Mathematical strategies to quantify exactly the forces acting on the nonaxisymmetric capillary bridges

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Abstract The didactic object of these developments on differential geometry of curves and surfaces is to present fine and convenient mathematical strategies, adapted to the study of capillary bridges from experimental data and relatively simple to use in practice. The common thread is to be able to calculate accurately in any situation the bending stress over the free surface Σ, represented mathematically by the integral of the Gaussian curvature over the surface (called the total curvature) and also to obtain an information concerning the capillary tension forces by term by term integrating the generalized Young-Laplace equation. We mainly develop three convenient mathematical tools for assessing the physical properties in the field of the axisymmetric or not capillary bridges with convex or nonconvex plane boundaries, according to the local wettability and roughness effects: the unit speed reparameterization (or by arc length) of a regular curve and in particular for surfaces of revolution, the Fenchel's theorem and the Gauss-Bonnet-Binet theorem that expresses a relation between the integral of the Gaussian curvature over the surface, the topology of the surface and the integrals of the geodesic curvatures which are directly linked to the wetting angle at the contact lines. We express also the resultant of the bending energy only with respect to the wetting angles at the contact line.

Keywords: Distortion of nonaxisymmetric capillary bridges · Mean and Gaussian curvatures impact · Euler characteristic · Generalized Young-Laplace equation · Bending effects · Fenchel's theorem in differential geometry · Gauss-Bonnet Theorem.· Geodesic curvature · Bending stress · Influence of the contact angles.

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1 Introduction

In this work, we present various results and complementary strategies of mathematical analysis that can be applied to concrete capillary bridges problems, concerning in a new way, the Gauss-Bonnet and Fenchel's theorems to establish various analytical formulas easy to use, concerning firstly the case of the rather delicate approach to nonaxisymmetric capillary bridges distortions. This article expands significantly on the partial results presented in [11], concerning the only surfaces of revolution (circular boundaries, which makes these cases much easier, especially for the explicit calculation of the total geodesic curvatures of the boundaries, rarely possible in practice by the integral calculus).

The key to achieving generalization is a direct consequence of the Fenchel's theorem in differential geometry [3] [20] : *the total curvature of any smooth closed convex plane curve equals* 2π, which avoids a lot of dead-end integration calculations. We also indicate how to calculate precisely the total curvature of generic expression $2k\pi$, $k \neq 1$ of any closed nonconvex plane curve, k being an integer, the index of the curve, the winding number. Consequently, in other words, the index k of a closed plane curve, convex or nonconvex, can be considered as the total curvature divided by 2π (the winding index in algebraic topology). In the specific context of this study, the Fenchel's theorem and the Gauss–Bonnet-Binet theorems are very strongly complementary, with the advantage of explicit calculative answers.

These developments relate in the first part to surfaces of revolution on the basis of an unit speed reparameterization (or by arc length) for a regular curve, in this case, the semi-meridian. For detailed presentations of the subject, the reader may refer to [3] p. 161-164, [25] p. 161-162 and also to [2][12][13][17].

A convenient conceptual bridge between differential geometry and kinematics is to imagine that the semi-meridian represents the successive positions of a moving object and from now on, we get the position of the object as a function of how far the object has traveled along the rotating curve. By this choice of calculation strategy, by reparameterization (always orientation-preserving), the speed is *ipso facto* unit and therefore, the speed and acceleration vectors are orthogonal. The Frenet-Serret apparatus is then very elegant and convenient to use.

To illustrate the convenience of the method, it is indicated that the calculation and the expression of the Gaussian curvature are very simplified. The calculation of the integral of the Gaussian curvature over the surface of the bridge (the total Gaussian curvature) is then direct and makes it possible to find again the analytical expression of the bending stress given by the subtle theorems of Gauss Bonnet and Fenchel (proportional to the sum of the cosine values of the two observed contact angles, possibly distinct). Moreover, it is also an accessible way to determine explicitly all the surfaces of revolution of constant Gaussian curvature.

It is also a convenient tool for calculating exactly the total curvature of the plane and smooth boundaries of the bridges, mainly in the nonconvex case (the concept of curvature κ (kappa) is indeed a way to measure how sharply a smooth curve turns).

