0 such that $a(T_0)=1$ is $T_0 = \sqrt{3}-1$. Then,

$$\alpha(t) := M_x(a(t),t) = 0, \quad \beta(t) := e^{\int_0^t \alpha(s)\,ds} = 1 \quad B(t) := \int_0^t \frac{1}{\beta(s)^2}\,ds = t,$$

leading to $\beta(t)\sqrt{B(t)} = \sqrt{t}$. We obtain that y^0 defined by (7) is given by

$$y^0(x,t) = \begin{cases} y_0, & x > \dfrac{t(t+2)}{2}, \\ v, & x < \dfrac{t(t+2)}{2}, \end{cases}$$

and

$$W^0(w,t) = \frac{y_0 - v}{2} erf\left(\frac{w}{2\sqrt{t}}\right) + \frac{y_0+v}{2}, \quad W^{1/2}(w,t) = 0, \quad w = \frac{2x - t(t+2)}{2\varepsilon^{1/2}}.$$

p^0 defined by (24) is given by $p^0(x,t) = W^0(w,t)$ while simply $p^{1/2}(x,t) = 0$. Then, we get

$$C_0(\tau,t) = W^0(2\tau,t), \quad C_{1/2}(z,\tau,t) = -zW^0_w(2\tau,t), \quad z = \frac{1-x}{\varepsilon}, \quad \tau = \frac{\sqrt{3}-1-t}{\varepsilon^{1/2}},$$

implying

$$P^0(x,t) = W^0(w,t) - W^0(1+t)\tau,t)e^{-(1+t)z}, \quad P^{1/2}(x,t) = -ze^{-(1+t)z}W^0_w((1+t)\tau,t).$$

Consequently, our approximation is given

$$\boxed{\begin{array}{l} P^\varepsilon(x,t) = W^0(w,t) - W^0((1+t)\tau,t)e^{-(1+t)z} - \varepsilon^{1/2}ze^{-(1+t)z}W^0_w((1+t)\tau,t), \\ w = \dfrac{2x-t(t+2)}{2\varepsilon^{1/2}}, z = \dfrac{1-x}{\varepsilon}, \quad \tau = \dfrac{\sqrt{3}-1-t}{\varepsilon^{1/2}}, \end{array}} \tag{52}$$

with here $W^0_w(w,t) = \frac{y_0-v}{\sqrt{\pi}}\frac{e^{-\frac{w^2}{2\sqrt{t}}}}{2\sqrt{t}}$.

5 Concluding remarks

We have performed an asymptotic analysis of the solution of an advection-diffusion parametrized by a small coefficient in front of the diffusion term. The analysis takes into account the intersection of several singular layers and leads to an approximation in an $\varepsilon^{1/2}$-neighborhood of the solution for the norm $L^\infty(0,T;L^2(0,1))$. This analysis extends [3] to non constant, strictly positive transport velocity, leading to much technical arguments. In particular, the solution in the internal layer (following the characteristic starting form $(0,0)$) solves a parabolic equation with non constant and unbounded coefficients related to the first derivatives of the velocity. The resulting solution is expressed in term of the heat kernel of the heat equation.

The $\varepsilon^{1/2}$ rate obtained for the final estimate is driven by our approximation W^0 in the internal layer, which depends on the choice non unique of the function g_0. The choice (19) we made here is simple but notably involves the properties $\lim_{t\to 0} W^0(\frac{-a(t)}{\varepsilon^{1/2}},t) = (y^0(0)+v(0))/2$ and $\lim_{x\to 0} W^0(\frac{x}{\varepsilon^{1/2}},0) = y^0(0)$. This gap at the point $(0,0)$ generates an artificial boundary layer in the approximation of the solution along the line $x=0$ above the characteristic $x-a(t)=0$ in the neighborhood of $t=0$. This boundary layer propagates inside the domain along the characteristic $x-a(t)=0$ and affects the quality of the approximation. By analogy with [3, Section 3] devoted to the constant velocity case, one may consider the more involved choice

$$g_0^\varepsilon(w) := \begin{cases} y_0(0), & w \geq 0, \\ v(0) + \dfrac{v(0)-y_0(0)}{2} e^{\frac{Fw}{\varepsilon^{1/2}}}, & w < 0, \end{cases} \tag{53}$$

with $F := \lim_{t\to 0^+} \frac{a(t)}{\beta(t)B(t)} = a'(0) > 0$ leading to $W_\varepsilon^0 = W^0 + U_\varepsilon^0$ with

$$U_\varepsilon^0(w,t) = \frac{v(0)-y_0(0)}{2} e^{\frac{Fw}{\beta(t)\varepsilon^{1/2}} + \frac{F^2 B(t)}{\varepsilon}} erfc\Big(\frac{w}{2\sqrt{B(t)}\beta(t)} + \frac{F\sqrt{B(t)}}{\varepsilon^{1/2}}\Big),$$

and W^0 defined by (20). The new function W_ε^0 still satisfies the asymptotic conditions $\lim_{w\to+\infty} W_\varepsilon^0(w,t) = y_0(0)$ and $\lim_{w\to-\infty} W_\varepsilon^0(w,t) = v(0)$ for all $t\in(0,T)$. Moreover, we now observe that

$$\lim_{t\to 0^+} W_0^\varepsilon\big(\frac{-a(t)}{\varepsilon^{1/2}},t\big) = v(0), \quad \lim_{x\to 0^+} W_\varepsilon^0\big(\frac{x}{\varepsilon^{1/2}},0\big) = y_0(0).$$

This allows to describe more accurately the discontinuity between the initial and Dirichlet conditions in the neighborhood of the point $(0,0)$. By analogy with [3], we may expect a $\mathcal{O}(\varepsilon^{3/4})$ rate, but this remains to be checked. Similarly, as in [3], in order to get a better rate, one would needs to consider additional terms in the developments, notably the outer term y^1 defined by (2.2) not used here, internal layer terms W^1, $W^{3/2}$ involving the heat kernel and boundary layer terms Y^1 and $Y^{3/2}$ solutions of ordinary differential equations with respect to the variable z. Whether or not we can perform the computations and get *a priori* estimates with these additional terms is an open issue.

In a futur work we plan to study the parabolic system describing the miscible displacement of compressible fluids in a porous medium. The displacement of one compressible fluid by another, completely miscible with the first, in a one-dimensional porous medium $\Omega=(0,1)$, assuming for simplicity that the two fluids have the same compressibility factor z, is described by the following differential system, see for instance [6],

$$z\partial_t p + \partial_x q = 0, \quad q = -\frac{\kappa}{\mu(c)}\partial_x p, \quad (x,t)\in Q_T, \tag{54}$$

$$\phi\partial_t c + q(x,t)\partial_x c - \partial_x\left(d(x,t)\partial_x c\right) = 0, \quad (x,t)\in Q_T, \tag{55}$$

to which initial and boundary conditions are added. The unknowns of system (54), (55) are the functions $p=p(x,t)$, $c=c(x,t)$ and $q=q(x,t)$; p is the pressure, c is the concentration of one of the two components of the fluid mixture, and q is the Darcy velocity. The function $\kappa=\kappa(x)$ is the permeability of the medium, the constant ϕ is the porosity, $\mu=\mu(c)$ is the concentration-dependent viscosity of the fluid mixture, $d=d(x,t)$ is a small term representing the molecular diffusion and dispersion in the porous medium. The viscosity

$\mu(c)$ is assumed to be determined by some mixing rule. See [4] for the existence of a weak solution to the one-dimensional equations governing compressible flows of m miscible components in a porous medium. Our aim is to perform an asymptotic analysis of system (54), (55) when d is small and the concentration c satisfies Dirichlet boundary conditions on $x = 0$ and $x = 1$.

