The De Rham, complex Hodge and p-adic Hodge realization functor on the derived category of relative motives over a field of characteristic zero

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Abstract

We introduce the categories of geometric complex mixed Hodge modules on algebraic varieties over a subfield $k \subset \mathbb{C}$, and for a prime number p, the categories of p-adic mixed Hodge modules on algebraic varieties over a subfield $k \subset \mathbb{C}_p$. We then give a complex Hodge realization functor on the derived category of relative motives over $k \subset \mathbb{C}$ and a p-adic Hodge realization functor on the derived category of relative motives over $k \subset \mathbb{C}_p$.

Contents

1 Introduction

Let k be a field of characteristic zero and S be a scheme of finite type over k. Let $DA_c(S)$ be the derived category of constructible motives over S. In a previous work ([10]) when $k = \mathbb{C}$ is the field of complex numbers, we built a Hodge realization functor :

$$\mathcal{F}_S^{Hdg} : \mathrm{DA}_c(S) \to D(MHM(S))$$

where $D(MHM(S))$ is the derived category of mixed Hodge modules introduced by Morihiko Saito. This functor commute to the sixth operations formalism and define a 2-functor morphism on the category of schemes over $\mathbb{C}$.

In this work, we extend this realization functor to the general case of any field k of characteristic zero embedded in $\mathbb{C}$ and we develop a p-adic analog of this realization.

The first step of the construction involves a rational version (i.e. over k) of the category of mixed Hodge modules. The key point of the construction is the existence of the Kashiwara-Malgrange V-filtration over k for regular holonomic $\mathcal{D}$-modules (theorem 35) which is proved by induction on dimension using the complex case, and Saito's theorem on the strictness and stability of the direct image for proper morphisms in the complex case. The mixed hodge modules over k are then the mixed Hodge modules whose regular holonomic sheaf is defined over k. To develop a p-adic analog, we also introduce the full subcategory of geometric mixed Hodge modules.

For simplicity, we assume S smooth and let $(M, F^\bullet M)$ be a filtered regular holonomic $\mathcal{D}_S$-module. We say $(M, F^\bullet M)$ is pure de Rham if it belongs to the full abelian category generated by the successive higher direct images of the structural $\mathcal{D}_X$-module (O_X, F_b) of S-schemes X proper over S and smooth over k, F_b being the trivial filtration.
Let $(M, F^\bullet M, W^\bullet M)$ be a bi-filtered regular holonomic $\mathcal{D}_S$-module ($F^\bullet M$ is the Hodge filtration and $W^\bullet M$ is the weight filtration). We say (M, F, W) is de Rham if the associated graded module $Gr_W(M, F)$ is a pure de Rham module and if the weight filtration is finite and satisfy an admissibility condition with respect to the Cartier divisors of S. The de Rham modules over $S \in \mathrm{Var}(k)$ are introduced in definition 58.

A constructible sheaf over S is given by a constructible sheaf K over the analytic complex space $S^{an}_{\mathbb{C}}$ such that there exist a stratification (S_i) of S over k such that $K_{|S^{an}_{i,\mathbb{C}}}$ is a $\mathbb{Q}$-local system.
We denote by $D_{c,k}(S^{an}_{\mathbb{C}}) \subset D(S^{an}_{\mathbb{C}})$ the full subcategory of the derived category of sheaves (or presheaves) on $S^{an}_{\mathbb{C}}$ whose cohomology sheaves are constructible over S and by $P_k(S^{an}_{\mathbb{C}}) := P(S^{an}_{\mathbb{C}}) \cap D_{c,k}(S^{an}_{\mathbb{C}})$ the full subcategory of perverse sheaves with constructible cohomology sheaves over S.

A geometric mixed Hodge module over S, assumed to be smooth for simplicity, is a triple $((M, F, W), (K, W), \alpha)$ where (M, F, W) is a de Rham module over S, $(K, W) \in P_k(S^{an}_{\mathbb{C}})$ is a filtered perverse sheaf over S and α is an isomorphism $\alpha : (K, W) \otimes \mathbb{C} \simeq DR(S)((M, W)^{an})$ compatible with the de Rham comparison theorem and where $DR(S)(M^{an})$ is the de Rham complex associated to M^{an}. The geometric mixed Hodge module over $S \in \mathrm{Var}(k)$ are introduced in definition 70.

Let $D(MHM_{gm,k,\mathbb{C}}(S))$ be the derived category of the category of complexes of geometric mixed Hodge modules over S. This category can be defined for any scheme S of finite type over k. We prove the following theorem :

Theorem 1. *Let* $\mathrm{Var}(k)$ *be the category of schemes of finite type over a subfield* $k \subset \mathbb{C}$. *Then :*

- *The categories* $D(MHM_{gm,k,\mathbb{C}}(S))$, *for* $S \in \mathrm{Var}(k)$, *are endowed with the formalism of the sixth operation.*
- *There exist a Hodge realization functor :*

$$\mathcal{F}^{Hdg}_S : \mathrm{DA}_c(S) \to D(MHM_{gm,k,\mathbb{C}}(S))$$

compatible with the sixth operations formalism.

Let p be a prime number, $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$ and let $k \subset K \subset \mathbb{C}_p$ be a subfield of a p-adic field K. Let S be a smooth scheme over k, $\mathbb{B}_{dR,S}$ be the sheaf of relative de Rham p-adic periods over the pro-étale site S^{an}_{proet} of the p-adic analytic Huber space S^{an} (introduced by Fontaine, Faltings and Scholze) and $C_{B_{dR}}(S)$ be the category of complexes of $\mathbb{B}_{dR,S}$-modules.

For any lisse $\mathbb{Q}_p$-sheaf L over S_{et}, the sheaf $L \otimes \mathbb{B}_{dR,S}$ over S^{an}_{proet} has a Poincaré resolution by the de Rham complex $\mathcal{O}\mathbb{B}_{dR,S} \otimes \Omega^\bullet_{S^{an}}$. We obtain a functor from the category of lisse $\mathbb{Q}_p$-sheaves over S_{et} to the category of complexes of $\mathbb{B}_{dR,S}$-modules. Using, a Beilinson's devissage by nearby and vanishing cycles functors, we extend this functor to perverse sheaves :

$$\mathbb{B}_{dr,S} : P_k(S_{et}) \to C_{B_{dR}}(S).$$

A geometric p-adic mixed Hodge module over S, assumed to be smooth for simplicity, is a triple $((M, F, W), (K, W), \alpha)$ where (M, F, W) is a de Rham module over S, (K, W) is a filtered perverse sheaf over S_{et} and

$$\alpha : \mathbb{B}_{dr,S}(K, W) \simeq F^0 DR(S)((O\mathbb{B}_{dR,S}, F) \otimes_{O_S} (M, F, W)^{an})$$

is an isomorphism of complexes of W-filtered $(\mathbb{B}_{dR,S}, G)$-modules over S^{an}_{proet}, compatible with the p-adic de Rham comparison theorem (Faltings and Scholze) and where $DR(S)$ is the de Rham complex associated to an analytic D_S-module and $G := \mathrm{Gal}(\bar{K}/K)$ is the Galois group of K. The geometric p-adic mixed Hodge module over $S \in \mathrm{Var}(k)$ are introduced in definition 88.

Let $D(MHM_{gm,k,\mathbb{C}_p}(S))$ be the derived category of the category of complexes of p-adic geometric Hodge modules over S. In fact $D(MHM_{gm,k,\mathbb{C}_p}(S))$ can be defined for any scheme S of finite type over k. We prove the following p-adic version of theorem 1 :

Theorem 2. *Let $k \hookrightarrow K \hookrightarrow \mathbb{C}_p$ be a subfield of a p-adic field. Then :*

- *The categories $D(MHM_{gm,k,\mathbb{C}_p}(S))$, for $S \in \mathrm{Var}(k)$, are endowed with the formalism of the sixth operation.*
- *There exist a p-adic Hodge realization functor :*

$$\mathcal{F}^{Hdg}_S : DA_c(S) \to D(MHM_{gm,k,\mathbb{C}_p}(S))$$

  compatible with the sixth operations formalism

The proof of the first part of the theorems 1 and 2 is similar to the proof of the classical complex case, the crucial point is to show in the p-adic case that the isomorphism α is functorial. We are then reduced to prove that the functor $\mathbb{B}_{dr}$ commute to direct images in the proper case (theorem 47). To do this, we use p-adic Hodge theory comparison theorems in the open case ([21]).

The proof of the second part of the theorems follows our strategy of [10] :
In the case $k \subset \mathbb{C}$, we first fully faithfully embed $D(MHM_{gm,k,\mathbb{C}}(S))$ into the fiber product of the derived category of bi-filtered regular holonomic $\mathcal{D}_S$-modules over k and the derived category of filtered constructible $\mathbb{Q}$-sheaves with k-rational stratification, and construct the realization functor inside this big category (definition 118), then we check that the image is contained in $DMHM_{gm,k,\mathbb{C}}(S)$ and commutes with the six operation (theorem 56).
In the case $k \subset K \subset \mathbb{C}_p$, we first fully faithfully embed $D(MHM_{gm,k,\mathbb{C}_p}(S))$ into the fiber product of the derived category of the category of bi-filtered regular holonomic $\mathcal{D}_S$-modules over k and the derived category of filtered constructible $\mathbb{Q}_p$-etale sheaves with k-rational stratification and construct the realization functor inside this big category (definition 120), then we check that the image is contained in $D(MHM_{gm,k,\mathbb{C}_p}(S))$ and commutes with the six operation (theorem 57).

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2 Preliminaries and Notations

2.1 Notations

- After fixing a universe, we denote by
  - Set the category of sets,
  - Top the category of topological spaces,
  - Ring the category of rings and cRing $\subset$ Ring the full subscategory of commutative rings,
  - TRing the category of topological rings and cTRing $\subset$ TRing the full subscategory of commutative topological rings,
  - RTop the category of ringed spaces,
    - whose set of objects is $\mathrm{RTop} := \{(X, O_X),\ X \in \mathrm{Top},\ O_X \in \mathrm{PSh}(X, \mathrm{Ring})\}$

- $*$ whose set of morphism is $\mathrm{Hom}((T,O_T),(S,O_S)) := \{((f:T\to S),(a_f:f^*O_S\to O_T))\}$

  and by $ts:\mathrm{RTop}\to\mathrm{Top}$ the forgetfull functor.

- RvTop the category of valued ringed spaces
  - $*$ whose set of objects is

    $$\mathrm{RvTop} := \{(X,O_X,(v_x,x\in X)),\ X\in\mathrm{Top},\ O_X\in\mathrm{PSh}(X,\mathrm{cTRing}),\ v_x\in\mathrm{Spv}(O_{X,x})\}$$

    where O_X is a sheaf of complete topological commutative ring for a non archimedean semi-norm, $\mathrm{Spv}(-)$ denote the set of continous valuations of a topological commutative ring for a non archimedean semi-norm
  - $*$ whose set of morphism is

    $$\mathrm{Hom}((T,O_T,(v_x,x\in T)),(S,O_S,(v_x,x\in S))) :=$$
    $$\left\{((f:T\to S),(a_f:f^*O_S\to O_T)),\ a_f^*(v_{f(x)})=v_x\,\text{for all}\,x\in T\right\}.$$

- Cat the category of small categories which comes with the forgetful functor $o:\mathrm{Cat}\to\mathrm{Fun}(\Delta^1,\mathrm{Set})$, where $\mathrm{Fun}(\Delta^1,\mathrm{Set})$ is the category of simplicial sets,
- RCat the category of ringed topos
  - $*$ whose set of objects is $\mathrm{RCat} := \{(\mathcal{X},O_X),\ \mathcal{X}\in\mathrm{Cat},\ O_X\in\mathrm{PSh}(\mathcal{X},\mathrm{Ring})\}$,
  - $*$ whose set of morphism is $\mathrm{Hom}((\mathcal{T},O_T),(\mathcal{S},O_S)) := \{((f:\mathcal{T}\to\mathcal{S}),(a_f:f^*O_S\to O_T)),\}$

  and by $tc:\mathrm{RCat}\to\mathrm{Cat}$ the forgetfull functor,
- AbCat a category consisting of a small set of abelian categories,
- TriCat a category constisting of a small set of triangulated categories.

- Let $F:\mathcal{C}\to\mathcal{C}'$ be a functor with $\mathcal{C},\mathcal{C}'\in\mathrm{Cat}$. For $X\in\mathcal{C}$, we denote by $F(X)\in\mathcal{C}'$ the image of X, and for $X,Y\in\mathcal{C}$, we denote by $F^{X,Y}:\mathrm{Hom}(X,Y)\to\mathrm{Hom}(F(X),F(Y))$ the corresponding map.

- For $\mathcal{C}\in\mathrm{Cat}$, we denote by $\mathcal{C}^{op}\in\mathrm{Cat}$ the opposite category whose set of object is the one of $\mathcal{C}$: $(\mathcal{C}^{op})^0=\mathcal{C}^0$, and whose morphisms are the morphisms of $\mathcal{C}$ with reversed arrows.

- Let $\mathcal{C}\in\mathrm{Cat}$. For $S\in\mathcal{C}$, we denote by $\mathcal{C}/S$ the category
  - whose set of objects $(\mathcal{C}/S)^0=\{X/S=(X,h)\}$ consist of the morphisms $h:X\to S$ with $X\in\mathcal{C}$,
  - whose set of morphism $\mathrm{Hom}(X'/S,X/S)$ between $X'/S=(X',h'),X/S=(X,h)\in\mathcal{C}/S$ consits of the morphisms $(g:X'\to X)\in\mathrm{Hom}(X',X)$ such that $h\circ g=h'$.

  We have then, for $S\in\mathcal{C}$, the canonical forgetful functor

  $$r(S):\mathcal{C}/S\to\mathcal{C},\ \ X/S\mapsto r(S)(X/S)=X,\ (g:X'/S\to X/S)\mapsto r(S)(g)=g$$

  and we denote again $r(S):\mathcal{C}\to\mathcal{C}/S$ the corresponding morphism of (pre)sites.
  - Let $F:\mathcal{C}\to\mathcal{C}'$ be a functor with $\mathcal{C},\mathcal{C}'\in\mathrm{Cat}$. Then for $S\in\mathcal{C}$, we have the canonical functor

    $$F_S:\mathcal{C}/S\to\mathcal{C}'/F(S),\ \ X/S\mapsto F(X/S)=F(X)/F(S),$$
    $$(g:X'/S\to X/S)\mapsto(F(g):F(X')/F(S)\to F(X)/F(S))$$

  - Let $\mathcal{S}\in\mathrm{Cat}$. Then, for a morphism $f:X'\to X$ with $X,X'\in\mathcal{S}$ we have the functor

    $$C(f):\mathcal{S}/X'\to\mathcal{S}/X,\ \ Y/X'=(Y,f_1)\mapsto C(f)(Y/X'):=(Y,f\circ f_1)\in\mathcal{S}/X,$$
    $$(g:Y_1/X'\to Y_2/X')\mapsto(C(f)(g):=g:Y_1/X\to Y_2/X)$$

– Let $\mathcal{S} \in \mathrm{Cat}$ a category which admits fiber products. Then, for a morphism $f : X' \to X$ with $X, X' \in \mathcal{S}$, we have the pullback functor

$$P(f) : \mathcal{S}/X \to \mathcal{S}/X', \ \ Y/X \mapsto P(f)(Y/X) := Y \times_X X'/X' \in \mathcal{S}/X',$$
$$(g : Y_1/X \to Y_2/X) \mapsto (P(f)(g) := (g \times I) : Y_1 \times_X X' \to Y_2 \times_X X')$$

which is right adjoint to $C(f) : \mathcal{S}/X' \to \mathcal{S}/X$, and we denote again $P(f) : \mathcal{S}/X' \to \mathcal{S}/X$ the corresponding morphism of (pre)sites.

- Let $\mathcal{C}, \mathcal{I} \in \mathrm{Cat}$. Assume that $\mathcal{C}$ admits fiber products. For $(S_\bullet) \in \mathrm{Fun}(\mathcal{I}^{op}, \mathcal{C})$, we denote by $\mathcal{C}/(S_\bullet) \in \mathrm{Fun}(\mathcal{I}, \mathrm{Cat})$ the diagram of category given by
  – for $I \in \mathcal{I}$, $\mathcal{C}/(S_\bullet)(I) := \mathcal{C}/S_I$,
  – for $r_{IJ} : I \to J$, $\mathcal{C}/(S_\bullet)(r_{IJ}) := P(r_{IJ}) : \mathcal{C}/S_I \to \mathcal{C}/S_J$, where we denoted again $r_{IJ} : S_J \to S_I$ the associated morphism in $\mathcal{C}$.

- Let $(F, G) : \mathcal{C} \leftrightarrows \mathcal{C}'$ an adjonction between two categories.
  – For $X \in C$ and $Y \in C'$, we consider the adjonction isomorphisms
    * $I(F, G)(X, Y) : \mathrm{Hom}(F(X), Y) \to \mathrm{Hom}(X, G(Y)),\ (u : F(X) \to Y) \mapsto (I(F, G)(X, Y)(u) : X \to G(Y))$
    * $I(F, G)(X, Y) : \mathrm{Hom}(X, G(Y)) \to \mathrm{Hom}(F(X), Y),\ (v : X \to G(Y)) \mapsto (I(F, G)(X, Y)(v) : F(X) \to Y)$.
  – For $X \in \mathcal{C}$, we denote by $\mathrm{ad}(F, G)(X) := I(F, G)(X, F(X))(I_{F(X)}) : X \to G \circ F(X)$.
  – For $Y \in \mathcal{C}'$ we denote also by $\mathrm{ad}(F, G)(Y) := I(F, G)(G(Y), Y)(I_{G(Y)}) : F \circ G(Y) \to Y$.

  Hence,
  – for $u : F(X) \to Y$ a morphism with $X \in C$ and $Y \in C'$, we have $I(F, G)(X, Y)(u) = G(u) \circ \mathrm{ad}(F, G)(X)$,
  – for $v : X \to G(Y)$ a morphism with $X \in C$ and $Y \in C'$, we have $I(F, G)(X, Y)(v) = \mathrm{ad}(F, G)(Y) \circ F(v)$.

- Let $\mathcal{C}$ a category.
  – We denote by $(\mathcal{C}, F)$ the category of filtered objects : $(X, F) \in (\mathcal{C}, F)$ is a sequence $(F^\bullet X)_{\bullet \in \mathbb{Z}}$ indexed by $\mathbb{Z}$ with value in $\mathcal{C}$ together with monomorphisms $a_p : F^p X \hookrightarrow F^{p-1} X \hookrightarrow X$.
  – We denote by $(\mathcal{C}, F, W)$ the category of bifiltered objects : $(X, F, W) \in (\mathcal{C}, F, W)$ is a sequence $(W^\bullet F^\bullet X)_{\bullet,\bullet} \in \mathbb{Z}^2$ indexed by $\mathbb{Z}^2$ with value in $\mathcal{C}$ together with monomorphisms $W^q F^p X \hookrightarrow F^{p-1} X$, $W^q F^p X \hookrightarrow W^{q-1} F^p X$.

- Let $\mathcal{A}$ an additive category.
  – We denote by $C(\mathcal{A}) := \mathrm{Fun}(\mathbb{Z}, \mathcal{A})$ the category of (unbounded) complexes with value in $\mathcal{A}$, where we have denoted $\mathbb{Z}$ the category whose set of objects is $\mathbb{Z}$, and whose set of morphism between $m, n \in \mathbb{Z}$ consists of one element (identity) if $n = m$, of one elemement if $n = m + 1$ and is $\emptyset$ in the other cases.
  – We have the full subcategories $C^b(\mathcal{A})$, $C^-(\mathcal{A})$, $C^+(\mathcal{A})$ of $C(\mathcal{A})$ consisting of bounded, resp. bounded above, resp. bounded below complexes.
  – We denote by $K(\mathcal{A}) := \mathrm{Ho}(C(\mathcal{A}))$ the homotopy category of $C(\mathcal{A})$ whose morphisms are equivalent homotopic classes of morphism and by $Ho : C(\mathcal{A}) \to K(\mathcal{A})$ the full homotopy functor. The category $K(\mathcal{A})$ is in the standard way a triangulated category.

- Let $\mathcal{A}$ an additive category.

- We denote by $C_{fil}(\mathcal{A}) \subset (C(\mathcal{A}), F) = C(\mathcal{A}, F)$ the full additive subcategory of filtered complexes of $\mathcal{A}$ such that the filtration is biregular : for $(A^\bullet, F) \in (C(\mathcal{A}), F)$, we say that F is biregular if $F^\bullet A^r$ is finite for all $r \in \mathbb{Z}$.
- We denote by $C_{2fil}(\mathcal{A}) \subset (C(\mathcal{A}), F, W) = C(\mathcal{A}, F, W)$ the full subcategory of bifiltered complexes of $\mathcal{A}$ such that the filtration is biregular.
- For $A^\bullet \in C(\mathcal{A})$, we denote by $(A^\bullet, F_b) \in (C(\mathcal{A}), F)$ the complex endowed with the trivial filtration (filtration bete) : $F^p A^n = 0$ if $p \geq n+1$ and $F^p A^n = A^n$ if $p \leq n$. Obviously, a morphism $\phi : A^\bullet \to B^\bullet$, with $A^\bullet, B^\bullet \in C(\mathcal{A})$ induces a morphism $\phi : (A^\bullet, F_b) \to (B^\bullet, F_b)$.
- For $(A^\bullet, F) \in C(\mathcal{A}, F)$, we denote by $(A^\bullet, F(r)) \in C(\mathcal{A}, F)$ the filtered complex where the filtration is given by $F^p(A^\bullet, F(r)) := F^{p+r}(A^\bullet, F)$.

- Let $\mathcal{A}$ be an abelian category. Then the additive category $(\mathcal{A}, F)$ is an exact category which admits kernel and cokernel (but is NOT an abelian category). A morphism $\phi : (M, F) \to (N, F)$ with $(M, F) \in (\mathcal{A}, F)$ is strict if the inclusion $\phi(F^n M) \subset F^n N \cap \mathrm{Im}(\phi)$ is an equality, i.e. if $\phi(F^n M) = F^n N \cap \mathrm{Im}(\phi)$.

- Let $\mathcal{A}$ be an abelian category.

  - For $(A^\bullet, F) \in C(\mathcal{A}, F)$, considering $a_p : F^p A^\bullet \hookrightarrow A^\bullet$ the structural monomorphism of of the filtration, we denote by, for $n \in \mathbb{N}$,

  $$H^n(A^\bullet, F) \in (\mathcal{A}, F), \ F^p H^n(A^\bullet, F) := \mathrm{Im}(H^n(a_p) : H^n(F^p A^\bullet) \to H^n(A^\bullet)) \subset H^n(A^\bullet)$$

  the filtration induced on the cohomology objects of the complex. In the case $(A^\bullet, F) \in C_{fil}(\mathcal{A})$, the spectral sequence $E_r^{p,q}(A^\bullet, F)$ associated to $(A^\bullet, F)$ converge to $\mathrm{Gr}_F^p H^{p+q}(A^\bullet, F)$, that is for all $p, q \in \mathbb{Z}$, there exist $r_{p+q} \in \mathbb{N}$, such that $E_s^{p,q}(A^\bullet, F) = \mathrm{Gr}_F^p H^{p+q}(A^\bullet, F)$ for all $s \leq r_{p+q}$.
  - A morphism $m : (A^\bullet, F) \to (B^\bullet, F)$ with $(A^\bullet, F), (B^\bullet, F) \in C(\mathcal{A}, F)$ is said to be a filtered quasi-isomorphism if for all $n, p \in \mathbb{Z}$,

  $$H^n \mathrm{Gr}_F^p(m) : H^n(\mathrm{Gr}_F^p A^\bullet) \xrightarrow{\sim} H^n(\mathrm{Gr}_F^p B^\bullet)$$

  is an isomorphism in $\mathcal{A}$. Consider a commutative diagram in $C(\mathcal{A}, F)$

  $$\begin{tikzcd}
  (A^\bullet, F) \arrow[r, "m"] \arrow[d, "\phi"'] & (B^\bullet, F) \arrow[r, "i_2"] \arrow[d, "\psi"'] & \mathrm{Cone}(m) = ((A^\bullet, F)[1] \oplus (B^\bullet, F), d, d' - m) \arrow[r, "p_1"] \arrow[d, "{(\phi[1], \psi)}"] & (A^\bullet, F)[1] \arrow[d, "{\phi[1]}"] \\
  (A'^\bullet, F) \arrow[r, "m'"] & (B'^\bullet, F) \arrow[r, "i_2"] & \mathrm{Cone}(m') = ((A'^\bullet, F)[1] \oplus (B'^\bullet, F), d, d' - m') \arrow[r, "p_1"] & (A'^\bullet, F)[1]
  \end{tikzcd}$$

  If ϕ and ψ are filtered quasi-isomorphisms, then $(\phi[1], \psi)$ is an filtered quasi-isomorphism. That is, the filtered quasi-isomorphism satisfy the 2 of 3 property for canonical triangles.

- Let $\mathcal{A}$ be an abelian category.

  - We denote by $D(\mathcal{A})$ the localization of $K(\mathcal{A})$ with respect to the quasi-isomorphisms and by $D : K(\mathcal{A}) \to D(\mathcal{A})$ the localization functor. The category $D(\mathcal{A})$ is a triangulated category in the unique way such that D a triangulated functor.
  - We denote by $D_{fil}(\mathcal{A})$ the localization of $K_{fil}(\mathcal{A})$ with respect to the filtered quasi-isomorphisms and by $D : K_{fil}(\mathcal{A}) \to D_{fil}(\mathcal{A})$ the localization functor.

- Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{Inj}(A) \subset A$ the full subcategory of injective objects, and by $\mathrm{Proj}(A) \subset A$ the full subcategory of projective objects.

- For $\mathcal{S} \in \mathrm{Cat}$ a small category, we denote by
  - $\mathrm{PSh}(\mathcal{S}) := \mathrm{PSh}(\mathcal{S}, \mathrm{Ab}) := \mathrm{Fun}(\mathcal{S}, \mathrm{Ab})$ the category of presheaves on $\mathcal{S}$, i.e. the category of presheaves of abelian groups on $\mathcal{S}$,
  - $K(\mathcal{S}) := K(\mathrm{PSh}(\mathcal{S})) = \mathrm{Ho}(C(\mathcal{S}))$ In particular, we have the full homotopy functor $Ho : C(\mathcal{S}) \to K(\mathcal{S})$,
  - $C_{(2)fil}(\mathcal{S}) := C_{(2)fil}(\mathrm{PSh}(\mathcal{S})) \subset C(\mathrm{PSh}(\mathcal{S}), F, W)$ the big abelian category of (bi)filtered complexes of presheaves on $\mathcal{S}$ with value in abelian groups such that the filtration is biregular, and $\mathrm{PSh}_{(2)fil}(\mathcal{S}) = (\mathrm{PSh}(\mathcal{S}), F, W)$,
  - $K_{fil}(\mathcal{S}) := K_{fil}(\mathrm{PSh}(\mathcal{S})) = \mathrm{Ho}(C_{fil}(\mathcal{S}))$,
  - $K_{fil,r}(\mathcal{S}) := K_{fil,r}(\mathrm{PSh}(\mathcal{S})) = \mathrm{Ho}_r(C_{fil}(\mathcal{S}))$, $K_{fil,\infty}(\mathcal{S}) := K_{fil,\infty}(\mathrm{PSh}(\mathcal{S})) = \mathrm{Ho}_\infty(C_{fil}(\mathcal{S}))$.

  For $f : \mathcal{T} \to \mathcal{S}$ a morphism a presite with $\mathcal{T}, \mathcal{S} \in \mathrm{Cat}$, given by the functor $P(f) : \mathcal{S} \to \mathcal{T}$, we will consider the adjonctions given by the direct and inverse image functors :

  - $(f^*, f_*) = (f^{-1}, f_*) : \mathrm{PSh}(\mathcal{S}) \leftrightarrows \mathrm{PSh}(\mathcal{T})$, which induces $(f^*, f_*) : C(\mathcal{S}) \leftrightarrows C(\mathcal{T})$, we denote, for $F \in C(\mathcal{S})$ and $G \in C(\mathcal{T})$ by

    $$\mathrm{ad}(f^*, f_*)(F) : F \to f_* f^* F \ , \ \mathrm{ad}(f^*, f_*)(G) : f^* f_* G \to G$$

    the adjonction maps,
  - $(f_*, f^\perp) : \mathrm{PSh}(\mathcal{T}) \leftrightarrows \mathrm{PSh}(\mathcal{S})$, which induces $(f_*, f^\perp) : C(\mathcal{T}) \leftrightarrows C(\mathcal{S})$, we denote for $F \in C(\mathcal{S})$ and $G \in C(\mathcal{T})$ by

    $$\mathrm{ad}(f_*, f^\perp)(F) : G \to f^\perp f_* G \ , \ \mathrm{ad}(f_*, f^\perp)(G) : f_* f^\perp F \to F$$

    the adjonction maps.

- For $(\mathcal{S}, O_S) \in \mathrm{RCat}$ a ringed topos, we denote by
  - $\mathrm{PSh}_{O_S}(\mathcal{S})$ the category of presheaves of O_S modules on $\mathcal{S}$, whose objects are $\mathrm{PSh}_{O_S}(\mathcal{S})^0 := \{(M, m), M \in \mathrm{PSh}(\mathcal{S}), m : M \otimes O_S \to M\}$, together with the forgetful functor $o : \mathrm{PSh}(\mathcal{S}) \to \mathrm{PSh}_{O_S}(\mathcal{S})$,
  - $C_{O_S}(\mathcal{S}) = C(\mathrm{PSh}_{O_S}(\mathcal{S}))$ the big abelian category of complexes of presheaves of O_S modules on $\mathcal{S}$,
  - $K_{O_S}(\mathcal{S}) := K(\mathrm{PSh}_{O_S}(\mathcal{S})) = \mathrm{Ho}(C_{O_S}(\mathcal{S}))$, in particular, we have the full homotopy functor $Ho : C_{O_S}(\mathcal{S}) \to K_{O_S}(\mathcal{S})$,
  - $C_{O_S(2)fil}(\mathcal{S}) := C_{(2)fil}(\mathrm{PSh}_{O_S}(\mathcal{S})) \subset C(\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of O_S modules on $\mathcal{S}$ such that the filtration is biregular and $\mathrm{PSh}_{O_S(2)fil}(\mathcal{S}) = (\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$,
  - $K_{O_S fil}(\mathcal{S}) := K_{fil}(\mathrm{PSh}_{O_S}(\mathcal{S})) = \mathrm{Ho}(C_{O_S fil}(\mathcal{S}))$,
  - $K_{O_S fil,r}(\mathcal{S}) := K_{fil,r}(\mathrm{PSh}_{O_S}(\mathcal{S})) = \mathrm{Ho}_r(C_{O_S fil}(\mathcal{S}))$, $K_{O_S fil,\infty}(\mathcal{S}) := K_{fil,\infty}(\mathrm{PSh}_{O_S}(\mathcal{S})) = \mathrm{Ho}_\infty(C_{O_S fil}(\mathcal{S}))$.

- For $\mathcal{S} \in \mathrm{Cat}$ a small category and $n \in \mathbb{N}$, $\mathrm{PSh}_{\mathbb{Z}/n\mathbb{Z}}(\mathcal{S}) \subset \mathrm{PSh}(\mathcal{S})$ is the full subcategory of n-torsion presheaves. The functor

  $$(-) \otimes \mathbb{Z}/n\mathbb{Z} : \mathrm{PSh}(\mathcal{S}) \to \mathrm{PSh}_{\mathbb{Z}/n\mathbb{Z}}(\mathcal{S}), F \mapsto F \otimes \mathbb{Z}/n\mathbb{Z}$$

  is right exact and its restriction the full subcategory $\mathrm{PSh}(\mathcal{S})_L \subset \mathrm{PSh}(\mathcal{S})$ of torsion free presheaves is exact. For $\mathcal{S} \in \mathrm{Cat}$ and $p \in \mathbb{N}$ a prime number,

  $$\mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{S}) \subset \mathrm{PSh}(\mathbb{N} \times \mathcal{S}) = \mathrm{PSh}(\mathcal{S}, \mathrm{Fun}(\mathbb{N}, \mathrm{Ab}))$$

is category whose objects are $(F_l)_{l\in\mathbb{N}}$ with $F_l \in \mathrm{PSh}_{\mathbb{Z}/p^l\mathbb{Z}}(\mathcal{S})$ such that $F_l \to F_{l+1}/p^l F_{l+1}$ is an isomorphism. We then have

$$C_{\mathbb{Z}_p}(\mathcal{S}) := C(\mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{S})) \subset C(\mathbb{N}\times\mathcal{S}) = \mathrm{PSh}(\mathcal{S},\mathrm{Fun}(\mathbb{N},C(\mathbb{Z}))).$$

We get the functor

$$(-)\otimes\mathbb{Z}_p : C(\mathcal{S}) \to C_{\mathbb{Z}_p}(\mathcal{S}), F \mapsto (F\otimes\mathbb{Z}/p^l\mathbb{Z})_{l\in\mathbb{N}}$$

which is right exact. For $\mathcal{S}\in\mathrm{Cat}$ a site with topology τ, we have the localization

$$D_{\mathbb{Z}_p}(\mathcal{S}) := \mathrm{Ho}_\tau\, C(\mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{S}))$$

of τ local equivalences of $C(\mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{S})) \subset \mathrm{PSh}(\mathcal{S},\mathrm{Fun}(\mathbb{N},C(\mathbb{Z})))$.

- For $\mathcal{S}_\bullet \in \mathrm{Fun}(\mathcal{I},\mathrm{Cat})$ a diagram of (pre)sites, with $\mathcal{I}\in\mathrm{Cat}$ a small category, we denote by $S_\bullet := \Gamma\mathcal{S}_\bullet \in \mathrm{Cat}$ the associated diagram category
  - whose objects are $\Gamma\mathcal{S}_\bullet^0 := \{(X_I,u_{IJ})_{I\in\mathcal{I}}\}$, with $X_I\in\mathcal{S}_I$, and for $r_{IJ}: I\to J$ with $I,J\in\mathcal{I}$, $u_{IJ}: X_J \to r_{IJ}(X_I)$ are morphism in $\mathcal{S}_J$ noting again $r_{IJ}:\mathcal{S}_I\to\mathcal{S}_J$ the associated functor,
  - whose morphism are $m=(m_I):(X_I,u_{IJ})\to(X'_I,v_{IJ})$ satisfying $v_{IJ}\circ m_I = r_{IJ}(m_J)\circ u_{IJ}$ in $\mathcal{S}_J$,

  We have then $\mathrm{PSh}(\mathcal{S}_\bullet)=\mathrm{PSh}(\Gamma\mathcal{S}_\bullet)$ the category of presheaves on $\mathcal{S}_\bullet$,
  - whose objects are $\mathrm{PSh}(\mathcal{S}_\bullet)^0 := \{(F_I,u_{IJ})_{I\in\mathcal{I}}\}$, with $F_I\in\mathrm{PSh}(\mathcal{S}_I)$, and for $r_{IJ}:I\to J$ with $I,J\in\mathcal{I}$, $u_{IJ}: F_I\to r_{IJ*}F_J$ are morphism in $\mathrm{PSh}(\mathcal{S}_I)$, noting again $r_{IJ}:\mathcal{S}_J\to\mathcal{S}_I$ the associated morphism of presite,
  - whose morphism are $m=(m_I):(F_I,u_{IJ})\to(G_I,v_{IJ})$ satisfying $v_{IJ}\circ m_I = r_{IJ*}m_J\circ u_{IJ}$ in $\mathrm{PSh}(\mathcal{S}_I)$,

  Let $\mathcal{I},\mathcal{I}'\in\mathrm{Cat}$ be small categories. Let $(f_\bullet,s):\mathcal{T}_\bullet\to\mathcal{S}_\bullet$ a morphism a diagrams of (pre)site with $\mathcal{T}_\bullet\in\mathrm{Fun}(\mathcal{I},\mathrm{Cat}),\mathcal{S}_\bullet\in\mathrm{Fun}(\mathcal{I}',\mathrm{Cat})$, which is by definition given by a functor $s:\mathcal{I}\to\mathcal{I}'$ and morphism of functor $P(f_\bullet):\mathcal{S}_{s(\bullet)} := \mathcal{S}_\bullet\circ s\to\mathcal{T}_\bullet$. Here, we denote for short, $\mathcal{S}_{s(\bullet)} := \mathcal{S}_\bullet\circ s \in \mathrm{Fun}(\mathcal{I},\mathrm{Cat})$. We have then, for $r_{IJ}:I\to J$ a morphism, with $I,J\in\mathcal{I}$, a commutative diagram in Cat

$$D_{fIJ} := \begin{array}{ccc} \mathcal{S}_{s(J)} & \xrightarrow{r^s_{IJ}} & \mathcal{S}_{s(I)} \\ {\scriptstyle f_J}\uparrow & & \uparrow{\scriptstyle f_I} \\ \mathcal{T}_J & \xrightarrow{r^t_{IJ}} & \mathcal{T}_I \end{array} .$$

  In particular we get the adjonction given by the direct and inverse image functors :

$$\begin{aligned}((f_\bullet,s)^*,(f_\bullet,s)^*) = ((f_\bullet,s)^{-1},(f_\bullet,s)_*) : \mathrm{PSh}(\mathcal{S}_{s(\bullet)}) &\leftrightarrows \mathrm{PSh}(\mathcal{T}_\bullet),\\ F=(F_I,u_{IJ}) &\mapsto (f_\bullet,s)^*F := (f_I^*F_I, T(D_{fIJ})(F_J)\circ f_I^*u_{IJ}),\\ G=(G_I,v_{IJ}) &\mapsto (f_\bullet,s)_*G := (f_{I*}G_I, f_{I*}v_{IJ}).\end{aligned}$$

- Let $\mathcal{I}\in\mathrm{Cat}$ a small category. For $(\mathcal{S}_\bullet,O_{S_\bullet})\in\mathrm{Fun}(\mathcal{I},\mathrm{RCat})$ a diagram of ringed topos, we denote by

$$(\mathcal{S}_\bullet,O_{S_\bullet}) := (\Gamma\mathcal{S}_\bullet, O_{\Gamma\mathcal{S}_\bullet})\in\mathrm{RCat}.$$

  We have then $\mathrm{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet) = \mathrm{PSh}_{O_{\Gamma\mathcal{S}_\bullet}}(\Gamma\mathcal{S}_\bullet)$ the category of presheaves of modules on $(\mathcal{S}_\bullet,O_{S_\bullet})$,
  - whose objects are $\mathrm{PSh}_{O_{S_\bullet}}(\mathcal{S}_\bullet)^0 := \{(F_I,u_{IJ})_{I\in\mathcal{I}}\}$, with $F_I\in\mathrm{PSh}_{O_{S_I}}(\mathcal{S}_I)$, and for $r_{IJ}:I\to J$ with $I,J\in\mathcal{I}$, $u_{IJ}:F_I\to r_{IJ*}F_J$ are morphism in $\mathrm{PSh}_{O_{S_I}}(\mathcal{S}_I)$, noting again $r_{IJ}:\mathcal{S}_J\to\mathcal{S}_I$ the associated morphism of presite,

– whose morphism are $m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ})$ satisfying $v_{IJ} \circ m_I = r_{IJ*}m_J \circ u_{IJ}$ in $\mathrm{PSh}_{O_{S_I}}(\mathcal{S}_I)$,

- For $A \in \mathrm{Ring}$, $\dim_K(A)$ denote the Krull dimension of A. For $\sigma : A \to B$ a morphism with $A, B \in \mathrm{cRing}$, we have the extention of scalar functor

$$\otimes_A B : (-) \otimes_A B : \mathrm{Mod}(A) \to \mathrm{Mod}(B),\ M \mapsto M \otimes_A B$$
$$(m : M' \to M) \mapsto (m_B := m \otimes I : M' \otimes_A B \to M \otimes_A B).$$

which is left ajoint to the restriction of scalar

$$Res_{A/B} : \mathrm{Mod}(B) \to \mathrm{Mod}(A),\ M = (M, a_M) \mapsto M = (M, a_M \circ \sigma),\ (m : M' \to M) \mapsto (m : M' \to M)$$

The adjonction maps are

  – for $M \in \mathrm{Mod}(A)$, the canonical map in $\mathrm{Mod}(A)$

$$n_{A/B}(M) : M \to M \otimes_A B,\ n_{A/B}(M)(m) := m \otimes 1,$$

  – for $M \in \mathrm{Mod}(B)$,

$$I \times \Delta_B : M \otimes_A B = M \otimes_B B \otimes_A B$$

  in $\mathrm{Mod}(B)$, where $\Delta_B : B \otimes_A B \to B$ is given by for $x, y \in B$, $\Delta_B(x, y) = x - y$.

Let $\sigma : A \to B$ a morphism with $A, B \in \mathrm{cRing}$. A module $M \in \mathrm{Mod}(B)$ is said to be defined over A if there exist a module $M_0 \in \mathrm{Mod}(A)$ and an isomorphism $M \simeq M_0 \otimes_A B$ in $\mathrm{Mod}(B)$. A module $M \in \mathrm{Mod}(B)$ is defined over A if and only if there exist a presentation of M, that is an exact sequence in $\mathrm{Mod}(B)$, $B^{\oplus^J} \xrightarrow{\phi} B^{\oplus^I} \to M \to 0$, such that $\phi \circ \sigma(A^{\oplus^J}) \subset A^{\oplus^I}$.

- For $f = (f, a_f) : (\mathcal{T}, O_T) \to (\mathcal{S}, O_S)$ a morphism of ringed topos with $(\mathcal{S}, O_S), (\mathcal{T}, O_T) \in \mathrm{RCat}$, $a_f : f^*O_S \to O_T$, we have the pull-back of presheaves of modules

$$f^{*mod} : \mathrm{PSh}_{O_S}(\mathcal{S}) \to \mathrm{PSh}_{O_T}(\mathcal{T}),\ M \mapsto f^{*mod}M := f^*M \otimes_{f^*O_S} O_T$$
$$(m : M' \to M) \mapsto (f^{*mod}M := f^*m \otimes I : f^{*mod}M' \to f^{*mod}M).$$

which is left ajoint to

$$f_* : \mathrm{PSh}_{O_T}(\mathcal{T}) \to \mathrm{PSh}_{O_S}(\mathcal{S}),\ M = (M, a_M) \mapsto f_*M = (f_*M, a_M \circ a_f),$$
$$(m : M' \to M) \mapsto (f_*m : f_*M' \to f_*M)$$

The adjonction maps are

  – for $M \in \mathrm{PSh}_{O_S}(\mathcal{S})$, the canonical map in $\mathrm{PSh}_{O_S}(\mathcal{S})$

$$\mathrm{ad}(f^{*mod}, f_*)(M) := n_{f^*O_S/O_T}(M) : M \to f_*f^{*mod}M = f_*f^*M \otimes_{f_*f^*O_S} f_*O_T,$$
$$n_{f^*O_S/O_T}(M)(m) := \mathrm{ad}(f_*, f^*)(M)(m) \otimes 1,$$

  – for $M \in \mathrm{PSh}_{O_T}(\mathcal{S})$, the canonical map

$$\mathrm{ad}(f^{*mod}, f_*)(M) := I \times \Delta_{O_T} : f^{*mod}f_*M = f^*f_*M \otimes_{O_T} O_T \otimes_{f^*O_S} O_T \to M,$$

  where $\Delta_{O_T} : O_T \otimes_{f^*O_S} O_T \to O_T$ is given by for $x, y \in \Gamma(T, O_T)$, $T \in \mathcal{T}$, $\Delta_{O_T}(x, y) = x - y$.

Let $f = (f, a_f) : (\mathcal{T}, O_T) \to (\mathcal{S}, O_S)$ a morphism of ringed topos with $(\mathcal{S}, O_S), (\mathcal{T}, O_T) \in \mathrm{RCat}$, $a_f : f^*O_S \to O_T$ A presheaf $M \in \mathrm{PSh}_{O_T}(\mathcal{T})$ is said to be defined over $(\mathcal{S}, O_S)$ if there exist a $M_0 \in\in \mathrm{PSh}_{O_S}(\mathcal{S})$ such that $M \simeq f^{*mod}M_0$ in $\mathrm{PSh}_{O_T}(\mathcal{T})$. For $M \in \mathrm{PSh}_{O_T}(\mathcal{T})$ quasi-coherent, M is locally defined over $(\mathcal{S}, O_S)$ if and only if there exists locally a presentation of M, that is an exact sequence in $\mathrm{PSh}_{O_T}(\mathcal{T}')$, $\mathcal{T}' \subset T$, $O_T^{\oplus^J} \xrightarrow{\phi} O_T^{\oplus^I} \to M \to 0$, such that $\phi \circ a_f(O_S^{\oplus^J}) \subset O_S^{\oplus^I}$.

- For $X \in \text{Top}$, we denote by $\dim_F(X)$ its Krull dimension and $\dim_L(X)$ its Lebegue dimension. Note that if X is Hausdorf $\dim_F(X) = 0$ and if X is everywhere not Hausdorf $\dim_L(X) = 0$. For $X \in \text{Top}$ and $x \in X$, we denote by $\dim_{F,x}(X) := inf_{x \in U} \dim_F(U)$ its Krull dimension at x and $\dim_{L,x}(X) := inf_{x \in U} \dim_L(U)$ its Lebegue dimension at x.
- Denote by $\text{Sch} \subset \text{RTop}$ the full subcategory of schemes. For $X \in \text{Sch}$, $\dim(X) := \dim_F(X)$. For $X = \operatorname{Spec} A \in \text{Sch}$ an affine scheme, $\dim(X) = \dim_K(A)$. For $X \in \text{Sch}$ and $x \in X$, $\dim_x(X) := \dim_{F,x}(X) = \dim(O_{X,x})$. A morphism $h : U \to S$ with $U, S \in \text{Sch}$ is said to be smooth if it is flat with smooth fibers geometric fibers. A morphism $r : U \to X$ with $U, X \in \text{Sch}$ is said to be etale if it is non ramified and flat. In particular an etale morphism $r : U \to X$ with $U, X \in \text{Sch}$ is smooth and quasi-finite (i.e. the fibers are either the empty set or a finite subset of X) For $X \in \text{Sch}$, we denote by
  - $\text{Sch}^{ft}/X \subset \text{Sch}/X$ the full subcategory consisting of objects $X'/X = (X', f) \in \text{Sch}/X$ such that $f : X' \to X$ is an morphism of finite type
  - $X^{et} \subset \text{Sch}^{ft}/X$ the full subcategory consisting of objects $U/X = (X, h) \in \text{Sch}/X$ such that $h : U \to X$ is an etale morphism.
  - $X^{sm} \subset \text{Sch}^{ft}/X$ the full subcategory consisting of objects $U/X = (X, h) \in \text{Sch}/X$ such that $h : U \to X$ is a smooth morphism.

  For a field k, we consider $\text{Sch}/k := \text{Sch}/\operatorname{Spec} k$ the category of schemes over $\operatorname{Spec} k$. We then denote by
  - $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k, i.e. schemes of finite type over k,
  - $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
  - $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

  For a morphism of field $\sigma : k \hookrightarrow K$, we have the extention of scalar functor

  $$\otimes_k K : \text{Sch}/k \to \text{Sch}/K, \ X \mapsto X_K := X_{K,\sigma} := X \otimes_k K, \ (f : X' \to X) \mapsto (f_K := f \otimes I : X'_K \to X_K).$$

  which is left ajoint to the restriction of scalar

  $$Res_{k/K} : \text{Sch}/K \to \text{Sch}/k, \ X = (X, a_X) \mapsto X = (X, \sigma \circ a_X), \ (f : X' \to X) \mapsto (f : X' \to X)$$

  The adjonction maps are
  - for $X \in \text{Sch}/k$, the projection $\pi_{k/K}(X) : X_K \to X$ in Sch/k, for $X = \cup_i X_i$ an affine open cover with $X_i = \operatorname{Spec}(A_i)$ we have by definition $\pi_{k/K}(X_i) = n_{k/K}(A_i)$,
  - for $X \in \text{Sch}/K$, $I \times \Delta_K : X \hookrightarrow X_K = X \times_K K \otimes_k K$ in Sch/K, where $\Delta_K : K \otimes_k K \to K$ is the diagonal which is given by for $x, y \in K$, $\Delta_K(x, y) = x - y$.

  The extention of scalar functor restrict to a functor

  $$\otimes_k K : \text{Var}(k) \to \text{Var}(K), \ X \mapsto X_K := X_{K,\sigma} := X \otimes_k K, \ (f : X' \to X) \mapsto (f_K := f \otimes I : X'_K \to X_K).$$

  and for $X \in \text{Var}(k)$ we have $\pi_{k/K}(X) : X_K \to X$ the projection in Sch/k. An algebraic variety $X \in \text{Var}(K)$ is said to be defined over k if there exists $X_0 \in \text{Var}(k)$ such that $X \simeq X_0 \otimes_k K$ in $\text{Var}(K)$. By definition,
  - for $X = \operatorname{Spec}(A) \in \text{Var}(K)$ an affine variety, X is defined over K if $A \in \text{Mod}(K)$ is defined over k, that is if $A = K[x_1, \ldots, x_N]/I$ is a presentation of A, $I =< f_1, \cdots f_r >\subset K[x_1, \ldots, x_N]$ with $f_1, \ldots, f_r \in k[x_1, \cdots, x_N]$ is generated by elements over k.

- for $X = \mathrm{Proj}(B) \in \mathrm{PVar}(K)$ an projective variety, X is defined over K if $B \in \mathrm{Mod}(K)$ is defined over k, that is if $B = K[x_0, \ldots, x_N]/I$ is a presentation of B with I generated by homogeneous elements, $I =< f_1, \cdots f_r >\subset K[x_0, \ldots, x_N]$ with $f_1, \ldots, f_r \in k[x_0, \cdots, x_N]$ homogeneous.

For $X = (X, a_X) \in \mathrm{Var}(k)$, we have $\mathrm{Sch}^{ft}/X = \mathrm{Var}(k)/X$ since for $f : X' \to X$ a morphism of schemes of finite type, $(X', a_X \circ f) \in \mathrm{Var}(k)$ is the unique structure of variety over k of $X' \in \mathrm{Sch}$ such that f becomes a morphism of algebraic varieties over k, in particular we have

- $X^{et} \subset \mathrm{Sch}^{ft}/X = \mathrm{Var}(k)/X$,
- $X^{sm} \subset \mathrm{Sch}^{ft}/X = \mathrm{Var}(k)/X$.

- Denote by $\mathrm{CW} \subset \mathrm{Top}$ the full subcategory of CW complexes, by $\mathrm{CS} \subset \mathrm{CW}$ the full subcategory of Δ complexes, by $\mathrm{TM}(\mathbb{R}) \subset \mathrm{CW}$ the full subcategory of topological (real) manifolds which admits a CW structure (a topological manifold admits a CW structure if it admits a differential structure) and by $\mathrm{Diff}(\mathbb{R}) \subset \mathrm{RTop}$ the full subcategory of differentiable (real) manifold.

- Denote by $\mathrm{AnSp}(\mathbb{C}) \subset \mathrm{RTop}$ the full subcategory of analytic spaces over $\mathbb{C}$, and by $\mathrm{AnSm}(\mathbb{C}) \subset \mathrm{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold). For $X \in \mathrm{AnSp}(\mathbb{C})$, we set $\dim(X) := 1/2 \dim_L(X)$, and for $x \in X$ $\dim_x(X) := 1/2 \dim_{L,x}(X)$. For $X \in \mathrm{AnSp}(\mathbb{C})$ and $x \in X$, there exist by Weirstrass preparation theorem a finite surjective morphism $r : X_x \to \mathbb{D}^n_0$ where $\mathbb{D}^n = D(0,1)^n \subset \mathbb{C}^n$ is the open ball and $\dim_x(X) := 1/2 \dim_{L,x}(X) = \dim_K(O_{X,x}) = n$. For $X \in \mathrm{AnSm}(\mathbb{C})$ and $x \in X$, there exist an isomorphism $r : X_x \xrightarrow{\sim} \mathbb{D}^n_0$, hence there exist a covering by open subsets $X = \cup_i X_i$ such that $r_i : X_i \xrightarrow{\sim} \mathbb{D}^{n_i}$. If $X \in \mathrm{AnSm}(\mathbb{C})$ is connected then $\dim(X) := \dim_L(X) = 2n$ where $r : X_x \xrightarrow{\sim} \mathbb{D}^n_0$ for $x \in X$. For $X \in \mathrm{AnSp}(\mathbb{C})$, $\dim(X) := \dim_L(X) = \dim_L(X_{reg})$ where $X_{reg} \subset X$ is the smooth locus of X. A morphism $h : U \to S$ with $U, S \in \mathrm{AnSp}(\mathbb{C})$ is said to be smooth if it is flat with smooth fibers. A morphism $r : U \to X$ with $U, X \in \mathrm{AnSp}(\mathbb{C})$ is said to be etale if it is non ramified and flat. For $X \in \mathrm{AnSp}(\mathbb{C})$, we denote by

  - $X^{et} \subset \mathrm{AnSp}(\mathbb{C})/X$ the full subcategory consisting of objects $U/X = (X, h) \in \mathrm{AnSp}(\mathbb{C})/X$ such that $h : U \to X$ is an etale morphism.
  - $X^{sm} \subset \mathrm{AnSp}(\mathbb{C})/X$ the full subcategory consisting of objects $U/X = (X, h) \in \mathrm{AnSp}(\mathbb{C})/X$ such that $h : U \to X$ is a smooth morphism.

  By the Weirstrass preparation theorem (or the implicit function theorem if U and X are smooth), a morphism $r : U \to X$ with $U, X \in \mathrm{AnSp}(\mathbb{C})$ is etale if and only if it is an isomorphism local. Hence for $X \in \mathrm{AnSp}(\mathbb{C})$, the morphism of site $\pi_X : X^{et} \to X$ is an isomorphism of site.

- For $V \in \mathrm{Var}(\mathbb{C})$, we denote by $V^{an} \in \mathrm{AnSp}(\mathbb{C})$ the complex analytic space associated to V with the usual topology induced by the usual topology of $\mathbb{C}^N$. For $W \in \mathrm{AnSp}(\mathbb{C})$, we denote by $W^{cw} \in \mathrm{AnSp}(\mathbb{C})$ the topological space given by W which is a CW complex. For simplicity, for $V \in \mathrm{Var}(\mathbb{C})$, we denote by $V^{cw} := (V^{an})^{cw} \in \mathrm{CW}$. We have then

  - the analytical functor $\mathrm{An} : \mathrm{Var}(\mathbb{C}) \to \mathrm{AnSp}(\mathbb{C})$, $\mathrm{An}(V) = V^{an}$,
  - the forgetful functor $\mathrm{Cw} = tp : \mathrm{AnSp}(\mathbb{C}) \to \mathrm{CW}$, $\mathrm{Cw}(W) = W^{cw}$,
  - the composite of these two functors $\widetilde{\mathrm{Cw}} = \mathrm{Cw} \circ \mathrm{An} : \mathrm{Var}(\mathbb{C}) \to \mathrm{CW}$, $\widetilde{\mathrm{Cw}}(V) = V^{cw}$.

- Let $S \in \mathrm{RTop}$. Let $S = \cup_{i \in L} S_i$ open cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \mathrm{RTop}$, $L \in \mathrm{Set}$. We denote by $(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(L)^{op}, \mathrm{RTop})$ the diagram given by for $I \in L$ $\tilde{S}_L := \Pi_{i \in I} \tilde{S}_I$ and for $I \subset J$, $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ is the projection. We have then open embeddings $j_I : S_I := \cap_{i \in I} S_i \hookrightarrow S$ and closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$. We consider the functor

$$T(S/(\tilde{S}_I)) : C(S) \to C(S/(\tilde{S}_I)) \hookrightarrow C((\tilde{S}_I)),\ K \mapsto T(S/(\tilde{S}_I))(K) := (i_{I*} j_I^* K, I).$$

- Let $k \subset \mathbb{C}$ a subfield. For $S \in \mathrm{Var}(k)$, let $S = \cup_i S_i$ affine open cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \mathrm{SmVar}(k)$ connected. We denote by $DR(S) := DR(S)^{[-]}$ the De Rham functor

$$DR(S) := DR(S)^{[-]} : C_{\mathcal{D}fil}(S/(\tilde{S}_I)) \to C(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}})),$$
$$((M_I, F), u_{IJ}) \mapsto DR(S)((M_I, F), u_{IJ}) := (DR(\tilde{S}^{an}_{I,\mathbb{C}})(M_I, F)[-d_{\tilde{S}_I}], DR(u_{IJ}))$$

- We denote by $\mathrm{AdSp} \subset \mathrm{RvTop}$ the full subcategory of adic spaces. By definition, for $X = (X, O_X, O_X^+) \in \mathrm{AdSp}$ there exist an open cover $X = \cup_i X_i$ such that

$$X_i = \mathrm{Spa}(R_i, R_i^+) := \left\{v \in \mathrm{Spv}(R_i),\, v(f) \leq 1 \,\text{for all}\, f \in R_i^+\right\}$$

with $R_i \in \mathrm{cTRing}$ for a non archimedean semi-norm and $R_i^+ \subset R_i^o \subset R_i$ a subring, where $R_i^o = \{f \in R_i s.t |f_i| \leq 1\}$. We then have

$$R_i^+ = \left\{f \in R_i,\, v(f) \leq 1 \,\text{for all}\, v \in \mathrm{Spa}(R_i, R_i^+)\right\}.$$

- For $K \subset \mathbb{C}_p$ a p-adic field, denote by $\mathrm{AnSp}(K) \subset \mathrm{RTop}$ the full subcategory of analytic spaces over K, By definition, for $X \in \mathrm{AnSp}(K)$ there exist an open cover $X = \cup_i X_i$ such that $X_i = \mathrm{Spv}(R_i)$ where $R_i \in \mathrm{cTRing}$ is a Tate algebra over K. Note that we have the map $m : X \to M(X)$ in RTop with $m_{|X_i} : X_i = \mathrm{Spv}(R_i) \to M(X_i) = \mathrm{Spec}(R_i)$ which sends a valuation v to its support $p = \{f \in R_i,\, v(f) = 0\}$ and where $M(X_i)$ is endowed with the standard G-topology. By definition, we have

  - the forgetfull functor $o_K : \mathrm{AdSp}/(K, K^+) \to \mathrm{AnSp}(K)$, such that $o_K(\mathrm{Spa}(R, R^+)) = \mathrm{Spv}(R)$,
  - the canonical functor $R_K : \mathrm{AnSp}(K) \to \mathrm{AdSp}/(K, O_K)$ such that $R_K(\mathrm{Spv}(R)) = \mathrm{Spa}(R, R^o)$.

  We denote by $\mathrm{AnSm}(K) \subset \mathrm{AnSp}(K)$ the full subcategory of smooth analytic spaces. For $X \in \mathrm{AnSp}(K)$, we set $\dim(X) := \dim_L(X)$, and for $x \in X$ $\dim_x(X) := \dim_{L,x}(X)$. For $X \in \mathrm{AnSp}(K)$ and $x \in X$, there exist by Weirstrass preparation theorem a finite surjective morphism $r : X_x \to \mathbb{D}^n_0$ where $\mathbb{D}^n = D(0,1)^n \subset K^n$ is the open ball and $\dim_x(X) := \dim_{L,x}(X) = \dim_K(O_{X,x}) = n$. For $X = \mathrm{Spv}(A) \in \mathrm{AnSp}(K)$ affinoid, $\dim(X) := \dim_L(X) = \dim_K(A)$ where for the last equality note that for $0 < r < 1$, $D(0,r)^n = \mathrm{Spv}(K < x_1, \cdots, x_n >_r) \subset D(0,1)^n = \mathrm{Spv}(K < x_1, \cdots, x_n >)$ is a rational open subset since the norm is ultrametric in contrast to the complex case. A morphism $h : U \to S$ with $U, S \in \mathrm{AnSp}(K)$ is said to be smooth if it is flat with smooth geometric fibers. A morphism $r : U \to X$ with $U, X \in \mathrm{AnSp}(K)$ is said to be etale if it is non ramified and flat. For $X \in \mathrm{AnSp}(K)$, we denote by

  - $X^{et} \subset \mathrm{AnSp}(K)/X$ the full subcategory consisting of objects $U/X = (X, h) \in \mathrm{AnSp}(K)/X$ such that $h : U \to X$ is an etale morphism.
  - $X^{sm} \subset \mathrm{AnSp}(K)/X$ the full subcategory consisting of objects $U/X = (X, h) \in \mathrm{AnSp}(K)/X$ such that $h : U \to X$ is a smooth morphism.

  For $X \in \mathrm{AnSp}(K)$, we have the morphism of site $\pi_X : X^{et} \to X$.

- Let $K \subset \mathbb{C}_p$ a p-adic field. For $V \in \mathrm{Var}(K)$, we denote by $V^{an} \in \mathrm{AnSp}(K)$. We have then the analytical functor $\mathrm{An} : \mathrm{Var}(K) \to \mathrm{AnSp}(K)$, $\mathrm{An}(V) = V^{an}$, $\mathrm{An}(f) = f^{an}$. We will also consider the canonical functor $R_K : \mathrm{AnSp}(K) \to \mathrm{AdSp}/(K, O_K)$, which sends by definition $X = \mathrm{Spv}(R)$ affinoid with R a Tate algebra over K to $X = \mathrm{Spa}(R, R^o)$ with $R^o := \{f \in R,\, |f|_p \leq 1\}$.

- Denote by Top^2 the category whose set of objects is

$$(\mathrm{Top}^2)^0 := \{(X, Z),\; Z \subset X \text{ closed}\} \subset \mathrm{Top} \times \mathrm{Top}$$

and whose set of morphism between $(X_1, Z_1), (X_2, Z_2) \in \mathrm{Top}^2$ is

$$\mathrm{Hom}_{\mathrm{Top}^2}((X_1, Z_1), (X_2, Z_2)) := \left\{(f : X_1 \to X_2), \text{ s.t. } Z_1 \subset f^{-1}(Z_2)\right\} \subset \mathrm{Hom}_{\mathrm{Top}}(X_1, X_2)$$

For $S \in \mathrm{Top}$, $\mathrm{Top}^2/S := \mathrm{Top}^2/(S, S)$ is then by definition the category whose set of objects is

$$(\mathrm{Top}^2/S)^0 := \{((X, Z), h), h : X \to S, \ Z \subset X \text{ closed }\} \subset \mathrm{Top}/S \times \mathrm{Top}$$

and whose set of morphisms between $(X_1, Z_1)/S = ((X_1, Z_1), h_1), (X_2, Z_2)/S = ((X_2, Z_2), h_2) \in \mathrm{Top}^2/S$ is the subset

$$\mathrm{Hom}_{\mathrm{Top}^2/S}((X_1, Z_1)/S, (X_2, Z_2)/S) := \\ \left\{(f : X_1 \to X_2), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\right\} \subset \mathrm{Hom}_{\mathrm{RTop}}(X_1, X_2)$$

We denote by

$$\mu_S : \mathrm{Top}^{2,pr}/S := \{((Y \times S, Z), p), p : Y \times S \to S, \ Z \subset Y \times S \text{ closed }\} \hookrightarrow \mathrm{Top}^2/S$$

the full subcategory whose objects are those with $p : Y \times S \to S$ a projection, and again $\mu_S : \mathrm{Top}^2/S \to \mathrm{Top}^{2,pr}/S$ the corresponding morphism of sites. We denote by

$$\mathrm{Gr}_S^{12} : \mathrm{Top}/S \to \mathrm{Top}^{2,pr}/S, \ X/S \mapsto \mathrm{Gr}_S^{12}(X/S) := (X \times S, \bar{X})/S, \\ (g : X/S \to X'/S) \mapsto \mathrm{Gr}_S^{12}(g) := (g \times I_S : (X \times S, \bar{X}) \to (X' \times S, \bar{X}'))$$

the graph functor, $X \hookrightarrow X \times S$ being the graph embedding (which is a closed embedding if X is separated), and again $\mathrm{Gr}_S^{12} : \mathrm{Top}^{2,pr}/S \to \mathrm{Top}/S$ the corresponding morphism of sites.

- Denote by RTop^2 the category whose set of objects is

$$(\mathrm{RTop}^2)^0 := \{((X, O_X), Z), \ Z \subset X \text{ closed}\} \subset \mathrm{RTop} \times \mathrm{Top}$$

and whose set of morphism between $((X_1, O_{X_1}), Z_1), ((X_2, O_{X_2}), Z_2) \in \mathrm{RTop}^2$ is

$$\mathrm{Hom}_{\mathrm{RTop}^2}(((X_1, O_{X_1}), Z_1), ((X_2, O_{X_2}), Z_2)) := \\ \left\{(f : (X_1, O_{X_1}) \to (X_2, O_{X_2})), \text{ s.t. } Z_1 \subset f^{-1}(Z_2)\right\} \subset \mathrm{Hom}_{\mathrm{RTop}}((X_1, O_{X_1}), (X_2, O_{X_2}))$$

For $(S, O_S) \in \mathrm{RTop}$, $\mathrm{RTop}^2/(S, O_S) := \mathrm{RTop}^2/((S, O_S), S)$ is then by definition the category whose set of objects is

$$(\mathrm{RTop}^2/(S, O_S))^0 := \\ \{(((X, O_X), Z), h), h : (X, O_X) \to (S, O_S), \ Z \subset X \text{ closed }\} \subset \mathrm{RTop}/(S, O_S) \times \mathrm{Top}$$

and whose set of morphisms between $(((X_1, O_{X_1}), Z_1), h_1), (((X_2, O_{X_2}), Z_2), h_2) \in \mathrm{RTop}^2/(S, O_S)$ is the subset

$$\mathrm{Hom}_{\mathrm{RTop}^2/(S,O_S)}(((X_1, O_{X_1}), Z_1)/(S, O_S), ((X_2, O_{X_2}), Z_2)/(S, O_S)) := \\ \left\{(f : (X_1, O_{X_1}) \to (X_2, O_{X_2})), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\right\} \\ \subset \mathrm{Hom}_{\mathrm{RTop}}((X_1, O_{X_1}), (X_2, O_{X_2}))$$

We denote by

$$\mu_S : \mathrm{RTop}^{2,pr}/S := \{(((Y \times S, q^* O_Y \otimes p^* O_S), Z), p), p : Y \times S \to S, \ Z \subset Y \times S \text{ closed }\} \hookrightarrow \mathrm{RTop}^2/S$$

the full subcategory whose objects are those with $p : Y \times S \to S$ is a projection, and again $\mu_S : \mathrm{RTop}^2/S \to \mathrm{RTop}^{2,pr}/S$ the corresponding morphism of sites. We denote by

$$\begin{aligned}
\mathrm{Gr}_S^{12} : \mathrm{RTop}/S \to \mathrm{RTop}^{2,pr}/S, \\
(X, O_X)/(S, O_S) \mapsto \mathrm{Gr}_S^{12}((X, O_X)/(S, O_S)) := ((X \times S, q^*O_X \otimes p^*O_S), \bar{X})/(S, O_S), \\
(g : (X, O_X)/(S, O_S) \to (X', O_{X'})/(S, O_S)) \mapsto \\
\mathrm{Gr}_S^{12}(g) := (g \times I_S : ((X \times S, q^*O_X \otimes p^*O_S), \bar{X}) \to ((X' \times S, q^*O_X \otimes p^*O_S), \bar{X}'))
\end{aligned}$$

the graph functor, $X \hookrightarrow X \times S$ being the graph embedding (which is a closed embedding if X is separated), $p : X \times S \to S$, $q : X \times S \to X$ the projections, and again $\mathrm{Gr}_S^{12} : \mathrm{RTop}^{2,pr}/S \to \mathrm{RTop}/S$ the corresponding morphism of sites.

- We denote by $\mathrm{Sch}^2 \subset \mathrm{RTop}^2$ the full subcategory such that the first factors are schemes. For a field k, we denote by $\mathrm{Sch}^2/k := \mathrm{Sch}^2/(\mathrm{Spec}\, k, \{\mathrm{pt}\})$ and by
  - $\mathrm{Var}(k)^2 \subset \mathrm{Sch}^2/k$ the full subcategory such that the first factors are algebraic varieties over k, i.e. schemes of finite type over k,
  - $\mathrm{PVar}(k)^2 \subset \mathrm{QPVar}(k)^2 \subset \mathrm{Var}(k)^2$ the full subcategories such that the first factors are quasi-projective varieties and projective varieties respectively,
  - $\mathrm{PSmVar}(k)^2 \subset \mathrm{SmVar}(k)^2 \subset \mathrm{Var}(k)^2$ the full subcategories such that the first factors are smooth varieties and smooth projective varieties respectively.

