SUPPLEMENTARY MATERIAL: STABLE SUMS TO INFER HIGH RETURN LEVELS OF MULTIVARIATE RAINFALL TIME SERIES

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Abstract. First, we complement the case study of heavy rainfall in France by implementing Pareto-based methods using declustering techniques. Second, we develop on the asymptotic theory of the stable sums method. To prove Theorem 6.1, we give a more general statement and prove the multivariate central limit theory of regularly varying time series with unit (tail)-index.

Keywords: Environmental time series; multivariate regular variation; stable distribution; stationary time series; cluster process.

A. Supplement on the case study of heavy rainfall in France

We review the case study presented in Section 5, on the France fall rainfall data set. For comparison, we also implement the Pareto-based methods in Section 4.4 for the radial component analysis.

A.1. **Implementation.** Recall that to study the 3-dimensional sample $(\mathbf{X}_1, \dots, \mathbf{X}_n)$ obtained from each region, we have implemented the stable sums method as a function of the number of order statistics k, with $k = 150, 250, 350, 450, 550$. Then we have obtained pairs $\widehat{\alpha}^n(k), b(k)$ to run Algorithm 1; see Section 5.1. For comparison, we also implement Pareto-based methods as a function of k as follows. We compute the peaks over threshold method for threshold levels $th(k) = X_{(k)}$ such that $X_{(1)} \geq X_{(2)} \geq \cdots \geq X_{(n)}$, and we compute the block maxima method for blocks of length $bl_{BM}(k) := n/k$.

A.2. **Analysis of the radial component.** We apply all three methods on the regional samples $(|\mathbf{X}_1|, \dots, |\mathbf{X}_n|)$ to compute confidence intervals of the 50 years return level of fall observations. The estimates are presented in Figure A.1. Rows correspond to different regions and the columns correspond to different methods. As mentioned, the stable sums method, illustrated in the third column of Figure A.1, gives robust estimates as a function of k. Also, as suggested by the simulation study in Section 4, the estimates obtained with the peaks over threshold method might underestimate return levels. We also see in Figure A.1 that the block maxima method varies strongly for different block length choices.

B. Supplement on the asymptotic theory

In the remaining, we detail on the asymptotics of the stable blocks method and we prove Theorem 6.1. We consider $(\mathbf{X}_t)_{t\in\mathbb{Z}}$ to be a regularly varying time series in $(\mathbb{R}^d, |\cdot|)$. We denote $(\mathbf{\Theta}_t)_{t\in\mathbb{Z}}$ spectral tail process of the series as in (6). Recall **AC**, **CS**, **MX**, the main assumptions of Theorem 6.1.

B.1. **Preliminaries.** We start by reviewing Proposition 3.2 in [5].

Lemma B.1. *Let $(\mathbf{X}_t)_{t\in\mathbb{Z}}$ be a regularly varying time series in $(\mathbb{R}^d, |\cdot|)$, with (tail)-index $\alpha > 0$, and spectral tail process $(\mathbf{\Theta}_t)_{t\in\mathbb{Z}}$. Consider a sequence (x_n) and assume $\mathbf{AC}((x_{k_n})^\alpha)$ holds. Then,*

$$\textstyle\sum_{t\in\mathbb{Z}} |\mathbf{\Theta}_t|^\alpha \;:=\; \|\mathbf{\Theta}_t\|_\alpha^\alpha < +\infty \quad a.s.$$

and $\mathbf{Q}_t = \mathbf{\Theta}_t / \|\mathbf{\Theta}_t\|_\alpha$, for $t \in \mathbb{Z}$, is well defined. We call $(\mathbf{Q}_t)_{t\in\mathbb{Z}}$ the cluster process of the series.

