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Résumé

This paper addresses difference flatness for structured LTI discrete-time systems. Two forms of necessary and sufficient conditions for an output to be a structural flat output are given. First, a preliminary result algebraically defines a flat output in terms of invariant zeros regardless whether an LTI system is structured or not. Next, the conditions are expressed in terms of graphical conditions to define a structural flat output. Checking for the graphical conditions calls for algorithms that have polynomial-time complexity and that are commonly used for digraphs. The tractability of the conditions is illustrated on several examples.

Key words: Structured linear systems, Flatness, Graph theory

1 Introduction

This paper is concerned with flatness of discrete-time structured dynamical systems. Flatness of discrete-time systems is usually called difference flatness. It has been first reported in (Sira-Ramirez and Agrawal, 2004; Fliess and Marquez, 2000). It acts as the discrete-time counterpart of differential flatness, introduced in (Fliess *et al.*, 1995), that applies for continuous-time systems. Let us recall that for a flat continuous-time system, flatness gives a complete parametrization of all system variables (inputs and states) in terms of a finite number of independent variables and a finite number of their time derivatives. Those variables are called flat outputs. For a flat discrete-time system, the state variables as well as the input can be written as a function of the flat output and its backward/forward shifts. This being the case, flatness is interesting for both control and state reconstruction perspectives. For control purposes, the parametrization of the input in terms of outputs of the system provides in a straightforward manner a constructive way to design a feedforward control to track a prescribed trajectory of the plant output. The reader may consult (Yong *et al.*, 2015) or Chapter 5 in the book (Sira-Ramirez and Agrawal, 2004) for illustrative examples in the case of LTI discrete-time systems. As for state reconstruction, the parametrization of the state in terms of outputs of the system provides in a straightforward manner a constructive way to design an unknown input state observer. Such an issue has been discussed in (Daafouz *et al.*, 2006) in a general statement or for example in (Shoukry *et al.*, 2015) in the context of cybersecurity where the state reconstruction allows for detecting sensor attacks.

Most of the definitions, including the ones given in (Sira-Ramirez and Agrawal, 2004; Yong *et al.*, 2015) dealing with LTI systems, call for backward flatness or forward flatness, *i.e.*, backward or forward shifts exclusively are involved in the expressions of the state and the input. However, more general definitions involving both backward and forward shifts have been recently proposed and motivated in (Guillot and Millérioux, 2020; Diwold *et al.*, 2021) for both linear and nonlinear systems. Difference flatness is motivated by the fact that some systems are intrinsically discrete (models of population growth, economy, biology, finance, discrete automata,. . .). Besides, it must be stressed that the property of flatness may not be preserved when a flat continuous-time system is discretized, even in the linear case. Hence, difference

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flatness for sampled-data systems should preferably be addressed directly within the discrete-time framework. Specific characterizations of flatness have been provided in the literature according to distinct classes of discrete-time systems as LTI systems (Sira-Ramirez and Agrawal, 2004; Yong *et al.*, 2015), switched linear systems (Millérioux and Daafouz, 2009), LPV systems (Parriaux and Millérioux, 2013), or more general classes of nonlinear systems (Guillot and Millérioux, 2020; Kaldmäe and Kotta, 2013; Sato, 2012; Kolar *et al.*, 2016a; Kolar *et al.*, 2016b).

As it turns out, a general framework based on structural and graph-oriented approaches has never been proposed so far to deal with difference flatness of structured LTI systems. And yet, those approaches have been used with success over the years to characterize many structural properties of dynamical systems like controllability, observability (including with unknown inputs), and identifiability. The reader may refer to the survey (Ramos *et al.*, 2020) that gives a exhaustive overview of the works and applications of structural analysis from the seminal paper (Dion *et al.*, 2003) to most recent ones. We can also mention extension of results to other classes of systems like descriptor systems (Clark *et al.* , 2017), bilinear systems (Boukhobza and Hamelin, 2007), switching systems (Boukhobza, 2012) or complex nonlinear networks (Kawano and Cao, 2019) to mention a few. An attempt to establish results on flatness had been proposed in (Boukhobza and Millérioux, 2016) but it was restricted to SISO systems and the approach was not suitable to tackle general LTI systems. Structural analysis allows to characterize properties independently of the exact values of the parameters and thus, to deal with systems of which the model equations are not known exactly. Furthermore, structural models usually involve equations derived from physical laws where the states are variables that get a physical meaning. Hence, structural properties are easily interpreted in terms of physical ones. In this respect, the applicability of the graph-oriented approaches is large and can also be efficient for sensor placements, reachability problems, reliability analysis, security in Cyber Physical Systems as in (Dakil *et al.*, 2015; Gracy *et al.*, 2020) but also in life sciences as biology (Liu and Linqiang, 2015) for example.

The aim of this paper is to propose a graph-oriented approach to address flatness for the class of structured LTI discrete-time systems. More specifically, necessary and sufficient graphical conditions for an output to be a structural flat output are given. These conditions can be checked by resorting to well-known algorithms, commonly used for finding successors and predecessors of vertex subsets, or for computing maximal linkings and essential vertices in a digraph. As a result, the proposed solution is simple to implement and has polynomial complexity.

The paper is organized as follows. Section 2 is devoted to the problem statement. The definitions of a difference flat output and a difference flat system are recalled and a preliminary result (Theorem 1) is established. It gives an algebraic characterization of a flat output in terms of invariant zeros. The result is quite general since it does not exclusively apply to structural systems. In Section 3, structured systems and the notion of structural flatness are introduced. Necessary background on graph-theoretic tools and recalls on digraph representation of LTI structured discrete-time systems are provided. In Section 4, the main result is established. It gives a necessary and sufficient condition (Theorem 2) for an output of an LTI system to be structurally flat. An equivalent characterization (Theorem 3) is also provided. In Section 5, the conditions are illustrated with some basic examples. Section 6 ends this paper with some concluding remarks and possible further work.

Standard notation : I_k, $(k \in \mathbb{N})$ stands for the $k-$dimensional identity matrix. For a vector z of dimension n $(n \in \mathbb{N})$, z_i with $i \in \{1, \ldots, n\}$ denotes its ith component. For a $m \times l$-dimensional matrix M (being m and l natural integers), $M(i, j)$ with $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, l\}$ denotes the entry of M located at row i and column j.

2 Problem statement

2.1 Difference flatness

Let us consider the discrete-time LTI system which admits the state space representation

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector and $u(k) \in \mathbb{R}^m$ is the control input, with n and m being positive integers. The matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the dynamical matrix and the input matrix, respectively.
Besides, let us consider for any integer $k \in \mathbb{N}$, the output of system (1) as the m-dimensional vector defined as

$$y(k) = Cx(k) + Du(k), \tag{2}$$

with suitable matrices $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$.
The system (1) with output $y(k)$ is square, which means that the number of control inputs m (dimension of $u(k)$) is equal to the number of outputs (dimension of $y(k)$). Whenever useful in the sequel, to get more compact notation, we introduce the forward shift operator ξ as $\xi y(k) := y(k+1)$, and similarly for $\xi x(k)$ and $\xi u(k)$. Then, for instance, the double shift forward and the backward shift can be described by $\xi^2 y(k) = y(k+2)$ and $\xi^{-1}y(k) = y(k-1)$, respectively.

