Introducing smooth amnesia to the memory of the Elephant Random Walk

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Abstract

This paper is devoted to the asymptotic analysis of the amnesic elephant random walk (AERW) using a martingale approach. More precisely, our analysis relies on asymptotic results for multidimensional martingales with matrix normalization. In the diffusive and critical regimes, we establish the almost sure convergence and the quadratic strong law for the position of the AERW. The law of iterated logarithm is given in the critical regime. The distributional convergences of the AERW to Gaussian processes are also provided. In the superdiffusive regime, we prove the distributional convergence as well as the mean square convergence of the AERW.

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1 Introduction

The Elephant Random Walk (ERW) is a discrete-time random walk, introduced by Schütz and Trimper [21] in the early 2000s. At first, the ERW was used in order to see how long-range memory affects the random walk and induces a crossover from a diffusive to superdiffusive behavior. It was referred to as the ERW in allusion to the traditional saying that elephants can always remember anywhere they have been. The elephant starts at the origin at time zero, $S_0 = 0$. At time $n = 1$, the elephant moves one step to the right with probability q and to the left with probability $1 - q$ for some q in $[0, 1]$. Afterwards, at time $n + 1$, the elephant chooses uniformly at random an integer k among the previous times $1, \ldots, n$. Then, it moves exactly in the same direction as that of time k with probability p or the opposite direction with the probability $1 - p$, where the parameter p stands for the memory parameter of the ERW. The position of the elephant at time $n + 1$ is given by

$$S_{n+1} = S_n + X_{n+1} \tag{1.1}$$

where X_{n+1} is the $(n + 1)$-th increment of the random walk, such that

$$X_{n+1} = \alpha_{n+1} X_{\beta_{n+1}} \tag{1.2}$$

where $\alpha_{n+1} \sim \mathcal{R}(p)$ and $\beta_{n+1} \sim \mathcal{U}(1, n)$ are mutually independent and independant of the past. The ERW shows three differents regimes depending on the location of its memory parameter p with respect to the critical value $p = 3/4$.

On the one hand, a wide literature is now available on the ERW in dimension $d = 1$ thanks to a variety of approaches. Baur and Bertoin [2] used the connection to Pólya-type urns as well as functional limit theorems for multitype branching processes due to Janson [16]. Bercu [3]

and Coletti et al. [11] used martingales to obtain the almost sure convergence and asymptotic normality, among other results. Kürsten [18] and Businger [9] used the construction of random trees with Bernoulli percolation. A strong law of large numbers and a central limit theorem for the position of the ERW, properly normalized, were established in the diffusive regime $p < 3/4$ and the critical regime $p = 3/4$, see [2, 3, 11, 10, 23]. In the superdiffusive regime $p > 3/4$, Bercu [3] proved that the limit of the position of the ERW is not Gaussian and Kubota and Takei [17] showed that the fluctuation of the ERW around this limit is Gaussian.
On the other hand, over the last years, various processes derivated from the ERW have recevied a lot of attention. Bercu and Laulin in [6] extended all the results of [3] to the multidimensional ERW (MERW) where $d \geq 1$ and to its center of mass [7] using a martingale approach, while Bertenghi used the connection [8] to Pólya-type urns for the MERW. The ERW with stops or minimal RW, changing in particular the distribution of α_n, has also been investigated [5, 4, 14, 20]. The ERW with reinforced memory has been studied by Baur [1] via the urn approach, and Laulin [19] using martingales.

The idea of this paper is to use the approach developped in [7] and [19] to study how changing the memory allows us to induce amnesia to the ERW. More precisely, the distribution of the memory β_n of our new variation of the ERW is such that the probability of choosing a fixed instant $k \in \mathbb{N}^*$ at time $n \geq k$ decreases approximatly with speed $(k/n)^\beta$ for some amnesia parameter $\beta \geq 0$.

The very interesting question of amnesic elephant random walk (AERW) has not been investigated a lot. Gut and Stadmüller [15, 13] studied variations of the memory for the special cases of ERW with delays or gradually increasing memory. In [15] the elephant could stop and only remember the first (and second) step it tooks. Consequently, it did not induced a phase transition. In [13], the elephant only remembered a portion of its past (recent or distant), this portion being fixed or depending on the time n, but was always "small".

The entire study we conduct below can be generalized when $\beta < 0$ is not an integer. This can be interpreted as cases where the elephant remembers more vividly the first steps it performed. When $\beta < -1$, it appears that the AERW only have one regime that is the diffusive regime. This observation is coherent with the work of Gut and Stadmüller [13].

The AERW will appear to be non-Markovian, as the reinfroced ERW. However, unlike the reinforced ERW, the AERW can not be studied using Pólya-type urns. The major change for the AERW is that the distribution of the memory β_n in equation (1.2) is no longer uniform but depends on the amnesia parameter $\beta \geq 0$. In this approach, the elephant chooses an instant according to β_{n+1} as follows,

$$\mathbb{P}(\beta_{n+1} = k) = \frac{(\beta+1)\Gamma(k+\beta)\Gamma(n)}{\Gamma(k)\Gamma(n+\beta+1)} = \frac{(\beta+1)}{n}\frac{\mu_k}{\mu_{n+1}} \quad \text{for } 1 \leq k \leq n, \tag{1.3}$$

where

$$\mu_n = \prod_{k=1}^{n-1}\Big(1 + \frac{\beta}{k}\Big) = \frac{\Gamma(n+\beta)}{\Gamma(n)\Gamma(\beta+1)}. \tag{1.4}$$

The case $\beta = 0$ corresponds to the traditionnal ERW. As β grows, the probability of choosing a recent instant gets bigger, see the illustrative Figure 1.