  In particular we have, for $S \in \mathrm{Var}(k)$, the graph functor

  $$\begin{aligned}
  \mathrm{Gr}_S^{12} : \mathrm{Var}(k)/S \to \mathrm{Var}(k)^{2,pr}/S,\ X/S \mapsto \mathrm{Gr}_S^{12}(X/S) := (X \times S, X)/S, \\
  (g : X/S \to X'/S) \mapsto \mathrm{Gr}_S^{12}(g) := (g \times I_S : (X \times S, X) \to (X' \times S, X'))
  \end{aligned}$$

  the graph embedding $X \hookrightarrow X \times S$ is a closed embedding since X is separated in the subcategory of schemes $\mathrm{Sch} \subset \mathrm{RTop}$, and again $\mathrm{Gr}_S^{12} : \mathrm{Var}(k)^{2,pr}/S \to \mathrm{Var}(k)/S$ the corresponding morphism of sites.

- We denote by $\mathrm{CW}^2 \subset \mathrm{Top}^2$ the full subcategory such that the first factors are CW complexes, by $\mathrm{TM}(\mathbb{R})^2 \subset \mathrm{CW}^2$ the full subcategory such that the first factors are topological (real) manifolds and by $\mathrm{Diff}(\mathbb{R})^2 \subset \mathrm{RTop}^2$ the full subcategory such that the first factors are differentiable (real) manifold.

2.2 The p-adic de Rham period sheaves for adic spaces over a p-adic field

Let p a prime number. For $X = (X, O_X, O_X^+) \in \mathrm{AdSp}/(K, K^+)$ an adic space over a p-adic field $K \subset \mathbb{C}_p$, we consider

- the map $W_X : \mathbb{A}_{inf,X} := W(\hat{O}_X^{b+}) \to \hat{O}_X^+$ where $\hat{O}_X$ denote the completion of O_X with respect to the ideal $pO_X \subset O_X$, b the tilting functor and W the Witt vectors,
- the map $W_X : \mathbb{B}_{inf,X} := W(\hat{O}_X^{b+})[p^{-1}] \to \hat{O}_X := \hat{O}_X^+[p^{-1}]$
- the integral period sheaf $\mathbb{B}_{dr,X}^+ := \varprojlim_{n \in \mathbb{N}} \mathbb{B}_{inf,X}/(\ker W_X)^n$ with the filtration $F^k\mathbb{B}_{dr,X}^+ := (\ker W_X)^k \mathbb{B}_{dr,X}^+ \subset \mathbb{B}_{dr,X}^+$.
- the period sheaf $\mathbb{B}_{dr,X} := \mathbb{B}_{dr,X}^+[t^{-1}]$ where t is a generator of the ideal $\ker W_X \subset \mathbb{B}_{dr,X}^+$ with the filtration $F^k\mathbb{B}_{dr,X} := \sum_j t^{-j} F^{k+j}\mathbb{B}_{dr,X}^+ \subset \mathbb{B}_{dr,X}$.

- the integral sheaf

$$O\mathbb{B}^+_{dr,X} := \varprojlim_{n\in\mathbb{N}} (O^+_X \hat{\otimes}_{W(O_K/pO_K)} \mathbb{A}_{inf,X}[p^{-1}])/(\ker I \otimes W_X)^n$$

with the filtration $F^k O\mathbb{B}^+_{dr,X} := (\ker W_X)^k O\mathbb{B}^+_{dr,X} \subset O\mathbb{B}^+_{dr,X}$.

- the period sheaf $O\mathbb{B}_{dr,X} := O\mathbb{B}^+_{dr,X}[t^{-1}]$ where t is a generator of the ideal $\ker W_X \subset O\mathbb{B}^+_{dr,X}$ with the filtration $F^k O\mathbb{B}_{dr,X} := \sum_j t^{-j} F^{k+j} O\mathbb{B}^+_{dr,X} \subset O\mathbb{B}_{dr,X}$.

2.3 The classical theorems for etale topology on schemes, for CW complexes, and the comparaison theorem with the analytic topologies for algebraic varieties over local fields

We first recall the smooth base change theorem

Theorem 3. *(i) Consider a commutative diagram in* Sch *which is cartesian*

$$\begin{tikzcd} X_T \arrow[r, "f'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

such that g is smooth or more generally locally acyclic. Let $F \in C(X^{et})$ be a torsion sheaf where we recall that $X^{et} \subset \mathrm{Sch}^{ft}/X$ is the small etale site. Then the transformation map (see [10] section 2) in $D(T^{et})$

$$T(f,g)(F) : g^* Rf_* F \to Rf'_* g'^* F$$

is an isomorphism.

(ii) Let k'/k an extention of field of characteristic zero. Let $f : X \to S$ a morphism in $\mathrm{Var}(k)$*. Let $F \in C(X^{et})$ be a torsion sheaf Then the transformation map (see [10] section 2) in $D(S^{et}_{k'})$*

$$T(f, \pi_{k/k'})(F) : \pi^*_{k/k'} Rf_* F \to Rf_{k'*} \pi^*_{k/k'} F$$

is an isomorphism where we recall (see section 2) $\pi_{k/k'} = \pi_{k/k'}(X) : X_{k'} \to X$ and $\pi_{k/k'} = \pi_{k/k'}(S) : S_{k'} \to S$ are the projections.

Proof. (i): Standard : see [22] for example.
(ii):Follows from (i). □

We now recall the proper base change theorem :

Theorem 4. *Consider a commutative diagram in* Sch *which is cartesian*

$$\begin{tikzcd} X_T \arrow[r, "f'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

such that f is proper. Let $F \in C(X^{et})$ be a torsion sheaf where we recall that $X^{et} \subset \mathrm{Sch}^{ft}/X$ is the small etale site. Then the transformation map (see [10] section 2) in $D(T^{et})$

$$T(f,g)(F) : g^* Rf_* F \to Rf'_* g'^* F$$

is an isomorphism.

Proof. Standard : see [22] for example. ☐

We deduce from the proper base change theorem the projection formula and the Kunneth formula for the cohomology of etale sheaves:

Theorem 5. *(i) Let $f : X \to S$ a proper morphism with $S, X \in \mathrm{Sch}$. Let $n \in \mathbb{N}$. Let $F \in C_{\mathbb{Z}/n\mathbb{Z}}(X^{et})$ and $G \in C_{\mathbb{Z}/n\mathbb{Z}}(S^{et})$ be n-torsion sheaves. Then the transformation map (see [10] section 2) in $D_{\mathbb{Z}/n\mathbb{Z}}(S^{et})$*

$$T(f, \otimes)(F, G) : Rf_*F \otimes^L_{\mathbb{Z}/n\mathbb{Z}} G \to Rf_*(F \otimes^L_{\mathbb{Z}/n\mathbb{Z}} f^*G)$$

is an isomorphism.

(ii) Let $f : X \to S$ a morphism with $S, X \in \mathrm{Sch}$. Let $n \in \mathbb{N}$. Let $F \in C_{\mathbb{Z}/n\mathbb{Z}}(X^{et})$ and $G \in C_{\mathbb{Z}/n\mathbb{Z}}(S^{et})$ be n-torsion sheaves. Then the transformation map (see [10] section 2) in $D_{\mathbb{Z}/n\mathbb{Z}}(S^{et})$ given by (i) and the open embedding case after taking a compactification of f

$$T_!(f, \otimes)(F, G) : Rf_!F \otimes^L_{\mathbb{Z}/n\mathbb{Z}} G \to Rf_!(F \otimes^L_{\mathbb{Z}/n\mathbb{Z}} f^*G)$$

is an isomorphism.

Proof. (i):Follows from theorem 4: see [22] for example.
(ii):Follows from (i) by taking a compactification $\bar{f} : \bar{X} \to \bar{S}$ of $f : X \to S$. ☐

Remark 1. *Let $f : X \to S$ a morphism with $S, X \in \mathrm{Sch}$. Let $n \in \mathbb{N}$. Let $F \in C_{\mathbb{Z}/n\mathbb{Z}}(X^{et})$ and $G \in C_{\mathbb{Z}/n\mathbb{Z}}(S^{et})$ be n-torsion sheaves. Then, if f is not proper, $H^k(Rf_*F \otimes^L_{\mathbb{Z}/n\mathbb{Z}} G)$ is NOT isomorphic in $\mathrm{Shv}_{\mathbb{Z}/n\mathbb{Z}}(S^{et})$ to $H^kRf_*(F \otimes^L_{\mathbb{Z}/n\mathbb{Z}} G)$ in general.*

Let $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ two morphisms with $X_1, X_2, S \in \mathrm{Sch}$. Denote $p_1 : X_1 \times_S X_2 \to X_1$ and $p_2 : X_1 \times_S X_2 \to X_2$ the base change maps. We have then

$$f_1 \otimes f_2 = f_1 \circ p_1 = f_2 \circ p_2 : X_1 \times_S X_2 \to S.$$

Let $F_1 \in C_{\mathbb{Z}/n\mathbb{Z}}(X_1^{et})$ and $F_2 \in C_{\mathbb{Z}/n\mathbb{Z}}(X_2^{et})$ be n-torsion sheaves. Then the canonical map in $C_{\mathbb{Z}/n\mathbb{Z}}(S^{et})$ (see [10] section 2)

$$\begin{aligned} T(f_1, f_2, \otimes)(F_1, F_2) : Rf_{1*}F_1 \otimes^L_{\mathbb{Z}/n\mathbb{Z}} Rf_{2*}F_2 &\xrightarrow{\mathrm{ad}(p_1^*, Rp_{1*})(F_1) \otimes \mathrm{ad}(p_2^*, Rp_{2*})(F_2)} \\ Rf_{1*}Rp_{1*}p_1^*F_1 \otimes^L_{\mathbb{Z}/n\mathbb{Z}} Rf_{2*}Rp_{2*}p_2^*F_2 &\xrightarrow{=} R(f_1 \otimes f_2)_*p_1^*F_1 \otimes^L_{\mathbb{Z}/n\mathbb{Z}} R(f_1 \otimes f_2)_*p_2^*F_2 \\ &\xrightarrow{T(\otimes, E)(-,-)} R(f_1 \times f_2)_*(p_1^*F \otimes^L_{\mathbb{Z}/n\mathbb{Z}} p_2^*F_2). \end{aligned}$$

Theorem 6. *(i) Let $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ two proper morphisms with $X_1, X_2, S \in \mathrm{Sch}$. Denote $p_1 : X_1 \times_S X_2 \to X_1$ and $p_2 : X_1 \times_S X_2 \to X_2$ the base change maps. Let $F_1 \in C_{\mathbb{Z}/n\mathbb{Z}}(X_1^{et})$ and $F_2 \in C_{\mathbb{Z}/n\mathbb{Z}}(X_2^{et})$ be n-torsion sheaves. Then the canonical map in $C_{\mathbb{Z}/n\mathbb{Z}}(S^{et})$ given above*

$$T(f_1, f_2, \otimes)(F_1, F_2) : Rf_{1*}F_1 \otimes^L_{\mathbb{Z}/n\mathbb{Z}} Rf_{2*}F_2 \to R(f_1 \times f_2)_*(p_1^*F \otimes^L_{\mathbb{Z}/n\mathbb{Z}} p_2^*F_2)$$

is an isomorphism.

(ii) Let $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ two morphisms with $X_1, X_2, S \in \mathrm{Sch}$. Denote $p_1 : X_1 \times_S X_2 \to X_1$ and $p_2 : X_1 \times_S X_2 \to X_2$ the base change maps. Let $F_1 \in C_{\mathbb{Z}/n\mathbb{Z}}(X_1^{et})$ and $F_2 \in C_{\mathbb{Z}/n\mathbb{Z}}(X_2^{et})$ be n-torsion sheaves. Then the canonical map in $C_{\mathbb{Z}/n\mathbb{Z}}(S^{et})$ given after taking compactification of f_1 and f_1 by the one of (i) for the compactifications and on the other hand by the open embedding case

$$T(f_1, f_2, \otimes)(F_1, F_2) : Rf_{1!}F_1 \otimes^L_{\mathbb{Z}/n\mathbb{Z}} Rf_{2!}F_2 \to R(f_1 \times f_2)_!(p_1^*F \otimes^L_{\mathbb{Z}/n\mathbb{Z}} p_2^*F_2)$$

is an isomorphism.

Proof. (i): Follows from theorem 4: see [22] for example.
(ii):Follows from (i) by taking a compactification $\bar{f}_1 : \bar{X}_1 \to \bar{S}$ of $f_1 : X_1 \to S$ and a compactification $\bar{f}_2 : \bar{X}_2 \to \bar{S}$ of $f_2 : X_2 \to S$. $\square$

Remark 2. *Let $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ two morphisms with $S, X_1, X_2 \in \mathrm{Sch}$. Let $n \in \mathbb{N}$. Denote $p_1 : X_1 \times_S X_2 \to X_1$ and $p_2 : X_1 \times_S X_2 \to X_2$ the base change maps. Let $F_1 \in C_{\mathbb{Z}/n\mathbb{Z}}(X_1^{et})$ and $F_2 \in C_{\mathbb{Z}/n\mathbb{Z}}(X_2^{et})$ be n-torsion sheaves. Then, if f_1 or f_2 is not proper, $H^k(Rf_{1*}F_1 \otimes^L_{\mathbb{Z}/n\mathbb{Z}} Rf_{2*}F_2)$ is NOT isomorphic in $\mathrm{Shv}_{\mathbb{Z}/n\mathbb{Z}}(S^{et})$ to $H^k R(f_1 \times f_2)_*(p_1^* F_1 \otimes^L_{\mathbb{Z}/n\mathbb{Z}} p_2^* F_2)$ in general.*

We now recall the comparaison theorems :

Theorem 7. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(\mathbb{C})$. Let $F \in C(X^{et})$ be a torsion sheaf where we recall that $X^{et} \subset \mathrm{Sch}^{ft}/X = \mathrm{Var}(\mathbb{C})/X$ is the small etale site. Then the transformation map (see [10] section 2) in $D(S^{an})$*

$$T(f, an)(F) : \mathrm{an}_S^* Rf_*F \to Rf_* \mathrm{an}_X^* F$$

is an isomorphism.

Proof. Standard, see [22] for example : follows from theorem 4 (i) and the open embedding case. $\square$

Theorem 8. *Let $K \subset \mathbb{C}_p$ be a p-adic field. Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(K)$. Let $F \in C_{\mathbb{Z}/n\mathbb{Z}}(X^{et})$ be an n-torsion sheaf where we recall that $X^{et} \subset \mathrm{Sch}^{ft}/X = \mathrm{Var}(\mathbb{C})/X$ is the small etale site. Then the transformation map (see [10] section 2) in $D(S^{an,et})$*

$$T(f, an)(F) : \mathrm{an}_S^* Rf_*F \to Rf_* \mathrm{an}_X^* F$$

is an isomorphism.

Proof. Standard, see [19] for example : follows from theorem 4 (i) and the open embedding case. $\square$

On the other hand for CW complexes, we have the followings :

Theorem 9. *Consider a commutative diagram in* CW *which is cartesian*

$$\begin{tikzcd} X_T \arrow[r, "f'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

such that f is proper. Let $F \in C(X)$. Then the transformation map (see [10] section 2) in $D(T)$

$$T(f, g)(F) : g^* Rf_*F \to Rf'_* g^{'*} F$$

is an isomorphism.

Proof. Standard. $\square$

Theorem 10. *(i) Let $f : X \to S$ a proper morphism with $S, X \in \mathrm{CW}$. Let $F \in C(X)$ and $G \in C(S)$. Then the transformation map (see [10] section 2) in $D(S)$*

$$T(f, \otimes)(F, G) : Rf_*F \otimes^L G \to Rf_*(F \otimes^L f^*G)$$

is an isomorphism.

(ii) Let $f : X \to S$ a morphism with $S, X \in \mathrm{CW}$. Let $F \in C(X)$ and $G \in C(S)$. Then the transformation map (see [10] section 2) in $D(S)$

$$T_!(f, \otimes)(F, G) : Rf_!F \otimes^L G \to Rf_!(F \otimes^L f^*G)$$

is an isomorphism.

Proof. Standard. $\square$

Theorem 11. *(i) Let $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ two proper morphisms with $X_1, X_2, S \in \mathrm{CW}$. Denote $p_1 : X_1 \times_S X_2 \to X_1$ and $p_2 : X_1 \times_S X_2 \to X_2$ the base change maps. Let $F_1 \in C(X_1)$ and $F_2 \in C(X_2)$. Then the canonical map in $C(S)$ given as above*

$$T(f_1, f_2, \otimes)(F_1, F_2) : Rf_{1*}F_1 \otimes^L Rf_{2*}F_2 \to R(f_1 \times f_2)_*(p_1^*F \otimes^L p_2^*F_2)$$

is an isomorphism.

(ii) Let $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ two morphisms with $X_1, X_2, S \in \mathrm{CW}$. Denote $p_1 : X_1 \times_S X_2 \to X_1$ and $p_2 : X_1 \times_S X_2 \to X_2$ the base change maps. Let $F_1 \in C(X_1)$ and $F_2 \in C(X_2)$. Then the canonical map in $C(S)$ given after taking compactification of f_1 and f_1 by the one of (i) for the compactifications and on the other hand by the open embedding case

$$T(f_1, f_2, \otimes)(F_1, F_2) : Rf_{1!}F_1 \otimes^L Rf_{2!}F_2 \to R(f_1 \times f_2)_!(p_1^*F \otimes^L p_2^*F_2)$$

is an isomorphism.

Proof. Standard. $\square$

2.4 Constructible and perverse sheaves on algebraic varieties over a subfield $k \subset \mathbb{C}$

Let $S \in \mathrm{AnSp}(\mathbb{C})$. We have

- the classical dual functor

$$\mathbb{D}^0_S : C(S) \to C(S), \quad K \mapsto \mathbb{D}^0_S K := \mathcal{H}om(LK, E_{usu}(\mathbb{Z}_S))$$

  which induces in the derived category

$$\mathbb{D}^0_S : D(S) \to D(S), \quad K \mapsto \mathbb{D}^0_S K := \mathcal{H}om(LK, E_{usu}(\mathbb{Z}_S)) = R\mathcal{H}om(K, \mathbb{Z}_S)$$

- the Verdier dual functor

$$\mathbb{D}^v_S : D(S) \to D(S), \quad K \mapsto \mathbb{D}^v_S K := R\mathcal{H}om(K, w_S)$$

  where $w_S := a_S^! \mathbb{Z}_S$ is the dualizing complex, $a_S : S \to \{\mathrm{pt}\}$ being the terminal map.

For $S \in \mathrm{AnSm}(\mathbb{C})$ smooth connected of dimension $\dim(S) = d_S$ we have $\mathbb{D}^v_S = \mathbb{D}^0_S[2d_S]$.

We recall the definition of constructible sheaves on algebraic varieties over a subfield $k \subset \mathbb{C}$:

Definition 1. *Let $k \subset \mathbb{C}$ a subfield. Let $S \in \mathrm{Var}(k)$.*

(i) A sheaf $K \in \mathrm{Shv}(S^{an}_{\mathbb{C}})$ is called constructible (with respect to a Zariski stratification over k) if there exists a stratification $S = \sqcup_\alpha S_\alpha$ with $l_\alpha : S_\alpha \hookrightarrow S$ locally closed subsets such that $l_\alpha^* K \in \mathrm{Shv}(S^{an}_{\alpha,\mathbb{C}})$ are (finite dimensional) local systems (for the usual topology) for all α. Note that we make the hypothesis that the strata S_α are defined over k.

(ii) We denote by

$$C_{c,k}(S^{an}_{\mathbb{C}}) \subset C(S^{an}_{\mathbb{C}}) \text{ and } D_{c,k}(S^{an}_{\mathbb{C}}) \subset D(S^{an}_{\mathbb{C}})$$

the full subcategories consisting of $K \in C(S^{an}_{\mathbb{C}})$ such that $a_{usu}H^n K \in \mathrm{Shv}(S^{an}_{\mathbb{C}})$ are constructible with respect to a Zariski stratification over k for all $n \in \mathbb{Z}$ (see (i)).

(ii)' We denote by

$$C_{fil,c,k}(S_{\mathbb{C}}^{an}) \subset C_{fil}(S_{\mathbb{C}}^{an}) \text{ and } D_{fil,c,k}(S_{\mathbb{C}}^{an}) \subset D_{fil}(S_{\mathbb{C}}^{an})$$

the full subcategories consisting of $(K,W) \in C_{fil}(S_{\mathbb{C}}^{an})$ such that $a_{usu}H^n \operatorname{Gr}_k^W K \in \operatorname{Shv}(S_{\mathbb{C}}^{an})$ are constructible with respect to a Zariski stratification over k for all $n,k \in \mathbb{Z}$ (see (i)).

(iii) We denote by

$$P_k(S_{\mathbb{C}}^{an}) := P(S_{\mathbb{C}}^{an}) \cap D_{c,k}(S_{\mathbb{C}}^{an}) \subset D_{c,k}(S_{\mathbb{C}}^{an}) \otimes \mathbb{Q} \subset D_{c,k}(S_{\mathbb{C}}^{an})$$

the full subcategory of perverse sheaves (which are by definition torsion free) whose cohomology sheaves are constructible with respect to a Zariski stratification (defined over k), see (ii).

(iii)' We denote by

$$P_{fil,k}(S_{\mathbb{C}}^{an}) := P_{fil}(S_{\mathbb{C}}^{an}) \cap D_{fil,c,k}(S_{\mathbb{C}}^{an}) \subset D_{fil,c,k}(S_{\mathbb{C}}^{an}) \otimes \mathbb{Q} \subset D_{fil,c,k}(S_{\mathbb{C}}^{an})$$

the full subcategory of filtered perverse sheaves (which are by definition torsion free) whose cohomology sheaves are constructible with respect to a Zariski stratification (defined over k), see (ii)'.

Theorem 12. *Let $k \subset \mathbb{C}$ a subfield.*

- *Let $S \in \operatorname{Var}(k)$. Then for $K \in D_{c,k}(S_{\mathbb{C}}^{an})$, $\mathbb{D}_S^v K \in D_{c,k}(S_{\mathbb{C}}^{an})$. For $K, K' \in D_{c,k}(S_{\mathbb{C}}^{an})$, $K \otimes^L K' \in D_{c,k}(S_{\mathbb{C}}^{an})$.*
- *Let $f : X \to S$ a morphism with $S, X \in \operatorname{Var}(k)$. Then for $K \in D_{c,k}(X_{\mathbb{C}}^{an})$, $Rf_*K \in D_{c,k}(S_{\mathbb{C}}^{an})$ and $Rf_!K = \mathbb{D}_S^v Rf_* \mathbb{D}_X^v K \in D_{c,k}(S_{\mathbb{C}}^{an})$.*
- *Let $f : X \to S$ a morphism with $S, X \in \operatorname{Var}(k)$. Then for $K \in D_{c,k}(S_{\mathbb{C}}^{an})$, $f^*K \in D_{c,k}(X_{\mathbb{C}}^{an})$ and $f^!K := \mathbb{D}_X^v f^* \mathbb{D}_S^v K \in D_{c,k}(X_{\mathbb{C}}^{an})$.*

For $S \in \operatorname{Var}(k)$, we have thus the full subcategory

$$\begin{aligned} D_{c,k,gm}(S_{\mathbb{C}}^{an}) : &= < Rf_*\mathbb{Z}_X, (f : X \to S) \in \operatorname{Var}(k) > \\ &= < Rf_*\mathbb{Z}_X, (f : X \to S) \in \operatorname{Var}(k) \text{ proper}, \ X \text{smooth} > \subset D_{c,k}(S_{\mathbb{C}}^{an}) \end{aligned}$$

where <> means the full triangulated category generated by.

Proof. Standard : follows from the fact that a morphism $f : X \to S$ with $X, S \in \operatorname{Var}(k)$ admits a Whitney stratification whose strata $X = \sqcup_{\alpha,\beta} X_{\alpha,\beta}$, $S = \sqcup_\alpha S_\alpha$ are Zariski locally closed subset, that is locally closed subvarieties defined over k. □

Definition 2. *Let $k \subset \mathbb{C}$ a subfield We have then, using theorem 12, for $S \in \operatorname{Var}(k)$, the full subcategory*

$$D_{fil,c,k,gm}(S_{\mathbb{C}}^{an}) :=< (K,W), \text{ s.t. } \operatorname{Gr}_n^W K \in D_{c,k,gm}(S_{\mathbb{C}}^{an}), \text{ for all} n \in \mathbb{Z} > \subset D_{fil,c,k}(S_{\mathbb{C}}^{an})$$

where <> means the full triangulated category generated by.

Let $k \subset \mathbb{C}$ a subfield.

Let $S \in \operatorname{Var}(k)$ and $D = V(s) \subset S$ a Cartier divisor. Denote $i : D \hookrightarrow S$ the closed embedding and $j : S^o := S \backslash D \hookrightarrow S$ the open embedding. Let $\pi : \tilde{S}_{\mathbb{C}}^{o,an} \to S_{\mathbb{C}}^{o,an}$ the universal covering. Denote by $T : \tilde{S}_{\mathbb{C}}^{o,an} \to \tilde{S}_{\mathbb{C}}^{o,an}$ the monodromy automorphism. We then consider,

- for $K \in C(S_{\mathbb{C}}^{o,an})$, the nearby cycle functor

  $$\psi_D K := i^* R(j \circ \pi)_* \pi^* K \in D(D_{\mathbb{C}}^{an}),$$

  we write again $\psi_D K := i_* \psi_D K \in D(S_{\mathbb{C}}^{an})$,

- for $K \in C(S^{o,an}_{\mathbb{C}})$, the vanishing cycle functor

$$\phi_D K := \mathrm{Cone}(\mathrm{ad}(j \circ \pi^*, j \circ \pi_*)(K) : i^*K \to i^*R(j \circ \pi)_*\pi^*K =: \psi_D K \in D(D^{an}_{\mathbb{C}})$$

together with the canonical map $c(\phi_D K) : \psi_D K \to \phi_D K$ in $D(D^{o,an}_{\mathbb{C}})$, we write again $\phi_D K := i_*\phi_D K \in D(S^{an}_{\mathbb{C}})$, for $K \in C_c(S^{o,an}_{\mathbb{C}})$ we have a canonical isomorphism in $D_c(D^{an}_{\mathbb{C}})$

$$T(D, \psi_D)(K) : \phi_D \mathbb{D}^v_S K \xrightarrow{\sim} \mathbb{D}^v_S \psi_D K[1]$$

- for $K \in C_c(S^{o,an}_{\mathbb{C}})$, the canonical morphisms in $D_c(D^{an}_{\mathbb{C}})$

$$can(K) := c(\phi_D K) : \psi_D K \to \phi_D K \ , \ var(K) := (0, T - I) : \phi_D K \to \psi_D K.$$

- for $K \in C(S^{o,an}_{\mathbb{C}})$, the maximal extension

$$x_{S^o/S}(K) := \mathrm{Cone}(\mathrm{ad}(i^*, i_*)(-) \circ \mathrm{ad}(\pi^*, \pi_*)(K) : Rj_*K \to i_*R(j \circ \pi)_*\pi^*K =: \psi_D K) \in D(S^{an}_{\mathbb{C}}).$$

For $K \in C_{c,k}(S^{o,an}_{\mathbb{C}})$, we have $\psi_D K \in C_{c,k}(D^{o,an}_{\mathbb{C}})$ since it preserve local systems hence $\phi_D K \in C_{c,k}(D^{o,an}_{\mathbb{C}})$ and $x_{S^o/S}K \in C_{c,k}(S^{o,an}_{\mathbb{C}})$.

Let $S \in \mathrm{Var}(k)$ and $D = V(s) \subset S$ a Cartier divisor. Denote $i : D \hookrightarrow S$ the closed embedding and $j : S^o := S \backslash D \hookrightarrow S$ the open embedding. Let $\pi : \tilde{S}^{o,an}_{\mathbb{C}} \to S^{o,an}_{\mathbb{C}}$ the universal covering.

- For $K, K' \in C_c(S^{o,an}_{\mathbb{C}})$, we have by (Verdier) duality and theorem 11 $\psi_D(K \otimes^L K') = \psi_D(K) \otimes^L \psi_D(K')$,
- For $K, K' \in C_c(S^{o,an}_{\mathbb{C}})$ we have by (Verdier) duality and the preceding point $\phi_D(K \otimes^L K') = \phi_D(K) \otimes^L \phi_D(K')$
- For $K, K' \in C(S^{o,an}_{\mathbb{C}})$, we have the tranformation map in $D(S^{o,an}_{\mathbb{C}})$

$$T(\otimes, \psi_D)(K, K') : (\psi_D K) \otimes^L K' \xrightarrow{I \otimes (\mathrm{ad}(i^*, i_*)(-) \circ \mathrm{ad}(\pi^*, \pi_*)(-))} \psi_D K \otimes^L \psi_D K' = \psi_D(K \otimes^L K').$$

- For $K, K' \in C_c(S^{o,an}_{\mathbb{C}})$ the tranformation map in $D_c(S^{o,an}_{\mathbb{C}})$

$$\begin{aligned} T(\otimes, \phi_D)(K, K') : \phi_D(K \otimes^L K') = \mathbb{D}^v_S \psi_D(\mathbb{D}^v_S K \otimes^L \mathbb{D}^v_S K') \\ \xrightarrow{\mathbb{D}^v_S T(\otimes, \psi_D)(\mathbb{D}^v_S K, \mathbb{D}^v_S K')} \mathbb{D}^v_S(\psi_D \mathbb{D}^v_S(K) \otimes^L \mathbb{D}^v_S K') = \phi_D K \otimes^L K'. \end{aligned}$$

We will use a version of a result of Beilison on perverse sheaves.

Definition 3. *Let $k \subset \mathbb{C}$ a subfield. Let $S \in \mathrm{Var}(k)$ and $D = V(s) \subset S$ a Cartier divisor. Denote $S^o := S \backslash D$. We denote by $P_k(S^{o,an}_{\mathbb{C}}) \times_J P_k(D^{an}_{\mathbb{C}})$ the category*

- *whose objects are (K', K'', u, v) where $K' \in P_k(S^{o,an}_{\mathbb{C}})$ and $K'' \in P_k(D^{an}_{\mathbb{C}})$ are perverse sheaves and $u : \psi_D K'^{an} \to K''^{an}$ and $v : K''^{an} \to \psi_D K'^{an}$ are morphism in $D_c(D^{an}_{\mathbb{C}})$ such that $v \circ u = T - I$,*
- *whose morphisms are $m = (m', m'') : (K'_1, K''_1, u_1, v_1) \to (K'_2, K''_2, u_2, v_2)$ such that $u_2 \circ \psi_D m' = m'' \circ u_1$ and $\psi_D m' \circ v_1 = v_2 \circ m''$.*

We give the following version of the well known theorem for perserse sheaves

Theorem 13. *Let $k \subset \mathbb{C}$ a subfield. Let $S \in \mathrm{Var}(k)$ and $D = V(s) \subset S$ a Cartier divisor. Denote $i : D \hookrightarrow S$ the closed embedding and $j : S^o := S \backslash D \hookrightarrow S$ the open embedding. Let $\pi : \tilde{S}^{o,an}_{\mathbb{C}} \to S^{o,an}_{\mathbb{C}}$ the universal covering. Denote by $T : \tilde{S}^{o,an}_{\mathbb{C}} \to \tilde{S}^{o,an}_{\mathbb{C}}$ the monodromy automorphism. Then the functor*

$$(j^*, \phi_D[-1], can, var) : P_k(S^{an}_{\mathbb{C}}) \to P_k(S^{o,an}_{\mathbb{C}}) \times_J P_k(D^{an}_{\mathbb{C}})$$

is an equivalence of category whose inverse is

$$P_k(S^{o,an}_{\mathbb{C}}) \times_J P_k(D^{an}_{\mathbb{C}}) \to P_k(S^{an}_{\mathbb{C}}),$$
$$(K', K'', u, v) \mapsto H^1(\psi_D K' \xrightarrow{(c(x_{S^o/S}(K')),u)} x_{S^o/S}(K') \oplus i_* K'' \xrightarrow{((0,T-I),v)} \psi_D K').$$

We denote, for $K \in P_k(S^{an}_{\mathbb{C}})$ by

$$Is(K) := (0, (\mathrm{ad}(j^*, j_*)(K), \mathrm{ad}(j \circ \pi^*, j \circ \pi_*)(K)), 0) :$$
$$K \xrightarrow{\sim} (\psi_D K \xrightarrow{(c(x_{S^o/S}(K)),can(K))} x_{S^o/S}(K) \oplus i_* \phi_D K \xrightarrow{((0,T-I),var(K))} \psi_D K)[-1]$$

the canonical isomorphism in $D_{c,k}(S^{an}_{\mathbb{C}})$.

Proof. Similar to the proof of [5] : follows from the fact that $P_k(S^{an}_{\mathbb{C}})$ form an abelian category stable by the nearby and vanishing cycle functors. □

In the filtered case, we will consider the weight monodromy filtration for open embeddings :

Definition 4. *Let $k \subset \mathbb{C}$ a subfield.*

(i) Let $S \in \mathrm{Var}(k)$ and $j : S^o \hookrightarrow S$ an open embedding such that $D := S \backslash S^o = V(s) \subset S$ is a Cartier divisor. Let $P_{fil,k}(S^{o,an}_{\mathbb{C}})^{ad,D} \subset P_{fil,k}(S^{o,an}_{\mathbb{C}})$ the full subcategory such that the relative weight monodromy filtration of W with respect to $D \subset S$ exists.

- *For $(K, W) \in P_{fil,k}(S^{o,an}_{\mathbb{C}})^{ad,D}$, we consider as in [25]*

$$j_{*w}(K, W) := (Rj_* K, W) \in P_{fil,k}(S^{an}_{\mathbb{C}}),\ W_k Rj_* K := < Rj_* W_k K, W(N)_k K > \subset Rj_* K$$

so that $j^* j_{*w}(K, W) = (K, W)$, where $W_k Rj_* K \subset Rj_* K$ is given by W and the weight monodromy filtration $W(N)$ of the universal cover $\pi : \tilde{S}^{o,an}_{\mathbb{C}} \to S^{o,an}_{\mathbb{C}}$. Note that a stratitification of $W_k Rj_* K$ is given by the closure of a stratification of $W_k K$ and $D := S \backslash S^o$.

- *For $(K, W) \in P_{fil,k}(S^{o,an}_{\mathbb{C}})^{ad,D}$, we consider*

$$j_{!w}(K, W) := \mathbb{D}^v_S j_{*w} \mathbb{D}^v_S (K, W) \in P_{fil,k}(S^{an}_{\mathbb{C}})$$

so that $j^* j_{!w}(K, W) = (K, W)$.

For $(K', W) \in P_{fil,k}(S^{an}_{\mathbb{C}})^{ad,D}$, there is, by construction,

- *a canonical map $\mathrm{ad}(j^*, j_{*w})(K', W) = \mathrm{ad}(j^*, j_*)(K') : (K', W) \to j_{*w} j^*(K', W)$ in $P_{fil,k}(S^{an}_{\mathbb{C}})$,*
- *a canonical map $\mathrm{ad}(j_{!w}, j^*)(K', W) = \mathrm{ad}(j_!, j^*)(K') : j_{!w} j^*(K', W) \to (K', W)$ in $P_{fil,k}(S^{an}_{\mathbb{C}})$.*

(ii) Let $S \in \mathrm{Var}(k)$. Let $j : S^o := S \backslash Z \hookrightarrow S$ an open embedding with $Z = V(\mathcal{I}) \subset S$ an arbitrary closed subset, $\mathcal{I} \subset O_S$ being an ideal subsheaf. Taking generators $\mathcal{I} = (s_1, \ldots, s_r)$, we get $Z = V(s_1, \ldots, s_r) = \cap_{i=1}^r Z_i \subset S$ with $Z_i = V(s_i) \subset S$, $s_i \in \Gamma(S, \mathcal{L}_i)$ and L_i a line bundle. Note that Z is an arbitrary closed subset, $d_Z \geq d_X - r$ needing not be a complete intersection. Denote by $j_I : S^{o,I} := \cap_{i \in I}(S \backslash Z_i) = S \backslash (\cup_{i \in I} Z_i) \xrightarrow{j^o_I} S^o \xrightarrow{j} S$ the open complementary embeddings, where $I \subset \{1, \cdots, r\}$. Denote

$$\mathcal{D}(Z/S) := \left\{(Z_i)_{i \in [1,\ldots r]}, Z_i \subset S, \cap Z_i = Z\right\}, Z'_i \subset Z_i$$

the flag category. Let $P_{fil,k}(S^{o,an}_{\mathbb{C}})^{ad,(Z_i)} \subset P_{fil,k}(S^{o,an}_{\mathbb{C}})$ the full subcategory such that the relative weight monodromy filtration of W with respect to the $Z_i \subset S$ exists. For $(K, W) \in C(P_{fil,k}(S^{o,an}_{\mathbb{C}}))^{ad,(Z_i)}$, we define by (i)

– *the (bi)-filtered complex of D_S-modules*

$$j_{*w}(K,W) := \varinjlim_{\mathcal{D}(Z/S)} \mathrm{Tot}_{cardI=\bullet}(j_{I*w}j_I^{o*}(K,W)) \in C(P_{fil,k}(S_{\mathbb{C}}^{an}))$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \mathrm{ad}(j_{IJ}^*, j_{IJ*w})(j_I^{o*}(K,W))$, $j_{IJ} : S^{oJ} \hookrightarrow S^{oI}$ being the open embedding, and $d_{IJ} = 0$ if $I \notin J$,

– *the (bi)-filtered complex of D_S-modules*

$$j_{!w}(K,W) := \varprojlim_{\mathcal{D}(Z/S)} \mathrm{Tot}_{cardI=-\bullet}(j_{I!w}j_I^{o*}(K,W)) = \mathbb{D}_S^v j_{*w}\mathbb{D}_S^v(K,W) \in C(P_{fil,k}(S_{\mathbb{C}}^{an})),$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \mathrm{ad}(j_{IJ!w}, j_{IJ}^*)(j_I^{o*}(K,W))$, $j_{IJ} : S^{oJ} \hookrightarrow S^{oI}$ being the open embedding, and $d_{IJ} = 0$ if $I \notin J$.

By definition, we have for $(K,W) \in C(P_{fil,k}(S_{\mathbb{C}}^{o,an}))^{ad,(Z_i)}$, $j^*j_{*w}(K,W) = (K,W)$ and $j^*j_{!w}(K,W) = (K,W)$. For $(K',W) \in C(P_{fil,k}(S_{\mathbb{C}}^{an}))^{ad,(Z_i)}$, there is, by (i),

– *a canonical map $\mathrm{ad}(j^*, j_{*w})(K',W) : (K',W) \to j_{*w}j^*(K',W)$ in $C(P_{fil,k}(S_{\mathbb{C}}^{an}))$,*

– *a canonical map $\mathrm{ad}(j_{!w}, j^*)(K',W) : j_{!w}j^*(K',W) \to (K',W)$ in $C(P_{fil,k}(S_{\mathbb{C}}^{an}))$.*

Definition 5. *Let $S \in \mathrm{Var}(k)$. Let $Z \subset S$ a closed subset. Denote by $j : S\backslash Z \hookrightarrow S$ the complementary open embedding.*

(i) We define using definition 4, the filtered Hodge support section functor

$$\Gamma_Z^w : C(P_{fil,k}(S_{\mathbb{C}}^{an})^{ad,(Z_i)}) \to C(P_{fil,k}(S_{\mathbb{C}}^{an})),$$
$$(K,W) \mapsto \Gamma_Z^w(K,W) := \mathrm{Cone}(\mathrm{ad}(j^*, j_{*w})(K,W) : (K,W) \to j_{*w}j^*(K,W))[-1],$$

together we the canonical map $\gamma_Z^w(K,W) : \Gamma_Z^w(K,W) \to (K,W)$.

(i)' Since $j_{*w} : C(P_{fil,k}(S_{\mathbb{C}}^{o,an})^{ad,(Z_i)}) \to C(P_{fil,k}(S_{\mathbb{C}}^{an}))$ is an exact functor, Γ_Z^w induces the functor

$$\Gamma_Z^w : D_{fil,c,k}(S_{\mathbb{C}}^{an})^{ad,(Z_i)} \to D_{fil,c,k}(S_{\mathbb{C}}^{an}),\ (K,W) \mapsto \Gamma_Z^w(K,W)$$

(ii) We define using definition 4, the dual filtered Hodge support section functor

$$\Gamma_Z^{\vee,w} : C(P_{fil,k}(S_{\mathbb{C}}^{an})^{ad,(Z_i)}) \to C(P_{fil,k}(S_{\mathbb{C}}^{an})),$$
$$(K,W) \mapsto \Gamma_Z^{\vee,w}(K,W) := \mathrm{Cone}(\mathrm{ad}(j_{!w}, j^*)(K,W) : j_{!w}, j^*(K,W) \to (K,W)),$$

together we the canonical map $\gamma_Z^{\vee,Hdg}(K,W) : (K,W) \to \Gamma_Z^{\vee,w}(K,W)$.

(ii)' Since $j_{!w} : C(P_{fil,k}(S_{\mathbb{C}}^{o,an})^{ad,(Z_i)}) \to C(P_{fil,k}(S_{\mathbb{C}}^{an}))$ is an exact functor, $\Gamma_Z^{\vee,w}$ induces the functor

$$\Gamma_Z^{\vee,w} : D_{fil,c,k}(S_{\mathbb{C}}^{an})^{ad,(Z_i)} \to D_{fil,c,k}(S_{\mathbb{C}}^{an}),\ (K,W) \mapsto \Gamma_Z^{\vee,w}(K,W)$$

Let $S \in \mathrm{Var}(k)$ and $D = V(s) \subset S$ a Cartier divisor. Denote $i : D \hookrightarrow S$ the closed embedding and $j : S^o := S\backslash D \hookrightarrow S$ the open embedding. Let $\pi : \tilde{S}_{\mathbb{C}}^{o,an} \to S_{\mathbb{C}}^{o,an}$ the universal covering. We then consider, for $(K,W) \in D_{fil,c}(S_{\mathbb{C}}^{o,an})^{ad,D} = \mathrm{Ho}(C(P_{fil,k}(S_{\mathbb{C}}^{o,an})^{ad,D})$,

- the filtered nearby cycle functor

$$\psi_D(K,W) := (\psi_D K, W) \in D_{fil,c}(D_{\mathbb{C}}^{an}),\ W_k(\psi_D(K,W)) :=< W_k\psi_D K, W(N)_k\psi_D K >\subset \psi_D K,$$

- the vanishing cycle functor

$$\phi_D(K,W) := \mathrm{Cone}(\mathrm{ad}(j\circ\pi^*, j\circ\pi_*)(K,W) : i^*(K,W) \to \psi_D(K,W)) \in D_{fil,c,k}(D^{an}_{\mathbb{C}}),$$

where the morphism

$$\mathrm{ad}(j\circ\pi^*, j\circ\pi_*)(K,W) : i^*(K,W) \to i^*R(j\circ\pi)^*(j\circ\pi)^*(K,W)$$

being compatible with the weight monodromy filtration induces the morphism

$$\mathrm{ad}(j\circ\pi^*, j\circ\pi_*)(K,W) : i^*(K,W) \to \psi_D(K,W),$$

- the canonical morphisms in $D_{fil,c,k}(D^{an}_{\mathbb{C}})$

$$can(K,W) := c(\phi_D(K,W)) : \psi_D(K,W) \to \phi_D(K,W),$$
$$var(K,W) := (0, T-I) : \phi_D(K,W) \to \psi_D(K,W),$$

- the maximal extension

$$x_{S^o/S}(K,W) := \mathrm{Cone}(\mathrm{ad}(i^*,i_*)(-)\circ\mathrm{ad}(\pi^*,\pi_*)(K,W) : j_{*w}(K,W) \to \psi_D(K,W)) \in D_{fil,c,k}(S^{an}_{\mathbb{C}}),$$

where the morphism

$$\mathrm{ad}(i^*,i_*)(-)\circ\mathrm{ad}(\pi^*,\pi_*)(K,W) : j_*(K,W) \to i_*i^*R(j\circ\pi)^*(j\circ\pi)^*(K,W)$$

being compatible with the weight monodromy filtration induces the morphism

$$\mathrm{ad}(i^*,i_*)(-)\circ\mathrm{ad}(\pi^*,\pi_*)(K,W) : j_{*w}(K,W) \to \psi_D(K,W).$$

Definition 6. *Let $k\subset\mathbb{C}$ a subfield.*

(i) Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(k)$. Consider the graph factorization $f : X \xrightarrow{l} X\times S \xrightarrow{p} S$ of f where l the the graph closed embedding and p is the projection. We have, using definition 5,

- *the inverse image functor*

$$f^{*w} : D_{fil,c,k}(S^{an}_{\mathbb{C}})^{ad,(\Gamma_{f,i})} \to D_{fil,c,k}(X^{an}_{\mathbb{C}}),\ (K,W) \mapsto f^{*w}(K,W) := l^*\Gamma^{\vee,w}_X p^*(K,W)$$

- *the exceptional inverse image functor*

$$f^{!w} : D_{fil,c,k}(S^{an}_{\mathbb{C}})^{ad,(\Gamma_{f,i})} \to D_{fil,c,k}(X^{an}_{\mathbb{C}}),\ (K,W) \mapsto f^{!w}(K,W) := l^*\Gamma^{w}_X p^*(K,W).$$

(ii) Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(k)$. Consider a compactification $f : X \hookrightarrow j\bar{X} \xrightarrow{\bar{f}} S$ of f with $\bar{X} \in \mathrm{Var}(k)$, j an open embedding and $\bar{f}$ a proper morphism. Denote $Z = \bar{X}\backslash X$. We have, using definition 4,

- *the direct image functor*

$$Rf_{*w} : D_{fil,c,k}(X^{an}_{\mathbb{C}})^{ad,(Z_i)} \to D_{fil,c,k}(S^{an}_{\mathbb{C}}),\ (K,W) \mapsto Rf_{*w}(K,W) := R\bar{f}_*j_{*w}(K,W)$$

- *the proper direct image functor*

$$Rf_{!w} : D_{fil,c,k}(X^{an}_{\mathbb{C}})^{ad,(Z_i)} \to D_{fil,c,k}(S^{an}_{\mathbb{C}}),\ (K,W) \mapsto Rf_{!w}(K,W) := R\bar{f}_*j_{!w}(K,W).$$

(iii) Let $S \in \mathrm{Var}(k)$. Denote by $\Delta_S : S \hookrightarrow S \times S$ the diagonal closed embedding and $p_1 : S \times S \to S$, $p_2 : S \times S \to S$ the projections. We have by (i) the functor

$$\otimes^{Lw} : D_{fil,c,k}(S^{an}_{\mathbb{C}})^{ad,(S_i)} \times D_{fil,c,k}(S^{an}_{\mathbb{C}})^{ad,(S_i)} \to D_{fil,c,k}(S^{an}_{\mathbb{C}}),$$
$$((K_1, W), (K_2, W)) \mapsto (K_1, W) \otimes^{L,w} (K_2, W) := \Delta_S^{!w}(p_1^*(K_1, W) \otimes^L p_2^*(K_2, W)).$$

Let $S \in \mathrm{Var}(k)$ and $D = V(s) \subset S$ a Cartier divisor. Denote $i : D \hookrightarrow S$ the closed embedding and $j : S^o := S \backslash D \hookrightarrow S$ the open embedding. In the filtered case, we get, for $(K, W) \in P_{fil,k}(S^{an}_{\mathbb{C}})$ the map in $D_{fil,c,k}(S^{an}_{\mathbb{C}})$

$$Is(K, W) := (0, (\mathrm{ad}(j^*, j_*)(K), \mathrm{ad}(j \circ \pi^*, j \circ \pi_*)(K)), 0) :$$
$$(K, W) \to (\psi_D(K, W) \xrightarrow{(c(x_{S^o/S}(K,W)), can(K,W))} x_{S^o/S}(K, W) \oplus i_*\phi_D(K, W)$$
$$\xrightarrow{((0, T-I), var(K,W))} \psi_D(K, W))[-1]$$

which is NOT an isomorphism in general (it leads to different W-filtration on perverse cohomology).

2.5 Constructible and perverse etale sheaves on algebraic varieties over a field k of charactersitic 0

Let k a field of caracteristic zero. Let $S \in \mathrm{Var}(k)$. We have

- the classical dual functor

$$\mathbb{D}^0_S : C(S^{et}) \to C(S^{et}), \ \ K \mapsto \mathbb{D}^0_S K := \mathcal{H}om(LK, E_{et}(\mathbb{Z}_S))$$

  which induces in the derived category

$$\mathbb{D}^0_S : D(S^{et}) \to D(S^{et}), \ \ K \mapsto \mathbb{D}^0_S K := \mathcal{H}om(LK, E_{et}(\mathbb{Z}_S)) = R\mathcal{H}om(K, \mathbb{Z}_S)$$

- the Verdier dual functor

$$\mathbb{D}^v_S : D(S^{et}) \to D(S^{et}), \ \ K \mapsto \mathbb{D}^v_S K := R\mathcal{H}om(K, w_S)$$

  where $w_S := a_S^! \mathbb{Z}_S$ is the dualizing complex, $a_S : S \to \{\mathrm{pt}\}$ being the terminal map.

For $S \in \mathrm{SmVar}(k)$ smooth connected of dimension $\dim(S) = d_S$ we have $\mathbb{D}^v_S = \mathbb{D}^0_S[2d_S]$.

We recall the definition of constructible etale sheaves on algebraic varieties over a field k of caracteristic zero:

Definition 7. *Let k a field of caracteristic zero. Let $S \in \mathrm{Var}(k)$. Let l a prime number. Recall (see section 2.1) that*

$$K = (K_n)_{n \in \mathbb{N}} \in \mathrm{Shv}_{\mathbb{Z}_l}(S^{et}) \subset \mathrm{PSh}(S^{et}, \mathrm{Fun}(\mathbb{N}, \mathrm{Ab}))$$

is a projective system with $K_n \in \mathrm{Shv}_{\mathbb{Z}/l^n\mathbb{Z}}(S^{et})$ such that $K_n \to K_{n+1}/l^n K_{n+1}$ is an isomorphism.

(i) A sheaf $K \in \mathrm{Shv}_{\mathbb{Z}_l}(S^{et})$ is called constructible if there exists a stratification $S = \sqcup_\alpha S_\alpha$ with $l_\alpha : S_\alpha \hookrightarrow S$ locally closed subsets such that $l_\alpha^* K \in \mathrm{Shv}_{\mathbb{Z}_l}(S^{et}_\alpha)$ are (finite dimensional) local systems (for the etale topology) for all α.

(ii) We denote by

$$C_{\mathbb{Z}_l,c,k}(S^{et}) \subset C_{\mathbb{Z}_l}(S^{et}) \text{ and } D_{\mathbb{Z}_l,c,k}(S^{et}) \subset D_{\mathbb{Z}_l}(S^{et})$$

the full subcategories consisting of $K \in C_{\mathbb{Z}_l}(S^{et})$ such that $a_{et} H^n K \in \mathrm{Shv}_{\mathbb{Z}_l}(S^{et})$ are constructible for all $n \in \mathbb{Z}$ (see (i)).

(ii)' We denote by
$$C_{\mathbb{Z}_l fil,c,k}(S^{et}) \subset C_{\mathbb{Z}_l fil}(S^{et}) \text{ and } D_{\mathbb{Z}_l fil,c,k}(S^{et}) \subset D_{\mathbb{Z}_l fil}(S^{et})$$
the full subcategories consisting of $(K,W) \in C_{\mathbb{Z}_l fil}(S^{et})$ such that $a_{et} \operatorname{Gr}^W_k H^n K \in \operatorname{Shv}_{\mathbb{Z}_l}(S^{et})$ are constructible for all $n,k \in \mathbb{Z}$ (see (i)).

(iii) We denote by
$$P_{\mathbb{Z}_l,k}(S^{et}) \subset D_{\mathbb{Z}_l,c,k}(S^{et}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \subset D_{\mathbb{Z}_l,c,k}(S^{et})$$
the full subcategory of perverse sheaves (which are by definition torsion free).

(iii)' We denote by
$$P_{\mathbb{Z}_l,fil,k}(S^{et}) \subset D_{\mathbb{Z}_l fil,c,k}(S^{et}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \subset D_{\mathbb{Z}_l fil,c,k}(S^{et})$$
the full subcategory of filtered perverse sheaves (which are by definition torsion free).

Let k/k' a field extention. Let $S \in \operatorname{Var}(k)$. Let l a prime number.

(i) A sheaf $K \in \operatorname{Shv}_{\mathbb{Z}_l}(S^{et}_{k'})$ is called constructible (over k) if there exists a stratification $S = \sqcup_\alpha S_\alpha$ with $l_\alpha : S_\alpha \hookrightarrow S$ locally closed subsets such that $l_\alpha^* K \in \operatorname{Shv}_{\mathbb{Z}_l}(S^{et}_\alpha)$ are (finite dimensional) local systems (for the etale topology) for all α.

(ii) We denote by
$$C_{\mathbb{Z}_l,c,k}(S^{et}_{k'}) \subset C_{\mathbb{Z}_l}(S^{et}_{k'}) \text{ and } D_{\mathbb{Z}_l,c,k}(S^{et}_{k'}) \subset D_{\mathbb{Z}_l}(S^{et}_{k'})$$
the full subcategories consisting of $K \in C_{\mathbb{Z}_l}(S^{et}_{k'})$ such that $a_{et} H^n K \in \operatorname{Shv}_{\mathbb{Z}_l}(S^{et}_{k'})$ are constructible over k for all $n \in \mathbb{Z}$ (see (i)).

(ii)' We denote by
$$C_{\mathbb{Z}_l fil,c,k}(S^{et}_{k'}) \subset C_{\mathbb{Z}_l fil}(S^{et}_{k'}) \text{ and } D_{\mathbb{Z}_l fil,c,k}(S^{et}_{k'}) \subset D_{\mathbb{Z}_l fil}(S^{et}_{k'})$$
the full subcategories consisting of $(K,W) \in C_{\mathbb{Z}_l fil}(S^{et}_{k'})$ such that $a_{et} \operatorname{Gr}^W_k H^n K \in \operatorname{Shv}_{\mathbb{Z}_l}(S^{et}_{k'})$ are constructible over k for all $n,k \in \mathbb{Z}$ (see (i)).

(iii) We denote by
$$P_{\mathbb{Z}_l,k}(S^{et}_{k'}) := P_{\mathbb{Z}_l,k'}(S^{et}_{k'}) \cap D_{\mathbb{Z}_l,c,k}(S^{et}_{k'}) \subset D_{\mathbb{Z}_l,c,k}(S^{et}_{k'})$$
the full subcategory of perverse sheaves whose stratification is defined over k.

(iii)' We denote by
$$P_{\mathbb{Z}_l,fil,k}(S^{et}_{k'}) := P_{\mathbb{Z}_l,fil,k'}(S^{et}_{k'}) \cap D_{\mathbb{Z}_l fil,c,k}(S^{et}_{k'}) \subset D_{\mathbb{Z}_l fil,c,k}(S^{et}_{k'})$$
the full subcategory of filtered perverse sheaves whose stratification is defined over k.

Theorem 14. *Let k a field of caracteristic zero. Let l a prime number.*

- *Let $S \in \operatorname{Var}(k)$. Then for $K \in D_{\mathbb{Z}_l,c,k}(S^{et})$, $\mathbb{D}^v_S K \in D_{\mathbb{Z}_l,c,k}(S^{et})$. For $K,K' \in D_{\mathbb{Z}_l,c,k}(S^{et})$, $K \otimes^L K' \in D_{\mathbb{Z}_l,c,k}(S^{et})$.*
- *Let $f : X \to S$ a morphism with $S, X \in \operatorname{Var}(k)$. Then for $K \in D_{\mathbb{Z}_l,c,k}(X^{et})$, $Rf_*K \in D_{\mathbb{Z}_l,c,k}(S^{et})$ and $Rf_!K = \mathbb{D}^v_S Rf_* \mathbb{D}^v_X K \in D_{\mathbb{Z}_l,c,k}(S^{et})$.*
- *Let $f : X \to S$ a morphism with $S, X \in \operatorname{Var}(k)$. Then for $K \in D_{\mathbb{Z}_l,c,k}(S^{et})$, $f^*K \in D_{\mathbb{Z}_l,c,k}(X^{et})$ and $f^!K := \mathbb{D}^v_X f^* \mathbb{D}^v_S K \in D_{\mathbb{Z}_l,c,k}(X^{et})$.*

We have then, for $S \in \operatorname{Var}(k)$, the full subcategory
$$\begin{aligned} D_{\mathbb{Z}_l,c,k,gm}(S^{et}) : &= < Rf_*\mathbb{Z}_X, (f : X \to S) \in \operatorname{Var}(k) > \\ &= < Rf_*\mathbb{Z}_X, (f : X \to S) \in \operatorname{Var}(k) \text{ proper}, \ X \text{smooth} > \subset D_{\mathbb{Z}_l,c,k}(S^{et}) \end{aligned}$$
where <> means the full triangulated category generated by.

Proof. Standard : see [22] for example. □

Definition 8. *We have then, using theorem 14, for $S \in \mathrm{Var}(k)$, the full subcategory*

$$D_{\mathbb{Z}_l fil,c,k,gm}(S^{et}) :=< (K,W),\ s.t.\ \mathrm{Gr}_n^W K \in D_{\mathbb{Z}_l,c,k,gm}(S^{et}),\ for\ all n \in \mathbb{Z} >\subset D_{\mathbb{Z}_l fil,c,k}(S^{et})$$

where <> means the full triangulated category generated by.

We now give, following Ayoub, the definition of the nearby and vanishing cycle functors for constructible etale sheaves :

Let k a field of characteristic zero. Let l a prime number. Let $S \in \mathrm{Var}(k)$ and $D = V(s) \subset S$ a Cartier divisor with $s \in \Gamma(S,L)$. Denote $i : D \hookrightarrow S$ the closed embedding and $j : S^o := S \backslash D \hookrightarrow S$ the open embedding. Let

$$\pi : \tilde{S}^o := \varinjlim_{n\in\mathbb{N}} \tilde{S}^o_n := \mathrm{Spec}(L[t]/(t^n - s)) \to S^o$$

the universal covering given by taking the n-th roots of s and $\mathcal{A}_{S^o} := (S^o \times_{S^o\times S^o} S^o) \in \mathrm{Fun}(\Delta^\bullet, \mathrm{Var}(k)^{sm}/S^o)$ the diagram of lattices (see [4]). We then consider,

- for $K \in C_{\mathbb{Z}_l}(S^{o,et})$, the nearby cycle functor

$$\begin{aligned} \psi_D K : &= i^* e(S^{et})_* R(j\circ\pi)_*\pi^*\mathcal{H}om(\mathcal{A}_{S^o}, e(S^{et})^*K) \\ &= i^* R(j\circ\pi)_*\pi^*\mathcal{H}om(\mathcal{A}^e_{S^o}, K) \in D_{\mathbb{Z}_l}(D^{et}), \end{aligned}$$

  together with the monodromy morphism $T : \psi_D K \to \psi_D K$ in $D_{\mathbb{Z}_l}(D^{et})$, where

$$\mathcal{A}^e_{S^o} := (\cdots \to Rp_{S^o!}\mathbb{Z}_{l,S^o\times S^o} \to \cdots) \in D_{\mathbb{Z}_l}(S^{o,et})$$

  and we write again $\psi_D K := i_*\psi_D K \in D_{\mathbb{Z}_l,c}(S^{et})$, note that by adjonction $(Rp_{S^o!}, p^*_{S^o})$ we have $e(S^{et})^*\psi_D K = \psi_D e(S^{et})^* K$,

- for $K \in C_{\mathbb{Z}_l}(S^{o,et})$, the vanishing cycle functor

$$\phi_D K := \mathrm{Cone}(\mathrm{ad}(j\circ\pi^*, j\circ\pi_*)(-)\circ ev(K) : i^*K \to \psi_D K \in D_{\mathbb{Z}_l}(D^{et})$$

  together with the canonical map $c(\phi_D K) : \psi_D K \to \phi_D K$ in $D_{\mathbb{Z}_l}(D^{et})$, we write again $\phi_D K := i_*\phi_D K \in D_{\mathbb{Z}_l}(S^{et})$, for $K \in C_{\mathbb{Z}_l,c,k}(S^{o,et})$ we have a canonical isomorphism in $D_{\mathbb{Z}_l,c,k}(D^{et})$

$$T(D,\psi_D)(K) : \phi_D \mathbb{D}^v_S K \xrightarrow{\sim} \mathbb{D}^v_S \psi_D K$$

- for $K \in C_{\mathbb{Z}_l,c,k}(S^{o,et})$, the canonical morphisms in $D_{\mathbb{Z}_l,c,k}(D^{et})$

$$can(K) := c(\phi_D K) : \psi_D K \to \phi_D K\ ,\ var(K) := ((0, T-I)) : \phi_D K \to \psi_D K,$$

- for $K \in C_{\mathbb{Z}_l}(S^{o,et})$, the maximal extension

$$\begin{gathered} x_{S^o/S}(K) := \mathrm{Cone}(\mathrm{ad}(i^*,i_*)(-)\circ\mathrm{ad}(\pi^*,\pi_*)(-)\circ ev(K) : \\ Rj_*K \to i_*R(j\circ\pi)_*\pi^*\mathcal{H}om(\mathcal{A}_{S^o}, K) =: \psi_D K) \in D_{\mathbb{Z}_l}(S^{et}). \end{gathered}$$

Let $S \in \mathrm{Var}(k)$ and $D = V(s) \subset S$ a Cartier divisor with $s \in \Gamma(S,L)$. Denote $i : D \hookrightarrow S$ the closed embedding and $j : S^o := S\backslash D \hookrightarrow S$ the open embedding. Let

$$\pi : \tilde{S}^o := \varinjlim_{n\in\mathbb{N}} \tilde{S}^o_n := \mathrm{Spec}(L[t]/(t^n - s)) \to S^o$$

the universal covering and $\mathcal{A}_{S^o} := (S^o \times_{S^o\times S^o} S^o) \in \mathrm{Fun}(\Delta^\bullet, \mathrm{Var}(k)^{sm}/S^o)$ the diagram of lattices (see [4]).

- For $K, K' \in C_{\mathbb{Z}_l,c,k}(S^{o,et})$, we have by (Verdier) duality and theorem 6(ii) $\psi_D(K \otimes^L K') = \psi_D(K) \otimes^L \psi_D(K')$,
- For $K, K' \in C_{\mathbb{Z}_l,c,k}(S^{o,et})$ we have by (Verdier) duality and the preceding point $\phi_D(K \otimes^L K') = \phi_D(K) \otimes^L \phi_D(K')$
- For $K, K' \in C_{\mathbb{Z}_l}(S^{o,et})$, we have the transformation map in $D_{\mathbb{Z}_l}(S^{o,et})$

$$T(\otimes, \psi_D)(K, K') : (\psi_D K) \otimes^L K' \xrightarrow{I \otimes (\mathrm{ad}(i^*, i_*)(-) \circ \mathrm{ad}(\pi^*, \pi_*)(-))} \psi_D K \otimes^L \psi_D K' = \psi_D(K \otimes^L K').$$

- For $K, K' \in C_{\mathbb{Z}_l,c,k}(S^{o,et})$ the transformation map in $D_{\mathbb{Z}_l,c,k}(S^{o,et})$

$$T(\otimes, \phi_D)(K, K') : \phi_D(K \otimes^L K') = \mathbb{D}^v_S \psi_D(\mathbb{D}^v_S K \otimes^L \mathbb{D}^v_S K')$$
$$\xrightarrow{\mathbb{D}^v_S T(\otimes, \psi_D)(\mathbb{D}^v_S K, \mathbb{D}^v_S K')} \mathbb{D}^v_S(\psi_D \mathbb{D}^v_S(K) \otimes^L \mathbb{D}^v_S K') = \phi_D K \otimes^L K'.$$

We have then the following :

Proposition 1. *Let k a field of characteristic zero. Let $S \in \mathrm{Var}(k)$ and $D \subset S$ a Cartier divisor. For $K \in P_{\mathbb{Z}_l,k}(S^{et})$, we have $\psi_D K[-1], \phi_D K[-1] \in P_{\mathbb{Z}_l,k}(S^{et})$.*

Proof. Take an embedding $k \hookrightarrow \mathbb{C}$ and consider the morphism of site

$$\mathrm{an}_S : S^{an}_{\mathbb{C}} = S^{an,et}_{\mathbb{C}} \to S^{et}_{\mathbb{C}} \xrightarrow{\pi_{k/\mathbb{C}}(S)} S^{et}$$

given by the analytical functor. By [1], we have a canonical isomorphism in $D_{c,k}(S^{an}_{\mathbb{C}}) \otimes \mathbb{Q}$

$$T(an, \psi) : \mathrm{an}^*_S \psi_D K \xrightarrow{\sim} \psi_D \mathrm{an}^*_S K$$

where we recall (see section 2) that $D_{c,k}(S^{an}_{\mathbb{C}}) \subset D(S^{an}_{\mathbb{C}})$ is the full subcategory consisting of classes of complexes of presheaves whose cohomology sheaves are constructible with respect to a Zariski stratification of S (in particular defined over k). Since

$$K = ((K_n)_{n \in \mathbb{N}}) \otimes \mathbb{Q}_p \in P_{\mathbb{Z}_l,k}(S^{et}),$$

$\mathrm{an}^*_S K \in P_k(S^{an}_{\mathbb{C}})$, where we recall (see section 2) that $P_k(S^{an}_{\mathbb{C}}) \subset D_{c,k}(S^{an}_{\mathbb{C}}) \otimes \mathbb{Q}$ is the full subcategory consisting of presheaves sheaves whose cohomology sheaves are constructible with respect to a Zariski stratification of S (defined over k). Thus by the complex case (see e.g. [23]) $\psi_D \mathrm{an}^*_S K[-1] \in P_k(S^{an}_{\mathbb{C}})$. Hence $\mathrm{an}^*_S \psi_D K[-1] \in P_k(S^{an}_{\mathbb{C}})$ and thus $\psi_D K[-1], \phi_D K[-1] \in P_{\mathbb{Z}_l}(S^{et})$. □

We will use the etale version of a result of Beilinson on perverse sheaves.

Definition 9. *Let k a field of characteristic zero. Let $S \in \mathrm{Var}(k)$ and $D = V(s) \subset S$ a Cartier divisor. Denote $S^o := S \backslash D$. We denote by $P_{\mathbb{Z}_l}(S^{o,et}) \times_J P_{\mathbb{Z}_l}(D^{et})$ the category*

- *whose objects are (K', K'', u, v) where $K' \in P_{\mathbb{Z}_l}(S^{o,et})$ and $K'' \in P_{\mathbb{Z}_l}(D^{et})$ are perverse sheaves and $u : \psi_D K'^{an} \to K''^{an}$ and $v : K''^{an} \to \psi_D K'^{an}$ are morphism in $D_c(D^{an,pet})$ such that $v \circ u = T - I$,*
- *whose morphisms are $m = (m', m'') : (K'_1, K''_1, u_1, v_1) \to (K'_2, K''_2, u_2, v_2)$ such that $u_2 \circ \psi_D m' = m'' \circ u_1$ and $\psi_D m' \circ v_1 = v_2 \circ m''$.*

We give the version for etale construcible sheaves of the well known theorem 13 for perserse sheaves

Theorem 15. *Let k a field of characteristic zero. Let $S \in \mathrm{Var}(k)$ and $D = V(s) \subset S$ a Cartier divisor. Denote $i : D \hookrightarrow S$ the closed embedding and $j : S^o := S\backslash D \hookrightarrow S$ the open embedding. Then the functor*

$$(j^*, \phi_D[-1], can, var) : P_{\mathbb{Z}_l,k}(S^{et}) \to P_{\mathbb{Z}_l,k}(S^{o,et}) \times_J P_{\mathbb{Z}_l,k}(D^{et})$$

is an equivalence of category whose inverse is

$$P_{\mathbb{Z}_l,k}(S^{o,et}) \times_J P_{\mathbb{Z}_l,k}(D^{et}) \to P_{\mathbb{Z}_l,k}(S^{et}),$$

$$(K', K'', u, v) \mapsto H^1(\psi_D K' \xrightarrow{c(x_{S^o/S}(K')),u)} x_{S^o/S}(K') \oplus i_* K'' \xrightarrow{((0,T-I),v)} \psi_D K').$$

We denote, for $K \in P_{\mathbb{Z}_l}(S^{et})$ by

$$Is(K) := (0, (\mathrm{ad}(j^*, j_*)(K), \mathrm{ad}((j \circ \pi)^*, (j \circ \pi)_*)(K)), 0) :$$

$$K \xrightarrow{\sim} (\psi_D K \xrightarrow{(c(x_{S^o/S}(K)),can(K))} x_{S^o/S}(K) \oplus i_* \phi_D K \xrightarrow{((0,T-I),var(K))} \psi_D K)[-1]$$

the canonical isomorphism in $D_{\mathbb{Z}_l,c,k}(S^{et})$.