Definition 1 (flat output) *The output y in (2) is said to be a flat output for the dynamical system (1) if there* *exists a non negative integer k_0 such that every variable of the system, i.e., the state $x(k)$ and the input $u(k)$, can be expressed as a function of $y(k)$, and a finite number of its backward and/or forward iterates, for $k \geq k_0$. In particular, there exist integers r_0, r_1, s_0 and s_1 such that $r_0 \leq r_1$ and $s_0 \leq s_1$, and matrices $F_r \in \mathbb{R}^{n\times m}, r_0 \leq r \leq r_1$, and $G_s \in \mathbb{R}^{m\times m}, s_0 \leq s \leq s_1$, such that for $k \geq k_0$,*

$$x(k) = \sum_{r=r_0}^{r_1} F_r y(k+r) \text{ and } u(k) = \sum_{s=s_0}^{s_1} G_s y(k+s),$$

where $x(k)$ and $u(k)$ satisfy (1), or in more compact notation

$$x(k) = F(\xi)y(k) \text{ and } u(k) = G(\xi)y(k), \tag{3}$$

where $F(z) = \sum_{r=r_0}^{r_1} F_r z^r$ and $G(z) = \sum_{s=s_0}^{s_1} G_s z^s$ are polynomial matrices with entries in the ring $\mathbb{R}[z, z^{-1}]$, consisting of polynomials with finitely many positive and/or negative powers of z.

Definition 2 (flat system) *The dynamical system (1) is difference flat if it admits a flat output.*

2.2 Some remarks on difference flatness and a basic example

The following remarks are in order.

Remark 1 *The term* ***difference flat*** *is used for discrete-time systems and must be distinguished from the term differentially flat that applies for continuous-time systems. However, since only discrete-time systems are under concern in this paper, we will hereafter without ambiguity leave out the adjective difference, and shortly use the notions flat output or flat system.*

Remark 2 *Flatness is equivalent to controllability for LTI systems (see (Sira-Ramirez and Agrawal, 2004; Fliess and Marquez, 2000)). Let us note that such an equivalence includes the particular class of non reversible LTI systems provided that we accept a state space transformation as pointed out in (Fliess, 1992) and is thereby also in accordance with the result given in (Guillot and Millérioux, 2020).*

Remark 3 *When only past or, respectively, future outputs are involved in* (3)*, the flatness is called backward or forward flatness, respectively. Involving altogether backward and forward shifts (see (Diwold et al., 2021; Guillot and Millérioux, 2020)) allows the consideration of outputs of any relative degree for SISO systems, or any inherent delay for MIMO systems.*

Example : Let us consider an LTI discrete-time system like (1) defined by :

$$\begin{cases} x_1(k+1) = ax_1(k) + bu(k) \\ x_2(k+1) = cu(k), \end{cases} \tag{4}$$

where a, b and c are constant real parameters and $b \neq 0$.
Case 1.1 : Consider the output defined as $y(k) = x_1(k)$. Such an output is flat because Equations (3) are fulfilled. Indeed, it holds that $x_1(k) = y(k)$ and $x_2(k) = cb^{-1}y(k) - acb^{-1}y(k-1)$ that define the polynomial matrix F and $u(k) = b^{-1}y(k+1) - b^{-1}ay(k)$ that defines G. Let us notice that backward and forward shifts in the output are involved. That corroborates Remark 3. Indeed, as it turns out, the relative degree of (4) with respect to $y(k)$ is equal to 1.

Case 1.2 : Consider the output defined as $y(k) = ax_1(k) + bu(k)$. Again, such an output is flat because Equations (3) are fulfilled. Indeed, we obtain $x_1(k) = y(k-1)$ and $x_2(k) = cb^{-1}y(k-1) - acb^{-1}y(k-2)$ that define the polynomial matrix F and $u(k) = b^{-1}y(k) - b^{-1}ay(k-1)$ that defines G. For such an output, only backward shifts are involved, the relative degree of (4) with respect to $y(k)$ is equal to zero.

Case 1.3 : Consider the output $y(k) = x_2(k)$. Such an output is not flat because, clearly, $x_1(k)$ cannot be exclusively expressed in terms of shifts in the output $y(k)$.

Remark 4 *This simple example illustrates that the parametrization of the state in terms of a finite number of shifted outputs is especially interesting for state reconstruction. Indeed, it is clear from this example that the state vector can be reconstructed despite unknown inputs. It also illustrates that the parametrization of the input gives explicitly the feedforward control that allows the tracking of a prescribed output trajectory.*

2.3 Algebraic characterization of a flat output

System (1) together with the output (2) defines an input-output system

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + Bu(k), \\ \quad y(k) = Cx(k) + Du(k). \end{cases} \tag{5}$$

Define the matrix

$$M(z) := \begin{pmatrix} A - zI & B \\ C & D \end{pmatrix}. \tag{6}$$

Matrix $M(z)$ can be seen as a square matrix with entries in the ring $\mathbb{R}[z, z^{-1}]$. Further, note that $\det M(z)$ is a polynomial in $\mathbb{R}[z, z^{-1}]$. Let us give the central result from which the graph-based conditions will be derived later on.

Theorem 1 *System (1) has a flat output in the form of (2) if and only if the combined system Σ, defined in (5), has no invariant zeros outside $z = 0$, i.e.,* $\det M(z) = \gamma z^{\nu}$ *with real $\gamma \neq 0$ and $\nu \in \mathbb{N}$, where $M(z)$ is defined in (6).*

Proof 1 *The proof is given in Appendix. It is constructive since from Equation* (16)*, the explicit expression of F and G involved in* (3) *are obtained in a straightforward manner. It extends the result established in (Yong et al., 2015) that was restricted to backward and forward flatness. No specific assumption is required on system* (1)*, in particular neither controllability, nor submersivity.*

3 Structural flatness and graph-theoretic tools

This section is devoted to the definition of structural flatness, that is flatness when system (5) is structured as detailed in next subsection. The proposed methodology to check whether an output is structurally flat, that is the main objective of this paper, will be jointly based on the algebraic result proved in Theorem 1 and a graph-oriented approach to derive structural conditions from this result (see Section 4). Thus, necessary background on graph-theoretic tools is also provided in this section.

3.1 Structured systems and flat outputs

A structural property, also said generic property, is a property that applies for a structured system, see (Dion *et al.*, 2003). More specifically, structural properties of system (5) are properties which are true for almost any value of the non-zero entries of the matrices A, B, C and D. System (5) is structured when its matrices of the state space representation are defined by their sparsity pattern. In other words, no specific values in A, B, C and D are considered but one must merely distinguish between the entries of A, B, C and D that are fixed zeros and the other ones. Null entries $A(i, j)$ (*resp.*, null entries $B(i, j)$, null entries $C(i, j)$, null entries $D(i, j)$) means that there is no relation (dynamical interaction) between the state $x_i(k+1)$ at time $k+1$ and the state $x_j(k)$ at time k (*resp.* the state $x_i(k+1)$ at time $k+1$ and the input $u_j(k)$ at time k, the output $y_i(k)$ at time k and the state $x_j(k)$ at time k, the output $y_i(k)$ at time k and the input $u_j(k)$ at time k). A given output of (5) in the form $y = Cx + Du$ is structurally flat, also said generically flat, if it is flat for the structured system (5), that is for almost any values of the non-zero entries of the matrices A, B, C and D. If such an output exists, the structured system (2) is said to be structurally flat. The objective of the paper is to provide conditions to check whether a given output in the form $y = Cx + Du$ is generically flat or not.