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A Appendix

A.1 Proof of Lemma 3.2

According to (30) and (31),

$$\begin{aligned} P^\varepsilon(x,t) = P^0(x,t) + \varepsilon^{1/2}P^{1/2}(x,t) =& y^0(x,t) + W^0(w,t) - y^0(a(t)^\pm,t) \\ &- \Big(y^0(1,t) + W^0(\tau M(1,t),t) - y^0(a(t)^\pm,t)\Big)e^{-M(1,t)z} \\ &+ \varepsilon^{1/2}\left(W^{1/2}(w,t) - y^0_x(a(t)^\pm,t)w\right) \\ &- \varepsilon^{1/2}\left(C_{1/2}(0,\tau,t) + W^0_w(\tau M(1,t),t)z\right)e^{-M(1,t)z}. \end{aligned}$$

Let us note here that the function $P^0 + \varepsilon^{1/2}P^{1/2}$ belongs to $C^1([0,1]\times(0,T])$, and that $(P^0 - W^0) + \varepsilon^{1/2}(P^{1/2} - W^{1/2})$ belongs to $C^1(\overline{Q_T})$. Then, from the regularity assumptions on y_0, v and M, $(P^0 - W^0) + \varepsilon^{1/2}(P^{1/2} - W^{1/2})$ belongs to $H^2(Q_T)$. We have

$$\begin{aligned} L_\varepsilon(W^0) &= W^0_t(w,t) - \frac{a'(t)}{\varepsilon^{1/2}}W^0_w(w,t) + \frac{M(x,t)}{\varepsilon^{1/2}}W^0_w(w,t) - W^0_{ww}(w,t) \\ &= W^0_t(w,t) + \frac{M(\varepsilon^{1/2}w + a(t),t) - M(a(t),t)}{\varepsilon^{1/2}}W^0_w(w,t) - W^0_{ww}(w,t). \end{aligned}$$

Using the Taylor expansion

$$M(\varepsilon^{1/2}w + a(t),t) - M(a(t),t) = \varepsilon^{1/2}wM_x(a(t),t) + \varepsilon\frac{w^2}{2}M_{xx}(\kappa_1,t), \tag{56}$$

with $inf\{a(t), a(t)+\varepsilon^{1/2}w\} < \kappa_1 < sup\{a(t), a(t)+\varepsilon^{1/2}w\}$, and equation (10) we get

$$L_\varepsilon(W^0) = \varepsilon^{1/2}\frac{w^2}{2}M_{xx}(\kappa_1,t)W^0_w(w,t).$$

By similar calculations we get

$$L_\varepsilon(W^{1/2}) = -\frac{w^2}{2}M_{xx}(a(t),t)W^0_w(w,t) + \varepsilon^{1/2}\frac{w^2}{2}M_{xx}(\kappa_1,t)W^{1/2}_w(w,t).$$

We observe that, when calculating $L_\varepsilon(P^0 + \varepsilon^{1/2}P^{1/2})$, it suffices to perform the calculation in $\Omega^+\cup\Omega^-$, where $\Omega^+ = \{(x,t)\in Q_T : x > a(t)\}$ and $\Omega^- = \{(x,t)\in Q_T : x < a(t)\}$. Straightforward calculations then give

$$L_\varepsilon\Big(y^0(x,t) - y^0(a(t)^\pm,t)\Big) = -\varepsilon y^0_{xx}(x,t). \tag{57}$$

Moreover,

$$L_\varepsilon(y^0_x(a(t)^\pm,t)w) = w\Big(y^0_{xt}(a(t)^\pm,t) + a'(t)y^0_{xx}(a(t)^\pm,t)\Big) - \frac{a'(t)}{\varepsilon^{1/2}}y^0_x(a(t)^\pm,t) + \frac{M(x,t)}{\varepsilon^{1/2}}y^0_x(a(t)^\pm,t),$$

leading, using (56), to

$$L_\varepsilon(y^0_x(a(t)^\pm,t)w) = w\Big(y^0_{xt}(a(t)^\pm,t) + M(a(t),t)y^0_{xx}(a(t)^\pm,t)\Big) + wM_x(a(t),t)\, y^0_x(a(t)^\pm,t) + \varepsilon^{1/2}\frac{w^2}{2}M_{xx}(\kappa_1,t)\, y^0_x(a(t)^\pm,t).$$

Differentiating equation $(6)_1$ with respect to x yields

$$y^0_{xt}(x,t) + M_x(x,t)\, y^0_x(x,t) + M(x,t)\, y^0_{xx}(x,t) = 0 \quad \text{for } (x,t) \in \Omega^+ \cup \Omega^-.$$

Letting $x \to a(t)^+$ and $x \to a(t)^-$, we get

$$y^0_{xt}(a(t)^\pm,t) + M_x(a(t),t)\, y^0_x(a(t)^\pm,t) + M(a(t),t)\, y^0_{xx}(a(t)^\pm,t) = 0.$$

Therefore

$$L_\varepsilon(y^0_x(a(t)^\pm,t)w) = \varepsilon^{1/2}\frac{w^2}{2}M_{xx}(\kappa_1,t)\, y^0_x(a(t)^\pm,t).$$

It results that

$$\begin{aligned} &L_\varepsilon\left(W^0(w,t) + \varepsilon^{1/2}\left(W^{1/2}(w,t) - y^0_x(a(t)^\pm,t)w\right)\right) \\ &= \varepsilon\frac{w^2}{2}M_{xx}(\kappa_1,t)\left(W^{1/2}_w(w,t) - y^0_x(a(t)^\pm,t)\right) + \varepsilon^{1/2}\frac{w^2}{2}W^0_w(w,t)\Big(M_{xx}(\kappa_1,t) - M_{xx}(a(t),t)\Big). \end{aligned} \tag{58}$$

We have

$$L_\varepsilon\left(y^0(1,t)e^{-M(1,t)z}\right) = y^0_t(1,t)e^{-M(1,t)z} - y^0(1,t)\Big(M_t(1,t) + M(1,t)M_x(\kappa_2,t)\Big)ze^{-M(1,t)z},$$

$$L_\varepsilon\left(y^0(a(t)^\pm,t)e^{-M(1,t)z}\right) = -y^0(a(t)^\pm,t)\Big(M_t(1,t) + M(1,t)M_x(\kappa_2,t)\Big)ze^{-M(1,t)z},$$

$$\begin{aligned} L_\varepsilon\left(W^0(\tau M(1,t),t)e^{-M(1,t)z}\right) &= \left(W^0_t(\tau M(1,t),t) + \left(\tau M_t(1,t)) - \frac{M(1,t)}{\varepsilon^{1/2}}\right)W^0_w(\tau M(1,t),t)\right)e^{-M(1,t)z} \\ &- \Big(M_t(1,t) + M(1,t)M_x(\kappa_2,t)\Big)W^0(\tau M(1,t),t)ze^{-M(1,t)z}, \end{aligned}$$

with $1 - \varepsilon z < \kappa_2 < 1$. We deduce that

$$\begin{aligned} &L_\varepsilon\left(\Big(y^0(1,t) + W^0(\tau M(1,t),t) - y^0(a(t)^\pm,t)\Big)e^{-M(1,t)z}\right) \\ &= y^0_t(1,t)e^{-M(1,t)z} - \Big(M_t(1,t) + M(1,t)M_x(\kappa_2,t)\Big)\Big(y^0(1,t) - y^0(a(t)^\pm,t)\Big)ze^{-M(1,t)z} \\ &\quad + \left(W^0_t(\tau M(1,t),t) + \left(\tau M_t(1,t) - \frac{M(1,t)}{\varepsilon^{1/2}}\right)W^0_w(\tau M(1,t),t)\right)e^{-M(1,t)z} \\ &\quad - \Big(M_t(1,t) + M(1,t)M_x(\kappa_2,t)W^0(\tau M(1,t),t)\Big)ze^{-M(1,t)z}. \end{aligned} \tag{59}$$

We also have

$$\begin{aligned} &L_\varepsilon\left(W^0_w(\tau M(1,t),t)ze^{-M(1,t)z}\right) \\ &\quad = \left(W^0_{wt}(\tau M(1,t),t) + \left(\tau M_t(1,t)) - \frac{M(1,t}{\varepsilon^{1/2}}\right)W^0_{ww}(\tau M(1,t),t)\right)ze^{-M(1,t)z} \\ &\quad + M_x(\kappa_2,t)W^0_w(\tau M(1,t),t)ze^{-M(1,t)z} \\ &\quad - \Big(M_t(1,t) + M(1,t)M_x(\kappa_2,t)\Big)W^0_w(\tau M(1,t),t)z^2e^{-M(1,t)z}, \end{aligned}$$

and

$$L_\varepsilon\left(C_{1/2}(0,\tau,t)e^{-M(1,t)z}\right) = \left(C_{1/2,t}(0,\tau,t) - \frac{C_{1/2,\tau}(0,\tau,t)}{\varepsilon^{1/2}}\right)e^{-M(1,t)z} - C_{1/2}(0,\tau,t)\Big(M_t(1,t) + M(1,t)M_x(\kappa_2,t)\Big)ze^{-M(1,t)z}, \tag{60}$$

where $C_{1/2,t}(0,\tau,t)$, $C_{1/2,\tau}(0,\tau,t)$ are given by

$$C_{1/2,t}(0,\tau,t) = W_t^{1/2}(\tau M(1,t),t) + \tau M_t(1,t)\left(W_w^{1/2}(\tau M(1,t),t) - y_x^0(a(t)^\pm,t)\right) + \tau M(1,t)M_x(a(t),t)y_x^0(a(t)^\pm,t), \tag{61}$$

and

$$C_{1/2,\tau}(0,\tau,t) = M(1,t)\left(W_w^{1/2}(\tau M(1,t),t) - y_x^0(a(t)^\pm,t)\right). \tag{62}$$

Collecting equalities (57)–(60) we obtain the lemma.