Proof. Similar to the proof of theorem 13. Follows from the fact that $P_{\mathbb{Z}_l,k}(S^{et}) \subset D_{\mathbb{Z}_l,c,k}(S^{et})$ is a full abelian subcategory (as the heart of the perverse t-structure) which is stable by the functors ϕ_D and ϕ_D for $D \subset S$ a Cartier divisor by proposition 1. □

In the filtered case, we will consider the weight monodromy filtration for open embeddings :

Definition 10. *Let k a field of characteristic zero. Let l a prime number.*

(i) Let $S \in \mathrm{Var}(k)$ and $j : S^o \hookrightarrow S$ an open embedding such that $D := S\backslash S^o = V(s) \subset S$ is a Cartier divisor. Let $P_{\mathbb{Z}_l fil,k}(S^{o,et})^{ad,D} \subset P_{fil,k}(S^{o,et})$ the full subcategory such that the relative weight monodromy filtration of W with respect to $D \subset S$ exists.

- *For $(K, W) \in P_{\mathbb{Z}_l fil,k}(S^{o,et})^{ad,D}$, we consider as in [25]*

  $$j_{*w}(K, W) := (Rj_* K, W) \in P_{\mathbb{Z}_l fil,k}(S^{et}),\ W_k Rj_* K :=< Rj_* W_k K, W(N)_k K >\subset Rj_* K$$

  so that $j^* j_{*w}(K, W) = (K, W)$, where $W_k Rj_* K \subset Rj_* K$ is given by W and the weight monodromy filtration $W(N)$ of the universal cover $\pi : \tilde{S}^o \to S^o$, see [4]. Note that a stratitification of $W_k Rj_* K$ is given by the closure of a stratification of $W_k K$ and $D := S\backslash S^o$.
- *For $(K, W) \in P_{\mathbb{Z}_l,fil,k}(S^{o,et})^{ad,D}$, we consider*

  $$j_{!w}(K, W) := \mathbb{D}^v_S j_{*w} \mathbb{D}^v_S(K, W) \in P_{\mathbb{Z}_l,fil}(S^{et})$$

  so that $j^* j_{!w}(K, W) = (K, W)$.

For $(K', W) \in P_{\mathbb{Z}_l fil,k}(S^{et})^{ad,D}$, there is, by construction,

- *a canonical map $\mathrm{ad}(j^*, j_{*w})(K', W) = \mathrm{ad}(j^*, j_*)(K') : (K', W) \to j_{*w} j^*(K', W)$ in $P_{\mathbb{Z}_l,fil,k}(S^{et})$,*
- *a canonical map $\mathrm{ad}(j_{!w}, j^*)(K', W) = \mathrm{ad}(j_!, j^*)(K') : j_{!w} j^*(K', W) \to (K', W)$ in $P_{\mathbb{Z}_l,fil,k}(S^{et})$.*

(ii) Let $S \in \mathrm{Var}(k)$. Let $j : S^o := S\backslash Z \hookrightarrow S$ an open embedding with $Z = V(\mathcal{I}) \subset S$ an arbitrary closed subset, $\mathcal{I} \subset O_S$ being an ideal subsheaf. Taking generators $\mathcal{I} = (s_1, \ldots, s_r)$, we get $Z = V(s_1, \ldots, s_r) = \cap_{i=1}^r Z_i \subset S$ with $Z_i = V(s_i) \subset S$, $s_i \in \Gamma(S, \mathcal{L}_i)$ and L_i a line bundle. Note that Z is an arbitrary closed subset, $d_Z \geq d_X - r$ needing not be a complete intersection. Denote by $j_I : S^{o,I} := \cap_{i\in I}(S\backslash Z_i) = S\backslash(\cup_{i\in I} Z_i) \xrightarrow{j_I^o} S^o \xrightarrow{j} S$ the open complementary embeddings, where $I \subset \{1, \cdots, r\}$. Denote

$$\mathcal{D}(Z/S) := \left\{(Z_i)_{i\in[1,\ldots r]}, Z_i \subset S, \cap Z_i = Z\right\}, Z'_i \subset Z_i$$

the flag category. Let $P_{\mathbb{Z}_l fil,k}(S^{o,et})^{ad,(Z_i)} \subset P_{\mathbb{Z}_l fil,k}(S^{o,et})$ the full subcategory such that the relative weight monodromy filtration of W with respect to the $Z_i \subset S$ exists. For $(K, W) \in C(P_{\mathbb{Z}_l,fil,k}(S^{o,et})^{ad,(Z_i)})$, we define by (i)

– the (bi)-filtered complex of D_S-modules

$$j_{*w}(K,W) := \varinjlim_{\mathcal{D}(Z/S)} \mathrm{Tot}_{cardI=\bullet}(j_{I*w}j_I^{o*}(K,W)) \in C(P_{\mathbb{Z}_l,fil}(S^{et}))$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \mathrm{ad}(j_{IJ}^*, j_{IJ*w})(j_I^{o*}(K,W))$, $j_{IJ} : S^{oJ} \hookrightarrow S^{oI}$ being the open embedding, and $d_{IJ} = 0$ if $I \notin J$,

– the (bi)-filtered complex of D_S-modules

$$j_{!w}(K,W) := \varprojlim_{\mathcal{D}(Z/S)} \mathrm{Tot}_{cardI=-\bullet}(j_{I!w}j_I^{o*}(K,W)) = \mathbb{D}_S^v j_{*w}\mathbb{D}_S^v(K,W) \in C(P_{\mathbb{Z}_l,fil}(S^{et})),$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \mathrm{ad}(j_{IJ!w}, j_{IJ}^*)(j_I^{o*}(K,W))$, $j_{IJ} : S^{oJ} \hookrightarrow S^{oI}$ being the open embedding, and $d_{IJ} = 0$ if $I \notin J$.

By definition, we have for $(K,W) \in C(P_{\mathbb{Z}_l,fil,k}(S^{o,et})^{ad,(Z_i)})$, $j^*j_{*w}(K,W) = (K,W)$ and $j^*j_{!w}(K,W) = (K,W)$. For $(K',W) \in C(P_{\mathbb{Z}_l,fil}(S^{et})^{ad,(Z_i)})$, there is, by (i),

– a canonical map $\mathrm{ad}(j^*, j_{*w})(K',W) : (K',W) \to j_{*w}j^*(K',W)$ in $C(P_{\mathbb{Z}_l,fil,k}(S^{et}))$,

– a canonical map $\mathrm{ad}(j_{!w}, j^*)(K',W) : j_{!w}j^*(K',W) \to (K',W)$ in $C(P_{\mathbb{Z}_l,fil,k}(S^{et}))$.

(iii) Let $S \in \mathrm{Var}(k)$. Let $j : S^o := S\backslash Z \hookrightarrow S$ an open embedding with $Z = V(\mathcal{I}) \subset S$ an arbitrary closed subset (over k), $\mathcal{I} \subset O_S$ being an ideal subsheaf. Let k/k' a field extension. For $(K,W) \in C(P_{\mathbb{Z}_l,fil,k}(S_{k'}^{o,et})^{ad,(Z_i)})$, (ii) gives

– the (bi)-filtered complex of D_S-modules

$$j_{*w}(K,W) := \varinjlim_{\mathcal{D}(Z/S)} \mathrm{Tot}_{cardI=\bullet}(j_{I*w}j_I^{o*}(K,W)) \in C(P_{\mathbb{Z}_l,fil,k}(S_{k'}^{et}))$$

– the (bi)-filtered complex of D_S-modules

$$j_{!w}(K,W) := \varprojlim_{\mathcal{D}(Z/S)} \mathrm{Tot}_{cardI=-\bullet}(j_{I!w}j_I^{o*}(K,W)) = \mathbb{D}_S^v j_{*w}\mathbb{D}_S^v(K,W) \in C(P_{\mathbb{Z}_l,fil}(S_{k'}^{et})),$$

By definition, we have for $(K,W) \in C(P_{\mathbb{Z}_l,fil,k}(S_{k'}^{o,et})^{ad,(Z_i)})$, $j^*j_{*w}(K,W) = (K,W)$ and $j^*j_{!w}(K,W) = (K,W)$. For $(K',W) \in C(P_{\mathbb{Z}_l,fil,k}(S_{k'}^{et})^{ad,(Z_i)})$, we have, see (ii),

– the canonical map $\mathrm{ad}(j^*, j_{*w})(K',W) : (K',W) \to j_{*w}j^*(K',W)$ in $C(P_{\mathbb{Z}_l,fil,k}(S_{k'}^{et}))$,

– the canonical map $\mathrm{ad}(j_{!w}, j^*)(K',W) : j_{!w}j^*(K',W) \to (K',W)$ in $C(P_{\mathbb{Z}_l,fil,k}(S_{k'}^{et}))$.

Definition 11. *Let $S \in \mathrm{Var}(k)$. Let $Z \subset S$ a closed subset. Denote by $j : S\backslash Z \hookrightarrow S$ the complementary open embedding. Let l a prime number.*

(i) We define using definition 10, the filtered Hodge support section functor

$$\Gamma_Z^w : C(P_{\mathbb{Z}_l fil,k}(S^{et})^{ad,(Z_i)}) \to C(P_{\mathbb{Z}_l fil,k}(S^{et})),$$
$$(K,W) \mapsto \Gamma_Z^w(K,W) := \mathrm{Cone}(\mathrm{ad}(j^*, j_{*w})(K,W) : (K,W) \to j_{*w}j^*(K,W))[-1],$$

together we the canonical map $\gamma_Z^w(K,W) : \Gamma_Z^w(K,W) \to (K,W)$. Since

$$j_{*w} : C(P_{\mathbb{Z}_l fil,k}(S^{o,et})^{ad,(Z_i)}) \to C(P_{\mathbb{Z}_l fil,k}(S^{et}))$$

is an exact functor, Γ_Z^w induces the functor

$$\Gamma_Z^w : D_{\mathbb{Z}_l fil,c,k}(S^{et})^{ad,(Z_i)} \to D_{\mathbb{Z}_l fil,c,k}(S^{et}),\ (K,W) \mapsto \Gamma_Z^w(K,W)$$

(ii) We define using definition 4, the dual filtered Hodge support section functor

$$\Gamma_Z^{\vee,w} : C(P_{\mathbb{Z}_l,fil}(S^{et})^{ad,(Z_i)}) \to C(P_{\mathbb{Z}_l,fil,k}(S^{et})),$$
$$(K,W) \mapsto \Gamma_Z^{\vee,w}(K,W) := \mathrm{Cone}(\mathrm{ad}(j_{!w}, j^*)(K,W) : j_{!w}, j^*(K,W) \to (K,W)),$$

together we the canonical map $\gamma_Z^{\vee,Hdg}(K,W) : (K,W) \to \Gamma_Z^{\vee,w}(K,W)$*. Since*

$$j_{!w} : C(P_{\mathbb{Z}_l,fil,k}(S^{o,et})^{ad,(Z_i)}) \to C(P_{\mathbb{Z}_l,fil,k}(S^{et}))$$

is an exact functor, $\Gamma_Z^{\vee,w}$ *induces the functor*

$$\Gamma_Z^{\vee,w} : D_{\mathbb{Z}_l fil,c,k}(S^{et})^{ad,(Z_i)} \to D_{\mathbb{Z}_l fil,c,k}(S^{et}),\ (K,W) \mapsto \Gamma_Z^{\vee,w}(K,W)$$

Let k/k' a field extension. Let $S \in \mathrm{Var}(k)$. Let $Z \subset S$ a closed subset (over k). Denote by $j : S\backslash Z \hookrightarrow S$ the complementary open embedding. Let l a prime number.

(i)' Then (i) gives the filtered Hodge support section functor

$$\Gamma_Z^{w} : C(P_{\mathbb{Z}_l,fil,k}(S_{k'}^{et})^{ad,(Z_i)}) \to C(P_{\mathbb{Z}_l,fil,k}(S_{k'}^{et})),$$
$$(K,W) \mapsto \Gamma_Z^{w}(K,W) := \mathrm{Cone}(\mathrm{ad}(j^*, j_{*w})(K,W) : (K,W) \to j_{*w}j^*(K,W))[-1],$$

together we the canonical map $\gamma_Z^{w}(K,W) : \Gamma_Z^{w}(K,W) \to (K,W)$*, which induces*

$$\Gamma_Z^{w} : D_{\mathbb{Z}_l fil,c,k}(S_{k'}^{et})^{ad,(Z_i)} \to D_{\mathbb{Z}_l fil,c,k}(S_{k'}^{et}),\ (K,W) \mapsto \Gamma_Z^{w}(K,W)$$

(ii)' Then, (ii) gives the dual filtered Hodge support section functor

$$\Gamma_Z^{\vee,w} : C(P_{\mathbb{Z}_l,fil,k}(S_{k'}^{et})^{ad,(Z_i)}) \to C(P_{\mathbb{Z}_l,fil,k}(S_{k'}^{et})),$$
$$(K,W) \mapsto \Gamma_Z^{\vee,w}(K,W) := \mathrm{Cone}(\mathrm{ad}(j_{!w}, j^*)(K,W) : j_{!w}, j^*(K,W) \to (K,W)),$$

together we the canonical map $\gamma_Z^{\vee,Hdg}(K,W) : (K,W) \to \Gamma_Z^{\vee,w}(K,W)$*, which induces*

$$\Gamma_Z^{\vee,w} : D_{\mathbb{Z}_l fil,c,k}(S_{k'}^{et})^{ad,(Z_i)} \to D_{\mathbb{Z}_l fil,c,k}(S_{k'}^{et}),\ (K,W) \mapsto \Gamma_Z^{\vee,w}(K,W)$$

Let $S \in \mathrm{Var}(k)$ and $D = V(s) \subset S$ a Cartier divisor. Denote $i : D \hookrightarrow S$ the closed embedding and $j : S^o := S\backslash D \hookrightarrow S$ the open embedding. Let $\pi : \tilde{S}_{\mathbb{C}}^{o,an} \to S_{\mathbb{C}}^{o,an}$ the universal covering. Let l a prime number. We then consider, for $(K,W) \in D_{\mathbb{Z}_l fil,c,k}(S^{o,et})^{ad,D} = \mathrm{Ho}(C(P_{\mathbb{Z}_l fil,k}(S^{o,et})^{ad,D})$,

- the filtered nearby cycle functor

$$\psi_D(K,W) := (\psi_D K, W) \in D_{\mathbb{Z}_l fil,c,k}(D^{et}),\ W_k(\psi_D(K,W)) :=< W_k\psi_D K, W(N)_k\psi_D K > \subset \psi_D K,$$

- the vanishing cycle functor

$$\phi_D(K,W) := \mathrm{Cone}(\mathrm{ad}(j\circ\pi^*, j\circ\pi_*)(K,W)\circ ev(K,W) : i^*(K,W) \to \psi_D(K,W)) \in D_{\mathbb{Z}_l fil,c}(D^{et}),$$

where the morphism

$$\mathrm{ad}(j\circ\pi^*, j\circ\pi_*)(K,W)\circ ev(K,W) : i^*(K,W) \to i^*R(j\circ\pi)_*\pi^*\mathcal{H}om(\mathcal{A}_{S^o}, (K,W))$$

being by definition compatible with the weight monodromy filtration induces the morphism

$$\mathrm{ad}(j\circ\pi^*, j\circ\pi_*)(K,W)\circ ev(K,W) : i^*(K,W) \to \psi_D(K,W)$$

- the canonical morphisms in $D_{\mathbb{Z}_l fil,c,k}(D^{et})$

$$can(K,W) := c(\phi_D(K,W)) : \psi_D(K,W) \to \phi_D(K,W),$$
$$var(K,W) := (0, T-I) : \phi_D(K,W) \to \psi_D(K,W),$$

- the maximal extension

$$x_{S^o/S}(K,W) := \mathrm{Cone}(\mathrm{ad}(i^*,i_*)(-) \circ \mathrm{ad}(\pi^*,\pi_*)(K,W) \circ ev(K,W) : j_{*w}(K,W) \to \psi_D(K,W)) \in D_{\mathbb{Z}_l fil,c,k}(S^{et}),$$

where the morphism

$$\mathrm{ad}(\pi^*,\pi_*)(K,W) \circ ev(K,W) : Rj_*(K,W) \to R(j\circ\pi)_*\pi^*\mathcal{H}om(\mathcal{A}_{S^o},(K,W))$$

being by definition compatible with the weight monodromy filtration induces the morphism

$$\mathrm{ad}(i^*,i_*)(-) \circ \mathrm{ad}(\pi^*,\pi_*)(K,W) \circ ev(K,W) : j_{*w}(K,W) \to \psi_D(K,W).$$

Let k/k' a field extension. For $(K,W) \in D_{\mathbb{Z}_l fil,c,k}(S^{o,et}_{k'})^{ad,D} = \mathrm{Ho}(C(P_{\mathbb{Z}_l fil,k}(S^{o,et}_{k'})^{ad,D})$, we get

- the filtered nearby cycle functor

$$\psi_D(K,W) := (\psi_D K, W) \in D_{\mathbb{Z}_l fil,c,k}(D^{et}_{k'}),\ W_k(\psi_D(K,W)) :=< W_k\psi_D K, W(N)_k\psi_D K >\subset \psi_D K,$$

- the vanishing cycle functor

$$\phi_D(K,W) := \mathrm{Cone}(\mathrm{ad}(j\circ\pi^*, j\circ\pi_*)(K,W)\circ ev(K,W) : i^*(K,W) \to \psi_D(K,W)) \in D_{\mathbb{Z}_l fil,c,k}(D^{et}_{k'}),$$

- the canonical morphisms in $D_{\mathbb{Z}_l fil,c,k}(D^{et}_{k'})$

$$can(K,W) := c(\phi_D(K,W)) : \psi_D(K,W) \to \phi_D(K,W),$$
$$var(K,W) := (0, T-I) : \phi_D(K,W) \to \psi_D(K,W),$$

- the maximal extension

$$x_{S^o/S}(K,W) := \mathrm{Cone}(\mathrm{ad}(i^*,i_*)(-) \circ \mathrm{ad}(\pi^*,\pi_*)(K,W) \circ ev(K,W) : j_{*w}(K,W) \to \psi_D(K,W)) \in D_{\mathbb{Z}_l fil,c,k}(S^{et}_{k'}).$$

Definition 12. *Let k a field of characteristic zero. Let l a prime number.*

(i) Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(k)$. Consider the graph factorization $f : X \xrightarrow{l} X\times S \xrightarrow{p} S$ of f where l the the graph closed embedding and p is the projection. We have, using definition 11,

- *the inverse image functor*

$$f^{*w} : D_{\mathbb{Z}_l fil,c,k}(S^{et})^{ad,(\Gamma_{f,i})} \to D_{\mathbb{Z}_l fil,c,k}(X^{et}),\ (K,W) \mapsto f^{*w}(K,W) := l^*\Gamma^{\vee,w}_X p^*(K,W)$$

- *the exceptional inverse image functor*

$$f^{!w} : D_{\mathbb{Z}_l fil,c,k}(S^{et})^{ad,(\Gamma_{f,i})} \to D_{\mathbb{Z}_l fil,c,k}(X^{et}),\ (K,W) \mapsto f^{!w}(K,W) := l^*\Gamma^{w}_X p^*(K,W).$$

(i)' Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(k)$. Consider the graph factorization $f : X \xrightarrow{l} X\times S \xrightarrow{p} S$ of f where l the the graph closed embedding and p is the projection. Let k/k' a field extension. We have, by (i),

– *the inverse image functor*

$$f^{*w}:D_{\mathbb{Z}_lfil,c,k}(S^{et}_{k'})^{ad,(\Gamma_{f,i})}\to D_{\mathbb{Z}_lfil,c,k}(X^{et}_{k'}),\ (K,W)\mapsto f^{*w}(K,W):=l^*\Gamma^{\vee,w}_X p^*(K,W)$$

– *the exceptional inverse image functor*

$$f^{!w}:D_{\mathbb{Z}_lfil,c,k}(S^{et}_{k'})^{ad,(\Gamma_{f,i})}\to D_{\mathbb{Z}_lfil,c,k}(X^{et}_{k'}),\ (K,W)\mapsto f^{!w}(K,W):=l^*\Gamma^{w}_X p^*(K,W).$$

(ii) Let $f:X\to S$ a morphism with $S,X\in\mathrm{Var}(k)$. Consider a compactification $f:X\hookrightarrow j\bar{X}\xrightarrow{\bar{f}}S$ of f with $\bar{X}\in\mathrm{Var}(k)$, j an open embedding and $\bar{f}$ a proper morphism. Denote $Z:=\bar{X}\backslash X$. We have, using definition 4,

– *the direct image functor*

$$Rf_{*w}:D_{\mathbb{Z}_lfil,c,k}(X^{et})^{ad,(Z_i)}\to D_{\mathbb{Z}_lfil,c,k}(S^{et}),\ (K,W)\mapsto Rf_{*w}(K,W):=R\bar{f}_*j_{*w}(K,W)$$

– *the proper direct image functor*

$$Rf_{!w}:D_{\mathbb{Z}_lfil,c,k}(X^{et})^{ad,(Z_i)}\to D_{\mathbb{Z}_lfil,c,k}(S^{et}),\ (K,W)\mapsto Rf_{!w}(K,W):=R\bar{f}_*j_{!w}(K,W).$$

(ii)' Let $f:X\to S$ a morphism with $S,X\in\mathrm{Var}(k)$. Consider a compactification $f:X\hookrightarrow j\bar{X}\xrightarrow{\bar{f}}S$ of f with $\bar{X}\in\mathrm{Var}(k)$, j an open embedding and $\bar{f}$ a proper morphism. Let k/k' a field extension. We have, by (ii),

– *the direct image functor*

$$Rf_{*w}:D_{\mathbb{Z}_lfil,c,k}(X^{et}_{k'})^{ad,(Z_i)}\to D_{\mathbb{Z}_lfil,c,k}(S^{et}_{k'}),\ (K,W)\mapsto Rf_{*w}(K,W):=R\bar{f}_*j_{*w}(K,W)$$

– *the proper direct image functor*

$$Rf_{!w}:D_{\mathbb{Z}_lfil,c,k}(X^{et}_{k'})^{ad,(Z_i)}\to D_{\mathbb{Z}_lfil,c,k}(S^{et}_{k'}),\ (K,W)\mapsto Rf_{!w}(K,W):=R\bar{f}_*j_{!w}(K,W).$$

(iii) Let $S\in\mathrm{Var}(k)$. Denote by $\Delta_S:S\hookrightarrow S\times S$ the diagonal closed embedding and $p_1:S\times S\to S$, $p_2:S\times S\to S$ the projections. We have by (i) the functor

$$\otimes^{Lw}:D_{\mathbb{Z}_lfil,c,k}(S^{et})^{ad,(S_i)}\times D_{\mathbb{Z}_lfil,c,k}(S^{et})^{ad,(S_i)}\to D_{\mathbb{Z}_lfil,c,k}(S^{et}),$$
$$((K_1,W),(K_2,W))\mapsto (K_1,W)\otimes^{L,w}(K_2,W):=\Delta_S^{!w}(p_1^*(K_1,W)\otimes^L p_2^*(K_2,W)).$$

Let $S\in\mathrm{Var}(k)$ and $D=V(s)\subset S$ a Cartier divisor. Denote $i:D\hookrightarrow S$ the closed embedding and $j:S^o:=S\backslash D\hookrightarrow S$ the open embedding. In the filtered case, we get, for $(K,W)\in P_{\mathbb{Z}_lfil,k}(S^{et})$ the map in $D_{\mathbb{Z}_lfil,c,k}(S^{et})$

$$Is(K,W):=(0,(\mathrm{ad}(j^*,j_*)(K),\mathrm{ad}(j\circ\pi^*,j\circ\pi_*)(K)),0):$$
$$(K,W)\to(\psi_D(K,W)\xrightarrow{(c(x_{S^o/S}(K,W)),can(K,W))}x_{S^o/S}(K,W)\oplus i_*\phi_D(K,W)$$
$$\xrightarrow{((0,T-I),var(K,W))}\psi_DK)[-1]$$

which is NOT an isomorphism in general (it leads to different W-filtration on perverse cohomology).

We recall the definition of constructible pro etale sheaves on algebraic varieties over a subfield $k\subset K$ of a p adic field:

Definition 13. *Let $k\subset K\subset\mathbb{C}_p$ a subfield of a p adic field. Let l a prime number. Let $S\in\mathrm{Var}(k)$.*

(i) A sheaf $K \in \mathrm{Shv}_{\mathbb{Z}_l}(S_K^{an,pet})$ is called constructible if there exists a stratification $S = \sqcup_\alpha S_\alpha$ with $l_\alpha : S_\alpha \hookrightarrow S$ locally closed subsets (defined over k) such that $l_\alpha^* K \in \mathrm{Shv}(S_\alpha^{an,pet})$ are (finite dimensional) local systems (for the etale topology) for all α.

(ii) We denote by

$$C_{\mathbb{Z}_l,c,k}(S^{an,pet}) \subset C_{\mathbb{Z}_l}(S^{an,pet}) \text{ and } D_{\mathbb{Z}_l,c,k}(S^{an,pet}) \subset D_{\mathbb{Z}_l}(S^{an,pet})$$

the full subcategories consisting of $K \in C(S_K^{an,pet})$ such that $a_{et}H^n K \in \mathrm{Shv}(S_K^{an,pet})$ are constructible for all $n \in \mathbb{Z}$.

(ii)' We denote by

$$C_{\mathbb{Z}_l fil,c,k}(S^{an,pet}) \subset C_{\mathbb{Z}_l fil}(S^{an,pet}) \text{ and } D_{\mathbb{Z}_l fil,c,k}(S^{an,pet}) \subset D_{\mathbb{Z}_l fil}(S^{an,pet})$$

the full subcategories consisting of $(K, W) \in C_{\mathbb{Z}_l fil}(S_K^{an,pet})$ such that $a_{et}H^n \operatorname{Gr}_k^W K \in \mathrm{Shv}_{\mathbb{Z}_l}(S_K^{an,pet})$ are constructible for all $n, k \in \mathbb{Z}$.

(iii) We denote by $P_{\mathbb{Z}_l,k}(S^{an,pet}) \subset D_{\mathbb{Z}_l,c,k}(S^{an,pet})$ the full subcategory of perverse sheaves.

(iii)' We denote by $P_{\mathbb{Z}_l,fil,k}(S^{an,pet}) \subset D_{\mathbb{Z}_l fil,c,k}(S^{an,pet})$ the full subcategory of filtered perverse sheaves.

Let $K \subset \mathbb{C}_p$ a p adic field. Let $S \in \mathrm{Var}(K)$ and $D = V(s) \subset S$ a Cartier divisor. Denote $i : D \hookrightarrow S$ the closed embedding and $j : S^o := S \backslash D \hookrightarrow S$ the open embedding. Let $\pi : \tilde{S}^{o,an} \to S^{o,an}$ the perfectoid universal covering (see [27]. We then consider,

- for $K \in C_{\mathbb{Z}_l}(S^{o,an,pet})$, the nearby cycle functor

$$\psi_D K := i^* R(j \circ \pi)_* \pi^* K = i^*(j \circ \pi)_* \pi^* K \in D_{\mathbb{Z}_l}(D^{an,pet}),$$

  we write again $\psi_D K := i_* \psi_D K \in D_{\mathbb{Z}_l}(S^{an,pet})$,

- for $K \in C_{\mathbb{Z}_l}(S^{o,an,pet})$, the vanishing cycle functor

$$\phi_D K := \mathrm{Cone}(\mathrm{ad}(j \circ \pi^*, j \circ \pi_*)(K) : i^* K \to \psi_D K \in D_{\mathbb{Z}_l}(D^{an,pet})$$

  together with the canonical map $c(\phi_D K) : \psi_D K \to \phi_D K$ in $D_{\mathbb{Z}_l}(S^{an,pet})$ we write again $\phi_D K := i_* \phi_D K \in D_{\mathbb{Z}_l}(S^{an,pet})$,

- for $K \in C_{\mathbb{Z}_l,c,k}(S^{o,an,pet})$, the canonical morphisms in $D_{\mathbb{Z}_l,c,k}(D^{an,pet})$

$$can(K) := c(\phi_D K) : \psi_D K \to \phi_D K \ , \ var(K) := (0, T - I) : \phi_D K \to \psi_D K,$$

- for $K \in C_{\mathbb{Z}_l,c}(S^{o,an,pet})$, the maximal extension

$$x_{S^o/S}(K) := \mathrm{Cone}(\mathrm{ad}(i^*, i_*)(-) \circ \mathrm{ad}(\pi^*, \pi_*)(K) : \\ Rj_* K \to i_* R(j \circ \pi)_* \pi^* K =: \psi_D K) \in D_{\mathbb{Z}_l}(S^{an,pet}).$$

Let $k \subset K \subset \mathbb{C}_p$ a subfield of a p adic field. Let $S \in \mathrm{Var}(k)$ and $D = V(s) \subset S$ a Cartier divisor. Denote by $S^o := S \backslash D$. By definition, we have for $K \in C_{\mathbb{Z}_l}(S^{o,et})$

$$\mathrm{an}_S^*(\psi_D K) = \psi_D(\mathrm{an}_S^* K) \in D_{\mathbb{Z}_l}(S_K^{an,pet}) \text{ and } \mathrm{an}_S^*(\phi_D K) = \phi_D(\mathrm{an}_S^* K) \in D_{\mathbb{Z}_l}(S_K^{an,pet}).$$

where $\mathrm{an}_S : S_K^{an,pet} \xrightarrow{\mathrm{an}_S} S_K^{et} \xrightarrow{\otimes_k K} S^{et}$ is the morphism of site induced by the analytical functor.

We then deduce from the algebraic case the following :

Corollary 1. *Let $k \subset K \subset \mathbb{C}_p$ a subfield of a p adic field. Let $S \in \mathrm{Var}(k)$ and $D = V(s) \subset S$ a Cartier divisor. Denote $i : D \hookrightarrow S$ the closed embedding and $j : S^o := S \backslash D \hookrightarrow S$ the open embedding.*

(i) For $K \in P_{\mathbb{Z}_l,k}(S^{et})$, we have $\psi_D \operatorname{an}_S^* K, \phi_D \operatorname{an}_S^* K \in P_{\mathbb{Z}_l,k}(S_K^{an,pet})$

(ii) We have, for $K \in P_{\mathbb{Z}_l,k}(S^{et})$, denoting again $K = \operatorname{an}_S^* K \in P_{\mathbb{Z}_l,k}(S_K^{an,pet})$,

$$Is(K) := (0, (\operatorname{ad}(j^*, j_*)(K), \operatorname{ad}((j \circ \pi)^*, (j \circ \pi)_*)(K)), 0) :$$

$$K \xrightarrow{\sim} (\psi_D K \xrightarrow{(c(x_{S^o/S}(K)), can(K))} x_{S^o/S}(K') \oplus i_* \phi_D K \xrightarrow{((0,T-I), var(K))} \psi_D K)[-1]$$

the canonical isomorphism in $D_{\mathbb{Z}_l,c,k}(S_K^{an,pet})$.

Proof. (i):Follows from proposition 1.
(ii):Follows immediately from theorem 15. □

2.6 Presheaves on the big Zariski site or on the big etale site

Let k a field of caracteristic 0. For $S \in \operatorname{Var}(k)$, we denote by $\rho_S : \operatorname{Var}(k)^{sm}/S \hookrightarrow \operatorname{Var}(k)/S$ be the full subcategory consisting of the objects $U/S = (U, h) \in \operatorname{Var}(k)/S$ such that the morphism $h : U \to S$ is smooth. That is, $\operatorname{Var}(k)^{sm}/S$ is the category

- whose objects are smooth morphisms $U/S = (U, h)$, $h : U \to S$ with $U \in \operatorname{Var}(k)$,
- whose morphisms $g : U/S = (U, h_1) \to V/S = (V, h_2)$ is a morphism $g : U \to V$ of complex algebraic varieties such that $h_2 \circ g = h_1$.

We denote again $\rho_S : \operatorname{Var}(k)/S \to \operatorname{Var}(k)^{sm}/S$ the associated morphism of site. We will consider

$$r^s(S) : \operatorname{Var}(k) \xrightarrow{r(S)} \operatorname{Var}(k)/S \xrightarrow{\rho_S} \operatorname{Var}(k)^{sm}/S$$

the composite morphism of site. For $S \in \operatorname{Var}(k)$, we denote by $\mathbb{Z}_S := \mathbb{Z}(S/S) \in \operatorname{PSh}(\operatorname{Var}(k)^{sm}/S)$ the constant presheaf By Yoneda lemma, we have for $F \in C(\operatorname{Var}(k)^{sm}/S)$, $\mathcal{H}om(\mathbb{Z}_S, F) = F$. For $f : T \to S$ a morphism, with $T, S \in \operatorname{Var}(k)$, we have the following commutative diagram of sites

$$\begin{tikzcd} \operatorname{Var}(k)/T \arrow[r, "\rho_T"] \arrow[d, "P(f)"] & \operatorname{Var}(k)^{sm}/T \arrow[d, "P(f)"] \\ \operatorname{Var}(k)/S \arrow[r, "\rho_S"] & \operatorname{Var}(k)^{sm}/S \end{tikzcd} \quad (1)$$

We denote, for $S \in \operatorname{Var}(k)$, the obvious morphism of sites

$$\tilde{e}(S) : \operatorname{Var}(k)/S \xrightarrow{\rho_S} \operatorname{Var}(k)^{sm}/S \xrightarrow{e(S)} \operatorname{Ouv}(S)$$

where $\operatorname{Ouv}(S)$ is the set of the Zariski open subsets of S, given by the inclusion functors $\tilde{e}(S) : \operatorname{Ouv}(S) \hookrightarrow \operatorname{Var}(k)^{sm}/S \hookrightarrow \operatorname{Var}(k)/S$. By definition, for $f : T \to S$ a morphism with $S, T \in \operatorname{Var}(k)$, the commutative diagram of sites (1) extend a commutative diagram of sites :

$$\begin{tikzcd} \tilde{e}(T) : \operatorname{Var}(k)/T \arrow[r, "\rho_T"] \arrow[d, "P(f)"] & \operatorname{Var}(k)^{sm}/T \arrow[r, "e(T)"] \arrow[d, "P(f)"] & \operatorname{Ouv}(T) \arrow[d, "P(f)"] \\ \tilde{e}(S) : \operatorname{Var}(k)/S \arrow[r, "\rho_S"] & \operatorname{Var}(k)^{sm}/S \arrow[r, "e(S)"] & \operatorname{Ouv}(S) \end{tikzcd} \quad (2)$$

- As usual, we denote by

$$(f^*, f_*) := (P(f)^*, P(f)_*) : C(\operatorname{Var}(k)^{sm}/S) \to C(\operatorname{Var}(k)^{sm}/T)$$

the adjonction induced by $P(f) : \operatorname{Var}(k)^{sm}/T \to \operatorname{Var}(k)^{sm}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjonction

$$(f^*, f_*) : C_{fil}(\operatorname{Var}(k)^{sm}/S) \leftrightarrows C_{fil}(\operatorname{Var}(k)^{sm}/T),\ f^*(G, F) := (f^*G, f^*F)$$

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$$(f^*, f_*) : C_{fil}(\mathrm{Var}(k)/S) \leftrightarrows C_{fil}(\mathrm{Var}(k)/T),\ f^*(G, F) := (f^*G, f^*F)$$

For $h : U \to S$ a smooth morphism with $U, S \in \mathrm{Var}(k)$, the pullback functor $P(h) : \mathrm{Var}(k)^{sm}/S \to \mathrm{Var}(k)^{sm}/U$ admits a left adjoint $C(h)(X \to U) = (X \to U \to S)$. Hence, $h^* : C(\mathrm{Var}(k)^{sm}/S) \to C(\mathrm{Var}(k)^{sm}/U)$ admits a left adjoint

$$h_\sharp : C(\mathrm{Var}(k)^{sm}/U) \to C(\mathrm{Var}(k)^{sm}/S),\ F \mapsto ((V, h_0) \mapsto \lim_{(V', h\circ h') \to (V, h_0)} F(V', h'))$$

Note that we have for $V/U = (V, h')$ with $h' : V \to U$ a smooth morphism we have $h_\sharp(\mathbb{Z}(V/U)) = \mathbb{Z}(V'/S)$ with $V'/S = (V', h\circ h')$. Hence, since projective presheaves are the direct summands of the representable presheaves, $h_\sharp$ sends projective presheaves to projective presheaves.

We have the support section functors of a closed embedding $i : Z \hookrightarrow S$ for presheaves on the big Zariski site.

Definition 14. *Let $i : Z \hookrightarrow S$ be a closed embedding with $S, Z \in \mathrm{Var}(k)$ and $j : S\backslash Z \hookrightarrow S$ be the open complementary subset.*

(i) We define the functor

$$\Gamma_Z : C(\mathrm{Var}(k)^{sm}/S) \to C(\mathrm{Var}(k)^{sm}/S),\ G^\bullet \mapsto \Gamma_Z G^\bullet := \mathrm{Cone}(\mathrm{ad}(j^*, j_*)(G^\bullet) : G^\bullet \to j_*j^*G^\bullet)[-1],$$

so that there is then a canonical map $\gamma_Z(G^\bullet) : \Gamma_Z G^\bullet \to G^\bullet$.

(ii) We have the dual functor of (i) :

$$\Gamma^\vee_Z : C(\mathrm{Var}(k)^{sm}/S) \to C(\mathrm{Var}(k)^{sm}/S),\ F \mapsto \Gamma^\vee_Z(F^\bullet) := \mathrm{Cone}(\mathrm{ad}(j_\sharp, j^*)(G^\bullet) : j_\sharp j^* G^\bullet \to G^\bullet),$$

together with the canonical map $\gamma^\vee_Z(G) : F \to \Gamma^\vee_Z(G)$.

(iii) For $F, G \in C(\mathrm{Var}(k)^{sm}/S)$, we denote by

$$I(\gamma, hom)(F, G) := (I, I(j_\sharp, j^*)(F, G)^{-1}) : \Gamma_Z \mathcal{H}om(F, G) \xrightarrow{\sim} \mathcal{H}om(\Gamma^\vee_Z F, G)$$

the canonical isomorphism given by adjonction.

Let $S_\bullet \in \mathrm{Fun}(\mathcal{I}, \mathrm{Var}(k))$ with $\mathcal{I} \in \mathrm{Cat}$, a diagram of algebraic varieties. It gives the diagram of sites $\mathrm{Var}(k)^2/S_\bullet \in \mathrm{Fun}(\mathcal{I}, \mathrm{Cat})$.

- Then $C_{fil}(\mathrm{Var}(k)/S_\bullet)$ is the category
  - whose objects $(G, F) = ((G_I, F)_{I\in\mathcal{I}}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\mathrm{Var}(k)/S_I)$, and $u_{IJ} : (G_I, F) \to r_{IJ*}(G_J, F)$ for $r_{IJ} : I \to J$, denoting again $r_{IJ} : S_I \to S_J$, are morphisms satisfying for $I \to J \to K$, $r_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\mathrm{Var}(k)/S_I)$,
  - the morphisms $m : ((G, F), u_{IJ}) \to ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

    $$m = (m_I : (G_I, F) \to (H_I, F))_{I\in\mathcal{I}}$$

    such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{fil}(\mathrm{Var}(k)/S_I)$.

- Then $C_{fil}(\mathrm{Var}(k)^{sm}/S_\bullet)$ is the category

- whose objects $(G,F)=((G_I,F)_{I\in\mathcal{I}},u_{IJ})$, with $(G_I,F)\in C_{fil}(\mathrm{Var}(k)^{sm}/S_I)$, and $u_{IJ}:(G_I,F)\to r_{IJ*}(G_J,F)$ for $r_{IJ}:I\to J$, denoting again $r_{IJ}:S_I\to S_J$, are morphisms satisfying for $I\to J\to K$, $r_{IJ*}u_{JK}\circ u_{IJ}=u_{IK}$ in $C_{fil}(\mathrm{Var}(k)^{sm}/S_I)$,
- the morphisms $m:((G,F),u_{IJ})\to((H,F),v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m=(m_I:(G_I,F)\to(H_I,F))_{I\in\mathcal{I}}$$

  such that $v_{IJ}\circ m_I=p_{IJ*}m_J\circ u_{IJ}$ in $C_{fil}(\mathrm{Var}(k)^{sm}/S_I)$.

As usual, we denote by

$$(f_\bullet^*,f_{\bullet*}):=(P(f_\bullet)^*,P(f_\bullet)_*):C(\mathrm{Var}(k)^{(sm)}/S_\bullet)\to C(\mathrm{Var}(k)^{(sm)}/T_\bullet)$$

the adjonction induced by $P(f_\bullet):\mathrm{Var}(k)^{(sm)}/T_\bullet\to\mathrm{Var}(k)^{(sm)}/S_\bullet$. Since the colimits involved in the definition of $f_\bullet^*=P(f_\bullet)^*$ are filtered, $f_\bullet^*$ also preserve monomorphism. Hence, we get an adjonction

$$(f_\bullet^*,f_{\bullet*}):C_{fil}(\mathrm{Var}(k)^{(sm)}/S_\bullet)\leftrightarrows C_{fil}(\mathrm{Var}(k)^{(sm)}/T_\bullet),$$
$$f_\bullet^*((G_I,F),u_{IJ}):=((f_I^*G_I,f_I^*F),T(f_I,r_{IJ})(-)\circ f_I^*u_{IJ}).$$

Let $S\in\mathrm{Var}(k)$. Let $S=\cup_{i=1}^l S_i$ an open affine cover and denote by $S_I=\cap_{i\in I}S_i$. Let $i_i:S_i\hookrightarrow\tilde{S}_i$ closed embeddings, with $\tilde{S}_i\in\mathrm{Var}(k)$. For $I\subset[1,\cdots l]$, denote by $\tilde{S}_I=\Pi_{i\in I}\tilde{S}_i$. We then have closed embeddings $i_I:S_I\hookrightarrow\tilde{S}_I$ and for $J\subset I$ the following commutative diagram

$$D_{IJ}=\quad\begin{tikzcd} S_I \arrow[r, "i_I"] & \tilde{S}_I \\ S_J \arrow[u, "j_{IJ}"] \arrow[r, "i_J"] & \tilde{S}_J \arrow[u, "p_{IJ}"'] \end{tikzcd}$$

where $p_{IJ}:\tilde{S}_J\to\tilde{S}_I$ is the projection and $j_{IJ}:S_J\hookrightarrow S_I$ is the open embedding so that $j_I\circ j_{IJ}=j_J$. This gives the diagram of algebraic varieties $(\tilde{S}_I)\in\mathrm{Fun}(\mathcal{P}(\mathbb{N}),\mathrm{Var}(k))$ which the diagram of sites $\mathrm{Var}(k)^{sm}/(\tilde{S}_I)\in\mathrm{Fun}(\mathcal{P}(\mathbb{N}),\mathrm{Cat})$. Denote by $m:\tilde{S}_I\backslash(S_I\backslash S_J)\hookrightarrow\tilde{S}_I$ the open embedding. Then $C_{fil}(\mathrm{Var}(k)^{sm}/(\tilde{S}_I))$ is the category

- whose objects $(G,F)=((G_I,F),u_{IJ})$ with $(G_I,F)\in C_{fil}(\mathrm{Var}(k)^{sm}/\tilde{S}_I)$, and $u_{IJ}:(G_I,F)\to p_{IJ*}(G_J,F)$ are morphisms satisfying for $I\subset J\subset K$, $p_{IJ*}u_{JK}\circ u_{IJ}=u_{IK}$ in $C_{fil}(\mathrm{Var}(k)^{sm}/\tilde{S}_I)$,
- the morphisms $m:((G,F),u_{IJ})\to((H,F),v_{IJ})$ being a family of morphisms of complexes,

$$m=(m_I:(G_I,F)\to(H_I,F))_{I\in\mathcal{I}}$$

  such that $v_{IJ}\circ m_I=p_{IJ*}m_J\circ u_{IJ}$ in $C_{fil}(\mathrm{Var}(k)^{sm}/\tilde{S}_I)$.

Similarly, $C_{fil}(\mathrm{Var}(k)^{sm}/(\tilde{S}_I)^{op})$ is the category

- whose objects $(G,F)=((G_I,F)_{I\subset[1,\cdots l]},u_{IJ})$, with $(G_I,F)\in C_{fil}(\mathrm{Var}(k)^{(sm)}/\tilde{S}_I)$, and $u_{IJ}:(G_J,F)\to p_{IJ}^*(G_I,F)$ for $I\subset J$, are morphisms satisfying for $I\subset J\subset K$, $p_{JK}^*u_{IJ}\circ u_{JK}=u_{IK}$ in $C_{fil}(\mathrm{Var}(k)^{(sm)}/\tilde{S}_K)$,
- the morphisms $m:((G,F),u_{IJ})\to((H,F),v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m=(m_I:(G_I,F)\to(H_I,F))_{I\subset[1,\cdots l]}$$

  such that $v_{IJ}\circ m_J=p_{IJ}^*m_I\circ u_{IJ}$ in $C_{fil}(\mathrm{Var}(k)^{(sm)}/\tilde{S}_J)$.

Definition 15. *Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^{l} S_i$ an open cover and denote by $S_I = \cap_{i\in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{Var}(k)$. We will denote by $C_{fil}(\mathrm{Var}(k)^{sm}/(S/(\tilde{S}_I))) \subset C_{fil}(\mathrm{Var}(k)^{sm}/(\tilde{S}_I))$ the full subcategory whose objects $(G,F) = ((G_I,F)_{I\subset[1,\cdots l]}, u_{IJ})$, with $(G_I,F) \in C_{fil,S_I}(\mathrm{Var}(k)^{sm}/\tilde{S}_I)$, and $u_{IJ} : m^*(G_I,F) \to m^* p_{IJ*}(G_J,F)$ for $I \subset J$, are ∞-filtered Zariski local equivalence,*

We now give the definition of the $\mathbb{A}^1$ local property :

Denote by

$$p_a : \mathrm{Var}(k)^{(sm)}/S \to \mathrm{Var}(k)^{(sm)}/S,\ X/S = (X,h) \mapsto (X\times\mathbb{A}^1)/S = (X\times\mathbb{A}^1, h\circ p_X),$$
$$(g : X/S \to X'/S) \mapsto ((g\times I_{\mathbb{A}^1}) : X\times\mathbb{A}^1/S \to X'\times\mathbb{A}^1/S)$$

the projection functor and again by $p_a : \mathrm{Var}(k)^{(sm)}/S \to \mathrm{Var}(k)^{(sm)}/S$ the corresponding morphism of site.

Definition 16. *Let $S \in \mathrm{Var}(k)$. Denote for short $\mathrm{Var}(k)^{(sm)}/S$ either the category $\mathrm{Var}(k)/S$ or the category $\mathrm{Var}(k)^{sm}/S$.*

(i0) A complex $F \in C(\mathrm{Var}(k)^{(sm)}/S)$ is said to be $\mathbb{A}^1$ homotopic if $\mathrm{ad}(p_a^*, p_{a*})(F) : F \to p_{a*}p_a^*F$ is an homotopy equivalence.

(i) A complex $F \in C(\mathrm{Var}(k)^{(sm)}/S)$ is said to be $\mathbb{A}^1$ invariant if for all $U/S \in \mathrm{Var}(k)^{(sm)}/S$,

$$F(p_U) : F(U/S) \to F(U\times\mathbb{A}^1/S)$$

is a quasi-isomorphism, where $p_U : U\times\mathbb{A}^1 \to U$ is the projection. Obviously, if a complex $F \in C(\mathrm{Var}(k)^{(sm)}/S)$ is $\mathbb{A}^1$ homotopic then it is $\mathbb{A}^1$ invariant.

(ii) Let τ a topology on $\mathrm{Var}(k)$. A complex $F \in C(\mathrm{Var}(k)^{(sm)}/S)$ is said to be $\mathbb{A}^1$ local for the topology τ, if for a (hence every) τ local equivalence $k : F \to G$ with k injective and $G \in C(\mathrm{Var}(k)^{(sm)}/S)$ τ fibrant, e.g. $k : F \to E_\tau(F)$, G is $\mathbb{A}^1$ invariant for all $n \in \mathbb{Z}$.

(iii) A morphism $m : F \to G$ with $F,G \in C(\mathrm{Var}(k)^{(sm)}/S)$ is said to an $(\mathbb{A}^1, et)$ local equivalence if for all $H \in C(\mathrm{Var}(k)^{(sm)}/S)$ which is $\mathbb{A}^1$ local for the etale topology

$$\mathrm{Hom}(L(m), E_{et}(H)) : \mathrm{Hom}(L(G), E_{et}(H)) \to \mathrm{Hom}(L(F), E_{et}(H))$$

is a quasi-isomorphism.

Denote $\square^* := \mathbb{P}^* \backslash \{1\}$

- Let $S \in \mathrm{Var}(k)$. For $U/S = (U,h) \in \mathrm{Var}(k)^{sm}/S$, we consider

$$\square^* \times U/S = (\square^*\times U, h\circ p) \in \mathrm{Fun}(\Delta, \mathrm{Var}(k)^{sm}/S).$$

For $F \in C^-(\mathrm{Var}(k)^{sm}/S)$, it gives the complex

$$C_*F \in C^-(\mathrm{Var}(k)^{sm}/S), U/S = (U,h) \mapsto C_*F(U/S) := \mathrm{Tot}\, F(\square^*\times U/S)$$

together with the canonical map $c_F := (0, I_F) : F \to C_*F$. For $F \in C(\mathrm{Var}(k)^{sm}/S)$, we get

$$C_*F := \mathrm{holim}_n\, C_*F^{\leq n} \in C(\mathrm{Var}(k)^{sm}/S),$$

together with the canonical map $c_F := (0, I_F) : F \to C_*F$. For $m : F \to G$ a morphism, with $F, G \in C(\mathrm{Var}(k)^{sm}/S)$, we get by functoriality the morphism $C_*m : C_*F \to C_*G$.

- Let $S \in \mathrm{Var}(k)$. For $U/S = (U,h) \in \mathrm{Var}(k)/S$, we consider

$$\square^* \times U/S = (\mathbb{A}^* \times U, h \circ p) \in \mathrm{Fun}(\Delta, \mathrm{Var}(k)/S).$$

For $F \in C^-(\mathrm{Var}(k)/S)$, it gives the complex

$$C_*F \in C^-(\mathrm{Var}(k)/S), U/S = (U,h) \mapsto C_*F(U/S) := \mathrm{Tot}\, F(\square^* \times U/S)$$

together with the canonical map $c = c(F) := (0, I_F) : F \to C_*F$. For $F \in C(\mathrm{Var}(k)/S)$, we get

$$C_*F := \mathrm{holim}_n\, C_*F^{\leq n} \in C(\mathrm{Var}(k)/S),$$

together with the canonical map $c_F := (0, I_F) : F \to C_*F$. For $m : F \to G$ a morphism, with $F, G \in C(\mathrm{Var}(k)/S)$, we get by functoriality the morphism $C_*m : C_*F \to C_*G$.

Proposition 2. *(i) Let $S \in \mathrm{Var}(k)$. Then for $F \in C(\mathrm{Var}(k)^{sm}/S)$, C_*F is $\mathbb{A}^1$ local for the etale topology and $c(F) : F \to C_*F$ is an equivalence $(\mathbb{A}^1, et)$ local.*

(ii) A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(k)^{(sm)}/S)$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if there exists

$$\{X_{1,\alpha}/S, \alpha \in \Lambda_1\}, \ldots, \{X_{r,\alpha}/S, \alpha \in \Lambda_r\} \subset \mathrm{Var}(k)^{(sm)}/S$$

such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(k)^{(sm)}/S))$

$$\begin{aligned}\mathrm{Cone}(m) \xrightarrow{\sim} \mathrm{Cone}(&\oplus_{\alpha\in\Lambda_1} \mathrm{Cone}(\mathbb{Z}(X_{1,\alpha} \times \mathbb{A}^1/S) \to \mathbb{Z}(X_{1,\alpha}/S)) \\ &\to \cdots \to \oplus_{\alpha\in\Lambda_r} \mathrm{Cone}(\mathbb{Z}(X_{r,\alpha} \times \mathbb{A}^1/S) \to \mathbb{Z}(X_{r,\alpha}/S)))\end{aligned}$$

Proof. Standard : see Ayoub's thesis for example. □

Definition-Proposition 1. *Let $S \in \mathrm{Var}(k)$.*

(i) With the weak equivalence the $(\mathbb{A}^1, et)$ local equivalence and the fibration the epimorphism with $\mathbb{A}^1_S$ local and etale fibrant kernels gives a model structure on $C(\mathrm{Var}(k)^{sm}/S)$: the left bousfield localization of the projective model structure of $C(\mathrm{Var}(k)^{sm}/S)$. We call it the projective $(\mathbb{A}^1, et)$ model structure.

(ii) With the weak equivalence the $(\mathbb{A}^1, et)$ local equivalence and the fibration the epimorphism with $\mathbb{A}^1_S$ local and etale fibrant kernels gives a model structure on $C(\mathrm{Var}(k)/S)$: the left bousfield localization of the projective model structure of $C(\mathrm{Var}(k)/S)$. We call it the projective $(\mathbb{A}^1, et)$ model structure.

Proof. See [12]. □

Proposition 3. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(k)$.*

(i) The adjonction $(g^*, g_*) : C(\mathrm{Var}(k)^{sm}/S) \leftrightarrows C(\mathrm{Var}(k)^{sm}/T)$ is a Quillen adjonction for the $(\mathbb{A}^1, et)$ projective model structure (see definition-proposition 1).

(i)' Let $h : U \to S$ a smooth morphism with $U, S \in \mathrm{Var}(k)$. The adjonction $(h_\sharp, h^*) : C(\mathrm{Var}(k)^{sm}/U) \leftrightarrows C(\mathrm{Var}(k)^{sm}/S)$ is a Quillen adjonction for the $(\mathbb{A}^1, et)$ projective model structure.

(i)" The functor $g^* : C(\mathrm{Var}(k)^{sm}/S) \to C(\mathrm{Var}(k)^{sm}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence.

(ii) The adjonction $(g^*, g_*) : C(\mathrm{Var}(k)/S) \leftrightarrows C(\mathrm{Var}(k)/T)$ is a Quillen adjonction for the $(\mathbb{A}^1, et)$ projective model structure (see definition-proposition 1).

(ii)' The adjonction $(g_\sharp, g^*) : C(\mathrm{Var}(k)/T) \leftrightarrows C(\mathrm{Var}(k)/S)$ is a Quillen adjonction for the $(\mathbb{A}^1, et)$ projective model structure.

(ii)" The functor $g^* : C(\mathrm{Var}(k)/S) \to C(\mathrm{Var}(k)/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence.

Proof. Standard : see [12] for example. □

Proposition 4. *Let $S \in \mathrm{Var}(k)$.*

(i) The adjonction $(\rho_S^*, \rho_{S*}) : C(\mathrm{Var}(k)^{sm}/S) \leftrightarrows C(\mathrm{Var}(k)/S)$ is a Quillen adjonction for the $(\mathbb{A}^1, et)$ projective model structure.

(ii) The functor $\rho_{S*} : C(\mathrm{Var}(k)/S) \to C(\mathrm{Var}(k)^{sm}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence.

Proof. Standard : see [12] for example. □

Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i\in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{Var}(k)$.

- For $(G_I, K_{IJ}) \in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)^{op})$ and $(H_I, T_{IJ}) \in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I))$, we denote

  $$\mathcal{H}om((G_I, K_{IJ}), (H_I, T_{IJ})) := (\mathcal{H}om(G_I, H_I), u_{IJ}((G_I, K_{IJ}), (H_I, T_{IJ}))) \in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I))$$

  with

  $$\begin{aligned} & u_{IJ}((G_I, K_{IJ})(H_I, T_{IJ})) : \mathcal{H}om(G_I, H_I) \\ & \xrightarrow{\mathrm{ad}(p_{IJ}^*, p_{IJ*})(-)} p_{IJ*} p_{IJ}^* \mathcal{H}om(G_I, H_I) \xrightarrow{T(p_{IJ}, hom)(-,-)} p_{IJ*} \mathcal{H}om(p_{IJ}^* G_I, p_{IJ}^* H_I) \\ & \xrightarrow{\mathcal{H}om(p_{IJ}^* G_I, T_{IJ})} p_{IJ*} \mathcal{H}om(p_{IJ}^* G_I, H_J) \xrightarrow{\mathcal{H}om(K_{IJ}, H_J)} p_{IJ*} \mathcal{H}om(G_J, H_J). \end{aligned}$$

  This gives in particular the functor

  $$\begin{aligned} \mathbb{D}^0_{(\tilde{S}_I)} &: C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)^{op}) \to C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)), \\ (H_I, T_{IJ}) &\mapsto (\mathbb{D}^0_{\tilde{S}_I} L H_I, T^d_{IJ}) := \mathcal{H}om((L H_I, T_{IJ}), (E_{et}(\mathbb{Z}_{\tilde{S}_I}), I_{IJ})). \end{aligned}$$

- For $(G_I, K_{IJ}) \in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I))$ and $(H_I, T_{IJ}) \in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)^{op})$, we denote

  $$\mathcal{H}om((G_I, K_{IJ}), (H_I, T_{IJ})) := (\mathcal{H}om(G_I, H_I), u_{IJ}((G_I, K_{IJ}), (H_I, T_{IJ}))) \in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)^{op})$$

  with

  $$\begin{aligned} & u_{IJ}((G_I, K_{IJ})(H_I, T_{IJ})) : \mathcal{H}om(G_J, H_J) \\ & \xrightarrow{\mathcal{H}om(\mathrm{ad}(p_{IJ}^*, p_{IJ*})(G_J), H_J)} \mathcal{H}om(p_{IJ}^* p_{IJ*} G_J, H_J) \xrightarrow{\mathcal{H}om(p_{IJ}^* K_{IJ}, H_J)} \mathcal{H}om(p_{IJ}^* G_I, H_J) \\ & \xrightarrow{\mathcal{H}om(p_{IJ}^* G_I, T_{IJ})} \mathcal{H}om(p_{IJ}^* G_I, p_{IJ}^* H_I) \xrightarrow{T(p_{IJ}, \mathrm{hom})(-,-)^{-1}} p_{IJ}^* \mathcal{H}om(G_I, H_I). \end{aligned}$$

  This gives in particular the functor

  $$\begin{aligned} \mathbb{D}^0_{(\tilde{S}_I)} &: C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)) \to C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)^{op}), \\ (H_I, T_{IJ}) &\mapsto (\mathbb{D}^0_{\tilde{S}_I} L H_I, T^d_{IJ}) := (\mathcal{H}om((L H_I, T_{IJ}), (E_{et}(\mathbb{Z}_{\tilde{S}_I}), I_{IJ}))). \end{aligned}$$

Definition 17. *Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i\in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{Var}(k)$.*

(i0) *A complex* $(F_I,u_{IJ})\in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I))$ *is said to be* $\mathbb{A}^1$ *homotopic if* $\mathrm{ad}(p_a^*,p_{a*})((F_I,u_{IJ})):(F_I,u_{IJ})\to p_{a*}p_a^*(F_I,u_{IJ})$ *is an homotopy equivalence.*

(i) *A complex* $(F_I,u_{IJ})\in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I))$ *is said to be* $\mathbb{A}^1$ *invariant if for all* $(X_I/\tilde{S}_I,s_{IJ})\in\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)$

$$(F_I(p_{X_I})):(F_I(X_I/\tilde{S}_I),F_J(s_{IJ})\circ u_{IJ}(-))\to(F_I(X_I\times\mathbb{A}^1/\tilde{S}_I),F_J(s_{IJ}\times I)\circ u_{IJ}(-))$$

is a quasi-isomorphism, where $p_{X_I}:X_I\times\mathbb{A}^1\to X_I$ *are the projection, and* $s_{IJ}:X_I\times\tilde{S}_{J\backslash I}/\tilde{S}_J\to X_J/\tilde{S}_J$. *Obviously a complex* $(F_I,u_{IJ})\in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I))$ *is* $\mathbb{A}^1$ *invariant if and only if all the* F_I *are* $\mathbb{A}^1$ *invariant.*

(ii) *Let* τ *a topology on* $\mathrm{Var}(k)$. *A complex* $F=(F_I,u_{IJ})\in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I))$ *is said to be* $\mathbb{A}^1$ *local for the* τ *topology induced on* $\mathrm{Var}(k)/(\tilde{S}_I)$, *if for an (hence every)* τ *local equivalence* $k:F\to G$ *with* k *injective and* $G=(G_I,v_{IJ})\in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I))$ τ *fibrant, e.g.* $k:(F_I,u_{IJ})\to(E_\tau(F_I),E(u_{IJ}))$, G *is* $\mathbb{A}^1$ *invariant.*

(iii) *A morphism* $m=(m_I):(F_I,u_{IJ})\to(G_I,v_{IJ})$ *with* $(F_I,u_{IJ}),(G_I,v_{IJ})\in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I))$ *is said to be an* $(\mathbb{A}^1,et)$ *local equivalence if for all* $(H_I,w_{IJ})\in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I))$ *which is* $\mathbb{A}^1$ *local for the etale topology*

$$(\mathrm{Hom}(L(m_I),E_{et}(H_I))):\mathrm{Hom}(L(G_I,v_{IJ}),E_{et}(H_I,w_{IJ}))\to\mathrm{Hom}(L(F_I,u_{IJ}),E_{et}(H_I,w_{IJ}))$$

is a quasi-isomorphism (of complexes of abelian groups). Obviously, if a morphism $m=(m_I):(F_I,u_{IJ})\to(G_I,v_{IJ})$ *with* $(F_I,u_{IJ}),(G_I,u_{IJ})\in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I))$ *is an* $(\mathbb{A}^1,et)$ *local equivalence, then all the* $m_I:F_I\to G_I$ *are* $(\mathbb{A}^1,et)$ *local equivalence.*

(iv) *A morphism* $m=(m_I):(F_I,u_{IJ})\to(G_I,v_{IJ})$ *with* $(F_I,u_{IJ}),(G_I,v_{IJ})\in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)^{op})$ *is said to be an* $(\mathbb{A}^1,et)$ *local equivalence if for all* $(H_I,w_{IJ})\in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)^{op})$ *which is* $\mathbb{A}^1$ *local for the etale topology*

$$(\mathrm{Hom}(L(m_I),E_{et}(H_I))):\mathrm{Hom}(L(G_I,v_{IJ}),E_{et}(H_I,w_{IJ}))\to\mathrm{Hom}(L(F_I,u_{IJ}),E_{et}(H_I,w_{IJ}))$$

is a quasi-isomorphism (of complexes of abelian groups). Obviously, if a morphism $m=(m_I):(F_I,u_{IJ})\to(G_I,v_{IJ})$ *with* $(F_I,u_{IJ}),(G_I,u_{IJ})\in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)^{op})$ *is an* $(\mathbb{A}^1,et)$ *local equivalence, then all the* $m_I:F_I\to G_I$ *are* $(\mathbb{A}^1,et)$ *local equivalence and for all* $(H_I,w_{IJ})\in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I))$,

$$(\mathrm{Hom}(L(m_I),E_{et}(H_I))):\mathrm{Hom}(L(G_I,v_{IJ}),E_{et}(H_I,w_{IJ}))\to\mathrm{Hom}(L(F_I,u_{IJ}),E_{et}(H_I,w_{IJ}))$$

is a quasi-isomorphism (of diagrams of complexes of abelian groups).

Proposition 5. *Let* $S\in\mathrm{Var}(k)$. *Let* $S=\cup_{i=1}^l S_i$ *an open affine cover and denote by* $S_I=\cap_{i\in I}S_i$. *Let* $i_i:S_i\hookrightarrow\tilde{S}_i$ *closed embeddings, with* $\tilde{S}_i\in\mathrm{Var}(k)$.

(i) *A morphism* $m:F\to G$ *with* $F,G\in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I))$ *is an* $(\mathbb{A}^1,et)$ *local equivalence if and only if there exists*

$$\left\{(X_{1,\alpha,I}/\tilde{S}_I,u^1_{IJ}),\alpha\in\Lambda_1\right\},\ldots,\left\{(X_{r,\alpha,I}/\tilde{S}_I,u^r_{IJ}),\alpha\in\Lambda_r\right\}\subset\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)$$

with $u^l_{IJ}:X_{l,\alpha,I}\times\tilde{S}_{J\backslash I}/\tilde{S}_J\to X_{l,\alpha,J}/\tilde{S}_J$, *such that we have in* $\mathrm{Ho}_{et}(C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)))$

$$\begin{aligned}\mathrm{Cone}(m)\xrightarrow{\sim}\mathrm{Cone}(&\oplus_{\alpha\in\Lambda_1}\mathrm{Cone}((\mathbb{Z}(X_{1,\alpha,I}\times\mathbb{A}^1/\tilde{S}_I),\mathbb{Z}(u^1_{IJ}\times I))\to(\mathbb{Z}(X_{1,\alpha,I}/\tilde{S}_I),\mathbb{Z}(u^1_{IJ})))\\&\to\cdots\to\oplus_{\alpha\in\Lambda_r}\mathrm{Cone}((\mathbb{Z}(X_{r,\alpha,I}\times\mathbb{A}^1/\tilde{S}_I),\mathbb{Z}(u^r_{IJ}\times I))\to(\mathbb{Z}(X_{r,\alpha,I}/\tilde{S}_I),\mathbb{Z}(u^r_{IJ}))))\end{aligned}$$

(ii) *A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)^{op})$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if there exists*

$$\left\{(X_{1,\alpha,I}/\tilde{S}_I, u^1_{IJ}), \alpha \in \Lambda_1\right\}, \ldots, \left\{(X_{r,\alpha,I}/\tilde{S}_I, u^r_{IJ}), \alpha \in \Lambda_r\right\} \subset \mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)^{op}$$

with $u^l_{IJ} : X_{l,\alpha,J}/\tilde{S}_J \to X_{l,\alpha,I} \times \tilde{S}_{J\setminus I}/\tilde{S}_J$, such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(k)^{(sm)}/(\tilde{S}_I)^{op}))$

$$\begin{aligned}\mathrm{Cone}(m) \xrightarrow{\sim} &\mathrm{Cone}(\oplus_{\alpha\in\Lambda_1} \mathrm{Cone}((\mathbb{Z}(X_{1,\alpha,I} \times \mathbb{A}^1/\tilde{S}_I), \mathbb{Z}(u^1_{IJ} \times I)) \to (\mathbb{Z}(X_{1,\alpha,I}/\tilde{S}_I), \mathbb{Z}(u^1_{IJ}))) \\ &\to \cdots \to \oplus_{\alpha\in\Lambda_r} \mathrm{Cone}((\mathbb{Z}(X_{r,\alpha,I} \times \mathbb{A}^1/\tilde{S}_I), \mathbb{Z}(u^r_{IJ} \times I)) \to (\mathbb{Z}(X_{r,\alpha,I}/\tilde{S}_I), \mathbb{Z}(u^r_{IJ}))))\end{aligned}$$

Proof. Standard. See Ayoub's thesis for example. □

- For $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$, we denote as usual (see [12] for example), $\mathbb{Z}^{tr}(X/S) \in \mathrm{PSh}(\mathrm{Var}(k)/S)$ the presheaf given by
  - for $X'/S \in \mathrm{Var}(k)/S$, with X' irreducible, $\mathbb{Z}^{tr}(X/S)(X'/S) := \mathcal{Z}^{fs/X}(X' \times_S X) \subset \mathcal{Z}_{d_{X'}}(X' \times_S X)$ which consist of algebraic cycles $\alpha = \sum_i n_i\alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times_S X)$ such that, denoting $\mathrm{supp}(\alpha) = \cup_i\alpha_i \subset X' \times_S X$ its support and $f' : X' \times_S X \to X'$ the projection, $f'_{|\mathrm{supp}(\alpha)} : \mathrm{supp}(\alpha) \to X'$ is finite surjective,
  - for $g : X_2/S \to X_1/S$ a morphism, with $X_1/S, X_2/S \in \mathrm{Var}(k)/S$,

    $$\mathbb{Z}^{tr}(X/S)(g) : \mathbb{Z}^{tr}(X/S)(X_1/S) \to \mathbb{Z}^{tr}(X/S)(X_2/S),\ \alpha \mapsto (g \times I)^{-1}(\alpha)$$

    with $g \times I : X_2 \times_S X \to X_1 \times_S X$, noting that, by base change, $f_{2|\mathrm{supp}((g\times I)^{-1}(\alpha))} : \mathrm{supp}((g \times I)^{-1}(\alpha)) \to X_2$ is finite surjective, $f_2 : X_2 \times_S X \to X_2$ being the projection.

- For $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$ and $r \in \mathbb{N}$, we denote as usual (see [12] for example), $\mathbb{Z}^{equir}(X/S) \in \mathrm{PSh}(\mathrm{Var}(k)/S)$ the presheaf given by
  - for $X'/S \in \mathrm{Var}(k)/S$, with X' irreducible, $\mathbb{Z}^{equir}(X/S)(X'/S) := \mathcal{Z}^{equir/X}(X' \times_S X) \subset \mathcal{Z}_{d_{X'}}(X' \times_S X)$ which consist of algebraic cycles $\alpha = \sum_i n_i\alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times_S X)$ such that, denoting $\mathrm{supp}(\alpha) = \cup_i\alpha_i \subset X' \times_S X$ its support and $f' : X' \times_S X \to X'$ the projection, $f'_{|\mathrm{supp}(\alpha)} : \mathrm{supp}(\alpha) \to X'$ is dominant, with fibers either empty or of dimension r,
  - for $g : X_2/S \to X_1/S$ a morphism, with $X_1/S, X_2/S \in \mathrm{Var}(k)/S$,

    $$\mathbb{Z}^{equir}(X/S)(g) : \mathbb{Z}^{equir}(X/S)(X_1/S) \to \mathbb{Z}^{equir}(X/S)(X_2/S),\ \alpha \mapsto (g \times I)^{-1}(\alpha)$$

    with $g \times I : X_2 \times_S X \to X_1 \times_S X$, noting that, by base change, $f_{2|\mathrm{supp}((g\times I)^{-1}(\alpha))} : \mathrm{supp}((g \times I)^{-1}(\alpha)) \to X_2$ is obviously dominant, with fibers either empty or of dimension r, $f_2 : X_2 \times_S X \to X_2$ being the projection.