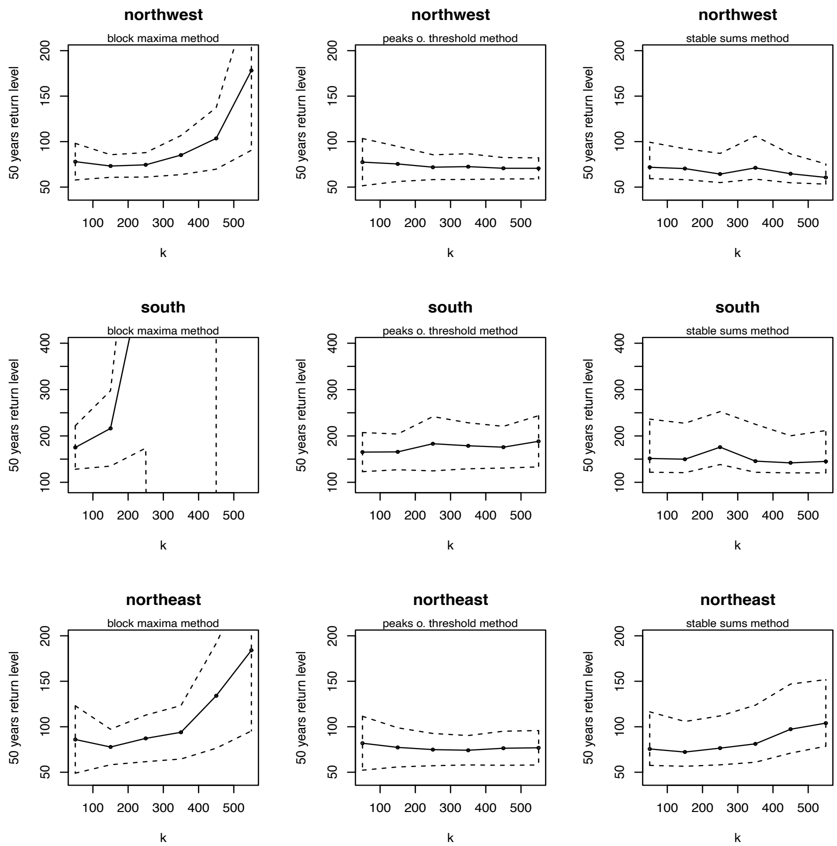

FIGURE A.1. Estimates of the 50 years return level of fall norm observations with confidence intervals. We write estimates as a function of k with the parametrization detailed in Section A.1.

Lemma 7.1 in [5] shows $c(\alpha) = 1$ in (2), which justifies the right-hand side of (4) for $p = \alpha$. We recall its statement below. The proof is deferred to Section C.1.

Lemma B.2. *Let* $(\mathbf{X}_t)_{t\in\mathbb{Z}}$ *be a regularly varying time series in* $(\mathbb{R}^d, |\cdot|)$, *with (tail)-index* $\alpha > 0$. *Let* (x_n) *satisfy* $\mathbf{AC}((x_{k_n})^\alpha), \mathbf{CS}((x_{k_n})^\alpha)$, *and* $n\mathbb{P}(|\mathbf{X}_0| > x_n) \to 0$, *then*

$$\lim_{n\to+\infty} \frac{\mathbb{P}\big((S_n(\alpha))^{1/\alpha} > x_n\big)}{n\,\mathbb{P}(|\mathbf{X}_0| > x_n)} \;=\; \mathbb{E}[\|\mathbf{Q}_t\|_\alpha^\alpha] \;=\; 1. \tag{B.1}$$

Lemma 7.1. in [5] states that in many cases, the limit of p-norms (with p instead of α at the left-hand side of (B.1)) equals $\mathbb{E}[\|\mathbf{Q}_t\|_p^\alpha]$. In this way, the choice $p = \alpha$ in (2) is a mindful strategy for handling the time dependencies. Indeed, the unit limit holds regardless of the underlying extremal time dependencies captured in $(\mathbf{Q}_t)$. The proof of Lemma B.2 uses arguments of large deviations of sums as the ones considered in [3, 4, 9].