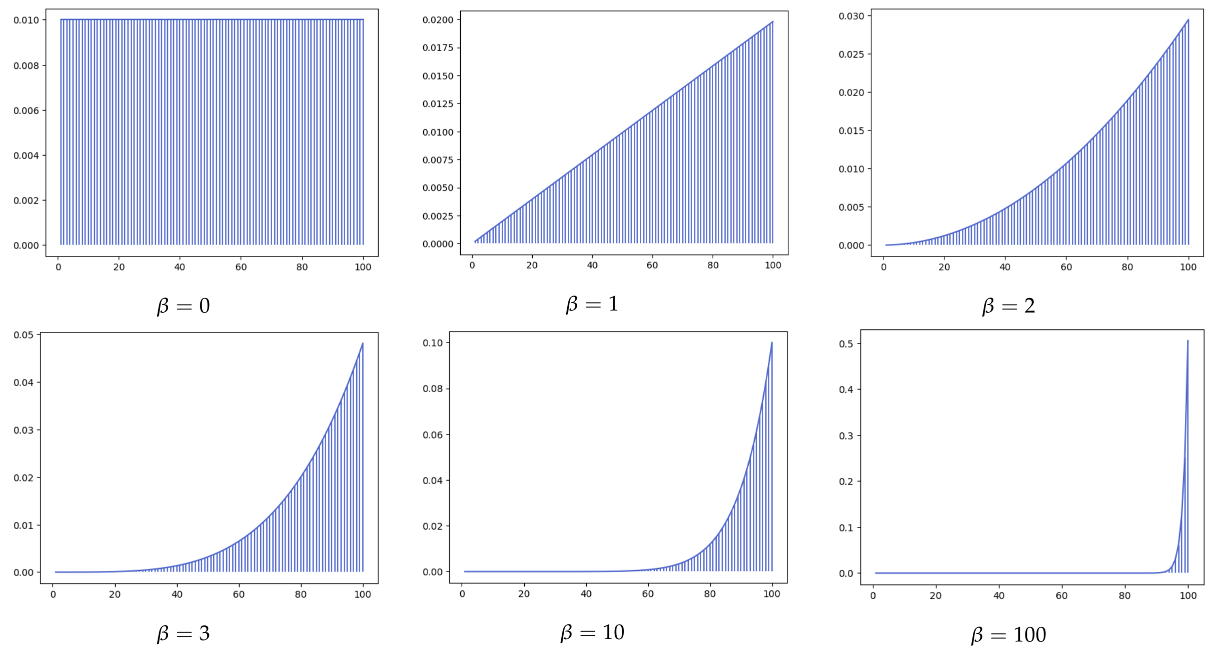

Figure 1: Mass function of the memory depending on the value of β.

We have by definition of the step X_{n+1} given in (1.2) and the distribution β_{n+1} (1.3) that

$$\begin{aligned}\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[\alpha_{n+1}]\mathbb{E}[X_{\beta_{n+1}} \mid \mathcal{F}_n] \\ &= (2p-1)\mathbb{E}\Big[\sum_{k=1}^{n} X_k \mathbb{1}_{\beta_{n+1}=k} \mid \mathcal{F}_n\Big] \\ &= \frac{(2p-1)(\beta+1)}{n\mu_{n+1}} \sum_{k=1}^{n} X_k \mu_k. \end{aligned} \tag{1.5}$$

Then, denote $a = 2p-1$ and

$$Y_n = \sum_{k=1}^{n} X_k \mu_k. \tag{1.6}$$

We deduce from (1.5) that

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \Big(1 + \frac{a(\beta+1)}{n}\Big) Y_n. \tag{1.7}$$

Hereafter, for any $n \geq 1$, let

$$\gamma_n = 1 + \frac{a(\beta+1)}{n} \tag{1.8}$$

and

$$a_n = \prod_{k=1}^{n-1} \gamma_k^{-1} = \frac{\Gamma(n)\Gamma(a(\beta+1)+1)}{\Gamma(n+a(\beta+1))}. \tag{1.9}$$

It follows from standard resultats on the Gamma function that

$$\lim_{n\to\infty} n^{a(\beta+1)} a_n = \Gamma(a(\beta+1)+1) \tag{1.10}$$

and

$$\lim_{n\to\infty} n^{-\beta}\mu_n = \Gamma(\beta+1). \tag{1.11}$$

Our strategy for proving asymptotic results for the AERW is as follows. On the one hand, the behavior of the position S_n is closely related to the one of the sequences (M_n) and (N_n) defined, for all $n \geq 0$, by

$$M_n = a_n Y_n \quad \text{and} \quad N_n = S_n + \frac{a(\beta+1)}{\beta - a(\beta+1)}\mu_n^{-1} Y_n. \tag{1.12}$$

We immediatly get from (1.7) and (1.9) that (M_n) is a locally square-integrable martingale adapted to $(\mathcal{F}_n)$. Moreover, we have from (1.5) that

$$E\Big[S_{n+1} + \frac{a(\beta+1)}{\beta - a(\beta+1)}\mu_{n+1}^{-1} Y_{n+1} \mid \mathcal{F}_n\Big] = S_n + \frac{a(\beta+1)}{\beta - a(\beta+1)}\mu_n^{-1} Y_n$$

which also means that (N_n) is also a locally square-integrable martingale adapted to $\mathcal{F}_n$. On the other hand, we can rewrite S_n as

$$S_n = N_n - \frac{a(\beta+1)}{\beta - a(\beta+1)}(\mu_n a_n)^{-1} M_n \tag{1.13}$$

and equation (1.13) allows us to establish the asymptotic behavior of the AERW via an extensive use of the martingale theory.

The main results of this paper are given in Section 2. We first investigate the diffusive regime and we establish the strong law of large numbers, the law of iterated logarithm and the quadratic strong law for the AERW. The functional central limit theorem is also provided. Next, we prove similar results in the critical regime. Finally, we establish a strong limit theorem in the superdiffusive regime. Our martingale approach is described in Section 3. Finally, we give some of the technical proofs in Section 4.

2 Main results

2.1 The diffusive regime

Our first result deals with the strong law of large numbers for the AERW in the diffusive regime where $p < \frac{4\beta+3}{4(\beta+1)}$. The following strong law for the AERW will be deduced from both the strong laws for (N_n) and (M_n).