[3] Werner Fenchel (1905-1988), Professor at the universities of Götingen and Copenhagen.

From there, we build an analytical toolbox for the *in situ* methodological evaluation of various properties of these capillary bridges. The hard part would be inverting the speed function that cannot usually be integrated explicitly in terms of usual functions. However, to apply the method, it is not necessary to explain the parameterization; the analytical formulas obtained only take into account the experimental data.

The general non-axisymmetric case requires the much more delicate but not insurmountable use of the Gauss-Bonnet theorem closely associated to the Fenchel's theorem in differential geometry. In a singular way, for a smooth surface of revolution whose regular generating curve does not cross the axis of rotation, we have at hand a classical method in differential geometry of surfaces which is well suited in this situation: the arc length (or unit speed) reparameterization of the rotating curve considered as a trajectory.

2 The analytical framework and the main objectives

2.1 About the generalized Young-Laplace equation, the bending forces and the resulting geometrical effects

The generalized Young-Laplace equation concerns the strong distortions for which the bending effects are modeled by an additional curvature-related term: the introduction of C_K, a multiplier coefficient of the Gaussian curvature K, at the dimension of a force and standing for the bending stress [1][10][15][34]. Under appropriate boundary conditions, the shape of the free interface is then described by the so-called generalized Young-Laplace equation, thus involving both mean and Gaussian curvatures. The resulting strongly nonlinear differential equation, at the downward vertical measurement x linked to the value Δp_0 at $x = 0$, (a spontaneous unknown value) comes in the following form that is detailed and justified more specifically in [10][11][12][15][16][18][22][28][32][33] and gratefully [6] for a very early initiating study:

$$\gamma \left(\frac{1}{\rho_c} + \frac{1}{N} \right) + C_K \frac{1}{\rho_c \; N} = \Delta p_0 - \Delta \rho \; gx \; , \qquad (1)$$

associated with adequate miscellaneous boundary conditions [18], where the force C_K divided by the area $\rho_c N$ stands *via* a pressure for the local bending stress, ρ_c and N being the principal radii of curvature (evaluated algebraically, positively when the curvature is turned into the interior of the capillary bridge) and the pressure deficiency is Δp_0 at $x = 0$.

It is assumed that the different cefficients, implicit unknown *a priori*, as Δp_0, resulting from the final equilibrium, have been previously identified *in situ* from exprerimental data, by solving a linear system, well posed and numerically stable (for example, thanks to a first integral translating a principle of conservation) [9][10][11][12][14].

It is extremely noteworthily [12][31], that this strongly nonlinear differential equation is mathematically isomorphic (the same structure) but with different variables and physical units, to the Gullstrand equation of geometrical optics, which relates the optic power P'_{op} of a thick lens (in diopters, the reciprocal of the equivalent focal length) to its geometry and the properties of the media. For example, the superficial tension γ is equivalent to the refractivity $\frac{n_1}{n_2} - 1$, where n_i is a refractive index, C_K is analogous to the expression $-\left(\frac{n_1}{n_2} - 1\right)^2 \frac{n_2}{n_1} d$, d the lens thickness and Δp corresponds to P'_{op}.

Shear or free energy problems and the longitudinal bending stress of ship hulls have an analogy with the subject [4][27][30][37].
The mathematical modeling and simulations of the petroleum engineering are also concerned by this theoretical topic, in order to obtain for media with periodic microstructure an "equivalent" macroscopic representation, by some statistical or homogenization methods [8], chapter 1.

The bending stress over the free surface Σ is then represented in the following integral form, at the dimension of a force:

$$\mathcal{E}_{bending\ stress} = C_K \int_\Sigma K \ d\Sigma \ ,$$

where K is the Gaussian curvature of the free surface Σ, intrinsic value, in particular independent of the choice of the unit normal vector, and the nondimensional integral is the total curvature[4].