- Let $S \in \mathrm{Var}(k)$. We denote by $\mathbb{Z}_S(d) := \mathbb{Z}^{equi0}(S \times \mathbb{A}^d/S)[-2d]$ the Tate twist. For $F \in C(\mathrm{Var}(k)/S)$, we denote by $F(d) := F \otimes \mathbb{Z}_S(d)$.

For $S \in \mathrm{Var}(k)$, let $\mathrm{Cor}(\mathrm{Var}(k)^{sm}/S)$ be the category

- whose objects are smooth morphisms $U/S = (U, h)$, $h : U \to S$ with $U \in \mathrm{Var}(k)$,
- whose morphisms $\alpha : U/S = (U, h_1) \to V/S = (V, h_2)$ is finite correspondence that is $\alpha \in \oplus_i\mathcal{Z}^{fs}(U_i \times_S V)$, where $U = \sqcup_i U_i$, with U_i connected (hence irreducible by smoothness), and $\mathcal{Z}^{fs}(U_i \times_S V)$ is the abelian group of cycle finite and surjective over U_i.

We denote by $\mathrm{Tr}(S) : \mathrm{Cor}(\mathrm{Var}(k)^{sm}/S) \to \mathrm{Var}(k)^{sm}/S$ the morphism of site given by the inclusion functor $\mathrm{Tr}(S) : \mathrm{Var}(k)^{sm}/S \hookrightarrow \mathrm{Cor}(\mathrm{Var}(k)^{sm}/S)$ It induces an adjonction

$$(\mathrm{Tr}(S)^* \, \mathrm{Tr}(S)_*) : C(\mathrm{Var}(k)^{sm}/S) \leftrightarrows C(\mathrm{Cor}(\mathrm{Var}(k)^{sm}/S))$$

A complex of preheaves $G \in C(\mathrm{Var}(k)^{sm}/S)$ is said to admit transferts if it is in the image of the embedding

$$\mathrm{Tr}(S)_* : C(\mathrm{Cor}(\mathrm{Var}(k)^{sm}/S) \hookrightarrow C(\mathrm{Var}(k)^{sm}/S),$$

that is $G = \mathrm{Tr}(S)_* \, \mathrm{Tr}(S)^* G$.

We will use to compute the algebraic Gauss-Manin realization functor the following

Theorem 16. *Let $\phi : F^\bullet \to G^\bullet$ an etale local equivalence with $F^\bullet, G^\bullet \in C(\mathrm{Var}(k)^{sm}/S)$. If $F^\bullet$ and $G^\bullet$ are $\mathbb{A}^1$ local and admit tranferts then $\phi : F^\bullet \to G^\bullet$ is a Zariski local equivalence. Hence if $F \in C(\mathrm{Var}(k)^{sm}/S)$ is $\mathbb{A}^1$ local and admits transfert*

$$k : E_{zar}(F) \to E_{et}(E_{zar}(F)) = E_{et}(F)$$

is a Zariski local equivalence.

Proof. See [12]. □

2.7 Presheaves on the big Zariski site or the big etale site of pairs

We recall the definition given in subsection 5.1 : For $S \in \mathrm{Var}(k)$, $\mathrm{Var}(k)^2/S := \mathrm{Var}(k)^2/(S,S)$ is by definition (see subsection 2.1) the category whose set of objects is

$$(\mathrm{Var}(k)^2/S)^0 := \{((X,Z),h), h : X \to S, \; Z \subset X \text{ closed } \} \subset \mathrm{Var}(k)/S \times \mathrm{Top}$$

and whose set of morphisms between $(X_1,Z_1)/S = ((X_1,Z_1),h_1), (X_1,Z_1)/S = ((X_2,Z_2),h_2) \in \mathrm{Var}(k)^2/S$ is the subset

$$\mathrm{Hom}_{\mathrm{Var}(k)^2/S}((X_1,Z_1)/S,(X_2,Z_2)/S) := \\ \left\{(f : X_2 \to X_2), \text{ s.t. } h_1 \circ f = h_2 \text{ and } Z_1 \subset f^{-1}(Z_2)\right\} \subset \mathrm{Hom}_{\mathrm{Var}(k)}(X_1,X_2)$$

The category $\mathrm{Var}(k)^2$ admits fiber products : $(X_1,Z_1) \times_{(S,Z)} (X_2,Z_2) = (X_1 \times_S X_2, Z_1 \times_Z Z_2)$. In particular, for $f : T \to S$ a morphism with $S,T \in \mathrm{Var}(k)$, we have the pullback functor

$$P(f) : \mathrm{Var}(k)^2/S \to \mathrm{Var}(k)^2/T, P(f)((X,Z)/S) := (X_T,Z_T)/T, P(f)(g) := (g \times_S f)$$

and we note again $P(f) : \mathrm{Var}(k)^2/T \to \mathrm{Var}(k)^2/S$ the corresponding morphism of sites.

We will consider in the construction of the filtered De Rham realization functor the full subcategory $\mathrm{Var}(k)^{2,sm}/S \subset \mathrm{Var}(k)^2/S$ such that the first factor is a smooth morphism : We will also consider, in order to obtain a complex of D modules in the construction of the filtered De Rham realization functor, the restriction to the full subcategory $\mathrm{Var}(k)^{2,pr}/S \subset \mathrm{Var}(k)^2/S$ such that the first factor is a projection :

Definition 18. *(i) Let $S \in \mathrm{Var}(k)$. We denote by*

$$\rho_S : \mathrm{Var}(k)^{2,sm}/S \hookrightarrow \mathrm{Var}(k)^2/S$$

the full subcategory consisting of the objects $(U,Z)/S = ((U,Z),h) \in \mathrm{Var}(k)^2/S$ such that the morphism $h : U \to S$ is smooth. That is, $\mathrm{Var}(k)^{2,sm}/S$ is the category

- *whose objects are $(U,Z)/S = ((U,Z),h)$, with $U \in \mathrm{Var}(k)$, $Z \subset U$ a closed subset, and $h : U \to S$ a smooth morphism,*

– *whose morphisms* $g : (U,Z)/S = ((U,Z),h_1) \to (U',Z')/S = ((U',Z'),h_2)$ *is a morphism* $g : U \to U'$ *of complex algebraic varieties such that* $Z \subset g^{-1}(Z')$ *and* $h_2 \circ g = h_1$.

We denote again $\rho_S : \mathrm{Var}(k)^2/S \to \mathrm{Var}(k)^{2,sm}/S$ *the associated morphism of site.We have*

$$r^s(S) : \mathrm{Var}(k)^2 \xrightarrow{r(S):=r(S,S)} \mathrm{Var}(k)^2/S \xrightarrow{\rho_S} \mathrm{Var}(k)^{2,sm}/S$$

the composite morphism of site.

(ii) Let $S \in \mathrm{Var}(k)$. *We will consider the full subcategory*

$$\mu_S : \mathrm{Var}(k)^{2,pr}/S \hookrightarrow \mathrm{Var}(k)^2/S$$

whose subset of object consist of those whose morphism is a projection to S *:*

$(\mathrm{Var}(k)^{2,pr}/S)^0 := \{((Y \times S, X), p),\ Y \in \mathrm{Var}(k),\ p : Y \times S \to S \textit{ the projection}\} \subset (\mathrm{Var}(k)^2/S)^0.$

(iii) We will consider the full subcategory

$$\mu_S : (\mathrm{Var}(k)^{2,smpr}/S) \hookrightarrow \mathrm{Var}(k)^{2,sm}/S$$

whose subset of object consist of those whose morphism is a smooth projection to S *:*

$(\mathrm{Var}(k)^{2,smpr}/S)^0 := \{((Y \times S, X), p),\ Y \in \mathrm{SmVar}(k),\ p : Y \times S \to S \textit{ the projection}\} \subset (\mathrm{Var}(k)^2/S)^0$

For $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(k)$, we have by definition, the following commutative diagram of sites

$$\begin{tikzcd}
\mathrm{Var}(k)^2/T \arrow[rr, "\mu_T"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"'] & & \mathrm{Var}(k)^{2,pr}/T \arrow[dr, "\rho_T"] \arrow[dd] & \\
 & \mathrm{Var}(k)^{2,sm}/T \arrow[rr, "\mu_T"] \arrow[dd] & & \mathrm{Var}(k)^{2,smpr}/T \arrow[dd, "P(f)"] \\
\mathrm{Var}(k)^2/S \arrow[rr, "\mu_S"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(k)^{2,pr}/S \arrow[dr, "\rho_S"] & \\
 & \mathrm{Var}(k)^{2,sm}/S \arrow[rr, "\mu_S"] & & \mathrm{Var}(k)^{2,smpr}/S
\end{tikzcd} \quad . \tag{3}$$

Recall we have (see subsection 2.1), for $S \in \mathrm{Var}(k)$, the graph functor

$$\mathrm{Gr}_S^{12} : \mathrm{Var}(k)/S \to \mathrm{Var}(k)^{2,pr}/S,\ X/S \mapsto \mathrm{Gr}_S^{12}(X/S) := (X \times S, X)/S,$$
$$(g : X/S \to X'/S) \mapsto \mathrm{Gr}_S^{12}(g) := (g \times I_S : (X \times S, X) \to (X' \times S, X'))$$

Note that Gr_S^{12} is fully faithfull. For $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(k)$, we have by definition, the following commutative diagram of sites

$$\begin{tikzcd}
\mathrm{Var}(k)^{2,pr}/T \arrow[rr, "\mathrm{Gr}_T^{12}"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"'] & & \mathrm{Var}(k)/T \arrow[dr, "\rho_T"] \arrow[dd] & \\
 & \mathrm{Var}(k)^{2,smpr}/T \arrow[rr, "\mathrm{Gr}_T^{12}"] \arrow[dd] & & \mathrm{Var}(k)^{sm}/T \arrow[dd, "P(f)"] \\
\mathrm{Var}(k)^{2,pr}/S \arrow[rr, "\mathrm{Gr}_S^{12}"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(k)/S \arrow[dr, "\rho_S"] & \\
 & \mathrm{Var}(k)^{2,sm}/S \arrow[rr, "\mathrm{Gr}_S^{12}"] & & \mathrm{Var}(k)^{sm}/S
\end{tikzcd} \quad . \tag{4}$$

where we recall that $P(f)((X,Z)/S) := ((X_T,Z_T)/T)$, since smooth morphisms are preserved by base change.

- As usual, we denote by
$$(f^*,f_*) := (P(f)^*,P(f)_*) : C(\mathrm{Var}(k)^{2,(sm)}/S) \to C(\mathrm{Var}(k)^{2,(sm)}/T)$$
the adjonction induced by $P(f) : \mathrm{Var}(k)^{2,(sm)}/T \to \mathrm{Var}(k)^{2,(sm)}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjonction
$$(f^*,f_*) : C_{fil}(\mathrm{Var}(k)^{2,(sm)}/S) \leftrightarrows C_{fil}(\mathrm{Var}(k)^{2,(sm)}/T),\ f^*(G,F) := (f^*G,f^*F)$$
For $S \in \mathrm{Var}(k)$, we denote by $\mathbb{Z}_S := \mathbb{Z}((S,S)/(S,S)) \in \mathrm{PSh}(\mathrm{Var}(k)^{2,(sm)}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\mathrm{Var}(k)^{2,(sm)}/S)$, $\mathcal{H}om(\mathbb{Z}_S,F) = F$.
- As usual, we denote by
$$(f^*,f_*) := (P(f)^*,P(f)_*) : C(\mathrm{Var}(k)^{2,(sm)pr}/S) \to C(\mathrm{Var}(k)^{2,(sm)pr}/T)$$
the adjonction induced by $P(f) : \mathrm{Var}(k)^{2,(sm)pr}/T \to \mathrm{Var}(k)^{2,(sm)pr}/S$. Since the colimits involved in the definition of $f^* = P(f)^*$ are filtered, f^* also preserve monomorphism. Hence, we get an adjonction
$$(f^*,f_*) : C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/S) \leftrightarrows C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/T),\ f^*(G,F) := (f^*G,f^*F)$$
For $S \in \mathrm{Var}(k)$, we denote by $\mathbb{Z}_S := \mathbb{Z}((S,S)/(S,S)) \in \mathrm{PSh}(\mathrm{Var}(k)^{2,sm}/S)$ the constant presheaf. By Yoneda lemma, we have for $F \in C(\mathrm{Var}(k)^{2,sm}/S)$, $\mathcal{H}om(\mathbb{Z}_S,F) = F$.
- For $h : U \to S$ a smooth morphism with $U,S \in \mathrm{Var}(k)$, $P(h) : \mathrm{Var}(k)^{2,sm}/S \to \mathrm{Var}(k)^{2,sm}/U$ admits a left adjoint
$$C(h) : \mathrm{Var}(k)^{2,sm}/U \to \mathrm{Var}(k)^{2,sm}/S,\ C(h)((U',Z'),h') = ((U',Z'),h \circ h').$$
Hence $h^* : C(\mathrm{Var}(k)^{2,sm}/S) \to C(\mathrm{Var}(k)^{2,sm}/U)$ admits a left adjoint
$$h_\sharp : C(\mathrm{Var}(k)^{2,sm}/U) \to C(\mathrm{Var}(k)^{2,sm}/S),$$
$$F \mapsto (h_\sharp F : ((U,Z),h_0) \mapsto \lim_{((U',Z'),h \circ h') \to ((U,Z),h_0)} F((U',Z')/U))$$
- For $h : X \to S$ a morphism with $X,S \in \mathrm{Var}(k)$, $P(h) : \mathrm{Var}(k)^2/S \to \mathrm{Var}(k)^2/X$ admits a left adjoint
$$C(h) : \mathrm{Var}(k)^2/X \to \mathrm{Var}(k)^2/S,\ C(h)((X',Z'),h') = ((X',Z'),h \circ h').$$
Hence $h^* : C(\mathrm{Var}(k)^2/S) \to C(\mathrm{Var}(k)^2/X)$ admits a left adjoint
$$h_\sharp : C(\mathrm{Var}(k)^2/X) \to C(\mathrm{Var}(k)^{2,sm}/S),$$
$$F \mapsto (h_\sharp F : ((X,Z),h_0) \mapsto \lim_{((X',Z'),h \circ h') \to ((X,Z),h_0)} F((X',Z')/X))$$
- For $p : Y \times S \to S$ a projection with $Y,S \in \mathrm{Var}(k)$ with Y smooth, $P(p) : \mathrm{Var}(k)^{2,smpr}/S \to \mathrm{Var}(k)^{2,smpr}/Y \times S$ admits a left adjoint
$$C(p) : \mathrm{Var}(k)^{2,smpr}/Y \times S \to \mathrm{Var}(k)^{2,smpr}/S,$$
$$C(p)((Y' \times S,Z'),p') = ((Y' \times S,Z'),p \circ p').$$
Hence $p^* : C(\mathrm{Var}(k)^{2,smpr}/S) \to C(\mathrm{Var}(k)^{2,smpr}/Y \times S)$ admits a left adjoint
$$p_\sharp : C(\mathrm{Var}(k)^{2,smpr}/Y \times S) \to C(\mathrm{Var}(k)^{2,smpr}/S),$$
$$F \mapsto (p_\sharp F : ((Y_0 \times S,Z),p_0) \mapsto \lim_{((Y' \times Y \times S,Z'),p \circ p') \to ((Y_0 \times S,Z),p_0)} F((Y' \times Y \times S,Z')/Y \times S))$$

- For $p : Y \times S \to S$ a projection with $Y, S \in \mathrm{Var}(k)$, $P(p) : \mathrm{Var}(k)^{2,pr}/S \to \mathrm{Var}(k)^{2,pr}/Y \times S$ admits a left adjoint

$$C(p) : \mathrm{Var}(k)^{2,pr}/Y \times S \to \mathrm{Var}(k)^{2,pr}/S,\ C(p)((Y' \times S, Z'), p') = ((Y' \times S, Z'), p \circ p').$$

Hence $p^* : C(\mathrm{Var}(k)^{2,pr}/S) \to C(\mathrm{Var}(k)^{2,pr}/Y \times S)$ admits a left adjoint

$$p_\sharp : C(\mathrm{Var}(k)^{2,pr}/Y \times S) \to C(\mathrm{Var}(k)^{2,pr}/S),$$
$$F \mapsto (p_\sharp F : ((Y_0 \times S, Z), p_0) \mapsto \lim_{((Y' \times Y \times S, Z'), p \circ p') \to ((Y_0 \times S, Z), p_0)} F((Y' \times Y \times S, Z')/Y \times S))$$

Let $S_\bullet \in \mathrm{Fun}(\mathcal{I}, \mathrm{Var}(k))$ with $\mathcal{I} \in \mathrm{Cat}$, a diagram of algebraic varieties. It gives the diagram of sites $\mathrm{Var}(k)^2/S_\bullet \in \mathrm{Fun}(\mathcal{I}, \mathrm{Cat})$.

- Then $C_{fil}(\mathrm{Var}(k)^{2,(sm)}/S_\bullet)$ is the category
  - whose objects $(G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\mathrm{Var}(k)^{2,(sm)}/S_I)$, and $u_{IJ} : (G_I, F) \to r_{IJ*}(G_J, F)$ for $r_{IJ} : I \to J$, denoting again $r_{IJ} : S_I \to S_J$, are morphisms satisfying for $I \to J \to K$, $r_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\mathrm{Var}(k)^{2,(sm)}/S_I)$,
  - the morphisms $m : ((G, F), u_{IJ}) \to ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,
  $$m = (m_I : (G_I, F) \to (H_I, F))_{I \in \mathcal{I}}$$
  such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{fil}(\mathrm{Var}(k)^{2,(sm)}/S_I)$.
- Then $C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/S_\bullet)$ is the category
  - whose objects $(G, F) = ((G_I, F)_{I \in \mathcal{I}}, u_{IJ})$, with $(G_I, F) \in C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/S_I)$, and $u_{IJ} : (G_I, F) \to r_{IJ*}(G_J, F)$ for $r_{IJ} : I \to J$, denoting again $r_{IJ} : S_I \to S_J$, are morphisms satisfying for $I \to J \to K$, $r_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{fil}(\mathrm{Var}(k)^{2,(sm)}/S_I)$,
  - the morphisms $m : ((G, F), u_{IJ}) \to ((H, F), v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,
  $$m = (m_I : (G_I, F) \to (H_I, F))_{I \in \mathcal{I}}$$
  such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/S_I)$.

For $s : \mathcal{I} \to \mathcal{J}$ a functor, with $\mathcal{I}, \mathcal{J} \in \mathrm{Cat}$, and $f_\bullet : T_\bullet \to S_{s(\bullet)}$ a morphism with $T_\bullet \in \mathrm{Fun}(\mathcal{J}, \mathrm{Var}(k))$ and $S_\bullet \in \mathrm{Fun}(\mathcal{I}, \mathrm{Var}(k))$, we have by definition, the following commutative diagrams of sites

$$\begin{tikzcd}
\mathrm{Var}(k)^2/T_\bullet \arrow[rr, "\mu_{T_\bullet}"] \arrow[dr, "\rho_{T_\bullet}"] \arrow[dd, "P(f_\bullet)"'] & & \mathrm{Var}(k)^{2,pr}/T_\bullet \arrow[dr, "\rho_{T_\bullet}"] \arrow[dd] & \\
& \mathrm{Var}(k)^{2,sm}/T_\bullet \arrow[rr, "\mu_{T_\bullet}", near start] \arrow[dd, "P(f_\bullet)", near end] & & \mathrm{Var}(k)^{2,smpr}/T_\bullet \arrow[dd, "P(f_\bullet)"] \\
\mathrm{Var}(k)^2/S_\bullet \arrow[rr, "\mu_{S_\bullet}", near end] \arrow[dr, "\rho_{S_\bullet}"] & & \mathrm{Var}(k)^{2,pr}/S_\bullet \arrow[dr, "\rho_{S_\bullet}"] & \\
& \mathrm{Var}(k)^{2,sm}/S_\bullet \arrow[rr, "\mu_{S_\bullet}"] & & \mathrm{Var}(k)^{2,smpr}/S_\bullet
\end{tikzcd} . \tag{5}$$

and

$$\begin{tikzcd}
\mathrm{Var}(k)^{2,pr}/T_\bullet \arrow[rr, "\mathrm{Gr}^{12}_{T_\bullet}"] \arrow[dr, "\rho_{T_\bullet}"] \arrow[dd, "P(f_\bullet)"'] & & \mathrm{Var}(k)/T \arrow[dr, "\rho_{T_\bullet}"] \arrow[dd, "P(f_\bullet)"] & \\
& \mathrm{Var}(k)^{2,smpr}/T_\bullet \arrow[rr, "\mathrm{Gr}^{12}_{T_\bullet}"] \arrow[dd, "P(f_\bullet)"] & & \mathrm{Var}(k)^{sm}/T_\bullet \arrow[dd, "P(f_\bullet)"] \\
\mathrm{Var}(k)^{2,pr}/S_\bullet \arrow[rr, "\mathrm{Gr}^{12}_{S_\bullet}"] \arrow[dr, "\rho_{S_\bullet}"] & & \mathrm{Var}(k)/S_\bullet \arrow[dr, "\rho_{S_\bullet}"] & \\
& \mathrm{Var}(k)^{2,sm}/S_\bullet \arrow[rr, "\mathrm{Gr}^{12}_{S_\bullet}"] & & \mathrm{Var}(k)^{sm}/S_\bullet
\end{tikzcd} . \tag{6}$$

Let $s : \mathcal{I} \to \mathcal{J}$ a functor, with $\mathcal{I}, \mathcal{J} \in \mathrm{Cat}$, and $f_\bullet : T_\bullet \to S_{s(\bullet)}$ a morphism with $T_\bullet \in \mathrm{Fun}(\mathcal{J}, \mathrm{Var}(k))$ and $S_\bullet \in \mathrm{Fun}(\mathcal{I}, \mathrm{Var}(k))$.

- As usual, we denote by

$$(f_\bullet^*, f_{\bullet*}) := (P(f_\bullet)^*, P(f_\bullet)_*) : C(\mathrm{Var}(k)^{2,(sm)}/S_\bullet) \to C(\mathrm{Var}(k)^{2,(sm)}/T_\bullet)$$

  the adjonction induced by $P(f_\bullet) : \mathrm{Var}(k)^{2,(sm)}/T_\bullet \to \mathrm{Var}(k)^{2,(sm)}/S_\bullet$. Since the colimits involved in the definition of $f_\bullet^* = P(f_\bullet)^*$ are filtered, $f_\bullet^*$ also preserve monomorphism. Hence, we get an adjonction

$$(f_\bullet^*, f_{\bullet*}) : C_{fil}(\mathrm{Var}(k)^{2,(sm)}/S_\bullet) \leftrightarrows C_{fil}(\mathrm{Var}(k)^{2,(sm)}/T_\bullet),$$
$$f_\bullet^*((G_I, F), u_{IJ}) := ((f_I^* G_I, f_I^* F), T(f_I, r_{IJ})(-) \circ f_I^* u_{IJ})$$

- As usual, we denote by

$$(f_\bullet^*, f_{\bullet*}) := (P(f_\bullet)^*, P(f_\bullet)_*) : C(\mathrm{Var}(k)^{2,(sm)pr}/S_\bullet) \to C(\mathrm{Var}(k)^{2,(sm)pr}/T_\bullet)$$

  the adjonction induced by $P(f_\bullet) : \mathrm{Var}(k)^{2,(sm)pr}/T_\bullet \to \mathrm{Var}(k)^{2,(sm)pr}/S_\bullet$. Since the colimits involved in the definition of $f_\bullet^* = P(f_\bullet)^*$ are filtered, $f_\bullet^*$ also preserve monomorphism. Hence, we get an adjonction

$$(f_\bullet^*, f_{\bullet*}) : C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/S_\bullet) \leftrightarrows C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/T_\bullet),$$
$$f_\bullet^*((G_I, F), u_{IJ}) := ((f_I^* G_I, f_I^* F), T(f_I, r_{IJ})(-) \circ f_I^* u_{IJ})$$

Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{Var}(k)$. For $I \subset [1, \cdots l]$, denote by $\tilde{S}_I = \Pi_{i \in I} \tilde{S}_i$. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I$ and for $J \subset I$ the following commutative diagram

$$D_{IJ} = \quad \begin{tikzcd}
S_I \arrow[r, "i_I"] & \tilde{S}_I \\
S_J \arrow[u, "j_{IJ}"] \arrow[r, "i_J"] & \tilde{S}_J \arrow[u, "p_{IJ}"']
\end{tikzcd}$$

where $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ is the projection and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. This gives the diagram of algebraic varieties $(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{Var}(k))$ which gives the diagram of sites $\mathrm{Var}(k)^2/(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{Cat})$. This gives also the diagram of algebraic varieties $(\tilde{S}_I)^{op} \in \mathrm{Fun}(\mathcal{P}(\mathbb{N})^{op}, \mathrm{Var}(k))$ which gives the diagram of sites $\mathrm{Var}(k)^2/(\tilde{S}_I)^{op} \in \mathrm{Fun}(\mathcal{P}(\mathbb{N})^{op}, \mathrm{Cat})$.

- Then $C_{fil}(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I))$ is the category

– whose objects $(G,F)=((G_I,F)_{I\subset[1,\cdots l]},u_{IJ})$, with $(G_I,F)\in C_{fil}(\mathrm{Var}(k)^{2,(sm)}/\tilde{S}_I)$, and $u_{IJ}:(G_I,F)\to p_{IJ*}(G_J,F)$ for $I\subset J$, are morphisms satisfying for $I\subset J\subset K$, $p_{IJ*}u_{JK}\circ u_{IJ}=u_{IK}$ in $C_{fil}(\mathrm{Var}(k)^{2,(sm)}/\tilde{S}_I)$,
– the morphisms $m:((G,F),u_{IJ})\to((H,F),v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m=(m_I:(G_I,F)\to(H_I,F))_{I\subset[1,\cdots l]}$$

such that $v_{IJ}\circ m_I=p_{IJ*}m_J\circ u_{IJ}$ in $C_{fil}(\mathrm{Var}(k)^{2,(sm)}/\tilde{S}_I)$.

- Then $C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I))$ is the category
  – whose objects $(G,F)=((G_I,F)_{I\subset[1,\cdots l]},u_{IJ})$, with $(G_I,F)\in C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/\tilde{S}_I)$, and $u_{IJ}:(G_I,F)\to p_{IJ*}(G_J,F)$ for $I\subset J$, are morphisms satisfying for $I\subset J\subset K$, $p_{IJ*}u_{JK}\circ u_{IJ}=u_{IK}$ in $C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/\tilde{S}_I)$,
  – the morphisms $m:((G,F),u_{IJ})\to((H,F),v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m=(m_I:(G_I,F)\to(H_I,F))_{I\subset[1,\cdots l]}$$

  such that $v_{IJ}\circ m_I=p_{IJ*}m_J\circ u_{IJ}$ in $C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/\tilde{S}_I)$.

- Then $C_{fil}(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I)^{op})$ is the category
  – whose objects $(G,F)=((G_I,F)_{I\subset[1,\cdots l]},u_{IJ})$, with $(G_I,F)\in C_{fil}(\mathrm{Var}(k)^{2,(sm)}/\tilde{S}_I)$, and $u_{IJ}:(G_J,F)\to p^*_{IJ}(G_I,F)$ for $I\subset J$, are morphisms satisfying for $I\subset J\subset K$, $p^*_{JK}u_{IJ}\circ u_{JK}=u_{IK}$ in $C_{fil}(\mathrm{Var}(k)^{2,(sm)}/\tilde{S}_K)$,
  – the morphisms $m:((G,F),u_{IJ})\to((H,F),v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m=(m_I:(G_I,F)\to(H_I,F))_{I\subset[1,\cdots l]}$$

  such that $v_{IJ}\circ m_J=p^*_{IJ}m_I\circ u_{IJ}$ in $C_{fil}(\mathrm{Var}(k)^{2,(sm)}/\tilde{S}_J)$.

- Then $C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I)^{op})$ is the category
  – whose objects $(G,F)=((G_I,F)_{I\subset[1,\cdots l]},u_{IJ})$, with $(G_I,F)\in C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/\tilde{S}_I)$, and $u_{IJ}:(G_J,F)\to p^*_{IJ}(G_I,F)$ for $I\subset J$, are morphisms satisfying for $I\subset J\subset K$, $p^*_{JK}u_{IJ}\circ u_{JK}=u_{IK}$ in $C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/\tilde{S}_K)$,
  – the morphisms $m:((G,F),u_{IJ})\to((H,F),v_{IJ})$ being (see section 2.1) a family of morphisms of complexes,

$$m=(m_I:(G_I,F)\to(H_I,F))_{I\subset[1,\cdots l]}$$

  such that $v_{IJ}\circ m_J=p^*_{IJ}m_I\circ u_{IJ}$ in $C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/\tilde{S}_J)$.

We now define the Zariski and the etale topology on $\mathrm{Var}(k)^2/S$.

Definition 19. *Let $S\in\mathrm{Var}(k)$.*

(i) Denote by τ a topology on $\mathrm{Var}(k)$, e.g. the Zariski or the etale topology. The τ covers in $\mathrm{Var}(k)^2/S$ of $(X,Z)/S$ are the families of morphisms

$$\{(c_i:(U_i,Z\times_X U_i)/S\to(X,Z)/S)_{i\in I},\ \textit{with}\ (c_i:U_i\to X)_{i\in I}\,\tau\ \textit{cover of}\,X\ \textit{in}\ \mathrm{Var}(k)\}$$

(ii) Denote by τ the Zariski or the etale topology on $\mathrm{Var}(k)$. The τ covers in $\mathrm{Var}(k)^{2,sm}/S$ of $(U,Z)/S$ are the families of morphisms

$$\{(c_i:(U_i,Z\times_U U_i)/S\to(U,Z)/S)_{i\in I},\ \textit{with}\ (c_i:U_i\to U)_{i\in I}\,\tau\ \textit{cover of}U\ \textit{in}\ \mathrm{Var}(k)\}$$

(iii) Denote by τ the Zariski or the etale topology on $\mathrm{Var}(k)$. The τ covers in $\mathrm{Var}(k)^{2,(sm)pr}/S$ of $(Y\times S,Z)/S$ are the families of morphisms

$$\{(c_i\times I_S:(U_i\times S,Z\times_{Y\times S}U_i\times S)/S\to(Y\times S,Z)/S)_{i\in I},\ \text{with } (c_i:U_i\to Y)_{i\in I}\ \tau \text{ cover of} Y \text{ in } \mathrm{Var}(k)\}$$

Will now define the $\mathbb{A}^1$ local property on $\mathrm{Var}(k)^2/S$.
Denote $\square^*:=\mathbb{P}^*_{\mathbb{C}}\backslash\{1\}$

- Let $S\in\mathrm{Var}(k)$. For $(X,Z)/S=((X,Z),h)\in\mathrm{Var}(k)^{2,(sm)}/S$, we consider
$$(\square^*\times X,\square^*\times Z)/S=((\square^*\times X,\square^*\times Z,h\circ p)\in\mathrm{Fun}(\Delta,\mathrm{Var}(k)^{2,(sm)}/S).$$
For $F\in C^-(\mathrm{Var}(k)^{2,(sm)}/S)$, it gives the complex
$$C_*F\in C^-(\mathrm{Var}(k)^{2,(sm)}/S),(X,Z)/S=((X,Z),h)\mapsto C_*F((X,Z)/S):=\mathrm{Tot}\,F((\square^*\times X,\square^*\times Z/S)$$
together with the canonical map $c_F:=(0,I_F):F\to C_*F$. For $F\in C(\mathrm{Var}(k)^{2,(sm)}/S)$, we get
$$C_*F:=\mathrm{holim}_n\,C_*F^{\leq n}\in C(\mathrm{Var}(k)^{2,(sm)}/S),$$
together with the canonical map $c_F:=(0,I_F):F\to C_*F$. For $m:F\to G$ a morphism, with $F,G\in C(\mathrm{Var}(k)^{2,(sm)}/S)$, we get by functoriality the morphism $C_*m:C_*F\to C_*G$.
- Let $S\in\mathrm{Var}(k)$. For $(Y\times S,Z)/S=((Y\times S,Z),h)\in\mathrm{Var}(k)^{2,(sm)pr}/S$, we consider
$$(\square^*\times Y\times S,\square^*\times Z)/S=(\square^*\times Y\times S,\square^*\times Z,h\circ p)\in\mathrm{Fun}(\Delta,\mathrm{Var}(k)/S).$$
For $F\in C^-(\mathrm{Var}(k)^{2,(sm)pr}/S)$, it gives the complex
$$C_*F\in C^-(\mathrm{Var}(k)^{2,(sm)pr}/S),$$
$$(Y\times S,Z)/S=((Y\times S,Z),h)\mapsto C_*F((Y\times S,Z)/S):=\mathrm{Tot}\,F(\square^*\times Y\times S,\square^*\times Z)/S)$$
together with the canonical map $c=c(F):=(0,I_F):F\to C_*F$. For $F\in C(\mathrm{Var}(k)^{2,(sm)pr}/S)$, we get
$$C_*F:=\mathrm{holim}_n\,C_*F^{\leq n}\in C(\mathrm{Var}(k)^{2,(sm)pr}/S),$$
together with the canonical map $c=c(F):=(0,I_F):F\to C_*F$. For $m:F\to G$ a morphism, with $F,G\in C(\mathrm{Var}(k)^{2,(sm)pr}/S)$, we get by functoriality the morphism $C_*m:C_*F\to C_*G$.
- Let $S\in\mathrm{Var}(k)$. Let $S=\cup_{i=1}^l S_i$ an open affine cover and denote by $S_I=\cap_{i\in I}S_i$. Let $i_i:S_i\hookrightarrow\tilde S_i$ closed embeddings, with $\tilde S_i\in\mathrm{Var}(k)$. For $F=(F_I,u_{IJ})\in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I))$, it gives the complex
$$C_*F=(C_*F_I,C_*u_{IJ})\in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I)),$$
together with the canonical map $c_F:=(0,I_F):F\to C_*F$.
- Let $S\in\mathrm{Var}(k)$. Let $S=\cup_{i=1}^l S_i$ an open affine cover and denote by $S_I=\cap_{i\in I}S_i$. Let $i_i:S_i\hookrightarrow\tilde S_i$ closed embeddings, with $\tilde S_i\in\mathrm{Var}(k)$. For $F=(F_I,u_{IJ})\in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I))$, it gives the complex
$$C_*F=(C_*F_I,C_*u_{IJ})\in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I)^{op}),$$
together with the canonical map $c_F:=(0,I_F):F\to C_*F$.
- Let $S\in\mathrm{Var}(k)$. Let $S=\cup_{i=1}^l S_i$ an open affine cover and denote by $S_I=\cap_{i\in I}S_i$. Let $i_i:S_i\hookrightarrow\tilde S_i$ closed embeddings, with $\tilde S_i\in\mathrm{Var}(k)$. For $F=(F_I,u_{IJ})\in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I))$, it gives the complex
$$C_*F=(C_*F_I,C_*u_{IJ})\in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I)),$$
together with the canonical map $c_F:=(0,I_F):F\to C_*F$.

- Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^{l} S_i$ an open affine cover and denote by $S_I = \cap_{i\in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{Var}(k)$. For $F = (F_I, u_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I))$, it gives the complex
$$C_*F = (C_*F_I, C_*u_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I)^{op}),$$
together with the canonical map $c_F := (0, I_F) : F \to C_*F$.

Let $S \in \mathrm{Var}(k)$. Denote for short $\mathrm{Var}(k)^{2,(sm)}/S$ either the category $\mathrm{Var}(k)^2/S$ or the category $\mathrm{Var}(k)^{2,sm}/S$. Denote by

$$\begin{aligned} p_a : \mathrm{Var}(k)^{2,(sm)}/S &\to \mathrm{Var}(k)^{2,(sm)}/S, \\ (X,Z)/S = ((X,Z),h) &\mapsto (X\times\mathbb{A}^1, Z\times\mathbb{A}^1)/S = ((X\times\mathbb{A}^1, Z\times\mathbb{A}^1, h\circ p_X), \\ (g:(X,Z)/S \to (X',Z')/S) &\mapsto ((g\times I_{\mathbb{A}^1}) : (X\times\mathbb{A}^1, Z\times\mathbb{A}^1)/S \to (X'\times\mathbb{A}^1, Z'\times\mathbb{A}^1)/S) \end{aligned}$$

the projection functor and again by $p_a : \mathrm{Var}(k)^{2,(sm)}/S \to \mathrm{Var}(k)^{2,(sm)}/S$ the corresponding morphism of site. Let $S \in \mathrm{Var}(k)$.Denote for short $\mathrm{Var}(k)^{2,(sm)}/S$ either the category $\mathrm{Var}(k)^2/S$ or the category $\mathrm{Var}(k)^{2,sm}/S$. Denote for short $\mathrm{Var}(k)^{2,(sm)pr}/S$ either the category $\mathrm{Var}(k)^{2,pr}/S$ or the category $\mathrm{Var}(k)^{2,smpr}/S$. Denote by

$$\begin{aligned} p_a : \mathrm{Var}(k)^{2,(sm)pr}/S &\to \mathrm{Var}(k)^{2,(sm)pr}/S, \\ (Y\times S,Z)/S = ((Y\times S,Z),p_S) &\mapsto (Y\times S\times\mathbb{A}^1, Z\times\mathbb{A}^1)/S = ((Y\times S\times\mathbb{A}^1, Z\times\mathbb{A}^1, p_S\circ p_{Y\times S}), \\ (g:(Y\times S,Z)/S \to (Y'\times S,Z')/S) &\mapsto ((g\times I_{\mathbb{A}^1}) : (Y\times S\times\mathbb{A}^1, Z\times\mathbb{A}^1)/S \to (Y'\times S\times\mathbb{A}^1, Z'\times\mathbb{A}^1)/S) \end{aligned}$$

the projection functor and again by $p_a : \mathrm{Var}(k)^{2,(sm)pr}/S \to \mathrm{Var}(k)^{2,(sm)pr}/S$ the corresponding morphism of site.

Definition 20. *(i0) A complex $F \in C(\mathrm{Var}(k)^{2,(sm)}/S)$ is said to be $\mathbb{A}^1$ homotopic if $\mathrm{ad}(p_a^*, p_{a*})(F) : F \to p_{a*}p_a^*F$ is an homotopy equivalence.*

(i0)' A complex $F \in C(\mathrm{Var}(k)^{2,(sm)pr}/S)$ is said to be $\mathbb{A}^1$ homotopic if $\mathrm{ad}(p_a^*, p_{a*})(F) : F \to p_{a*}p_a^*F$ is an homotopy equivalence.

(i) A complex $F \in C(\mathrm{Var}(k)^{2,(sm)}/S)$, is said to be $\mathbb{A}^1$ invariant if for all $(X,Z)/S \in \mathrm{Var}(k)^{2,(sm)}/S$

$$F(p_X) : F((X,Z)/S) \to F((X\times\mathbb{A}^1, (Z\times\mathbb{A}^1))/S)$$

is a quasi-isomorphism, where $p_X : (X\times\mathbb{A}^1, (Z\times\mathbb{A}^1)) \to (X,Z)$ is the projection. Obviously, if a complex $F \in C(\mathrm{Var}(k)^{2,(sm)}/S)$ is $\mathbb{A}^1$ homotopic, then it is $\mathbb{A}^1$ invariant.

(i)' A complex $G \in C(\mathrm{Var}(k)^{2,(sm)pr}/S)$, is said to be $\mathbb{A}^1$ invariant if for all $(Y\times S, Z)/S \in \mathrm{Var}(k)^{2,(sm)pr}/S$

$$G(p_{Y\times S}) : G((Y\times S,Z)/S) \to G((Y\times\mathbb{A}^1\times S, (Z\times\mathbb{A}^1))/S)$$

is a quasi-isomorphism of abelian group. Obviously, if a complex $F \in C(\mathrm{Var}(k)^{2,(sm)pr}/S)$ is $\mathbb{A}^1$ homotopic, then it is $\mathbb{A}^1$ invariant.

(ii) Let τ a topology on $\mathrm{Var}(k)$. A complex $F \in C(\mathrm{Var}(k)^{2,(sm)}/S)$ is said to be $\mathbb{A}^1$ local for the τ topology induced on $\mathrm{Var}(k)^2/S$, if for an (hence every) τ local equivalence $k : F \to G$ with k injective and $G \in C(\mathrm{Var}(k)^{2,(sm)}/S)$ τ fibrant, e.g. $k : F \to E_\tau(F)$, G is $\mathbb{A}^1$ invariant.

(ii)' Let τ a topology on $\mathrm{Var}(k)$. A complex $F \in C(\mathrm{Var}(k)^{2,(sm)pr}/S)$ is said to be $\mathbb{A}^1$ local for the τ topology induced on $\mathrm{Var}(k)^{2,pr}/S$, if for an (hence every) τ local equivalence $k : F \to G$ with k injective and $G \in C(\mathrm{Var}(k)^{2,(sm)pr}/S)$ τ fibrant, e.g. $k : F \to E_\tau(F)$, G is $\mathbb{A}^1$ invariant.

(iii) *A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(k)^{2,(sm)}/S)$ is said to an $(\mathbb{A}^1, et)$ local equivalence if for all $H \in C(\mathrm{Var}(k)^{2,(sm)}/S)$ which is $\mathbb{A}^1$ local for the etale topology*

$$\mathrm{Hom}(L(m), E_{et}(H)) : \mathrm{Hom}(L(G), E_{et}(H)) \to \mathrm{Hom}(L(F), E_{et}(H))$$

is a quasi-isomorphism.

(iii)' *A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(k)^{2,(sm)pr}/S)$ is said to an $(\mathbb{A}^1, et)$ local equivalence if for all $H \in C(\mathrm{Var}(k)^{2,(sm)pr}/S)$ which is $\mathbb{A}^1$ local for the etale topology*

$$\mathrm{Hom}(L(m), E_{et}(H)) : \mathrm{Hom}(L(G), E_{et}(H)) \to \mathrm{Hom}(L(F), E_{et}(H))$$

is a quasi-isomorphism.

Proposition 6. *(i) Let $S \in \mathrm{Var}(k)$. Then for $F \in C(\mathrm{Var}(k)^{2,(sm)}/S)$, C_*F is $\mathbb{A}^1$ local for the etale topology and $c(F) : F \to C_*F$ is an equivalence $(\mathbb{A}^1, et)$ local.*

(i)' Let $S \in \mathrm{Var}(k)$. Then for $F \in C(\mathrm{Var}(k)^{2,(sm)pr}/S)$, C_*F is $\mathbb{A}^1$ local for the etale topology and $c(F) : F \to C_*F$ is an equivalence $(\mathbb{A}^1, et)$ local.

(ii) A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(k)^{2,(sm)}/S)$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if $a_{et}H^nC_* \mathrm{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.

(ii)' A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(k)^{2,(sm)pr}/S)$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if $a_{et}H^nC_* \mathrm{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.

(iii) A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(k)^{2,(sm)}/S)$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if there exists

$$\{(X_{1,\alpha}, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \ldots, \{(X_{r,\alpha}, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \mathrm{Var}(k)^{2,(sm)}/S$$

such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(k)^{2,(sm)}/S))$

$$\begin{aligned}\mathrm{Cone}(m) \xrightarrow{\sim} \mathrm{Cone}(&\oplus_{\alpha\in\Lambda_1} \mathrm{Cone}(\mathbb{Z}((X_{1,\alpha} \times \mathbb{A}^1, Z_{1,\alpha} \times \mathbb{A}^1)/S) \to \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)) \\ &\to \cdots \to \oplus_{\alpha\in\Lambda_r} \mathrm{Cone}(\mathbb{Z}((X_{r,\alpha} \times \mathbb{A}^1, Z_{r,\alpha} \times \mathbb{A}^1)/S) \to \mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S)))\end{aligned}$$

(iii)' A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(k)^{2,(sm)pr}/S)$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if there exists

$$\{(Y_{1,\alpha} \times S, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \ldots, \{(Y_{r,\alpha} \times S, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \mathrm{Var}(k)^{2,(sm)pr}/S$$

such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(k)^{2,(sm)}/S))$

$$\begin{aligned}\mathrm{Cone}(m) \xrightarrow{\sim} \mathrm{Cone}(&\oplus_{\alpha\in\Lambda_1} \mathrm{Cone}(\mathbb{Z}((Y_{1,\alpha} \times \mathbb{A}^1 \times S, Z_{1,\alpha} \times \mathbb{A}^1)/S) \to \mathbb{Z}((Y_{1,\alpha} \times S, Z_{1,\alpha})/S)) \\ &\to \cdots \to \oplus_{\alpha\in\Lambda_r} \mathrm{Cone}(\mathbb{Z}((Y_{r,\alpha} \times \mathbb{A}^1 \times S, Z_{r,\alpha} \times \mathbb{A}^1)/S) \to \mathbb{Z}((Y_{r,\alpha} \times S, Z_{r,\alpha})/S)))\end{aligned}$$

Proof. Standard : see Ayoub's thesis section 4 for example. Indeed, for (iii), by definition, if $\mathrm{Cone}(m)$ is of the given form, then it is an equivalence $(\mathbb{A}^1, et)$ local, on the other hand if m is an equivalence $(\mathbb{A}^1, et)$ local, we consider the commutative diagram

$$\begin{tikzcd} F \arrow[r, "c(F)"] \arrow[d, "m"] & C_*F \arrow[d, "c_*m"] \\ G \arrow[r, "c(G)"] & C_*G \end{tikzcd}$$

to deduce that $\mathrm{Cone}(m)$ is of the given form. $\square$

Definition-Proposition 2. *Let $S \in \mathrm{Var}(k)$.*

(i) With the weak equivalence the $(\mathbb{A}^1, et)$ local equivalence and the fibration the epimorphism with $\mathbb{A}^1_S$ local and etale fibrant kernels gives a model structure on $C(\mathrm{Var}(k)^{2,(sm)}/S)$: the left bousfield localization of the projective model structure of $C(\mathrm{Var}(k)^{2,(sm)}/S)$. We call it the projective $(\mathbb{A}^1, et)$ model structure.

(ii) With the weak equivalence the $(\mathbb{A}^1, et)$ local equivalence and the fibration the epimorphism with $\mathbb{A}^1_S$ local and etale fibrant kernels gives a model structure on $C(\mathrm{Var}(k)^{2,(sm)pr}/S)$: the left bousfield localization of the projective model structure of $C(\mathrm{Var}(k)^{2,(sm)pr}/S)$. We call it the projective $(\mathbb{A}^1, et)$ model structure.

Proof. Similar to the proof of proposition 1. □

We have, similarly to the case of single varieties the following :

Proposition 7. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(k)$.*

(i) The adjonction $(g^*, g_*) : C(\mathrm{Var}(k)^{2,(sm)}/S) \leftrightarrows C(\mathrm{Var}(k)^{2,(sm)}/T)$ is a Quillen adjonction for the projective $(\mathbb{A}^1, et)$ model structure (see definition-proposition 2)

(i)' The functor $g^* : C(\mathrm{Var}(k)^{2,(sm)}/S) \to C(\mathrm{Var}(k)^{2,(sm)}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence.

(ii) The adjonction $(g^*, g_*) : C(\mathrm{Var}(k)^{2,(sm)pr}/S) \leftrightarrows C(\mathrm{Var}(k)^{2,(sm)pr}/T)$ is a Quillen adjonction for the projective $(\mathbb{A}^1, et)$ model structure (see definition-proposition 2)

(ii)' The functor $g^* : C(\mathrm{Var}(k)^{2,(sm)pr}/S) \to C(\mathrm{Var}(k)^{2,(sm)pr}/T)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence.

Proof. (i):Follows immediately from definition. (i)': Since the functor g^* preserve epimorphism and also monomorphism (the colimits involved being filetered), g^* sends quasi-isomorphism to quasi-isomorphism. Hence it preserve Zariski and etale local equivalence. The fact that it preserve $(\mathbb{A}^1, et)$ local equivalence then follows similarly to the single case by the fact that g_* preserve by definition $\mathbb{A}^1$ equivariant presheaves. (ii) and (ii)': Similar to (i) and (i)'. □

Proposition 8. *Let $S \in \mathrm{Var}(k)$.*

(i) The adjonction $(\rho_S^*, \rho_{S*}) : C(\mathrm{Var}(k)^{2,sm}/S) \leftrightarrows C(\mathrm{Var}(k)^2/S)$ is a Quillen adjonction for the $(\mathbb{A}^1, et)$ projective model structure.

(i)' The functor $\rho_{S*} : C(\mathrm{Var}(k)^2/S) \to C(\mathrm{Var}(k)^{2,sm}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence.

(ii) The adjonction $(\rho_S^*, \rho_{S*}) : C(\mathrm{Var}(k)^{2,smpr}/S) \leftrightarrows C(\mathrm{Var}(k)^{2,pr}/S)$ is a Quillen adjonction for the $(\mathbb{A}^1, et)$ projective model structure.

(ii)' The functor $\rho_{S*} : C(\mathrm{Var}(k)^{2,pr}/S) \to C(\mathrm{Var}(k)^{2,smpr}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence.

Proof. Similar to the proof of proposition 4. □

Proposition 9. *Let $S \in \mathrm{Var}(k)$.*

(i) The adjonction $(\mu_S^*, \mu_{S*}) : C(\mathrm{Var}(k)^{2,pr}/S) \leftrightarrows C(\mathrm{Var}(k)^2/S)$ is a Quillen adjonction for the $(\mathbb{A}^1, et)$ projective model structure.

(i)' The functor $\mu_{S*} : C(\mathrm{Var}(k)^2/S) \to C(\mathrm{Var}(k)^{2,pr}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence.

(ii) The adjonction $(\mu_S^*, \mu_{S*}) : C(\mathrm{Var}(k)^{2,smpr}/S) \leftrightarrows C(\mathrm{Var}(k)^{2,pr}/S)$ is a Quillen adjonction for the $(\mathbb{A}^1, et)$ projective model structure.

(ii)' The functor $\mu_{S*} : C(\mathrm{Var}(k)^{2,sm}/S) \to C(\mathrm{Var}(k)^{2,smpr}/S)$ sends quasi-isomorphism to quasi-isomorphism, sends equivalence Zariski local to equivalence Zariski local, and equivalence etale local to equivalence etale local, sends $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence.

Proof. Similar to the proof of proposition 4. Indeed, for (i)' or (ii)', if $m : F \to G$ with $F, G \in C(\mathrm{Var}(k)^{2,(sm)})$ is an equivalence $(\mathbb{A}^1, et)$ local then (see proposition 6), there exists

$$\{(X_{1,\alpha}, Z_{1,\alpha})/S, \alpha \in \Lambda_1\}, \ldots, \{(X_{r,\alpha}, Z_{r,\alpha})/S, \alpha \in \Lambda_r\} \subset \mathrm{Var}(k)^{2,(sm)}/S$$

such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(k)^{2,(sm)}/S))$

$$\begin{aligned}\mathrm{Cone}(m) &\xrightarrow{\sim} \mathrm{Cone}(\oplus_{\alpha\in\Lambda_1} \mathrm{Cone}(\mathbb{Z}((X_{1,\alpha} \times \mathbb{A}^1, Z_{1,\alpha} \times \mathbb{A}^1)/S) \to \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S))\\ &\to \cdots \to \oplus_{\alpha\in\Lambda_r} \mathrm{Cone}(\mathbb{Z}((X_{r,\alpha} \times \mathbb{A}^1, Z_{r,\alpha} \times \mathbb{A}^1)/S) \to \mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S)))\\ &\xrightarrow{\sim} \mathrm{Cone}(\mathrm{Cone}(\oplus_{\alpha\in\Lambda_1}\mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S) \otimes \mathbb{Z}(\mathbb{A}^1, \mathbb{A}^1)/S \to \oplus_{\alpha\in\Lambda_1}\mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S))\\ &\to \cdots \to \mathrm{Cone}(\oplus_{\alpha\in\Lambda_r}\mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S) \otimes \mathbb{Z}((\mathbb{A}^1, \mathbb{A}^1)/S) \to \oplus_{\alpha\in\Lambda_r}\mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S)),\end{aligned}$$

this gives in $\mathrm{Ho}_{et}(C(\mathrm{Var}(k)^{2,(sm)pr}/S))$

$$\begin{aligned}&\mathrm{Cone}(\mu_{S*}m) \xrightarrow{\sim} \mathrm{Cone}(\\ &\mathrm{Cone}((L\mu_{S*} \oplus_{\alpha\in\Lambda_1} \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S)) \otimes \mathbb{Z}((\mathbb{A}^1, \mathbb{A}^1)/S) \to (L\mu_{S*} \oplus_{\alpha\in\Lambda_1} \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S))\\ &\to \cdots \to \mathrm{Cone}((L\mu_{S*} \oplus_{\alpha\in\Lambda_r} \mathbb{Z}((X_{r,\alpha}, Z_{r,\alpha})/S)) \otimes \mathbb{Z}((\mathbb{A}^1, \mathbb{A}^1)/S) \to (L\mu_{S*} \oplus_{\alpha\in\Lambda_r} \mathbb{Z}((X_{1,\alpha}, Z_{1,\alpha})/S))))\end{aligned}$$

hence $\mu_{S*}m : \mu_{S*}F \to \mu_{S*}G$ is an equivalence $(\mathbb{A}^1, et)$ local. □

We also have

Proposition 10. *Let $S \in \mathrm{Var}(k)$.*

(i) The adjonction $(\mathrm{Gr}_S^{12*}, \mathrm{Gr}_{S*}^{12}) : C(\mathrm{Var}(k)/S) \leftrightarrows C(\mathrm{Var}(k)^{2,pr}/S)$ is a Quillen adjonction for the $(\mathbb{A}^1, et)$ projective model structure.

(ii) The adjonction $(\mathrm{Gr}_S^{12*}\,\mathrm{Gr}_{S*}^{12} : C(\mathrm{Var}(k)^{sm}/S) \leftrightarrows C(\mathrm{Var}(k)^{2,smpr}/S)$ is a Quillen adjonction for the $(\mathbb{A}^1, et)$ projective model structure.

Proof. Immediate from definition. □

- For $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$ and $Z \subset X$ a closed subset, we denote $\mathbb{Z}^{tr}((X, Z)/S) \in \mathrm{PSh}(\mathrm{Var}(k)^2/S)$ the presheaf given by
  - for $(X', Z')/S \in \mathrm{Var}(k)^2/S$, with X' irreducible,

    $$\mathbb{Z}^{tr}((X, Z)/S)((X', Z')/S) := \left\{\alpha \in \mathcal{Z}^{fs/X}(X' \times_S X), s.t.p_X(p_{X'}^{-1}(Z')) \subset Z\right\} \subset \mathcal{Z}_{d_{X'}}(X' \times_S X)$$

  - for $g : (X_2, Z_2)/S \to (X_1, Z_1)/S$ a morphism, with $(X_1, Z_1)/S, (X_2, Z_2)/S \in \mathrm{Var}(k)^2/S$,

    $$\mathbb{Z}^{tr}((X, Z)/S)(g) : \mathbb{Z}^{tr}((X, Z)/S)((X_1, Z_1)/S) \to \mathbb{Z}^{tr}((X, Z)/S)((X_2, Z_2)/S),\ \alpha \mapsto (g \times I)^{-1}(\alpha)$$

    with $g \times I : X_2 \times_S X \to X_1 \times_S X$.

- For $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$, $Z \subset X$ a closed subset and $r \in \mathbb{N}$, we denote $\mathbb{Z}^{equir}((X,Z)/S) \in \mathrm{PSh}(\mathrm{Var}(k)^2/S)$ the presheaf given by
  - for $(X',Z')/S \in \mathrm{Var}(k)^2/S$, with X' irreducible,
  $$\mathbb{Z}^{equir}((X,Z)/S)((X',Z')/S) := \left\{\alpha \in \mathcal{Z}^{equir/X}(X' \times_S X), s.t.p_X(p_{X'}^{-1}(Z'))\right\} \subset \mathcal{Z}_{d_{X'}}(X' \times_S X)$$
  - for $g : (X_2,Z_2)/S \to (X_1,Z_1)/S$ a morphism, with $(X_1,Z_1)/S, (X_2,Z_2)/S \in \mathrm{Var}(k)^2/S$,
  $$\mathbb{Z}^{equir}((X,Z)/S)(g) : \mathbb{Z}^{equir}((X,Z)/S)((X_1,Z_1)/S) \to \mathbb{Z}^{equir}((X,Z)/S)((X_2,Z_2)/S), \; \alpha \mapsto (g \times I)^{-1}(\alpha)$$
  with $g \times I : X_2 \times_S X \to X_1 \times_S X$.
- Let $S \in \mathrm{Var}(k)$. We denote by $\mathbb{Z}_S(d) := \mathbb{Z}^{equi0}((S \times \mathbb{A}^d, S \times \mathbb{A}^d)/S)[-2d]$ the Tate twist. For $F \in C(\mathrm{Var}(k)^2/S)$, we denote by $F(d) := F \otimes \mathbb{Z}_S(d)$.

For $S \in \mathrm{Var}(k)$, let $\mathrm{Cor}(\mathrm{Var}(k)^{2,(sm)}/S)$ be the category

- whose objects are those of $\mathrm{Var}(k)^{2,(sm)}/S$, i.e. $(X,Z)/S = ((X,Z),h)$, $h : X \to S$ with $X \in \mathrm{Var}(k)$, $Z \subset X$ a closed subset,
- whose morphisms $\alpha : (X',Z)/S = ((X',Z),h_1) \to (X,Z)/S = ((X,Z),h_2)$ is finite correspondence that is $\alpha \in \oplus_i \mathbb{Z}^{tr}((X_i,Z)/S)((X',Z')/S)$, where $X' = \sqcup_i X'_i$, with X'_i connected, the composition being defined in the same way as the morphism $\mathrm{Cor}(\mathrm{Var}(k)^{(sm)}/S)$.

We denote by $\mathrm{Tr}(S) : \mathrm{Cor}(\mathrm{Var}(k)^{2,(sm)}/S) \to \mathrm{Var}(k)^{2,(sm)}/S$ the morphism of site given by the inclusion functor $\mathrm{Tr}(S) : \mathrm{Var}(k)^{2,(sm)}/S \hookrightarrow \mathrm{Cor}(\mathrm{Var}(k)^{2,(sm)}/S)$ It induces an adjonction
$$(\mathrm{Tr}(S)^* \mathrm{Tr}(S)_*) : C(\mathrm{Var}(k)^{2,(sm)}/S) \leftrightarrows C(\mathrm{Cor}(\mathrm{Var}(k)^{2,(sm)}/S))$$
A complex of preheaves $G \in C(\mathrm{Var}(k)^{2,(sm)}/S)$ is said to admit transferts if it is in the image of the embedding
$$\mathrm{Tr}(S)_* : C(\mathrm{Cor}(\mathrm{Var}(k)^{2,(sm)}/S) \hookrightarrow C(\mathrm{Var}(k)^{2,(sm)}/S),$$
that is $G = \mathrm{Tr}(S)_* \mathrm{Tr}(S)^* G$. We then have the full subcategory $\mathrm{Cor}(\mathrm{Var}(k)^{2,(sm)pr}/S) \subset \mathrm{Cor}(\mathrm{Var}(k)^{2,(sm)}/S)$ consisting of the objects of $\mathrm{Var}(k)^{2,(sm)pr}/S)$. We have the adjonction
$$(\mathrm{Tr}(S)^* \mathrm{Tr}(S)_*) : C(\mathrm{Var}(k)^{2,(sm)pr}/S) \leftrightarrows C(\mathrm{Cor}(\mathrm{Var}(k)^{2,(sm)pr}/S))$$
A complex of preheaves $G \in C(\mathrm{Var}(k)^{2,(sm)pr}/S)$ is said to admit transferts if it is in the image of the embedding
$$\mathrm{Tr}(S)_* : C(\mathrm{Cor}(\mathrm{Var}(k)^{2,(sm)pr}/S) \hookrightarrow C(\mathrm{Var}(k)^{2,(sm)pr}/S),$$
that is $G = \mathrm{Tr}(S)_* \mathrm{Tr}(S)^* G$.

Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^l S_i$ an open affine cover and denote by $S_I = \cap_{i \in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{Var}(k)$.

- For $(G_I, K_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I)^{op})$ and $(H_I, T_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I))$, we denote
  $$\mathcal{H}om((G_I,K_{IJ}),(H_I,T_{IJ})) := (\mathcal{H}om(G_I,H_I), u_{IJ}((G_I,K_{IJ}),(H_I,T_{IJ}))) \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I))$$
  with
  $$\begin{aligned} u_{IJ}((G_I,K_{IJ})(H_I,T_{IJ})) : \mathcal{H}om(G_I,H_I) \\ \xrightarrow{\mathrm{ad}(p_{IJ}^*,p_{IJ*})(-)} p_{IJ*}p_{IJ}^*\mathcal{H}om(G_I,H_I) \xrightarrow{T(p_{IJ},hom)(-,-)} p_{IJ*}\mathcal{H}om(p_{IJ}^*G_I, p_{IJ}^*H_I) \\ \xrightarrow{\mathcal{H}om(p_{IJ}^*G_I,T_{IJ})} p_{IJ*}\mathcal{H}om(p_{IJ}^*G_I,H_J) \xrightarrow{\mathcal{H}om(K_{IJ},H_J)} p_{IJ*}\mathcal{H}om(G_J,H_J). \end{aligned}$$
  This gives in particular the functor
  $$\mathbb{D}^{12}_{(\tilde{S}_I)} : C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I)^{op}) \to C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I)),$$
  $$(H_I,T_{IJ}) \mapsto \mathbb{D}^{12}_{(\tilde{S}_I)} L(H_I,T_{IJ}) := \mathcal{H}om((LH_I,T^q_{IJ}),(E_{et}\mathbb{Z}_{\tilde{S}_I},I_{IJ})) = (\mathbb{D}^{12}_{\tilde{S}_I} LH_I, T^d_{IJ})$$

- For $(G_I,K_{IJ})\in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I)^{op})$ and $(H_I,T_{IJ})\in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I))$, we denote

  $$\mathcal{H}om((G_I,K_{IJ}),(H_I,T_{IJ})):=(\mathcal{H}om(G_I,H_I),u_{IJ}((G_I,K_{IJ}),(H_I,T_{IJ})))\in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I))$$

  with

  $$\begin{aligned}u_{IJ}((G_I,K_{IJ})(H_I,T_{IJ})):\mathcal{H}om(G_I,H_I)\\ \xrightarrow{\mathrm{ad}(p_{IJ}^*,p_{IJ*})(-)} p_{IJ*}p_{IJ}^*\mathcal{H}om(G_I,H_I)\xrightarrow{T(p_{IJ},hom)(-,-)} p_{IJ*}\mathcal{H}om(p_{IJ}^*G_I,p_{IJ}^*H_I)\\ \xrightarrow{\mathcal{H}om(p_{IJ}^*G_I,T_{IJ})} p_{IJ*}\mathcal{H}om(p_{IJ}^*G_I,H_J)\xrightarrow{\mathcal{H}om(K_{IJ},H_J)} p_{IJ*}\mathcal{H}om(G_J,H_J).\end{aligned}$$

  This gives in particular the functor

  $$\mathbb{D}^{12}_{(\tilde S_I)}:C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I)^{op})\to C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I)),(H_I,T_{IJ})\mapsto \mathbb{D}^{12}_{(\tilde S_I)}L(H_I,T_{IJ})$$

The functors p_a naturally extend to functors

$$\begin{aligned}p_a:\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I)\to\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I),\\ ((X,Z)/\tilde S_I,u_{IJ})\mapsto((X\times\mathbb{A}^1,Z\times\mathbb{A}^1)/\tilde S_I,u_{IJ}\times I),\\ (g:((X,Z)/\tilde S_I,u_{IJ})\to((X',Z')/\tilde S_I,u_{IJ}))\mapsto\\ ((g\times I_{\mathbb{A}^1}):((X\times\mathbb{A}^1,Z\times\mathbb{A}^1)/\tilde S_I,u_{IJ}\times I)\to((X'\times\mathbb{A}^1,Z'\times\mathbb{A}^1)/\tilde S_I,u_{IJ}\times I))\end{aligned}$$

the projection functor and again by $p_a:\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I)\to\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I)$ the corresponding morphism of site, and

$$\begin{aligned}p_a:\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I)\to\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I),\\ ((Y\times\tilde S_I,Z)/\tilde S_I,u_{IJ})\mapsto((Y\times\tilde S_I\times\mathbb{A}^1,Z\times\mathbb{A}^1)/\tilde S_I,u_{IJ}\times I),\\ (g:((Y\times\tilde S_I,Z)/\tilde S_I,u_{IJ})\to((Y'\times\tilde S_I,Z')/\tilde S_I,u_{IJ}))\mapsto\\ ((g\times I_{\mathbb{A}^1}):((Y\times\tilde S_I\times\mathbb{A}^1,Z\times\mathbb{A}^1)/\tilde S_I,u_{IJ}\times I),((Y'\times\tilde S_I\times\mathbb{A}^1,Z'\times\mathbb{A}^1)/\tilde S_I,u_{IJ}\times I)),\end{aligned}$$

the projection functor and again by $p_a:\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I)\to\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I)$ the corresponding morphism of site. These functors also gives the morphisms of sites $p_a:\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I)^{op}\to\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I)^{op}$ and $p_a:\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I)^{op}\to\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I)^{op}$.

Definition 21. *Let $S\in\mathrm{Var}(k)$. Let $S=\cup_{i=1}^l S_i$ an open affine cover and denote by $S_I=\cap_{i\in I}S_i$. Let $i_i:S_i\hookrightarrow\tilde S_i$ closed embeddings, with $\tilde S_i\in\mathrm{Var}(k)$.*

- *(i0) A complex $(F_I,u_{IJ})\in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I))$ is said to be $\mathbb{A}^1$ homotopic if $\mathrm{ad}(p_a^*,p_{a*})((F_I,u_{IJ})):(F_I,u_{IJ})\to p_{a*}p_a^*(F_I,u_{IJ})$ is an homotopy equivalence.*
- *(i0)' A complex $(F_I,u_{IJ})\in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde S_I))$ is said to be $\mathbb{A}^1$ homotopic if $\mathrm{ad}(p_a^*,p_{a*})((F_I,u_{IJ})):(F_I,u_{IJ})\to p_{a*}p_a^*(F_I,u_{IJ})$ is an homotopy equivalence.*
- *(i) A complex $(F_I,u_{IJ})\in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I))$ is said to be $\mathbb{A}^1$ invariant if for all $((X_I,Z_I)/\tilde S_I,s_{IJ})\in\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I)$*

  $$(F_I(p_{X_I})):(F_I((X_I,Z_I)/\tilde S_I),F_J(s_{IJ})\circ u_{IJ}(-)\to(F_I((X_I\times\mathbb{A}^1,(Z_I\times\mathbb{A}^1))/\tilde S_I),F_J(s_{IJ}\times I)\circ u_{IJ}(-))$$

  is a quasi-isomorphism, where $p_{X_I}:(X_I\times\mathbb{A}^1,(Z_I\times\mathbb{A}^1))\to(X_I,Z_I)$ are the projection, and $s_{IJ}:(X_I\times\tilde S_{J\setminus I},Z_I)/\tilde S_J\to(X_J,Z_J)/\tilde S_J$. Obviously a complex $(F_I,u_{IJ})\in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde S_I))$ is $\mathbb{A}^1$ invariant if and only if all the F_I are $\mathbb{A}^1$ invariant.