Theorem 2.1. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \quad a.s. \tag{2.1}$$

The almost sure rate of convergence for the AERW is as follows, for

$$\sigma_\beta^2 = \frac{2\beta+1-a}{(1-a)(1+2\beta-2a(\beta+1))}.$$

Theorem 2.2. *We have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=1}^{n} \frac{S_k^2}{k^2} = \sigma_\beta^2 \quad a.s. \tag{2.2}$$

Hereafter, we are interested in the distributional convergence of the AERW, which holds in the Skorokhod space $D([0,\infty[)$ of right-continuous functions with left-hand limits.

Theorem 2.3. *The following convergence in distribution in $D([0,\infty[)$ holds*

$$\Big(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}},\ t \geq 0\Big) \Longrightarrow \big(W_t,\ t \geq 0\big) \tag{2.3}$$

where $\big(W_t,\ t\geq 0\big)$ is a real-valued centered Gaussian process starting from the origin with covariance

$$\begin{aligned} \mathbb{E}[W_s W_t] = {}& \frac{a(1+\beta)(1-a)+a\beta}{(2(\beta+1)(1-a)-1)(a-\beta(1-a))(1-a)} s \Big(\frac{t}{s}\Big)^{a-\beta(1-a)} \\ &+ \frac{\beta}{(\beta(1-a)-a)(1-a)} s \end{aligned} \tag{2.4}$$

for $0 < s \leq t$. In particular, we have

$$\frac{S_n}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma_\beta^2\right). \tag{2.5}$$

Remark 2.4. *When $\beta = 0$ we find again the results from [2] for the ERW*

$$\Big(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}},\ t \geq 0\Big) \Longrightarrow \big(W_t,\ t \geq 0\big)$$

where $\big(W_t,\ t\geq 0\big)$ is a real-valued mean-zero Gaussian process starting from the origin and

$$\mathbb{E}[W_s W_t] = \frac{1}{1-2a} s \Big(\frac{t}{s}\Big)^a.$$

2.2 The critical regime

Hereafter, we investigate the critical regime where $p = \frac{4\beta+3}{4(\beta+1)}$. It is interesting to notice that, when β is really large (or $\beta \to \infty$) the critical regime is reached for the memory parameter $p = 1$. Hence, the greater β is, the more there are values of the memory parameter p for which the AERW stays in the diffusive regime; but whatever the value of β, we still observe a phase transition.

Theorem 2.5. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}\log n} = 0 \quad a.s. \tag{2.6}$$

The almost sure rates of convergence for the AERW are as follows.

Theorem 2.6. *We have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} \frac{S_k^2}{(k\log k)^2} = (2\beta+1)^2 \quad a.s. \tag{2.7}$$

In addition, we also have the law of iterated logarithm

$$\limsup_{n\to\infty} \frac{S_n^2}{2n\log n \log\log\log n} = (2\beta+1)^2 \quad a.s. \tag{2.8}$$

Once again, our next result concerns the asymptotic normality of the AERW.

Theorem 2.7. *The following convergence in distribution in $D([0,\infty[)$ holds*

$$\Big(\frac{S_{\lfloor n^t \rfloor}}{\sqrt{n^t \log n}},\ t \geq 0\Big) \Longrightarrow (2\beta+1)\big(B_t,\ t \geq 0\big) \tag{2.9}$$

where $(B_t,\ t \geq 0)$ is a one-dimensional standard Brownian motion. In particular, we have the asymptotic normality

$$\frac{S_n}{\sqrt{n \log n}} \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0, (2\beta+1)^2\Big). \tag{2.10}$$

2.3 The superdiffusive regime

Finally, we focus our attention on the superdiffusive regime where $p > \frac{4\beta+3}{4(\beta+1)}$.

Theorem 2.8. *We have the following distributional convergence in $D([0,\infty[)$*

$$\Big(\frac{S_{\lfloor nt \rfloor}}{n^{a(\beta+1)}},\ t \geq 0\Big) \Longrightarrow (\Lambda_t,\ t \geq 0) \tag{2.11}$$

where the limiting $\Lambda_t = t^{a(\beta+1)} L_\beta$, L_β being some non-denegerate random variable. In particular, we have

$$\lim_{n\to\infty} \frac{S_n}{n^{a(\beta+1)-\beta}} = L_\beta \quad a.s. \tag{2.12}$$

where the limiting L_β is a non-degenerate random variable. We also have the mean square convergence

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big|\frac{S_n}{n^{a(\beta+1)-\beta}} - L_\beta\Big|^2\Big] = 0. \tag{2.13}$$

Remark 2.9. *The expected value of L_β is*

$$\mathbb{E}[L_\beta] = \frac{a(\beta+1)(2q-1)\Gamma(\beta+1)}{\big(a(\beta+1)-\beta\big)\Gamma\big(a(\beta+1)+1\big)} \tag{2.14}$$

while its second order moment is given by

$$\mathbb{E}\big[L_\beta^2\big] = \frac{a^2(\beta+1)^2\Gamma(\beta+1)^2\Gamma\big(2(a-1)(\beta+1)+1\big)}{\big(a(\beta+1)-\beta\big)^2\Gamma\big((2a-1)(\beta+1)+1\big)^2}. \tag{2.15}$$

When $\beta = 0$ we find again the expected values for the ERW from [3]

$$\mathbb{E}[L] = \frac{2q-1}{\Gamma(a+1)} \quad \text{and} \quad \mathbb{E}[L^2] = \frac{1}{(2a-1)\Gamma(2a)}.$$