Knowing the exact value of the bending stress, on would deduce from the generalized Young-Laplace equation by a fine numerical integration of the right-hand side, the capillary tension values.
Concerning the capillary tension forces, by term by term integrating over the free surface Σ the generalized Young-Laplace equation, we have for example the relationship between various forces:

$$\gamma \int_\Sigma \left(\frac{1}{\rho_c} + \frac{1}{N}\right) d\Sigma = -C_K \int_\Sigma K \ d\Sigma \ + \int_\Sigma (\Delta p_0 - \Delta\rho \ gx) \ d\Sigma,$$

with the particular situation:

$$\gamma \int_\Sigma \left(\frac{1}{\rho_c} + \frac{1}{N}\right) d\Sigma = -C_K \int_\Sigma K \ d\Sigma + \Delta p_0 \text{ area } (\Sigma)$$

when neglecting gravity effects.

This would allow to have a reasoned opinion on the relative importance of the bending forces; according to an objective criterion, either by relative value or by intrinsic value.

2.2 Homotopic surfaces and Euler characteristic in geometry and algebraic topology.

Recall that the Euler characteristic (or Euler-Poincaré characteristic) is a topological invariant, an integer that describes, according to precise axiomatic principles, the shape or a structure of a topological space regardless of how it is bent according to the formula:

[4] For example, the total curvature of the catenoid whose axis is of infinite length is -4π, the total curvature of the sphere of radius r is 4π and the torus 0.

number of vertices−number of edges+ number of faces with the property of invariance by homeomorphy. It is commonly denoted by χ or $\chi(M)$. As examples for surfaces in homological algebra, we have $\chi(M) = 2$ for a sphere, $\chi(M) = 4$ for two spheres (not connected), $\chi(M) = 0$ for a torus and $\chi(M) = -2$ for a two-holed torus.

To speak very figuratively, quite approximately, the Euler-Poincaré characteristic is an integer, invariant when the size and the shape of a geometrical object change by an effect of a "plastic" deformation. One will find more axiomatic definitions!

This invariance property makes it a providential tool in the context of this study on the bending effects, associated to the Gauss–Bonnet theorem, a deep relationship between surfaces in differential geometry, connecting the Gaussian curvature of a surface to its Euler characteristic.

The Euler characteristic of the right cylinder is zero, thus so is that of the cylinder with one or two boundaries. These following free surfaces with two circular boundaries and whose meridian is an arc of Delaunay roulette are considered topologically equivalent (same common topological genus), because it is possible to continuously move one to obtain the other: portion of concave or convex, catenoid or unduloid (the right cylinder being the transition case). Accordingly, these axisymmetric surfaces have in common the same Euler characteristic, in this case, the value zero. It is the same for their continuous axisymmetric smooth deformations by distorting effect of bending or gravity [22][26].

2.3 The Gauss-Bonnet or Gauss-Bonnet-Binet theorems

The Gauss-Bonnet theorem (Binet is often forgotten[5]) is reputed to be one of the most profound and elegant results of the study of surfaces [2][3][5]. It has no surprisingly many applications in Physics. It is used in sectors of activity where the problems of bending beams surely arises (civil engineering, naval architecture, shell theory to predict the stress and the displacement arising in an elastic shell, [5], *etc...*).

In fact, it unexpectedly links two completely different ways of studying a surface: one geometric, the other topological.

Indeed, for any compact, boundaryless two-dimensional Riemannian manifold Σ, the integral of the Gaussian curvature K over the entire manifold with respect to area measure is 2π times the Euler characteristic of Σ, also called the Euler number of the manifold, *i.e.*

$$\int_{\Sigma} K \, d\Sigma = 2\pi\chi(\Sigma).$$

For example, for a sphere Σ of radius R in $\mathbb{R}^3$, it comes:

$$\int_{\Sigma} K \, d\Sigma = \frac{1}{R^2} \, 4\pi R^2 = 4\pi \quad \text{and here } \chi(\Sigma) = 2.$$

Suppose now that M is a compact two-dimensional Riemannian manifold with a boundary δM and let k_g the signed geodesic curvature of δM. Then, in nondimensional writing,

[5] Because of the Binet's independent work, but Bonnet credits Binet, *cf.* Tome III, correspondance de l'Ecole Polytechnique.

$$\int_M K\ dM + \int_{\delta M} k_g\ ds = 2\pi\chi\left(M\right).$$

We recall that the geodesic curvature k_g, of an arbitrary curve at a point P on a smooth surface, is defined as the curvature at P of the orthogonal projection of the curve onto the plane tangent to the surface at P and we have:

$$k_g = k cos\ \theta, \tag{2}$$

where θ is the angle between the osculating plane of C and the tangent plane Q at point P. It is easy to see that $cos\ \theta$ is nothing else than the local contact angle for wetting problems and capillary bridges.