A.2 An equality

Lemma A.1. *Let K, β and B be the functions defined in Lemma 2.2. Let $\gamma(\tau) := \beta^2(\tau)B(\tau)$ and*

$$f(\xi,\tau) := \frac{1}{2}\frac{1}{\sqrt{\pi}}\frac{1}{\gamma(\tau)^{1/2}}M_{xx}(a(\tau),\tau)\xi^2 e^{-\frac{\xi^2}{4\gamma(\tau)}}, \quad \tau\in(0,t), \xi\in\mathbb{R}.$$

For all $(w,t)\in\mathbb{R}\times(0,T)$,

$$\int_0^t\int_{\mathbb{R}} K(w,t;\xi,\tau)f(\xi,\tau)\,d\xi d\tau = \frac{1}{\pi}\frac{e^{-\frac{w^2}{4\beta^2(t)B(t)}}}{\beta^2(t)B^2(t)\sqrt{B(t)}}\int_0^t M_{xx}(a(\tau),\tau)\sqrt{B(\tau)}\Big(B(\tau)w^2 + 2\beta^2(t)B^2(t) - 2\beta^2(t)B(t)B(\tau)\Big)\times \times\sqrt{B(t)-B(\tau)}d\tau. \tag{63}$$

Proof. We have, for all $(w,t)\in\mathbb{R}\times(0,T)$,

$$\int_{\mathbb{R}} K(w,t;\xi,\tau)f(\xi,\tau)\,d\xi = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-v^2} f\left(\frac{\beta(\tau)}{\beta(t)}w + 2\beta(\tau)\sqrt{(B(t)-B(\tau))}\,v,\tau\right)dv = \frac{1}{\pi}\frac{1}{2\gamma(\tau)^{1/2}}M_{xx}(a(\tau),\tau)\int_{\mathbb{R}} e^{-v^2}\widetilde{f}\left(\frac{\beta(\tau)}{\beta(t)}w + 2\beta(\tau)\sqrt{B(t)-B(\tau)}\,v,\tau\right)dv,$$

with

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-v^2}\widetilde{f}\left(\frac{\beta(\tau)}{\beta(t)}w + 2\beta(\tau)\sqrt{B(t)-B(\tau)}\,v,\tau\right)dv \\
&= \int_{\mathbb{R}} e^{-v^2}e^{-\frac{\left(\frac{\beta(\tau)}{\beta(t)}w + 2\beta(\tau)\sqrt{B(t)-B(\tau)}\,v\right)^2}{4\gamma(\tau)}}\left(\frac{\beta(\tau)}{\beta(t)}w + 2\beta(\tau)\sqrt{B(t)-B(\tau)}\,v\right)^2 dv \\
&= 2e^{-\frac{w^2}{4\beta(t)^2B(t)}}\beta(\tau)\sqrt{B(\tau)}\Big(B(\tau)w^2 + 2\beta^2(t)B^2(t) - 2\beta^2(t)B(t)B(\tau)\Big)\frac{\sqrt{\beta^2(\tau)(B(t)-B(\tau))B(\tau)}}{\beta^2(t)B^2(t)\sqrt{B(t)}} \\
&= 2e^{-\frac{w^2}{4\beta(t)^2B(t)}}\beta^2(\tau)B(\tau)\Big(B(\tau)w^2 + 2\beta^2(t)B^2(t) - 2\beta^2(t)B(t)B(\tau)\Big)\frac{\sqrt{B(t)-B(\tau)}}{\beta^2(t)B^2(t)\sqrt{B(t)}},
\end{aligned}$$

leading to (63). □

A.3 Proof of Lemma 3.3

Let $s \in (0,T]$ be arbitrary. We estimate the $L^1(0,s;L^2(0,1))$ norm of each term I_i^ε of the expression $L_\varepsilon(P^\varepsilon) = \sum_{i=1}^5 I_i^\varepsilon$, given by Lemma 3.2. In particular, we use that

$$\|z(x)^n e^{-M(1,t)z(x)}\|_{L^\infty(0,s;L^2(\Omega))} = \mathcal{O}(\varepsilon^{1/2}) \text{ for } n = 0,1 \text{ and } z(x) = \frac{1-x}{\varepsilon}. \tag{64}$$

Formula (63) will also be used several times. To avoid repetition, some points of the proof are shortened.

(a) Estimate of $\|I_1^\varepsilon\|_{L^1(0,s;L^2(\Omega))}$. Clearly,

$$\|I_1^\varepsilon\|_{L^1(0,s;L^2(\Omega))} \leq c\varepsilon. \tag{65}$$

(b) Estimate of $\|I_2^\varepsilon\|_{L^1(0,s;L^2(\Omega))}$.

(b.1) Estimate of $\left\|\varepsilon^{1/2}\Big(M_{xx}(\kappa_1,t) - M_{xx}(a(t),t)\Big)\frac{w^2}{2}W_w^0(w,t)\right\|_{L^1(0,s;L^2(\Omega))}$. Let us denote

$$J_1 := \varepsilon \int_0^s \int_0^1 \left| \frac{1}{2}\left(\frac{x-a(t)}{\varepsilon^{1/2}}\right)^2 W_w^0\left(\frac{x-a(t)}{\varepsilon^{1/2}}, t\right)\right|^2 dxdt.$$

Using the change of variable $\frac{x-a(t)}{\varepsilon^{1/2}} = w$, there holds that

$$J_1 = \varepsilon^{3/2} \int_0^s \int_{-\frac{a(t)}{\varepsilon^{1/2}}}^{\frac{1-a(t)}{\varepsilon^{1/2}}} \left|\frac{w^2}{2}W_w^0(w,t)\right|^2 dwdt \leq \varepsilon^{3/2}\int_0^s\int_{-\infty}^{+\infty}\left|\frac{w^2}{2}W_w^0(w,t)\right|^2 dwdt,$$

provided that the last integral is finite. Using the explicit expression

$$W_w^0(w,t) = \frac{y_0(0)-v(0)}{2}\frac{2}{\sqrt{\pi}}\left(\frac{1}{2\beta(t)\sqrt{B(t)}}\right)e^{-\frac{w^2}{4\beta(t)^2 B(t)}},$$

and the change of variable $\frac{w}{2\beta(t)\sqrt{B(t)}} = v$, there holds that

$$J_1 \leq \varepsilon^{3/2}\int_0^s\int_{-\infty}^{+\infty} 8\beta(t)^2 B(t)^{3/2}\left|\left(\frac{c^+-c^-}{2}\right)\frac{1}{\sqrt{\pi}}v^2 e^{-v^2}\right|^2 dvdt \leq c\varepsilon^{3/2}.$$

We conclude that

$$\left\|\varepsilon^{1/2}\Big(M_{xx}(\kappa_1,t) - M_{xx}(a(t),t)\Big)\frac{w^2}{2}W_w^0(w,t)\right\|_{L^1(0,s;L^2(\Omega))} \leq c\varepsilon^{3/4}. \tag{66}$$

(b.2) Estimate of $\left\|\varepsilon\frac{w^2}{2}M_{xx}(\kappa_1,t)\left(W_w^{1/2}(w,t) - y_x^0(a(t)^\pm,t)\right)\right\|_{L^1(0,s;L^2(\Omega))}$. Let us denote

$$J_2 := \varepsilon^2\int_0^s\int_0^1\left|\frac{1}{2}\left(\frac{x-a(t)}{\varepsilon^{1/2}}\right)^2\left(W_w^{1/2}\left(\frac{x-a(t)}{\varepsilon^{1/2}},t\right) - y_x^0(a(t)^\pm,t)\right)\right|^2 dxdt.$$

Using the change of variable $\frac{x-a(t)}{\varepsilon^{1/2}} = w$, there holds that

$$\begin{aligned} J_2 &= \varepsilon^{5/2}\int_0^s\int_{-\frac{a(t)}{\varepsilon^{1/2}}}^{\frac{1-a(t)}{\varepsilon^{1/2}}}\left|\frac{w^2}{2}\left(W_w^{1/2}(w,t) - y_x^0(a(t)^\pm,t)\right)\right|^2 dwdt \\ &\leq \varepsilon^{5/2}\int_0^s\int_{-\infty}^{+\infty} w^4\left|W_w^{1/2}(w,t) - y_x^0(a(t)^\pm,t)\right|^2 dwdt, \end{aligned}$$

provided that the last integral is finite. We have from (23)

$$W_w^{1/2}(w,t) = \widetilde{g}_{1/2,w}(w,t) + \int_0^t \int_{\mathbb{R}} K_w(w,t;\xi,\tau)\Big(f_{1/2}(\xi,\tau) - g^\star_{1/2}(\xi,\tau)\Big)\, d\xi d\tau, \quad w\in\mathbb{R}, t\in(0,T),$$

where $\widetilde{g}_{1/2}$ is defined by (21) and $g^\star_{1/2}$ is defined by (22). Writing

$$W_w^{1/2}(w,t) - y_x^0(a(t)^\pm,t) = \Big(W_w^{1/2}(w,t) - \widetilde{g}_{1/2,w}(w,t)\Big) + \Big(\widetilde{g}_{1/2,w}(w,t) - y_x^0(a(t)^\pm,t)\Big),$$

and noting that $\widetilde{g}_{1/2,w}(w,t) - y_x^0(a(t)^\pm,t) = 0$ for $|w|\geq 2$, we get

$$J_2 \leq \varepsilon^{5/2} \int_0^s \int_{-\infty}^{+\infty} w^4 \left| W_w^{1/2}(w,t) - \widetilde{g}_{1/2,w}(w,t)\right|^2 dwdt + \varepsilon^{5/2}.$$