Illustration :
Let us consider again the LTI discrete-time system defined by (4). The output $y(k) = x_1(k)$ is a structured flat output. Indeed, $\det M(z) = b\lambda$ and Theorem 1 is fulfilled for any constant real parameters a, b and c and $b \neq 0$. On the other hand, for real numbers λ_1 and λ_2, the output $y(k) = \lambda_1 x_1(k) + \lambda_2 x_2(k)$ is not structurally flat. Indeed, it follows that generically $\lambda_2 \neq 0$ and $\lambda_1 b + \lambda_2 c \neq 0$ and $\det M(z) = \lambda(b\lambda_1 + c\lambda_2) - ac\lambda_2$. Thus, Theorem 1 implies that the output $y(k)$ is not structurally flat.

Before proceeding further, the next subsection aims at recalling necessary background on digraphs.

3.2 Graph-theoretic tools

Digraph $\mathcal{G}(\Sigma)$
A digraph $\mathcal{G}(\Sigma)$ describing the structured linear system Σ is the combination of a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$. The vertices represent the state, input and output components of Σ, while the edges describe the relations between these variables. One has $\mathcal{V} = \mathbf{X} \cup \mathbf{U} \cup \mathbf{Y}$, where $\mathbf{X}$ is the set of state vertices defined as $\mathbf{X} = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$, $\mathbf{U}$ is the set of input vertices $\mathbf{U} = \{\mathbf{u_1}, \ldots, \mathbf{u_m}\}$, and $\mathbf{Y}$ is the set of output vertices $\mathbf{Y} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y_m}\}$. The edge set is $\mathcal{E} = \mathcal{E}_A \cup \mathcal{E}_B \cup \mathcal{E}_C \cup \mathcal{E}_D$, with $\mathcal{E}_A = \{(\mathbf{x_i}, \mathbf{x_j}) \,|A(i, j) \neq 0\}$, $\mathcal{E}_B = \{(\mathbf{x_i}, \mathbf{u_j}) \,|B(i, j) \neq 0\}$, $\mathcal{E}_C = \{(\mathbf{y_i}, \mathbf{x_j}) \,|C(i, j) \neq 0\}$ and $\mathcal{E}_D = \{(\mathbf{y_i}, \mathbf{u_j}) \,|D(i, j) \neq 0\}$.

Path and related definitions
In the sequel, we will denote by $\mathbf{v}$ or $\mathbf{v_j}$ a vertex of digraph $\mathcal{G}(\Sigma)$, regardless whether it is an input, state or output vertex. Useful definitions are given below.

A directed path $\mathbf{P}$ is a sequence of successive edges directed in the same direction which connect a sequence of vertices. It is said that the path $\mathbf{P}$ covers a vertex if this vertex is the begin or the end vertex of one of the edges of $\mathbf{P}$. Two paths are disjoint if they have no common vertex. The length of a directed path $\mathbf{P}$ is equal to the number of edges involved in $\mathbf{P}$. We denote by $\ell(\mathbf{v_i}, \mathbf{v_j})$ the minimal length of a path connecting $\mathbf{v_i}$ to $\mathbf{v_j}$. A simple path is a directed path where every vertex occurs only once in the path. A cycle is a simple path linking a vertex $\mathbf{v_i}$ to $\mathbf{v_i}$ having length $\ell(\mathbf{v_i}, \mathbf{v_i}) > 0$.

Linkings
The following definitions apply for two sets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$. A simple path $\mathbf{P}$ is said to be a $\mathcal{V}_1$-$\mathcal{V}_2$ path if its begin vertex belongs to $\mathcal{V}_1$ and its end vertex belongs to $\mathcal{V}_2$. If the only vertices of $\mathbf{P}$ belonging to $\mathcal{V}_1 \cup \mathcal{V}_2$ are its begin and its end vertices, then $\mathbf{P}$ is a direct $\mathcal{V}_1$-$\mathcal{V}_2$ path. A $\mathcal{V}_1$-$\mathcal{V}_2$ linking is a set of disjoint $\mathcal{V}_1$-$\mathcal{V}_2$ paths. The number of these paths is called the cardinality, or the size of the linking. Note that there are possibly several maximum linkings, but by definition they all have the same size $\eta\,(\mathcal{V}_1, \mathcal{V}_2)$. The number of maximum linkings is

denoted by $n_{max}(\mathcal{V}_1,\mathcal{V}_2)$. The length of a maximal $\mathcal{V}_1$-$\mathcal{V}_2$ linking is the sum of the length of each of its disjoint $\mathcal{V}_1$-$\mathcal{V}_2$ paths. $\mu(\mathcal{V}_1,\mathcal{V}_2)$ is the minimal number of vertices covered by a maximum $\mathcal{V}_1$-$\mathcal{V}_2$ linking.

The vertices that are covered by all maximum $\mathcal{V}_1$-$\mathcal{V}_2$ linkings are called the essential vertices of the $\mathcal{V}_1$-$\mathcal{V}_2$ linkings. These vertices constitute a specific subset denoted, $V_{ess}(\mathcal{V}_1,\mathcal{V}_2)$, which is defined as $V_{ess}(\mathcal{V}_1,\mathcal{V}_2) \overset{def}{=} \{\mathbf{v}\in\mathcal{V}\ |\mathbf{v}$ is covered by any maximum $\mathcal{V}_1$-$\mathcal{V}_2$ linking$\}$.
Otherwise characterized, if $V^i_{max}(\mathcal{V}_1,\mathcal{V}_2)$ denotes the set of vertices of the i-th $\mathcal{V}_1$-$\mathcal{V}_2$ maximum linking ($i=1,\ldots,n_{max}(\mathcal{V}_1,\mathcal{V}_2)$), then $V_{ess}(\mathcal{V}_1,\mathcal{V}_2) = \bigcap_{i=1}^{n_{max}(\mathcal{V}_1,\mathcal{V}_2)} V^i_{max}(\mathcal{V}_1,\mathcal{V}_2)$.