We want to obtain in any situation an explicit expression of this integral, of simple and immediate use for the experimenter, and highlight the determining parameters and their respective influence.

3 The aim and effective convenience of the arc length reparameterization strategy

3.1 Surface of revolution

The rather paradoxical aspect of this method is that it is only very rarely easy to get in practice an explicit calculation formula. However, it leads to general quantitative results in the form of analytical formulas very easy to use from the experimental data, via a very convenient expression of the Gaussian curvature of a surface of revolution (the speed and acceleration vectors are then orthogonal).'How complexity leads to simplicity...'

To illustrate what we are talking about, let us consider a smooth curve of the half-plan $\{y > 0,\ z = 0\}$ parametrized by the arc length. The surface of revolution resulting in $\mathbb{R}^3$ from the rotation of the curve around the x-axis, ψ being the angle of rotation, is parametrized by:

$$M\left((s,\psi)\right) = \left(x\left(s\right),\ y\left(s\right)\cos\psi,\ y\left(s\right)\sin\left(\psi\right),\right)\ \ 0 \leq \psi \leq 2\pi,\ 0 \leq s \leq L.$$

As the meridian portion is parameterized by arc length, we have *ipso facto* the following remarkable and convenient relations and convenient expressions for angular (in radians) and trigonometric values as well as for the Gaussian curvature of the surfaces of revolution:

$$x'^2 + y'^2 = 1 \ \text{ at any point}, \tag{3}$$

and therefore, by differentiating, the orthogonality relationship

$$x'x'' + y'y^{''} = 0\ ,$$

that is to say that $T.\dfrac{dT}{ds} = 0$ where the dot denotes the scalar product of $\mathbb{R}^3$ and where $T(s) = \dfrac{dM(s)}{ds}$ is the unit tangent vector to the curve $M(s) = M(s, 0)$.

The Gaussian curvature K of the surface of revolution has then the very convenient expression (see [2], p.162, eq. (9)):

$$K(s,\theta) = -\frac{y''(s)}{y(s)}.$$

In this context, the expression of the mean curvature H of a surface of revolution is less attractive (see [2], p.162, eq. (11)).

A remarkable illustrative example of the arc length reparameterization strategy is the determination of the axisymmetric surfaces of constant Gaussian curvature[6] (procedure indicated in [2], p.162, eq. (9)).We consider then the classical differential equation, linear, of the second order, homogeneous:

$$y''(s) + K\ y(s) - 0\ , \quad 0 \leq s \leq L,$$

with the three following cases: $K < 0$, $K = 0$, $K > 0$. Then, introducing the general form of the corresponding solutions in y, we consider the resulting differential equations

$$x'^2 = 1 - y'^2\ , \quad 0 \leq s \leq L \tag{4}$$

resulting from (3).

Moreover, φ being the angle that the tangent to the profile curve makes with the x-axis, we have the following relationships:

$$\sin\varphi = \frac{y'(s)}{\sqrt{x'^2(s) + y'^2(s)}}\ , \ i.e. \quad \sin\varphi = \ y'(s) \quad \text{and} \quad cos\varphi = x'(s).$$

To compute the global bending stress in this context, we have to consider succesively:

$$\mathcal{E}_{bending\ stress} = C_k \int_M K\ dM = C_k \int\int -\frac{y''(s)}{y(s)} y(s)\ d\psi ds$$

and therefore

$$C_k \int\int -y''(s)\, d\psi ds = -2\pi C_k\ \left(y'(L) - y'(0)\right)$$

so that

$$\mathcal{E}_{bending\ stress} = 2\pi(C_k\left(\sin\left(\varphi\left(L\right)\right) - \sin\left(\varphi(0)\right)\right),$$

φ being the angle that the tangent to the profile curve makes with the $x-$axis, the axis of rotation.