Let us show that the integral $\widetilde{J}_2 := \int_0^s \int_{-\infty}^{+\infty} w^4 \left| W_w^{1/2}(w,t) - \widetilde{g}_{1/2,w}(w,t)\right|^2 dwdt$ is finite. We have

$$\widetilde{J}_2 = \int_0^s \int_{-\infty}^{+\infty} w^4 \left| \int_0^t \int_{-\infty}^{+\infty} K_w(w,t;\xi,\tau)\Big(f_{1/2}(\xi,\tau) - g^\star_{1/2}(\xi,\tau)\Big) d\xi d\tau\right|^2 dwdt.$$

Consider the integral $\widetilde{J}_2^1 := \int_0^s \int_{-\infty}^{+\infty} w^4 \left| \int_0^t \int_{-\infty}^{+\infty} K_w(w,t;\xi,\tau) f_{1/2}(\xi,\tau)\, d\xi d\tau\right|^2 dwdt$. Differentiating identity (63) in w gives

$$\int_0^t \int_{-\infty}^{+\infty} K_w(w,t;\xi,\tau) f_{1/2}(\xi,\tau)\, d\xi d\tau = A_1(w,t) + A_2(w,t),$$

with

$$\begin{aligned} A_1(w,t) :=& \frac{v(0)-y_0(0)}{\pi} \frac{(-w)e^{-\frac{w^2}{4\beta^2(t)B(t)}}}{2\beta^4(t)B^3(t)\sqrt{B(t)}} \times \\ &\times \int_0^t M_{xx}(a(\tau),\tau)\sqrt{B(\tau)}\Big(B(\tau)w^2 + 2\beta^2(t)B^2(t) - 2\beta^2(t)B(t)B(\tau)\Big)\sqrt{B(t)-B(\tau)}d\tau, \end{aligned} \tag{67}$$

$$A_2(w,t) := \frac{v(0)-y_0(0)}{\pi} \frac{2w\, e^{-\frac{w^2}{4\beta(t)^2B(t)}}}{\beta^2(t)B^2(t)\sqrt{B(t)}} \int_0^t M_{xx}(a(\tau),\tau)\sqrt{B(\tau)}B(\tau)\sqrt{B(t)-B(\tau)}d\tau. \tag{68}$$

It results that

$$\widetilde{J}_2^1 \leq 2\int_0^s \int_{-\infty}^{+\infty} w^4\big(\left|A_1(w,t)\right|^2 + \left|A_2(w,t)\right|^2\big)\, dwdt. \tag{69}$$

Using the change of variable $\frac{w}{2\beta(t)\sqrt{B(t)}} = v$, there holds that

$$\begin{aligned} \int_0^s \int_{-\infty}^{+\infty} w^4 \left|A_1(w,t)\right|^2 dwdt =& \int_0^s \int_{-\infty}^{+\infty} 32\left(\frac{v(0)-y_0(0)}{\pi}\right)^2 \frac{v^6 e^{-2v^2}}{\beta(t)B^{7/2}(t)} \times \\ \times \bigg| \int_0^t & M_{xx}(a(\tau),\tau)\sqrt{B(\tau)}\Big(B(\tau)(4\beta^2(t)B(t)v^2) + 2\beta^2(t)B^2(t) - 2\beta^2(t)B(t)B(\tau)\Big)\times \\ &\times \sqrt{B(t)-B(\tau)}d\tau\bigg|^2 dvdt, \end{aligned}$$

then, using that B is strictly increasing, it results that

$$\begin{aligned} &\int_0^s \int_{-\infty}^{+\infty} w^4 \left|A_1(w,t)\right|^2 dwdt \\ &\leq c\int_0^s \int_{-\infty}^{+\infty} \beta^3(t)B^{3/2}(t)\left(\int_0^t |M_{xx}(a(\tau),\tau)|\sqrt{B(t)-B(\tau)}d\tau\right)^2 v^6(v^2+1)^2 e^{-2v^2} dvdt, \end{aligned}$$

hence $\int_0^s \int_{-\infty}^{+\infty} w^4 \big|A_1(w,t)\big|^2 \, dwdt < \infty$. Similarly,

$$\int_0^s \int_{-\infty}^{+\infty} w^4 \big|A_2(w,t)\big|^2 \, dwdt$$
$$\le c \int_0^s \int_{-\infty}^{+\infty} \beta^3(t) B^{3/2}(t) \left(\int_0^t |M_{xx}(a(\tau),\tau)| \sqrt{B(t)-B(\tau)} d\tau \right)^2 v^6 e^{-2v^2} \, dvdt,$$

hence $\int_0^s \int_{-\infty}^{+\infty} w^4 \big|A_2(w,t)\big|^2 \, dwdt < \infty$. It results from (69) that the integral $\widetilde{J}_2^1$ is finite.

Consider the integral $\widetilde{J}_2^2 := \int_0^s \int_{-\infty}^{+\infty} w^4 \left| \int_0^t \int_{-\infty}^{+\infty} K_w(w,t;\xi,\tau) g^\star_{1/2}(\xi,\tau) \, d\xi d\tau \right|^2 \, dwdt$. Since $g^\star_{1/2}(\xi,\tau) = 0$ for $|w| \ge 2$ we have

$$\widetilde{J}_2^2 = \int_0^s \int_{-\infty}^{+\infty} w^4 \left| \int_0^t \int_{-2}^{+2} K_w(w,t;\xi,\tau) g^\star_{1/2}(\xi,\tau) \, d\xi d\tau \right|^2 \, dwdt.$$

Explicitly,

$$K_w(w,t;\xi,\tau) = \frac{-\left(\frac{w}{\beta(t)} - \frac{\xi}{\beta(\tau)}\right)}{4\sqrt{\pi}\beta(t)\beta(\tau)(B(t)-B(\tau))^{3/2}} e^{-\frac{\left(\frac{w}{\beta(t)} - \frac{\xi}{\beta(\tau)}\right)^2}{4(B(t)-B(\tau))}}, \quad (w,\xi) \in \mathbb{R}^2, \ 0 \le \tau < t \le T,$$

then, using the change of variable $\frac{\frac{\xi}{\beta(\tau)} - \frac{w}{\beta(t)}}{2\sqrt{(B(t)-B(\tau))}} = v$ gives

$$\int_{-2}^{+2} K_w(w,t;\xi,\tau) g^\star_{1/2}(\xi,\tau) \, d\xi$$
$$= \frac{1}{4\sqrt{\pi}\beta(t)\beta(\tau)(B(t)-B(\tau))} \int_{a(w,t,\tau)}^{b(w,t,\tau)} 2ve^{-v^2} g^\star_{1/2}\left(\frac{\beta(\tau)}{\beta(t)} w + 2\beta(\tau)\sqrt{B(t)-B(\tau)}\, v, \tau \right) dv,$$

with $a(w,t,\tau) := \frac{-2 - \frac{\beta(\tau)}{\beta(t)} w}{2\beta(\tau)\sqrt{B(t)-B(\tau)}}$ and $b(w,t,\tau) := \frac{2 - \frac{\beta(\tau)}{\beta(t)} w}{2\beta(\tau)\sqrt{B(t)-B(\tau)}}$. Since $g^\star_{1/2}(\xi,\tau)$ is uniformly bounded, we have

$$\left| \int_{-2}^{+2} K_w(w,t;\xi,\tau) g^\star_{1/2}(\xi,\tau) \, d\xi \right| \le \frac{c}{\beta(t)\beta(\tau)(B(t)-B(\tau))} \int_{a(w,t,\tau)}^{b(w,t,\tau)} 2|v| e^{-v^2} \, dv.$$

There are two positive constants c_1 and c_2 such that $c_1 \le \frac{\beta(\tau)}{\beta(t)} \le c_2, \quad 0 \le \tau < t \le T$. Let $w_1 = \frac{4}{c_1}$, and $w_2 = -\frac{4}{c_2}$. We observe that:

$$w \ge w_1 \Longrightarrow a(w,t,\tau) \le b(w,t,\tau) \le 0; \quad w \le -w_2 \Longrightarrow b(w,t,\tau) \ge a(w,t,\tau) \ge 0,$$

then deduce that:

$$\text{for } w \ge w_1, \quad \int_{a(w,t,\tau)}^{b(w,t,\tau)} 2|v| e^{-v^2} \, dv = e^{-b(w,t,\tau)^2} - e^{-a(w,t,\tau)^2};$$
$$\text{for } w \le -w_2, \quad \int_{a(w,t,\tau)}^{b(w,t,\tau)} 2|v| e^{-v^2} \, dv = e^{-a(w,t,\tau)^2} - e^{-b(w,t,\tau)^2}.$$