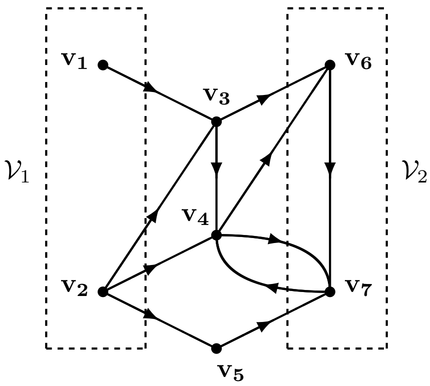

FIGURE 1. Example of digraph

The previous definitions are illustrated by means of the digraph depicted in Figure 1. We consider the sets of vertices $\mathcal{V}_1=\{\mathbf{v_1},\mathbf{v_2}\}$ and $\mathcal{V}_2=\{\mathbf{v_6},\mathbf{v_7}\}$. The simple paths $\mathbf{v_1}\to\mathbf{v_3}\to\mathbf{v_4}\to\mathbf{v_7}$ and $\mathbf{v_1}\to\mathbf{v_3}\to\mathbf{v_6}\to\mathbf{v_7}$ are $\mathcal{V}_1$-$\mathcal{V}_2$ paths, but only the first path is a direct one. $\mathbf{v_1}\to\mathbf{v_3}\to\mathbf{v_4}\to\mathbf{v_7}$, or $\mathbf{v_2}\to\mathbf{v_5}\to\mathbf{v_7}$ are $\mathcal{V}_1$-$\mathcal{V}_2$ linkings with cardinality equal to 1. An example of $\mathcal{V}_1$-$\mathcal{V}_2$ linking with cardinality equal to 2 is $\{\mathbf{v_1}\to\mathbf{v_3}\to\mathbf{v_6},\mathbf{v_2}\to\mathbf{v_4}\to\mathbf{v_7}\}$. The set $\{\mathbf{v_1}\to\mathbf{v_3}\to\mathbf{v_6},\ \mathbf{v_2}\to\mathbf{v_3}\to\mathbf{v_6}\}$ is not a $\mathcal{V}_1$-$\mathcal{V}_2$ linking because its paths are not disjoint. The maximum number of disjoint $\mathcal{V}_1$-$\mathcal{V}_2$ paths is equal to 2. Hence, the maximum linkings are of size $\eta(\mathcal{V}_1,\mathcal{V}_2)=2$. The number of $\mathcal{V}_1$-$\mathcal{V}_2$ maximum linkings is 3. They are $\{\mathbf{v_1}\to\mathbf{v_3}\to\mathbf{v_6},\mathbf{v_2}\to\mathbf{v_4}\to\mathbf{v_7}\}$, $\{\mathbf{v_1}\to\mathbf{v_3}\to\mathbf{v_6},\mathbf{v_2}\to\mathbf{v_5}\to\mathbf{v_7}\}$, $\{\mathbf{v_1}\to\mathbf{v_3}\to\mathbf{v_4}\to\mathbf{v_6},\mathbf{v_2}\to\mathbf{v_5}\to\mathbf{v_7}\}$. The respective lengths are 4,4 and 5. The maximum $\mathcal{V}_1$-$\mathcal{V}_2$ linkings with the minimal number of vertices $\mu(\mathcal{V}_1,\mathcal{V}_2)=6$ are $\{\mathbf{v_1}\to\mathbf{v_3}\to\mathbf{v_6},\mathbf{v_2}\to\mathbf{v_4}\to\mathbf{v_7}\}$ and $\{\mathbf{v_1}\to\mathbf{v_3}\to\mathbf{v_6},\mathbf{v_2}\to\mathbf{v_5}\to\mathbf{v_7}\}$. In the graph used here, $V_{ess}(\mathcal{V}_1,\mathcal{V}_2)=\{\mathbf{v_1},\mathbf{v_2},\mathbf{v_3},\mathbf{v_6},\mathbf{v_7}\}$

4 Necessary and sufficient conditions for an output to be generically flat

4.1 Structural flatness based on invariant zeros

The algebraic characterization of flatness has been given in Theorem 1 does not exclusively apply for structured systems. From this characterization and using a set of known results relating the graph of a structured system with its generic structure (rank, finite and infinite zeros) (Dion *et al.*, 2003; van der Woude *et al.*, 2003), we are able to give a necessary and sufficient graph condition for an output to be generically flat.

Theorem 2 *Consider the structured linear discrete-time system Σ described by (5). The output denoted by $y(k)\in\mathbb{R}^m$ associated to a specific vertex set $\mathbf{Y}$ is a structural flat output if and only if, in the associated digraph $\mathcal{G}(\Sigma)$, the following both conditions hold :*

(1) $\eta(\mathbf{U},\mathbf{Y})=m$, i.e., the size of a maximal $(\mathbf{U},\mathbf{Y})$ linking in $\mathcal{G}(\Sigma)$ is the number of inputs.

(2) $\alpha=\beta$, where α is the minimal number of vertices in $\mathbf{X}$ contained in a size m linking from $\mathbf{U}$ to $\mathbf{Y}$, and β is the maximal number of vertices in $\mathbf{X}$ contained in the disjoint union of a size m linking from $\mathbf{U}$ to $\mathbf{Y}$ and a cycle family in $\mathbf{X}$.

Proof 2 *In the digraph $\mathcal{G}(\Sigma)$, decompose the state vertex set $\mathbf{X}$ in four non-intersecting subsets $\mathbf{X_1},\mathbf{X_2},\mathbf{X_3},\mathbf{X_4}$ as follows :*

- *$\mathbf{X_1}$ is the set of state vertices $\mathbf{x_i}$ such that there is no path from $\mathbf{U}$ to $\mathbf{x_i}$, and there is no path from $\mathbf{x_i}$ to $\mathbf{Y}$.*
- *$\mathbf{X_2}$ is the set of state vertices $\mathbf{x_i}$ such that there is no path from $\mathbf{U}$ to $\mathbf{x_i}$, but there is path from $\mathbf{x_i}$ to $\mathbf{Y}$.*
- *$\mathbf{X_3}$ is the set of state vertices $\mathbf{x_i}$ such that there is a path from $\mathbf{U}$ to x_i, and there is no path from $\mathbf{x_i}$ to $\mathbf{Y}$.*
- *$\mathbf{X_4}$ is the set of state vertices $\mathbf{x_i}$ such that there is path from $\mathbf{U}$ to $\mathbf{x_i}$, and there is a path from $\mathbf{x_i}$ to $\mathbf{Y}$, i.e., $\mathbf{X_4}$ is composed of the state vertices which belong to an input-output path.*

After a possible renumbering of the states with respect to the previous Kalman-like decomposition, the matrix $M(z)$ has the following form.

$$M(z)=\begin{pmatrix} A_{11}-zI_1 & A_{12} & 0 & 0 & 0\\ 0 & A_{22}-zI_2 & 0 & 0 & 0\\ A_{31} & A_{32} & A_{33}-zI_3 & A_{34} & B_3\\ 0 & A_{42} & 0 & A_{44}-zI_4 & B_4\\ 0 & C_2 & 0 & C_4 & D \end{pmatrix}, \tag{7}$$

where I_i is the identity matrix of size n_i, n_i being the cardinality of the set $\mathbf{X_i}$, for $i=1,\ldots,4$. The graph of

the matrix A_{ii}, for $i = 1, \ldots, 3$, is made of the nodes of $\mathbf{X_i}$ and edges between nodes of $\mathbf{X_i}$ in $\mathcal{G}(\Sigma)$.
From the particular form of $M(z)$, it can be seen that

$$\det M(z) = \det(A_{11} - zI_1)\det(A_{22} - zI_2)\det(A_{33} - zI_3)\det(\bar{M}(z))$$

where $\bar{M}(z) = \begin{pmatrix} A_{44} - zI_4 & B_4 \\ C_4 & D \end{pmatrix}$.

The condition of Theorem 1 can then be considered separately in Equation (7). For $i = 1, \ldots, 3$, $\det(A_{ii} - zI_i)$ is the characteristic polynomial of matrix A_{ii}. It is then a non-zero polynomial of which roots are the eigenvalues of A_{ii}. It is known that there are eigenvalues of a structured square matrix that are generically nonzero if and only if the corresponding graph contains cycles (Reinschke, 1988).
It remains to study $\det(\bar{M}(z))$. Notice that $\bar{M}(z)$ is a system matrix associated with the input set $\mathbf{U}$, output set $\mathbf{Y}$, and state set $\mathbf{X_4}$, containing all state vertices which belong to an input-output path. Hence, the corresponding system $\bar{\Sigma}$ is a square system for which all state vertices belong to an input-output path. This type of system has been extensively studied in the literature. First, the generic rank of $\bar{M}(z)$ is n_4 plus the size of a maximal linking between $\mathbf{U}$ and $\mathbf{Y}$ in the graph $\mathcal{G}(\bar{\Sigma})$ (van der Woude, 1991). Since the vertices of $\mathbf{X_i}$, for $i = 1, \ldots, 3$, do not belong to an input-output path, it is equivalent to say that $\det(\bar{M}(z)) \neq 0$, and therefore $\det(M(z)) \neq 0$, if and only if the size of a maximal $(\mathbf{U}, \mathbf{Y})$ linking in $\mathcal{G}(\Sigma)$ is equal to the number of inputs, i.e., is equal to m. The system is then a square and invertible.
From Theorem 5.1 in (van der Woude, 2000), for a square invertible system as $\bar{\Sigma}$, for which all state vertices belong to an input-output path, the generic number of invariant zeros is equal to n_4 minus the minimal number of vertices in $\mathbf{X_4}$ contained in a size m linking from $\mathbf{U}$ to $\mathbf{Y}$. Under the same conditions, from proposition 3.5, (van der Woude et al., 2003), the generic number of invariant zeros at $z = 0$ is equal to n_4 minus the maximal number of vertices in $\mathbf{X_4}$ contained in the disjoint union of a size m linking from $\mathbf{U}$ to $\mathbf{Y}$ and a cycle family in $\mathbf{X_4}$.
Assume that the two conditions of Theorem 2 are satisfied. Since the input-output paths are the same in $\mathcal{G}(\Sigma)$ and in $\mathcal{G}((\bar{\Sigma})$, from the previous observations, $\det(\bar{M}(z))$ is a nonzero polynomial with roots in zero. Moreover, since $\alpha = \beta$, no cycle may exist in the set of state vertices out of an input-output path. Therefore, for $i = 1, \ldots, 3$, $\det(A_{ii} - zI_i)$ is a nonzero polynomial with only roots, i.e., eigenvalues, in zero. In conclusion, $\det M(z)$ is a nonzero polynomial with roots in zero, and from Theorem 1, the system Σ is generically flat.
These arguments may be reversed to prove that conditions (1) and (2) of Theorem 2 are necessary for an output to be flat.