3 A two-dimensional martingale approach

In order to investigate the asymptotic behavior of (S_n), we introduce the two-dimensional martingale $(\mathcal{M}_n)$ defined by

$$\mathcal{M}_n = \begin{pmatrix} N_n \\ M_n \end{pmatrix} \tag{3.1}$$

where (M_n) and (N_n) are the two locally square-integrable martingales introduced in (1.12). As for the CMERW and the RERW, the main difficulty we face is that the predictable quadratic variations of (M_n) and (N_n) increase to infinity with two different speeds. A matrix normalization will again be necessary to establish the asymptotic behavior of the AERW. We will alternatively study $(\mathcal{M}_n)$, (M_n) or (N_n). Denote the martingale increment $\varepsilon_{n+1} = Y_{n+1} - \gamma_n Y_n$. We obtain that

$$\begin{aligned}
\Delta \mathcal{M}_{n+1} &= \mathcal{M}_{n+1} - \mathcal{M}_n \\
&= \begin{pmatrix} S_{n+1} - S_n + \frac{a(\beta+1)}{\beta - a(\beta+1)} \big(\frac{Y_{n+1}}{\mu_{n+1}} - \frac{Y_n}{\mu_n} \big) \\ a_{n+1} Y_{n+1} - a_n Y_n \end{pmatrix} \\
&= \begin{pmatrix} \big(1 + \frac{a(\beta+1)}{\beta - a(\beta+1)}\big) X_{n+1} - \frac{a(\beta+1)}{(\beta - a(\beta+1))\mu_{n+1}} \frac{\beta}{n} Y_n \\ a_{n+1} \varepsilon_{n+1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\beta}{(\beta - a(\beta+1))\mu_{n+1}} \big(Y_n + X_{n+1}\mu_{n+1} - (\gamma_n - 1) Y_n \big) \\ a_{n+1} \varepsilon_{n+1}. \end{pmatrix}
\end{aligned}$$

Consequently

$$\Delta \mathcal{M}_{n+1} = \begin{pmatrix} \frac{\beta}{(\beta - a(\beta+1))\mu_{n+1}} \\ a_{n+1} \end{pmatrix} \varepsilon_{n+1}.$$

We also obtain that

$$\begin{aligned}
\mathbb{E}[\varepsilon_{n+1}^2 \mid \mathcal{F}_n] &= \mathbb{E}[Y_{n+1}^2 \mid \mathcal{F}_n] - \gamma_n^2 Y_n^2 \\
&= Y_n^2 + 2(\gamma_n - 1) Y_n^2 + \mu_{n+1}^2 - \gamma_n^2 Y_n^2 \\
&= \mu_{n+1}^2 - (\gamma_n - 1)^2 Y_n^2.
\end{aligned} \tag{3.2}$$

Therefore, we deduce that

$$\mathbb{E}\big[(\Delta \mathcal{M}_{n+1})(\Delta \mathcal{M}_{n+1})^T \mid \mathcal{F}_n\big] = \\
(\mu_{n+1}^2 - (\gamma_n - 1)^2 Y_n^2) \begin{pmatrix} \big(\frac{\beta}{(\beta - a(\beta+1))\mu_{n+1}}\big)^2 & \frac{\beta a_{n+1}}{(\beta - a(\beta+1))\mu_{n+1}} \\ \frac{\beta a_{n+1}}{(\beta - a(\beta+1))\mu_{n+1}} & a_{n+1}^2 \end{pmatrix}.$$

We are now able to compute the quadratic variation of $\mathcal{M}_n$

$$\langle \mathcal{M} \rangle_n = \sum_{k=0}^{n-1} \begin{pmatrix} \big(\frac{\beta}{\beta - a(\beta+1)}\big)^2 & \frac{\beta a_{k+1}\mu_{k+1}}{\beta - a(\beta+1)} \\ \frac{\beta a_{k+1}\mu_{k+1}}{\beta - a(\beta+1)} & (a_{k+1}\mu_{k+1})^2 \end{pmatrix} - \xi_n \tag{3.3}$$

where

$$\xi_n = \sum_{k=0}^{n-1} (\gamma_k - 1)^2 Y_k^2 \begin{pmatrix} \big(\frac{\beta}{(\beta - a(\beta+1))}\big)^2 & \frac{\beta a_{k+1}\mu_{k+1}}{(\beta - a(\beta+1))} \\ \frac{\beta a_{k+1}\mu_{k+1}}{(\beta - a(\beta+1))} & (a_{k+1}\mu_{k+1})^2 \end{pmatrix}.$$

Hereafter, we immediatly deduce from (3.3) that

$$\langle M \rangle_n = \sum_{k=1}^{n} (a_k \mu_k)^2 - \zeta_n \quad \text{where} \quad \zeta_n = \sum_{k=1}^{n} a_k^2 (\gamma_k - 1)^2 Y_k^2 \tag{3.4}$$

and

$$\langle N \rangle_n = \Big(\frac{\beta}{\beta - a(\beta+1)}\Big)^2 n. \tag{3.5}$$

The asympotic behavior of M_n is closely related to the one of

$$w_n = \sum_{k=1}^{n} (a_k \mu_k)^2 \tag{3.6}$$

as one can observe that we always have $\langle M \rangle_n \leq w_n$ and that ζ_n is negligeable when compared to w_n. Consequently, it follows from the definitions of (a_n) and (μ_n)that we have three regimes of behavior for (M_n). In the diffusive regime where is $p < \frac{4\beta+3}{4(\beta+1)}$ or $a < 1 - \frac{1}{2(\beta+1)}$,

$$\lim_{n \to \infty} \frac{w_n}{n^{1-2(a(\beta+1)-\beta)}} = \ell \quad \text{where} \quad \ell = \frac{1}{1 + 2(\beta - a(\beta+1))} \Big(\frac{\Gamma(a(\beta+1)+1)}{\Gamma(\beta+1)} \Big)^2. \tag{3.7}$$

In the critical regime where $p = \frac{4\beta+3}{4(\beta+1)}$ or $a = 1 - \frac{1}{2(\beta+1)}$,

$$\lim_{n \to \infty} \frac{w_n}{\log n} = \Big(\frac{\Gamma(\beta + 1 + \frac{1}{2})}{\Gamma(\beta+1)} \Big)^2. \tag{3.8}$$