B.2. **Proof Theorem 6.1.** We now focus on showing central limit theory of the α-power partial sums of $(\mathbf{X}_t)_{t\in\mathbb{Z}}$. To prove Theorem 6.1, we first give a general statement in Theorem B.3 concerning multivariate central limit theory (see Section C.2 for its proof). Our proof of Theorem 6.1 then follows straightforwardly. In our proof of Theorem 6.1, we remove the assumption (4.10) in [2] and (**CT**) in Theorem 3.1. [1], treating the recentering term.

Moreover, Theorem 6.1 shows (3) holds for $p = \alpha$. For the $p-$power partial sums theory with $p/\alpha \in (0,1) \cup (1,2)$, we refer to Proposition 4.4. in [5].

Theorem B.3. *Let $(\mathbf{Z}_t)_{t\in\mathbb{Z}}$ be a regularly varying time series in $(\mathbb{R}^d, |\cdot|)$, with unit (tail)-index. Consider non-negative and real sequences (a_n), (d_n) such that $n\,\mathbb{P}(|\mathbf{Z}_0| > a_n) \to 1$, and $d_n := \mathbb{E}[|\mathbf{Z}_t|\mathbb{1}(|\mathbf{Z}_t| \le a_n)]$. Assume $\mathbf{AC}(a_n)$, $\mathbf{CS}(a_n)$, $\mathbf{MX}(k_n)$ hold for the multivariate series. Then,*

$$\textstyle\sum_{t=1}^n (\mathbf{Z}_t - d_n)/a_n \xrightarrow{d} \xi_1$$

where ξ_1 is a d-variate stable distribution with stable parameter one. Actually, the series admits a cluster process $(\mathbf{Q}^Z_t)_{t\in\mathbb{Z}}$, $\|\mathbf{Q}^Z\|_1 = 1$ a.s.

(see Lemma B.1). Then, for $\mathbf{u} \in \mathbb{R}^d$,

(B.2)

$$\log \mathbb{E}\big[\exp\big\{i\mathbf{u}^\top \xi_1\big\}\big]$$
$$= \int_0^\infty \mathbb{E}\big[\exp\big\{i\mathbf{u}^\top \textstyle\sum_{t\in\mathbb{Z}} y\mathbf{Q}_t^Z\big\} - 1 - i\sin\big(\mathbf{u}^\top \textstyle\sum_{t\in\mathbb{Z}} y\mathbf{Q}_t^Z\big)\big]\, d(-y^{-1})$$
$$+ i\mu(\mathbf{u})\,.$$

The last term is a location parameter given by

$$\mu(\mathbf{u}) = \int_1^\infty \mathbb{E}\big[\sin\big(\mathbf{u}^\top \textstyle\sum_{t\in\mathbb{Z}} y\mathbf{Q}_t^Z\big)$$
$$- \sin\big(\mathbf{u}^\top \textstyle\sum_{t\in\mathbb{Z}} y\mathbf{Q}_t^Z - \mathbf{u}^\top \textstyle\sum_{t\in\mathbb{Z}} \underline{y\mathbf{Q}_t^Z}_1\big)\big]\, d(-y^{-1}),$$

where $\underline{\mathbf{x}}_1 := \mathbf{x}\mathbb{1}(|\mathbf{x}| > 1)$.

Theorem 6.1 follows straightforwardly from Theorem B.3 for univariate time series.

Proof of Theorem 6.1. Let $(Z_t)_{t\in\mathbb{Z}}$ be defined by $Z_t = |\mathbf{X}_t|^\alpha$, $t \in \mathbb{Z}$. By Lemma B.1, it admits a non-negative spectral process $(Q_t^Z)_{t\in\mathbb{Z}}$ satisfying $\sum_{t\in\mathbb{Z}} Q_t^Z = 1$ a.s. Then, Theorem B.3 entails