4.2 Equivalent characterization

We propose now an equivalent formulation of the flatness conditions of Theorem 2.

Theorem 3 *Consider the structured linear discrete-time system Σ described by Equation (5). The output denoted by $y(k) \in \mathbb{R}^m$, associated to set of vertices $\mathbf{Y}$, is generically a flat output if and only if, in the associated digraph $\mathcal{G}(\Sigma)$, the following three conditions hold :*

(1) $\eta(\mathbf{U}, \mathbf{Y}) = m$.
(2) All the maximum $\mathbf{U}$-$\mathbf{Y}$ linkings have the same length.
(3) Every cycle in the digraph $\mathcal{G}(\Sigma)$ covers at least an element of $V_{ess}(\mathbf{U}, \mathbf{Y})$.

Proof 3 *Condition (1) is the same for both Theorem 2 and Theorem 3. It guarantees a generic invertibility property. Next, consider a disjoint union of a size m linking from $\mathbf{U}$ to $\mathbf{Y}$ and a cycle family in $\mathbf{X}$. If the conditions (2) and (3) of Theorem 3 hold, the linking cannot contain a cycle because of condition (3), and the number of state vertices in it is the same as for the minimal length linking by condition (2). Therefore, $\alpha = \beta$. Conversely, $\alpha = \beta$ clearly implies conditions (2) and (3) of Theorem 3.*

Remark 5 *An intuitive explanation of those conditions is the following. First, the characterization of flat outputs of a system in terms of invariant zeros that should not be distinct from zero is equivalent to state that the left inverse system has a trivial dynamics. In other words, the left inverse system has a finite memory which in turn, explains why, for a flat system, the state and the input are expressed in terms of a finite number of shifted outputs. To obtain such a finite memory property, the location of the cycles in the graph plays a central role. A "bad" location of cycles would induce an infinite memory for the left inverse system. That's why Condition (2) compare two sets of vertices with and without considering the cycles. Condition (1) is equivalent to the left invertibility of the system that guarantees the existence of the left inverse system, a necessary condition for an output to be flat.*

4.3 Practical consideration and complexity

Checking the conditions of Theorem 2, in order to know if a specific set of vertices $\mathbf{Y}$, is a structural flat output, implies the computation of :

— The size of a maximal $(\mathbf{U}, \mathbf{Y})$ linking in $\mathcal{G}(\Sigma)$,
— The minimal number α of vertices in $\mathbf{X}$ contained in a size m linking from $\mathbf{U}$ to $\mathbf{Y}$,
— The maximal number β of vertices in $\mathbf{X}$ contained in the disjoint union of a size m linking from $\mathbf{U}$ to $\mathbf{Y}$ and a cycle family in $\mathbf{X}$.

Several papers dealt with these computational aspects, using flow techniques as in (Yamado, 1988) or linear programming. In (Commault *et al.*, 2002), the authors proposed a unifying approach to these problems while reducing them to maximum matching problems with maximal cost on a bipartite graph associated with the graph $\mathcal{G}(\Sigma)$. The complexity of the maximum matching problems with maximal cost is $O(N^3)$, where N is the number of nodes in the bipartite graph (Edmonds and Karp, 1972). Since the number of nodes in the bipartite graph associated with $\mathcal{G}(\Sigma)$ is $N = 2(n+m)$, and $m \le n$, it follows that the conditions of Theorem 2 can be checked in $O(n^3)$.

5 Examples

Examples 1 aim at illustrating the structural flatness property based on the state space representation of a system and on its digraph counterpart. In particular, they show how, after having characterized a flat output, the parametrization in terms of shifted outputs defined by Equations (3) can be obtained. Examples 2 only focus on the digraph characterization and address the case where a flat output results from a linear combination of states.

5.1 Examples 1

5.1.1 Examples 1.1 : basic example

Let us consider again the simple example described by Equation (4). The corresponding digraph $\mathcal{G}$ is depicted on Figure 2. From this digraph, it holds that

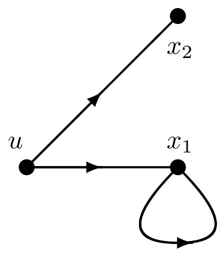

FIGURE 2. Digraph of Example 1.1

$y(k) = x_1(k)$ is a flat output since condition *(1)* and condition *(2)* of Theorem 2 are fulfilled with $m = 1$, $\alpha = \beta = 1$. The explicit expression of F and G given in Section 2.2 can be recovered by taking into account Equation (16) in Appendix that reads :

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} = M^{-1}(\xi) = (b\xi)^{-1} \begin{pmatrix} 0 & 0 & b\xi \\ c & -b & -c(a-\xi) \\ \xi & 0 & -(a-\xi)\xi \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix} \tag{8}$$

with $M(\xi)$, as defined in Equation (6)

$$M(\xi) = \begin{pmatrix} a-\xi & 0 & b \\ 0 & -\xi & c \\ 1 & 0 & 0 \end{pmatrix}$$

Indeed, from (8) , we have that $x_1(k) = y(k)$ and $x_2(k) = cb^{-1}y(k) - acb^{-1}y(k-1)$ that define F and $u(k) = b^{-1}y(k+1) - b^{-1}ay(k)$ that defines G. Finally, let us notice that $\det M(z) = z$. It corroborates that $y(k)$ is a flat output according to Theorem 1.