In the superdiffusive regime where $p > \frac{4\beta+3}{4(\beta+1)}$ or $a > 1 - \frac{1}{2(\beta+1)}$,

$$\lim_{n \to \infty} w_n = \sum_{k=1}^{\infty} \Big(\frac{\Gamma(a(\beta+1)+1)\Gamma(k+\beta)}{\Gamma(k + a(\beta+1))\Gamma(\beta+1)} \Big)^2 < +\infty. \tag{3.9}$$

4 Proofs of the main results

4.1 The diffusive regime

Lemma 4.1. *Let (V_n) be the sequence of positive definite diagonal matrices of order 2 given by*

$$V_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 0 \\ 0 & \frac{a(\beta+1)}{\beta - a(\beta+1)} (a_n \mu_n)^{-1} \end{pmatrix}. \tag{4.1}$$

Let $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ such that

$$v^T V_n \mathcal{M}_n = \frac{S_n}{\sqrt{n}}. \tag{4.2}$$

The quadratic variation of $\langle \mathcal{M} \rangle_n$ satisfies in the diffusive regime where is $a < 1 - \frac{1}{2(\beta+1)}$,

$$\lim_{n \to \infty} V_n \langle \mathcal{M} \rangle_n V_n^T = V \quad a.s. \tag{4.3}$$

where the matrix V is given by

$$V = \frac{1}{(\beta - a(\beta+1))^2} \begin{pmatrix} \beta^2 & \frac{a\beta}{1-a} \\ \frac{a\beta}{1-a} & \frac{a^2(\beta+1)^2}{1+2\beta-2a(\beta+1)}. \end{pmatrix} \tag{4.4}$$

Remark 4.2. *Following the same steps as in the proof of Lemma 4.1, we find that in the critical regime $a = 1 - \frac{1}{2(\beta+1)}$, the sequence of normalization matrices (V_n) has to be replaced by*

$$W_n = \frac{1}{\sqrt{n \log n}} \begin{pmatrix} 1 & 0 \\ 0 & (2\beta+1)(a_n \mu_n)^{-1} \end{pmatrix}. \tag{4.5}$$

The limit matrix V also need to be replaced by

$$W = (2\beta+1)^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.6}$$

Proof of Lemma 4.1. We obtain from Theorem 2.1, equations (1.10) and (3.7) that

$$\begin{aligned}
&\lim_{n\to\infty} V_n \langle \mathcal{M}\rangle_n V_n^T \\
&= \lim_{n\to\infty} \frac{1}{n} \begin{pmatrix} \sum_{k=0}^{n-1} \left(\frac{\beta}{(\beta-a(\beta+1))}\right)^2 & \frac{a(\beta+1)\beta}{(\beta-a(\beta+1))^2 a_n\mu_n} \sum_{k=0}^{n-1} a_{k+1}\mu_{k+1} \\ \frac{a(\beta+1)\beta}{(\beta-a(\beta+1))^2 a_n\mu_n} \sum_{k=0}^{n-1} a_{k+1}\mu_{k+1} & \left(\frac{a(\beta+1)}{(\beta-a(\beta+1))a_n\mu_n}\right)^2 \sum_{k=0}^{n-1} (a_{k+1}\mu_{k+1})^2 \end{pmatrix} \\
&= \frac{1}{(\beta-a(\beta+1))^2} \begin{pmatrix} \beta^2 & \frac{a(\beta+1)\beta}{\beta+1-a(\beta+1)} \\ \frac{a(\beta+1)\beta}{\beta+1-a(\beta+1)} & \frac{a(\beta+1)^2}{2(\beta-a(\beta+1))+1} \end{pmatrix}
\end{aligned}$$

which is exactly what we wanted to prove.

■

Proof of Theorem 2.1. We shall make extensive use of the strong law of large numbers for martingales given, e.g. by theorem 1.3.24 of [12]. First, we have for (M_n) that for any $\gamma > 0$,

$$M_n^2 = O\big((\log w_n)^{1+\gamma} w_n\big) \quad \text{a.s.}$$

which by definition of M_n and as a_n is asymptotically equivalent to $n^{-a(\beta+1)}$ and w_n is asymptotically equivalent to $n^{1+2(\beta-a(\beta+1))}$ ensures that

$$\frac{Y_n^2}{n^2} = O\Big((\log n)^{1+\gamma} \frac{n^{1+2(\beta-a(\beta+1))}}{n^{2(1-a(\beta+1))}}\Big) \quad \text{a.s.}$$

Finally as μ_n is asymptotically equivalent to n^{β}, we obtain that

$$\frac{Y_n^2}{(\mu_n n)^2} = O\Big(\frac{(\log n)^{1+\gamma}}{n}\Big) \quad \text{a.s.}$$

which reduces to

$$\lim_{n\to\infty} \frac{Y_n}{\mu_n n} = 0 \quad \text{a.s.} \tag{4.7}$$

We now focus our attention on (N_n). By the same token as before, we have that for any $\gamma > 0$,

$$N_n^2 = O\big((\log n)^{1+\gamma} n\big) \quad \text{a.s.}$$

which by definition of (N_n) gives us

$$\frac{\left(S_n - \frac{a(\beta+1)}{\beta-a(\beta+1)}\mu_n^{-1} Y_n\right)^2}{n^2} = O\Big(\frac{(\log n)^{1+\gamma}}{n}\Big) \quad \text{a.s.}$$

and we conclude that

$$\lim_{n\to\infty} \frac{S_n}{n} - \frac{a(\beta+1)}{\beta-a(\beta+1)} \frac{Y_n}{\mu_n n} = 0 \quad \text{a.s.} \tag{4.8}$$

This achieves the proof of Theorem 2.1 as the convergences (4.7) and (4.8) hold almost surely.

■

Proof of Theorem 2.3. In order to apply Theorem A.2 from [19], we must verify that (H.1), (H.2) and (H.3) are satisfied.