Let us quote that this case of a surface of revolution around x-axis may correspond to a capillary bridge between two parallel planes at $x = 0$ and $x = L$. With the notations of [9][11], the wetting angles at the triple lines are given by $\theta_1 = -\varphi(L)$ and $\theta_2 = \varphi(0)$. Therefore, we recover the expression of the bending stress of [11] obtained in the general case for a surface of revolution (not necessarily with constant Gauss curvature) from the Gauss-Bonnet integration theorem:

[6] Problem studied more extensively by Gaston Darboux 1890. Among the solutions, surfaces are found that look like a hyperboloid.

$$\mathcal{E}_{bending\ stress} = C_k \int_M K\ dM = -2\pi C_k \left(\cos\theta_1 + \cos\theta_2\right),$$

with here the Euler characteristic $\chi(M) = 0$ and when the total geodesic curvature at the boundaries is $2\pi\left(\cos\theta_1 + \cos\theta_2\right)$.

3.2 The conclusive Fenchel's theorem

In the case of a portion of an ellipse, the parameterization involves elliptic integrals, rarely possible to explain in practice, so that the parameters would have to be sought numerically (spline interpolation). This computational difficulty is overcame by knowing the Fenchel's theorem, which shows the complementarity of the three methods.

To well illustrate the interest of Fenchel's theorem associated with the theorems of Gauss and Bonnet, let us consider a reparameterization by arc length of the curve defining the surface of revolution.

Close plane curve

According to Fenchel's theorem[7] (1929), the value of the total curvature

$$\int_\Gamma k(s)ds$$

of any smooth closed space curve Γ is at least 2π , *i.e* $\int_\Gamma k(s)ds \geq 2\pi$. The equality holds if and only if the curve is a convex plane curve. In other words, the average curvature of a closed convex plane curve equals $2\pi/L$, where L is the length (the perimeter) of the curve[8].

By the Fenchel's theorem, without calculation of primitive functions, often tedious or ineffective, we deduce directly, for any closed convex plane curve Γ (*i.e.* the curve is the boundary of a convex set in the Euclidean plane), that

$$\int_\Gamma k(s)ds = 2\pi.$$

In the case of a closed nonconvex plane curve, we are led to conclude by defining of the notion of the winding index, a topological argument, in what follows.

Open plane curve

Let us give some classical preliminary elements of differential geometry related to smooth boundaries of surfaces, parametrized by arc length. The curvature of a plane curve parametrized

[7] The Fary-Milnor theorem concerning the total curvature of the knotted closed curves does not seem appropriate for the subject of this study.

[8] For a given arc of a plane curve, the local average curvature quantifies the ratio of the change in inclination of the tangent to the curve over the arc length.

by arc length is the rate of turning of the tangent line with respect to an *ad hoc* frame along the curve.

Let $\varphi(s)$ be the angle of inclination of the unit tangent vector $T = T(s)$ with respect to a fixed frame of reference, for instance $x-$axis. Considered then as a rate of turning for the tangent line when one moves along the curve at unit speed, the curvature $k(s)$ becomes

$$k(s) = \frac{d\varphi}{ds}(s) = \varphi'(s).$$

It follows that the total curvature of a smooth curve C is then given by the formula depending only of the initial and final states:

$$\int_C k(s)ds = \varphi(ending) - \varphi(starting), \text{ (in radians).} \tag{5}$$

For a piecewise smooth curve parametrized by arc length, then we need to deal with the exterior angles at the corners according to the orientation of the curve in the turning motion. However, up to now, to our knowledge such capillary bridges with non convex or open contact line have are not considered in literature.

3.3 General case for axisymmetric capillary bridges of revolution

Let us come back to the general case of axisymmetric capillary bridges of revolution whose profile have one or more inflection points. The methods of Euler's characteristic associated to the Gauss–Bonnet theorem apply immediately to these cases. These axisymmetric surfaces have in common the same Euler characteristic, in this case, the value zero. It then comes with adjusted data (cite [11] for more details):

$$\mathcal{E}_{bending\ stress} = C_k \int_\Sigma K \ d\Sigma = -2\pi C_k \left(\cos\theta_1 + \cos\theta_2\right).$$

In summary, it should be kept in mind that the value of the bending stress depends, besides physical constants, only on the observed values of the contact angles, whereas these angles result in part implicitly from the final equilibrium of the device.
The essential point here is there are no major issues for generalization of the bending stress calculation to the non-axisymmetric cases, as specified below.