5.1.2 Examples 1.2 : practical example

First, let us consider the structured system of the form (1) described by

$$x(k+1) = \begin{pmatrix} \lambda_1 & \lambda_2 & 0 \\ \lambda_3 & \lambda_4 & \lambda_5 \\ 0 & \lambda_6 & \lambda_7 \end{pmatrix} x(k) + \begin{pmatrix} \lambda_8 & 0 \\ 0 & 0 \\ 0 & \lambda_9 \end{pmatrix} \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix} \tag{9}$$

where $\lambda_i \in \mathbb{R}$ ($i \in \{1,\dots,9\}$) are possibly non-zero parameters.
Let us consider an output of the form (2) reading

$$y(k) = \begin{pmatrix} 0 & \lambda_{10} & 0 \\ 0 & 0 & \lambda_{11} \end{pmatrix} x(k) \tag{10}$$

where λ_{10} and λ_{11} are possibly non-zero real parameters. The digraph associated to this system is given in Figure 3. After inspection, it turns out that $m = 2$,

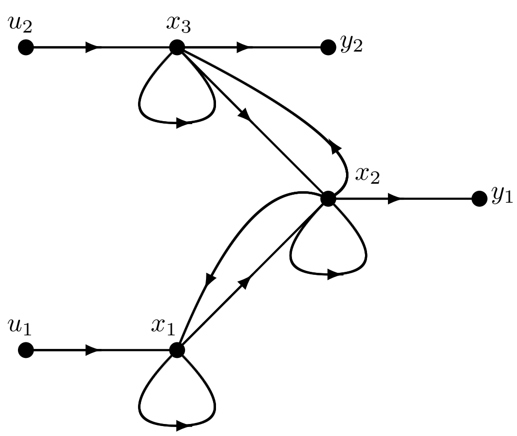

FIGURE 3. Digraph of Example 1.2

$\alpha = 3$, $\beta = 3$ and thus, according to Theorem 2, the output (10) is generically flat.

The practical model considered below and studied in (Sira-Ramirez and Agrawal, 2004) is an instantiation of the structured system (9). The model describes a thin slab homogeneous material subject to two external temperature control sources u_1 and u_2 at the left and right boundaries. The dynamic model of the temperature in three points of a spatial discretization of the slab of material involve three state variables $c(k)$, $b(k)$ and $a(k)$. The state space description, with state vector $x(k) = (c(k)\ b(k)\ a(k))^T$ reads :

$$x(k+1) = \begin{pmatrix} 1-2p & p & 0 \\ p & 1-2p & p \\ 0 & p & 1-2p \end{pmatrix} x(k) + \begin{pmatrix} p & 0 \\ 0 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix} \tag{11}$$

with p a real parameter.
Since it is an instantiation of a structured system, although the parameters λ_i, $i = 1, \cdots, 9$ are not free (they dependent one another through parameter p), the output

$$y(k) = (b(k)\ a(k))^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x(k),$$

is a flat output.
The explicit expressions of F and G in (3) can be obtained taking into account Equation (16) in Appendix. They respectively read

$$x(k) = \begin{pmatrix} -a(k) + \frac{1}{p} b(k+1) + \frac{2p-1}{p} b(k) \\ b(k) \\ a(k) \end{pmatrix} \tag{12}$$

$$u(k) = \begin{pmatrix} \frac{1}{p^2}\Big(b(k+2) - pa(k+1) - p(2p-1)a(k) + \\ 2(2p-1)b(k+1) - (p^2 - (2p-1)^2)b(k)\Big) \\ \\ \frac{1}{p}\Big(a(k+1) + (2p-1)a(k) - pb(k)\Big) \end{pmatrix} \tag{13}$$

5.2 Example 2

Let us consider a structural system associated to the digraph depicted in Figure 4.

Let us consider $\mathbf{U} = \{\mathbf{u}_1, \mathbf{u}_2\}$. It can be noticed that according to condition *(1)* of Theorem 2, the cardinality

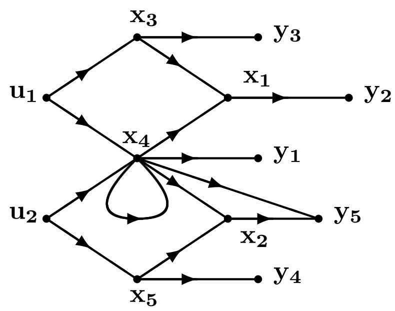

FIGURE 4. Digraph of Example 2

of $\mathbf{U}$ being 2, only sets $\mathbf{Y}$ of two vertices are admissible. Also notice that y_5 is a generic linear combination of x_2 and x_4. Generic linear combination of any number of state components or inputs can be addressed in a similar way because the pattern of the structured matrices C or D is totally free.

Table 1 collects the values of m, α and β for all pairs of output vertices.

	$\{\mathbf{y_1},\mathbf{y_2}\}$	$\{\mathbf{y_1},\mathbf{y_3}\}$	$\{\mathbf{y_1},\mathbf{y_4}\}$	$\{\mathbf{y_1},\mathbf{y_5}\}$	$\{\mathbf{y_2},\mathbf{y_3}\}$
m	2	2	2	2	2
α	3	2	2	3	3
β	3	2	2	3	3
	$\{\mathbf{y_2},\mathbf{y_4}\}$	$\{\mathbf{y_2},\mathbf{y_5}\}$	$\{\mathbf{y_3},\mathbf{y_4}\}$	$\{\mathbf{y_3},\mathbf{y_5}\}$	$\{\mathbf{y_4},\mathbf{y_5}\}$
m	2	2	2	2	2
α	3	3	2	3	2
β	4	5	3	4	3

TABLE 1: α and β with respect to distinct pairs of output vertices for the system associated to the digraph depicted in Figure 4.

Let us inspect Table 1 and apply Theorem 2. First, let us notice that for all the pairs, Condition *(1)* is fulfilled. Condition *(2)* is fulfilled for the pairs $\{\mathbf{y_1},\mathbf{y_2}\}$, $\{\mathbf{y_1},\mathbf{y_3}\}$, $\{\mathbf{y_1},\mathbf{y_4}\}$, $\{\mathbf{y_1},\mathbf{y_5}\}$ and $\{\mathbf{y_2},\mathbf{y_3}\}$ that are, consequently, flat output. The remaining pairs are not flat outputs. Indeed, for example, for the pair $\{\mathbf{y_2},\mathbf{y_5}\}$, $\alpha = 3$ because there are two $\mathbf{U}$-$\{\mathbf{y_2},\mathbf{y_5}\}$ linkings, which are $\{\mathbf{u_1} \to \mathbf{x_3} \to \mathbf{x_1} \to \mathbf{y_2}, \mathbf{u_2} \to \mathbf{x_4} \to \mathbf{y_5}\}$ and $\{\mathbf{u_1} \to \mathbf{x_3} \to \mathbf{x_1} \to \mathbf{y_2}, \mathbf{u_2} \to \mathbf{x_5} \to \mathbf{x_2} \to \mathbf{y_5}\}$. The first one involves three $\mathbf{X}$ vertices and the second one involves four $\mathbf{X}$ vertices. So the minimal number of state vertices is equal to 3. On the other hand, $\beta = 5$ because there is one cycle covering $\mathbf{x_4}$ and this cycle is disjoint from the linking $\{\mathbf{u_1} \to \mathbf{x_3} \to \mathbf{x_1} \to \mathbf{y_2}, \mathbf{u_2} \to \mathbf{x_5} \to \mathbf{x_2} \to \mathbf{y_5}\}$ that involves four $\mathbf{X}$ vertices. Let us notice that even without the cycle, the pair $\{\mathbf{y_2},\mathbf{y_5}\}$ would not be flat because in such a case, $\beta = 4 \neq \alpha$.
The same reasoning applies for the other pairs ($\{\mathbf{y_2}, \mathbf{y_4}\}$,$\{\mathbf{y_3}, \mathbf{y_5}\}$ and $\{\mathbf{y_4}, \mathbf{y_5}\}$) for which $\alpha < \beta$.