(i)' *A complex* $(G_I, u_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I))$ *is said to be* $\mathbb{A}^1$ *invariant if for all* $((Y\times\tilde{S}_I, Z_I)/\tilde{S}_I, s_{IJ}) \in \mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I)$

$$(G_I(p_{Y\times\tilde{S}_I})) : (G_I((Y \times \tilde{S}_I, Z_I)/\tilde{S}_I), G_J(s_{IJ}) \circ u_{IJ}(-)) \to (G_I((Y \times \tilde{S}_I \times \mathbb{A}^1, (Z_I \times \mathbb{A}^1))/\tilde{S}_I), G_J(s_{IJ} \times I) \circ u_{IJ}(-))$$

is a quasi-isomorphism. Obviously a complex $(G_I, u_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I))$ *is* $\mathbb{A}^1$ *invariant if and only if all the* G_I *are* $\mathbb{A}^1$ *invariant.*

(ii) *Let* τ *a topology on* $\mathrm{Var}(k)$*. A complex* $F = (F_I, u_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I))$ *is said to be* $\mathbb{A}^1$ *local for the* τ *topology induced on* $\mathrm{Var}(k)^2/(\tilde{S}_I)$*, if for an (hence every)* τ *local equivalence* $k : F \to G$ *with* k *injective and* $G = (G_I, v_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I))$ τ *fibrant, e.g.* $k : (F_I, u_{IJ}) \to (E_\tau(F_I), E(u_{IJ}))$*,* G *is* $\mathbb{A}^1$ *invariant.*

(ii)' *Let* τ *a topology on* $\mathrm{Var}(k)$*. A complex* $F = (F_I, u_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I))$ *is said to be* $\mathbb{A}^1$ *local for the* τ *topology induced on* $\mathrm{Var}(k)^2/(\tilde{S}_I)$*, if for an (hence every)* τ *local equivalence* $k : F \to G$ *with* k *injective and* $G = (G_I, u_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I))$ τ *fibrant, e.g.* $k : (F_I, u_{IJ}) \to (E_\tau(F_I), E(u_{IJ}))$*,* G *is* $\mathbb{A}^1$ *invariant.*

(iii) *A morphism* $m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ})$ *with* $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I))$ *is said to be an* $(\mathbb{A}^1, et)$ *local equivalence if for all* $H = (H_I, w_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I))$ *which is* $\mathbb{A}^1$ *local for the etale topology*

$$(\mathrm{Hom}(L(m_I), E_{et}(H_I))) : \mathrm{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \to \mathrm{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

is a quasi-isomorphism (of complexes of abelian groups). Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ})$ *with* $(F_I, u_{IJ}), (G_I, u_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I))$ *is an* $(\mathbb{A}^1, et)$ *local equivalence, then all the* $m_I : F_I \to G_I$ *are* $(\mathbb{A}^1, et)$ *local equivalence.*

(iii)' *A morphism* $m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ})$ *with* $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I))$ *is said to be an* $(\mathbb{A}^1, et)$ *local equivalence if for all* $(H_I, w_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I))$ *which is* $\mathbb{A}^1$ *local for the etale topology*

$$(\mathrm{Hom}(L(m_I), E_{et}(H_I))) : \mathrm{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \to \mathrm{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

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(iv) *A morphism* $m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ})$ *with* $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I)^{op})$ *is said to be an* $(\mathbb{A}^1, et)$ *local equivalence if for all* $H = (H_I, w_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I)^{op})$ *which is* $\mathbb{A}^1$ *local for the etale topology*

$$(\mathrm{Hom}(L(m_I), E_{et}(H_I))) : \mathrm{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \to \mathrm{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

is a quasi-isomorphism (of complexes of abelian groups). Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ})$ *with* $(F_I, u_{IJ}), (G_I, u_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I)^{op})$ *is an* $(\mathbb{A}^1, et)$ *local equivalence, then all the* $m_I : F_I \to G_I$ *are* $(\mathbb{A}^1, et)$ *local equivalence and for all* $H = (H_I, w_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I))$ *which is* $\mathbb{A}^1$ *local for the etale topology*

$$(\mathrm{Hom}(L(m_I), E_{et}(H_I))) : \mathrm{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \to \mathrm{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

is a quasi-isomorphism (of diagrams of complexes of abelian groups) *(iv)' A morphism $m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, v_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I)^{op})$ is said to be an $(\mathbb{A}^1, et)$ local equivalence if for all $(H_I, w_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I)^{op})$ which is $\mathbb{A}^1$ local for the etale topology*

$$(\mathrm{Hom}(L(m_I), E_{et}(H_I))) : \mathrm{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \to \mathrm{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

is a quasi-isomorphism (of complexes of abelian groups). Obviously, if a morphism $m = (m_I) : (F_I, u_{IJ}) \to (G_I, v_{IJ})$ with $(F_I, u_{IJ}), (G_I, u_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I)^{op})$ is an $(\mathbb{A}^1, et)$ local equivalence, then all the $m_I : F_I \to G_I$ are $(\mathbb{A}^1, et)$ local equivalence and for all $(H_I, w_{IJ}) \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I))$ which is $\mathbb{A}^1$ local for the etale topology

$$(\mathrm{Hom}(L(m_I), E_{et}(H_I))) : \mathrm{Hom}(L(G_I, v_{IJ}), E_{et}(H_I, w_{IJ})) \to \mathrm{Hom}(L(F_I, u_{IJ}), E_{et}(H_I, w_{IJ}))$$

is a quasi-isomorphism (of diagrams of complexes of abelian groups).

Proposition 11. *Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^{l} S_i$ an open affine cover and denote by $S_I = \cap_{i\in I} S_i$. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{Var}(k)$.*

(i) Then for $F \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I)^{op})$, C_*F is $\mathbb{A}^1$ local for the etale topology and $c(F) : F \to C_*F$ is an equivalence $(\mathbb{A}^1, et)$ local.

(i)' Then for $F \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I)^{op})$, C_*F is $\mathbb{A}^1$ local for the etale topology and $c(F) : F \to C_*F$ is an equivalence $(\mathbb{A}^1, et)$ local.

(ii) A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I)^{op})$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if $a_{et}H^n C_* \mathrm{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.

(ii)' A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I)^{op})$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if $a_{et}H^n C_* \mathrm{Cone}(m) = 0$ for all $n \in \mathbb{Z}$.

(iii) A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I)^{op})$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if there exists

$$\left\{((X_{1,\alpha,I}, Z_{1,\alpha,I})/\tilde{S}_I, u^1_{IJ}), \alpha \in \Lambda_1\right\}, \ldots, \left\{((X_{r,\alpha,I}, Z_{r,\alpha,I})/\tilde{S}_I, u^r_{IJ}), \alpha \in \Lambda_r\right\} \subset \mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I)^{op}$$

with

$$u^l_{IJ} : (X_{l,\alpha,J}, Z_{l,\alpha,J})/\tilde{S}_J \to (X_{l,\alpha,I} \times \tilde{S}_{J\setminus I}, Z_{l,\alpha,I} \times \tilde{S}_{J\setminus I})/\tilde{S}_J$$

such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I)^{op}))$

$$\begin{aligned} &\mathrm{Cone}(m) \xrightarrow{\sim} \mathrm{Cone}(\\ &\oplus_{\alpha\in\Lambda_1} \mathrm{Cone}((\mathbb{Z}((X_{1,\alpha,I} \times \mathbb{A}^1, Z_{1,\alpha,I} \times \mathbb{A}^1)/\tilde{S}_I), \mathbb{Z}(u^1_{IJ} \times I)) \to (\mathbb{Z}((X_{1,\alpha,I}, Z_{1,\alpha,I})/\tilde{S}_I), \mathbb{Z}(u^1_{IJ}))) \\ &\to \cdots \to \\ &\oplus_{\alpha\in\Lambda_r} \mathrm{Cone}((\mathbb{Z}((X_{r,\alpha,I} \times \mathbb{A}^1, Z_{r,\alpha,I} \times \mathbb{A}^1)/\tilde{S}_I), \mathbb{Z}(u^r_{IJ} \times I)) \to (\mathbb{Z}((X_{r,\alpha,I}, Z_{r,\alpha,I})/\tilde{S}_I), \mathbb{Z}(u^r_{IJ})))) \end{aligned}$$

(iii)' A morphism $m : F \to G$ with $F, G \in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I)^{op})$ is an $(\mathbb{A}^1, et)$ local equivalence if and only if there exists

$$\left\{((Y_{1,\alpha,I} \times \tilde{S}_I, Z_{1,\alpha,I})/\tilde{S}_I, u^1_{IJ}), \alpha \in \Lambda_1\right\}, \ldots, \left\{((Y_{r,\alpha,I} \times \tilde{S}_I, Z_{r,\alpha,I})/\tilde{S}_I, u^r_{IJ}), \alpha \in \Lambda_r\right\} \subset \mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I)$$

with

$$u^l_{IJ} : (Y_{l,\alpha,J} \times \tilde{S}_J, Z_{l,\alpha,J})/\tilde{S}_J \to (Y_{l,\alpha,I} \times \tilde{S}_J, Z_{l,\alpha,I} \times \tilde{S}_{J\setminus I})/\tilde{S}_J$$

such that we have in $\mathrm{Ho}_{et}(C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I)^{op}))$

$$\mathrm{Cone}(m) \xrightarrow{\sim} \mathrm{Cone}(\oplus_{\alpha\in\Lambda_1}$$
$$\mathrm{Cone}((\mathbb{Z}((Y_{1,\alpha,I}\times\mathbb{A}^1\times\tilde{S}_I, Z_{1,\alpha,I}\times\mathbb{A}^1)/\tilde{S}_I),\mathbb{Z}(u^1_{IJ}\times I))\to(\mathbb{Z}((Y_{1,\alpha,I}\times S, Z_{1,\alpha,I})/\tilde{S}_I),\mathbb{Z}(u_{IJ})))$$
$$\to\cdots\to\oplus_{\alpha\in\Lambda_r}$$
$$\mathrm{Cone}((\mathbb{Z}((Y_{r,\alpha,I}\times\mathbb{A}^1\times\tilde{S}_I, Z_{r,\alpha,I}\times\mathbb{A}^1)/\tilde{S}_I),\mathbb{Z}(u^r_{IJ}\times I))\to(\mathbb{Z}((Y_{r,\alpha,I}\times\tilde{S}_I, Z_{r,\alpha})/\tilde{S}_I),\mathbb{Z}(u^r_{IJ})))$$

(iv) A similar statement then (iii) holds for equivalence $(\mathbb{A}^1, et)$ *local* $m : F\to G$ *with* $F,G\in C(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I))$

(iv)' A similar statement then (iii) holds for equivalence $(\mathbb{A}^1, et)$ *local* $m : F\to G$ *with* $F,G\in C(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I))$

Proof. Similar to the proof of proposition 6. See Ayoub's thesis for example. □

In the filtered case we also consider :

Definition 22. *Let* $S\in\mathrm{Var}(k)$. *Let* $S=\cup_{i=1}^l S_i$ *an open affine cover and denote by* $S_I=\cap_{i\in I}S_i$. *Let* $i_i : S_i\hookrightarrow\tilde{S}_i$ *closed embeddings, with* $\tilde{S}_i\in\mathrm{SmVar}(k)$.

(i) A filtered complex $(G,F)\in C_{fil}(\mathrm{Var}(k)^{2,(sm)}/S)$ *is said to be r-filtered* $\mathbb{A}^1$ *homotopic if* $\mathrm{ad}(p_a^*,p_{a*})(G,F) : (G,F)\to p_{a*}p_a^*(G,F)$ *is an r-filtered homotopy equivalence.*

(i)' A filtered complex $(G,F)\in C_{fil}(\mathrm{Var}(k)^{2,(sm)}/(\tilde{S}_I))$ *is said to be r-filtered* $\mathbb{A}^1$ *homotopic if* $\mathrm{ad}(p_a^*,p_{a*})(G,F) : (G,F)\to p_{a*}p_a^*(G,F)$ *is an r-filtered homotopy equivalence.*

(ii) A filtered complex $(G,F)\in C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/S)$ *is said to be r-filtered* $\mathbb{A}^1$ *homotopic if* $\mathrm{ad}(p_a^*,p_{a*})(G,F) : (G,F)\to p_{a*}p_a^*(G,F)$ *is an r-filtered homotopy equivalence.*

(ii)' A filtered complex $(G,F)\in C_{fil}(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I))$ *is said to be r-filtered* $\mathbb{A}^1$ *homotopic if* $\mathrm{ad}(p_a^*,p_{a*})(G,F) : (G,F)\to p_{a*}p_a^*(G,F)$ *is an r-filtered homotopy equivalence.*

We will use to compute the algebraic De Rahm realization functor the followings

Theorem 17. *Let* $S\in\mathrm{Var}(k)$.

(i) Let $\phi : F^\bullet\to G^\bullet$ *an etale local equivalence with* $F^\bullet,G^\bullet\in C(\mathrm{Var}(k)^{2,sm}/S)$. *If* $F^\bullet$ *and* $G^\bullet$ *are* $\mathbb{A}^1$ *local and admit tranferts then* $\phi : F^\bullet\to G^\bullet$ *is a Zariski local equivalence. Hence if* $F\in C(\mathrm{Var}(k)^{2,sm}/S)$ *is* $\mathbb{A}^1$ *local and admits transfert*

$$k : E_{zar}(F)\to E_{et}(E_{zar}(F)) = E_{et}(F)$$

is a Zariski local equivalence.

(ii) Let $\phi : F^\bullet\to G^\bullet$ *an etale local equivalence with* $F^\bullet,G^\bullet\in C(\mathrm{Var}(k)^{2,smpr}/S)$. *If* $F^\bullet$ *and* $G^\bullet$ *are* $\mathbb{A}^1$ *local and admit tranferts then* $\phi : F^\bullet\to G^\bullet$ *is a Zariski local equivalence. Hence if* $F\in C(\mathrm{Var}(k)^{2,smpr}/S)$ *is* $\mathbb{A}^1$ *local and admits transfert*

$$k : E_{zar}(F)\to E_{et}(E_{zar}(F)) = E_{et}(F)$$

is a Zariski local equivalence.

Proof. Similar to the proof of theorem 16. □

Theorem 18. *Let* $S\in\mathrm{Var}(k)$. *Let* $S=\cup_{i=1}^l S_i$ *an open affine cover and denote by* $S_I=\cap_{i\in I}S_i$. *Let* $i_i : S_i\hookrightarrow\tilde{S}_i$ *closed embeddings, with* $\tilde{S}_i\in\mathrm{Var}(k)$.

(i) *Let $\phi : (F^{\bullet}, F) \to (G^{\bullet}, F)$ a filtered etale local equivalence with $(F^{\bullet}, F), (G^{\bullet}, F) \in C_{fil}(\mathrm{Var}(k)^{2,sm}/S)$. If $(F^{\bullet}, F)$ and $(G^{\bullet}, F)$ are r-filtered $\mathbb{A}^1$ homotopic and admit tranferts then $\phi : (F^{\bullet}, F) \to (G^{\bullet}, F)$ is an r-filtered Zariski local equivalence. Hence if $(G, F) \in C(\mathrm{Var}(k)^{2,sm}/S)$ is r-filtered $\mathbb{A}^1$ homotopic and admits transfert*

$$k : E_{zar}(G, F) \to E_{et}(E_{zar}(G, F)) = E_{et}(G, F)$$

is an r-filtered Zariski local equivalence.

(i)' *Let $\phi : (F^{\bullet}, F) \to (G^{\bullet}, F)$ a filtered etale local equivalence with $((F^{\bullet}_I, F), u_{IJ}), ((G^{\bullet}_I, F), v_{IJ}) \in C_{fil}(\mathrm{Var}(k)^{2,sm}/(\tilde{S}_I))$. If $((F^{\bullet}, F), u_{IJ})$ and $((G^{\bullet}, F), v_{IJ})$ are r-filtered $\mathbb{A}^1$ homotopic and admit tranferts then $\phi : ((F^{\bullet}, F), u_{IJ}) \to ((G^{\bullet}, F), v_{IJ})$ is an r-filtered Zariski local equivalence. Hence if $((G_I, F), u_{IJ}) \in C(\mathrm{Var}(k)^{2,sm}/S)$ is r-filtered $\mathbb{A}^1$ homotopic and admits transfert*

$$k : (E_{zar}(G_I, F), u_{IJ}) \to (E_{et}(E_{zar}(G_I, F)), u_{IJ}) = (E_{et}(G, F), u_{IJ})$$

is an r-filtered Zariski local equivalence.

(ii) *Let $\phi : (F^{\bullet}, F) \to (G^{\bullet}, F)$ a filtered etale local equivalence with $(F^{\bullet}, F), (G^{\bullet}, F) \in C_{fil}(\mathrm{Var}(k)^{2,smpr}/S)$. If $F^{\bullet}$ and $G^{\bullet}$ are r-filtered $\mathbb{A}^1$ homotopic and admit tranferts then $\phi : (F^{\bullet}, F) \to (G^{\bullet}, F)$ is an r-filtered Zariski local equivalence. Hence if $(G, F) \in C(\mathrm{Var}(k)^{2,smpr}/S)$ is r-filtered $\mathbb{A}^1$ homotopic and admits transfert*

$$k : E_{zar}(F) \to E_{et}(E_{zar}(F)) = E_{et}(F)$$

is an r-filtered Zariski local equivalence.

(ii)' *Let $\phi : (F^{\bullet}, F) \to (G^{\bullet}, F)$ a filtered etale local equivalence with $((F^{\bullet}_I, F), u_{IJ}), ((G^{\bullet}_I, F), v_{IJ}) \in C_{fil}(\mathrm{Var}(k)^{2,smpr}/(\tilde{S}_I))$. If $((F^{\bullet}, F), u_{IJ})$ and $((G^{\bullet}, F), v_{IJ})$ are r-filtered $\mathbb{A}^1$ homotopic and admit tranferts then $\phi : ((F^{\bullet}, F), u_{IJ}) \to ((G^{\bullet}, F), v_{IJ})$ is an r-filtered Zariski local equivalence. Hence if $((G_I, F), u_{IJ}) \in C(\mathrm{Var}(k)^{2,smpr}/S)$ is r-filtered $\mathbb{A}^1$ homotopic and admits transfert*

$$k : (E_{zar}(G_I, F), u_{IJ}) \to (E_{et}(E_{zar}(G_I, F)), u_{IJ}) = (E_{et}(G, F), u_{IJ})$$

is an r-filtered Zariski local equivalence.

Proof. Similar to the proof of theorem 17. □

2.8 The Borel-Moore Corti-Hanamura resolution functors R^{CH}, $\hat{R}^{CH}$, R^{0CH}, and $\hat{R}^{0CH}$

Let k a field of caracteristic zero.

Definition 23. (i) *Let $X_0 \in \mathrm{Var}(k)$ and $Z \subset X_0$ a closed subset. A desingularization of (X_0, Z) is a pair of complex varieties $(X, D) \in \mathrm{Var}^2(k))$, together with a morphism of pair of varieties $\epsilon : (X, D) \to (X_0, \Delta)$ with $Z \subset \Delta$ such that*

- *$X \in \mathrm{SmVar}(k)$ and $D := \epsilon^{-1}(\Delta) = \epsilon^{-1}(Z) \cup (\cup_i E_i) \subset X$ is a normal crossing divisor*
- *$\epsilon : X \to X_0$ is a proper modification with discriminant Δ, that is $\epsilon : X \to X_0$ is proper and $\epsilon : X \backslash D \xrightarrow{\sim} X \backslash \Delta$ is an isomorphism.*

(ii) *Let $X_0 \in \mathrm{Var}(k)$ and $Z \subset X_0$ a closed subset such that $X_0 \backslash Z$ is smooth. A strict desingularization of (X_0, Z) is a pair of complex varieties $(X, D) \in \mathrm{Var}^2(\mathbb{C}))$, together with a morphism of pair of varieties $\epsilon : (X, D) \to (X_0, Z)$ such that*

- *$X \in \mathrm{SmVar}(k)$ and $D := \epsilon^{-1}(Z) \subset X$ is a normal crossing divisor*
- *$\epsilon : X \to X_0$ is a proper modification with discriminant Z, that is $\epsilon : X \to X_0$ is proper and $\epsilon : X \backslash D \xrightarrow{\sim} X \backslash Z$ is an isomorphism.*

We have the following well known resolution of singularities of complex algebraic varieties and their functorialities :

Theorem 19. *(i) Let $X_0 \in \mathrm{Var}(k)$ and $Z \subset X_0$ a closed subset. There exists a desingularization of (X_0, Z), that is a pair of complex varieties $(X, D) \in \mathrm{Var}^2(k))$, together with a morphism of pair of varieties $\epsilon : (X, D) \to (X_0, \Delta)$ with $Z \subset \Delta$ such that*

- *$X \in \mathrm{SmVar}(k)$ and $D := \epsilon^{-1}(\Delta) = \epsilon^{-1}(Z) \cup (\cup_i E_i) \subset X$ is a normal crossing divisor*
- *$\epsilon : X \to X_0$ is a proper modification with discriminant Δ, that is $\epsilon : X \to X_0$ is proper and $\epsilon : X \backslash D \xrightarrow{\sim} X \backslash \Delta$ is an isomorphism.*

(ii) Let $X_0 \in \mathrm{PVar}(k)$ and $Z \subset X_0$ a closed subset such that $X_0 \backslash Z$ is smooth. There exists a strict desingularization of (X_0, Z), that is a pair of complex varieties $(X, D) \in \mathrm{PVar}^2(k))$, together with a morphism of pair of varieties $\epsilon : (X, D) \to (X_0, Z)$ such that

- *$X \in \mathrm{PSmVar}(k)$ and $D := \epsilon^{-1}(Z) \subset X$ is a normal crossing divisor*
- *$\epsilon : X \to X_0$ is a proper modification with discriminant Z, that is $\epsilon : X \to X_0$ is proper and $\epsilon : X \backslash D \xrightarrow{\sim} X \backslash Z$ is an isomorphism.*

Proof. (i):Standard. See [23] for example.
(ii):Follows immediately from (i). □

We use this theorem to construct a resolution of a morphism by Corti-Hanamura morphisms, we will need these resolution in the definition of the filtered De Rham realization functor :

Definition-Proposition 3. *(i) Let $h : V \to S$ a morphism, with $V, S \in \mathrm{Var}(k)$. Let $\bar{S} \in \mathrm{PVar}(k)$ be a compactification of S.*

- *There exist a compactification $\bar{X}_0 \in \mathrm{PVar}(\mathbb{C})$ of V such that $h : V \to S$ extend to a morphism $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$. Denote by $\bar{Z} = \bar{X}_0 \backslash V$. We denote by $j : V \hookrightarrow \bar{X}_0$ the open embedding and by $i_0 : \bar{Z} \hookrightarrow \bar{X}_0$ the complementary closed embedding. We then consider $X_0 := \bar{f}_0^{-1}(S) \subset \bar{X}_0$ the open subset, $f_0 := \bar{f}_{0|X_0} : X_0 \to S$, $Z = \bar{Z} \cap X_0$, and we denote again $j : V \hookrightarrow X_0$ the open embedding and by $i_0 : Z \hookrightarrow X_0$ the complementary closed embedding.*
- *In the case V is smooth, we take, using theorem 19(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$, with $\bar{X} \in \mathrm{PSmVar}(\mathbb{C})$ and $\bar{D} = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. We denote by $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X} = \bar{X}_{c(\bullet)}$ the morphism of simplicial varieties given by the closed embeddings $i_I : \bar{D}_I = \cap_{i \in I} \bar{D}_i \hookrightarrow \bar{X}$. Then the morphisms $\bar{f} := \bar{f}_0 \circ \bar{\epsilon} : \bar{X} \to \bar{S}$ and $\bar{f}_{D_\bullet} := \bar{f} \circ i_\bullet : \bar{D}_\bullet \to \bar{S}$ are projective since $\bar{X}$ and $\bar{D}_I$ are projective varieties. We then consider $(X, D) := \bar{\epsilon}^{-1}(X_0, Z)$, $\epsilon := \bar{\epsilon}_{|X} : (X, D) \to (X_0, Z)$ We denote again by $i_\bullet : D_\bullet \hookrightarrow X = X_{c(\bullet)}$ the morphism of simplicial varieties given by the closed embeddings $i_I : D_I = \cap_{i \in I} D_i \hookrightarrow X$. Then the morphisms $f := f_0 \circ \epsilon : X \to S$ and $f_{D_\bullet} := f \circ i_\bullet : D_\bullet \to S$ are projective since $\bar{f} : \bar{X}_0 \to \bar{S}$ is projective.*

(ii) Let $g : V'/S \to V/S$ a morphism, with $V'/S = (V', h'), V/S = (V, h) \in \mathrm{Var}(k)/S$

- *Take (see (i)) a compactification $\bar{X}_0 \in \mathrm{PVar}(\mathbb{C})$ of V such that $h : V \to S$ extend to a morphism $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$. Denote by $\bar{Z} = \bar{X}_0 \backslash V$. Then, there exist a compactification $\bar{X}'_0 \in \mathrm{PVar}(\mathbb{C})$ of V' such that $h' : V' \to S$ extend to a morphism $\bar{f}'_0 = \bar{h}'_0 : \bar{X}'_0 \to \bar{S}$, $g : V' \to V$ extend to a morphism $\bar{g}_0 : \bar{X}'_0 \to \bar{X}_0$ and $\bar{f}_0 \circ \bar{g}_0 = \bar{f}'_0$ that is $\bar{g}_0$ is gives a morphism $\bar{g}_0 : \bar{X}'_0/\bar{S} \to \bar{X}_0/\bar{S}$. Denote by $\bar{Z}' = \bar{X}'_0 \backslash V'$. We then have the following commutative diagram*

$$\begin{tikzcd}
V \arrow[r, "j"] & \bar{X}_0 & & \bar{Z} \arrow[ll, "i"'] \\
V' \arrow[u, "\bar{g}"] \arrow[r, "j'"] & \bar{X}'_0 \arrow[u, "\bar{g}_0"] & \bar{Z}' \arrow[l, "i'"'] & \bar{g}_0{}^{-1}(\bar{Z}) : i'_{g,0} \arrow[l, "i''_{g,0}"'] \arrow[u, "\bar{g}'"]
\end{tikzcd}$$

It gives the following commutative diagram

$$\begin{tikzcd} V \arrow[r, "j"] & X_0 := \bar{f}_0^{-1}(S) & & Z \arrow[ll, "i"'] \\ V' \arrow[u, "g"] \arrow[r, "j'"] & X'_0 := \bar{f}_0'^{-1}(S) \arrow[u, "\bar{g}_0"] & Z' \arrow[l, "i'"'] & \bar{g}_0^{-1}(Z) : i'_{g,0} \arrow[l, "i''_{g,0}"'] \arrow[u, "\bar{g}'_0"'] \end{tikzcd}$$

– *In the case V and V' are smooth, we take using theorem 19 a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of $(\bar{X}_0, \bar{Z})$. Then there exist a strict desingularization $\bar{\epsilon}'_{\bullet} : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ of $(\bar{X}'_0, \bar{Z}')$ and a morphism $\bar{g} : \bar{X}' \to \bar{X}$ such that the following diagram commutes*

$$\begin{tikzcd} \bar{X}'_0 \arrow[r, "\bar{g}_0"] & \bar{X}_0 . \\ \bar{X}' \arrow[u, "\bar{\epsilon}'"] \arrow[r, "\bar{g}"] & \bar{X} \arrow[u, "\bar{\epsilon}"'] \end{tikzcd}$$

We then have the following commutative diagram in $\mathrm{Fun}(\Delta, \mathrm{Var}(k))$

$$\begin{tikzcd} V = V_{c(\bullet)} \arrow[r, "j"] & \bar{X} = \bar{X}_{c(\bullet)} & & \bar{D}_{s_g(\bullet)} \arrow[ll, "i_\bullet"'] \\ V' = V'_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j'"] & \bar{X}' = \bar{X}'_{c(\bullet)} \arrow[u, "\bar{g}"] & \bar{D}'_{\bullet} \arrow[l, "i'_\bullet"'] & \bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g\bullet}"'] \arrow[u, "\bar{g}'_\bullet"'] \end{tikzcd}$$

where $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X}_\bullet$ the morphism of simplicial varieties given by the closed embeddings $i_n : \bar{D}_n \hookrightarrow \bar{X}_n$, and $i'_\bullet : \bar{D}'_\bullet \hookrightarrow \bar{X}'_\bullet$ the morphism of simplicial varieties given by the closed embeddings $i'_n : \bar{D}'_n \hookrightarrow \bar{X}'_n$. It gives the commutative diagram in $\mathrm{Fun}(\Delta, \mathrm{Var}(k))$

$$\begin{tikzcd} V = V_{c(\bullet)} \arrow[r, "j"] & X := \bar{\epsilon}^{-1}(X_0) = X_{c(\bullet)} & & D_{s_g(\bullet)} \arrow[ll, "i_\bullet"'] \\ V' = V'_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j'"] & X' := \bar{\epsilon}'^{,-1}(X'_0) = X'_{c(\bullet)} \arrow[u, "\bar{g}"] & D'_{\bullet} \arrow[l, "i'_\bullet"'] & \bar{g}^{-1}(D_{s_g(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g\bullet}"'] \arrow[u, "\bar{g}'_\bullet"'] \end{tikzcd}$$

Proof. (i): Let $\bar{X}_{00} \in \mathrm{PVar}(\mathbb{C})$ be a compactification of V. Let $l_0 : \bar{X}_0 = \bar{\Gamma}_h \hookrightarrow \bar{X}_{00} \times \bar{S}$ be the closure of the graph of h and $\bar{f}_0 := p_{\bar{S}} \circ l_0 : \bar{X}_0 \hookrightarrow \bar{X}_{00} \times \bar{S} \to \bar{S}$, $\epsilon_{\bar{X}_0} := p_{\bar{X}_{00}} \circ l_0 : \bar{X}_0 \hookrightarrow \bar{X}_{00} \times \bar{S} \to \bar{X}_{00}$ be the restriction to $\bar{X}_0$ of the projections. Then, $\bar{X} \in \mathrm{PVar}(\mathbb{C})$, $\epsilon_{\bar{X}_0} : \bar{X}_0 \to \bar{X}_{00}$ is a proper modification which does not affect the open subset $V \subset \bar{X}_0$, and $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ is a compactification of h.
(ii): There is two things to prove:

- Let $\bar{f}_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : V \to S$ and $\bar{f}'_{00} : \bar{X}'_{00} \to \bar{S}$ a compactification of $h' : V' \to S$ (see (i)). Let $l_0 : \bar{X}'_0 \hookrightarrow \bar{\Gamma}_g \subset \bar{X}'_{00} \times_{\bar{S}} \bar{X}_0$ be the closure of the graph of g, $\bar{f}'_0 := (\bar{f}'_{00}, \bar{f}_0) \circ l_0 : \bar{X}'_0 \hookrightarrow \bar{X}'_{00} \times_S \bar{X}_0 \to \bar{S}$ and $\bar{g}_0 := p_{\bar{X}_0} \circ l_0 : \bar{X}'_0 \hookrightarrow \bar{X}'_{00} \times_{\bar{S}} \bar{X}_0 \to \bar{X}_0$, $\epsilon_{\bar{X}'_{00}} := p_{\bar{X}'_0} \circ i : \bar{X}'_0 \hookrightarrow \bar{X}'_{00} \times_{\bar{S}} \bar{X}_0 \to \bar{X}'_{00}$ be the restriction to X of the projections. Then $\epsilon_{\bar{X}'_{00}} : \bar{X}'_0 \to \bar{X}'_{00}$ is a proper modification which does not affect the open subset $V' \subset \bar{X}'_0$, $\bar{f}'_0 : \bar{X}'_0 \to \bar{S}$ is an other compactification of $h' : V' \to S$ and $\bar{g}_0 : \bar{X}'_0 \to \bar{X}_0$ is a compactification of g.

- In the case V and V' are smooth, we take, using theorem 19, a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$. Take then, using theorem 19, a strict desingularization $\bar{\epsilon}'_1 : (\bar{X}', \bar{D}') \to (\bar{X} \times_{\bar{X}_0} \bar{X}'_0, \bar{X} \times_{\bar{X}_0} \bar{Z}')$ of the pair $(\bar{X} \times_{\bar{X}_0} \bar{X}'_0, \bar{X} \times_{\bar{X}_0} \bar{Z}')$. We consider then following commutative diagram whose square is cartesian :

$$\begin{tikzcd} & & \bar{X}'_0 \arrow[r, "\bar{g}_0"] & X_0 \\ & & \bar{X}\times_{\bar{X}_0}\bar{X}'_0 \arrow[u, "\bar{\epsilon}'_0"] \arrow[r, "\bar{g}'_0"] & \bar{X} \arrow[u, "\epsilon"'] \\ \bar{X}' \arrow[urru, "\epsilon'"] \arrow[urr, "\bar{\epsilon}'_1"] \arrow[urrr, "\bar{g}"'] \end{tikzcd}$$

and $\bar{\epsilon}' := \bar{\epsilon}'_0 \circ \bar{\epsilon}'_1 : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ is a strict desingularization of the pair $(\bar{X} \times_{\bar{X}_0} \bar{X}'_0, \bar{X} \times_{\bar{X}_0} \bar{Z}')$.

□

Let $S \in \mathrm{Var}(k)$. Recall we have the dual functor

$$\mathbb{D}^0_S : C(\mathrm{Var}(k)/S) \to C(\mathrm{Var}(k)/S),\ F \mapsto \mathbb{D}^0_S(F) := \mathcal{H}om(F, E_{et}(\mathbb{Z}(S/S)))$$

which induces the functor

$$L\mathbb{D}_S : D_\tau(\mathrm{Var}(k)/S) \to D_\tau(\mathrm{Var}(k)/S),\ F \mapsto L\mathbb{D}_S(F) := \mathbb{D}^0_S(LF) := \mathcal{H}om(LF, E_{et}(\mathbb{Z}(S/S)))$$

with τ a topology on $\mathrm{Var}(k)$.

We will use the following resolutions of representable presheaves by Corti-Hanamura presheaves and their the functorialities.

Definition 24. *(i) Let $h : U \to S$ a morphism, with $U, S \in \mathrm{Var}(k)$ and U smooth. Take, see definition-proposition 3, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $\bar{Z} = \bar{X}_0 \backslash U$. Take, using theorem 19(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$, with $\bar{X} \in \mathrm{PSmVar}(k)$ and $\bar{D} := \epsilon^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. We denote by $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X} = \bar{X}_{c(\bullet)}$ the morphism of simplicial varieties given by the closed embeddings $i_I : \bar{D}_I = \cap_{i\in I}\bar{D}_i \hookrightarrow \bar{X}$ We denote by $j : U \hookrightarrow \bar{X}$ the open embedding and by $p_S : \bar{X} \times S \to S$ and $p_S : U \times S \to S$ the projections. Considering the graph factorization $\bar{f} : \bar{X} \xrightarrow{\bar{l}} \bar{X} \times \bar{S} \xrightarrow{p_{\bar{S}}} \bar{S}$ of $\bar{f} : \bar{X} \to \bar{S}$, where $\bar{l}$ is the graph embedding and $p_{\bar{S}}$ the projection, we get closed embeddings $l := \bar{l} \times_{\bar{S}} S : X \hookrightarrow \bar{X} \times S$ and $l_{D_I} := \bar{D}_I \times_{\bar{X}} l : D_I \hookrightarrow \bar{D}_I \times S$. We then consider the following map in $C(\mathrm{Var}(k)^2/S)$*

$$\begin{aligned} r_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) &: R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \\ &\xrightarrow{:=} p_{S*}E_{et}(\mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), u_{IJ}) \to \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S))) \\ &\xrightarrow{p_{S*}E_{et}(0,k\circ\mathrm{ad}((j\times I)^*,(j\times I)_*)(\mathbb{Z}((\bar{X}\times S,X)/\bar{X}\times S)))} p_{S*}E_{et}(\mathbb{Z}((U \times S, U)/U \times S)) =: \mathbb{D}^{12}_S(\mathbb{Z}(U/S)). \end{aligned}$$

Note that $\mathbb{Z}((\bar{D}_I \times S, D_I)/\bar{X} \times S)$ and $\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S))$ are obviously $\mathbb{A}^1$ invariant. Note that $r_{(X,D)/S}$ is NOT an equivalence $(\mathbb{A}^1, et)$ local by proposition 8 since $\rho_{\bar{X}\times S*}\mathbb{Z}((\bar{D}_\bullet\times S, D_\bullet)/\bar{X}\times S) = 0$ and $\rho_{\bar{X}\times S*}\,\mathrm{ad}((j \times I)^*, (j \times I)_*)(\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)))$ is not an equivalence $(\mathbb{A}^1, et)$ local.

(ii) Let $g : U'/S \to U/S$ a morphism, with $U'/S = (U', h'), U/S = (U, h) \in \mathrm{Var}(k)/S$, with U and U' smooth. Take, see definition-proposition 3(ii),a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}$ of $h : U \to S$ and a compactification $\bar{f}'_0 = \bar{h}' : \bar{X}'_0 \to \bar{S}$ of $h' : U' \to S$ such that $g : U'/S \to U/S$ extend to a morphism $\bar{g}_0 : \bar{X}'_0/\bar{S} \to \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0\backslash U$ and $\bar{Z}' = \bar{X}'_0\backslash U'$. Take, see definition-proposition 3(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of $(\bar{X}_0, \bar{Z})$, a strict desingularization $\bar{\epsilon}'_\bullet : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ of $(\bar{X}'_0, \bar{Z}')$ and a morphism $\bar{g} : \bar{X}' \to \bar{X}$ such that the following diagram commutes

$$\begin{tikzcd} \bar{X}'_0 \arrow[r, "\bar{g}_0"] & \bar{X}_0 \\ \bar{X}' \arrow[u, "\bar{\epsilon}'"] \arrow[r, "\bar{g}"] & \bar{X} \arrow[u, "\bar{\epsilon}"'] \end{tikzcd} .$$

We then have, see definition-proposition 3(ii), the following commutative diagram in $\mathrm{Fun}(\Delta, \mathrm{Var}(k))$

$$\begin{tikzcd} U = U_{c(\bullet)} \arrow[r, "j"] & \bar{X} = \bar{X}_{c(\bullet)} & & \bar{D}_{s_g(\bullet)} \arrow[ll, "i_\bullet"'] \\ U' = U'_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j'"] & \bar{X}' = \bar{X}'_{c(\bullet)} \arrow[u, "\bar{g}"] & \bar{D}'_\bullet \arrow[l, "i'_\bullet"'] & \bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g\bullet}"'] \arrow[u, "\bar{g}'_\bullet"] \end{tikzcd} \tag{7}$$

Denote by $p_S : \bar{X} \times S \to S$ *and* $p'_S : \bar{X}' \times S \to S$ *the projections We then consider the following map in* $C(\mathrm{Var}(k)^2/S)$

$$\begin{aligned} &R^{CH}_S(g) : R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \xrightarrow{:=} \\ &p_{S*}E_{et}(\mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}((\bar{D}_{s_g(\bullet)} \times S, D_{s_g(\bullet)})/\bar{X} \times S), u_{IJ}) \to \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S))) \\ &\xrightarrow{T((\bar{g}\times I),E)(-)\circ p_{S*}\,\mathrm{ad}((\bar{g}\times I)^*,(\bar{g}\times I)_*)(-)} \\ &p'_{S*}E_{et}(\mathrm{Cone}(\mathbb{Z}(i'_{g\bullet} \times I) : \\ &(\mathbb{Z}((\bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) \times S, \bar{g}^{-1}(D_{s_g(\bullet)}))/\bar{X}' \times S), u_{IJ}) \to \mathbb{Z}((\bar{X}' \times S, X')/\bar{X}' \times S)))) \\ &\xrightarrow{p'_{S*}E_{et}(\mathbb{Z}(i''_{g\bullet}\times I),I)} \\ &p'_{S*}E_{et}(\mathrm{Cone}(\mathbb{Z}(i'_\bullet \times I) : ((\mathbb{Z}((\bar{D}'_\bullet \times S, D'_\bullet))/\bar{X}' \times S), u_{IJ}) \to \mathbb{Z}((\bar{X}' \times S, X')/\bar{X}' \times S))) \\ &\xrightarrow{=:} R_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S)) \end{aligned}$$

Then by the diagram (7) and adjonction, the following diagram in $C(\mathrm{Var}(k)^2/S)$ *obviously commutes*

$$\begin{tikzcd} R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \arrow[r, "r_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))"] \arrow[d, "R^{CH}_S(g)"'] & p_{S*}E_{et}(\mathbb{Z}((U \times S, U)/U \times S)) =: \mathbb{D}^{12}_S(\mathbb{Z}(U/S)) \arrow[d, "D^{12}_S(g) := T(g\times I,E)(-)\circ\mathrm{ad}((g\times I)^*,(g\times I)_*)(E_{et}(\mathbb{Z}((U\times S,U)/U\times S)))"] \\ R_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S)) \arrow[r, "r_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S))"] & p'_{S*}E_{et}(\mathbb{Z}((U' \times S, U')/U' \times S)) =: \mathbb{D}^{12}_S(\mathbb{Z}(U'/S)) \end{tikzcd}.$$

(iii) *For* $g_1 : U''/S \to U'/S$, $g_2 : U'/S \to U/S$ *two morphisms with* $U''/S = (U', h''), U'/S = (U', h'), U/S = (U, h) \in \mathrm{Var}(k)/S$, *with* U, U' *and* U'' *smooth. We get from (i) and (ii) a compactification* $\bar{f} = \bar{h} : \bar{X} \to \bar{S}$ *of* $h : U \to S$, *a compactification* $\bar{f}' = \bar{h}' : \bar{X}' \to \bar{S}$ *of* $h' : U' \to S$, *and a compactification* $\bar{f}'' = \bar{h}'' : \bar{X}'' \to \bar{S}$ *of* $h'' : U'' \to S$, *with* $\bar{X}, \bar{X}', \bar{X}'' \in \mathrm{PSmVar}(k)$, $\bar{D} := \bar{X} \backslash U \subset \bar{X}$ $\bar{D}' := \bar{X}' \backslash U' \subset \bar{X}'$, *and* $\bar{D}'' := \bar{X}'' \backslash U'' \subset \bar{X}''$ *normal crossing divisors, such that* $g_1 : U''/S \to U'/S$ *extend to* $\bar{g}_1 : \bar{X}''/\bar{S} \to \bar{X}'/\bar{S}$, $g_2 : U'/S \to U/S$ *extend to* $\bar{g}_2 : \bar{X}'/\bar{S} \to \bar{X}/\bar{S}$, *and*

$$R^{CH}_S(g_2 \circ g_1) = R^{CH}_S(g_1) \circ R^{CH}_S(g_2) : R_{(\bar{X},\bar{D})/S} \to R_{(\bar{X}'',\bar{D}'')/S}$$

(iv) *For*

$$Q^* := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_\beta/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with U^*_α *smooth, we get from (i), (ii) and (iii) the map in* $C(\mathrm{Var}(k)^2/S)$

$$\begin{aligned} r^{CH}_S(Q^*) : R^{CH}(Q^*) := (\cdots \to \oplus_{\beta\in\Lambda^{n-1}} \varinjlim_{(\bar{X}^{n-1}_\beta,\bar{D}^{n-1}_\beta)/S} R_{(\bar{X}^{n-1}_\beta,\bar{D}^{n-1}_\beta)/S}(\mathbb{Z}(U^{n-1}_\beta/S)) \\ \xrightarrow{(R^{CH}_S(g^n_{\alpha,\beta}))} \oplus_{\alpha\in\Lambda^n} \varinjlim_{(\bar{X}^n_\alpha,\bar{D}^n_\alpha)/S} R_{(\bar{X}^n_\alpha,\bar{D}^n_\alpha)/S}(\mathbb{Z}(U^n_\alpha/S)) \to \cdots) \to \mathbb{D}^{12}_S(Q^*), \end{aligned}$$

where for $(U^n_\alpha, h^n_\alpha) \in \mathrm{Var}(k)/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \to \bar{S}$ of $h_\alpha : U_\alpha \to S$ with $\bar{X}_\alpha \in \mathrm{PSmVar}(k)$ and $\bar{D}_\alpha := \bar{X}_\alpha \backslash U_\alpha$ a normal crossing divisor. For $m = (m^*) : Q^*_1 \to Q^*_2$ a morphism with

$$Q^*_1 := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_{1,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_{1,\beta}/S) \to \cdots),$$

$$Q^*_2 := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_{2,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_{2,\beta}/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

complexes of (maybe infinite) direct sum of representable presheaves with $U^*_{1,\alpha}$ and $U^*_{2,\alpha}$ smooth, we get again from (i), (ii) and (iii) a commutative diagram in $C(\mathrm{Var}(k)^2/S)$

$$\begin{tikzcd} R^{CH}(Q^*_2) \arrow[r, "r^{CH}_S(Q^*_2)"] \arrow[d, "R^{CH}_S(m):=(R^{CH}_S(m^*))"'] & \mathbb{D}^{12}_S(Q^*_2) \arrow[d, "\mathbb{D}^{12}_S(m):=(\mathbb{D}^{12}_S(m^*))"] \\ R^{CH}(Q^*_1) \arrow[r, "r^{CH}_S(Q^*_1)"] & \mathbb{D}^{12}_S(Q^*_1) \end{tikzcd} .$$

- Let $S \in \mathrm{Var}(k)$ For $(h, m, m') = (h^*, m^*, m'^*) : Q^*_1[1] \to Q^*_2$ an homotopy with $Q^*_1, Q^*_2 \in C(\mathrm{Var}(k)/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U^*_{1,\alpha}$ and $U^*_{2,\alpha}$ smooth,

  $$(R^{CH}_S(h), R^{CH}_S(m), R^{CH}_S(m')) = (R^{CH}_S(h^*), R^{CH}_S(m^*), R^{CH}_S(m'^*)) : R^{CH}(Q^*_2)[1] \to R^{CH}(Q^*_1)$$

  is an homotopy in $C(\mathrm{Var}(k)^2/S)$ using definition 24 (iii). In particular if $m : Q^*_1 \to Q^*_2$ with $Q^*_1, Q^*_2 \in C(\mathrm{Var}(k)/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U^*_{1,\alpha}$ and $U^*_{2,\alpha}$ smooth is an homotopy equivalence, then $R^{CH}_S(m) : R^{CH}(Q^*_2) \to R^{CH}(Q^*_1)$ is an homotopy equivalence.

- Let $S \in \mathrm{SmVar}(k)$. Let $F \in \mathrm{PSh}(\mathrm{Var}(k)^{sm}/S)$. Consider

  $$q : LF := (\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\mathbb{Z}(U_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\mathbb{Z}(U_\alpha/S) \to \cdots) \to F$$

  the canonical projective resolution given in subsection 2.3.3. Note that the U_α are smooth since S is smooth and h_α are smooth morphism. Definition 24(iv) gives in this particular case the map in $C(\mathrm{Var}(k)^2/S)$

  $$r^{CH}_S(\rho^*_S LF) : R^{CH}(\rho^*_S LF) := (\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S} \varinjlim_{(\bar{X}_\alpha,\bar{D}_\alpha)/S} R_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S))$$
  $$\xrightarrow{(R^{CH}_S(g^n_{\alpha,\beta}))} \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S} \varinjlim_{(\bar{X}_\alpha,\bar{D}_\alpha)/S} R_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \to \cdots) \to \mathbb{D}^{12}_S(\rho^*_S LF),$$

  where for $(U_\alpha, h_\alpha) \in \mathrm{Var}(k)^{sm}/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \to \bar{S}$ of $h_\alpha : U_\alpha \to S$ with $\bar{X}_\alpha \in \mathrm{PSmVar}(k)$ and $\bar{D}_\alpha := \bar{X}_\alpha\backslash U_\alpha$ a normal crossing divisor. Definition 24(iv) gives then by functoriality in particular, for $F = F^\bullet \in C(\mathrm{Var}(k)^{sm}/S)$, the map in $C(\mathrm{Var}(k)^2/S)$

  $$r^{CH}_S(\rho^*_S LF) = (r^{CH}_S(\rho^*_S LF^*)) : R^{CH}(\rho^*_S LF) \to \mathbb{D}^{12}_S(\rho^*_S LF).$$

- Let $g : T \to S$ a morphism with $T, S \in \mathrm{SmVar}(k)$. Let $h : U \to S$ a smooth morphism with $U \in \mathrm{Var}(k)$. Consider the cartesian square

  $$\begin{tikzcd} U_T \arrow[r, "h'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ U \arrow[r, "h"] & S \end{tikzcd}$$

Note that U is smooth since S and h are smooth, and U_T is smooth since T and h' are smooth. Take, see definition-proposition 3(ii),a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}$ of $h : U \to S$ and a compactification $\bar{f}'_0 = g \bar{\circ} h' : \bar{X}'_0 \to \bar{S}$ of $g \circ h' : U' \to S$ such that $g' : U_T/S \to U/S$ extend to a morphism $\bar{g}'_0 : \bar{X}'_0/\bar{S} \to \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \backslash U$ and $\bar{Z}' = \bar{X}'_0 \backslash U_T$. Take, see definition-proposition 3(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of $(\bar{X}_0, \bar{Z})$, a desingularization $\bar{\epsilon}'_\bullet : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ of $(\bar{X}'_0, \bar{Z}')$ and a morphism $\bar{g}' : \bar{X}' \to \bar{X}$ such that the following diagram commutes

$$\begin{tikzcd} \bar{X}'_0 \arrow[r, "\bar{g}'_0"] & \bar{X}_0 \\ \bar{X}' \arrow[u, "\bar{\epsilon}'"] \arrow[r, "\bar{g}'"] & \bar{X} \arrow[u, "\bar{\epsilon}"'] \end{tikzcd} .$$

We then have, see definition-proposition 3(ii), the following commutative diagram in $\mathrm{Fun}(\Delta, \mathrm{Var}(k))$

$$\begin{tikzcd} U = U_{c(\bullet)} \arrow[r, "j"] & \bar{X} = \bar{X}_{c(\bullet)} & & \bar{D}_{s_{g'}(\bullet)} \arrow[ll, "i_\bullet"'] \\ U_T = U_{T,c(\bullet)} \arrow[u, "g'"] \arrow[r, "j'"] & \bar{X}' = X'_{c(\bullet)} \arrow[u, "\bar{g}'"] & \bar{D}'_\bullet \arrow[l, "i'_\bullet"'] & \bar{g}^{-1}(\bar{D}_{s_{g'}(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g'\bullet}"'] \arrow[u, "(\bar{g}')'_\bullet"'] \end{tikzcd}$$

We then consider the following map in $C(\mathrm{Var}(k)^2/T)$, see definition 24(ii)

$$T(g, R^{CH})(\mathbb{Z}(U/S)) : g^* R_{(\bar{X}, \bar{D})/S}(\mathbb{Z}(U/S))$$
$$\xrightarrow{g^* R_S^{CH}(g')} g^* R_{(\bar{X}', \bar{D}')/S}(\mathbb{Z}(U_T/S)) = g^* g_* R_{(\bar{X}', \bar{D}')/T}(\mathbb{Z}(U_T/T))$$
$$\xrightarrow{\mathrm{ad}(g^*, g_*)(R_{(\bar{X}', \bar{D}')/T}(\mathbb{Z}(U_T/T)))} R_{(\bar{X}', \bar{D}')/T}(\mathbb{Z}(U_T/T))$$

For

$$Q^* := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U^n_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U^{n-1}_\beta/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with $h^n_\alpha : U^n_\alpha \to S$ smooth, we get the map in $C(\mathrm{Var}(k)^2/T)$

$$T(g, R^{CH})(Q^*) : g^* R^{CH}(Q^*) = (\cdots \to \oplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}^n_\alpha, \bar{D}^n_\alpha)/S} g^* R_{(\bar{X}^n_\alpha, \bar{D}^n_\alpha)/S}(\mathbb{Z}(U^n_\alpha/S)) \to \cdots)$$
$$\xrightarrow{(T(g, R^{CH})(\mathbb{Z}(U^n_\alpha/S)))} (\cdots \to \oplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}^{n\prime}_\alpha, \bar{D}^{n\prime}_\alpha)/T} R_{(\bar{X}^{n\prime}_\alpha, \bar{D}^{n\prime}_\alpha)/T}(\mathbb{Z}(U^n_{\alpha,T}/S)) \to \cdots) =: R^{CH}(g^* Q^*).$$

Let $F \in \mathrm{PSh}(\mathrm{Var}(k)^{sm}/S)$. Consider

$$q : LF := (\cdots \to \oplus_{(U_\alpha, h_\alpha) \in \mathrm{Var}(k)^{sm}/S} \mathbb{Z}(U_\alpha/S) \to \cdots) \to F$$

the canonical projective resolution given in subsection 2.3.3. We then get in particular the map in $C(\mathrm{Var}(k)^2/T)$

$$T(g, R^{CH})(\rho_S^* LF) : g^* R^{CH}(\rho_S^* LF) =$$
$$(\cdots \to \oplus_{(U_\alpha, h_\alpha) \in \mathrm{Var}(k)^{sm}/S} \varinjlim_{(\bar{X}_\alpha, \bar{D}_\alpha)/S} g^* R_{(\bar{X}_\alpha, \bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \to \cdots) \xrightarrow{(T(g, R^{CH})(\mathbb{Z}(U_\alpha/S)))}$$
$$(\cdots \to \oplus_{(U_\alpha, h_\alpha) \in \mathrm{Var}(k)^{sm}/S} \varinjlim_{(\bar{X}'_\alpha, \bar{D}'_\alpha)/T} R_{(\bar{X}'_\alpha, \bar{D}'_\alpha)/T}(\mathbb{Z}(U_{\alpha,T}/S)) \to \cdots) =: R^{CH}(\rho_T^* g^* LF).$$

By functoriality, we get in particular for $F = F^\bullet \in C(\mathrm{Var}(k)^{sm}/S)$, the map in $C(\mathrm{Var}(k)^2/T)$

$$T(g, R^{CH})(\rho_S^* LF) : g^* R^{CH}(\rho_S^* LF) \to R^{CH}(\rho_T^* g^* LF).$$

- Let $S_1, S_2 \in \mathrm{SmVar}(k)$ and $p : S_1 \times S_2 \to S_1$ the projection. Let $h : U \to S_1$ a smooth morphism with $U \in \mathrm{Var}(k)$. Consider the cartesian square

$$\begin{tikzcd} U \times S_2 \arrow[r, "h\times I"] \arrow[d, "p'"] & S_1 \times S_2 \arrow[d, "p"] \\ U \arrow[r, "h"] & S_1 \end{tikzcd}$$

Take, see definition-proposition 3(i),a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}_1$ of $h : U \to S_1$. Then $\bar{f}_0 \times I : \bar{X}_0 \times S_2 \to \bar{S}_1 \times S_2$ is a compactification of $h \times I : U \times S_2 \to S_1 \times S_2$ and $p' : U \times S_2 \to U$ extend to $\bar{p}'_0 := p_{X_0} : \bar{X}_0 \times S_2 \to \bar{X}_0$. Denote $Z = X_0 \backslash U$. Take see theorem 19(i), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$. We then have the following commutative diagram in $\mathrm{Fun}(\Delta, \mathrm{Var}(k))$ whose squares are cartesian

$$\begin{tikzcd} U = U_{c(\bullet)} \arrow[r, "j"] & \bar{X} & \bar{D}_\bullet \arrow[l, "i_\bullet"'] \\ U \times S_2 = (U \times S_2)_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j\times I"] & \bar{X} \times S_2 \arrow[u, "\bar{p}' := p_{\bar{X}}"] & \bar{D}_\bullet \times S_2 \arrow[l, "i'_\bullet"'] \arrow[u, "\bar{p}'_\bullet"] \end{tikzcd} \tag{8}$$

Then the map in $C(\mathrm{Var}(k)^2/S_1 \times S_2)$

$$T(p, R^{CH})(\mathbb{Z}(U/S_1)) : p^* R_{(\bar{X},\bar{D})/S_1}(\mathbb{Z}(U/S_1)) \xrightarrow{\sim} R_{(\bar{X}\times S_2, \bar{D}_\bullet \times S_2)/S_1\times S_2}(\mathbb{Z}(U \times S_2/S_1 \times S_2))$$

is an isomorphism. Hence, for $Q^* \in C(\mathrm{Var}(k)/S_1)$ a complex of (maybe infinite) direct sum of representable presheaves of smooth morphism, the map in $C(\mathrm{Var}(k)^2/S_1 \times S_2)$

$$T(p, R^{CH})(Q^*) : p^* R^{CH}(Q^*) \xrightarrow{\sim} R^{CH}(p^* Q^*)$$

is an isomorphism. In particular, for $F \in C(\mathrm{Var}(k)^{sm}/S_1)$ the map in $C(\mathrm{Var}(k)^2/S_1 \times S_2)$

$$T(p, R^{CH})(\rho^*_{S_1} LF) : p^* R^{CH}(\rho^*_{S_1} LF) \xrightarrow{\sim} R^{CH}(\rho^*_{S_1\times S_2} p^* LF)$$

is an isomorphism.

- Let $h_1 : U_1 \to S$, $h_2 : U_2 \to S$ two morphisms with $U_1, U_2, S \in \mathrm{Var}(k)$, U_1, U_2 smooth. Denote by $p_1 : U_1 \times_S U_2 \to U_1$ and $p_2 : U_1 \times_S U_2 \to U_2$ the projections. Take, see definition-proposition 3(i)), a compactification $\bar{f}_{10} = \bar{h}_1 : \bar{X}_{10} \to \bar{S}$ of $h_1 : U_1 \to S$ and a compactification $\bar{f}_{20} = \bar{h}_2 : \bar{X}_{20} \to \bar{S}$ of $h_2 : U_2 \to S$. Then,

  - $\bar{f}_{10} \times \bar{f}_{20} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \to S$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \to S$.
  - $\bar{p}_{10} := p_{X_{10}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \to \bar{X}_{10}$ is a compactification of $p_1 : U_1 \times_S U_2 \to U_1$.
  - $\bar{p}_{20} := p_{X_{20}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \to \bar{X}_{20}$ is a compactification of $p_2 : U_1 \times_S U_2 \to U_2$.

  Denote $\bar{Z}_1 = \bar{X}_{10} \backslash U_1$ and $\bar{Z}_2 = \bar{X}_{20} \backslash U_2$. Take, see theorem 19(i), a strict desingularization $\bar{\epsilon}_1 : (\bar{X}_1, \bar{D}) \to (\bar{X}_{10}, Z_1)$ of the pair $(\bar{X}_{10}, \bar{Z}_1)$ and a strictdesingularization $\bar{\epsilon}_2 : (\bar{X}_2, \bar{E}) \to (\bar{X}_{20}, Z_2)$ of the pair $(\bar{X}_{20}, \bar{Z}_2)$. Take then a strict desingularization

$$\bar{\epsilon}_{12} : ((\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F}) \to (\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (D \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$$

of the pair $(\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (\bar{D} \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$. We have then the following commutative diagram

$$\begin{tikzcd}
 & \bar{X}_1 \arrow[r, "\bar{f}_1"] & \bar{S} \\
 & \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \arrow[u, "\bar{p}_2"] \arrow[r, "\bar{p}_1"] & \bar{X}_2 \arrow[u, "\bar{f}_2"'] \\
(\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \arrow[uur, "(\bar{p}_2)^N"] \arrow[ur, "\epsilon_{12}"'] \arrow[urr, "(\bar{p}_1)^N"] & &
\end{tikzcd}$$

and

- $\bar{f}_1 \times \bar{f}_2 : \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \to \bar{S}$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \to S$.
- $(\bar{p}_1)^N := \bar{p}_1 \circ \epsilon_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \to \bar{X}_1$ is a compactification of $p_1 : U_1 \times_S U_2 \to U_1$.
- $(\bar{p}_2)^N := \bar{p}_2 \circ \epsilon_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \to \bar{X}_2$ is a compactification of $p_2 : U_1 \times_S U_2 \to U_2$.

We have then the morphism in $C(\mathrm{Var}(k)^2/S)$

$$T(\otimes, R_S^{CH})(\mathbb{Z}(U_1/S), \mathbb{Z}(U_2/S)) := R_S^{CH}(p_1) \otimes R_S^{CH}(p_2) :$$
$$R_{(\bar{X}_1, \bar{D})/S}(\mathbb{Z}(U_1/S)) \otimes R_{(X_2, E))/S}(\mathbb{Z}(U_2/S)) \xrightarrow{\sim} R_{(\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F})/S}(\mathbb{Z}(U_1 \times_S U_2/S))$$

For

$$Q_1^* := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{1,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{1,\beta}^{n-1}/S) \to \cdots),$$
$$Q_2^* := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U_{2,\alpha}^n/S) \xrightarrow{(\mathbb{Z}(g_{\alpha,\beta}^n))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U_{2,\beta}^{n-1}/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

complexes of (maybe infinite) direct sum of representable presheaves with U_α^* smooth, we get the morphism in $C(\mathrm{Var}(k)^2/S)$

$$T(\otimes, R_S^{CH})(Q_1^*, Q_2^*) : R^{CH}(Q_1^*) \otimes R^{CH}(Q_2^*) \xrightarrow{(T(\otimes, R_S^{CH})(\mathbb{Z}(U_{1,\alpha}^m), \mathbb{Z}(U_{2,\beta}^n))} R^{CH}(Q_1^* \otimes Q_2^*)).$$

For $F_1, F_2 \in C(\mathrm{Var}(k)^{sm}/S)$, we get in particular the morphism in $C(\mathrm{Var}(k)^2/S)$

$$T(\otimes, R_S^{CH})(\rho_S^* LF_1, \rho_S^* LF_2) : R^{CH}(\rho_S^* LF_1) \otimes R^{CH}(\rho_S^* LF_2) \to R^{CH}(\rho_S^*(LF_1 \otimes LF_2)).$$

Definition 25. *Let $h : U \to S$ a morphism, with $U, S \in \mathrm{Var}(k)$, U irreducible. Take, see definition-proposition 3, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $\bar{Z} = \bar{X}_0 \backslash U$. Take, using theorem 19, a desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \Delta)$ of the pair $(\bar{X}_0, \Delta)$, $\bar{Z} \subset \Delta$, with $\bar{X} \in \mathrm{PSmVar}(k)$ and $\bar{D} := \bar{\epsilon}^{-1}(\Delta) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. Denote $d_X := \dim(\bar{X}) = \dim(U)$.*

(i) The cycle $(\Delta_{\bar{D}_\bullet} \times S) \subset \bar{D}_\bullet \times \bar{D}_\bullet \times S$ induces by the diagonal $\Delta_{\bar{D}_\bullet} \subset \bar{D}_\bullet \times \bar{D}_\bullet$ gives the morphism in $C(\mathrm{Var}(k)^2/S)$

$$[\Delta_{\bar{D}_\bullet}] \in \mathrm{Hom}(\mathbb{Z}^{tr}((\bar{D}_\bullet \times S, D_\bullet)/S), p_{S*}E_{et}(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S)(d_X)[2d_X])) \xrightarrow{\sim}$$
$$\mathrm{Hom}(\mathbb{Z}((\bar{D}_\bullet \times S \times \bar{X}, D_\bullet)/\bar{X} \times S),$$
$$\mathbb{Z}^{tr}((\bar{D}_\bullet \times S \times \mathbb{P}^{d_X}, D_\bullet \times \mathbb{P}^{d_X})/\bar{X} \times S)/\mathbb{Z}^{tr}((-) \times \mathbb{P}^{d_X - 1}, (-) \times \mathbb{P}^{d_X - 1}))$$
$$\subset H^0(\mathcal{Z}_{d_{D_\bullet} + d_S}(\square^* \times \bar{D}_\bullet \times \bar{D}_\bullet \times S), s.t. \alpha_*(\times D_\bullet) = D_\bullet)$$

(ii) The cycle $(\Delta_{\bar{X}} \times S) \subset \bar{X} \times \bar{X} \times S$ induces by the diagonal $\Delta_{\bar{X}} \subset \bar{X} \times \bar{X}$ gives the morphism in $C(\mathrm{Var}(k)^2/S)$

$$[\Delta_{\bar{X}}] \in \mathrm{Hom}(\mathbb{Z}^{tr}((\bar{X} \times S, X)/S), p_{S*}E_{et}(\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)(d_X)[2d_X])) \xrightarrow{\sim}$$
$$\mathrm{Hom}(\mathbb{Z}((\bar{X} \times S \times \bar{X}, X)/\bar{X} \times S),$$
$$\mathbb{Z}^{tr}((\bar{X} \times S \times \mathbb{P}^{d_X}, X \times \mathbb{P}^{d_X})/\bar{X} \times S)/\mathbb{Z}^{tr}((-) \times \mathbb{P}^{d_X - 1}, (-) \times \mathbb{P}^{d_X - 1}))$$
$$\subset H^0(\mathcal{Z}_{d_X + d_S}(\square^* \times \bar{X} \times \bar{X} \times S), s.t. \alpha_*(\times X) = X)$$

Let $h : U \to S$ a morphism, with $U, S \in \mathrm{Var}(k)$, U smooth connected (hence irreducible by smoothness). Take, see definition-proposition 3, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $\bar{Z} = \bar{X}_0 \backslash U$. Take, using theorem 19(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$ with $\bar{X} \in \mathrm{PSmVar}(k)$ and $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. Denote $d_X := \dim(\bar{X}) = \dim(U)$.

(iii) We get from (i) and (ii) the morphism in $C(\mathrm{Var}(k)^2/S)$

$$\begin{aligned}T(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)) := ([\Delta_{\bar{D}_\bullet}], [\Delta_{\bar{X}}]) :\\ \mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}^{tr}((\bar{D}_\bullet \times S, D_\bullet)/S), u_{IJ}) \to \mathbb{Z}^{tr}((\bar{X} \times S, X)/S)) \to\\ p_{S*}E_{et}(\mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), u_{IJ}) \to\\ \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)))(d_X)[2d_X] =: R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X]\end{aligned}$$

(iii)' which gives the map in $C(\mathrm{Var}(k)^{2,smpr}/S)$

$$\begin{aligned}T^{\mu,q}(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)) :\\ \mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}^{tr}((\bar{D}_\bullet \times S, D_\bullet)/S), u_{IJ}) \to \mathbb{Z}^{tr}((\bar{X} \times S, X)/S)) =\\ L\rho_{S*}\mu_{S*}\,\mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}^{tr}((\bar{D}_\bullet \times S, D_\bullet)/S), u_{IJ}) \to \mathbb{Z}^{tr}((\bar{X} \times S, X)/S))\\ \xrightarrow{L\rho_{S*}\mu_{S*}T(p_{S\sharp},p_{S*})(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X}\times S), \mathbb{Z}((\bar{X}\times S, X)/\bar{X}\times S))} L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X]\end{aligned}$$

Proposition 12. *Let $h : U \to S$ a morphism, with $U, S \in \mathrm{Var}(k)$, U irreducible. Take, see definition-proposition 3, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $\bar{Z} = \bar{X}_0 \backslash U$. Take, using theorem 19(ii), a desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \Delta)$ of the pair $(\bar{X}_0, \Delta)$, $\bar{Z} \subset \Delta$ with $\bar{X} \in \mathrm{PSmVar}(k)$ and $\bar{D} := \bar{\epsilon}^{-1}(\Delta) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. Denote $d_X := \dim(\bar{X}) = \dim(U)$.*

(i) The morphism

$$[\Delta_{\bar{D}_\bullet}] : \mathbb{Z}^{tr}((\bar{D}_\bullet \times S, D_\bullet)/S), \to p_{S*}E_{et}(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S)(d_X)[2d_X])$$

given in definition 25(i) is an equivalence $(\mathbb{A}^1, et)$ local.

(ii) The morphism

$$[\Delta_{\bar{X}}] : \mathbb{Z}^{tr}((\bar{X} \times S, X)/S), \to p_{S*}E_{et}(\mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)(d_X)[2d_X])$$

given in definition 25(ii) is an equivalence $(\mathbb{A}^1, et)$ local.

Let $h : U \to S$ a morphism, with $U, S \in \mathrm{Var}(k)$, U smooth connected (hence irreducible by smoothness). Take, see definition-proposition 3, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $\bar{Z} = \bar{X}_0 \backslash U$. Take, using theorem 19(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$, with $\bar{X} \in \mathrm{PSmVar}(k)$ and $\bar{D} := \bar{\epsilon}^{-1}(Z) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor.

(iii) The morphism

$$\begin{aligned}T(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)) := ([\Delta_{\bar{D}_\bullet}], [\Delta_{\bar{X}}]) :\\ \mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}^{tr}((\bar{D}_\bullet \times S, D_\bullet)/S), u_{IJ}) \to \mathbb{Z}^{tr}((\bar{X} \times S, X)/S)) \to\\ p_{S*}E_{et}(\mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X} \times S), u_{IJ}) \to\\ \mathbb{Z}((\bar{X} \times S, X)/\bar{X} \times S)))(d_X)[2d_X] =: R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X]\end{aligned}$$

given in definition 25(iii)' is an equivalence $(\mathbb{A}^1, et)$ local.