For $w \ge w_1$, using the inequality $-2 - \frac{\beta(\tau)}{\beta(t)} w \le -\frac{\beta(\tau)}{\beta(t)} w$, we deduce that $e^{-a(w,t,\tau)^2} \le e^{-\frac{(c_1 w)^2}{4\beta^2(t)(B(t)-B(\tau))}}$. Then there is a positive constant c_3 such that

$$\frac{1}{\beta(t)\beta(\tau)(B(t)-B(\tau))} e^{-a(w,t,\tau)^2} \le c_3 e^{-\frac{(c_1 w)^2}{8\beta^2(t)(B(t)-B(\tau))}}. \tag{70}$$

We also have, for $w \ge w_1$, $\left(2 - \frac{\beta(\tau)}{\beta(t)} w\right)^2 \ge \left(\frac{1}{2}\frac{\beta(\tau)}{\beta(t)} w\right)^2 \ge \left(\frac{1}{2} c_1 w\right)^2$, then we deduce that

$$\frac{1}{\beta(t)\beta(\tau)(B(t)-B(\tau))} e^{-b(w,t,\tau)^2} \le c_3 e^{-\frac{(c_1 w)^2}{8\beta^2(t)(B(t)-B(\tau))}}. \tag{71}$$

Inequalities (70) and (71) hold similarly for $w \in (-\infty, w_2)$ so that, for $w \in (-\infty, w_2) \cup (w_1, \infty)$,

$$\left| \int_0^t \int_{-2}^{+2} K_w(w,t;\xi,\tau) g^\star_{1/2}(\xi,\tau)\, d\xi d\tau \right| \le c \int_0^t e^{-\frac{(c_1 w)^2}{8\beta^2(t)(B(t)-B(\tau))}}\, d\tau.$$

Using the Cauchy-Schwarz inequality, there holds that

$$\left| \int_0^t \int_{-2}^{+2} K_w(w,t;\xi,\tau) g^\star_{1/2}(\xi,\tau)\, d\xi d\tau \right|^2 \le cT \int_0^t e^{\left(-\frac{(c_1 w)^2}{4\beta^2(t)(B(t)-B(\tau))}\right)}\, d\tau,$$

then

$$\int_{-\infty}^{+\infty} |w|^4 \left| \int_0^t \int_{-2}^{+2} K_w(w,t;\xi,\tau) g^\star_{1/2}(\xi,\tau)\, d\xi d\tau \right|^2 dw \le cT \int_0^t \int_{-\infty}^{+\infty} |w|^4 e^{-\frac{(c_1 w)^2}{4\beta^2(t)(B(t)-B(\tau))}}\, d\tau\, dw.$$

Using the change of variable $\frac{w}{2\beta(t)\sqrt{(B(t)-B(\tau))}} = v$, we get that

$$\int_0^t \int_{-\infty}^{+\infty} |w|^4 e^{-\frac{(c_1 w)^2}{4\beta^2(t)(B(t)-B(\tau))}}\, d\tau\, dw = \int_0^t 2^5 \beta(t)^5 (B(t)-B(\tau))^{5/2} \int_{-\infty}^{+\infty} |v|^4 e^{-c_1^2 v^2}\, d\tau\, dv,$$

then clearly, $\widetilde{J}_2^2 < \infty$. Using the Young inequality, we deduce that $\widetilde{J}_2 < \infty$, then conclude that $J_2 \le c\varepsilon^{5/2}$, and

$$\left\| \varepsilon \frac{w^2}{2} M_{xx}(\kappa_1,t) \left(W_w^{1/2}(w,t) - y_x^0(a(t)^\pm,t) \right) \right\|_{L^1(0,s;L^2(\Omega))} \le c\varepsilon^{5/4}. \tag{72}$$

It results from (66) and (72) that

$$\|I_2^\varepsilon\|_{L^1(0,s;L^2(\Omega))} \le c\varepsilon^{3/4}. \tag{73}$$

(c) Estimate of $\|I_3^\varepsilon\|_{L^1(0,s;L^2(\Omega))}$.
(c.1) Using (64) there holds that

$$\left\| y_t^0(1,t) e^{-M(1,t)z} \right\|_{L^1(0,s;L^2(\Omega))} \le c\varepsilon^{1/2},$$

$$\left\| M(1,t) M_x(\kappa_2,t) \right) \left(y^0(1,t) - y^0(a(t)^\pm,t) \right) z e^{-M(1,t)z} \Big\|_{L^1(0,s;L^2(\Omega))} \le c\varepsilon^{1/2}.$$

(c.2) From the explicit expression of W^0, see (20), we get the uniform bound $|W^0(\tau M(1,t),t)| \le \max\{|v(0)|, |y_0(0)|\}$ then, using (64), there holds that

$$\left\| \left(M_t(1,t) + M(1,t) M_x(\kappa_2,t) \right) W^0(\tau M(1,t),t) z e^{-M(1,t)z} \right\|_{L^2(0,s;L^2(\Omega))} \le c\varepsilon^{1/2}.$$

(c.3) Let $\gamma(t) := \beta^2(t) B(t)$. By a direct calculation we have, for all $C \in \mathbb{R}$,

$$W_t^0(w,t) + C W_w^0(w,t) = \frac{v(0) - y_0(0)}{2} \frac{1}{2\sqrt{\pi}} \frac{2w\alpha(t)\gamma(t) + w - 2C\gamma(t)}{\gamma(t)^{3/2}} e^{-\frac{w^2}{4\gamma(t)}}. \tag{74}$$

Taking $w = \tau M(1,t) = \frac{T_1 - t}{\varepsilon^{1/2}} M(1,t)$ and $C = \left(\tau M_t(1,t) - \frac{M(1,t)}{\varepsilon^{1/2}} \right) = \left(\frac{T_1 - t}{\varepsilon^{1/2}} M_t(1,t) - \frac{M(1,t)}{\varepsilon^{1/2}} \right)$, we get

$$\|W_t^0(w,t) + C W_w^0(w,t)\|_{L^1(0,s)} = C\mathcal{O}(\varepsilon^{1/2}) + \mathcal{O}(\epsilon) = \mathcal{O}(1),$$

then

$$\left\| \left(W_t^0(\tau M(1,t),t) + \left(\tau M_t(1,t)) - \frac{M(1,t)}{\varepsilon^{1/2}} \right) W_w^0(\tau M(1,t),t) \right) e^{-M(1,t)z} \right\|_{L^1(0,s;L^2(\Omega))} \le c\varepsilon^{1/2}.$$

From the previous estimate and the estimates in (c.1) and (c.2) we deduce that

$$\|I_3^\varepsilon\|_{L^1(0,s;L^2(\Omega))} \le c\varepsilon^{1/2}. \tag{75}$$

(d) Estimate of $\|I_4^\varepsilon\|_{L^1(0,s;L^2(\Omega))}$.
(d.1) A direct calculation gives

$$W^0_{wt}(w,t) + CW^0_{ww}(w,t) = \frac{v(0)-y_0(0)}{2}\frac{1}{4\sqrt{\pi}}\frac{-2w^2\alpha(t)\gamma(t) - w^2 + 2wC\gamma(t) + 4\gamma(t)^2\alpha(t) + 2\gamma(t)}{\gamma(t)^{5/2}} e^{-\frac{w^2}{4\gamma(t)}}. \tag{76}$$

Taking $w = \tau M(1,t) = \frac{T_1-t}{\varepsilon^{1/2}}M(1,t)$ and $C = \left(\tau M_t(1,t) - \frac{M(1,t)}{\varepsilon^{1/2}}\right) = \left(\frac{T_1-t}{\varepsilon^{1/2}}M_t(1,t) - \frac{M(1,t)}{\varepsilon^{1/2}}\right)$, we get

$$\|W^0_{wt}(\tau M(1,t),t) + CW^0_{ww}(\tau M(1,t),t)\|_{L^1(0,s)} = \mathcal{O}(\varepsilon^{1/2}) + C\mathcal{O}(\epsilon) = \mathcal{O}(\varepsilon^{1/2}),$$

hence

$$\left\|\varepsilon^{1/2}\left(W^0_{wt}(\tau M(1,t),t) + \left(\tau M_t(1,t) - \frac{M(1,t)}{\varepsilon^{1/2}}\right)W^0_{ww}(\tau M(1,t),t)\right) ze^{-M(1,t)z}\right\|_{L^1(0,s;L^2(\Omega))} \le c\varepsilon^{3/2}.$$