Remark 6 *The following remark is devoted to Theorem 3.*

The set $\{\mathbf{y_1}, \mathbf{y_2}\}$ *defines a flat output* $y(k) = (x_1(k), x_2(k))$. *Indeed, let us consider* $\mathbf{Y} = \{\mathbf{y_1}, \mathbf{y_2}\}$ *and recall that* $card(\mathbf{U}) = 2$. *First, there is only one* $\mathbf{U}-\mathbf{Y}$ *maximum linkings. It is* $\{\mathbf{u_1} \rightarrow \mathbf{x_3} \rightarrow \mathbf{x_1} \rightarrow \mathbf{y_2}, \mathbf{u_2} \rightarrow \mathbf{x_4} \rightarrow \mathbf{y_1}\}$. *The size of the* $\mathbf{U}-\mathbf{Y}$ *maximum linking is* $\eta(\mathbf{U}, \mathbf{Y}) = 2$ *and consequently Condition (1) is fulfilled. Next, the length of the* $\mathbf{U}-\mathbf{Y}$ *maximum linking is* 5. *Since the* $\mathbf{U}-\mathbf{Y}$ *maximum linking is unique, it is clear that Condition (2) is fulfilled. Finally, the set* $V_{ess}(\mathbf{U}, \mathbf{Y}) = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{x_1}, \mathbf{x_3}, \mathbf{x_4}\}$. *There is one cycle which covers* $\mathbf{x_4}$, *which is an element of* $V_{ess}(\mathbf{U}, \mathbf{Y})$, *as it is the element* $\mathbf{x_4}$. *Hence, Condition (3) is fulfilled.*

6 Conclusion

Necessary and sufficient conditions for an output of a LTI structured system to be a flat output have been proposed. They are first expressed in terms of algebraic conditions involving the notion of invariant zeros. Then, the conditions have been recast in terms of graphical conditions. They can be checked by resorting to well-known algorithms of polynomial time complexity. To go further, a more challenging task will be an exact and exhaustive characterization or construction of all the possible sets of flat outputs. Indeed, the presented result clearly depends on the chosen outputs. From this perspective, it can be suspected that the result presented in this paper is a relevant starting point.

Appendix : proof of Theorem 1

For convenience, let us notice that Equations (1) and (2) can jointly be described by

$$\begin{pmatrix} A-\xi I & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}, \tag{14}$$

where ξ is the shift operator. Let us define

$$M(\xi) := \begin{pmatrix} A-\xi I & B \\ C & D \end{pmatrix}. \tag{15}$$

Since in the problem of checking for flat outputs, both types of shift are allowed, the goal is to find two linear combinations of finitely many shifts of $\mathbf{y}$, in forward and/or backward direction, that coincide with $\mathbf{x}$ and $\mathbf{u}$, respectively. Hence, the flat output checking problem amounts to finding polynomial matrices $F(z)$ and $G(z)$, containing positive and/or negative powers of z, such that $x(k) = F(\xi)y(k)$, $u(k) = G(\xi)y(k)$ for all $k \in \mathbb{Z}$, or, alternatively, such that $\mathbf{x} = F(\xi)\mathbf{y}$ and $\mathbf{u} = G(\xi)\mathbf{y}$.
When existing, the matrices $F(z)$ and $G(z)$ can be obtained by solving $\mathbf{x}$ and $\mathbf{u}$ in equation (14) from $\mathbf{y}$. For this, elementary operations involving powers of z and z^{-1} can be used. These are operations in the ring $\mathbb{R}[z, z^{-1}]$, consisting of polynomials with finitely many positive and/or negative powers of z.
Note that for polynomials $p(z), q(z) \in \mathbb{R}[z, z^{-1}]$, there holds that $p(z)q(z) = 1$ for all $z \in \mathbb{C}$ if and only if $p(z) = \gamma z^\nu$ and $q(z) = \gamma^{-1}z^{-\nu}$, with $\gamma \neq 0$ and $\nu \in \mathbb{Z}$. Indeed, write $p(z) = z^\nu \tilde{p}(z)$ and $q(z) = z^\mu \tilde{q}(z)$, with $\nu, \mu \in \mathbb{Z}$ and $\tilde{p}(z), \tilde{q}(z)$ polynomials with non-negative powers of z for which $\tilde{p}(0) = \gamma \neq 0$ and $\tilde{q}(0) = \rho \neq 0$. Then $p(z)q(z) = z^{\nu+\mu}\tilde{p}(z)\tilde{q}(z) = 1$ for all $z \in \mathbb{C}$ if and only if $\nu + \mu = 0$, $\gamma\rho = 1$ and $\tilde{p}(z)\tilde{q}(z)$ has no zeros at all. The latter means that both $\tilde{p}(z)$ and $\tilde{q}(z)$ have no zeros at all and are non-trivial polynomials of order zero, *i.e.*, are non-zero constants. Hence, $\tilde{p}(z) = \tilde{p}(0) = \gamma$ and $\tilde{q}(z) = \tilde{q}(0) = \rho$ for all $z \in \mathbb{C}$, so that $p(z) = \gamma z^\nu$ and $q(z) = \gamma^{-1}z^{-\nu}$ with $\gamma \neq 0$ and $\nu \in \mathbb{Z}$.
Next note that $M(z)$ can be seen as a matrix with entries in $\mathbb{R}[z, z^{-1}]$.
Further, note that $\det M(z)$ is a polynomial in $\mathbb{R}[z, z^{-1}]$. Then it follows easily from the above observation and Cramer's rule that, when $M(z)$ is invertible, its inverse has all entries in $\mathbb{R}[z, z^{-1}]$ if and only if $\det M(z) = \gamma z^\nu$ with $\gamma \neq 0$ and $\nu \in \mathbb{N}$.
If $M(z)$ is invertible in the above sense, *i.e.*, $M^{-1}(z)$ has entries in $\mathbb{R}[z, z^{-1}]$, the matrices $F(z)$ and $G(z)$ can be obtained by the following observation.

$$\begin{pmatrix} F(\xi) \\ G(\xi) \end{pmatrix} \mathbf{y} = \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} = M^{-1}(\xi) \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix} = M^{-1}(\xi) \begin{pmatrix} 0 \\ I \end{pmatrix} \mathbf{y}. \tag{16}$$

From this, it is clear that $F(z)$ and $G(z)$ follow from the last block column of $M^{-1}(z)$.
Hence, considering system (1), the output given by (2) is flat, whenever $\det M(z) = \gamma z^\nu$, with real $\gamma \neq 0$ and $\nu \in \mathbb{N}$. In other words, the system given by (1) has flat outputs in the form of (2) if the combination of system and outputs has no invariant zeros outside $z = 0$.
It turns out that the converse is true as well, *i.e.*, if the output (2) is flat output for system (1), then $\det M(z) = \gamma z^\nu$, with real $\gamma \neq 0$ and $\nu \in \mathbb{N}$. To prove this statement, assume that $\det M(\lambda) = 0$ for some $\lambda \neq 0$. (This can happen when there are invariant zeros located outside $z = 0$, or when $\det M(z) = 0$ for all $z \in \mathbb{C}$, in which case it makes no sense to talk about invariant zeros.) Then rank $M(\lambda) < n + m$, and there exists a (complex) vector $\begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that $M(\lambda) \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. In particular, it follows that $\lambda\bar{x} = A\bar{x} + B\bar{u}$ and $C\bar{x} + D\bar{u} = 0$. Now define $\tilde{x}(k) = \lambda^k \bar{x}, \tilde{u}(k) = z^k \bar{u}$ for all $k \in \mathbb{Z}$. Note that at least one of the two $\tilde{x}(k)$ or $\tilde{u}(k)$ is nonzero for all $k \in \mathbb{Z}$. Further, it follows easily that $\tilde{x}(k+1) = A\tilde{x}(k) + B\tilde{u}(k)$, and $\tilde{y}(k) = C\tilde{x}(k) + D\tilde{u}(k) = 0$ for all $k \in \mathbb{Z}$. Hence, $\{(\tilde{x}(k), \tilde{u}(k)) | k \in \mathbb{Z}\}$ forms a nontrivial solution pair of system (14), but in (16) no matrices $F(z)$ and $G(z)$ of suitable dimensions with entries in $\mathbb{R}[z, z^{-1}]$ exist such that $\tilde{x}(k) = F(\xi)\tilde{y}(k)$ and $\tilde{u}(k) = G(\xi)\tilde{y}(k)$,