It would be interesting to reconsider, in taking into account the bending effects, the resulting contact angles and the elegant Gauss-Bonnet theorem, a new analytical framework for evaluating the cohesion effects of coalescence between saddle shaped capillary bridges [14].

For didactic purposes, we limited ourselves to the detailed case of axisymmetric capillary bridges to show the interest of the simultaneous use of the Gauss-Binet-Bonnet formula and of the topological notion of Euler characteristic to evaluate the importance of the bending stress. The obtained result clearly shows in an explicit way, the major role of the contact angles values after distortion effects, eventually distinct.

3.4 The prevaling roles of the contact angles values and of the convexity/nonconvexity of the boundaries.

All the factors determining the contact angle have consecutively an influence on the bending stress (surface roughness and heterogeneity, influence of gravity, contact angle hysteresis, [16],[35], [36], *etc*).

It is well known that the contact angle value is determined by the balance between adhesive and cohesive forces on the rigid supports. As the tendency of a drop to spread out over a flat, solid surface increases, the contact angle decreases. Thus, the contact angle provides an inverse measure of wettability.
In this context, the case of the right cylinders is still a borderline case.

A contact angle less than $\frac{\pi}{2}$(low contact angle) usually indicates that wetting of the surface is very favorable, and the fluid will spread over a large area of the surface. Contact angles greater than $\frac{\pi}{2}$ (high contact angle) generally mean that wetting of the surface is unfavorable.
It should be quoted that a certain number of terms of the generalized Young-Laplace equation are spontaneous values, resulting from instantaneous equilibrium, and are therefore implicit unknowns. This is a difficulty for the mathematical resolution of this nonlinear differential boundary problem.
In addition, the Fenchel's theorem sheds light on the importance of the convexity or the nonconvexity of the outer edges in calculating exactly the value of the total curvatures.
Consequently, even a limited displacement of the surface boundary can modify the bending stress, by local modifications of the contact angles or affecting the local curvature of the outline curve and then, the total geodesic curvature. The contact angle hysteresis can also be significant.

4 The general case and its implementation

In the general cases of nonaxisymmetric capillary bridges between two supports, possibly of distinct natures, the method remains applicable in principle. The difficulty is not conceptual in dealing with the general case but rather calculative. We must then, in any given case, engage in a delicate exercise in differential and analytical geometries to explicitly calculate the total signed geodesic curvature of the boundaries by the classical methods of analytical geometry.

The calculation procedure is as follows.

At any point P of the border liquid-solid, one considers the tangent plane in P to the free surface (that supposes an adequate local regularity). One then considers the orthogonal projection of each edge into this tangent plane. The curvature in P of the projected curve is then calculated, what introduces the important role of the cosine of the local contact angle. To make a metaphor from the point of view of the kinematics, remember that for a unit speed curve on a smooth surface, the geodesic curvature is the length of the surface tangential component of acceleration. The geodesics correspond to zero values.

The curvature k of the closed curve C at point P is related to the geodesic curvature k_g at P by the relationship:

$$kg = k\ cos\ \theta$$

where θ is the local contact angle, *i.e.* the angle between the osculating plane of C and the tangent plane Q at point P.

When the contact angle is constant on the considered contact surface, we have the particularly simple relationship:

$$\int_{\Gamma_i} k_g(s)\ ds = cos\theta \int_{\Gamma_i} k(s)\ ds.$$

The special situation of heterogeneous contact surfaces

When the contact angles are separately variable on each of the contact surfaces, *i.e .* $\theta = \theta_1(M)$ and $\theta = \theta_2(M)$ according to the physical conditions of the two surfaces (non ideal smooth surfaces), the integral along each boundary, corresponding to the total geodesic curvature, in fact, of the kind

$$\int_{\Gamma_i} cos\theta_i(M((s))\ \ k(s)\ ds,$$

is more complicated to calculate with computational prediction of wetting (at our knowledge, an open problem for the probably most realistic case). The use of a mean theorem would likely be imprecise (effects of surface roughness).