$$\textstyle\sum_{t=1}^n (Z_t - d_n)/a_n \xrightarrow{d} \xi_1$$

for a univariate unit stable limit ξ_1, such that $\mathbb{P}(|\mathbf{X}_0|^\alpha > a_n) \to 1$, as $n \to +\infty$, and $d_n = \mathbb{E}[|\mathbf{X}_0|^\alpha \mathbb{1}(|\mathbf{X}_0| \le a_n)]$. The limit ξ_1 has log-characteristic function given by, for $u \in \mathbb{R}$,

$$\log \mathbb{E}[\exp\{i\,u\,\xi_1\}]$$
$$= \int_0^\infty \mathbb{E}\big[\exp\big\{i\,uy\big\} - 1 - i\sin(uy)\big]\, d(-y^{-1}) + i\mu(u)\,,$$

Finally, following the argument lines in section XVII.2 of Feller [6], we deduce the skewness parameter of the stable limit ξ_1 verifies $\beta = 1$. □

C. Auxiliary proofs

C.1. **Proof of Lemma B.2.** We rely on telescopic sum arguments from [7, 8]; see [1]. For all $\epsilon > 0$, $\delta > 0$,

$$\begin{aligned}
&\mathbb{P}(S_n(\alpha) > x_n^\alpha) \\
&= \mathbb{P}\big(S_n(\alpha) > x_n^\alpha,\ \textstyle\sum_{t=1}^n |\mathbf{X}_t|^\alpha \mathbb{1}_{\{|\mathbf{X}_t| \le \epsilon\, x_n\}} < \delta\, x_n^\alpha\big) \\
&+ \mathbb{P}\big(S_n(\alpha) > x_n^\alpha,\ \textstyle\sum_{t=1}^n |\mathbf{X}_t|^\alpha \mathbb{1}_{\{|\mathbf{X}_t| \le \epsilon\, x_n\}} > \delta\, x_n^\alpha\big).
\end{aligned}$$

Referring to condition **CS**, the probability term above satisfies

$$\begin{aligned}
&\mathbb{P}\big(\textstyle\sum_{t=1}^n |\mathbf{X}_t|^\alpha \mathbb{1}_{\{|\mathbf{X}_t| > \epsilon\, x_n\}} > x_n^\alpha\big) \\
&\le \mathbb{P}(S_n(\alpha) > x_n^\alpha) \\
&\le \mathbb{P}\big(\textstyle\sum_{t=1}^n |\mathbf{X}_t|^\alpha \mathbb{1}_{\{|\mathbf{X}_t| > \epsilon\, x_n\}} > (1-\delta) x_n^\alpha\big) \\
&+ o(n\, \mathbb{P}(|\mathbf{X}_0| > x_n)).
\end{aligned}$$

Hence, to show (B.1) it suffices to prove that for all $\delta > 0$ the following relation holds

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} \lim_{n \to +\infty} \frac{\mathbb{P}\big(\sum_{t=1}^n |\mathbf{X}_t|^\alpha \mathbb{1}_{\{|\mathbf{X}_t| > \epsilon\, x_n\}} > (1-\delta) x_n^\alpha\big)}{n\, \mathbb{P}(|\mathbf{X}_0| > x_n)} = 1. \tag{C.3}$$

Using the so-called telescopic sum argument we have

$$\begin{aligned}
&\mathbb{P}\big(\textstyle\sum_{t=1}^n |\mathbf{X}_t|^\alpha \mathbb{1}_{\{|\mathbf{X}_t| > \epsilon\, x_n\}} > (1-\delta) x_n^\alpha\big) \\
&= \textstyle\sum_{j=1}^{n-1} \Big\{ \mathbb{P}\big(\textstyle\sum_{t=1}^{j+1} |\mathbf{X}_t|^\alpha \mathbb{1}_{\{|\mathbf{X}_t| > \epsilon\, x_n\}} > (1-\delta) x_n^\alpha\,,\ |\mathbf{X}_1| > \epsilon\, x_n\big) \\
&- \mathbb{P}\big(\textstyle\sum_{t=2}^{j+1} |\mathbf{X}_t|^\alpha \mathbb{1}_{\{|\mathbf{X}_t| > \epsilon\, x_n\}} > (1-\delta) x_n^\alpha\,,\ |\mathbf{X}_1| > \epsilon\, x_n\big) \Big\} \\
&+ \mathbb{P}(|\mathbf{X}_1|^\alpha > (1-\delta) x_n^\alpha).
\end{aligned}$$