(d.2) From $W^0_w(w,t) = \frac{y_0(0)-v(0)}{2}\frac{1}{\sqrt{\pi}}\frac{1}{\gamma(t)^{1/2}}e^{-\frac{w^2}{4\gamma(t)}}$ with $w = \tau M(1,t) = \frac{T_1-t}{\varepsilon^{1/2}}M(1,t)$, we get

$$\|W^0_w(\tau M(1,t),t)\|_{L^1(0,s)} = \mathcal{O}(\varepsilon^{1/2}), \tag{77}$$

then

$$\left\|\varepsilon^{1/2}M_x(\kappa_2,t)W^0_w(\tau M(1,t),t)ze^{-M(1,t)z}\right\|_{L^1(0,s;L^2(\Omega))} \le c\varepsilon^{3/2},$$

$$\left\|\varepsilon^{1/2}\Big(M_t(1,t) + M(1,t)M_x(\kappa_2,t)\Big)W^0_w(\tau M(1,t),t)z^2e^{-M(1,t)z}\right\|_{L^1(0,s;L^2(\Omega))} \le c\varepsilon^{3/2}.$$

From the two last estimates and that in (d.1) we deduce that

$$\|I_4^\varepsilon\|_{L^1(0,s;L^2(\Omega))} \le c\varepsilon^{3/2}. \tag{78}$$

(e) Estimate of $\|I_5^\varepsilon\|_{L^1(0,s;L^2(\Omega))}$.
(e.1) Using (64) we deduce that

$$\left\|\tau M(1,t)M_x(a(t),t)y^0_x(a(t)^\pm,t)\right\|_{L^1(0,s)} = \mathcal{O}(\varepsilon^{-1/2}).$$

(e.2) Estimate of $\left\|C_{1/2}(0,\tau,t)\right\|_{L^1(0,s)}$. We use that

$$W^{1/2}(w,t) - y^0_x(a(t)^\pm,t)w = \Big(W^{1/2}(w,t) - \widetilde{g}_{1/2}(w,t)\Big) + \Big(\widetilde{g}_{1/2}(w,t) - y^0_x(a(t)^\pm,t)w\Big),$$

with $\Big(\widetilde{g}_{1/2}(w,t) - y^0_x(a(t)^\pm,t)w\Big) = 0$ for $|w| \ge 2$. Taking $w = \frac{T_1-t}{\varepsilon^{1/2}}M(1,t)$ in (63) there holds that

$$J_3^1(t) := \int_0^t\int_{-\infty}^{+\infty} K\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t;\xi,\sigma\right) f_{1/2}(\xi,\sigma)\,d\xi d\sigma = \frac{v(0)-y_0(0)}{\pi}\frac{e^{-\frac{\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t)\right)^2}{4\beta^2(t)B(t)}}}{\beta^2(t)B^2(t)\sqrt{B(t)}}\times$$
$$\times\int_0^t M_{xx}(a(\sigma),\sigma)\sqrt{B(\sigma)}\Bigg(B(\sigma)\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t)\right)^2 + 2\beta^2(t)B^2(t) - 2\beta^2(t)B(t)B(\sigma)\Bigg)\times$$
$$\times\sqrt{B(t)-B(\sigma)}d\sigma.$$

Consequently,

$$\left|J_3^1(t)\right| \le cT\, \frac{e^{-\frac{\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t)\right)^2}{4\beta(t)^2B(t)}}}{\sqrt{B(t)}} \left(\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t)\right)^2+1\right) \le cT\, e^{-\frac{\left(\frac{T_1-t}{\varepsilon^{1/2}}M_1\right)^2}{2\beta_2^2B_2}},$$

where we used the bounds $0 < M_1 \le M(x,t)$, $B(t) \le B_2$ and $\beta(t) \le \beta_2$ for all $(x,t) \in Q_T$. Then

$$\int_0^{T_1} \left|J_3^1(t)\right|^2 dt \le cT^2 \int_0^{T_1} e^{-\frac{\left(\frac{T_1-t}{\varepsilon^{1/2}}M_1\right)^2}{\beta_2^2B_2}} dt.$$

Using the change of variable $\frac{T_1-t}{\varepsilon^{1/2}} = \tau$, there holds that

$$\int_0^{T_1} \left|J_3^1(t)\right|^2 dt \le cT^2\varepsilon^{1/2} \int_0^{+\infty} e^{-\frac{(\tau M_1)^2}{\beta_2^2B_2}} d\tau \le cT^2\varepsilon^{1/2}.$$

When $s \ge T_1$, we derive similarly the estimate $\int_{T_1}^s \left|J_3^1(t)\right|^2 dt \le cT^2\varepsilon^{1/2}$, then get

$$\int_0^s \left|J_3^1(t)\right|^2 dt \le cT^2\varepsilon^{1/2}.$$

Let $J_3^2(t) := \int_0^t \int_{-\infty}^{+\infty} K\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t;\xi,\sigma\right) g_{1/2}^\star(\xi,\sigma)\, d\xi d\sigma$. We show similarly that

$$\int_0^s \left|J_3^2(t)\right|^2 dt \le cT^2\varepsilon^{1/2},$$

then derive the estimate

$$\left\|W^{1/2}(\tau M(1,t),t) - \tau M(1,t) y_x^0(a(t)^\pm,t)\right\|_{L^1(0,s)} \le cT^2\varepsilon^{1/2},$$

then

$$\left\|C_{1/2}(0,\tau,t)\right\|_{L^1(0,s)} = \mathcal{O}(\varepsilon^{1/2}),$$

and the first term of I_5 satisfies the estimate

$$\left\|\varepsilon^{1/2}\Big(M_t(1,t) + M(1,t)M_x(\kappa_2,t)\Big)C_{1/2}(0,\tau,t) z e^{-M(1,t)z}\right\|_{L^1(0,s;L^2(\Omega))} \le c\varepsilon^{3/2}.$$

(e.3) Estimate of $\left\|W_t^{1/2}(\tau M(1,t),t)\right\|_{L^1(0,s)}$. Let us denote $F_1(w,t) := \frac{v(0)-y_0(0)}{\pi} \frac{e^{-\frac{w^2}{4\beta^2(t)B(t)}}}{\beta^2(t)B^2(t)\sqrt{B(t)}}$,

$$F_2(w,t) := \int_0^t M_{xx}(a(\sigma),\sigma)\sqrt{B(\sigma)}\Big(B(\sigma)w^2 + 2\beta^2(t)B^2(t) - 2\beta^2(t)B(t)B(\sigma)\Big)\sqrt{B(t)-B(\sigma)}d\sigma,$$

and $F(w,t) := F_1(w,t)F_2(w,t)$. We have from (63), for all $(w,t) \in \mathbb{R}\times(0,T)$,

$$\int_0^t\int_{\mathbb{R}} K(w,t;\xi,\sigma) f_{1/2}(\xi,\sigma)\, d\xi d\sigma = F(w,t).$$

Differentiating the previous equality in t yields

$$\int_0^t\int_{\mathbb{R}} K_t(w,t;\xi,\sigma) f_{1/2}(\xi,\sigma)\, d\xi d\sigma + f_{1/2}(w,t) = F_t(w,t).$$

We have

$$F_{1,t}(w,t) = \frac{v(0)-y_0(0)}{\pi} e^{-\frac{w^2}{4\beta^2(t)B(t)}} \left(\frac{w^2\Big(2\beta'(t)B(t)+\beta(t)B'(t)\Big)}{4\beta^5(t)B^{9/2}(t)} - \frac{2\beta'(t)B(t)+5/2B'(t)\beta(t)}{\beta^3(t)B^{7/2}(t)}\right),$$

$$F_{2,t}(w,t) = \int_0^t M_{xx}(a(\sigma),\sigma)\sqrt{B(\sigma)}\times$$
$$\times\Big(4\beta(t)\beta'(t)B^2(t) + 4B(t)B'(t)\beta^2(t) - 4\beta(t)\beta'(t)B(t)B(\sigma) - 2\beta^2(t)B'(t)B(\sigma)\Big)\sqrt{B(t)-B(\sigma)}\,d\sigma$$
$$+\int_0^t M_{xx}(a(\sigma),\sigma)\sqrt{B(\sigma)}\Big(B(\sigma)w^2 + 2\beta^2(t)B^2(t) - 2\beta^2(t)B(t)B(\sigma)\Big)\frac{1}{2}\left(B(t)-B(\sigma)\right)^{-1/2}B'(t)d\sigma.$$

Differentiating (23) in t gives

$$W_t^{1/2}(w,t) = \widetilde{g}_{1/2,t}(w,t) + \int_0^t\int_{\mathbb{R}} K_t(w,t;\xi,\sigma)\Big(f_{1/2}(\xi,\sigma) - g^\star_{1/2}(\xi,\sigma)\Big)\,d\xi d\sigma + f_{1/2}(w,t) - g^\star_{1/2}(w,t).$$