(H.1) We have from (4.3) and the fact that $a_{\lfloor nt \rfloor}$ is asymtotically equivalent to $t^{-a(\beta+1)}a_n$ that

$$V_n \langle \mathcal{M} \rangle_{\lfloor nt \rfloor} V_n^T \underset{n\to\infty}{\longrightarrow} V_t \quad \text{a.s.}$$

where

$$V_t = \frac{1}{(\beta - a(\beta+1))^2} \begin{pmatrix} \beta^2 t & \dfrac{a\beta}{1-a} t^{1+\beta-a(\beta+1)} \\ \dfrac{a\beta}{1-a} t^{1+\beta-a(\beta+1)} & \dfrac{a^2(\beta+1)^2}{1+2\beta-2a(\beta+1)} t^{1+2\beta-2a(\beta+1)} \end{pmatrix}.$$

(H.2) In order to verify that Lindeberg's condition is satisfied, we start by deducing from (1.12) together with (3.1) and V_n given by (4.1) that for all $1 \leq k \leq n$

$$V_n \Delta \mathcal{M}_k = \frac{1}{(\beta - a(\beta+1))\sqrt{n}\mu_n} \begin{pmatrix} \beta \frac{\mu_n}{\mu_k} \\ a \frac{a_k}{a_n} \end{pmatrix} \varepsilon_k$$

which implies that

$$\|V_n \Delta \mathcal{M}_k\|^2 = \frac{1}{(\beta - a(\beta+1))^2 n} \Big(\frac{\beta^2}{\mu_k^2} + \frac{a^2 a_k^2}{(a_n \mu_n)^2} \Big) \varepsilon_k^2. \tag{4.9}$$

Consequently, we obtain that for all $\varepsilon > 0$,

$$\sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \mid \mathcal{F}_{k-1}\big] \leq \frac{1}{\varepsilon^2} \sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^4 \mid \mathcal{F}_{k-1}\big]. \tag{4.10}$$

It follows from (1.10) that

$$a_n^{-2} \sum_{k=1}^{n} a_k^2 = O(n) \quad \text{and} \quad a_n^{-4} \sum_{k=1}^{n} a_k^4 = O(n).$$

Hence, using that the sequence (ε_n) is bounded

$$\sup_{1 \leq k \leq n} |\varepsilon_k| \leq \sup_{1 \leq k \leq n} (\beta+2)\mu_k \leq (\beta+2)\mu_n \quad \text{a.s.} \tag{4.11}$$

we find that

$$\sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^4 \mid \mathcal{F}_{k-1}\big] = O\Big(\frac{1}{n}\Big) \quad \text{a.s.}$$

which ensures that Lindeberg's condition (H.2) holds almost surely, that is for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \mid \mathcal{F}_{k-1}\big] = 0 \quad \text{a.s.} \tag{4.12}$$

Since $V_n V_{\lfloor nt \rfloor}^{-1}$ converges, we immediatly obtain that

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \mid \mathcal{F}_{k-1}\big] &\leq \lim_{n\to\infty} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^4\big] \\ &\leq \lim_{n\to\infty} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\big[\|(V_n V_{\lfloor nt \rfloor}^{-1}) V_{\lfloor nt \rfloor} \Delta \mathcal{M}_k\|^4\big] \\ &= 0 \quad \text{a.s.} \end{aligned}$$

(H.3) In this particular case, we have $V_t = tK_1 + t^{\alpha_2}K_2 + t^{\alpha_3}K_3$ where

$$\alpha_2 = 1 - a(\beta+1) > 0 \quad \text{and} \quad \alpha_3 = 1 - 2a(\beta+1) > 0$$

as $a < 1 - \frac{1}{2(\beta+1)}$, and the matrix are symmetric

$$K_1 = \frac{\beta^2}{(\beta - a(\beta+1))^2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ K_2 = \frac{a\beta}{(1-a)(\beta - a(\beta+1))^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$K_3 = \frac{a^2(\beta+1)^2}{(1+2\beta-2a(\beta+1))(\beta - a(\beta+1))^2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently, we obtain that

$$\big(V_n\mathcal{M}_{\lfloor nt \rfloor},\ t \geq 0\big) \Longrightarrow \big(\mathcal{B}_t,\ t \geq 0\big)$$

where $\mathcal{B}$ is defined as in Theorem A.2 from [19]. Finally, using the fact that $S_{\lfloor nt \rfloor}$ is asymptotically equivalent to $N_{\lfloor nt \rfloor} + t^{\beta - a(\beta+1)}\frac{a(\beta+1)}{\beta - a(\beta+1)}(\mu_n a_n)^{-1}M_{\lfloor nt \rfloor}$, and multiplying by $u_t = \begin{pmatrix} 1 \\ t^{a(\beta+1)-\beta} \end{pmatrix}$, we conclude

$$\Big(\frac{1}{\sqrt{n}}S_{\lfloor nt \rfloor},\ t \geq 0\Big) \Longrightarrow \big(W_t,\ t \geq 0\big) \tag{4.13}$$

where $W_t = u_t^T\mathcal{B}_t$. It only remains to compute the covariance function of (W_t) that is for $0 \leq s \leq t$

$$\begin{aligned}
\mathbb{E}\big[W_sW_t\big] &= u_s^T\mathbb{E}\big[\mathcal{B}_s\mathcal{B}_t^T\big]u_t \\
&= u_s^TV_su_t \\
&= u_s^T\big(sK_1 + s^{1+\beta-a(\beta+1)}K_2 + s^{1+2\beta-2a(\beta+1)}K_3\big)u_t \\
&= \frac{\beta^2}{(\beta - a(\beta+1))^2}s + \frac{a\beta s^{1+\beta-a(\beta+1)}}{(1-a)(\beta - a(\beta+1))^2}\big(s^{a(\beta+1)-\beta} + t^{a(\beta+1)-\beta}\big) \\
&\quad + \frac{a^2(\beta+1)^2}{(1+2\beta-2a(\beta+1))(\beta-a(\beta+1))^2}s^{1+2\beta-2a(\beta+1)}(st)^{a(\beta+1)-\beta} \\
&= \frac{a(1+\beta)(1-a) + a\beta}{(2(\beta+1)(1-a)-1)(a-\beta(1-a))(1-a)}s\Big(\frac{t}{s}\Big)^{a-\beta(1-a)} \\
&\quad + \frac{\beta}{(\beta(1-a)-a)(1-a)}s.
\end{aligned}$$