Then, condition **AC** yields for all $K > 0$,

$$= \sum_{j=K}^{n-1}\Big\{\mathbb{P}\big(\sum_{t=1}^{K}|\mathbf{X}_t|^\alpha \mathbb{1}_{\{|\mathbf{X}_t|>\epsilon\, x_n\}} > (1-\delta)x_n^\alpha\,,\ |\mathbf{X}_1| > \epsilon\, x_n\big)$$

$$- \mathbb{P}\big(\sum_{t=2}^{K}|\mathbf{X}_t|^\alpha \mathbb{1}_{\{|\mathbf{X}_t|>\epsilon\, x_n\}} > (1-\delta)x_n^\alpha\,,\ |\mathbf{X}_1| > \epsilon\, x_n\big)\Big\}$$

$$+ \mathbb{P}(|\mathbf{X}_1|^\alpha > (1-\delta)x_n^\alpha) + o(n\,\mathbb{P}(|\mathbf{X}_0| > x_n)).$$

Then, writing $(\mathbf{\Theta}_t)_{t\in\mathbb{Z}}$ as in (6), we obtain

$$I := \lim_{n\to+\infty} \frac{\mathbb{P}\big(\sum_{t=1}^{n}|\mathbf{X}_t|^\alpha \mathbb{1}_{\{|\mathbf{X}_t|>\epsilon\, x_n\}} > (1-\delta)x_n^\alpha\big)}{n\,\mathbb{P}(|\mathbf{X}_0| > x_n)}$$

$$= \lim_{\epsilon\downarrow 0}\lim_{K\to+\infty}\big\{\epsilon^{-\alpha}\int_1^\infty \mathbb{P}\big(\sum_{t=0}^{K}|\epsilon y\mathbf{\Theta}_t|^\alpha \mathbb{1}_{\{y|\mathbf{\Theta}_t|>1\}} > (1-\delta)\big)$$

$$- \mathbb{P}\big(\sum_{t=1}^{K}|\epsilon y\mathbf{\Theta}_t|^\alpha \mathbb{1}_{\{y|\mathbf{\Theta}_t|>1\}} > (1-\delta)\big)d(-y^{-\alpha})\big\}.$$

To take the limit as n goes to infinity, notice that the points of discontinuity are contained in $\cup_{t=1}^{K}\{Y|\mathbf{\Theta}_t| = 1\}$, which has zero probability. Then, we take the limit as $K \to +\infty$ within the integral, which is justified by monotone convergence. Furthermore, the change of coordinates $u = \epsilon\, y$ entails

$$I = \lim_{\epsilon\downarrow 0}\big\{\int_\epsilon^\infty \mathbb{P}\big(\sum_{t=0}^{\infty}|y\mathbf{\Theta}_t|^\alpha \mathbb{1}_{\{y|\mathbf{\Theta}_t|>\epsilon\}} > (1-\delta)\big)$$

$$- \mathbb{P}\Big(\sum_{t=1}^{\infty}|y\mathbf{\Theta}_t|^\alpha \mathbb{1}_{\{y|\mathbf{\Theta}_t|>\epsilon\}} > (1-\delta)\big)d(-y^{-\alpha})\big\}.$$

Using again monotone convergence at each term, we can take the limit as ϵ goes to zero. As a result we obtain asymptotic equivalence with the term below

$$I \sim \int_0^\infty \mathbb{P}\big(\sum_{t=0}^{\infty}|y\mathbf{\Theta}_t|^\alpha > (1-\delta)\big)$$

$$- \mathbb{P}\big(\sum_{t=1}^{\infty}|y\mathbf{\Theta}_t|^\alpha > (1-\delta)\big)d(-y^{-\alpha})$$

$$= (1-\delta)^{-1}\mathbb{E}\big[\sum_{t=0}^{\infty}|\mathbf{\Theta}_t|^\alpha - \sum_{t=1}^{\infty}|\mathbf{\Theta}_t|^\alpha\big] \ = \ (1-\delta)^{-1}.$$

In the last step we use that $|\boldsymbol{\Theta}_0| = 1$ a.s.. Finally, we conclude taking the limit as δ goes to zero that the relation (C.3) holds, and this concludes the proof.