The general case of homogeneous contact surfaces

When multiplied by the coefficient $(-C_K)$ at the dimension of a force, the dimensionless integral of these curvature values along the reunion of the two contact edges gives finally the value of the resulting bending stress by the Fenchel's theorem (the cornerstone of the method).

The three possible scenarios then arise according to the geometry of the boundaries (closed plane convex or nonconvex curves) are the following, the surfaces having in common, without loss of generality, the same Euler characteristic, in this illustrative case, the value zero.

By introducing the contact angles θ_1 and θ_2 (in radians) on each outline of contact surfaces, we proved that, at least theoretically, the wettability being evaluated, here, by constant contact angles, separately on each contact support.

Case 1:
The boundaries are two closed plane convex curves. Then,

$$\mathcal{E}_{bending\ stress} = C_k \int_M K\ dM = -2\pi C_k\ \left(\cos\theta_1 + \cos\theta_2\right).$$

Case 2:
The boundaries are two closed plane curves, one convex and the other nonconvex. Then,

$$\mathcal{E}_{bending\ stress} = C_k \int_M K\ dM = -C_k\ \left(2k_1\pi\cos\theta_1 + 2\pi\cos\theta_2\right)\ ,$$

the observed integer k_1 , $k_1 \geq 2$, being the winding number of the nonconvex curve (the winding index in algebraic topology).

Case 3:
The boundaries are two closed disjoint plane curves, nonconvex. Then,

$$\mathcal{E}_{bending\ stress} = C_k \int_M K\ dM = -C_k\ \left(2k_1\pi\cos\theta_1 + 2k_2\pi\cos\theta_2\right),$$

k_1 and k_2 being the integers, ≥ 2, winding numbers of the curves, observed and known *in situ.*

In the rather theoretical case, where the value of the Euler characteristic is non-zero, it should be necessary to write:

$$\mathcal{E}_{bending\ stress} = C_k \int_M K\ dM = 2\pi\chi(M) - C_k\ \left(2k_1\pi\cos\theta_1 + 2k_2\pi\cos\theta_2\right).$$

According to these theoretical calculations, the bending stress is practically not existing in the neighborhood where the contact angle is close to $\frac{\pi}{2}$radian. Then, at the contact, the surface is locally similar to a right cylinder whose outline is a closed plane curve, convex or nonconvex. This last topological aspect of convexity is here non-essential (the non-convexity modifies in a known and controlled way the total curvature).

It must be emphasized that, when the contact angles are separately variable on each of the contact surfaces according to the physical conditions of the two surfaces (non ideal smooth surfaces), the integral along each boundary, corresponding to the total geodesic curvature of the plane and closed boundaries, seems a serious difficulty to explain. The question might interest specialists in differential geometry.

5 Conclusion

We then see that the developments obtained here, generalization of formulas relating to surfaces of revolution, result from concepts in differential geometry and geometric analysis with applications to Lagrangian Mechanics, without resorting to differential calculus and integral calculus, sometimes out of concrete practice for problems involving elliptic integrals, for example, requiring numerically computing.

The methods of Euler's characteristic, associated to the Gauss–Bonnet-Binet theorem and the strongly complementary Fenchel's theorem apply immediately to the cases of the non-axisymmetric surfaces, with explicit, easy-to-use, results formulations.

We proved that in the general way, the value of the bending stress depends, besides physical constants, only on the observed values of the contact angles, whereas these angles result in part implicitly from the final equilibrium of the device. Therefore, all the factors determining the contact angle have an influence on the bending stress (surface roughness and heterogeneity, influence of gravity, contact angle hysteresis.

It would be interesting to reconsider, in taking into account these new results concerning the bending effects, the important role of the contact curves geometry and the Gauss-Bonnet and Fenchel theorems, an analytical framework for reassessing the cohesion effects of coalescence between saddle shaped capillary bridges [14]. Finally, by creating a support material having a nonconvex region with high wettability and a complementary region with very low wettability, the experimenter could illustrate the theory by experimentation.

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