■

Proof of Theorem 2.2. We need to check that all the hypotheses of Theorem A.3 in [19] are satisfied. Thanks to Lemma 4.1, hypothesis (H.1) holds almost surely. We also immediately obtain from (4.12) that (H.2) is verified almost surely when $t = 1$.
Hereafter, we need to verify (H.4) is satisfied in the special case $\beta = 2$ that is

$$\sum_{n=1}^{\infty} \frac{1}{\big(\log(\det V_n^{-1})^2\big)^2}\mathbb{E}\big[\|V_n\Delta\mathcal{M}_n\|^4\big|\mathcal{F}_{n-1}\big] < \infty \quad \text{a.s.}$$

We immediately have from (4.1)

$$\det V_n^{-1} = \frac{\beta - a(\beta+1)}{a(\beta+1)} a_n \mu_n \sqrt{n}. \tag{4.14}$$

Hence, we obtain from (1.10) and (4.14) that

$$\lim_{n\to\infty} \frac{\log(\det V_n^{-1})^2}{\log n} = 1 + 2\beta - 2a(\beta+1). \tag{4.15}$$

Therefore, we can replace $\log(\det V_n^{-1})^2$ by $\log n$ in (4.1). Hereafter, we obtain from (4.9) and (4.11) that

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^2} \mathbb{E}\big[\|V_n \Delta\mathcal{M}_n\|^4 \big| \mathcal{F}_{n-1}\big] = O\Big(\sum_{n=1}^{\infty} \frac{1}{(n\log n)^2}\Big). \tag{4.16}$$

Thus, (4.16) guarentees that (H.4) is verified. We are now going to apply the quadratic strong law given by Theorem A.3 in [19]. We get from equation (4.15) that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \Big(\frac{(\det V_k)^2 - (\det V_{k+1})^2}{(\det V_k)^2}\Big) V_k \mathcal{M}_k \mathcal{M}_k^T V_k^T = \big(1 + 2\beta - 2a(\beta+1)\big) V \quad \text{a.s.} \tag{4.17}$$

However, we obtain from (1.10) and (4.14) that

$$\lim_{n\to\infty} n\Big(\frac{(\det V_n)^2 - (\det V_{n+1})^2}{(\det V_n)^2}\Big) = 1 + 2\beta - 2a(\beta+1). \tag{4.18}$$

Finally, we can deduce from (4.2), (4.17) and (4.18) that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{S_k^2}{k^2} = v^T V v \quad \text{a.s.} \tag{4.19}$$

which, together with

$$v^T V v = \frac{2\beta + 1 - a}{(1-a)(1 + 2\beta - 2a(\beta+1))} \tag{4.20}$$

completes the proof of Theorem 2.2.

■

4.2 The critical regime

The proofs of Theorems 2.5 and 2.7 follows essentially the same lines as the ones in the diffusive regimes, provided one exchange V_n with W_n. Hence, they shall not be explicited here.

Proof of Theorem 2.6. The proof of the quadratic strong law (2.7) is left to the reader as it follows essentially the same lines as that of (2.2). The only minor change is that the matrix V_n has to be replaced by the matrix W_n defined in (4.5). We shall now proceed to the proof of the law of iterated logarithm given by (2.8). On the one hand, it follows from (1.10) and (3.7) that

$$\sum_{n=1}^{+\infty} \frac{a_n^4}{w_n^2} < \infty. \tag{4.21}$$

Moreover, we have from (3.4) and (3.5) that

$$\lim_{n\to\infty} \frac{\langle M\rangle_n}{w_n} = 1 \quad \text{a.s.} \quad \text{and} \quad \lim_{n\to\infty} \frac{\langle N\rangle_n}{n} = \Big(\frac{\beta}{\beta - a(\beta+1)}\Big)^2 \quad \text{a.s.}$$

Consequently, we deduce from the law of iterated logarithm for martingales due to Stout [22], see also Corollary 6.4.25 in [12], that (M_n) satisfies when $a = 1 - 1/2(\beta+1)$

$$\begin{aligned}\limsup_{n\to\infty} \frac{M_n}{(2w_n \log\log w_n)^{1/2}} &= -\liminf_{n\to\infty} \frac{M_n}{(2w_n \log\log w_n)^{1/2}} \\ &= 1 \quad \text{a.s.}\end{aligned}$$

However, as $a_n w_n^{-1/2}$ is asymptotically equivalent to $(n^{2\beta+1}\log n)^{-1/2}$, we immediately obtain from (3.8) that

$$\begin{aligned}\limsup_{n\to\infty} \frac{Y_n}{(2n^{2\beta+1}\log n \log\log\log n)^{1/2}} &= -\liminf_{n\to\infty} \frac{Y_n}{(2n^{2\beta+1}\log n \log\log\log n)^{1/2}} \\ \limsup_{n\to\infty} \frac{n^{-\beta} Y_n}{(2n\log n \log\log\log n)^{1/2}} &= -\liminf_{n\to\infty} \frac{n^{-\beta} Y_n}{(2n\log n \log\log\log n)^{1/2}} \\ &= 1 \quad \text{a.s.}\end{aligned} \tag{4.22}$$