C.2. **Proof of Theorem B.3.**

Proof. Let $(\mathbf{Z}_t)_{t\in\mathbb{Z}}$ be an $\mathbb{R}^d$–valued regularly varying time series with index equal to one. We denote partial sums by $\mathbf{S}_n := \sum_{t=1}^n \mathbf{Z}_t$, and we introduce the truncation notation, for $\epsilon > 0$,

$$\begin{aligned}\overline{\mathbf{S}_n/a_n}^{\epsilon} &:= \textstyle\sum_{t=1}^n \mathbf{Z}_t/a_n \mathbb{1}_{\{|\mathbf{Z}_t|\le \epsilon\, a_n\}},\\ \underline{\mathbf{S}_n/a_n}_{\epsilon} &:= \textstyle\sum_{t=1}^n \mathbf{Z}_t/a_n \mathbb{1}_{\{|\mathbf{Z}_t|> \epsilon\, a_n\}}.\end{aligned}$$

More generally, for $\mathbf{x}\in\mathbb{R}^d$, we write $\underline{\mathbf{x}}_\epsilon$ for $\mathbf{x}\mathbb{1}_{|\mathbf{x}|>\epsilon}$, and $\overline{\mathbf{x}}^\epsilon$ for $\mathbf{x}\mathbb{1}_{|\mathbf{x}|\le\epsilon}$. We also consider a truncation of the centering sequence (d_n) as

$$\underline{d_n/a_n}_{\epsilon} = \mathbb{E}[|\mathbf{Z}_0/a_n|\mathbb{1}_{\{\epsilon\, a_n<|\mathbf{Z}_0|\le a_n\}}], \quad n\in\mathbb{N}.$$

To simplify we denote the cluster process $(\mathbf{Q}_t^Z)_{t\in\mathbb{Z}}$ by $(\mathbf{Q}_t)$, taking values in $(\mathbb{R}^d, |\cdot|)$.

To begin, notice the multivariate mixing condition $\mathbf{MX}(k_n)$ implies there exists a sequence $k := k_n \to +\infty$, as $n\to+\infty$, such that for $\mathbf{u}\in\mathbb{R}^d$,

$$\begin{aligned} II &= \mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top(\mathbf{S}_n/a_n - n\,d_n/a_n)\big\}\big]\\ &\sim \mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top(\mathbf{S}_k/a_n - k\,d_n/a_n)\big\}\big]^{\lfloor n/k\rfloor}.\end{aligned}$$

as $n\to+\infty$. Then, taking the logarithm at both sides yields

$$\begin{aligned}\log II &\sim \frac{n}{k}\log\mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top(\mathbf{S}_k/a_n - k\,d_n/a_n)\big\}\big]\\ &\sim \frac{\mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top(\mathbf{S}_k/a_n - k\,d_n/a_n)\big\}\big]-1}{k\,\mathbb{P}(|\mathbf{Z}_0|>a_n)}.\end{aligned}$$

This step is granted since $k\, d_n/a_n \to 0$, and also $\mathbf{S}_k/a_n \overset{\mathbb{P}}{\to} 0$, as $n \to +\infty$. Moreover,

$$\log II \;\sim\; \frac{\mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top(\underline{\mathbf{S}_k/a_n}_\epsilon - k\,\underline{d_n/a_n}_\epsilon)\big\}\big] - 1}{k\,\mathbb{P}(|\mathbf{Z}_0| > a_n)}.$$

This last relation follows by condition **CS**, since the the exponential function is bounded by $|e^{ix} - e^{iy}| \le |x-y| \wedge 1$, for all $x, y \in \mathbb{R}$.