(iii)' The morphism

$$T^{\mu,q}(p_{S\sharp},p_{S*})(\mathbb{Z}((\bar{D}_\bullet\times S,D_\bullet)/\bar{X}\times S),\mathbb{Z}((\bar{X}\times S,X)/\bar{X}\times S)):$$
$$\mathrm{Cone}(\mathbb{Z}(i_\bullet\times I):(\mathbb{Z}^{tr}((\bar{D}_\bullet\times S,D_\bullet)/S),u_{IJ})\to\mathbb{Z}^{tr}((\bar{X}\times S,X)/S))$$
$$\to L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X]$$

given in definition 25(iii)' is an equivalence $(\mathbb{A}^1,et)$ *local.*

Proof. (i): By Yoneda lemma, it is equivalent to show that for every morphism $g:T\to S$ with $T\in\mathrm{Var}(k)$ and every closed subset $E\subset T$, the composition morphism

$$[\Delta_{\bar{D}_\bullet}]:\mathrm{Hom}^\bullet(\mathbb{Z}((T,E)/S),C_*\mathbb{Z}^{tr}((\bar{D}_\bullet\times S,D_\bullet)/S))\xrightarrow{\mathrm{Hom}^\bullet(\mathbb{Z}((T,E)/S),C_*\Delta_{\bar{D}_\bullet})}$$
$$\mathrm{Hom}^\bullet(\mathbb{Z}((T,E)/S),p_{S*}E_{et}(\mathbb{Z}((\bar{D}_\bullet\times S,D_\bullet)/\bar{X}\times S)(d_X)[2d_X]))$$

is a quasi-isomorphism of abelian groups. But this map is the composite

$$\mathrm{Hom}^\bullet(\mathbb{Z}((T,E)/S),\mathbb{Z}^{tr}((\bar{D}_\bullet\times S,D_\bullet)/S))\xrightarrow{[\Delta_{\bar{D}_\bullet}]}$$
$$\mathrm{Hom}^\bullet(\mathbb{Z}((T,E)/S),p_{S*}E_{et}(\mathbb{Z}((\bar{D}_\bullet\times S,D_\bullet)/\bar{X}\times S)(d_X)[2d_X]))\xrightarrow{\sim}$$
$$\mathrm{Hom}^\bullet(\mathbb{Z}((T\times\bar{X},E)/S\times\bar{X}),$$
$$C_*\mathbb{Z}^{tr}((\bar{D}_\bullet\times S\times\mathbb{P}^{d_X},D_\bullet\times\mathbb{P}^{d_X})/\bar{X}\times S)/\mathbb{Z}^{tr}((-)\times\mathbb{P}^{d_X-1},(-)\times\mathbb{P}^{d_X-1}))$$

which is clearly a quasi-isomorphism.
(ii): Similar to (i).
(iii):Follows from (i) and (ii).
(iii)':Follows from (iii) and the fact that μ_{S*} preserve $(\mathbb{A}^1,et)$ local equivalence (see proposition 9) and the fact that ρ_{S*} preserve $(\mathbb{A}^1,et)$ local equivalence (see proposition 8). □

Definition 26. *(i) Let $h:U\to S$ a morphism, with $U,S\in\mathrm{Var}(k)$, U smooth. Take, see definition-proposition 3, $\bar{f}_0=\bar{h}_0:\bar{X}_0\to\bar{S}$ a compactification of $h:U\to S$ and denote by $\bar{Z}=\bar{X}_0\backslash U$. Take, using theorem 19(ii), a strict desingularization $\bar{\epsilon}:(\bar{X},\bar{D})\to(\bar{X}_0,\bar{Z})$ of the pair $(\bar{X}_0,\bar{Z})$, with $\bar{X}\in\mathrm{PSmVar}(k)$ and $\bar{D}:=\bar{\epsilon}^{-1}(\bar{Z})=\cup_{i=1}^s\bar{D}_i\subset\bar{X}$ a normal crossing divisor. We will consider the following canonical map in $C(\mathrm{Var}(k)^{sm}/S)$*

$$T_{(\bar{X},\bar{D})/S}(U/S):\mathrm{Gr}^{12}_{S*}L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))\xrightarrow{q}\mathrm{Gr}^{12}_{S*}\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))$$
$$\xrightarrow{r_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))}\mathrm{Gr}^{12}_{S*}\rho_{S*}\mu_{S*}p_{S*}E_{et}(\mathbb{Z}((U\times S,U)/U\times S))\xrightarrow{l(U/S)}h_*E_{et}(\mathbb{Z}(U/U))=:\mathbb{D}^0_S(\mathbb{Z}(U/S))$$

where, for $h':V\to S$ a smooth morphism with $V\in\mathrm{Var}(k)$,

$$l^{00}(U/S)(V/S):\mathbb{Z}((U\times S,U)/U\times S)(V\times U\times S,V\times_S U/U\times S)\to\mathbb{Z}(U/U)(V\times_S U),\alpha\mapsto\alpha_{|V\times_S U}$$

which gives

$$l^0(U/S)(V/S):E^0_{et}(\mathbb{Z}((U\times S,U)/U\times S))(V\times U\times S,V\times_S U/U\times S)\to E^0_{et}(\mathbb{Z}(U/U))(V\times_S U),$$

and by induction

$$\tau^{\leq i}l(U/S):\mathrm{Gr}^{12}_{S*}\rho_{S*}\mu_{S*}p_{S*}E^{\leq i}_{et}(\mathbb{Z}((U\times S,U)/U\times S))\to h_*E^{\leq i}_{et}(\mathbb{Z}(U/U))$$

where $\tau^{\leq i}$ is the cohomological truncation.

(ii) Let $g:U'/S\to U/S$ a morphism, with $U'/S=(U',h'),U/S=(U,h)\in\mathrm{Var}(k)/S$, U,U' smooth. Take, see definition-proposition 3(ii),a compactification $\bar{f}_0=\bar{h}:\bar{X}_0\to\bar{S}$ of $h:U\to S$ and a compactification $\bar{f}'_0=\bar{h}':\bar{X}'_0\to S$ of $h':U'\to S$ such that $g:U'/S\to U/S$ extend to a morphism

$\bar{g}_0 : \bar{X}'_0/\bar{S} \to \bar{X}_0/\bar{S}$. *Denote* $\bar{Z} = \bar{X}_0 \backslash U$ *and* $\bar{Z}' = \bar{X}'_0 \backslash U'$. *Take, see definition-proposition 3(ii), a strict desingularization* $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ *of* $(\bar{X}_0, \bar{Z})$, *a strict desingularization* $\bar{\epsilon}'_\bullet : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ *of* $(\bar{X}'_0, \bar{Z}')$ *and a morphism* $\bar{g} : \bar{X}' \to \bar{X}$ *such that the following diagram commutes*

$$\begin{tikzcd} \bar{X}'_0 \arrow[r, "\bar{g}_0"] & \bar{X}_0 \\ \bar{X}' \arrow[u, "\bar{\epsilon}'"] \arrow[r, "\bar{g}"] & \bar{X} \arrow[u, "\bar{\epsilon}"'] \end{tikzcd}.$$

Then by the diagram given in definition 24(ii), the following diagram in $C(\mathrm{Var}(k)^{sm}/S)$ *obviously commutes*

$$\begin{tikzcd} \mathrm{Gr}^{12}_{S*} L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \arrow[r, "T_{(\bar{X},\bar{D})/S}(U/S)"] \arrow[d, "R^{CH}_S(g)"'] & h_*E_{et}(\mathbb{Z}(U/U)) := \mathbb{D}^0_S(\mathbb{Z}(U/S)) \arrow[d, "T(g,E)(-)\circ\mathrm{ad}(g^*,g_*)(E_{et}(\mathbb{Z}(U/U))):=\mathbb{D}^0_S(g)"] \\ \mathrm{Gr}^{12}_{S*} L\rho_{S*}\mu_{S*}R_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S)) \arrow[r, "T_{(\bar{X}',\bar{D}')/S}(U'/S)"] & h'_*E_{et}(\mathbb{Z}(U'/U')) := \mathbb{D}^0_S(\mathbb{Z}(U'/S)) \end{tikzcd}$$

where $l(U/S)$ *are* $l(U'/S)$ *are the maps given in (i).*

(iii) Let $S \in \mathrm{SmVar}(k)$. *Let* $F \in C(\mathrm{Var}(k)^{sm}/S)$. *We get from (i) and (ii) morphisms in* $C(\mathrm{Var}(k)^{sm}/S)$

$$T^{CH}_S(LF) : \mathrm{Gr}^{12}_{S*} L\rho_{S*}\mu_{S*}R_{(\bar{X}^*,\bar{D}^*)/S}(\rho^*_S LF)$$
$$\xrightarrow{r^{CH}_S(LF)} \mathrm{Gr}^{12}_{S*} L\rho_{S*}\mu_{S*}\mathbb{D}^{12}_S(\rho^*_S LF) \xrightarrow{l(L(F)} \mathbb{D}^0_S(L(F))$$

Lemma 1. *(i) Let* $h : U \to S$ *a morphism, with* $U, S \in \mathrm{Var}(k)$, U *smooth. Take, see definition-proposition 3,* $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ *a compactification of* $h : U \to S$ *and denote by* $\bar{Z} = \bar{X}_0 \backslash U$. *Take, using theorem 19(ii), a strict desingularization* $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ *of the pair* $(\bar{X}_0, \bar{Z})$, *with* $\bar{X} \in \mathrm{PSmVar}(k)$ *and* $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ *a normal crossing divisor. Then the map in* $C(\mathrm{Var}(k)^{2,smpr}/S)$

$$\mathrm{ad}(\mathrm{Gr}^{12*}_S, \mathrm{Gr}^{12}_{S*})(L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))) \circ q :$$
$$\mathrm{Gr}^{12*}_S L\,\mathrm{Gr}^{12}_{S*} L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \to L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))$$

is an equivalence $(\mathbb{A}^1, et)$ *local.*

(ii) Let $S \in \mathrm{SmVar}(k)$. *Let* $F \in C(\mathrm{Var}(k)^{sm}/S)$. *Then the map in* $C(\mathrm{Var}(k)^{2,smpr}/S)$

$$\mathrm{ad}(\mathrm{Gr}^{12*}_S, \mathrm{Gr}^{12}_{S*})(L\rho_{S*}\mu_{S*}R_{(\bar{X}^*,\bar{D}^*)/S}(\rho^*_S LF)) \circ q :$$
$$\mathrm{Gr}^{12*}_S L\,\mathrm{Gr}^{12}_{S*} L\rho_{S*}\mu_{S*}R_{(\bar{X}^*,\bar{D}^*)/S}(\rho^*_S LF) \to L\rho_{S*}\mu_{S*}R_{(\bar{X}^*,\bar{D}^*)/S}(\rho^*_S LF)$$

is an equivalence $(\mathbb{A}^1, et)$ *local.*

Proof. (i): Follows from proposition 12.
(ii): Follows from (i). □

Definition 27. *(i) Let* $h : U \to S$ *a morphism, with* $U, S \in \mathrm{Var}(k)$ *and* U *smooth. Take, see definition-proposition 3,* $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ *a compactification of* $h : U \to S$ *and denote by* $\bar{Z} = \bar{X}_0 \backslash U$. *Take, using theorem 19(ii), a strict desingularization* $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ *of the pair* $(\bar{X}_0, \bar{Z})$, *with* $\bar{X} \in \mathrm{PSmVar}(k)$ *and* $\bar{D} := \epsilon^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ *a normal crossing divisor. We denote by* $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X} = \bar{X}_{c(\bullet)}$ *the morphism of simplicial varieties given by the closed embeddings* $i_I : \bar{D}_I = \cap_{i \in I} \bar{D}_i \hookrightarrow \bar{X}$ *We denote by* $j : U \hookrightarrow \bar{X}$ *the open embedding and by* $p_S : \bar{X} \times S \to S$

and $p_S : U \times S \to S$ the projections. Considering the graph factorization $\bar{f} : \bar{X} \xrightarrow{\bar{l}} \bar{X} \times \bar{S} \xrightarrow{p_{\bar{S}}} \bar{S}$ of $\bar{f} : \bar{X} \to \bar{S}$, where $\bar{l}$ is the graph embedding and $p_{\bar{S}}$ the projection, we get closed embeddings $l := \bar{l} \times_{\bar{S}} S : X \hookrightarrow \bar{X} \times S$ and $l_{D_I} := \bar{D}_I \times_{\bar{X}} l : D_I \hookrightarrow \bar{D}_I \times S$. We then consider the map in $C(\mathrm{Var}(k)^{2,smpr}/S)$

$$\begin{aligned} &T(\hat{R}^{CH}, R^{CH})(\mathbb{Z}(U/S)) : \hat{R}_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \\ &\xrightarrow{:=} \mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}^{tr}((\bar{D}_\bullet \times S, D_\bullet)/S), u_{IJ}) \to \mathbb{Z}^{tr}((\bar{X} \times S, X)/S))(-d_X)[-2d_X] \\ &\xrightarrow{T^{\mu,q}(p_{S\sharp}, p_{S*})(\mathbb{Z}((\bar{D}_\bullet \times S, D_\bullet)/\bar{X}\times S), \mathbb{Z}((\bar{X}\times S, X)/\bar{X}\times S))(-d_X)[-2d_X]} \\ &L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)). \end{aligned}$$

given in definition 25(iii).

(ii) Let $g : U'/S \to U/S$ a morphism, with $U'/S = (U', h'), U/S = (U, h) \in \mathrm{Var}(k)/S$, with U and U' smooth. Take, see definition-proposition 3(ii),a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}$ of $h : U \to S$ and a compactification $\bar{f}'_0 = \bar{h}' : \bar{X}'_0 \to \bar{S}$ of $h' : U' \to S$ such that $g : U'/S \to U/S$ extend to a morphism $\bar{g}_0 : \bar{X}'_0/\bar{S} \to \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \backslash U$ and $\bar{Z}' = \bar{X}'_0 \backslash U'$. Take, see definition-proposition 3(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of $(\bar{X}_0, \bar{Z})$, a strict desingularization $\bar{\epsilon}'_\bullet : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ of $(\bar{X}'_0, \bar{Z}')$ and a morphism $\bar{g} : \bar{X}' \to \bar{X}$ such that the following diagram commutes

$$\begin{tikzcd} \bar{X}'_0 \arrow[r, "\bar{g}_0"] & \bar{X}_0 \\ \bar{X}' \arrow[u, "\bar{\epsilon}'"] \arrow[r, "\bar{g}"] & \bar{X} \arrow[u, "\bar{\epsilon}"'] \end{tikzcd} .$$

We then have, see definition-proposition 3(ii), the diagram (7) in $\mathrm{Fun}(\Delta, \mathrm{Var}(k))$

$$\begin{tikzcd} U = U_{c(\bullet)} \arrow[r, "j"] & \bar{X} = \bar{X}_{c(\bullet)} & & \bar{D}_{s_g(\bullet)} \arrow[ll, "i_\bullet"'] \\ U' = U'_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j'"] & \bar{X}' = \bar{X}'_{c(\bullet)} \arrow[u, "\bar{g}"] & \bar{D}'_\bullet \arrow[l, "i'_\bullet"'] & \bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g\bullet}"'] \arrow[u, "\bar{g}'_\bullet"] \end{tikzcd}$$

Consider

$$\begin{aligned} [\Gamma_{\bar{g}}]^t \in \mathrm{Hom}(&\mathbb{Z}^{tr}((\bar{X} \times S, X)/S)(-d_X)[-2d_X], \mathbb{Z}^{tr}((\bar{X}' \times S, X')/S)(-d_{X'})[-2d_{X'}]) \\ &\xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}^{tr}((\bar{X} \times \mathbb{A}^{d_{X'}} \times S, X \times \mathbb{A}^{d_{X'}})/S), \\ &\mathbb{Z}_{tr}((\bar{X}' \times \mathbb{P}^{d_X} \times S, X' \times \mathbb{P}^{d_X})/S)/\mathbb{Z}_{tr}((-) \times \mathbb{P}^{d_X - 1}, (-) \times \mathbb{P}^{d_X - 1})) \end{aligned}$$

the morphism given by the transpose of the graph $\Gamma_g \subset X' \times_S X$ of $\bar{g} : \bar{X}' \to \bar{X}$. Then, since $i_\bullet \circ \bar{g}'_\bullet = \bar{g} \circ i''_{g\bullet} = \bar{g} \circ i' \circ \circ i'_{g\bullet}$, we have the factorization

$$\begin{aligned} &[\Gamma_g]^t \circ \mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}^{tr}((\bar{D}_{s_g(\bullet)} \times S, D_{s_g(\bullet)})/S), u_{IJ})(-d_X)[-2d_X] \\ &\xrightarrow{[\Gamma_{\bar{g}'_\bullet}]^t} (\mathbb{Z}^{tr}((\bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) \times S, \bar{g}^{-1}(D_{s_g(\bullet)}))/S), u_{IJ})(-d_{X'})[-2d_{X'}]) \\ &\xrightarrow{\mathbb{Z}(i'_{g\bullet} \times I)} \mathbb{Z}^{tr}((\bar{X}' \times S, X')/S)(-d_{X'})[-2d_{X'}]. \end{aligned}$$

with

$$\begin{aligned} &[\Gamma_{\bar{g}'_\bullet}]^t \in \mathrm{Hom}((\mathbb{Z}^{tr}((\bar{D}_{s_g(\bullet)} \times \mathbb{A}^{d_{X'}} \times S, D_{s_g(\bullet)} \times \mathbb{A}^{d_{X'}})/S), u_{IJ}), \\ &(\mathbb{Z}_{tr}((\bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) \times \mathbb{P}^{d_X} \times S, \bar{g}^{-1}(D_{s_g(\bullet)}) \times \mathbb{P}^{d_X})/S), u_{IJ})/\mathbb{Z}^{tr}((-) \times \mathbb{P}^{d_X - 1}, (-) \times \mathbb{P}^{d_X - 1})). \end{aligned}$$

We then consider the following map in $C(\mathrm{Var}(k)^{2,pr}/S)$

$$\hat{R}^{CH}_S(g) : \hat{R}_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \xrightarrow{:=}$$
$$\mathrm{Cone}(\mathbb{Z}(i_\bullet \times I) : (\mathbb{Z}^{tr}((\bar{D}_{s_g(\bullet)} \times S, D_{s_g(\bullet)})/S), u_{IJ}) \to \mathbb{Z}^{tr}((\bar{X} \times S, X)/S))(-d_X)[-2d_X]$$
$$\xrightarrow{([\Gamma_{\bar{g}'_\bullet}]^t, [\Gamma_{\bar{g}}]^t)}$$
$$\mathrm{Cone}(\mathbb{Z}(i'_{g\bullet} \times I) :$$
$$(\mathbb{Z}^{tr}((\bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) \times S, \bar{g}^{-1}(D_{s_g(\bullet)})/S), u_{IJ}) \to \mathbb{Z}^{tr}((\bar{X}' \times S, X')/S)))(-d_{X'})[-2d_{X'}]$$
$$\xrightarrow{(\mathbb{Z}(i''_{g\bullet} \times I), I)(-d_{X'})[-2d_{X'}]}$$
$$\mathrm{Cone}(\mathbb{Z}(i'_\bullet \times I) : ((\mathbb{Z}^{tr}((\bar{D}'_\bullet \times S, D'_\bullet)/S), u_{IJ}) \to \mathbb{Z}^{tr}((\bar{X}' \times S, X')/S)))(-d_{X'})[-2d_{X'}]$$
$$\xrightarrow{=:} \hat{R}_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S))$$

Then the following diagram in $C(\mathrm{Var}(k)^{2,smpr}/S)$ *commutes by definition*

$$\begin{tikzcd}
\hat{R}_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \arrow[r, "T(\hat{R}^{CH}, R^{CH})(\mathbb{Z}(U/S))"] \arrow[d, "\hat{R}^{CH}_S(g)"'] & L\rho_{S*}\mu_{S*}R_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \arrow[d, "L\rho_{S*}\mu_{S*}R^{CH}_S(g)"] \\
\hat{R}_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S)) \arrow[r, "T(\hat{R}^{CH}, R^{CH})(\mathbb{Z}(U'/S))"] & L\rho_{S*}\mu_{S*}R_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S))
\end{tikzcd} .$$

(iii) *For* $g_1 : U''/S \to U'/S$, $g_2 : U'/S \to U/S$ *two morphisms with* $U''/S = (U', h''), U'/S = (U', h'), U/S = (U, h) \in \mathrm{Var}(k)/S$, *with* U, U' *and* U'' *smooth. We get from (i) and (ii) a compactification* $\bar{f} = \bar{h} : \bar{X} \to \bar{S}$ *of* $h : U \to S$, *a compactification* $\bar{f}' = \bar{h}' : \bar{X}' \to \bar{S}$ *of* $h' : U' \to S$, *and a compactification* $\bar{f}'' = \bar{h}'' : \bar{X}'' \to \bar{S}$ *of* $h'' : U'' \to S$, *with* $\bar{X}, \bar{X}', \bar{X}'' \in \mathrm{PSmVar}(k)$, $\bar{D} := \bar{X} \backslash U \subset \bar{X}$ $\bar{D}' := \bar{X}' \backslash U' \subset \bar{X}'$, *and* $\bar{D}'' := \bar{X}'' \backslash U'' \subset \bar{X}''$ *normal crossing divisors, such that* $g_1 : U''/S \to U'/S$ *extend to* $\bar{g}_1 : \bar{X}''/\bar{S} \to \bar{X}'/\bar{S}$, $g_2 : U'/S \to U/S$ *extend to* $\bar{g}_2 : \bar{X}'/\bar{S} \to \bar{X}/\bar{S}$, *and*

$$\hat{R}^{CH}_S(g_2 \circ g_1) = \hat{R}^{CH}_S(g_1) \circ \hat{R}^{CH}_S(g_2) : \hat{R}_{(\bar{X},\bar{D})/S} \to \hat{R}_{(\bar{X}'',\bar{D}'')/S}$$

(iv) *For*

$$Q^* := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U^n_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U^{n-1}_\beta/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with U^*_α *smooth, we get from (i),(ii) and (iii) the map in* $C(\mathrm{Var}(k)^{2,smpr}/S)$

$$T(\hat{R}^{CH}, R^{CH})(Q^*) : \hat{R}^{CH}(Q^*) := (\cdots \to \oplus_{\beta \in \Lambda^{n-1}} \varinjlim_{(\bar{X}^{n-1}_\beta, \bar{D}^{n-1}_\beta)/S} \hat{R}_{(\bar{X}^{n-1}_\beta, \bar{D}^{n-1}_\beta)/S}(\mathbb{Z}(U^{n-1}_\beta/S))$$
$$\xrightarrow{(\hat{R}^{CH}_S(g^n_{\alpha,\beta}))} \oplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}^n_\alpha, \bar{D}^n_\alpha)/S} \hat{R}_{(\bar{X}^n_\alpha, \bar{D}^n_\alpha)/S}(\mathbb{Z}(U^n_\alpha/S)) \to \cdots) \to L\rho_{S*}\mu_{S*}R^{CH}(Q^*),$$

where for $(U^n_\alpha, h^n_\alpha) \in \mathrm{Var}(k)/S$, *the inductive limit run over all the compactifications* $\bar{f}_\alpha : \bar{X}_\alpha \to \bar{S}$ *of* $h_\alpha : U_\alpha \to S$ *with* $\bar{X}_\alpha \in \mathrm{PSmVar}(k)$ *and* $\bar{D}_\alpha := \bar{X}_\alpha \backslash U_\alpha$ *a normal crossing divisor. For* $m = (m^*) : Q^*_1 \to Q^*_2$ *a morphism with*

$$Q^*_1 := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U^n_{1,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U^{n-1}_{1,\beta}/S) \to \cdots),$$
$$Q^*_2 := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U^n_{2,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U^{n-1}_{2,\beta}/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

complexes of (maybe infinite) direct sum of representable presheaves with $U^*_{1,\alpha}$ and $U^*_{2,\alpha}$ smooth, we get again from (i),(ii) and (iii) a commutative diagram in $C(\mathrm{Var}(k)^{2,smpr}/S)$

$$\begin{tikzcd} \hat{R}^{CH}(Q^*_2) \arrow[r, "T(\hat{R}^{CH}_S, R^{CH}_S)(Q^*_2)"] \arrow[d, "\hat{R}^{CH}_S(m):=(\hat{R}^{CH}_S(m^*))"'] & L\rho_{S*}\mu_{S*}R^{CH}(Q^*_2) \arrow[d, "L\rho_{S*}\mu_{S*}R^{CH}_S(m):=L\rho_{S*}\mu_{S*}(R^{CH}_S(m^*))"] \\ \hat{R}^{CH}(Q^*_1) \arrow[r, "T(\hat{R}^{CH}_S, R^{CH}_S)(Q^*_1)"] & L\rho_{S*}\mu_{S*}R^{CH}(Q^*_1) \end{tikzcd}.$$

- Let $S \in \mathrm{Var}(k)$ For $(h, m, m') = (h^*, m^*, m^{'*}) : Q^*_1[1] \to Q^*_2$ an homotopy with $Q^*_1, Q^*_2 \in C(\mathrm{Var}(k)/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U^*_{1,\alpha}$ and $U^*_{2,\alpha}$ smooth,

$$(\hat{R}^{CH}_S(h), \hat{R}^{CH}_S(m), \hat{R}^{CH}_S(m')) = (\hat{R}^{CH}_S(h^*), \hat{R}^{CH}_S(m^*), \hat{R}^{CH}_S(m^{'*})) : R^{CH}(Q^*_2)[1] \to R^{CH}(Q^*_1)$$

  is an homotopy in $C(\mathrm{Var}(k)^{2,smpr}/S)$ using definition 27 (iii). In particular if $m : Q^*_1 \to Q^*_2$ with $Q^*_1, Q^*_2 \in C(\mathrm{Var}(k)/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U^*_{1,\alpha}$ and $U^*_{2,\alpha}$ smooth is an homotopy equivalence, then $\hat{R}^{CH}_S(m) : \hat{R}^{CH}(Q^*_2) \to \hat{R}^{CH}(Q^*_1)$ is an homotopy equivalence.

- Let $S \in \mathrm{SmVar}(k)$. Let $F \in \mathrm{PSh}(\mathrm{Var}(k)^{sm}/S)$. Consider

$$q : LF := (\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\mathbb{Z}(U_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\mathbb{Z}(U_\alpha/S) \to \cdots) \to F$$

  the canonical projective resolution given in subsection 2.3.3. Note that the U_α are smooth since S is smooth and h_α are smooth morphism. Definition 27(iv) gives in this particular case the map in $C(\mathrm{Var}(k)^2/S)$

$$T(\hat{R}^{CH}_S, R^{CH}_S)(\rho^*_S LF) : \hat{R}^{CH}(\rho^*_S LF) := (\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S} \varinjlim_{(\bar{X}_\alpha,\bar{D}_\alpha)/S} \hat{R}_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S))$$

$$\xrightarrow{(\hat{R}^{CH}_S(g^n_{\alpha,\beta}))} \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S} \varinjlim_{(\bar{X}_\alpha,\bar{D}_\alpha)/S} \hat{R}_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \to \cdots) \to L\rho_{S*}\mu_{S*}R^{CH}(\rho^*_S LF),$$

  where for $(U_\alpha, h_\alpha) \in \mathrm{Var}(k)^{sm}/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \to \bar{S}$ of $h_\alpha : U_\alpha \to S$ with $\bar{X}_\alpha \in \mathrm{PSmVar}(k)$ and $\bar{D}_\alpha := \bar{X}_\alpha \backslash U_\alpha$ a normal crossing divisor. Definition 27(iv) gives then by functoriality in particular, for $F = F^\bullet \in C(\mathrm{Var}(k)^{sm}/S)$, the map in $C(\mathrm{Var}(k)^{2,smpr}/S)$

$$T(\hat{R}^{CH}_S, R^{CH}_S)(\rho^*_S LF) : \hat{R}^{CH}(\rho^*_S LF) \to L\rho_{S*}\mu_{S*}R^{CH}(\rho^*_S LF).$$

- Let $g : T \to S$ a morphism with $T, S \in \mathrm{SmVar}(k)$. Let $h : U \to S$ a smooth morphism with $U \in \mathrm{Var}(k)$. Consider the cartesian square

$$\begin{tikzcd} U_T \arrow[r, "h'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ U \arrow[r, "h"] & S \end{tikzcd}$$

  Note that U is smooth since S and h are smooth, and U_T is smooth since T and h' are smooth. Take, see definition-proposition 3(ii),a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}$ of $h : U \to S$ and a compactification $\bar{f}'_0 = g \bar{\circ} h' : \bar{X}'_0 \to \bar{S}$ of $g \circ h' : U' \to S$ such that $g' : U_T/S \to U/S$ extend to a morphism $\bar{g}'_0 : \bar{X}'_0/\bar{S} \to \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \backslash U$ and $\bar{Z}' = \bar{X}'_0 \backslash U_T$. Take, see definition-proposition 3(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of $(\bar{X}_0, \bar{Z})$, a desingularization

$\bar{\epsilon}'_{\bullet} : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ of $(\bar{X}'_0, \bar{Z}')$ and a morphism $\bar{g}' : \bar{X}' \to \bar{X}$ such that the following diagram commutes

$$\begin{tikzcd}
\bar{X}'_0 \arrow[r, "\bar{g}'_0"] & \bar{X}_0 \\
\bar{X}' \arrow[u, "\bar{\epsilon}'"] \arrow[r, "\bar{g}'"] & \bar{X} \arrow[u, "\bar{\epsilon}"']
\end{tikzcd} .$$

We then have, see definition-proposition 3(ii), the following commutative diagram in $\mathrm{Fun}(\Delta, \mathrm{Var}(k))$

$$\begin{tikzcd}
U = U_{c(\bullet)} \arrow[r, "j"] & \bar{X} = \bar{X}_{c(\bullet)} & & \bar{D}_{s_{g'}(\bullet)} \arrow[ll, "i_{\bullet}"'] \\
U_T = U_{T,c(\bullet)} \arrow[u, "g'"] \arrow[r, "j'"] & \bar{X}' = X'_{c(\bullet)} \arrow[u, "\bar{g}'"] & \bar{D}'_{\bullet} \arrow[l, "i'_{\bullet}"'] & \bar{g}^{-1}(\bar{D}_{s_{g'}(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g'_{\bullet}}"'] \arrow[u, "(\bar{g}')'_{\bullet}"']
\end{tikzcd}$$

We then consider the following map in $C(\mathrm{Var}(k)^{2,pr}/T)$,

$$\begin{aligned}
&T(g, \hat{R}^{CH})(\mathbb{Z}(U/S)) : g^* \hat{R}_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \\
&\xrightarrow{:=} g^* \operatorname{Cone}(\mathbb{Z}(i_{\bullet} \times I) : (\mathbb{Z}^{tr}((\bar{D}_{\bullet} \times S, D_{\bullet})/S), u_{IJ}) \to \mathbb{Z}^{tr}((\bar{X} \times S, X)/S))(-d_X)[-2d_X] \\
&\xrightarrow{T(g,L)(-) \circ T(g,c)(-)} \\
&\operatorname{Cone}(\mathbb{Z}(i'_{g\bullet} \times I) : (\mathbb{Z}^{tr}((\bar{D}_{\bullet} \times T, \bar{g}^{-1}(D_{s_g(\bullet)})/T), u_{IJ}) \to \mathbb{Z}^{tr}((\bar{X} \times T, X')/T)))(-d_X)[-2d_X] \\
&\xrightarrow{([\Gamma_{\bar{g}'_{\bullet}}]^t, [\Gamma_{\bar{g}}]^t)} \\
&\operatorname{Cone}(\mathbb{Z}(i'_{g\bullet} \times I) : \\
&(\mathbb{Z}^{tr}((\bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) \times T, \bar{g}^{-1}(D_{s_g(\bullet)})/T), u_{IJ}) \to \mathbb{Z}^{tr}((\bar{X}' \times T, X')/T)))(-d_{X'})[-2d_{X'}] \\
&\xrightarrow{(\mathbb{Z}(i''_{g\bullet} \times I), I)(-d_{X'})[-2d_{X'}]} \\
&\operatorname{Cone}(\mathbb{Z}(i'_{\bullet} \times I) : ((\mathbb{Z}^{tr}((\bar{D}'_{\bullet} \times T, D'_{\bullet})/T), u_{IJ}) \to \mathbb{Z}^{tr}((\bar{X}' \times S, X')/T)))(-d_{X'})[-2d_{X'}] \\
&\xrightarrow{=:} \hat{R}_{(\bar{X}',\bar{D}')/T}(\mathbb{Z}(U_T/T))
\end{aligned}$$

For

$$Q^* := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U^n_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U^{n-1}_\beta/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with $h^n_\alpha : U^n_\alpha \to S$ smooth, we get the map in $C(\mathrm{Var}(k)^{2,smpr}/T)$

$$\begin{aligned}
&T(g, \hat{R}^{CH})(Q^*) : g^* \hat{R}^{CH}(Q^*) = (\cdots \to \oplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}^n_\alpha, \bar{D}^n_\alpha)/S} g^* \hat{R}_{(\bar{X}^n_\alpha, \bar{D}^n_\alpha)/S}(\mathbb{Z}(U^n_\alpha/S)) \to \cdots) \\
&\xrightarrow{(T(g, \hat{R}^{CH})(\mathbb{Z}(U^n_\alpha/S)))} (\cdots \to \oplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}^{n\prime}_\alpha, \bar{D}^{n\prime}_\alpha)/T} \hat{R}_{(\bar{X}^{n\prime}_\alpha, \bar{D}^{n\prime}_\alpha)/T}(\mathbb{Z}(U^n_{\alpha,T}/S)) \to \cdots) =: \hat{R}^{CH}(g^* Q^*)
\end{aligned}$$

together with the commutative diagram in $C(\mathrm{Var}(k)^{2,smpr}/T)$

$$\begin{tikzcd}
g^* \hat{R}^{CH}(Q^*) \arrow[r, "T(g, \hat{R}^{CH})(Q^*)"] \arrow[d, "g^* T(\hat{R}^{CH}_S, R^{CH}_S)(Q^*)"'] & \hat{R}^{CH}(g^* Q^*) \arrow[d, "T(\hat{R}^{CH}_T, R^{CH}_T)(g^* Q)"] \\
g^* L\rho_{S*}\mu_{S*} R^{CH}(Q^*) \arrow[r, "T(g, R^{CH})(Q^*) \circ T(g,\mu)(-) \circ T(g,\rho)(-) \circ T(g,L)(-)"] & L\rho_{T*}\mu_{T*} R^{CH}(g^* Q^*)
\end{tikzcd} .$$

Let $F \in \mathrm{PSh}(\mathrm{Var}(k)^{sm}/S)$. Consider

$$q : LF := (\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\mathbb{Z}(U_\alpha/S) \to \cdots) \to F$$

the canonical projective resolution given in subsection 2.3.3. We then get in particular the map in $C(\mathrm{Var}(k)^{2,smpr}/T)$

$$T(g,\hat{R}^{CH})(\rho_S^*LF) : g^*\hat{R}^{CH}(\rho_S^*LF) =$$
$$(\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S} \varinjlim_{(\bar{X}_\alpha,\bar{D}_\alpha)/S} g^*\hat{R}_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \to \cdots) \xrightarrow{(T(g,\hat{R}^{CH})(\mathbb{Z}(U_\alpha/S)))}$$
$$(\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S} \varinjlim_{(\bar{X}'_\alpha,\bar{D}'_\alpha)/T} \hat{R}_{(\bar{X}'_\alpha,\bar{D}'_\alpha)/T}(\mathbb{Z}(U_{\alpha,T}/S)) \to \cdots) =: \hat{R}^{CH}(\rho_T^*g^*LF),$$

and by functoriality, we get in particular for $F = F^\bullet \in C(\mathrm{Var}(k)^{sm}/S)$, the map in $C(\mathrm{Var}(k)^{2,smpr}/T)$

$$T(g,\hat{R}^{CH})(\rho_S^*LF) : g^*\hat{R}^{CH}(\rho_S^*LF) \to \hat{R}^{CH}(\rho_T^*g^*LF)$$

together with the commutative diagram in $C(\mathrm{Var}(k)^{2,smpr}/T)$

$$\begin{tikzcd} g^*\hat{R}^{CH}(\rho_S^*LF) \arrow[r, "T(g,\hat{R}^{CH})(\rho_S^*LF)"] \arrow[d, "g^*T(\hat{R}_S^{CH},R_S^{CH})(\rho_S^*LF)"'] & \hat{R}^{CH}(\rho_T^*g^*LF) \arrow[d, "T(\hat{R}_T^{CH},R_T^{CH})(\rho_T^*g^*LF)"] \\ g^*L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*LF) \arrow[r, "L\rho_{T*}\mu_{T*}T(g,R^{CH})(\rho_S^*LF)\circ T(g,\mu)(-)\circ T(g,\rho)(-)\circ T(g,L)(-)"] & L\rho_{T*}\mu_{T*}R^{CH}(\rho_T^*g^*LF) \end{tikzcd}.$$

- Let $S_1, S_2 \in \mathrm{SmVar}(k)$ and $p : S_1 \times S_2 \to S_1$ the projection. Let $h : U \to S_1$ a smooth morphism with $U \in \mathrm{Var}(k)$. Consider the cartesian square

$$\begin{tikzcd} U \times S_2 \arrow[r, "h\times I"] \arrow[d, "p'"] & S_1 \times S_2 \arrow[d, "p"] \\ U \arrow[r, "h"] & S_1 \end{tikzcd}$$

Take, see definition-proposition 3(i),a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}_1$ of $h : U \to S_1$. Then $\bar{f}_0 \times I : \bar{X}_0 \times S_2 \to \bar{S}_1 \times S_2$ is a compactification of $h \times I : U \times S_2 \to S_1 \times S_2$ and $p' : U \times S_2 \to U$ extend to $\bar{p}'_0 := p_{X_0} : \bar{X}_0 \times S_2 \to \bar{X}_0$. Denote $Z = X_0 \backslash U$. Take see theorem 19(i), a strict desingularization $\bar{\epsilon} : (\bar{X},\bar{D}) \to (\bar{X}_0,\bar{Z})$ of the pair $(\bar{X}_0,\bar{Z})$. We then have the commutative diagram (8) in $\mathrm{Fun}(\Delta,\mathrm{Var}(k))$ whose squares are cartesian

$$\begin{tikzcd} U = U_{c(\bullet)} \arrow[r, "j"] & \bar{X} & \bar{D}_\bullet \arrow[l, "i_\bullet"'] \\ U \times S_2 = (U\times S_2)_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j\times I"] & \bar{X} \times S_2 \arrow[u, "\bar{p}' := p_{\bar{X}}"'] & \bar{D}_\bullet \times S_2 \arrow[l, "i'_\bullet"'] \arrow[u, "\bar{p}'_\bullet"'] \end{tikzcd}$$

Then the map in $C(\mathrm{Var}(k)^{2,smpr}/S_1 \times S_2)$

$$T(p,\hat{R}^{CH})(\mathbb{Z}(U/S_1)) : p^*\hat{R}_{(\bar{X},\bar{D})/S_1}(\mathbb{Z}(U/S_1)) \xrightarrow{\sim} \hat{R}_{(\bar{X}\times S_2,\bar{D}_\bullet\times S_2)/S_1\times S_2}(\mathbb{Z}(U \times S_2/S_1 \times S_2))$$

is an isomorphism. Hence, for $Q^* \in C(\mathrm{Var}(k)/S_1)$ a complex of (maybe infinite) direct sum of representable presheaves of smooth morphism, the map in $C(\mathrm{Var}(k)^{2,smpr}/S_1 \times S_2)$

$$T(p,\hat{R}^{CH})(Q^*) : p^*\hat{R}^{CH}(Q^*) \xrightarrow{\sim} \hat{R}^{CH}(p^*Q^*)$$

is an isomorphism. In particular, for $F\in C(\mathrm{Var}(k)^{sm}/S_1)$ the map in $C(\mathrm{Var}(k)^{2,smpr}/S_1\times S_2)$

$$T(p,\hat{R}^{CH})(\rho_{S_1}^*LF):p^*\hat{R}^{CH}(\rho_{S_1}^*LF)\xrightarrow{\sim}\hat{R}^{CH}(\rho_{S_1\times S_2}^*p^*LF)$$

is an isomorphism.

- Let $h_1:U_1\to S$, $h_2:U_2\to S$ two morphisms with $U_1,U_2,S\in\mathrm{Var}(k)$, U_1,U_2 smooth. Denote by $p_1:U_1\times_S U_2\to U_1$ and $p_2:U_1\times_S U_2\to U_2$ the projections. Take, see definition-proposition 3(i)), a compactification $\bar{f}_{10}=\bar{h}_1:\bar{X}_{10}\to\bar{S}$ of $h_1:U_1\to S$ and a compactification $\bar{f}_{20}=\bar{h}_2:\bar{X}_{20}\to\bar{S}$ of $h_2:U_2\to S$. Then,

  - $\bar{f}_{10}\times\bar{f}_{20}:\bar{X}_{10}\times_{\bar{S}}\bar{X}_{20}\to S$ is a compactification of $h_1\times h_2:U_1\times_S U_2\to S$.
  - $\bar{p}_{10}:=p_{X_{10}}:\bar{X}_{10}\times_{\bar{S}}\bar{X}_{20}\to\bar{X}_{10}$ is a compactification of $p_1:U_1\times_S U_2\to U_1$.
  - $\bar{p}_{20}:=p_{X_{20}}:\bar{X}_{10}\times_{\bar{S}}\bar{X}_{20}\to\bar{X}_{20}$ is a compactification of $p_2:U_1\times_S U_2\to U_2$.

  Denote $\bar{Z}_1=\bar{X}_{10}\backslash U_1$ and $\bar{Z}_2=\bar{X}_{20}\backslash U_2$. Take, see theorem 19(i), a strict desingularization $\bar{\epsilon}_1:(\bar{X}_1,\bar{D})\to(\bar{X}_{10},Z_1)$ of the pair $(\bar{X}_{10},\bar{Z}_1)$ and a strictdesingularization $\bar{\epsilon}_2:(\bar{X}_2,\bar{E})\to(\bar{X}_{20},Z_2)$ of the pair $(\bar{X}_{20},\bar{Z}_2)$. Take then a strict desingularization

$$\bar{\epsilon}_{12}:((\bar{X}_1\times_{\bar{S}}\bar{X}_2)^N,\bar{F})\to(\bar{X}_1\times_{\bar{S}}\bar{X}_2,(D\times_{\bar{S}}\bar{X}_2)\cup(\bar{X}_1\times_{\bar{S}}\bar{E}))$$

  of the pair $(\bar{X}_1\times_{\bar{S}}\bar{X}_2,(\bar{D}\times_{\bar{S}}\bar{X}_2)\cup(\bar{X}_1\times_{\bar{S}}\bar{E}))$. We have then the following commutative diagram

$$\begin{tikzcd}
 & \bar{X}_1 \arrow[r, "\bar{f}_1"] & \bar{S} \\
 & \bar{X}_1\times_{\bar{S}}\bar{X}_2 \arrow[u, "\bar{p}_2"] \arrow[r, "\bar{p}_1"] & \bar{X}_2 \arrow[u, "\bar{f}_2"'] \\
(\bar{X}_1\times_{\bar{S}}\bar{X}_2)^N \arrow[uur, "(\bar{p}_2)^N"] \arrow[ur, "\bar{\epsilon}_{12}"] \arrow[urr, "(\bar{p}_1)^N"'] & &
\end{tikzcd}$$

  and

  - $\bar{f}_1\times\bar{f}_2:\bar{X}_1\times_{\bar{S}}\bar{X}_2\to\bar{S}$ is a compactification of $h_1\times h_2:U_1\times_S U_2\to S$.
  - $(\bar{p}_1)^N:=\bar{p}_1\circ\epsilon_{12}:(\bar{X}_1\times_{\bar{S}}\bar{X}_2)^N\to\bar{X}_1$ is a compactification of $p_1:U_1\times_S U_2\to U_1$.
  - $(\bar{p}_2)^N:=\bar{p}_2\circ\epsilon_{12}:(\bar{X}_1\times_{\bar{S}}\bar{X}_2)^N\to\bar{X}_2$ is a compactification of $p_2:U_1\times_S U_2\to U_2$.

  We have then the morphism in $C(\mathrm{Var}(k)^{2,smpr}/S)$

$$T(\otimes,\hat{R}^{CH}_S)(\mathbb{Z}(U_1/S),\mathbb{Z}(U_2/S)):=\hat{R}^{CH}_S(p_1)\otimes\hat{R}^{CH}_S(p_2):$$
$$\hat{R}_{(\bar{X}_1,\bar{D})/S}(\mathbb{Z}(U_1/S))\otimes\hat{R}_{(X_2,E))/S}(\mathbb{Z}(U_2/S))\xrightarrow{\sim}\hat{R}_{(\bar{X}_1\times_{\bar{S}}\bar{X}_2)^N,\bar{F})/S}(\mathbb{Z}(U_1\times_S U_2/S))$$

  For

$$Q_1^*:=(\cdots\to\oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_{1,\alpha}/S)\xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))}\oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_{1,\beta}/S)\to\cdots),$$
$$Q_2^*:=(\cdots\to\oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_{2,\alpha}/S)\xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))}\oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_{2,\beta}/S)\to\cdots)\in C(\mathrm{Var}(k)/S)$$

  complexes of (maybe infinite) direct sum of representable presheaves with U^*_α smooth, we get the morphism in $C(\mathrm{Var}(k)^{2,smpr}/S)$

$$T(\otimes,\hat{R}^{CH}_S)(Q_1^*,Q_2^*):\hat{R}^{CH}(Q_1^*)\otimes R^{CH}(Q_2^*)\xrightarrow{(T(\otimes,\hat{R}^{CH}_S)(\mathbb{Z}(U^m_{1,\alpha}),\mathbb{Z}(U^n_{2,\beta}))}\hat{R}^{CH}(Q_1^*\otimes Q_2^*))$$

, together with the commutative diagram in $C(\mathrm{Var}(k)^{2,smpr}/S)$

$$\begin{tikzcd}
\hat{R}^{CH}(Q_1^*)\otimes R^{CH}(Q_2^*) \arrow[r, "T(\otimes,\hat{R}_S^{CH})(Q_1^*,Q_2)"] \arrow[d, "T(\hat{R}_S^{CH},R_S^{CH})(Q_1^*)\otimes T(\hat{R}_S^{CH},R_S^{CH})(Q_2^*)"'] & \hat{R}^{CH}(Q_1^*\times Q_2^*) \arrow[d, "T(\hat{R}_S^{CH},R_S^{CH})(Q_1^*\otimes Q_2^*)"] \\
L\rho_{S*}\mu_{S*}(R^{CH}(Q_1^*)\otimes R^{CH}(Q_2^*)) \arrow[r, "L\rho_{S*}\mu_{S*}T(\otimes,R_S^{CH})(Q_1^*,Q_2)"] & L\rho_{S*}\mu_{S*}R^{CH}(Q_1^*\otimes Q_2)
\end{tikzcd}.$$

For $F_1, F_2 \in C(\mathrm{Var}(k)^{sm}/S)$, we get in particular the morphism in $C(\mathrm{Var}(k)^2/S)$

$$T(\otimes, R_S^{CH})(\rho_S^* LF_1, \rho_S^* LF_2) : R^{CH}(\rho_S^* LF_1) \otimes R^{CH}(\rho_S^* LF_2) \to R^{CH}(\rho_S^*(LF_1 \otimes LF_2))$$

together with the commutative diagram in $C(\mathrm{Var}(k)^{2,smpr}/S)$

$$\begin{tikzcd}
\hat{R}^{CH}(\rho_S^*LF_1)\otimes R^{CH}(\rho_S^*LF_2) \arrow[r, "T(\otimes,\hat{R}_S^{CH})(\rho_S^*LF_1,\rho_S^*LF_2)"] \arrow[d, "T(\hat{R}_S^{CH},R_S^{CH})(\rho_S^*LF_1)\otimes T(\hat{R}_S^{CH},R_S^{CH})(\rho_S^*LF_2)"'] & \hat{R}^{CH}(\rho_S^*LF_1\times\rho_S^*LF_2) \arrow[d, "T(\hat{R}_S^{CH},R_S^{CH})(\rho_S^*LF_1\otimes\rho_S^*"] \\
L\rho_{S*}\mu_{S*}(R^{CH}(\rho_S^*LF_1)\otimes R^{CH}(\rho_S^*LF_2)) \arrow[r, "L\rho_{S*}\mu_{S*}T(\otimes,R_S^{CH})(\rho_S^*LF_1,\rho_S^*LF_2)"] & L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*LF_1\times\rho_S^*LF_2)
\end{tikzcd}$$

For $S \in \mathrm{Var}(k)$, we will use rather the functors R_S^{0CH} and $\hat{R}_S^{0CH}$ since we are working in the image of the graph functor $\mathrm{Gr}_S^{12} : \mathrm{Var}(k)/S \to \mathrm{Var}(k)^2/S$. We have the full subcategory $\mathrm{SmVar}(k)/S \subset \mathrm{Var}(k)/S$ whose objects are morphisms $f : X \to S$ with $X \in \mathrm{SmVar}(k)$. Then $\mathrm{Gr}_S^{12}(\mathrm{SmVar}(k)/S) \subset \mathrm{Var}(k)^{2,smpr}/S$. If $S \in \mathrm{SmVar}(k)$, we have the factorization of morphism of site

$$\mathrm{Gr}_S^{12} : \mathrm{Var}(k)^{2,smpr}/S \xrightarrow{\mathrm{Gr}_S^{12}} \mathrm{SmVar}(k)/S \xrightarrow{\rho_S} \mathrm{Var}(k)^{sm}/S.$$

Definition 28. *(i) Let $h : U \to S$ a morphism, with $U, S \in \mathrm{Var}(k)$ and U smooth. Take, see definition-proposition 3, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $\bar{Z} = \bar{X}_0 \backslash U$. Take, using theorem 19(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$, with $\bar{X} \in \mathrm{PSmVar}(k)$ and $\bar{D} := \epsilon^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. We denote by $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X} = \bar{X}_{c(\bullet)}$ the morphism of simplicial varieties given by the closed embeddings $i_I : \bar{D}_I = \cap_{i \in I} \bar{D}_i \hookrightarrow \bar{X}$ We denote by $j : U \hookrightarrow \bar{X}$ the open embedding. We then consider the following map in $C(\mathrm{Var}(k)/S)$*

$$\begin{aligned}
&r^0_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) : R^0_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \\
&\xrightarrow{:=} \bar{f}_* E_{et}(\mathrm{Cone}(\mathbb{Z}(i_\bullet) : (\mathbb{Z}((\bar{D}_\bullet)/\bar{X}), u_{IJ}) \to \mathbb{Z}((\bar{X}/\bar{X})))) \\
&\xrightarrow{\bar{f}_* E_{et}(0, k \circ \mathrm{ad}(j^*, j_*)(\mathbb{Z}(\bar{X}/\bar{X})))} h_* E_{et}(\mathbb{Z}(U/U)) =: \mathbb{D}_S^0(\mathbb{Z}(U/S)).
\end{aligned}$$

Note that $\mathbb{Z}(\bar{D}_I/\bar{X})$ and $\mathbb{Z}(\bar{X}/\bar{X})$ are obviously $\mathbb{A}^1$ invariant. Note that $r_{(X,D)/S}$ is NOT an equivalence $(\mathbb{A}^1, et)$ local by proposition 4 since $\rho_{\bar{X}*}\mathbb{Z}(\bar{D}_\bullet/\bar{X}) = 0$, and $\rho_{\bar{X}*}\,\mathrm{ad}(j^*, j_*)(\mathbb{Z}(\bar{X}/\bar{X}))$ is not an equivalence $(\mathbb{A}^1, et)$ local.

(ii) Let $g : U'/S \to U/S$ a morphism, with $U'/S = (U', h'), U/S = (U, h) \in \mathrm{Var}(k)/S$, with U and U' smooth. Take, see definition-proposition 3(ii),a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}$ of $h : U \to S$ and a compactification $\bar{f}'_0 = \bar{h}' : \bar{X}'_0 \to \bar{S}$ of $h' : U' \to S$ such that $g : U'/S \to U/S$ extend to a morphism $\bar{g}_0 : \bar{X}'_0/\bar{S} \to \bar{X}_0/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \backslash U$ and $\bar{Z}' = \bar{X}'_0 \backslash U'$. Take, see definition-proposition 3(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of $(\bar{X}_0, \bar{Z})$, a strict desingularization $\bar{\epsilon}'_\bullet : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ of $(\bar{X}'_0, \bar{Z}')$ and a morphism $\bar{g} : \bar{X}' \to \bar{X}$ such that the following diagram commutes

$$\begin{tikzcd}
\bar{X}'_0 \arrow[r, "\bar{g}_0"] & \bar{X}_0 \\
\bar{X}' \arrow[u, "\bar{\epsilon}'"] \arrow[r, "\bar{g}"] & \bar{X} \arrow[u, "\bar{\epsilon}"']
\end{tikzcd}.$$

We then have, see definition-proposition 3(ii), the commutative diagram (7) in $\mathrm{Fun}(\Delta,\mathrm{Var}(k))$

$$\begin{tikzcd}
U=U_{c(\bullet)} \arrow[r, "j"] & \bar{X}=\bar{X}_{c(\bullet)} & & \bar{D}_{s_g(\bullet)} \arrow[ll, "i_\bullet"'] \\
U'=U'_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j'"] & \bar{X}'=\bar{X}'_{c(\bullet)} \arrow[u, "\bar{g}"] & \bar{D}'_\bullet \arrow[l, "i'_\bullet"'] & \bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g\bullet}"'] \arrow[u, "\bar{g}'_\bullet"']
\end{tikzcd}$$

We then consider the following map in $C(\mathrm{Var}(k)/S)$

$$\begin{aligned}
R^{0CH}_S(g) : R^0_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) &\xrightarrow{:=} \\
\bar{f}_*E_{et}(\mathrm{Cone}(\mathbb{Z}(i_\bullet) : (\mathbb{Z}((\bar{D}_{s_g(\bullet)})/\bar{X}), u_{IJ}) \to \mathbb{Z}(\bar{X}/\bar{X}))) &\xrightarrow{T(\bar{g},E)(-)\circ p_{S*}\mathrm{ad}(\bar{g}^*,\bar{g}_*)(-)} \\
\bar{f}'_*E_{et}(\mathrm{Cone}(\mathbb{Z}(i'_{g\bullet}) : (\mathbb{Z}((\bar{g}^{-1}(\bar{D}_{s_g(\bullet)}))/\bar{X}'), u_{IJ}) \to \mathbb{Z}((\bar{X}'/\bar{X}'))) &\xrightarrow{\bar{f}'_*E_{et}(\mathbb{Z}(i''_{g\bullet}),I)} \\
\bar{f}'_*E_{et}(\mathrm{Cone}(\mathbb{Z}(i'_\bullet) : (\mathbb{Z}(\bar{D}'_\bullet/\bar{X}'), u_{IJ}) \to \mathbb{Z}(\bar{X}'/\bar{X}'))) & \\
\xrightarrow{=:} R^0_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S)) &
\end{aligned}$$

Then by the diagram (7) and adjonction, the following diagram in $C(\mathrm{Var}(k)/S)$ *obviously commutes*

$$\begin{tikzcd}
R^0_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \arrow[r, "r_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))"] \arrow[d, "R^{0CH}_S(g)"'] & h_*E_{et}(\mathbb{Z}(U/U) =: \mathbb{D}^0_S(\mathbb{Z}(U/S)) \arrow[d, "D_S(g):=T(g,E)(-)\circ\mathrm{ad}(g^*,g_*)(E_{et}(\mathbb{Z}(U/U)))"] \\
R^0_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S)) \arrow[r, "r^0_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S))"] & h'_*E_{et}(\mathbb{Z}(U'/U')) =: \mathbb{D}^0_S(\mathbb{Z}(U'/S))
\end{tikzcd}.$$

(iii) *For* $g_1 : U''/S \to U'/S$, $g_2 : U'/S \to U/S$ *two morphisms with* $U''/S = (U', h''), U'/S = (U', h'), U/S = (U, h) \in \mathrm{Var}(k)/S$, *with* U, U' *and* U'' *smooth. We get from (i) and (ii) a compactification* $\bar{f} = \bar{h} : \bar{X} \to \bar{S}$ *of* $h : U \to S$, *a compactification* $\bar{f}' = \bar{h}' : \bar{X}' \to \bar{S}$ *of* $h' : U' \to S$, *and a compactification* $\bar{f}'' = \bar{h}'' : \bar{X}'' \to \bar{S}$ *of* $h'' : U'' \to S$, *with* $\bar{X}, \bar{X}', \bar{X}'' \in \mathrm{PSmVar}(k)$, $\bar{D} := \bar{X}\backslash U \subset \bar{X}$ $\bar{D}' := \bar{X}'\backslash U' \subset \bar{X}'$, *and* $\bar{D}'' := \bar{X}''\backslash U'' \subset \bar{X}''$ *normal crossing divisors, such that* $g_1 : U''/S \to U'/S$ *extend to* $\bar{g}_1 : \bar{X}''/\bar{S} \to \bar{X}'/\bar{S}$, $g_2 : U'/S \to U/S$ *extend to* $\bar{g}_2 : \bar{X}'/\bar{S} \to \bar{X}/\bar{S}$, *and*

$$R^{0CH}_S(g_2\circ g_1) = R^{0CH}_S(g_1)\circ R^{0CH}_S(g_2) : R^0_{(\bar{X},\bar{D})/S} \to R^0_{(\bar{X}'',\bar{D}'')/S}$$

(iv) *For*

$$Q^* := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_\beta/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with U^*_α *smooth, we get from (i), (ii) and (iii) the map in* $C(\mathrm{Var}(k)/S)$

$$\begin{aligned}
r^{0CH}_S(Q^*) : R^{0CH}(Q^*) := (\cdots \to \oplus_{\beta\in\Lambda^{n-1}} \varinjlim_{(\bar{X}^{n-1}_\beta,\bar{D}^{n-1}_\beta)/S} R^0_{(\bar{X}^{n-1}_\beta,\bar{D}^{n-1}_\beta)/S}(\mathbb{Z}(U^{n-1}_\beta/S)) \\
\xrightarrow{(R^{CH}_S(g^n_{\alpha,\beta}))} \oplus_{\alpha\in\Lambda^n} \varinjlim_{(\bar{X}^n_\alpha,\bar{D}^n_\alpha)/S} R^0_{(\bar{X}^n_\alpha,\bar{D}^n_\alpha)/S}(\mathbb{Z}(U^n_\alpha/S)) \to \cdots) \to \mathbb{D}_S(Q^*),
\end{aligned}$$

where for $(U^n_\alpha, h^n_\alpha) \in \mathrm{Var}(k)/S$, *the inductive limit run over all the compactifications* $\bar{f}_\alpha : \bar{X}_\alpha \to \bar{S}$ *of* $h_\alpha : U_\alpha \to S$ *with* $\bar{X}_\alpha \in \mathrm{PSmVar}(k)$ *and* $\bar{D}_\alpha := \bar{X}_\alpha\backslash U_\alpha$ *a normal crossing divisor. For* $m =$

$(m^*) : Q^*_1 \to Q^*_2$ a morphism with

$$Q^*_1 := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_{1,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_{1,\beta}/S) \to \cdots),$$
$$Q^*_2 := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_{2,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_{2,\beta}/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

complexes of (maybe infinite) direct sum of representable presheaves with $U^*_{1,\alpha}$ and $U^*_{2,\alpha}$ smooth, we get again from (i), (ii) and (iii) a commutative diagram in $C(\mathrm{Var}(k)/S)$

$$\begin{tikzcd}
R^{0CH}(Q^*_2) \arrow[r, "r^{0CH}_S(Q^*_2)"] \arrow[d, "R^{0CH}_S(m):=(R^{0CH}_S(m^*))"'] & \mathbb{D}^0_S(Q^*_2) \arrow[d, "\mathbb{D}_S(m):=(\mathbb{D}^0_S(m^*))"] \\
R^{0CH}(Q^*_1) \arrow[r, "r^{0CH}_S(Q^*_1)"] & \mathbb{D}^0_S(Q^*_1)
\end{tikzcd}.$$

(v) Let

$$Q^* := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_{\alpha}/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_{\beta}/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with U^*_α smooth, we have by definition

$$\mathrm{Gr}^{12*}_S R^{0CH}(Q^*) = R^{CH}(Q^*) \in C(\mathrm{Var}(k)^2/S).$$

- Let $S \in \mathrm{Var}(k)$ For $(h, m, m') = (h^*, m^*, m'^*) : Q^*_1[1] \to Q^*_2$ an homotopy with $Q^*_1, Q^*_2 \in C(\mathrm{Var}(k)/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U^*_{1,\alpha}$ and $U^*_{2,\alpha}$ smooth,

  $$(R^{0CH}_S(h), R^{0CH}_S(m), R^{0CH}_S(m')) = (R^{0CH}_S(h^*), R^{0CH}_S(m^*), R^{0CH}_S(m'^*)) : R^{0CH}(Q^*_2)[1] \to R^{0CH}(Q^*_1)$$

  is an homotopy in $C(\mathrm{Var}(k)/S)$ using definition 28 (iii). In particular if $m : Q^*_1 \to Q^*_2$ with $Q^*_1, Q^*_2 \in C(\mathrm{Var}(k)/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U^*_{1,\alpha}$ and $U^*_{2,\alpha}$ smooth is an homotopy equivalence, then $R^{0CH}_S(m) : R^{0CH}(Q^*_2) \to R^{0CH}(Q^*_1)$ is an homotopy equivalence.

- Let $S \in \mathrm{SmVar}(k)$. Let $F \in \mathrm{PSh}(\mathrm{Var}(k)^{sm}/S)$. Consider

  $$q : LF := (\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\mathbb{Z}(U_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\mathbb{Z}(U_\alpha/S) \to \cdots) \to F$$

  the canonical projective resolution given in subsection 2.3.3. Note that the U_α are smooth since S is smooth and h_α are smooth morphism. Definition 28(iv) gives in this particular case the map in $C(\mathrm{Var}(k)/S)$

  $$r^{0CH}_S(\rho^*_S LF) : R^{0CH}(\rho^*_S LF) := (\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S} \varinjlim_{(\bar{X}_\alpha,\bar{D}_\alpha)/S} R^0_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S))$$
  $$\xrightarrow{(R^{0CH}_S(g^n_{\alpha,\beta}))} \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S} \varinjlim_{(\bar{X}_\alpha,\bar{D}_\alpha)/S} R^0_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \to \cdots) \to \mathbb{D}^0_S(\rho^*_S LF),$$

  where for $(U_\alpha, h_\alpha) \in \mathrm{Var}(k)^{sm}/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \to \bar{S}$ of $h_\alpha : U_\alpha \to S$ with $\bar{X}_\alpha \in \mathrm{PSmVar}(\mathbb{C})$ and $\bar{D}_\alpha := \bar{X}_\alpha\backslash U_\alpha$ a normal crossing divisor. Definition 28(iv) gives then by functoriality in particular, for $F = F^\bullet \in C(\mathrm{Var}(k)^{sm}/S)$, the map in $C(\mathrm{Var}(k)/S)$

  $$r^{0CH}_S(\rho^*_S LF) = (r^{0CH}_S(\rho^*_S LF^*)) : R^{0CH}(\rho^*_S LF) \to \mathbb{D}^0_S(\rho^*_S LF).$$

- Let $g : T \to S$ a morphism with $T, S \in \mathrm{SmVar}(k)$. Let $h : U \to S$ a smooth morphism with $U \in \mathrm{Var}(k)$. Consider the cartesian square

$$\begin{tikzcd}
U_T \arrow[r, "h'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\
U \arrow[r, "h"] & S
\end{tikzcd}$$

Note that U is smooth since S and h are smooth, and U_T is smooth since T and h' are smooth. Take, see definition-proposition 3(ii),a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}$ of $h : U \to S$. Take, see definition-proposition 3(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of $(\bar{X}_0, \bar{Z})$. Then $\bar{f}'_0 = g \bar{\circ} h' : \bar{X}_T \to \bar{T}$ is a compactification of $g \circ h' : U_T \to S$ such that $g' : U_T/S \to U/S$ extend to a morphism $\bar{g}'_0 : \bar{X}_T/\bar{S} \to \bar{X}/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \backslash U$ and $\bar{Z}' = \bar{X}_T \backslash U_T$. Take, see definition-proposition 3(ii), a strict desingularization $\epsilon'_\bullet : (\bar{X}', \bar{D}') \to (\bar{X}_T, \bar{Z}')$ of $(\bar{X}_T, \bar{Z}')$. Denote $\bar{g}' = \bar{g}'_0 \circ \epsilon'_\bullet : \bar{X}' \to \bar{X}$. We then have, see definition-proposition 3(ii), the following commutative diagram in $\mathrm{Fun}(\Delta, \mathrm{Var}(k))$

$$\begin{tikzcd}
U = U_{c(\bullet)} \arrow[r, "j"] & \bar{X} = \bar{X}_{c(\bullet)} & & \bar{D}_{s_{g'}(\bullet)} \arrow[ll, "i_\bullet"'] \\
U_T = U_{T,c(\bullet)} \arrow[r, "j'"] \arrow[u, "g'"] & \bar{X}_T = X_{T,c(\bullet)} \arrow[u, "\bar{g}'_0"] & & \bar{g}'^{-1}(\bar{D}_{s_{g'}(\bullet)}) \arrow[ll, "i_{g\bullet}"'] \arrow[u, "(\bar{g}')'_\bullet"'] \\
U_T = U_{T,c(\bullet)} \arrow[r, "j'"] \arrow[u, "g'"] & \bar{X}' = X'_{c(\bullet)} \arrow[u, "\bar{g}'"] & \bar{D}'_\bullet \arrow[l, "i'_\bullet"'] & \bar{g}'^{-1}(\bar{D}_{s_{g'}(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g'\bullet}"'] \arrow[u, "\epsilon'_\bullet"']
\end{tikzcd}$$

We then consider the following map in $C(\mathrm{Var}(k)/T)$, see definition 28(ii)

$$\begin{aligned}
& T(g, R^{0CH})(\mathbb{Z}(U/S)) : g^* R^0_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \\
& \xrightarrow{g^* R^{0CH}_S(g')} g^* R^0_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U_T/S)) = g^* g_* R^0_{(\bar{X}',\bar{D}')/T}(\mathbb{Z}(U_T/T)) \\
& \xrightarrow{\mathrm{ad}(g^*, g_*)(R^0_{(\bar{X}',\bar{D}')/T}(\mathbb{Z}(U_T/T)))} R^0_{(\bar{X}',\bar{D}')/T}(\mathbb{Z}(U_T/T))
\end{aligned}$$

For

$$Q^* := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U^n_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U^{n-1}_\beta/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with $h^n_\alpha : U^n_\alpha \to S$ smooth, we get the map in $C(\mathrm{Var}(k)/T)$

$$\begin{aligned}
& T(g, R^{0CH})(Q^*) : g^* R^{0CH}(Q^*) = (\cdots \to \oplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}^n_\alpha, \bar{D}^n_\alpha)/S} g^* R^0_{(\bar{X}^n_\alpha, \bar{D}^n_\alpha)/S}(\mathbb{Z}(U^n_\alpha/S)) \to \cdots) \\
& \xrightarrow{(T(g, R^{0CH})(\mathbb{Z}(U^n_\alpha/S)))} (\cdots \to \oplus_{\alpha \in \Lambda^n} \varinjlim_{(\bar{X}^{n\prime}_\alpha, \bar{D}^{n\prime}_\alpha)/T} R^0_{(\bar{X}^{n\prime}_\alpha, \bar{D}^{n\prime}_\alpha)/T}(\mathbb{Z}(U^n_{\alpha,T}/S)) \to \cdots) =: R^{CH}(g^* Q^*).
\end{aligned}$$

Let $F \in \mathrm{PSh}(\mathrm{Var}(k)^{sm}/S)$. Consider

$$q : LF := (\cdots \to \oplus_{(U_\alpha, h_\alpha) \in \mathrm{Var}(k)^{sm}/S} \mathbb{Z}(U_\alpha/S) \to \cdots) \to F$$

the canonical projective resolution given in subsection 2.3.3. We then get in particular the map in

$C(\mathrm{Var}(k)/T)$

$$T(g,R^{0CH})(\rho_S^*LF):g^*R^{0CH}(\rho_S^*LF)=$$
$$(\cdots\to\oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\varinjlim_{(\bar X_\alpha,\bar D_\alpha)/S}g^*R^0_{(\bar X_\alpha,\bar D_\alpha)/S}(\mathbb{Z}(U_\alpha/S))\to\cdots)\xrightarrow{(T(g,R^{0CH})(\mathbb{Z}(U_\alpha/S)))}$$
$$(\cdots\to\oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\varinjlim_{(\bar X'_\alpha,\bar D'_\alpha)/T}R^0_{(\bar X'_\alpha,\bar D'_\alpha)/T}(\mathbb{Z}(U_{\alpha,T}/S))\to\cdots)=:R^{CH}(\rho_T^*g^*LF).$$

By functoriality, we get in particular for $F=F^\bullet\in C(\mathrm{Var}(k)^{sm}/S)$, the map in $C(\mathrm{Var}(k)/T)$

$$T(g,R^{0CH})(\rho_S^*LF):g^*R^{0CH}(\rho_S^*LF)\to R^{0CH}(\rho_T^*g^*LF).$$

- Let $S_1,S_2\in\mathrm{SmVar}(k)$ and $p:S_1\times S_2\to S_1$ the projection. Let $h:U\to S_1$ a smooth morphism with $U\in\mathrm{Var}(k)$. Consider the cartesian square

$$\begin{tikzcd} U\times S_2 \arrow[r, "h\times I"] \arrow[d, "p'"] & S_1\times S_2 \arrow[d, "p"] \\ U \arrow[r, "h"] & S_1 \end{tikzcd}$$

Take, see definition-proposition 3(i),a compactification $\bar f_0=\bar h:\bar X_0\to\bar S_1$ of $h:U\to S_1$. Then $\bar f_0\times I:\bar X_0\times S_2\to\bar S_1\times S_2$ is a compactification of $h\times I:U\times S_2\to S_1\times S_2$ and $p':U\times S_2\to U$ extend to $\bar p'_0:=p_{X_0}:\bar X_0\times S_2\to\bar X_0$. Denote $Z=X_0\backslash U$. Take see theorem 19(i), a strict desingularization $\bar\epsilon:(\bar X,\bar D)\to(\bar X_0,\bar Z)$ of the pair $(\bar X_0,\bar Z)$. We then have the commutative diagram (8) in $\mathrm{Fun}(\Delta,\mathrm{Var}(k))$ whose squares are cartesian

$$\begin{tikzcd} U=U_{c(\bullet)} \arrow[r, "j"] & \bar X & \bar D_\bullet \arrow[l, "i_\bullet"'] \\ U\times S_2=(U\times S_2)_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j\times I"] & \bar X\times S_2 \arrow[u, "\bar p':=p_{\bar X}"] & \bar D_\bullet\times S_2 \arrow[l, "i'_\bullet"'] \arrow[u, "\bar p'_\bullet"] \end{tikzcd}$$

Then the map in $C(\mathrm{Var}(k)/S_1\times S_2)$

$$T(p,R^{0CH})(\mathbb{Z}(U/S_1)):p^*R^0_{(\bar X,\bar D)/S_1}(\mathbb{Z}(U/S_1))\xrightarrow{\sim}R^0_{(\bar X\times S_2,\bar D_\bullet\times S_2)/S_1\times S_2}(\mathbb{Z}(U\times S_2/S_1\times S_2))$$

is an isomorphism. Hence, for $Q^*\in C(\mathrm{Var}(k)/S_1)$ a complex of (maybe infinite) direct sum of representable presheaves of smooth morphism, the map in $C(\mathrm{Var}(k)/S_1\times S_2)$

$$T(p,R^{0CH})(Q^*):p^*R^{0CH}(Q^*)\xrightarrow{\sim}R^{0CH}(p^*Q^*)$$

is an isomorphism. In particular, for $F\in C(\mathrm{Var}(k)^{sm}/S_1)$ the map in $C(\mathrm{Var}(k)/S_1\times S_2)$

$$T(p,R^{0CH})(\rho_{S_1}^*LF):p^*R^{0CH}(\rho_{S_1}^*LF)\xrightarrow{\sim}R^{0CH}(\rho_{S_1\times S_2}^*p^*LF)$$

is an isomorphism.