where $\tilde{y}(k) = C\tilde{x}(k) + D\tilde{u}(k)$. This is because $\tilde{y}(k) = 0$ for all $k \in \mathbb{Z}$, whereas either $\tilde{x}(k)$, $\tilde{u}(k)$, or both, are nonzero for all $k \in \mathbb{Z}$. Hence, the output given by (2) can not be a flat output for the system given by (1). Conversely, if (2) is a flat output for system (1), then it is necessary that $\det M(z) = \gamma z^{\nu}$, with real $\gamma \neq 0$ and $\nu \in \mathbb{N}$. $\square$

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Références

Boukhobza, T. (2012). Sensor location for discrete mode observability of switching structured linear systems with unknown input. *Automatica, 48*(7), 1262–1272.

Boukhobza, T., & Hamelin, F. (2007). Observability analysis for structured bilinear Systems : a graph-theoretic approach. *Automatica, 43*(11), 1968–1974.

Boukhobza, T.,& Millérioux, G. (2016). Graphical characterization of the set of all flat outputs for structured linear discrete-time systems. *6th Symposium on System Structure and Control, (SSSC 2016)*, Istanbul, Turkey.

Clark, A., Alomair, B., Bushnell, L. ,& Poovendran, R. (2017). Input selection for performance and controllability of structured linear descriptor systems. *SIAM Journal on Control and Optimization 55*(1), 457–485.

Commault, C., Dion, J.,& van der Woude, J. W. (2002). Characterization of generic properties of linear structured systems for efficient computations. *Kybernetika 38*, 503–520.

Daafouz, J., Fliess, M.,& Millérioux, G. (2006). Une approche intrinsèque des observateurs linéaires à entrées inconnues. *In : Proc. of the Conférence Internationale Francophone d'Automatique,* Bordeaux, France.

Dakil, M., Simon, C. ,& Boukhobza, T. (2015). Generic methodology for the probabilistic reliability assessment of some structural properties : A graph theoretical approach. *International Journal of Systems Science 46*(10), 1825–1838.

Dion, J-M., Commault, C., & Van der Woude, J. W. (2003). Generic properties and control of linear structured systems : A survey. *Automatica, 39*(7), 1125–1144.

Diwold, J., Kolar, B. ,& Schöberl, M. (2022). A trajectory-based approach to discrete-time flatness. *IEEE Control Systems Letters 6*, 289–294.

Edmonds, J.,& Karp, R. (1972). Theoretical improvements in algorithmic efficiency for network flow problems. *Journal of the ACM 19*(2), 248—264.

Fliess, M. (1992). Reversible linear and nonlinear discrete-time dynamics. *IEEE Trans. Automat. Contr. 37*(8), 1144–1153.

Fliess, M., Levine, J., Martin, P.,& Rouchon, P. (1995). Flatness and defect of non-linear systems : introductory theory and examples. *International Journal of Control 61*(6), 1327–1361.

Fliess, M.,& Marquez, R. (2000). Toward a module theoretic approach to discrete-time linear predictive control. *International Journal of Control 73*, 606–623.

Gracy, S., Milosevic,& J., Sandberg, H. (2020). Actuator security index for structured systems. In Proceedings of American Control Conference (ACC'2020).

Guillot, P.,& Millérioux, G. (2020). Flatness and submersivity of discrete-time dynamical systems. *IEEE Control Systems Letters 4*(2), 337–342.

Kaldmäe, A.,& Kotta, U. (2013). On flatness of discrete-time nonlinear systems. In Proceedings of the 9th IFAC Symposium on Nonlinear Control Systems. Toulouse, France.experiments. IEEE Transactions on Signal Processing, 63(13).

Kawano, Y.,& Cao, M. (2019). Structural accessibility and its applications to complex networks governed by nonlinear balance equations. *IEEE Transactions on Automatic Control 64* (11), 4607–4614.

Kolar, B., Kaldmäe, A., Schöberl, M., Kotta, U.,& Schlacher, K.(2016a). Construction of flat outputs of nonlinear discrete-time systems in a geometric and an algebraic framework. *In Proceedings of the 10th Non-Linear Control Systems (NOLCOS 2016)*, Monterey, USA.

Kolar, B., Schöberl, M.,& Schlacher, K. (2016b). A decomposition procedure for the construction of flat outputs of discrete-time nonlinear control systems. *In Proceedings of the 22nd International Symposium on Mathematical Theory of Networks and Systems*, Minneapolis, USA.

Liu, X.,& Linqiang, P. (2015). Identifying driver nodes in the human signalling network using structural controllability analysis. *IEEE/ACM Transactions on Computational Biology and Bioinformatics 12*(2), 467–472.

Millérioux, G.,& Daafouz, J. (2009). Flatness of switched linear discrete-time systems. *IEEE Trans. on Automatic Control 54*(3), 615–619.

Parriaux, J.,& Millérioux, G. (2013). Nilpotent semigroups for the characterization of flat outputs of switched linear and LPV discrete-time systems. *Systems and Control Letters 62*(8), 679–685.

Ramos, G., Aguiar, A. P.,& Pequito, S. D. (2020). Structural systems theory : an overview of the last 15 years. *ArXiv abs/2008.11223*.

Reinschke, K. J. (1988). *Multivariable Control. A Graph Theoretic Approach.*. Springer-Verlag. New York, U.S.A.

Sato, K. (2012). On an algorithm for checking whether or not a nonlinear discrete-time system is difference flat. *In Proceedings of 20th International Symposium on Mathematical Theory of Networks and Systems.* Melbourne, Australia.

Shoukry, Y., Nuzzo, P., Bezzo, N., Sangiovanni-Vincentelli, A. L., Seshia, S. A., &Tabuada, P. (2015). Secure state reconstruction in differentially flat systems under sensor attacks using satisfiability modulo theory solving. *In Proceedings of the 54th IEEE Conference on Decision and Control (CDC)*, Osaka, Japan.

Sira-Ramirez, H., & Agrawal, S. K. (2004). Differentially Flat Systems. *Marcel Dekker*, New York. USA

van der Woude, J. W. (1991). On the structure at infinity of a structured system. *Linear Algebra Appl. 148*, 145–169.

van der Woude, J. W. (2000). The generic number of invariant zeros of a structured linear system. *SIAM Journal of Control and Optimization, 38*(1), 1–21.

van der Woude, J. W. Commault, C. & Dion, J. M. (2003) Zero orders and dimensions of some invariant subspaces in linear structured systems. *Mathematics of Control, Signals and Systems, 16* (2-3), 225–237.

Yamada, T. (1988). A network flow algorithm to find an elementary I/O,matching. *Networks 18*, 105–109.

Yong, S. Z., Paden, B., & Frazzoli, E. (2015). Computational methods for MIMO flat linear systems : Flat output characterization, test and tracking control. *Proceedings of the American Control Conference (ACC'2015)*. Chicago, USA.