Then

$$W_t^{1/2}(w,t) = \widetilde{g}_{1/2,t}(w,t) + F_t(w,t) - \int_0^t\int_{\mathbb{R}} K_t(w,t;\xi,\sigma)g^\star_{1/2}(\xi,\sigma)\,d\xi d\sigma - g^\star_{1/2}(w,t). \tag{79}$$

Arguing as in (e.2), using the bounds $0 < M_1 \le M(x,t)$, $B(t) \le B_2$ and $\beta(t) \le \beta_2$ for all $(x,t) \in Q_T$, we find that

$$|F_1(w,t)| \le ce^{-\frac{w^2}{4\beta_2^2 B_2}},\quad |F_{1,t}(w,t)| \le ce^{-\frac{w^2}{4\beta_2^2 B_2}}(w^2+1),\quad |F_2(w,t)| \le c(w^2+1),$$
$$|F_{2,t}(w,t)| \le c + c(w^2+1)\int_0^t (B(t)-B(\sigma))^{-1/2}B'(t)\,d\sigma$$
$$= c + c(w^2+1)\int_0^t (B(t)-B(\sigma))^{-1/2}B'(\sigma)\frac{B'(t)}{B'(\sigma)}\,d\sigma \le c(w^2+1).$$

It results that

$$|F_t(w,t)| \le ce^{-\frac{w^2}{4\beta_2^2 B_2}}(w^2+1) \le \tilde{c}e^{-\frac{w^2}{4\beta_2^2 B_2}}, \tag{80}$$

for some constant $\tilde{c} > c$. Taking $w = \frac{T_1-t}{\varepsilon^{1/2}}M(1,t)$ in (79) we have, for ε small enough,

$$W_t^{1/2}\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t\right) = F_t\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t\right) - \int_0^t\int_{\mathbb{R}} K_t\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t;\xi,\sigma\right)g^\star_{1/2}(\xi,\tau)\Big)\,d\xi d\sigma. \tag{81}$$

Using inequality (80) and the change of variable $\frac{T_1-t}{\varepsilon^{1/2}} = \tau$, there holds that

$$\int_0^{T_1}\left|F_t\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t\right)\right|dt \le c\varepsilon^{1/2}\int_0^{+\infty} e^{-\frac{(\tau M_1)^2}{\beta_2^2 B_2}}\,d\tau \le c\varepsilon^{1/2}.$$

When $s \ge T_1$, we derive similarly the estimate $\int_{T_1}^s \left|F_t\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t\right)\right|dt \le c\varepsilon^{1/2}$, then get

$$\int_0^s\left|F_t\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t\right)\right|dt \le c\varepsilon^{1/2}. \tag{82}$$

Consider the term $\int_0^t\int_{\mathbb{R}} K_t(w,t;\xi,\sigma)g^\star_{1/2}(\xi,\sigma)\,d\xi d\sigma$. We have $K_t(w,t;\xi,\sigma) = R(w,t;\xi,\sigma)\,e^{-\frac{\left(\frac{w}{\beta(t)}-\frac{\xi}{\beta(\sigma)}\right)^2}{4(B(t)-B(\sigma))}}$, with

$$R(w,t;\xi,\sigma) = -\frac{1}{\beta(\sigma)}\frac{2\pi B'(t)}{(4\pi(B(t)-B(\sigma)))^{3/2}}$$
$$-\frac{1}{\beta(\sigma)\sqrt{4\pi(B(t)-B(\sigma))}}\left(\frac{\frac{-w\beta'(t)}{\beta^2(t)}\left(\frac{w}{\beta(t)}-\frac{\xi}{\beta(\sigma)}\right)}{2(B(t)-B(\sigma))} - \frac{B'(t)\left(\frac{w}{\beta(t)}-\frac{\xi}{\beta(\sigma)}\right)^2}{4(B(t)-B(\sigma))^2}\right).$$

Using the change of variable $\frac{\frac{\xi}{\beta(\sigma)}-\frac{w}{\beta(t)}}{2\sqrt{(B(t)-B(\sigma))}}=v$, there holds that

$$\int_{-2}^{+2} K_t(w,t;\xi,\sigma)g_{1/2}^{\star}(\xi,\sigma)\,d\xi$$
$$= 2\beta(\sigma)\sqrt{B(t)-B(\sigma)}\int_{a(w,t,\sigma)}^{b(w,t,\sigma)} e^{-v^2} R\left(w,t;\frac{\beta(\sigma)}{\beta(t)}w+2\beta(\sigma)\sqrt{B(t)-B(\sigma)}\,v,\sigma\right)\times$$
$$\times\, g_{1/2}^{\star}\left(w,t;\frac{\beta(\sigma)}{\beta(t)}w+2\beta(\sigma)\sqrt{B(t)-B(\sigma)}\,v,\sigma\right)\,dv,$$

with $a(w,t,\sigma)=\frac{-2-\frac{\beta(\sigma)}{\beta(t)}w}{2\beta(\sigma)\sqrt{B(t)-B(\sigma)}}$ and $b(w,t,\sigma)=\frac{2-\frac{\beta(\sigma)}{\beta(t)}w}{2\beta(\sigma)\sqrt{B(t)-B(\sigma)}}$. We have

$$2\beta(\sigma)\sqrt{B(t)-B(\sigma)}\;R\left(w,t;\frac{\beta(\sigma)}{\beta(t)}w+2\beta(\sigma)\sqrt{B(t)-B(\sigma)}\,v,\sigma\right)$$
$$=\frac{1}{\sqrt{\pi}}\left(\frac{B'(t)(2v^2-1)}{2(B(t)-B(\sigma))}-\frac{\frac{vw\beta'(t)}{\beta^2(t)}}{\sqrt{B(t)-B(\sigma)}}\right),$$

then

$$\int_{-2}^{+2} K_t(w,t;\xi,\sigma)g_{1/2}^{\star}(\xi,\sigma)\,d\xi=\frac{1}{\sqrt{\pi}}\int_{a(w,t,\sigma)}^{b(w,t,\sigma)} e^{-v^2} g_{1/2}^{\star}\left(\frac{\beta(\sigma)}{\beta(t)}w+2\beta(\sigma)\sqrt{(B(t)-B(\sigma))}\,v,\sigma\right)\times$$
$$\left(\frac{B'(t)(2v^2-1)}{2(B(t)-B(\sigma))}-\frac{\frac{vw\beta'(t)}{\beta^2(t)}}{\sqrt{B(t)-B(\sigma)}}\right)dv.$$

Using that $g_{1/2}^{\star}(\xi,\tau)$ is uniformly bounded, there holds that

$$\int_{-2}^{+2}\left|K_t(w,t;\xi,\sigma)g_{1/2}^{\star}(\xi,\sigma)\right|d\xi\le c\int_{a(w,t,\sigma)}^{b(w,t,\sigma)} e^{-v^2}\left(\frac{B'(t)(2v^2+1)}{2(B(t)-B(\sigma))}+\frac{\frac{|vw|\beta'(t)}{\beta^2(t)}}{\sqrt{B(t)-B(\sigma)}}\right)dv.$$

Let us denote

$$A_1(t,\sigma):=\frac{B'(t)}{2(B(t)-B(\sigma))},\quad A_2(t,\sigma):=\frac{\frac{\beta'(t)}{\beta^2(t)}}{\sqrt{B(t)-B(\sigma)}},\quad A_3(t,\sigma):=b(w,t,\sigma)-a(w,t,\sigma),$$

so that

$$\int_{-2}^{+2}\left|K_t(w,t;\xi,\sigma)g_{1/2}^{\star}(\xi,\sigma)\right|d\xi\le c\int_{a(w,t,\sigma)}^{b(w,t,\sigma)} e^{-v^2}\left(A_1(t,\sigma)(2v^2+1)+A_2(t,\sigma)|vw|\right)dv.$$

Using the inequalities $e^{-v^2}(2v^2+1)\le ce^{-\frac{v^2}{2}}$, $e^{-v^2}|v|\le ce^{-\frac{v^2}{2}}$, we have

$$\int_{-2}^{+2}\left|K_t(w,t;\xi,\sigma)g_{1/2}^{\star}(\xi,\sigma)\right|d\xi\le c\big(A_1(t,\sigma)+A_2(t,\sigma)|w|\big)\int_{a(w,t,\sigma)}^{b(w,t,\sigma)} e^{-\frac{v^2}{2}}\,dv$$
$$\le c\big(A_1(t,\sigma)+A_2(t,\sigma)|w|\big)A_3(t,\sigma)e^{-\frac{a(w,t,\sigma)^2}{4}}.$$