- Let $h_1:U_1\to S$, $h_2:U_2\to S$ two morphisms with $U_1,U_2,S\in\mathrm{Var}(k)$, U_1,U_2 smooth. Denote by $p_1:U_1\times_S U_2\to U_1$ and $p_2:U_1\times_S U_2\to U_2$ the projections. Take, see definition-proposition 3(i)), a compactification $\bar f_{10}=\bar h_1:\bar X_{10}\to\bar S$ of $h_1:U_1\to S$ and a compactification $\bar f_{20}=\bar h_2:\bar X_{20}\to\bar S$ of $h_2:U_2\to S$. Then,
  - $\bar f_{10}\times\bar f_{20}:\bar X_{10}\times_{\bar S}\bar X_{20}\to S$ is a compactification of $h_1\times h_2:U_1\times_S U_2\to S$.

- $\bar{p}_{10} := p_{X_{10}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \to \bar{X}_{10}$ is a compactification of $p_1 : U_1 \times_S U_2 \to U_1$.
- $\bar{p}_{20} := p_{X_{20}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \to \bar{X}_{20}$ is a compactification of $p_2 : U_1 \times_S U_2 \to U_2$.

Denote $\bar{Z}_1 = \bar{X}_{10} \backslash U_1$ and $\bar{Z}_2 = \bar{X}_{20} \backslash U_2$. Take, see theorem 19(i), a strict desingularization $\bar{\epsilon}_1 : (\bar{X}_1, \bar{D}) \to (\bar{X}_{10}, Z_1)$ of the pair $(\bar{X}_{10}, \bar{Z}_1)$ and a strictdesingularization $\bar{\epsilon}_2 : (\bar{X}_2, \bar{E}) \to (\bar{X}_{20}, Z_2)$ of the pair $(\bar{X}_{20}, \bar{Z}_2)$. Take then a strict desingularization

$$\bar{\epsilon}_{12} : ((\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F}) \to (\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (D \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$$

of the pair $(\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (\bar{D} \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$. We have then the following commutative diagram

$$\begin{tikzcd}
 & \bar{X}_1 \arrow[r, "\bar{f}_1"] & \bar{S} \\
 & \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \arrow[u, "\bar{p}_2"] \arrow[r, "\bar{p}_1"] & \bar{X}_2 \arrow[u, "\bar{f}_2"'] \\
(\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \arrow[uur, "(\bar{p}_2)^N"] \arrow[ur, "\bar{\epsilon}_{12}"] \arrow[urr, "(\bar{p}_1)^N"'] & &
\end{tikzcd}$$

and

- $\bar{f}_1 \times \bar{f}_2 : \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \to \bar{S}$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \to S$.
- $(\bar{p}_1)^N := \bar{p}_1 \circ \epsilon_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \to \bar{X}_1$ is a compactification of $p_1 : U_1 \times_S U_2 \to U_1$.
- $(\bar{p}_2)^N := \bar{p}_2 \circ \epsilon_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \to \bar{X}_2$ is a compactification of $p_2 : U_1 \times_S U_2 \to U_2$.

We have then the morphism in $C(\mathrm{Var}(k)/S)$

$$T(\otimes, R_S^{0CH})(\mathbb{Z}(U_1/S), \mathbb{Z}(U_2/S)) := R_S^{0CH}(p_1) \otimes R_S^{CH}(p_2) : \\ R^0_{(\bar{X}_1, \bar{D})/S}(\mathbb{Z}(U_1/S)) \otimes R^0_{(X_2, E))/S}(\mathbb{Z}(U_2/S)) \xrightarrow{\sim} R^0_{(\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F})/S}(\mathbb{Z}(U_1 \times_S U_2/S))$$

For

$$Q_1^* := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U^n_{1,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U^{n-1}_{1,\beta}/S) \to \cdots),$$

$$Q_2^* := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U^n_{2,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U^{n-1}_{2,\beta}/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

complexes of (maybe infinite) direct sum of representable presheaves with U^*_α smooth, we get the morphism in $C(\mathrm{Var}(k)/S)$

$$T(\otimes, R_S^{0CH})(Q_1^*, Q_2^*) : R^{0CH}(Q_1^*) \otimes R^{0CH}(Q_2^*) \xrightarrow{(T(\otimes, R_S^{0CH})(\mathbb{Z}(U^m_{1,\alpha}), \mathbb{Z}(U^n_{2,\beta})))} R^{0CH}(Q_1^* \otimes Q_2^*)).$$

For $F_1, F_2 \in C(\mathrm{Var}(k)^{sm}/S)$, we get in particular the morphism in $C(\mathrm{Var}(k)/S)$

$$T(\otimes, R_S^{0CH})(\rho_S^* LF_1, \rho_S^* LF_2) : R^{0CH}(\rho_S^* LF_1) \otimes R^{0CH}(\rho_S^* LF_2) \to R^{0CH}(\rho_S^*(LF_1 \otimes LF_2)).$$

Definition 29. *Let $h : U \to S$ a morphism, with $U, S \in \mathrm{Var}(k)$, U irreducible. Take, see definition-proposition 3, $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ a compactification of $h : U \to S$ and denote by $\bar{Z} = \bar{X}_0 \backslash U$. Take, using theorem 19, a desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \Delta)$ of the pair $(\bar{X}_0, \Delta)$, $\bar{Z} \subset \Delta$, with $\bar{X} \in \mathrm{PSmVar}(k)$ and $\bar{D} := \bar{\epsilon}^{-1}(\Delta) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ a normal crossing divisor. Denote $d_X := \dim(\bar{X}) = \dim(U)$.*

(i) The diagonal $\Delta_{\bar{D}_\bullet} \subset \bar{D}_\bullet \times \bar{D}_\bullet$ induces the morphism in $C(\mathrm{Var}(k)/S)$

$$[\Delta_{\bar{D}_\bullet}] \in \mathrm{Hom}(\mathbb{Z}^{tr}(\bar{D}_\bullet/S), \bar{f}_* E_{et}(\mathbb{Z}(\bar{D}_\bullet/\bar{X})(d_X)[2d_X])) \xrightarrow{\sim} \\ \mathrm{Hom}(\mathbb{Z}(\bar{D}_\bullet \times_S \bar{X}/\bar{X}), \mathbb{Z}^{tr}(\bar{D}_\bullet \times \mathbb{P}^{d_X}/\bar{X})/\mathbb{Z}^{tr}(\bar{D}_\bullet \times \mathbb{P}^{d_X - 1}/\bar{X})) \\ \subset H^0(\mathcal{Z}_{d_{D_\bullet}}(\square^* \times \bar{D}_\bullet \times_S \bar{D}_\bullet))$$

(ii) The cycle $\Delta_{\bar{X}} \subset \bar{X} \times_S \bar{X}$ *induces by the morphism in* $C(\mathrm{Var}(k)/S)$

$$\begin{aligned}[\Delta_{\bar{X}}] \in \mathrm{Hom}(\mathbb{Z}^{tr}(\bar{X}/S), \bar{f}_* E_{et}(\mathbb{Z}(\bar{X}/\bar{X})(d_X)[2d_X])) \xrightarrow{\sim} \\ \mathrm{Hom}(\mathbb{Z}(\bar{X} \times_S \bar{X}/\bar{X}), \mathbb{Z}^{tr}(\bar{X} \times \mathbb{P}^{d_X}/\bar{X})/\mathbb{Z}^{tr}(\bar{X} \times \mathbb{P}^{d_X-1}/\bar{X})) \\ \subset H^0(\mathcal{Z}_{d_X}(\square^* \times \bar{X} \times_S \bar{X}))\end{aligned}$$

Let $h : U \to S$ *a morphism, with* $U, S \in \mathrm{Var}(k)$*,* U *smooth connected (hence irreducible by smoothness). Take, see definition-proposition 3,* $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ *a compactification of* $h : U \to S$ *and denote by* $\bar{Z} = \bar{X}_0 \backslash U$*. Take, using theorem 19(ii), a strict desingularization* $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ *of the pair* $(\bar{X}_0, \bar{Z})$ *with* $\bar{X} \in \mathrm{PSmVar}(k)$ *and* $\bar{D} := \bar{\epsilon}^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ *a normal crossing divisor. Denote* $d_X := \dim(\bar{X}) = \dim(U)$*. We get from (i) and (ii) the morphism in* $C(\mathrm{Var}(k)/S)$

$$\begin{aligned}T(\bar{f}_\sharp, \bar{f}_*)(\mathbb{Z}(D_\bullet/\bar{X}), \mathbb{Z}(\bar{X}/\bar{X})) := ([\Delta_{\bar{D}_\bullet}], [\Delta_{\bar{X}}]) : \\ \mathrm{Cone}(\mathbb{Z}(i_\bullet) : (\mathbb{Z}^{tr}(\bar{D}_\bullet/S), u_{IJ}) \to \mathbb{Z}^{tr}(\bar{X}/S)) \to \\ \bar{f}_* E_{et}(\mathrm{Cone}(\mathbb{Z}(i_\bullet) : (\mathbb{Z}(\bar{D}_\bullet/\bar{X}), u_{IJ}) \to \mathbb{Z}(\bar{X}/\bar{X})))(d_X)[2d_X] \\ =: R^0_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S))(d_X)[2d_X].\end{aligned}$$

Definition 30. *(i) Let* $h : U \to S$ *a morphism, with* $U, S \in \mathrm{Var}(k)$ *and* U *smooth. Take, see definition-proposition 3,* $\bar{f}_0 = \bar{h}_0 : \bar{X}_0 \to \bar{S}$ *a compactification of* $h : U \to S$ *and denote by* $\bar{Z} = \bar{X}_0 \backslash U$*. Take, using theorem 19(ii), a strict desingularization* $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ *of the pair* $(\bar{X}_0, \bar{Z})$*, with* $\bar{X} \in \mathrm{PSmVar}(k)$ *and* $\bar{D} := \epsilon^{-1}(\bar{Z}) = \cup_{i=1}^s \bar{D}_i \subset \bar{X}$ *a normal crossing divisor. We denote by* $i_\bullet : \bar{D}_\bullet \hookrightarrow \bar{X} = \bar{X}_{c(\bullet)}$ *the morphism of simplicial varieties given by the closed embeddings* $i_I : \bar{D}_I = \cap_{i \in I} \bar{D}_i \hookrightarrow \bar{X}$ *We denote by* $j : U \hookrightarrow \bar{X}$ *the open embedding. We then consider the map in* $C(\mathrm{Var}(k)/S)$

$$\begin{aligned}T(\hat{R}^{0CH}, R^{0CH})(\mathbb{Z}(U/S)) : \hat{R}^0_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \\ \xrightarrow{:=} \mathrm{Cone}(\mathbb{Z}(i_\bullet) : (\mathbb{Z}^{tr}(D_\bullet/S), u_{IJ}) \to \mathbb{Z}^{tr}(X/S))(-d_X)[-2d_X] \\ \xrightarrow{T(\bar{f}_\sharp, \bar{f}_*)(\mathbb{Z}(\bar{D}_\bullet/\bar{X}), \mathbb{Z}(\bar{X}/\bar{X}))(-d_X)[-2d_X]} \\ R^0_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)).\end{aligned}$$

given in definition 25(iii).

(ii) Let $g : U'/S \to U/S$ *a morphism, with* $U'/S = (U', h'), U/S = (U, h) \in \mathrm{Var}(k)/S$*, with* U *and* U' *smooth. Take, see definition-proposition 3(ii),a compactification* $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}$ *of* $h : U \to S$ *and a compactification* $\bar{f}'_0 = \bar{h}' : \bar{X}'_0 \to \bar{S}$ *of* $h' : U' \to S$ *such that* $g : U'/S \to U/S$ *extend to a morphism* $\bar{g}_0 : \bar{X}'_0/\bar{S} \to \bar{X}_0/\bar{S}$*. Denote* $\bar{Z} = \bar{X}_0 \backslash U$ *and* $\bar{Z}' = \bar{X}'_0 \backslash U'$*. Take, see definition-proposition 3(ii), a strict desingularization* $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ *of* $(\bar{X}_0, \bar{Z})$*, a strict desingularization* $\bar{\epsilon}'_\bullet : (\bar{X}', \bar{D}') \to (\bar{X}'_0, \bar{Z}')$ *of* $(\bar{X}'_0, \bar{Z}')$ *and a morphism* $\bar{g} : \bar{X}' \to \bar{X}$ *such that the following diagram commutes*

$$\begin{tikzcd} \bar{X}'_0 \arrow[r, "\bar{g}_0"] & \bar{X}_0 \\ \bar{X}' \arrow[u, "\bar{\epsilon}'"] \arrow[r, "\bar{g}"] & \bar{X} \arrow[u, "\bar{\epsilon}"'] \end{tikzcd} .$$

We then have, see definition-proposition 3(ii), the diagram (7) in $\mathrm{Fun}(\Delta, \mathrm{Var}(k))$

$$\begin{tikzcd} U = U_{c(\bullet)} \arrow[r, "j"] & \bar{X} = \bar{X}_{c(\bullet)} & & \bar{D}_{s_g(\bullet)} \arrow[ll, "i_\bullet"'] \\ U' = U'_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j'"] & \bar{X}' = \bar{X}'_{c(\bullet)} \arrow[u, "\bar{g}"] & \bar{D}'_\bullet \arrow[l, "i'_\bullet"'] & \bar{g}^{-1}(\bar{D}_{s_g(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g\bullet}"'] \arrow[u, "\bar{g}'_\bullet"'] \end{tikzcd}$$

Consider

$$\begin{aligned}[\Gamma_{\bar{g}}]^t \in \mathrm{Hom}(\mathbb{Z}^{tr}(\bar{X}/S)(-d_X)[-2d_X], \mathbb{Z}^{tr}(\bar{X}'/S)(-d_{X'})[-2d_{X'}]) \\ \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}^{tr}(\bar{X}\times\mathbb{P}^{d_X}/S)/\mathbb{Z}_{tr}(\bar{X}\times\mathbb{P}^{d_X-1}/S), \\ \mathbb{Z}_{tr}(\bar{X}'\times\mathbb{P}^{d_{X'}}/S)/\mathbb{Z}_{tr}(\bar{X}'\times\mathbb{P}^{d_{X'}-1}/S)\end{aligned}$$

the morphism given by the transpose of the graph $\Gamma_g \subset X'\times_S X$ of $\bar{g}:\bar{X}'\to\bar{X}$. Then, since $i_\bullet\circ\bar{g}'_\bullet = \bar{g}\circ i''_{g\bullet} = \bar{g}\circ i'\circ\circ i'_{g\bullet}$, we have the factorization

$$\begin{aligned}[\Gamma_g]^t\circ\mathbb{Z}(i_\bullet) : (\mathbb{Z}^{tr}(\bar{D}_{s_g(\bullet)}/S), u_{IJ})(-d_X)[-2d_X] \\ \xrightarrow{[\Gamma_{\bar{g}'_\bullet}]^t} (\mathbb{Z}^{tr}(\bar{g}^{-1}(\bar{D}_{s_g(\bullet)})/S), u_{IJ})(-d_{X'})[-2d_{X'}] \\ \xrightarrow{\mathbb{Z}(i'_{g\bullet})} \mathbb{Z}^{tr}(\bar{X}'/S)(-d_{X'})[-2d_{X'}].\end{aligned}$$

with

$$\begin{aligned}[\Gamma_{\bar{g}'_\bullet}]^t \in \mathrm{Hom}((\mathbb{Z}^{tr}(\bar{D}_{s_g(\bullet)}\times\mathbb{P}^{d_X}/S), u_{IJ})/(\mathbb{Z}^{tr}(\bar{D}_{s_g(\bullet)}\times\mathbb{P}^{d_{X-1}}/S), u_{IJ}), \\ (\mathbb{Z}_{tr}(\bar{g}^{-1}(\bar{D}_{s_g(\bullet)})\times\mathbb{P}^{d_{X'}}/S), u_{IJ})/(\mathbb{Z}_{tr}(\bar{g}^{-1}(\bar{D}_{s_g(\bullet)})\times\mathbb{P}^{d_{X'-1}}/S), u_{IJ})).\end{aligned}$$

We then consider the following map in $C(\mathrm{Var}(k)/S)$

$$\begin{aligned}\hat{R}^{0CH}_S(g) : \hat{R}^0_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \xrightarrow{:=} \\ \mathrm{Cone}(\mathbb{Z}(i_\bullet) : (\mathbb{Z}^{tr}(\bar{D}_{s_g(\bullet)}/S), u_{IJ}) \to \mathbb{Z}^{tr}(\bar{X}/S)(-d_X)[-2d_X] \\ \xrightarrow{([\Gamma_{\bar{g}'_\bullet}]^t, [\Gamma_{\bar{g}}]^t)} \\ \mathrm{Cone}(\mathbb{Z}(i'_{g\bullet}) : (\mathbb{Z}^{tr}(\bar{g}^{-1}(\bar{D}_{s_g(\bullet)})/S), u_{IJ}) \to \mathbb{Z}^{tr}(\bar{X}'/S))(-d_{X'})[-2d_{X'}] \\ \xrightarrow{(\mathbb{Z}(i''_{g\bullet}), I)(-d_{X'})[-2d_{X'}]} \\ \mathrm{Cone}(\mathbb{Z}(i'_\bullet) : ((\mathbb{Z}^{tr}(\bar{D}'_\bullet/S), u_{IJ}) \to \mathbb{Z}^{tr}(\bar{X}'/S))(-d_{X'})[-2d_{X'}] \\ \xrightarrow{=:} \hat{R}^0_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S))\end{aligned}$$

Then the following diagram in $C(\mathrm{Var}(k)/S)$ commutes by definition

$$\begin{tikzcd}
\hat{R}^0_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \arrow[r, "T(\hat{R}^{0CH}, R^{0CH})(\mathbb{Z}(U/S))"] \arrow[d, "\hat{R}^{0CH}_S(g)"'] & R^0_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \arrow[d, "R^{0CH}_S(g)"] \\
\hat{R}^0_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S)) \arrow[r, "T(\hat{R}^{0CH}, R^{0CH})(\mathbb{Z}(U'/S))"] & R^0_{(\bar{X}',\bar{D}')/S}(\mathbb{Z}(U'/S))
\end{tikzcd} .$$

(iii) *For $g_1 : U''/S \to U'/S$, $g_2 : U'/S \to U/S$ two morphisms with $U''/S = (U', h''), U'/S = (U', h'), U/S = (U, h) \in \mathrm{Var}(k)/S$, with U, U' and U'' smooth. We get from (i) and (ii) a compactification $\bar{f} = \bar{h} : \bar{X} \to \bar{S}$ of $h : U \to S$, a compactification $\bar{f}' = \bar{h}' : \bar{X}' \to \bar{S}$ of $h' : U' \to S$, and a compactification $\bar{f}'' = \bar{h}'' : \bar{X}'' \to \bar{S}$ of $h'' : U'' \to S$, with $\bar{X}, \bar{X}', \bar{X}'' \in \mathrm{PSmVar}(k)$, $\bar{D} := \bar{X}\backslash U \subset \bar{X}$ $\bar{D}' := \bar{X}'\backslash U' \subset \bar{X}'$, and $\bar{D}'' := \bar{X}''\backslash U'' \subset \bar{X}''$ normal crossing divisors, such that $g_1 : U''/S \to U'/S$ extend to $\bar{g}_1 : \bar{X}''/\bar{S} \to \bar{X}'/\bar{S}$, $g_2 : U'/S \to U/S$ extend to $\bar{g}_2 : \bar{X}'/\bar{S} \to \bar{X}/\bar{S}$, and*

$$\hat{R}^{0CH}_S(g_2\circ g_1) = \hat{R}^{0CH}_S(g_1)\circ\hat{R}^{0CH}_S(g_2) : \hat{R}^0_{(\bar{X},\bar{D})/S} \to \hat{R}^0_{(\bar{X}'',\bar{D}'')/S}$$

(iv) *For*

$$Q^* := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_\beta/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with U^*_α smooth, we get from (i),(ii) and (iii) the map in $C(\mathrm{Var}(k)/S)$

$$T(\hat{R}^{0CH},R^{0CH})(Q^*):\hat{R}^{0CH}(Q^*):=(\cdots\to\oplus_{\beta\in\Lambda^{n-1}}\varinjlim_{(\bar{X}^{n-1}_\beta,\bar{D}^{n-1}_\beta)/S}\hat{R}^0_{(\bar{X}^{n-1}_\beta,\bar{D}^{n-1}_\beta)/S}(\mathbb{Z}(U^{n-1}_\beta/S))$$

$$\xrightarrow{(\hat{R}^{0CH}_S(g^n_{\alpha,\beta}))}\oplus_{\alpha\in\Lambda^n}\varinjlim_{(\bar{X}^n_\alpha,\bar{D}^n_\alpha)/S}\hat{R}^0_{(\bar{X}^n_\alpha,\bar{D}^n_\alpha)/S}(\mathbb{Z}(U^n_\alpha/S))\to\cdots)\to R^{0CH}(Q^*),$$

where for $(U^n_\alpha,h^n_\alpha)\in\mathrm{Var}(k)/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha:\bar{X}_\alpha\to\bar{S}$ of $h_\alpha:U_\alpha\to S$ with $\bar{X}_\alpha\in\mathrm{PSmVar}(k)$ and $\bar{D}_\alpha:=\bar{X}_\alpha\backslash U_\alpha$ a normal crossing divisor. For $m=(m^*):Q^*_1\to Q^*_2$ a morphism with

$$Q^*_1:=(\cdots\to\oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_{1,\alpha}/S)\xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))}\oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_{1,\beta}/S)\to\cdots),$$

$$Q^*_2:=(\cdots\to\oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_{2,\alpha}/S)\xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))}\oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_{2,\beta}/S)\to\cdots)\in C(\mathrm{Var}(k)/S)$$

complexes of (maybe infinite) direct sum of representable presheaves with $U^*_{1,\alpha}$ and $U^*_{2,\alpha}$ smooth, we get again from (i),(ii) and (iii) a commutative diagram in $C(\mathrm{Var}(k)/S)$

$$\begin{tikzcd}
\hat{R}^{0CH}(Q^*_2) \arrow[r, "T(\hat{R}^{0CH}_S,R^{0CH}_S)(Q^*_2)"] \arrow[d, "\hat{R}^{0CH}_S(m):=(\hat{R}^{0CH}_S(m^*))"'] & R^{0CH}(Q^*_2) \arrow[d, "(R^{0CH}_S(m^*))"] \\
\hat{R}^{0CH}(Q^*_1) \arrow[r, "T(\hat{R}^{0CH}_S,R^{0CH}_S)(Q^*_1)"] & R^{0CH}(Q^*_1)
\end{tikzcd}.$$

(v) *Let*

$$Q^*:=(\cdots\to\oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_\alpha/S)\xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))}\oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_\beta/S)\to\cdots)\in C(\mathrm{Var}(k)/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with U^*_α smooth, we have by definition

$$\mathrm{Gr}^{12*}_S\hat{R}^{0CH}(Q^*)=\hat{R}^{CH}(Q^*)\in C(\mathrm{Var}(k)^{2,smpr}/S).$$

- Let $S\in\mathrm{Var}(k)$ For $(h,m,m')=(h^*,m^*,m^{'*}):Q^*_1[1]\to Q^*_2$ an homotopy with $Q^*_1,Q^*_2\in C(\mathrm{Var}(k)/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U^*_{1,\alpha}$ and $U^*_{2,\alpha}$ smooth,

  $$(\hat{R}^{0CH}_S(h),\hat{R}^{0CH}_S(m),\hat{R}^{0CH}_S(m'))=(\hat{R}^{0CH}_S(h^*),\hat{R}^{0CH}_S(m^*),\hat{R}^{0CH}_S(m^{'*})):R^{0CH}(Q^*_2)[1]\to R^{0CH}(Q^*_1)$$

  is an homotopy in $C(\mathrm{Var}(k)/S)$ using definition 30 (iii). In particular if $m:Q^*_1\to Q^*_2$ with $Q^*_1,Q^*_2\in C(\mathrm{Var}(k)/S)$ complexes of (maybe infinite) direct sum of representable presheaves with $U^*_{1,\alpha}$ and $U^*_{2,\alpha}$ smooth is an homotopy equivalence, then $\hat{R}^{0CH}_S(m):\hat{R}^{0CH}(Q^*_2)\to\hat{R}^{0CH}(Q^*_1)$ is an homotopy equivalence.

- Let $S\in\mathrm{SmVar}(k)$. Let $F\in\mathrm{PSh}(\mathrm{Var}(k)^{sm}/S)$. Consider

  $$q:LF:=(\cdots\to\oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\mathbb{Z}(U_\alpha/S)\xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))}\oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\mathbb{Z}(U_\alpha/S)\to\cdots)\to F$$

  the canonical projective resolution given in subsection 2.3.3. Note that the U_α are smooth since S is smooth and h_α are smooth morphism. Definition 30(iv) gives in this particular case the map in $C(\mathrm{Var}(k)/S)$

  $$T(\hat{R}^{0CH}_S,R^{0CH}_S)(\rho^*_S LF):\hat{R}^{0CH}(\rho^*_S LF):=(\cdots\to\oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\varinjlim_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}\hat{R}^0_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S))$$

  $$\xrightarrow{(\hat{R}^{0CH}_S(g^n_{\alpha,\beta}))}\oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\varinjlim_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}\hat{R}^0_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S))\to\cdots)\to R^{0CH}(\rho^*_S LF),$$

where for $(U_\alpha, h_\alpha) \in \mathrm{Var}(k)^{sm}/S$, the inductive limit run over all the compactifications $\bar{f}_\alpha : \bar{X}_\alpha \to \bar{S}$ of $h_\alpha : U_\alpha \to S$ with $\bar{X}_\alpha \in \mathrm{PSmVar}(k)$ and $\bar{D}_\alpha := \bar{X}_\alpha \backslash U_\alpha$ a normal crossing divisor. Definition 30(iv) gives then by functoriality in particular, for $F = F^\bullet \in C(\mathrm{Var}(k)^{sm}/S)$, the map in $C(\mathrm{Var}(k)/S)$

$$T(\hat{R}^{0CH}_S, R^{0CH}_S)(\rho_S^* LF) : \hat{R}^{0CH}(\rho_S^* LF) \to R^{0CH}(\rho_S^* LF).$$

- Let $g : T \to S$ a morphism with $T, S \in \mathrm{SmVar}(k)$. Let $h : U \to S$ a smooth morphism with $U \in \mathrm{Var}(k)$. Consider the cartesian square

$$\begin{tikzcd} U_T \arrow[r, "h'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ U \arrow[r, "h"] & S \end{tikzcd}$$

Note that U is smooth since S and h are smooth, and U_T is smooth since T and h' are smooth. Take, see definition-proposition 3(ii),a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}$ of $h : U \to S$. Take, see definition-proposition 3(ii), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of $(\bar{X}_0, \bar{Z})$. Then $\bar{f}'_0 = g \bar{\circ} h' : \bar{X}_T \to \bar{T}$ is a compactification of $g \circ h' : U_T \to S$ such that $g' : U_T/S \to U/S$ extend to a morphism $\bar{g}'_0 : \bar{X}_T/\bar{S} \to \bar{X}/\bar{S}$. Denote $\bar{Z} = \bar{X}_0 \backslash U$ and $\bar{Z}' = \bar{X}_T \backslash U_T$. Take, see definition-proposition 3(ii), a strict desingularization $\epsilon'_\bullet : (\bar{X}', \bar{D}') \to (\bar{X}_T, \bar{Z}')$ of $(\bar{X}_T, \bar{Z}')$. Denote $\bar{g}' = \bar{g}'_0 \circ \epsilon'_\bullet : \bar{X}' \to \bar{X}$. We then have, see definition-proposition 3(ii), the following commutative diagram in $\mathrm{Fun}(\Delta, \mathrm{Var}(k))$

$$\begin{tikzcd} U = U_{c(\bullet)} \arrow[r, "j"] & \bar{X} = \bar{X}_{c(\bullet)} & & \bar{D}_{s_{g'}(\bullet)} \arrow[ll, "i_\bullet"'] \\ U_T = U_{T,c(\bullet)} \arrow[r, "j'"] \arrow[u, "g'"] & \bar{X}_T = X_{T,c(\bullet)} \arrow[u, "\bar{g}'_0"] & & \bar{g}'^{-1}(\bar{D}_{s_{g'}(\bullet)}) \arrow[ll, "i_{g\bullet}"'] \arrow[u, "(\bar{g}')'_\bullet"'] \\ U_T = U_{T,c(\bullet)} \arrow[r, "j'"] \arrow[u, "g'"] & \bar{X}' = X'_{c(\bullet)} \arrow[u, "\bar{g}'"] & \bar{D}'_\bullet \arrow[l, "i'_\bullet"'] & \bar{g}'^{-1}(\bar{D}_{s_{g'}(\bullet)}) : i'_{g\bullet} \arrow[l, "i''_{g'\bullet}"'] \arrow[u, "\epsilon'_\bullet"'] \end{tikzcd}$$

We then consider the following map in $C(\mathrm{Var}(k)/T)$,

$$\begin{aligned} T(g, \hat{R}^{0CH})(\mathbb{Z}(U/S)) : g^* \hat{R}^0_{(\bar{X},\bar{D})/S}(\mathbb{Z}(U/S)) \\ \xrightarrow{:=} g^* \mathrm{Cone}(\mathbb{Z}(i_\bullet) : (\mathbb{Z}^{tr}(\bar{D}_\bullet/S), u_{IJ}) \to \mathbb{Z}^{tr}(\bar{X}/S))(-d_X)[-2d_X] \\ \xrightarrow{=} \\ \mathrm{Cone}(\mathbb{Z}(i_{g\bullet}) : (\mathbb{Z}^{tr}(\bar{g}^{-1}(D_{s_g(\bullet)}))/T), u_{IJ}) \to \mathbb{Z}^{tr}(\bar{X}_T/T))(-d_X)[-2d_X] \\ \xrightarrow{(\mathbb{Z}(i''_{g\bullet}), [\Gamma_{\epsilon'}]^t)} \\ \mathrm{Cone}(\mathbb{Z}(i'_\bullet) : ((\mathbb{Z}^{tr}(\bar{D}'_\bullet/T), u_{IJ}) \to \mathbb{Z}^{tr}((\bar{X}'/T)))(-d_{X'})[-2d_{X'}] \\ \xrightarrow{=:} \hat{R}^0_{(\bar{X}',\bar{D}')/T}(\mathbb{Z}(U_T/T)) \end{aligned}$$

For

$$Q^* := (\cdots \to \oplus_{\alpha \in \Lambda^n} \mathbb{Z}(U^n_\alpha/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta \in \Lambda^{n-1}} \mathbb{Z}(U^{n-1}_\beta/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

a complex of (maybe infinite) direct sum of representable presheaves with $h^n_\alpha : U^n_\alpha \to S$ smooth, we get the map in $C(\mathrm{Var}(k)/T)$

$$T(g,\hat{R}^{0CH})(Q^*) : g^*\hat{R}^{0CH}(Q^*) = (\cdots \to \oplus_{\alpha\in\Lambda^n} \varinjlim_{(\bar{X}^n_\alpha,\bar{D}^n_\alpha)/S} g^*\hat{R}^0_{(\bar{X}^n_\alpha,\bar{D}^n_\alpha)/S}(\mathbb{Z}(U^n_\alpha/S)) \to \cdots)$$

$$\xrightarrow{(T(g,\hat{R}^{0CH})(\mathbb{Z}(U^n_\alpha/S)))} (\cdots \to \oplus_{\alpha\in\Lambda^n} \varinjlim_{(\bar{X}^{n\prime}_\alpha,\bar{D}^{n\prime}_\alpha)/T} \hat{R}_{(\bar{X}^{n\prime}_\alpha,\bar{D}^{n\prime}_\alpha)/T}(\mathbb{Z}(U^n_{\alpha,T}/S)) \to \cdots) =: \hat{R}^{CH}(g^*Q^*)$$

together with the commutative diagram in $C(\mathrm{Var}(k)/T)$

$$\begin{tikzcd} g^*\hat{R}^{0CH}(Q^*) \arrow[r, "T(g,\hat{R}^{0CH})(Q^*)"] \arrow[d, "g^*T(\hat{R}^{0CH}_S,R^{0CH}_S)(Q^*)"'] & \hat{R}^{0CH}(g^*Q^*) \arrow[d, "T(\hat{R}^{0CH}_T,R^{0CH}_T)(g^*Q)"] \\ g^*R^{0CH}(Q^*) \arrow[r, "T(g,R^{0CH})(Q^*)"] & R^{0CH}(g^*Q^*) \end{tikzcd}.$$

Let $F \in \mathrm{PSh}(\mathrm{Var}(k)^{sm}/S)$. Consider

$$q : LF := (\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S}\mathbb{Z}(U_\alpha/S) \to \cdots) \to F$$

the canonical projective resolution given in subsection 2.3.3. We then get in particular the map in $C(\mathrm{Var}(k)/T)$

$$T(g,\hat{R}^{0CH})(\rho_S^*LF) : g^*\hat{R}^{0CH}(\rho_S^*LF) =$$
$$(\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S} \varinjlim_{(\bar{X}_\alpha,\bar{D}_\alpha)/S} g^*\hat{R}_{(\bar{X}_\alpha,\bar{D}_\alpha)/S}(\mathbb{Z}(U_\alpha/S)) \to \cdots) \xrightarrow{(T(g,\hat{R}^{0CH})(\mathbb{Z}(U_\alpha/S)))}$$
$$(\cdots \to \oplus_{(U_\alpha,h_\alpha)\in\mathrm{Var}(k)^{sm}/S} \varinjlim_{(\bar{X}'_\alpha,\bar{D}'_\alpha)/T} \hat{R}_{(\bar{X}'_\alpha,\bar{D}'_\alpha)/T}(\mathbb{Z}(U_{\alpha,T}/S)) \to \cdots) =: \hat{R}^{0CH}(\rho_T^*g^*LF),$$

and by functoriality, we get in particular for $F = F^\bullet \in C(\mathrm{Var}(k)^{sm}/S)$, the map in $C(\mathrm{Var}(k)/T)$

$$T(g,\hat{R}^{0CH})(\rho_S^*LF) : g^*\hat{R}^{0CH}(\rho_S^*LF) \to \hat{R}^{0CH}(\rho_T^*g^*LF)$$

together with the commutative diagram in $C(\mathrm{Var}(k)/T)$

$$\begin{tikzcd} g^*\hat{R}^{0CH}(\rho_S^*LF) \arrow[r, "T(g,\hat{R}^{0CH})(\rho_S^*LF)"] \arrow[d, "g^*T(\hat{R}^{0CH}_S,R^{0CH}_S)(\rho_S^*LF)"'] & \hat{R}^{0CH}(\rho_T^*g^*LF) \arrow[d, "T(\hat{R}^{0CH}_T,R^{0CH}_T)(\rho_T^*g^*LF)"] \\ g^*L\rho_{S*}\mu_{S*}R^{CH}(\rho_S^*LF) \arrow[r, "T(g,R^{0CH})(\rho_S^*LF)"] & R^{CH}(\rho_T^*g^*LF) \end{tikzcd}.$$

- Let $S_1, S_2 \in \mathrm{SmVar}(k)$ and $p : S_1\times S_2 \to S_1$ the projection. Let $h : U \to S_1$ a smooth morphism with $U \in \mathrm{Var}(k)$. Consider the cartesian square

$$\begin{tikzcd} U\times S_2 \arrow[r, "h\times I"] \arrow[d, "p'"] & S_1\times S_2 \arrow[d, "p"] \\ U \arrow[r, "h"] & S_1 \end{tikzcd}$$

Take, see definition-proposition 3(i),a compactification $\bar{f}_0 = \bar{h} : \bar{X}_0 \to \bar{S}_1$ of $h : U \to S_1$. Then $\bar{f}_0 \times I : \bar{X}_0 \times S_2 \to \bar{S}_1 \times S_2$ is a compactification of $h\times I : U\times S_2 \to S_1\times S_2$ and $p' : U\times S_2 \to U$ extend to $\bar{p}'_0 := p_{X_0} : \bar{X}_0\times S_2 \to \bar{X}_0$. Denote $Z = X_0\backslash U$. Take see theorem 19(i), a strict desingularization $\bar{\epsilon} : (\bar{X}, \bar{D}) \to (\bar{X}_0, \bar{Z})$ of the pair $(\bar{X}_0, \bar{Z})$. We then have the commutative diagram (8) in $\mathrm{Fun}(\Delta, \mathrm{Var}(k))$ whose squares are cartesian

$$\begin{tikzcd}
U = U_{c(\bullet)} \arrow[r, "j"] & \bar{X} & \bar{D}_\bullet \arrow[l, "i_\bullet"'] \\
U \times S_2 = (U \times S_2)_{c(\bullet)} \arrow[u, "g"] \arrow[r, "j \times I"] & \bar{X} \times S_2 \arrow[u, "\bar{p}' := p_{\bar{X}}"'] & \bar{D}_\bullet \times S_2 \arrow[l, "i'_\bullet"'] \arrow[u, "\bar{p}'_\bullet"']
\end{tikzcd}$$

Then the map in $C(\mathrm{Var}(k)/S_1 \times S_2)$

$$T(p, \hat{R}^{0CH})(\mathbb{Z}(U/S_1)) : p^* \hat{R}^0_{(\bar{X}, \bar{D})/S_1}(\mathbb{Z}(U/S_1)) \xrightarrow{\sim} \hat{R}^0_{(\bar{X} \times S_2, \bar{D}_\bullet \times S_2)/S_1 \times S_2}(\mathbb{Z}(U \times S_2/S_1 \times S_2))$$

is an isomorphism. Hence, for $Q^* \in C(\mathrm{Var}(k)/S_1)$ a complex of (maybe infinite) direct sum of representable presheaves of smooth morphism, the map in $C(\mathrm{Var}(k)/S_1 \times S_2)$

$$T(p, \hat{R}^{0CH})(Q^*) : p^* \hat{R}^{0CH}(Q^*) \xrightarrow{\sim} \hat{R}^{0CH}(p^* Q^*)$$

is an isomorphism. In particular, for $F \in C(\mathrm{Var}(k)^{sm}/S_1)$ the map in $C(\mathrm{Var}(k)/S_1 \times S_2)$

$$T(p, \hat{R}^{0CH})(\rho_{S_1}^* LF) : p^* \hat{R}^{0CH}(\rho_{S_1}^* LF) \xrightarrow{\sim} \hat{R}^{0CH}(\rho_{S_1 \times S_2}^* p^* LF)$$

is an isomorphism.

- Let $h_1 : U_1 \to S$, $h_2 : U_2 \to S$ two morphisms with $U_1, U_2, S \in \mathrm{Var}(k)$, U_1, U_2 smooth. Denote by $p_1 : U_1 \times_S U_2 \to U_1$ and $p_2 : U_1 \times_S U_2 \to U_2$ the projections. Take, see definition-proposition 3(i)), a compactification $\bar{f}_{10} = \bar{h}_1 : \bar{X}_{10} \to \bar{S}$ of $h_1 : U_1 \to S$ and a compactification $\bar{f}_{20} = \bar{h}_2 : \bar{X}_{20} \to \bar{S}$ of $h_2 : U_2 \to S$. Then,

  - $\bar{f}_{10} \times \bar{f}_{20} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \to S$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \to S$.
  - $\bar{p}_{10} := p_{X_{10}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \to \bar{X}_{10}$ is a compactification of $p_1 : U_1 \times_S U_2 \to U_1$.
  - $\bar{p}_{20} := p_{X_{20}} : \bar{X}_{10} \times_{\bar{S}} \bar{X}_{20} \to \bar{X}_{20}$ is a compactification of $p_2 : U_1 \times_S U_2 \to U_2$.

  Denote $\bar{Z}_1 = \bar{X}_{10} \backslash U_1$ and $\bar{Z}_2 = \bar{X}_{20} \backslash U_2$. Take, see theorem 19(i), a strict desingularization $\bar{\epsilon}_1 : (\bar{X}_1, \bar{D}) \to (\bar{X}_{10}, Z_1)$ of the pair $(\bar{X}_{10}, \bar{Z}_1)$ and a strictdesingularization $\bar{\epsilon}_2 : (\bar{X}_2, \bar{E}) \to (\bar{X}_{20}, Z_2)$ of the pair $(\bar{X}_{20}, \bar{Z}_2)$. Take then a strict desingularization

  $$\bar{\epsilon}_{12} : ((\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N, \bar{F}) \to (\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (D \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$$

  of the pair $(\bar{X}_1 \times_{\bar{S}} \bar{X}_2, (\bar{D} \times_{\bar{S}} \bar{X}_2) \cup (\bar{X}_1 \times_{\bar{S}} \bar{E}))$. We have then the following commutative diagram

  $$\begin{tikzcd}
  & \bar{X}_1 \arrow[r, "\bar{f}_1"] & \bar{S} \\
  & \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \arrow[u, "\bar{p}_2"] \arrow[r, "\bar{p}_1"] & \bar{X}_2 \arrow[u, "\bar{f}_2"'] \\
  (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \arrow[ruu, "(\bar{p}_2)^N"] \arrow[ru, "\epsilon_{12}"] \arrow[rru, "(\bar{p}_1)^N"] & &
  \end{tikzcd}$$

  and

  - $\bar{f}_1 \times \bar{f}_2 : \bar{X}_1 \times_{\bar{S}} \bar{X}_2 \to \bar{S}$ is a compactification of $h_1 \times h_2 : U_1 \times_S U_2 \to S$.
  - $(\bar{p}_1)^N := \bar{p}_1 \circ \epsilon_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \to \bar{X}_1$ is a compactification of $p_1 : U_1 \times_S U_2 \to U_1$.

- $(\bar{p}_2)^N := \bar{p}_2 \circ \epsilon_{12} : (\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N \to \bar{X}_2$ is a compactification of $p_2 : U_1 \times_S U_2 \to U_2$.

We have then the morphism in $C(\mathrm{Var}(k)/S)$

$$T(\otimes, \hat{R}^{0CH}_S)(\mathbb{Z}(U_1/S), \mathbb{Z}(U_2/S)) := \hat{R}^{0CH}_S(p_1) \otimes \hat{R}^{0CH}_S(p_2) : \hat{R}^0_{(\bar{X}_1,\bar{D})/S}(\mathbb{Z}(U_1/S)) \otimes \hat{R}^0_{(X_2,E))/S}(\mathbb{Z}(U_2/S)) \xrightarrow{\sim} \hat{R}^0_{(\bar{X}_1 \times_{\bar{S}} \bar{X}_2)^N,\bar{F})/S}(\mathbb{Z}(U_1 \times_S U_2/S))$$

For

$$Q_1^* := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_{1,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_{1,\beta}/S) \to \cdots),$$

$$Q_2^* := (\cdots \to \oplus_{\alpha\in\Lambda^n}\mathbb{Z}(U^n_{2,\alpha}/S) \xrightarrow{(\mathbb{Z}(g^n_{\alpha,\beta}))} \oplus_{\beta\in\Lambda^{n-1}}\mathbb{Z}(U^{n-1}_{2,\beta}/S) \to \cdots) \in C(\mathrm{Var}(k)/S)$$

complexes of (maybe infinite) direct sum of representable presheaves with U^*_α smooth, we get the morphism in $C(\mathrm{Var}(k)/S)$

$$T(\otimes, \hat{R}^{0CH}_S)(Q_1^*, Q_2^*) : \hat{R}^{0CH}(Q_1^*) \otimes R^{0CH}(Q_2^*) \xrightarrow{(T(\otimes,\hat{R}^{CH}_S)(\mathbb{Z}(U^m_{1,\alpha}),\mathbb{Z}(U^n_{2,\beta}))} \hat{R}^{0CH}(Q_1^* \otimes Q_2^*))$$

, together with the commutative diagram in $C(\mathrm{Var}(k)/S)$

$$\begin{tikzcd} \hat{R}^{0CH}(Q_1^*) \otimes R^{0CH}(Q_2^*) \arrow[r, "T(\otimes,\hat{R}^{0CH}_S)(Q_1^*,Q_2)"] \arrow[d, "T(\hat{R}^{0CH}_S,R^{CH}_S)(Q_1^*)\otimes T(\hat{R}^{0CH}_S,R^{0CH}_S)(Q_2^*)"'] & \hat{R}^{0CH}(Q_1^* \times Q_2^*) \arrow[d, "T(\hat{R}^{0CH}_S,R^{0CH}_S)(Q_1^*\otimes Q_2^*)"] \\ R^{0CH}(Q_1^*) \otimes R^{0CH}(Q_2^*) \arrow[r, "T(\otimes,R^{0CH}_S)(Q_1^*,Q_2)"] & R^{0CH}(Q_1^* \otimes Q_2) \end{tikzcd}.$$

For $F_1, F_2 \in C(\mathrm{Var}(k)^{sm}/S)$, we get in particular the morphism in $C(\mathrm{Var}(k)/S)$

$$T(\otimes, R^{0CH}_S)(\rho_S^*LF_1, \rho_S^*LF_2) : R^{0CH}(\rho_S^*LF_1) \otimes R^{0CH}(\rho_S^*LF_2) \to R^{0CH}(\rho_S^*(LF_1 \otimes LF_2))$$

together with the commutative diagram in $C(\mathrm{Var}(k)/S)$

$$\begin{tikzcd} \hat{R}^{0CH}(\rho_S^*LF_1) \otimes R^{0CH}(\rho_S^*LF_2) \arrow[r, "T(\otimes,\hat{R}^{0CH}_S)(\rho_S^*LF_1,\rho_S^*LF_2)"] \arrow[d, "T(\hat{R}^{0CH}_S,R^{0CH}_S)(\rho_S^*LF_1)\otimes T(\hat{R}^{0CH}_S,R^{0CH}_S)(\rho_S^*LF_2)"'] & \hat{R}^{0CH}(\rho_S^*LF_1 \times \rho_S^*LF_2) \arrow[d, "T(\hat{R}^{0CH}_S,R^{0CH}_S)(\rho_S^*LF_1\otimes\rho_S^*LF_2)"] \\ R^{0CH}(\rho_S^*LF_1) \otimes R^{0CH}(\rho_S^*LF_2) \arrow[r, "T(\otimes,R^{0CH}_S)(\rho_S^*LF_1,\rho_S^*LF_2)"] & R^{0CH}(\rho_S^*LF_1 \times \rho_S^*LF_2) \end{tikzcd}$$

3 Triangulated category of motives

3.1 Definition and the six functor formalism

The category of motives is obtained by inverting the $(\mathbb{A}^1_S, et)$ equivalence. Hence the $\mathbb{A}^1_S$ local complexes of presheaves plays a key role.

Definition 31. *The derived category of motives of complex algebraic varieties over S is the category*

$$\mathrm{DA}(S) := \mathrm{Ho}_{\mathbb{A}^1_S,et}(C(\mathrm{Var}(k)^{sm}/S)),$$

which is the localization of the category of complexes of presheaves on $\mathrm{Var}(k)^{sm}/S$ with respect to $(\mathbb{A}^1_S, et)$ local equivalence and we denote by

$$D(\mathbb{A}^1_S, et) := D(\mathbb{A}^1_S) \circ D(et) : C(\mathrm{Var}(k)^{sm}/S) \to \mathrm{DA}(S)$$

the localization functor. We have $\mathrm{DA}^-(S) := D(\mathbb{A}^1_S, et)(\mathrm{PSh}(\mathrm{Var}(k)^{sm}/S, C^-(\mathbb{Z}))) \subset \mathrm{DA}(S)$ the full subcategory consisting of bounded above complexes.

Definition 32. *The stable derived category of motives of complex algebraic varieties over S is the category*

$$\mathrm{DA}_{st}(S) := \mathrm{Ho}_{\mathbb{A}^1_S,et}(C_\Sigma(\mathrm{Var}(k)^{sm}/S)),$$

which is the localization of the category of $\mathbb{G}_{mS}$-spectra ($\Sigma F^\bullet = F^\bullet \otimes \mathbb{G}_{mS}$) of complexes of presheaves on $\mathrm{Var}(k)^{sm}/S$ with respect to $(\mathbb{A}^1_S, et)$ local equivalence. The functor

$$\Sigma^\infty : C(\mathrm{Var}(k)^{sm}/S) \hookrightarrow C_\Sigma(\mathrm{Var}(k)^{sm}/S)$$

induces the functor $\Sigma^\infty : \mathrm{DA}(S) \to \mathrm{DA}_{st}(S)$.

We have all the six functor formalism by [12]. We give a list of the operation we will use :

- For $f : T \to S$ a morphism with $S, T \in \mathrm{Var}(k)$, the adjonction

$$(f^*, f_*) : C(\mathrm{Var}(k)^{sm}/S) \leftrightarrows C(\mathrm{Var}(k)^{sm}/T)$$

  is a Quillen adjonction which induces in the derived categories (f^* derives trivially), (f^*, Rf_*) : $\mathrm{DA}(S) \leftrightarrows \mathrm{DA}(T)$.

- For $h : V \to S$ a smooth morphism with $V, S \in \mathrm{Var}(k)$, the adjonction

$$(h_\sharp, h^*) : C(\mathrm{Var}(k)^{sm}/V) \leftrightarrows C(\mathrm{Var}(k)^{sm}/S)$$

  is a Quillen adjonction which induces in the derived categories (h^* derive trivially) $(Lh_\sharp, h^*) =:$ $\mathrm{DA}(V) \leftrightarrows \mathrm{DA}(S)$.

- For $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \mathrm{Var}(k)$,

$$(i_*, i^!) := (i_*, i^\perp) : C(\mathrm{Var}(k)^{sm}/Z) \leftrightarrows C(\mathrm{Var}(k)^{sm}/S)$$

  is a Quillen adjonction, which induces in the derived categories (i_* derive trivially) $(i_*, Ri^!)$: $\mathrm{DA}(Z) \leftrightarrows \mathrm{DA}(S)$. The fact that i_* derive trivially (i.e. send $(\mathbb{A}^1, et)$ local equivalence to $(\mathbb{A}^1, et)$ local equivalence is proved in [4].

- For $S \in \mathrm{Var}(k)$, the adjonction given by the tensor product of complexes of abelian groups and the internal hom of presheaves

$$((\cdot \otimes \cdot), \mathcal{H}om^\bullet(\cdot, \cdot)) : C(\mathrm{Var}(k)^{sm}/S)^2 \to C(\mathrm{Var}(k)^{sm}/S),$$

  is a Quillen adjonction, which induces in the derived category

$$,((\cdot \otimes^L \cdot), R\mathcal{H}om^\bullet(\cdot, \cdot)) : \mathrm{DA}(S)^2 \to \mathrm{DA}(S),$$

- Let $M, N \in \mathrm{DA}(S)$, $Q^\bullet$ projectively cofibrant such that $M = D(\mathbb{A}^1, et)(Q^\bullet)$, and $G^\bullet$ be $\mathbb{A}^1$ local for the etale topology such that $N = D(\mathbb{A}^1, et)(G^\bullet)$. Then,

$$R\mathcal{H}om^\bullet(M, N) = \mathcal{H}om^\bullet(Q^\bullet, E(G^\bullet)) \in \mathrm{DA}(S). \tag{9}$$

  This is well defined since if $s : Q_1 \to Q_2$ is a etale local equivalence,

$$\mathcal{H}om(s, E(G)) : \mathcal{H}om(Q_1, E(G)) \to \mathcal{H}om(Q_2, E(G))$$

  is a etale local equivalence for $1 \leq i \leq l$.

We get from the first point 2 functors :

- The 2-functor $C(\mathrm{Var}(k)^{sm}/\cdot) : \mathrm{Var}(k) \to Ab\,\mathrm{Cat}$, given by
$$S \mapsto C(\mathrm{Var}(k)^{sm}/S)\ ,\ (f : T \to S) \mapsto (f^* : C(\mathrm{Var}(k)^{sm}/S) \to C(\mathrm{Var}(k)^{sm}/T)).$$
- The 2-functor $\mathrm{DA}(\cdot) : \mathrm{Var}(k) \to \mathrm{TriCat}$, given by
$$S \mapsto \mathrm{DA}(S)\ ,\ (f : T \to S) \mapsto (f^* : \mathrm{DA}(S) \to \mathrm{DA}(T)).$$

The main theorem is the following :

Theorem 20. *[4][12] The 2-functor* $\mathrm{DA}(\cdot) : \mathrm{Var}(k) \to \mathrm{TriCat}$*, given by*
$$S \mapsto \mathrm{DA}(S)\ ,\ (f : T \to S) \mapsto (f^* : \mathrm{DA}(S) \to \mathrm{DA}(T))$$
is a 2-homotopic functor ([4])

From theorem 20, we get in particular

- For $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(k)$, there by theorem 20 is also a pair of adjoint functor
$$(f_!, f^!) : \mathrm{DA}(S) \leftrightarrows \mathrm{DA}(T)$$
  - with $f_! = Rf_*$ if f is proper,
  - with $f^! = f^*[d]$ if f is smooth of relative dimension d.

  For $h : U \to S$ a smooth morphism with $U, S \in \mathrm{Var}(k)$ irreducible, have, for $M \in \mathrm{DA}(U)$, an isomorphism
$$Lh_\sharp M \to h_! M[d], \tag{10}$$
  in $\mathrm{DA}(S)$.
- The 2-functor $S \in \mathrm{Var}(k) \mapsto \mathrm{DA}(S)$ satisfy the localization property, that is for $i : Z \hookrightarrow X$ a closed embedding with $Z, X \in \mathrm{Var}(k)$, denote by $j : S\backslash Z \hookrightarrow S$ the open complementary subset, we have for $M \in \mathrm{DA}(S)$ a distinguish triangle in $\mathrm{DA}(S)$
$$j_\sharp j^* M \xrightarrow{\mathrm{ad}(j_\sharp, j^*)(M)} M \xrightarrow{\mathrm{ad}(i^*, i_*)(M)} i_* i^* M \to j_\sharp j^* M[1]$$
  equivalently,
  - the functor
$$(i^*, j^*) : \mathrm{DA}(S) \xrightarrow{\sim} \mathrm{DA}(Z) \times \mathrm{DA}(S\backslash Z)$$
    is conservative,
  - and for $F \in C(\mathrm{Var}(k)^{sm}/Z)$, the adjonction map $\mathrm{ad}(i^*, i_*)(F) : i^* i_* F \to F$ is an equivalence Zariski local, hence for $M \in \mathrm{DA}(S)$, the induced map in the derived category
$$\mathrm{ad}(i^*, i_*)(M) : i^* i_* M \xrightarrow{\sim} M$$
    is an isomorphism.
- For $f : X \to S$ a proper map, $g : T \to S$ a morphism, with $T, X, S \in \mathrm{Var}(k)$, and $M \in \mathrm{DA}(X)$,
$$T(f, g)(M) : g^* Rf_* M \to Rf'_* \tilde{g}'^* M$$
  is an isomorphism in $\mathrm{DA}(T)$ if f is proper.

Definition 33. *The derived category of extended motives of complex algebraic varieties over S is the category*
$$\underline{\mathrm{DA}}(S) := \mathrm{Ho}_{\mathbb{A}^1_S, et}(C(\mathrm{Var}(k)/S)),$$
which is the localization of the category of complexes of presheaves on $\mathrm{Var}(k)/S$ *with respect to* $(\mathbb{A}^1_S, et)$ *local equivalence and we denote by*
$$D(\mathbb{A}^1_S, et) := D(\mathbb{A}^1_S) \circ D(et) : C(\mathrm{Var}(k)/S) \to \underline{\mathrm{DA}}(S)$$
the localization functor. We have $\underline{\mathrm{DA}}^-(S) := D(\mathbb{A}^1_S, et)(\mathrm{PSh}(\mathrm{Var}(k)/S, C^-(\mathbb{Z}))) \subset \underline{\mathrm{DA}}(S)$ *the full subcategory consisting of bounded above complexes.*

3.2 Constructible motives and resolution of a motive by Corti-Hanamura motives

We now give the definition of the motives of morphisms $f : X \to S$ which are constructible motives and the definition of the category of Corti-Hanamura motives.

Definition 34. *Let $S \in \mathrm{Var}(k)$,*

- *the homological motive functor is $M(/S) : \mathrm{Var}(k)/S \to \mathrm{DA}(S)$, $(f : X \to S) \mapsto M(X/S) := f_! f^! M(S/S)$,*
- *the cohomological motive functor is $M^\vee(/S) : \mathrm{Var}(k)/S \to \mathrm{DA}(S)$, $(f : X \to S) \mapsto M(X/S)^\vee := Rf_* M(X/X) := f_* E_{et}(\mathbb{Z}_X)$,*
- *the Borel-Moore motive functor is $M^{BM}(/S) : \mathrm{Var}(k)/S \to \mathrm{DA}(S)$, $(f : X \to S) \mapsto M^{BM}(X/S) := f_! M(X/X)$,*
- *the (homological) motive with compact support functor is $M_c(/S) : \mathrm{Var}(k)/S \to \mathrm{DA}(S)$, $(f : X \to S) \mapsto M_c(X/S) := Rf_* f^! M(S/S)$.*

Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$. Assume that there exist a factorization $f : X \xrightarrow{i} Y \times S \xrightarrow{p} S$, with $Y \in \mathrm{SmVar}(k)$, $i : X \hookrightarrow Y$ is a closed embedding and p the projection. Then,

$$Q(X/S) := p_\sharp \Gamma_X^\vee \mathbb{Z}_{Y \times S} \in C(\mathrm{Var}(k)^{sm}/S)$$

is projective, admits transfert, and satisfy $D(\mathbb{A}^1_S, et)(Q(X/S)) = M(X/S)$.

Definition 35. *(i) Let $S \in \mathrm{Var}(k)$. We define the full subcategory*

$$\begin{aligned} \mathrm{DA}_c(S) : &= < Rf_* \mathbb{Z}_X, (f : X \to S) \in \mathrm{Var}(k) > \\ &= < Rf_* \mathbb{Z}_X, (f : X \to S) \in \mathrm{Var}(k),\ \mathit{proper},\ X\ \mathit{smooth} \subset \mathrm{DA}(S) \end{aligned}$$

where $<, >$ denoted the full triangulated category generated by.

(ii) Let $X, S \in \mathrm{Var}(k)$. If $f : X \to S$ is proper (but not necessary smooth) and X is smooth, $M(X/S)$ is said to be a Corti-Hanamura motive and we have by above in this case $M(X/S) = M^{BM}(X/S)[c] = M(X/S)^\vee[c]$, with $c = \mathrm{codim}(X, X \times S)$ where $f : X \hookrightarrow X \times S \to S$. We denote by

$$\mathcal{CH}(S) = \{M(X/S)\}^{pa}_{\{X/S=(X,f), f pr, X sm\}} \subset DM(S)$$

the full subcategory which is the pseudo-abelian completion of the full subcategory whose objects are Corti-Hanamura motives.

(iii) We denote by

$$\mathcal{CH}^0(S) \subset \mathcal{CH}(S)$$

the full subcategory which is the pseudo-abelian completion of the full subcategory whose objects are Corti-Hanamura motives $M(X/S)$ such that the morphism $f : X \to S$ is projective.

For bounded above motives, we have

Theorem 21. *Let $S \in \mathrm{Var}(k)$.*

(i) There exists a unique weight structure ω on $\mathrm{DA}^-(S)$ such that $\mathrm{DA}^-(S)^{\omega=0} = \mathcal{CH}(S)$

(ii) There exist a well defined functor

$$W(S) : \mathrm{DA}^-(S) \to K^-(\mathcal{CH}(S)) \; , \; W(S)(M) = [M^{(\bullet)}]$$

where $M^{(\bullet)} \in C^-(\mathcal{CH}(S))$ is a bounded above weight complex, such that if $m \in \mathbb{Z}$ is the highest weight, we have a generalized distinguish triangle for all $i \leq m$

$$T_i : M^{(i)}[i] \to M^{(i+1)}[(i+1)] \to \cdots \to M^{(m)}[m] \to M^{w \geq i} \tag{11}$$

Moreover the maps $w(M)^{(\geq i)} : M^{\geq i} \to M$ induce an isomorphism $w(M) : holim_i M^{\geq i} \xrightarrow{\sim} M$ in $\mathrm{DA}^-(S)$.

(iii) Denote by $Chow(S)$ the category of Chow motives, which is the pseudo-abelian completion of the category

- *whose set of objects consist of the $X/S = (X, f) \in \mathrm{Var}(k)/S$ such that f is proper and X is smooth,*
- *whose set of morphisms between X_1/S and X_2/S is $\mathrm{CH}^{d_1}(X_1 \times_S X_2)$, and the composition law is given in [13].*

We have then a canonical functor $CH_S : Chow(S) \hookrightarrow \mathrm{DA}(S)$, with $CH_S(X/S) := M(X/S) := Rf_*\mathbb{Z}(X/X)$, which is a full embedding whose image is the category $\mathcal{CH}(S)$.

Proof. (i): The category $\mathrm{DA}(S)$ is clearly weakly generated by $\mathcal{CH}(S)$. Moreover $\mathcal{CH}(S) \subset \mathrm{DA}(S)$ is negative. Hence, the result follows from [8] theorem 4.3.2 III.
(ii): Follows from (i) by standard fact of weight structure on triangulated categories. See [8] theorem 3.2.2 and theorem 4.3.2 V for example.
(iii): See [16]. □

Theorem 22. *Let $S \in \mathrm{Var}(k)$.*

(i) There exists a unique weight structure ω on $\mathrm{DA}^-(S)$ such that $\mathrm{DA}^-(S)^{\omega=0} = \mathcal{CH}^0(S)$

(ii) There exist a well defined functor

$$W(S) : \mathrm{DA}^-(S) \to K^-(\mathcal{CH}^0(S)) \; , \; W(S)(M) = [M^{(\bullet)}]$$

where $M^{(\bullet)} \in C^-(\mathcal{CH}^0(S))$ is a bounded above weight complex, such that if $m \in \mathbb{Z}$ is the highest weight, we have a generalized distinguish triangle for all $i \leq m$

$$T_i : M^{(i)}[i] \to M^{(i+1)}[(i+1)] \to \cdots \to M^{(m)}[m] \to M^{w \geq i} \tag{12}$$

Moreover the maps $w(M)^{(\geq i)} : M^{\geq i} \to M$ induce an isomorphism $w(M) : holim_i M^{\geq i} \xrightarrow{\sim} M$ in $\mathrm{DA}^-(S)$.

Proof. Similar to the proof of theorem 21. □

Corollary 2. *Let $S \in \mathrm{Var}(k)$. Let $M \in \mathrm{DA}(S)$. Then there exist $(F, W) \in C_{fil}(\mathrm{Var}(k)^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F) = M$ and $D(\mathbb{A}^1, et)(\mathrm{Gr}_p^W F) \in \mathcal{CH}^0(S)$.*

Proof. By theorem 22, there exist, by induction, for $i \in \mathbb{Z}$, a distinguish triangle in $\mathrm{DA}(S)$

$$T_i : M^{(i)}[i] \xrightarrow{m_i} M^{(i+1)} \xrightarrow{m_{i+1}} \cdots \xrightarrow{m_{m-1}} M^{(m)}[m] \to M^{w \geq i} \tag{13}$$

with $M^{(j)}[j] \in \mathcal{CH}^0(S)$ and $w(M) : holim_i M^{\geq i} \xrightarrow{\sim} M$ in $\mathrm{DA}^-(S)$. For $i \in \mathbb{Z}$, take $(F_j)_{j \geq i}, F_{w \geq i} \in C(\mathrm{Var}(k)^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F_j) = M^{(j)}[j]$, $D(\mathbb{A}^1, et)(F_{w \geq i}) = M^{w \geq i}$ and such that we have in $C(\mathrm{Var}(k)^{sm}/S)$,

$$F_{w \geq i} = \mathrm{Cone}(F_i \xrightarrow{m_i} F_{i+1} \xrightarrow{m_{i+1}} \cdots \xrightarrow{m_{m-1}} F_m) \tag{14}$$

where $m_j : F_j \to F_{j+1}$ are morphisms in $C(\mathrm{Var}(k)^{sm}/S)$ such that $D(\mathbb{A}^1, et)(m_j) = m_j$. Now set $F = \mathrm{holim}_i F_{w \geq i} \in C(\mathrm{Var}(k)^{sm}/S)$ and $W_i F := F_{w \geq i} \hookrightarrow F$, so that $(F, W) \in C_{fil}(\mathrm{Var}(k)^{sm}/S)$ satisfy $D(\mathbb{A}^1, et)(\mathrm{Gr}_p^W F) = M^{(p)}[p] \in \mathcal{CH}^0(S)$. □

4 The (filtered) D modules and the (filtered) De Rham functor on algebraic varieties over a field k of characteristic zero

4.1 The D-modules on smooth algebraic varieties over a field k of characteristic zero and their functorialities

Let k a field of characteristic zero.

For $S = (S, O_S) \in \mathrm{SmVar}(k)$, we denote by

- $D_S := D(O_S) \subset \mathcal{H}om_{\mathbb{C}_S}(O_S, O_S)$ the subsheaf consisting of differential operators. By a D_S module, we mean a left D_S module.
- we denote by
  - $\mathrm{PSh}_{\mathcal{D}}(S)$ the abelian category of Zariski presheaves on S with a structure of left D_S module, and by $\mathrm{PSh}_{\mathcal{D},c}(S) \subset \mathrm{PSh}_{\mathcal{D}}(S)$ the full subcategory whose objects are coherent sheaves of left D_S modules,
  - $\mathrm{PSh}_{\mathcal{D}^{op}}(S)$ the abelian category of Zariski presheaves on S with a structure of right D_S module, and by $\mathrm{PSh}_{\mathcal{D}^{op},c}(S) \subset \mathrm{PSh}_{\mathcal{D}^{op}}(S)$ the full subcategories whose objects are coherent sheaves of right D_S modules,
- we denote by
  - $C_{\mathcal{D}}(S) = C(\mathrm{PSh}_{\mathcal{D}}(S))$ the category of complexes of Zariski presheaves on S with a structure of D_S module,
  - $C_{\mathcal{D}^{op}}(S) = C(\mathrm{PSh}_{\mathcal{D}^{op}}(S))$ the category of complexes of Zariski presheaves on S with a structure of right D_S module,
- in the filtered case we have
  - $C_{\mathcal{D}(2)fil}(S) \subset C(\mathrm{PSh}_{\mathcal{D}}(S), F, W) := C(\mathrm{PSh}_{D(O_S)}(S), F, W)$ the category of (bi)filtered complexes of algebraic D_S modules such that the filtration is biregular, $D_{\mathcal{D}(2)fil}(S) := \mathrm{Ho}_{zar}(C_{\mathcal{D}(2)fil}(S))$ its localization with respect to filtered Zariski local equivalence, and more generally $D_{\mathcal{D}(2)fil,r}(S) := \mathrm{Ho}_{zar}(C_{\mathcal{D}(2)fil}(S))$ its localization with respect to r-filtered Zariski local equivalence for $r = 1, \cdots, \infty$,
  - $C_{\mathcal{D}0fil}(S) \subset C_{\mathcal{D}fil}(S)$ the full subcategory such that the filtration is a filtration by D_S submodule (which is stronger then Griffitz transversality), $C_{\mathcal{D}(1,0)fil}(S) \subset C_{\mathcal{D}2fil}(S)$ the full subcategory such that W is a filtration by D_S submodules, $D_{\mathcal{D}(1,0)fil}(S) := \mathrm{Ho}_{zar}(C_{\mathcal{D}(1,0)fil}(S))$ its localization with respect to filtered Zariski local equivalence, and more generally $D_{\mathcal{D}(1,0)fil,r}(S) := \mathrm{Ho}_{zar}(C_{\mathcal{D}(1,0)fil}(S))$ its localization with respect to r-filtered Zariski local equivalence for $r = 1, \cdots, \infty$,
  - $C_{\mathcal{D}^{op}(2)fil}(S) \subset C(\mathrm{PSh}_{\mathcal{D}^{op}}(S), F, W) := C(\mathrm{PSh}_{D(O_S)^{op}}(S), F, W)$ the category of (bi)filtered complexes of algebraic (resp. analytic) right D_S modules such that the filtration is biregular, as in the left case we consider the subcategories as above.