The law of iterated logarithm for martingales also allow us to find that (N_n) satisfies

$$\begin{aligned}\limsup_{n\to\infty} \frac{N_n}{(2n\log\log n)^{1/2}} &= -\liminf_{n\to\infty} \frac{N_n}{(2n\log\log n)^{1/2}} \\ &= \sqrt{4\beta^2} \quad \text{a.s.}\end{aligned}$$

which ensures that

$$\limsup_{n\to\infty} \frac{N_n}{(2n\log n\log\log\log n)^{1/2}} = 0 \quad \text{a.s.}$$

Hence, we deduce from (1.13) and (4.22) that

$$\begin{aligned}\limsup_{n\to\infty} \frac{S_n}{(2n\log n\log\log\log n)^{1/2}} &= \limsup_{n\to\infty} \frac{N_n + (2\beta+1)(\mu_n a_n)^{-1} M_n}{(2n\log n\log\log\log n)^{1/2}} \\ &= \limsup_{n\to\infty} (2\beta+1) \frac{Y_n}{(2n^{2\beta+1}\log n\log\log\log n)^{1/2}} \\ &= -\liminf_{n\to\infty} (2\beta+1) \frac{Y_n}{(2n^{2\beta+1}\log n\log\log\log n)^{1/2}} \\ &= -\liminf_{n\to\infty} \frac{S_n}{(2n\log n\log\log\log n)^{1/2}}.\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}\limsup_{n\to\infty} \frac{S_n^2}{2n\log n\log\log\log n} &= \limsup_{n\to\infty} (2\beta+1)^2 \frac{Y_n^2}{2n\log n\log\log\log n} \\ &= (2\beta+1)^2\end{aligned}$$

which immediately leads to (2.8), thus completing the proof of Theorem 2.6.

■

4.3 The superdiffusive regime

Proof of Theorem 2.8. Hereafter, we shall again make extensive use of the strong law of large numbers for martingales given, e.g. by theorem 1.3.24 of [12] in order to prove (2.12). When $a > 1 - \frac{1}{2(\beta+1)}$, we have from (3.9) that w_n converges. Hence, as $\langle M \rangle_n \leq w_n$, we clealy have that $\langle M \rangle_\infty < \infty$ almost surely and we can conclude that

$$\lim_{n\to\infty} M_n = M \quad \text{a.s.} \quad \text{where} \quad M = \sum_{k=1}^{\infty} a_k \varepsilon_k$$

which by definition of M_n, and as a_n is asymptotically equivalent to $\Gamma(a(\beta+1)+1)n^{-a(\beta+1)}$, ensures that

$$\lim_{n\to\infty} \frac{Y_n}{n^{a(\beta+1)}} = Y \quad \text{a.s.} \quad \text{where} \quad Y = \frac{1}{\Gamma(a(\beta+1)+1)} M. \tag{4.23}$$

Moreover, we still have that for any $\gamma > 0$,

$$N_n^2 = O\big((\log n)^{1+\gamma} n\big) \quad \text{a.s.}$$

which by definition of N_n gives us for all $t \geq 0$

$$\frac{\big(S_n + \frac{a(\beta+1)}{\beta - a(\beta+1)}(\mu_n)^{-1} Y_n\big)^2}{n^{2a(\beta+1)-2\beta}} = O\Big(\frac{(\log n)^{1+\gamma}}{n^{2a(\beta+1)-2\beta-1}}\Big) \quad \text{a.s.}$$

We know that $a > 1 - \frac{1}{2(\beta+1)}$ in the superdiffusive regime, which ensures that $2a(\beta+1) - 2\beta - 1 > 0$. Then, we obtain thanks to (1.11) and (4.7) that for all $t \geq 0$

$$\lim_{n\to\infty} \frac{S_{\lfloor nt \rfloor}}{\lfloor nt \rfloor^{a(\beta+1)-\beta}} + \frac{a(\beta+1)}{\beta - a(\beta+1)} \frac{Y_{\lfloor nt \rfloor}}{\lfloor nt \rfloor^{a(\beta+1)}} = 0 \quad \text{a.s.} \tag{4.24}$$

The convergences (4.23) and (4.24) hold almost surely and $\lfloor nt \rfloor$ is asymptotically equivalent to nt which implies

$$\lim_{n\to\infty} \frac{S_{\lfloor nt \rfloor}}{n^a(\beta+1)} = t^{a(\beta+1)} L_\beta \quad \text{a.s.} \tag{4.25}$$

Finally, the fact that (4.25) holds almost surely ensures that it also holds for the finite-dimensional distributions, and we obtain (2.11) with $\Lambda_t = t^{a(\beta+1)} L_\beta$ and $L_\beta = \frac{a(\beta+1)}{a(\beta+1)-\beta} Y$.

We shall now proceed to the proof of the mean square convergence (2.13). On the one hand, as $M_0 = 0$ we have from (3.4) that

$$\mathbb{E}\big[M_n^2\big] = \mathbb{E}\big[\langle M \rangle_n\big] \leq w_n.$$

Hence, we obtain from (3.9) that

$$\sup_{n\geq 1} \mathbb{E}\big[M_n^2\big] < \infty$$

which ensures that the martingale (M_n) is bounded in $\mathbb{L}^2$. Therefore, we have the mean square convergence

$$\lim_{n\to\infty} \mathbb{E}\big[|M_n - M|^2\big] = 0$$

which implies that

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big|\frac{Y_n}{n^{a(\beta+1)}} - Y\Big|^2\Big] = 0. \tag{4.26}$$

On the other hand, for any $n \geq 0$, the martingale (N_n) satisfies

$$\mathbb{E}[N_n^2] = \mathbb{E}[\langle N \rangle_n] \leq \Big(\frac{\beta}{\beta - a(\beta+1)}\Big)^2 n$$

and since $a(\beta+1) - \beta > \frac{1}{2}$ we obtain

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big|\frac{N_n}{n^{a(\beta+1)-\beta}}\Big|^2\Big] = 0. \tag{4.27}$$

Finally, we obtain the mean square convergence (2.13) from (4.26) and (4.27) and we achieve the proof of Theorem 2.8.

■

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