Moreover, a Taylor expansion yields

$$\begin{aligned}
&\Big|\big(\mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top(\underline{\mathbf{S}_k/a_n}_\epsilon - k\,\underline{d_n/a_n}_\epsilon)\big\}\big] - 1\big) \\
&\qquad - \big(\mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top\,\underline{\mathbf{S}_k/a_n}_\epsilon\big\}\big] - 1 - i\mathbb{E}[\sin(\mathbf{u}^\top\,\overline{\underline{\mathbf{S}_k/a_n}_\epsilon}^1)]\big)\Big| \\
&\qquad\qquad = O\big(k\,\underline{d_n/a_n}_\epsilon \mathbb{E}[|\underline{\mathbf{S}_k/a_n}_\epsilon|\mathbb{1}(|\underline{\mathbf{S}_k/a_n}_\epsilon| \le 1)]\big),
\end{aligned}$$

and $|k\,\underline{d_n/a_n}_\epsilon \mathbb{E}[|\underline{\mathbf{S}_k/a_n}_\epsilon|\mathbb{1}(|\underline{\mathbf{S}_k/a_n}_\epsilon| \le 1)]|/k\,\mathbb{P}(|\mathbf{Z}_0| > a_n) \to 0$, as n goes to infinity. Thus, the asymptotic equivalence below holds.

$$\begin{aligned}
&\log\mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top(\underline{\mathbf{S}_n)/a_n}_\epsilon - n\,\underline{d_n/a_n}_\epsilon)\big\}\big] \\
&\sim\; \frac{\mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top\,\underline{\mathbf{S}_k/a_n}_\epsilon\big\} - 1 - i\sin(u\,\overline{\underline{\mathbf{S}_k/a_n}_\epsilon}^1)\big]}{k\,\mathbb{P}(|\mathbf{Z}_0| > a_n)}.
\end{aligned}$$

as $n \to +\infty$. Furthermore,

$$\begin{aligned}
&\mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top\,\underline{\mathbf{S}_k/a_n}_\epsilon\big\} - 1 - i\sin(\mathbf{u}^\top\,\overline{\underline{\mathbf{S}_k/a_n}_\epsilon}^1)\big] \\
&=\; \mathbb{E}\big[\big(\exp\big\{i\,\mathbf{u}^\top\,\underline{\mathbf{S}_k/a_n}_\epsilon\big\} - 1)\big)\mathbb{1}_{\{|\mathbf{S}_k| > \epsilon\,a_n\}}\big] \\
&-\; \mathbb{E}\big[i\sin(\mathbf{u}^\top\,\overline{\underline{\mathbf{S}_k/a_n}_\epsilon}^1)\mathbb{1}_{\{|\mathbf{S}_k| > \epsilon\,a_n\}}\big].
\end{aligned}$$

Then, conditioning to the event $\{|\mathbf{S}_k| > \epsilon\,a_n\}$, we use the limit relation in (B.1) and Proposition 4.2. in [5] and take the limit as n goes to infinity in the above expression. Hence,

$$\frac{\mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top\underline{\mathbf{S}_k/a_n}_\epsilon\big\}-1-i\sin(\mathbf{u}^\top\,\overline{\underline{\mathbf{S}_k/a_n}_\epsilon}^1)\big]}{k\mathbb{P}(|\mathbf{Z}_0|>a_n)}$$
$$\sim\ \int_0^\infty\mathbb{E}\big[\exp\big\{i\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}}\underline{y\mathbf{Q}_t}_\epsilon\big\}-1-i\sin\big(\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}}(\overline{\underline{y\mathbf{Q}_t}_\epsilon}^1)\big)\big]d(-y^{-1}),$$

where $(\mathbf{Q}_t)_{t\in\mathbb{Z}}$ is the cluster process of the stationary process $(\mathbf{Z}_t)$. In particular, it takes values in $\mathbb{R}^\mathbb{Z}$ and verifies $\sum_{t\in\mathbb{Z}}|\mathbf{Q}_t| = 1$ with probability one.