Let us take $w=\frac{T_1-t}{\varepsilon^{1/2}}M(1,t)$ in the previous inequality. For ε small enough and $t\le T_1$ we have $a(w,t,\sigma)<0$, then

$$\big(A_1(t,\sigma)+A_2(t,\sigma)|w|\big)A_3(t,\sigma)e^{-\frac{a(w,t,\sigma)^2}{2}}\le c\big(|w|+1\big)e^{-\frac{a(w,t,\sigma)^2}{4}}.$$

Then

$$\int_{-2}^{+2}\left|K_t\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t;\xi,\sigma\right)g^{\star}_{1/2}(\xi,\sigma)\right|d\xi \le c\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t)+1\right)e^{-\frac{a\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t,\sigma\right)^2}{4}}.$$

Integrating in σ, we have

$$\int_0^t\int_{-2}^{+2}\left|K_t\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t;\xi,\sigma\right)g^{\star}_{1/2}(\xi,\sigma)\right|d\xi d\sigma \le c\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t)+1\right)\int_0^t e^{-\frac{a\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t,\sigma\right)^2}{4}}d\sigma. \tag{83}$$

Integrating in t and using the change of variable $\frac{T_1-t}{\varepsilon^{1/2}}=\tau$, there holds that

$$\begin{aligned}
&\int_0^{T_1}\int_0^t\int_{-2}^{+2}\left|K_t\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t;\xi,\sigma\right)g^{\star}_{1/2}(\xi,\sigma)\right|d\xi d\sigma dt\\
&\le c\varepsilon^{1/2}\int_0^{\frac{T_1}{\varepsilon^{1/2}}}(\tau M(1,t)+1)\int_0^t e^{-\frac{a(\tau M(1,t),t,\sigma)^2}{4}}d\sigma d\tau\\
&\le c\varepsilon^{1/2}\int_0^{+\infty}(\tau M(1,t)+1)\int_0^t e^{-\frac{a(\tau M(1,t),t,\sigma)^2}{4}}d\sigma d\tau\\
&\le c\varepsilon^{1/2}.
\end{aligned}$$

When $s\ge T_1$, we derive similarly the estimate $\int_{T_1}^s\int_{-2}^{+2}\left|K_t\left(\frac{T_1-t}{\varepsilon^{1/2}}M(1,t),t;\xi,\sigma\right)g^{\star}_{1/2}(\xi,\sigma)\right|d\xi d\sigma\le c\varepsilon^{1/2}$, then get

$$\int_0^s\left|\int_0^t\int_{\mathbb{R}}K_t(w,t;\xi,\sigma)g^{\star}_{1/2}(\xi,\sigma)\,d\xi d\sigma\right|dt\le c\varepsilon^{1/2}.$$

We conclude that

$$\left\|W_t^{1/2}(\tau M(1,t),t)\right\|_{L^1(0,s)}\le c\varepsilon^{1/2}.$$

(e.4) Estimate of $\left\|\tau M_t(1,t)\left(W_w^{1/2}(\tau M(1,t),t)-y_x^0(a(t)^{\pm},t)\right)\right\|_{L^1(0,s)}$. Let us denote

$$J_4(t):=W_w^{1/2}\left(\frac{T_1-t}{\varepsilon^{1/2}}M_t(1,t),t\right)-y_x^0(a(t)^{\pm},t).$$

We have from (23)

$$W_w^{1/2}(w,t)=\widetilde{g}_{1/2,w}(w,t)+\int_0^t\int_{\mathbb{R}}K_w(w,t;\xi,\tau)\Big(f_{1/2}(\xi,\tau)-g^{\star}_{1/2}(\xi,\tau)\Big)\,d\xi d\tau,\quad w\in\mathbb{R},t\in(0,T).$$

Writing

$$W_w^{1/2}(w,t)-y_x^0(a(t)^{\pm},t)=\Big(W_w^{1/2}(w,t)-\widetilde{g}_{1/2,w}(w,t)\Big)+\Big(\widetilde{g}_{1/2,w}(w,t)-y_x^0(a(t)^{\pm},t)\Big),$$

then taking $w=\frac{T_1-t}{\varepsilon^{1/2}}M_t(1,t)$ we have, for ε small enough,

$$W_w^{1/2}\left(\frac{T_1-t}{\varepsilon^{1/2}}M_t(1,t),t\right)-y_x^0(a(t)^{\pm},t)=W_w^{1/2}\left(\frac{T_1-t}{\varepsilon^{1/2}}M_t(1,t),t\right)-\widetilde{g}_{1/2,w}\left(\frac{T_1-t}{\varepsilon^{1/2}}M_t(1,t),t\right).$$

Then

$$W_w^{1/2}\left(\frac{T_1-t}{\varepsilon^{1/2}}M_t(1,t),t\right)-y_x^0(a(t)^{\pm},t)=\int_0^t\int_{\mathbb{R}}K_w\left(\frac{T_1-t}{\varepsilon^{1/2}}M_t(1,t),t;\xi,\tau\right)\Big(f_{1/2}(\xi,\tau)-g^{\star}_{1/2}(\xi,\tau)\Big)d\xi d\tau. \tag{84}$$

We have from (b.2)

$$J_4^1(t) := \int_0^t \int_{-\infty}^{+\infty} K_w \left(\frac{T_1 - t}{\varepsilon^{1/2}} M_t(1,t), t; \xi, \tau \right) f_{1/2}(\xi, \tau)\, d\xi d\tau$$
$$= A_1 \left(\frac{T_1 - t}{\varepsilon^{1/2}} M_t(1,t), t \right) + A_2 \left(\frac{T_1 - t}{\varepsilon^{1/2}} M_t(1,t), t \right),$$

where $A_1(w,t)$ and $A_2(w,t)$ are defined by (67) and (68). We have $|A_1(w,t)| + |A_2(w,t)| \leq c e^{-\frac{w^2}{2\beta_2^2 B_2}}$ where we used the bounds $0 < M_1 \leq M(x,t)$, $B(t) \leq B_2$ and $\beta(t) \leq \beta_2$ for all $(x,t) \in Q_T$. Then

$$\int_0^{T_1} \left| J_4^1(t) \right| dt \leq c \int_0^{T_1} e^{-\frac{\left(\frac{T_1 - t}{\varepsilon^{1/2}} M_1 \right)^2}{2\beta_2^2 B_2}} dt \leq c\varepsilon^{1/2} \int_0^{+\infty} e^{-\frac{(\tau M_1)^2}{2\beta_2^2 B_2}} d\tau \leq c\varepsilon^{1/2},$$

using the change of variable $\frac{T_1 - t}{\varepsilon^{1/2}} = \tau$. When $s \geq T_1$, we derive similarly the estimate $\int_{T_1}^s \left| J_4^1(t) \right|^2 dt \leq c\varepsilon^{1/2}$, then get $\int_0^s \left| J_4^1(t) \right| dt \leq c\varepsilon^{1/2}$. Let $J_4^2(t) = \int_0^t \int_{-\infty}^{+\infty} K\left(\frac{T_1 - t}{\varepsilon^{1/2}} M(1,t), t; \xi, \sigma \right) g_{1/2}^\star(\xi, \sigma)\, d\xi d\sigma$. We show similarly that $\int_0^s \left| J_4^2(t) \right| dt \leq c\varepsilon^{1/2}$, then derive the estimates

$$\left\| W_w^{1/2}(\tau M(1,t), t) - y_x^0(a(t)^\pm, t) \right\|_{L^1(0,s)} \leq c\varepsilon^{1/2},$$
$$\left\| \tau M_t(1,t) \left(W_w^{1/2}(\tau M(1,t), t) - y_x^0(a(t)^\pm, t) \right) \right\|_{L^1(0,s)} \leq c.$$

(e.5) Estimate of $\left\| \varepsilon^{1/2} C_{1/2,t}(0,\tau,t) e^{-M(1,t)z} \right\|_{L^1(0,s;L^2(\Omega))}$. Using (64) and the estimates in (e.3) and (e.4), we deduce that

$$\left\| \varepsilon^{1/2} C_{1/2,t}(0,\tau,t) e^{-M(1,t)z} \right\|_{L^1(0,s;L^2(\Omega))} \leq c\varepsilon^{1/2}.$$

(e.6) We have from (e.4) that $\left\| W_w^{1/2}(\tau M(1,t), t) - y_x^0(a(t)^\pm, t) \right\|_{L^1(0,s)} \leq c\varepsilon^{1/2}$, then, using (64) we deduce that

$$\left\| C_{1/2,\tau}(0,\tau,t) e^{-M(1,t)z} \right\|_{L^1(0,s;L^2(\Omega))} \leq c\varepsilon^{1/2}.$$

From the previous estimate, the last estimate in (e.2), and the estimate in (e.5) we deduce that

$$\| I_5^\varepsilon \|_{L^1(0,s;L^2(\Omega))} \leq c\varepsilon^{1/2}. \tag{85}$$

Collecting estimates (65), (73), (75), (78), and (85) we get the desired result.

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