Furthermore, let $\delta > 0$ and let's divide the integral above on the events $\{y > \delta\}$ and $\{y \le \delta\}$. On the event $\{y \le \delta\}$, for $\delta < 1$,

$$\int_0^\delta\mathbb{E}\big[\exp\big\{i\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}}\underline{y\mathbf{Q}_t}_\epsilon\big\}-1-i\sin\big(\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}}\overline{\underline{y\mathbf{Q}_t}_\epsilon}^1\big)\big]\,d(-y^{-1})$$
$$=\int_0^\delta\mathbb{E}\big[\exp\big\{i\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}}\underline{y\mathbf{Q}_t}_\epsilon\big\}-1-i\sin\big(\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}}\underline{y\mathbf{Q}_t}_\epsilon\big)\big]\,d(-y^{-1}).$$

Recall the inequality

$$|\exp\{iz\}-1-i\sin(z)|\ \le\ |z|^2$$

for all $z \in \mathbb{R}$. Then, the integral above is bounded in absolute value by

$$\int_0^\delta\mathbb{E}\big[\big|\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}}|y\mathbf{Q}_t|\big|^2\big]d(-y^{-1})$$
$$\le\ \delta\mathbb{E}\big[\big|\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}}|\mathbf{Q}_t|\big|^2\big]\ =\ \delta|\mathbf{u}|^2<+\infty.$$

We conclude that

$$\log\mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top\,(\mathbf{S}_k/a_n-n\,d_n/a_n)\big\}\big]$$
$$\sim\ \lim_{\delta\to 0}\int_\delta^\infty\mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}}y\mathbf{Q}_t\big\}-1-i\sin\big(\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}}\overline{y\mathbf{Q}_t}^1\big)\big]d(-y^{-1}).$$

We now write the term above as the sum of two integrals as follows

$$\lim_{\delta\to 0}\int_\delta^\infty \mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}} y\mathbf{Q}_t\big\} - 1 - i\sin\big(\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}} y\mathbf{Q}_t\big)\big]\,d(-y^{-1})$$
$$+\,i\,\mu(\mathbf{u})$$
$$= III + i\mu(\mathbf{u})$$

where

$$\mu(\mathbf{u}) = \int_1^\infty \mathbb{E}\big[\sin\big(\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}} y\mathbf{Q}_t\big)$$
$$-\ \sin\big(\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}} y\mathbf{Q}_t - \mathbf{u}^\top\sum_{t\in\mathbb{Z}}\underline{y\mathbf{Q}_{t}}_1\big)\big]\,d(-y^{-1})$$

To simplify the expression above, we recall the trigonometric relation

$$\sin(p) - \sin(p-q) \ = \ 2\sin(p/2)\cos(p-(q/2)),$$

for $p, q \in \mathbb{R}$. Then, $\mu(\mathbf{u})$ is bounded in absolute value by one since $\int_1^{+\infty} y^{-2} = 1$. We interpret the term $\mu(\mathbf{u})$ as a location parameter.

Finally, using the bound previously derived, we can take the limit as δ goes to 0 in III which yields

$$\log\mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top(\mathbf{S}_k/a_n - n\,d_n/a_n)\big\}\big]$$
$$\sim \int_0^\infty \mathbb{E}\big[\exp\big\{i\,\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}} y\mathbf{Q}_t\big\} - 1 - i\sin\big(\mathbf{u}^\top\textstyle\sum_{t\in\mathbb{Z}} y\mathbf{Q}_t\big)\big]\,d(-y^{-1})$$
$$+\,i\,\mu(\mathbf{u})$$

as $n \to +\infty$. This shows the limit relation from equation (B.2), and this concludes the proof. □

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