Definition 36. *An $X \in \mathrm{SmVar}(k)$ is said to be D-affine if the following two condition hold:*

(i) The global section functor $\Gamma(X, \cdot) : \mathcal{Q}Coh_{\mathcal{D}}(X) \to Mod(\Gamma(X, D_X))$ is exact.

(ii) If $\Gamma(X, M) = 0$ for $M \in \mathcal{Q}Coh_{\mathcal{D}}(X)$, then $M = 0$.

Proposition 13. *If $X \in \mathrm{SmVar}(k)$ is D-affine, then :*

(i) Any $M \in \mathcal{Q}Coh_{\mathcal{D}}(X)$ is generated by its global sections.

(ii) The functor $\Gamma(X,\cdot) : \mathcal{Q}Coh_{\mathcal{D}}(X) \to Mod(\Gamma(X,D_X))$ is an equivalence of category whose inverse is $L \in Mod(\Gamma(X,D_X)) \mapsto D_X \otimes_{\Gamma(X,D_X)} L \in \mathcal{Q}Coh_{\mathcal{D}}(X)$.

(iii) We have $\Gamma(X,\cdot)(\mathcal{C}oh_{\mathcal{D}}(X)) = Mod(\Gamma(X,D_X))_f$, that is the global sections of a coherent D_X module is a finite module over the differential operators on X.

Proof. Similar to the complex case : see [18]. □

Let $f : X \to S$ be a morphism with $X, S \in \mathrm{SmVar}(k)$, Then, we recall from [10] section 4.1, the transfers modules

- $(D_{X\to S}, F^{ord}) := f^{*mod}(D_S, F^{ord}) := f^*(D_S, F^{ord}) \otimes_{f^*O_S} (O_X, F_b)$ which is a left D_X module and a left and right f^*D_S module
- $(D_{X\leftarrow S}, F^{ord}) := (K_X, F_b) \otimes_{O_X} (D_{X\to S}, F^{ord}) \otimes_{f^*O_S} f^*(K_S, F_b)$. which is a right D_X module and a left and right f^*D_S module.

Proposition 14. *Let $i : Z \hookrightarrow S$ be a closed embedding with $Z, S \in \mathrm{SmVar}(k)$. Then, $D_{Z\to S} = i^*D_S/D_S\mathcal{I}_Z$ and it is a locally free (left) D_Z module. Similarly, $D_{Z\leftarrow S} = i^*D_S/\mathcal{I}_Z D_S$ and it is a locally free right D_Z module.*

Proof. Similar to the complex case : see [18]. □

- Let $S \in \mathrm{SmVar}(k)$.
  - For $M \in C_{\mathcal{D}}(S)$, we have the canonical projective resolution $q : L_D(M) \to M$ of complexes of D_S modules.
  - Let τ a topology on S. For $M \in C_{\mathcal{D}}(S)$, there exist a unique strucure of D_S module on the flasque presheaves $E^i_\tau(M)$ such that $E_\tau(M) \in C_{\mathcal{D}}(S)$ (i.e. is a complex of D_S modules) and that the map $k : M \to E(M)$ is a morphism of complexes of D_S modules.
- Let $S \in \mathrm{SmVar}(k)$. For $M \in C_{\mathcal{D}(op)}(S)$, $N \in C(S)$, we will consider the induced D module structure (right D_S module in the case one is a left D_S module and the other one is a right one) on the presheaf $M \otimes N := M \otimes_{\mathbb{Z}_S} N$ (see section 2). We get the bifunctor
$$C(S) \times C_{\mathcal{D}}(S) \to C_{\mathcal{D}}(S), (M, N) \mapsto M \otimes N$$
- Let $S \in \mathrm{SmVar}(k)$. For $M, N \in C_{\mathcal{D}(op)}(S)$, $M \otimes_{O_S} N$ has a canonical structure of D_S modules (right D_S module in the case one is a left D_S module and the other one is a right one) given by (in the left case) for $S^o \subset S$ an open subset,
$$m \otimes n \in \Gamma(S^o, M \otimes_{O_S} N), \gamma \in \Gamma(S^o, D_S),\ \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes \gamma.n$$
This gives the bifunctor
$$C_{\mathcal{D}(op)}(S)^2 \to C_{\mathcal{D}(op)}(S), (M, N) \mapsto M \otimes_{O_S} N$$
More generally, let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$. Assume S smooth. For $M, N \in C_{f^*\mathcal{D}(op)}(X)$, $M \otimes_{f^*O_S} N$ (see section 2), has a canonical structure of f^*D_S modules (right f^*D_S module in the case one is a left f^*D_S module and the other one is a right one) given by (in the left case) for $X^o \subset X$ an open subset,
$$m \otimes n \in \Gamma(X^o, M \otimes_{f^*O_S} N), \gamma \in \Gamma(X^o, f^*D_S),\ \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes \gamma.n$$
This gives the bifunctor
$$C_{f^*\mathcal{D}(op)}(X)^2 \to C_{f^*\mathcal{D}(op)}(X), (M, N) \mapsto M \otimes_{f^*O_S} N$$

- Let $S \in \mathrm{SmVar}(k)$. The internal hom bifunctor

$$\mathcal{H}om(\cdot,\cdot) := \mathcal{H}om_{\mathbb{Z}_S}(\cdot,\cdot) : C(S)^2 \to C(S)$$

  induces a bifunctor

$$\mathcal{H}om(\cdot,\cdot) := \mathcal{H}om_{\mathbb{Z}_S}(\cdot,\cdot) : C(S) \times C_{\mathcal{D}}(S) \to C_{\mathcal{D}}(S)$$

  such that, for $F \in C(S)$ and $G \in C_{\mathcal{D}}(S)$, the D_S structure on $\mathcal{H}om^{\bullet}(F,G)$ is given by

$$\gamma \in \Gamma(S^o, D_S) \longmapsto (\phi \in \mathrm{Hom}^p(F^{\bullet}_{|S^o}, G_{|S^o}) \mapsto (\gamma \cdot \phi : \alpha \in F^{\bullet}(S^o) \mapsto \gamma \cdot \phi^p(S^o)(\alpha))$$

  where $\phi^p(S^o)(\alpha) \in \Gamma(S^o, G)$.

- Let $S \in \mathrm{SmVar}(k)$. For $M, N \in C_{\mathcal{D}}(S)$, $\mathcal{H}om_{O_S}(M,N)$, has a canonical structure of D_S modules given by for $S^o \subset S$ an open subset and $\phi \in \Gamma(S^o, \mathcal{H}om(M, O_S))$, $\gamma \in \Gamma(S^o, D_S)$, $(\gamma.\phi)(m) := \gamma.(\phi(m)) - \phi(\gamma.m)$ This gives the bifunctor

$$\mathrm{Hom}^{\bullet}_{O_S}(-,-) : C_{\mathcal{D}}(S)^2 \to C_{\mathcal{D}}(S)^{op}, (M,N) \mapsto \mathcal{H}om^{\bullet}_{O_S}(M,N)$$

- Let $S \in \mathrm{SmVar}(k)$. We have the bifunctors
  - $\mathrm{Hom}^{\bullet}_{D_S}(-,-) : C_{\mathcal{D}}(S)^2 \to C(S)$, $(M,N) \mapsto \mathcal{H}om^{\bullet}_{D_S}(M,N)$, and if N is a bimodule (i.e. has a right D_S module structure whose opposite coincide with the left one), $\mathcal{H}om_{D_S}(M,N) \in C_{\mathcal{D}^{op}}(S)$ given by for $S^o \subset S$ an open subset and $\phi \in \Gamma(S^o, \mathcal{H}om(M,N))$, $\gamma \in \Gamma(S^o, D_S)$, $(\phi.\gamma)(m) := (\phi(m)).\gamma$
  - $\mathrm{Hom}_{D_S}(-,-) : C_{\mathcal{D}^{op}}(S)^2 \to C(S)$, $(M,N) \mapsto \mathcal{H}om_{D_S}(M,N)$ and if N is a bimodule, $\mathcal{H}om_{D_S}(M,N) \in C_{\mathcal{D}}(S)$

  For $M \in C_{\mathcal{D}}(S)$, we get in particular the dual with respect $\mathbb{D}_S$,

$$\mathbb{D}_S M := \mathcal{H}om_{D_S}(M, D_S) \in C_{\mathcal{D}}(S) \ ; \ \mathbb{D}^K_S M := \mathcal{H}om_{D_S}(M, D_S) \otimes_{O_S} \mathbb{D}^O_S w(K_S)[d_S] \in C_{\mathcal{D}}(S)$$

  and we have canonical map $d : M \to \mathbb{D}^2_S M$. This functor induces in the derived category, for $M \in D_{\mathcal{D}}(S)$,

$$L\mathbb{D}_S M := R\mathcal{H}om_{D_S}(L_D M, D_S) \otimes_{O_S} \mathbb{D}^O_S w(K_S)[d_S] = \mathbb{D}^K_S L_D M \in D_{\mathcal{D}}(S).$$

  where $\mathbb{D}^O_S w(S) : \mathbb{D}^O_S w(K_S) \to \mathbb{D}^O_S K_S = K_S^{-1}$ is the dual of the Koczul resolution of the canonical bundle (proposition 32), and the canonical map $d : M \to L\mathbb{D}^2_S M$.

- Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(k)$. For $N \in C_{\mathcal{D},f^*\mathcal{D}}(X)$ and $M \in C_{\mathcal{D}}(X)$, $N \otimes_{D_X} M$ has the canonical f^*D_S module structure given by, for $X^o \subset X$ an open subset,

$$\gamma \in \Gamma(X^o, f^*D_S), m \in \Gamma(X^o, M), n \in \Gamma(X^o, N), \ \gamma.(n \otimes m) = (\gamma.n) \otimes m.$$

  This gives the functor

$$C_{\mathcal{D},f^*\mathcal{D}}(X) \times C_{\mathcal{D}}(X) \to C_{f^*\mathcal{D}}(X), \ (M,N) \mapsto M \otimes_{D_X} N$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{SmVar}(k)$, Then, for $M \in C_{\mathcal{D}}(S)$, $O_X \otimes_{f^*O_S} f^*M$ has a canonical D_X module structure given by given by, for $X^o \subset X$ an open subset,

$$m \otimes n \in \Gamma(X^o, O_X \otimes_{f^*O_S} f^*M), \gamma \in \Gamma(X^o, D_X), \ \gamma.(m \otimes n) := (\gamma.m) \otimes n - m \otimes df(\gamma).n.$$

  This gives the inverse image functor

$$f^{*mod} : \mathrm{PSh}_{\mathcal{D}}(S) \to \mathrm{PSh}_{\mathcal{D}}(X), \ \ M \mapsto f^{*mod} M := O_X \otimes_{f^*O_S} f^*M = D_{X \to S} \otimes_{f^*D_S} f^*M$$

which induces in the derived category the functor

$$Lf^{*mod}: D_{\mathcal{D}}(S) \to D_{\mathcal{D}}(X), \quad M \mapsto Lf^{*mod}M := O_X \otimes^L_{f^*O_S} f^*M = O_X \otimes_{f^*O_S} f^*L_D M,$$

We will also consider the shifted inverse image functor

$$Lf^{*mod[-]} := Lf^{*mod}[d_S - d_X] : D_{\mathcal{D}}(S) \to D_{\mathcal{D}}(X).$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{SmVar}(k)$. For $M \in C_{\mathcal{D}}(X)$, $D_{X \leftarrow S} \otimes_{D_X} M$ has the canonical f^*D_S module structure given above. Then, the direct image functor

$$f^0_{*mod} : \mathrm{PSh}_{\mathcal{D}}(X) \to \mathrm{PSh}_{\mathcal{D}}(S), \quad M \mapsto f_{*mod}M := f_*(D_{X \leftarrow S} \otimes_{D_X} M)$$

  induces in the derived category the functor

$$\int_f = Rf_{*mod} : D_{\mathcal{D}}(X) \to D_{\mathcal{D}}(S), \quad M \mapsto \int_f M = Rf_*(D_{X \leftarrow S} \otimes^L_{D_X} M).$$

The functorialities given above induce :

- Let $S \in \mathrm{SmVar}(k)$. For $(M, F) \in C_{fil}(S)$ and $(N, F) \in C_{fil}(S)$, recall that

$$F^p((M, F) \otimes (N, F)) := \mathrm{Im}(\oplus_q F^q M \otimes F^{p-q} N \to M \otimes N)$$

  This gives the functor

$$(\cdot, \cdot) : C_{fil}(S) \times C_{\mathcal{D}fil}(S) \to C_{\mathcal{D}fil}(S)\ ,\ ((M, F), (N, F)) \mapsto (M, F) \otimes (N, F).$$

  It induces in the derived categories by taking r-projective resolutions the bifunctors, for $r = 1, \ldots, \infty$,

$$(\cdot, \cdot) : D_{\mathcal{D}fil,r}(S) \times D_{fil,r}(S) \to D_{\mathcal{D}fil,r}(S)\ ,\ ((M, F), (N, F)) \mapsto (M, F) \otimes^L (N, F) = L_D(M, F) \otimes (N, F).$$

- Let $S \in \mathrm{SmVar}(k)$. For $(M, F) \in C_{O_S fil}(S)$ and $(N, F) \in C_{O_S fil}(S)$, recall that

$$F^p((M, F) \otimes_{O'_S} (N, F)) := \mathrm{Im}(\oplus_q F^q M \otimes_{O'_S} F^{p-q} N \to M \otimes_{O'_S} N)$$

  It induces in the derived categories by taking r-projective resolutions the bifunctors, for $r = 1, \ldots, \infty$,

$$(\cdot, \cdot) : D_{\mathcal{D}fil,r}(S)^2 \to D_{\mathcal{D}fil,r}(S)\ ,\ ((M, F), (N, F)) \mapsto (M, F) \otimes^L_{O_S} (N, F).$$

  More generally, let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$. Assume S smooth. We have the bifunctors

$$(\cdot, \cdot) : D_{f^*\mathcal{D}fil,r}(X)^2 \to D_{f^*\mathcal{D}fil,r}(X),$$
$$((M, F), (N, F)) \mapsto (M, F) \otimes^L_{f^*O_S} (N, F) = (M, F) \otimes_{f^*O_S} L_{f^*D}(N, F).$$

- Let $S \in \mathrm{SmVar}(k)$. The hom functor induces the bifunctor

$$\mathrm{Hom}(-, -) : C_{\mathcal{D}fil}(S) \times C_{fil}(S) \to C_{\mathcal{D}(1,0)fil}(S), ((M, W), (N, F)) \mapsto \mathcal{H}om((M, W), (N, F)).$$

- Let $S \in \mathrm{SmVar}(k)$. The hom functor induces the bifunctor

$$\mathrm{Hom}_{O_S}(-, -) : C_{\mathcal{D}fil}(S)^2 \to C_{\mathcal{D}2fil}(S), ((M, W), (N, F)) \mapsto \mathcal{H}om_{O_S}((M, W), (N, F)).$$

- Let $S\in\mathrm{SmVar}(k)$. The hom functor induces the bifunctors
  - $\mathrm{Hom}_{D_S}(-,-):C_{\mathcal{D}fil}(S)^2\to C_{2fil}(S)$, $((M,W),(N,F))\mapsto\mathcal{H}om_{D_S}((M,W),(N,F))$,
  - $\mathrm{Hom}_{D_S}(-,-):C_{\mathcal{D}^{op}fil}(S)^2\to C_{2fil}(S)$, $((M,W),(N,F))\mapsto\mathcal{H}om_{D_S}((M,W),(N,F))$.

  We get the filtered dual

  $\mathbb{D}_S^K(\cdot):C_{\mathcal{D}(2)fil}(S)\to C_{\mathcal{D}(2)fil}(S)^{op},\ (M,F)\mapsto\mathbb{D}_S^K(M,F):=\mathcal{H}om_{D_S}((M,F),D_S)\otimes_{O_S}\mathbb{D}_S^Ow(K_S)[d_S]$

  together with the canonical map $d(M,F):(M,F)\to\mathbb{D}_S^{2,K}(M,F)$. Of course $\mathbb{D}_S^K(\cdot)(C_{\mathcal{D}(1,0)fil}(S))\subset C_{\mathcal{D}(1,0)fil}(S)$. It induces in the derived categories $D_{\mathcal{D}fil,r}(S)$, for $r=1,\ldots,\infty$, the functors

  $$L\mathbb{D}_S(\cdot):D_{\mathcal{D}(2)fil,r}(S)\to D_{\mathcal{D}(2)fil,r}(S)^{op},(M,F)\mapsto L\mathbb{D}_S(M,F):=\mathbb{D}_S^KL_D(M,F).$$

  together with the canonical map $d(M,F):L_D(M,F)\to\mathbb{D}_S^{2,K}L_D(M,F)$.

- Let $f:X\to S$ be a morphism with $X,S\in\mathrm{SmVar}(k)$. Then, the inverse image functor

  $$f^{*mod}:C_{\mathcal{D}(2)fil}(S)\to C_{\mathcal{D}(2)fil}(X),$$
  $$(M,F)\mapsto f^{*mod}(M,F):=(O_X,F_b)\otimes_{f^*O_S}f^*(M,F)=(D_{X\to S},F^{ord})\otimes_{f^*D_S}f^*(M,F),$$

  induces in the derived categories the functors, for $r=1,\ldots,\infty$ (resp. $r\in(1,\ldots\infty)^2$),

  $$Lf^{*mod}:D_{\mathcal{D}(2)fil,r}(S)\to D_{\mathcal{D}(2)fil,r}(X),$$
  $$(M,F)\mapsto Lf^{*mod}M:=(O_X,F_b)\otimes^L_{f^*O_S}f^*(M,F)=(O_X,F_b)\otimes_{f^*O_S}f^*L_D(M,F).$$

  Of course $f^{*mod}(C_{\mathcal{D}(1,0)fil}(S))\subset C_{\mathcal{D}(1,0)fil}(X)$. Note that
  - If the M is a complex of locally free O_S modules, then $Lf^{*mod}(M,F)=f^{*mod}(M,F)$ in $D_{\mathcal{D}(2)fil,\infty}(S)$.
  - If the Gr_F^pM are complexes of locally free O_S modules, then $Lf^{*mod}(M,F)=f^{*mod}(M,F)$ in $D_{\mathcal{D}(2)fil}(S)$.

  We will consider also the shifted inverse image functors

  $$Lf^{*mod[-]}:=Lf^{*mod}[d_S-d_X]:D_{\mathcal{D}(2)fil,r}(S)\to D_{\mathcal{D}(2)fil,r}(X).$$

- Let $f:X\to S$ be a morphism with $X,S\in\mathrm{SmVar}(k)$. Then,the direct image functor

  $f^{00}_{*mod}:(\mathrm{PSh}_{\mathcal{D}}(X),F)\to(\mathrm{PSh}_{\mathcal{D}}(S),F),\ \ (M,F)\mapsto f_{*mod}(M,F):=f_*((D_{S\leftarrow X},F^{ord})\otimes_{D_X}(M,F))$

  induces in the derived categories by taking r-injective resolutions the functors, for $r=1,\ldots,\infty$,

  $$\int_f=Rf_{*mod}:D_{\mathcal{D}(2)fil,r}(X)\to D_{\mathcal{D}(2)fil,r}(S),(M,F)\mapsto\int_f(M,F)=Rf_*((D_{S\leftarrow X},F^{ord})\otimes^L_{D_X}(M,F)).$$

  Let $f_1:X\to Y$ and $f_2:Y\to S$ two morphism with $X,Y,S\in\mathrm{SmVar}(\mathbb{C})$ or with $X,Y,S\in\mathrm{AnSm}(\mathbb{C})$. We have, for $(M,F)\in C_{\mathcal{D}fil}(X)$, the canonical transformation map in $D_{\mathcal{D}(2)fil,r}(S)$

  $$T(\int_{f_2}\circ\int_{f_1},\int_{f_2\circ f_1})(M,F):$$
  $$\int_{f_2}\int_{f_1}(M,F):=Rf_{2*}((D_{Y\leftarrow S},F^{ord})\otimes^L_{D_Y}Rf_{1*}((D_{X\leftarrow Y},F^{ord})\otimes^L_{D_X}(M,F)))$$
  $$\xrightarrow{T(f_1,\otimes)(-,-)}Rf_{2*}Rf_{1*}(f_1^*(D_{Y\leftarrow S},F^{ord})\otimes^L_{D_Y}((D_{X\leftarrow Y},F^{ord})\otimes^L_{D_X}(M,F)))$$
  $$\xrightarrow{\sim}Rf_{2*}Rf_{1*}((f_1^*(D_{Y\leftarrow S},F^{ord})\otimes^L_{D_Y}(D_{X\leftarrow Y},F^{ord}))\otimes^L_{D_X}(M,F))$$
  $$\xrightarrow{\sim}Rf_{2*}Rf_{1*}((D_{X\leftarrow S},F^{ord})\otimes^L_{D_X}(M,F)):=\int_{f_2\circ f_1}(M,F)$$

- Let $f : X \to S$ be a morphism with $X, S \in \mathrm{SmVar}(k)$. Then the functor

$$f^{\hat{*}mod} : C_{\mathcal{D}2fil}(S) \to C_{\mathcal{D}2fil}(X), (M, F) \mapsto f^{\hat{*}mod}(M, F) := \mathbb{D}_X^K L_D f^{*mod} L_D \mathbb{D}_S^K (M, F)$$

induces in the derived categories the exceptional inverse image functors, for $r = 1, \ldots, \infty$ (resp. $r \in (1, \ldots \infty)^2$),

$$Lf^{\hat{*}mod} : D_{\mathcal{D}(2)fil,r}(S) \to D_{\mathcal{D}(2)fil,r}(X),$$
$$(M, F) \mapsto Lf^{\hat{*}mod}(M, F) := L\mathbb{D}_X Lf^{*mod} L\mathbb{D}_S(M, F) := f^{\hat{*}mod} L_D(M, F).$$

Of course $f^{\hat{*}mod}(C_{\mathcal{D}(1,0)fil}(S)) \subset C_{\mathcal{D}(1,0)fil}(X)$. We will also consider the shifted exceptional inverse image functors

$$Lf^{\hat{*}mod[-]} := Lf^{\hat{*}mod}[d_S - d_X] : D_{\mathcal{D}(2)fil,r}(S) \to D_{\mathcal{D}(2)fil,r}(X).$$

- Let $S_1, S_2 \in \mathrm{SmVar}(k)$. Consider $p : S_1 \times S_2 \to S_1$ the projection. Since p is a projection, we have a canonical embedding $p^* D_{S_1} \hookrightarrow D_{S_1 \times S_2}$. For $(M, F) \in C_{\mathcal{D}(2)fil}(S_1 \times S_2)$, (M, F) has a canonical $p^* D_{S_1}$ module structure. Moreover, with this structure, for $(M_1, F) \in C_{\mathcal{D}(2)fil}(S_1)$

$$\mathrm{ad}(p^{*mod}, p)(M_1, F) : (M_1, F) \to p_* p^{*mod}(M_1, F)$$

is a map of complexes of D_{S_1} modules, and for $(M_{12}, F) \in C_{\mathcal{D}(2)fil}(S_1 \times S_2))$

$$\mathrm{ad}(p^{*mod}, p)(M_{12}, F) : p^{*mod} p_*(M_{12}, F) \to (M_{12}, F)$$

is a map of complexes of $D_{S_1 \times S_2}$ modules.

Proposition 15. *(i) Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \mathrm{SmVar}(k)$.*

- *Let $(M, F) \in C_{\mathcal{D}(2)fil,r}(S)$. Then $(f_2 \circ f_1)^{*mod}(M, F) = f_1^{*mod} f_2^{*mod}(M, F)$.*
- *Let $(M, F) \in D_{\mathcal{D}(2)fil,r}(S)$. Then $L(f_2 \circ f_1)^{*mod}(M, F) = Lf_1^{*mod}(Lf_2^{*mod}(M, F))$.*

(ii) Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \mathrm{SmVar}(k)$. Let $M \in D_{\mathcal{D}}(X)$. Then,

$$T(\int_{f_2} \circ \int_{f_1}, \int_{f_2 \circ f_1})(M) : \int_{f_2} \int_{f_1} (M) \xrightarrow{\sim} \int_{f_2 \circ f_1} (M)$$

is an isomorphism in $D_{\mathcal{D}}(S)$ (i.e. if we forget filtration).

(iii) Let $i_0 : Z_2 \hookrightarrow Z_1$ and $i_1 : Z_1 \hookrightarrow S$ two closed embedding, with $Z_2, Z_1, S \in \mathrm{SmVar}(k)$. Let $(M, F) \in C_{\mathcal{D}(2)fil}(Z_2)$. Then, $(i_1 \circ i_0)_{*mod}(M, F) = i_{1*mod}(i_{0*mod}(M, F))$ in $C_{\mathcal{D}(2)fil}(S)$.

Proof. Similar to the complex case : see [10]. □

Proposition 16. *For $X \in \mathrm{SmVar}(k)$, we have for $(M, F), (N, F) \in C_{O_X fil}(X)$ or $(M, F), (N, F) \in C_{\mathcal{D}fil}(X)$, Denote by $\Delta_X : X \hookrightarrow X \times X$ the diagonal closed embedding and $p_1 : X \times X \to X$, $p_2 : X \times X \to X$ the projections. We have*

$$(M, F) \otimes_{O_X} (N, F) = \Delta_X^{*mod}(p_1^{*mod}(M, F) \otimes_{O_{X \times X}} p_2^{*mod}(N, F))$$

Proof. Similar to the complex case : see [18]. □

Let $i : Z \hookrightarrow S$ a closed embedding, with $Z, S \in \mathrm{SmVar}(k)$. We have the functor

$$i^\sharp : C_{\mathcal{D}fil}(S) \to C_{\mathcal{D}fil}(Z), (M, F) \mapsto i^\sharp(M, F) := \mathcal{H}om_{i^* D_S}((D_{S \leftarrow Z}, F^{ord}), i^*(M, F))$$

where the (left) D_Z module structure on $i^\sharp M$ comes from the right module structure on $D_{S \leftarrow Z}$, resp. O_Z. We denote by

- for $(M,F)\in C_{\mathcal{D}fil}(S)$, the canonical map in $C_{\mathcal{D}fil}(S)$

$$\mathrm{ad}(i_{*mod},i^{\sharp})(M,F):i_{*mod}i^{\sharp}(M,F):=i_*(\mathcal{H}om_{i^*D_S}((D_{S\leftarrow Z},F^{ord}),i^*(M,F))\otimes_{D_Z}(D_{S\leftarrow Z},F^{ord}))$$
$$\to(M,F),\phi\otimes P\mapsto\phi(P)$$

- for $(N,F)\in C_{\mathcal{D}fil}(Z)$, the canonical map in $C_{\mathcal{D}fil}(Z)$

$$\mathrm{ad}(i_{*mod},i^{\sharp})(N,F):(N,F)\to i^{\sharp}i_{*mod}(N,F):=\mathcal{H}om_{i^*D_S}(D_{S\leftarrow Z},i^*i_*((N,F)\otimes_{D_Z}(D_{S\leftarrow Z},F^{ord})))$$
$$n\mapsto(P\mapsto n\otimes P)$$

The functor $i^{\sharp}$ induces in the derived category the functor :

$$Ri^{\sharp}:D_{\mathcal{D}(2)fil,r}(S)\to D_{\mathcal{D}(2)fil,r}(Z),\ (M,F)\mapsto$$
$$Ri^{\sharp}(M,F):=R\mathcal{H}om_{i^*D_S}((D_{Z\leftarrow S},F^{ord}),i^*(M,F))=\mathcal{H}om_{i^*D_S}((D_{Z\leftarrow S},F^{ord}),E(i^*(M,F))).$$

Proposition 17. *Let $i:Z\hookrightarrow S$ a closed embedding, with $Z,S\in\mathrm{SmVar}(k)$. The functor i_{*mod} : $C_{\mathcal{D}}(Z)\to C_{\mathcal{D}}(S)$ admit a right adjoint which is the functor $i^{\sharp}:C_{\mathcal{D}}(S)\to C_{\mathcal{D}}(Z)$ and*

$$\mathrm{ad}(i_{*mod},i^{\sharp})(N):N\to i^{\sharp}i_{*mod}N\ \ \textit{and}\ \ \mathrm{ad}(i_{*mod},i^{\sharp})(M):i_{*mod}i^{\sharp}M\to M$$

are the adjonction maps.

Proof. Similar to the complex case : see [18]. □

One of the main results in D modules is Kashiwara equivalence :

Theorem 23. *Let $i:Z\hookrightarrow S$ a closed embedding with $Z,S\in\mathrm{SmVar}(k)$.*

(i) The functor $i_{*mod}:\mathcal{Q}Coh_{\mathcal{D}}(Z)\to\mathcal{Q}Coh_{\mathcal{D},Z}(S)$ is an equivalence of category whose inverse is given by $i^{\sharp}:=a_{\tau}i^{\sharp}:\mathcal{Q}Coh_{\mathcal{D}}(S)\to\mathcal{Q}Coh_{\mathcal{D}}(Z)$. That is, for $M\in QCoh_{\mathcal{D},Z}(S)$ and $N\in QCoh_{\mathcal{D}}(Z)$, the adjonction maps

$$\mathrm{ad}(i_{*mod},i^{\sharp})(M):i_{*mod}i^{\sharp}M\xrightarrow{\sim}M\ ,\ \mathrm{ad}(i_{*mod},i^{\sharp})(N):i^{\sharp}i_{*mod}N\xrightarrow{\sim}N$$

are isomorphisms.

(ii) The functor $\int_i=i_{*mod}:D_{\mathcal{D}}(Z)\to D_{\mathcal{D},Z}(S)$ is an equivalence of category whose inverse is given by $Ri^{\sharp}:D_{\mathcal{D}}(S)\to D_{\mathcal{D}}(Z)$. That is, for $M\in D_{\mathcal{D},Z}(S)$ and $N\in D_{\mathcal{D}}(Z)$, the adjonction maps

$$\mathrm{ad}(\int_i,Ri^{\sharp})(M):i_{*mod}Ri^{\sharp}M\xrightarrow{\sim}M\ ,\ \mathrm{ad}(\int_i,Ri^{\sharp})(N):Ri^{\sharp}i_{*mod}N\xrightarrow{\sim}N$$

are isomorphisms.

Proof. Similar to the complex case : see [18] :(ii) follows from (i). □

Lemma 2. *Let $i:Z\hookrightarrow S$ a closed embedding with $Z,S\in\mathrm{Var}(k)$. Denote by $j:U:=S\backslash Z\hookrightarrow Z$ the open complementary embedding. Then, if i is a locally complete intersection embedding (e.g. if Z,S are smooth), we have for $M\in C_{O_U}(U)$ quasi-coherent, $Li^{*mod}Rj_*M=0$.*

Proof. Similar to the complex case : see [10]. □

We deduce from theorem 23(i) and lemma 2 the localization for D-modules for a closed embedding of smooth algebraic varieties:

Theorem 24. *Let $i : Z \hookrightarrow S$ a closed embedding with $Z, S \in \mathrm{SmVar}(k)$. Denote by $c = \mathrm{codim}(Z, S)$. Then, for $M \in C_{\mathcal{D}}(S)$, we have by Kashiwara equivalence the following map in $C_{\mathcal{D}}(S)$:*

$$\mathcal{K}_{Z/S}(M) : \Gamma_Z E(M) \xrightarrow{\mathrm{ad}(i_{*mod}, i^{\sharp})(-)^{-1}} i_{*mod} i^{\sharp} \Gamma_Z E(M)$$
$$\xrightarrow{\gamma_Z(-)} i_{*mod} i^{\sharp}(E(M)) \xrightarrow{\mathcal{H}om(q_K, E(i^* M)) \circ \mathcal{H}om(O_Z, T(i,E)(M))} i_{*mod} K^{\vee}_{i^* O_S}(O_Z) \otimes_{i^* O_S} M$$

which is an equivalence Zariski local. It gives the isomorphism in $D_{\mathcal{D}}(S)$

$$\mathcal{K}_{Z/S}(M) : R\Gamma_Z M \to i_{*mod} K^{\vee}_{i^* O_S}(O_Z) = i_{*mod} L i^{*mod} M[c]$$

Proof. Follows from theorem 23 and lemma 2. □

Let k a field of caracteristic zero. Let $S \in \mathrm{SmVar}(k)$. Let $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$ a coherent D_S module so that it admits a good filtration (M, F) for the filtered ring (D_S, F^{ord}). We then have the characteristic variety

$$Ch(M) := \mathrm{supp}(cc(\mathrm{Gr}^F M)) \subset T_S$$

which is the support of the characteristic cycle $cc(\mathrm{Gr}^F M) \in \mathcal{Z}(T_S)$ of the coherent sheaf $\mathrm{Gr}^F M \in \mathrm{Shv}_c(T_S)$. Since for two good filtration (M, F) and (M, F') there exist $r, s \in \mathbb{Z}$ satisfying $F'^{i} M \subset F^{i-r} M \subset F'^{i-s} M$ for all i, $cc(\mathrm{Gr}^F M) \in \mathcal{Z}(T_S)$ and $Ch(M) \in T_S$ does NOT depend on the choice of a good filtration F.

For $k \subset k'$ a subfield of characteristic zero and $S \in \mathrm{SmVar}(k)$, we have by definition

$$cc(\mathrm{Gr}^F(\pi_{k/k'}(S)^{*mod} M)) = cc(\mathrm{Gr}^F M) \otimes_k k' \in \mathcal{Z}(T_{S_{k'}})$$

and thus

$$Ch(\pi_{k/k'}(S)^{*mod} M) = Ch(M)_{k'} \subset T_{S_{k'}}$$

with $S_{k'} := S \otimes_k k'$, since if (M, F) is a good filtration then $(\pi_{k/k'}(S)^{*mod} M, \pi_{k/k'}(S)^{*mod} F)$ is a good filtration, $\pi_{k/k'}(S) : S_{k'} := S \otimes_k k' \to S$ being the projection (see section 2).

We have the following proposition :

Proposition 18. *Let k a field of characteristic zero.*

- *(i) Let $i : Z \hookrightarrow S$ a closed embedding with $S, Z \in \mathrm{SmVar}(k)$. Let $M \in \mathrm{PSh}_{\mathcal{D},c}(Z)$ a coherent D_Z module so that it admits a good filtration (M, F) for the filtered ring (D_S, F^{ord}). Then $i_{*mod}(M, F)$ is a good filtration for the filtered ring (D_Z, F^{ord}) and $Ch(i_{*mod} M) = di(Ch(M)) \subset T_S$ where $d_i : T_Z \hookrightarrow T_{S|Z} := p_S^{-1}(Z) \hookrightarrow T_S$ is the closed embedding where the first embedding is given by the differential of $i : i^* O_S \to O_Z$.*
- *(ii) Let $S \in \mathrm{SmVar}(k)$. Let $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$ a coherent D_S module so that it admits a good filtration (M, F) for the filtered ring (D_S, F^{ord}). Let $Ch(M) = \cup_l C^l_M$ with C^l_M the irreducible components of $Ch(M)$. Then $\dim(C^l_M) \leq \dim(S)$ for all l.*

Proof. (i): Similar to the proof of [18].
(ii) Let $S = \cup_i S_i$ an open affine cover. Since by definition $Ch(j_i^* M) = Ch(M) \cap p_S^{-1}(S_i)$, it is enought to prove the result for a smooth affine variety. So, let $S' \in \mathrm{SmVar}(k)$ affine and $i : S' \hookrightarrow \mathbb{A}^n_k$ a closed embedding. By (i) $Ch(i_{*mod} M) = di(Ch(M))$ so the result follows from [14]. □

Definition 37. *Let k a field of characteristic zero.*

- *(i) Let $S \in \mathrm{SmVar}(k)$ connected (hence irreducible since S is smooth). A coherent D_S module $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$ is called holonomic if all the irreducible components C^l_M of $\mathrm{supp}(Ch(M)) = \cup_l C^l_M \in T_S$ are of dimension $\dim(C^l_M) = \dim(S)$.*
- *(ii) Let $S \in \mathrm{SmVar}(k)$. Then $S = \sqcup_i S_i$ with $S_i \in \mathrm{SmVar}(k)$ connected. A coherent D_S module $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$ is called holonomic if $j_i^* M \in \mathrm{PSh}_{\mathcal{D},c}(S_i)$ is holonomic for all i.*

Let $k \subset k'$ a subfield. Consider the projection $\pi := \pi_{k/k'}(S) : S_{k'} \to S$. By definition $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$ is holonomic if and only if $\pi^{*mod}M \in \mathrm{PSh}_{\mathcal{D},c}(S_{k'})$ is holonomic since $Ch(M)_{k'} = Ch(\pi^{*mod}M) \subset T_{S_{k'}}$. In particular considering an embedding $\sigma : k \subset \mathbb{C}$, $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$ is holonomic if and only if $\pi_{k/\mathbb{C}}(S)^{*mod}M \in \mathrm{PSh}_{\mathcal{D},c}(S_{\mathbb{C}})$ is holonomic.

Proposition 19. *Let $S \in \mathrm{SmVar}(k)$.*

(i) Consider an exact sequence in $\mathrm{PSh}_{\mathcal{D},c}(S)$

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

Then $M_2 \in \mathrm{PSh}_{\mathcal{D},h}(S)$ if and only if $M_1, M_3 \in \mathrm{PSh}_{\mathcal{D},h}(S)$.

(ii) An holonomic module $M \in \mathrm{PSh}_{\mathcal{D},h}(S)$ has finite length.

Proof. Similar to the complex case : See [14] or [18]. □

Let $S \in \mathrm{SmVar}(k)$. A locally free O_S module with a structure of D_S module is called an integrable connexion. We denote by $\mathrm{Vect}_{\mathcal{D}}(S) \subset \mathrm{PSh}_{\mathcal{D},c}(S)$ the full subcategory whose set of objects consists of integrable connexions. By definition, an integrable connexion $M \in \mathrm{Vect}_{\mathcal{D}}(S)$ is holonomic since $Ch(M) = i_0(S) \subset T_S$ where $i_0 : S \hookrightarrow T_S$, $i_0(s) = (s, 0)$ is the zero section embedding. Hence $\mathrm{Vect}_{\mathcal{D}}(S) \subset \mathrm{PSh}_{\mathcal{D},h}(S)$.

Proposition 20. *Let k a field of characteristic zero. Let $S \in \mathrm{SmVar}(k)$.*

(i) A coherent D_S module $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$ which is a coherent O_S module is a locally free O_S module.

(ii) An holonomic D_S module $M \in \mathrm{PSh}_{\mathcal{D},h}(S)$ is generically an integrable connexion, that is there exists an open subset $j : S^o \subset S$ such that $M_{|S^o} := j^*M \in \mathrm{Vect}_{\mathcal{D}}(S^o)$.

Proof. (i): Similar to the complex case : see [18].
(ii): Similar to the complex case : follows from (i) since there exist an open subset $S^o \subset S$ such that $ch(M) \cap p^{-1}(S^o) = T_{S^o}S^o$ where $T_S S \subset T_S$ is the zero section. □

Let k a field of characteristic zero. Let $S \in \mathrm{SmVar}(k)$

- we consider
  - the full subcategories

    $$C_{\mathcal{D},h}(S) \subset C_{\mathcal{D},c}(S) \subset C_{\mathcal{D}}(S) \text{ and } D_{\mathcal{D},h}(S) \subset D_{\mathcal{D},c}(S) \subset D_{\mathcal{D}}(S)$$

    consisting of complexes of presheaves M such that $a_\tau H^n(M)$ are coherent, resp. holonomic, sheaves of D_S modules, a_τ being the sheaftification functor for the Zariski topology,
  - the full subcategories

    $$C_{\mathcal{D}^{op},h}(S) \subset C_{\mathcal{D}^{op},c}(S) \subset C_{\mathcal{D}^{op}}(S) \text{ and } D_{\mathcal{D}^{op},h}(S) \subset D_{\mathcal{D}^{op},c}(S) \subset D_{\mathcal{D}^{op}}(S)$$

    the full subcategories consisting of complexes of presheaves M such that $a_\tau H^n(M)$ are coherent, resp. holonomic, sheaves of right D_S modules,
- in the filtered case we have
  - the full subcategories

    $$C_{\mathcal{D}(2)fil,h}(S) \subset C_{\mathcal{D}(2)fil,c}(S) \subset C_{\mathcal{D}(2)fil}(S), \text{ and } D_{\mathcal{D}(2)fil,h}(S) \subset D_{\mathcal{D}(2)fil,c}(S) \subset D_{\mathcal{D}(2)fil}(S),$$

    consisting of filtered complexes of presheaves (M, F) such that $a_\tau H^n(M, F)$ are filtered coherent, resp. filtered holonomic, sheaves of D_S modules, that is $a_\tau H^n(M)$ are coherent, resp.

holonomic sheaves of D_S modules and F induces a good filtration on $a_\tau H^n(M)$ (in particular $F^p a_\tau H^n(M) \subset a_\tau H^n(M)$ are coherent sub O_S modules), the full subcategories

$$C_{\mathcal{D}(1,0)fil,h}(S) = C_{\mathcal{D}2fil,h}(S) \cap C_{\mathcal{D}(1,0)fil}(S) \subset C_{\mathcal{D}2fil,h}(S), \text{ and}$$
$$D_{\mathcal{D}(1,0)fil,h}(S) = D_{\mathcal{D}2fil,h}(S) \cap D_{\mathcal{D}(1,0)fil}(S) \subset D_{\mathcal{D}2fil,h}(S),$$

consisting of filtered complexes of presheaves (M, F, W) such that $a_\tau H^n(M, F)$ are filtered holonomic sheaves of D_S modules and such that $W^p M \subset M$ are D_S submodules (recall that the O_S submodules $F^p M \subset M$ are NOT D_S submodules but satisfy by definition $md : F^r D_S \otimes F^p M \subset F^{p+r} M$),

- and similarly the full subcategories

$$C_{\mathcal{D}^{op}(2)fil,h}(S) \subset C_{\mathcal{D}^{op}(2)fil,c}(S) \subset C_{\mathcal{D}^{op}(2)fil}(S),$$

the full subcategories consisting of filtered complexes of presheaves (M, F) such that $a_\tau H^n(M, F)$ are filtered coherent, resp. filtered holonomic, sheaves of right D_S modules.

Let $S \in \mathrm{Var}(k)$. Let $Z \subset S$ a closed subset. Denote by $j : S \backslash Z \hookrightarrow S$ the open embedding. We denote by $C_{\mathcal{D}fil,h,Z}(S) \subset C_{\mathcal{D}fil,h}(S)$ the full subcategory consisting of $(M, F) \in C_{\mathcal{D}fil,h}(S)$ such that $j^* \mathrm{Gr}_F^p M \in C_{O_S}(S)$ is acyclic for all $p \in \mathbb{Z}$.

Proposition 21. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(k)$. Then,*

(i) For $(M, F) \in C_{\mathcal{D}(2)fil,h}(S)$, we have $L\mathbb{D}_S(M, F) \in D_{\mathcal{D}(2)fil,h}(S)$.

(ii) For $(M, F) \in C_{\mathcal{D}(2)fil,h}(S)$, we have $Lf^{*mod}(M, F) \in D_{\mathcal{D}(2)fil,h}(X)$ and $Lf^{\hat{*}mod}(M, F) \in D_{\mathcal{D}(2)fil,h}(X)$.

(iii) For $M \in C_{\mathcal{D},h}(X)$, we have $\int_f M \in D_{\mathcal{D},h}(S)$ and $\int_{f!} M := L\mathbb{D}_S \int_f L\mathbb{D}_X \in D_{\mathcal{D},h}(S)$.

(iv) If f is proper, for $(M, F) \in C_{\mathcal{D}(2)fil,h}(X)$, we have $\int_f (M, F) \in D_{\mathcal{D}(2)fil,h}(S)$.

(v) For $(M, F), (N, F) \in C_{\mathcal{D}(2)fil,h}(S)$, $(M, F) \otimes^L_{O_S} (N, F) \in D_{\mathcal{D}(2)fil,h}(S)$

Proof. Similar to the proof of the complex case in [18] or simply follows from the complex case since $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$ is holonomic if and only if $\pi_{k/\mathbb{C}}(S)^{*mod} M \in \mathrm{PSh}_{\mathcal{D},c}(S_\mathbb{C})$ is holonomic. Note that (v) follows from (ii) by proposition 16. □

Proposition 22. *Let $S \in \mathrm{SmVar}(k)$. For $M \in C_{\mathcal{D},c}(S)$, the canonical map $d(M) : M \to \mathbb{D}_S^2 L_D M$ is an equivalence Zariski local*

Proof. Standard. □

Proposition 23. *Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \mathrm{SmVar}(k)$. Let $M \in C_{\mathcal{D},h}(S)$. Then, we have $L(f_2 \circ f_1)^{\hat{*}mod} M = Lf_1^{\hat{*}mod}(Lf_2^{\hat{*}mod} M)$ in $D_{\mathcal{D},h}(X)$.*

Proof. Follows from proposition 15 (i), proposition 21 and proposition 22, or directly from the complex case. □

Theorem 25. *Let k a field of characteristic zero. Let $S \in \mathrm{SmVar}(k)$.*

(i) Let $M \in D_{\mathcal{D},c}(S)$. Then $M \in D_{\mathcal{D},h}(S)$ if and only if there exist a finite sequence

$$S = S_0 \supset S_1 \supset \cdots \supset S_r \supset S_{r+1} = \emptyset$$

such that for all l, $S_l \backslash S_{l+1}$ is smooth and $H^k(i_l^{*mod} M) \in \mathrm{Vect}_{\mathcal{D}}(S_l \backslash S_{l+1})$ are integrable connexion for all $k \in \mathbb{Z}$, $i_l : S_l \backslash S_{l+1} \hookrightarrow S$ being the locally closed embedding.

(ii) Let $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$. Then $M \in \mathrm{PSh}_{\mathcal{D},h}(S)$ if and only if there exist a finite sequence

$$S = S_0 \supset S_1 \supset \cdots \supset S_r \supset S_{r+1} = \emptyset$$

such that for all l, $S_l \backslash S_{l+1}$ is smooth and $i_l^{*mod} M \in \mathrm{Vect}_{\mathcal{D}}(S_l \backslash S_{l+1})$ is an integrable connexion, $i_l : S_l \backslash S_{l+1} \hookrightarrow S$ being the locally closed embedding.

Proof. (i): Similar to the complex case : follows from proposition 20, theorem 24 and proposition 21. (ii):It is a particular case of (i). □

Let k a field of characteristic zero.

- Let $C \in \mathrm{SmVar}(k)$ connected (hence irreducible). An algebraic meromorphism connexion $M = (M, \nabla) \in \mathrm{Mod}(K_C, D(O_{C,s}))$ at $s \in C$ is a K_C module M endowed with a k linear map $\nabla : M \to \Omega^1_{C,s} \otimes_{O_{C,s}} M$ such that $\nabla(fm) = d_f \otimes m + f\nabla(m)$ for $f \in K_C$ where $K_C := Frac(O_{C,s})$ is the field of fraction of C,
- Let $S \in \mathrm{SmVar}(k)$. An algebraic meromorphism connexion $M = (M, \nabla) \in \mathrm{PSh}_{O_S(*D),D_S}(S)$ along a (Cartier divisor) $D \subset S$ is a coherent $O_S(*D)$ module which has a structure of D_S module. In particular, $M_{|S\backslash D} \in \mathrm{Vect}_{\mathcal{D}}(S\backslash D)$ is an integrable connexion since it is a $D_{S\backslash D}$ module which is a coherent $O_{S\backslash D}$ module.

Lemma 3. *Let $S \in \mathrm{SmVar}(k)$. Let $D \subset S$ a (Cartier) divisor. Denote by $j : S^o := S\backslash D \hookrightarrow S$ the open embedding. Then, the restriction*

$$j^* : \mathrm{PSh}_{O_S(*D),D_S}(S) \to \mathrm{Vect}_{\mathcal{D}}(S^o)$$

is an equivalence of category whose inverse is

$$j_* : \mathrm{Vect}_{\mathcal{D}}(S^o) \to \mathrm{PSh}_{O_S(*D),D_S}(S).$$

By proposition 21, we get a full subcategory $\mathrm{PSh}_{O_S(*D),D_S}(S) \subset \mathrm{PSh}_{\mathcal{D},h}(S)$.

Proof. Standard fact on coherent $O_S(*D) = j_* O_{S^o}$ module. □

We now give the definition of the regularity of integrable connexions and holonomic D_S-modules on $S \in \mathrm{SmVar}(k)$: We first define it for integrable connexion and holonomic D_C-module for $C \in \mathrm{SmVar}(k)$ a smooth algebraic curve over k.

Definition 38. *Let k a field of characteristic zero.*

(i) Let $C \in \mathrm{SmVar}(k)$ connected (hence irreducible). An algebraic meromorphism connexion $M = (M, \nabla) \in \mathrm{Mod}(K_C, D(O_{C,s}))$ at $s \in C$ is called regular if there exists a finitely generated $O_{C,s}$ module $L \subset M$ such that $M = K_{C,s}L$ and $x\nabla(L) \subset \Omega^1_{C,s} \otimes_{O_{C,s}} L$ for some local parameter $x \in O_{C,s}$. We call such an $L \subset M$ an integral lattice.

(ii) Let $C \in \mathrm{SmVar}(k)$. An integrable connexion $M = (M, \nabla) \in \mathrm{Vect}_{\mathcal{D}}(C)$ is called regular if for any smooth compactification $\bar{C} \in \mathrm{PSmVar}(k)$ of C with $j : C \hookrightarrow \bar{C}$ denoting the open embedding the algebraic meromorphic connexion

$$j_* M := (j_* M, j_* \nabla) \in \mathrm{PSh}_{O_{\bar{C}}(*\bar{C}\backslash C),D_{\bar{C}}}(\bar{C})$$

is regular at all $s \in \bar{C}$, that is for all $s \in \bar{C}$ the algebraic meromorphic connexion $(j_*M)_s = ((j_*M)_s, (j_*\nabla)_s) \in \mathrm{Mod}(K_C, D(O_{\bar{C},s}))$ at $s \in \bar{C}$ is regular (see (i)).

(iii) Let $C \in \mathrm{SmVar}(k)$. Let $M \in \mathrm{PSh}_{\mathcal{D},h}(C)$ an holonomic D_C module. Then by proposition 20, there exist an open subset $l : C^o \subset C$ such that $M_{|C^o} := l^* M \in \mathrm{Vect}_{\mathcal{D}}(C^o)$ is an integrable connexion. We say that M is regular if $l^* M \in \mathrm{Vect}_{\mathcal{D}}(C^o)$ is regular (see (ii)).

We have the following :

Proposition 24. *Let $C \in \mathrm{SmVar}(k)$ connected (hence irreducible). Consider an algebraic meromorphism connexion $M = (M, \nabla) \in \mathrm{Mod}(K_C, D(O_{C,s}))$ at $s \in C$. The following are equivalent :*

(i) M is regular

(ii) For any $m \in M$, there exists a finitly generated $O_{C,s}$ submodule $L \subset M$ such that $x\nabla(L) \subset L$.

(iii) For any $m \in M$ there exist a polynomial $F(t) = t^m + a_1 t^{m-1} + \cdots + a_m \in O_{C,s}[t]$ such that $F(x\nabla)(m) = 0$.

Proof. Similar to the complex case. □

We have then the following lemma

Lemma 4. *(i) Let $k : C \to C'$ a morphism with $C, C' \in \mathrm{SmVar}(k)$ smooth algebraic curves.*

- *Let $M \in \mathrm{PSh}_{\mathcal{D},h}(C)$. Then M is regular if and only if $H^k \int_k M \in \mathrm{PSh}_{\mathcal{D},h}(C')$ are regular for all k.*
- *Let $N \in \mathrm{PSh}_{\mathcal{D},h}(C')$. Then M is regular if and only if $H^k Lk^{*mod} N \in \mathrm{PSh}_{\mathcal{D},h}(C)$ are regular for all k.*

(ii) Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. Let $C \in \mathrm{SmVar}(k)$ and $M \in \mathrm{PSh}_{\mathcal{D},h}(C)$. Then M is regular if and only if $\pi_{k/\mathbb{C}}(C)^{*mod} M \in \mathrm{PSh}_{\mathcal{D},h}(C_{\mathbb{C}})$ is regular

Proof. (i):Similar to the complex case : see [18].
(ii): Follows from the fact that for $l : C^o \hookrightarrow C$ an open subset such that $l^*M \in \mathrm{Vect}_{\mathcal{D}}(C^o)$ is an integral connexion and $s \in \bar{C}$ with $\bar{C} \in \mathrm{SmVar}(k)$ a compactification of C, $j : C^o \hookrightarrow C \hookrightarrow \bar{C}$, if $L \subset (j_* l^* M)_s$ is an integral lattice then $\pi_{k/\mathbb{C}}(\bar{C})^{*mod} L \subset (\pi_{k/\mathbb{C}}(\bar{C})^{*mod} j_* l^* M)_s$ is an integral lattice, and conversely if $L' \subset (\pi_{k/\mathbb{C}}(\bar{C})^{*mod} j_* l^* M)_s$ is an integral lattice then $L' \cap (j_* l^* M)_s \subset (j_* l^* M)_s$ is an integral lattice the canonical map

$$(j_* n_{O_{C^o}/O_{C^o_{\mathbb{C}}}}(l^*M))_s : (j_* l^* M)_s \hookrightarrow \pi_{k/\mathbb{C}}(\bar{C})^{*mod} (j_* l^* M)_s, m \mapsto m \otimes 1$$

being injective since l^*M is a locally free O_{C^o} module. □

For integral connexions and holonomic D_S modules on $S \in \mathrm{SmVar}(k)$ an algebraic variety of arbitrary dimesion over k, we define it by the case of curves

Definition 39. *Let k a field of characteristic zero.*

(i) Let $S \in \mathrm{SmVar}(k)$. An algebraic meromorphism connexion $M = (M, \nabla) \in \mathrm{PSh}_{O_S(*D), D_S}(S)$ along a (Cartier divisor) $D \subset S$ is called regular, if for all morphism $i_C : C \to X$ with $C \in \mathrm{SmVar}(k)$ a smooth curve and all $s = D \cap i_C(C)$, the meromorphic connexion $(i_C^{*mod} M, \nabla) \in \mathrm{Mod}(K_C, D(O_{C,s}))$ is regular (see definition 38).

(ii) Let $S \in \mathrm{SmVar}(k)$. An integrable connexion $M = (M, \nabla) \in \mathrm{Vect}_{\mathcal{D}}(S)$ is called regular if for any smooth compactification $\bar{S} \in \mathrm{PSmVar}(k)$ of S with $D := \bar{S} \backslash S \subset \bar{S}$ a (Cartier) divisor, the algebraic meromorphic connexion $(j_* M) = ((j_* M), (j_* \nabla)) \in \mathrm{PSh}_{O_{\bar{S}}(*D), D_{\bar{S}}}(\bar{S})$ along $D \subset S$ is regular, where $j : S \hookrightarrow \bar{S}$ is the open embedding (see (i)).

(iii) Let $S \in \mathrm{SmVar}(k)$. An holonomic D_S module $M \in \mathrm{PSh}_{\mathcal{D},h}(S)$ is called regular if for all morphism $i_C : C \to S$ with $C \in \mathrm{SmVar}(k)$, $i_C^{*mod} M \in \mathrm{PSh}_{\mathcal{D},h}(C)$ is regular (see definition 38).

(iv) Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. Let $S \in \mathrm{SmVar}(k)$ and $M \in \mathrm{PSh}_{\mathcal{D},h}(S)$. Consider the projection $\pi_{k/\mathbb{C}}(S) : S \to S_{\mathbb{C}}$. Then by lemma 4(ii), if $\pi_{k/\mathbb{C}}(S)^{*mod}M \in \mathrm{PSh}_{\mathcal{D},h}(S_{\mathbb{C}})$ is regular then $M \in \mathrm{PSh}_{\mathcal{D},h}(S)$ is regular. So, let k a field of characteristic zero. Let $S \in \mathrm{SmVar}(k)$ and $M \in \mathrm{PSh}_{\mathcal{D},h}(S)$. We say that M is regular in the strong sense if

$$\pi_{k/\mathbb{C}}(S_0)^{*mod}M_0 \in \mathrm{PSh}_{\mathcal{D},h}(S_{0\mathbb{C}})$$

is regular, where $k_0 \subset k$ is a subfield of finite transcandence degree over $\mathbb{Q}$ such that S and M are defined that is $S = S_{0k} := S_0 \otimes_{k_0} k$ with $S_0 \in \mathrm{SmVar}(k_0)$ and $M = \pi_{k_0/k}(S_0)^{*mod}M_0$ with $M_0 \in \mathrm{PSh}_{\mathcal{D},h}(S_0)$, and we take an embedding $\sigma : k_0 \hookrightarrow \mathbb{C}$. This definition does NOT depend on the choice of the subfield k_0 and the embedding $\sigma : k_0 \hookrightarrow \mathbb{C}$.

For $S \in \mathrm{SmVar}(k)$, we denote $\mathrm{PSh}_{\mathcal{D},rh}(S) \subset \mathrm{PSh}_{\mathcal{D},h}(S)$ the full subcategory consisting of holonomic D_S modules $M \in \mathrm{PSh}_{\mathcal{D},h}(S)$ regular in the strong sense (see (iv)).

We have then the following easy proposition :

Proposition 25. *Let $S \in \mathrm{SmVar}(k)$. Consider an exact sequence in $\mathrm{PSh}_{\mathcal{D},h}(S)$*

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

(i) Then, M_2 is regular if and only if M_1 and M_3 are regular

(ii) Then, $M_2 \in \mathrm{PSh}_{\mathcal{D},rh}(S)$ if and only if $M_1, M_3 \in \mathrm{PSh}_{\mathcal{D},rh}(S)$.

Proof. (i):Similar to the complex case : by definition we are reduced to the case of integrable connexions on curves. But for $0 \to M_1 \to M_2 \xrightarrow{q} M_3 \to 0$ an exact sequence of integrable connexions on a curve $C \in \mathrm{SmVar}(k)$ with compactification $\bar{C} \in \mathrm{PSmVar}(k)$, M_1 and M_3 are regular at $s \in \bar{C}$ if and only if M_2 is regular at s by proposition 24 (use (iii)).
(ii):Follows by definition from the complex case which is a particular case of (i). □

Let k a field of characteristic zero. Let $S \in \mathrm{SmVar}(k)$

- we consider
  - the full subcategories

$$C_{\mathcal{D},rh}(S) \subset C_{\mathcal{D},h}(S) \text{ and } D_{\mathcal{D},rh}(S) \subset D_{\mathcal{D},h}(S)$$

    consisting of complexes of presheaves M such that $a_\tau H^n(M) \in \mathrm{PSh}_{\mathcal{D},rh}(S)$ (see definition 39), a_τ being the sheaftification functor for the Zariski topology,
  - the full subcategories

$$C_{\mathcal{D}^{op},rh}(S) \subset C_{\mathcal{D}^{op},h}(S) \text{ and } D_{\mathcal{D}^{op},rh}(S) \subset D_{\mathcal{D}^{op},h}(S)$$

    the full subcategories consisting of complexes of presheaves M such that $a_\tau H^n(M)^{op} \in \mathrm{PSh}_{\mathcal{D},rh}(S)$,
- in the filtered case we have
  - the full subcategories

$$C_{\mathcal{D}(2)fil,rh}(S) \subset C_{\mathcal{D}(2)fil,h}(S), \text{ and } D_{\mathcal{D}(2)fil,rh}(S) \subset D_{\mathcal{D}(2)fil,h}(S),$$

    consisting of filtered complexes of presheaves (M, F) such that $a_\tau H^n(M) \in \mathrm{PSh}_{\mathcal{D},rh}(S)$ (see definition 39), the full subcategories

$$C_{\mathcal{D}(1,0)fil,rh}(S) = C_{\mathcal{D}2fil,rh}(S) \cap C_{\mathcal{D}(1,0)fil}(S) \subset C_{\mathcal{D}2fil,rh}(S), \text{ and}$$
$$D_{\mathcal{D}(1,0)fil,rh}(S) = D_{\mathcal{D}2fil,rh}(S) \cap D_{\mathcal{D}(1,0)fil}(S) \subset D_{\mathcal{D}2fil,rh}(S),$$

    consisting of filtered complexes of presheaves $(M, F, W) \in C_{\mathcal{D}(1,0)fil,h}(S)$ such that $a_\tau H^n(M) \in \mathrm{PSh}_{\mathcal{D},rh}(S)$ (see definition 39)

– and similarly the full subcategories

$$C_{\mathcal{D}^{op}(2)fil,rh}(S) \subset C_{\mathcal{D}^{op}(2)fil,h}(S), \text{ and } D_{\mathcal{D}^{op}(2)fil,rh}(S) \subset D_{\mathcal{D}^{op}(2)fil,h}(S)$$

the full subcategories consisting of filtered complexes of presheaves $(M,F) \in C_{\mathcal{D}^{op}(2)fil,h}(S)$ such that $a_\tau H^n(M)^{op} \in \mathrm{PSh}_{\mathcal{D},rh}(S)$.

Let $S \in \mathrm{SmVar}(k)$. Let $Z \subset S$ a closed subset. Denote by $j : S\backslash Z \hookrightarrow S$ the open embedding. We denote by $C_{\mathcal{D}fil,Z,rh}(S) \subset C_{\mathcal{D}fil,rh}(S)$ the full subcategory consisting of $(M,F) \in C_{\mathcal{D}fil,rh}(S)$ such that $j^* \mathrm{Gr}_F^p M \in C_{O_S}(S)$ is acyclic for all $p \in \mathbb{Z}$.

We now give an equivalent definition of regular holonomic D_S modules together with a result on stability by direct, inverse image and duality.

Let $S \in \mathrm{SmVar}(k)$. For each $M \in \mathrm{PSh}_{\mathcal{D},h}(S)$ there exists by proposition 19(ii) a finite sequence of holonomic submodules

$$0 = M_{r+1} \subset M_r \subset \cdots \subset M_1 \subset M_0 = M$$

such that $M_i/M_{i+1} \in \mathrm{PSh}_{\mathcal{D},h}(S)$ is simple.

Definition 40. *Let $k : Z^o \hookrightarrow S$ a locally closed embedding with $S, Z^o \in \mathrm{SmVar}(k)$, and assume k is affine. We define for $M \in \mathrm{PSh}_{\mathcal{D},h}(Z^o)$ the minimal extension*

$$L_{Z^o/S}(M) := T(k_!, k_*)(\int_{k!} M) \subset \int_k M$$

where $T(k_!, k_*)(M) : \int_{k!} M \to \int_k M$ is given by, using a factorization of k by open embeddings and proper morphisms

- *the adjonction map $T(j_!, j_*)(N) := \mathrm{ad}(j^*, j_*)(j_! N) : j_! N \to j_* N$ for open embeddings $j : X^o \hookrightarrow X$ with $X \in \mathrm{SmVar}(k)$,*
- *the trace map on proper morphisms.*

By proposition 19(i), $L_{Z^o/S}(M) \in \mathrm{PSh}_{\mathcal{D},h}(S)$ is holonomic.

Theorem 26. *(i) Let $k : Z^o \hookrightarrow S$ a locally closed embedding with $S, Z^o \in \mathrm{SmVar}(k)$, and assume k is affine. Let $M \in \mathrm{PSh}_{\mathcal{D},h}(Z^o)$. If M is simple, then $L_{Z^o/S}(M)$ is also simple, and is the unique simple submodule of $\int_k M$ and the unique quotient module of $\int_{k!} M$.*

(ii) Let $S \in \mathrm{SmVar}(k)$. Let $M \in \mathrm{PSh}_{\mathcal{D},h}(S)$. If M is a simple D_S module then there exist $k : Z^o \hookrightarrow S$ a locally closed embedding with $Z \in \mathrm{SmVar}(k)$, k affine, such that $M \simeq L_{Z^o/S}(N)$ with $N \in \mathrm{Vect}_{\mathcal{D}}(Z^o)$ a simple integral connexion.

(iii) Let $k : Z^o \hookrightarrow S$, $k' : Z'^o$ locally closed embeddings with $S, Z^o, Z'^o \in \mathrm{SmVar}(k)$, k, k' affine. Let $N \in \mathrm{Vect}_{\mathcal{D}}(Z^o)$ and $N \in \mathrm{Vect}_{\mathcal{D}}(Z'^o)$ simple integral connexions. Then $L_{Z^o/S}(N) \simeq L_{Z'^o/S}(N')$ in $\mathrm{PSh}_{\mathcal{D}}(S)$ if and only if $\bar{Z}^o = \bar{Z}'^o$ and $N_{|U} \simeq N'_{|U}$ for an open dense subset $U \subset Z^o \cap Z'^o$.

Proof. Similar to the complex case : see [18]. □

Theorem 27. *Let $S \in \mathrm{SmVar}(k)$. Let $M \in \mathrm{PSh}_{\mathcal{D},h}(S)$. Take by proposition 19(ii) a finite sequence of holonomic submodules*

$$0 = M_{r+1} \subset M_r \subset \cdots \subset M_1 \subset M_0 = M$$

such that $M_i/M_{i+1} \in \mathrm{PSh}_{\mathcal{D},h}(S)$ is simple. By theorem 26 there exist locally closed embeddings $k_i : Z_i^o \hookrightarrow S$ with $Z_i^o \in \mathrm{SmVar}(k)$ and $N_i \in \mathrm{Vect}_{\mathcal{D}}(Z_i^o)$ simple integrable connexion such that $M_i \simeq L_{S_i^o/S}(N_i)$ in $\mathrm{PSh}_{\mathcal{D}}(S)$. Then M is regular if and only if the simple integral connexions $N_i \in \mathrm{Vect}_{\mathcal{D}}(Z_i^o)$ are regular (see definition 39).

Proof. Similar to the proof of the complex case in [18]. □

Theorem 28. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(k)$. Then,*

(i) For $(M,F) \in C_{\mathcal{D}(2)fil,rh}(S)$, we have $L\mathbb{D}_S(M,F) \in D_{\mathcal{D}(2)fil,rh}(S)$.

(ii) For $(M,F) \in C_{\mathcal{D}(2)fil,rh}(S)$, we have $Lf^{*mod}(M,F) \in D_{\mathcal{D}(2)fil,rh}(X)$ and $Lf^{\hat{*}mod}(M,F) \in D_{\mathcal{D}(2)fil,rh}(X)$.

(iii) For $M \in C_{\mathcal{D},rh}(X)$, we have $\int_f M \in D_{\mathcal{D},rh}(S)$. and $\int_{f!} M := L\mathbb{D}_S \int_f L\mathbb{D}_X \in D_{\mathcal{D},rh}(S)$.

(iv) If f is proper, for $(M,F) \in C_{\mathcal{D}(2)fil,rh}(X)$, we have $\int_f (M,F) \in D_{\mathcal{D}(2)fil,rh}(S)$.

(v) For $(M,F),(N,F) \in C_{\mathcal{D}(2)fil,rh}(S)$, $(M,F) \otimes^L_{O_S} (N,F) \in D_{\mathcal{D}(2)fil,rh}(S)$

Proof. Follows by definition from the complex case : (i),(ii) and (iii): See [18].
(iv): Follows from (iii) and stability of coherent O_X-modules by direct image of proper morphism $f : X \to S$.
(v):Follows from (ii) by proposition 16. □

4.2 The D modules on singular algebraic varieties over a field k of characteristic zero

In this subsection by defining the category of complexes of filtered D-modules in the singular case and there functorialities.

4.2.1 Definition

In all this subsection, we fix the notations: Let k a field of characteristic zero. For $S \in \mathrm{Var}(k)$, we denote by $S = \cup_{i=1}^l S_i$ an open cover such that there exits closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We have then closed embeddings $i_I : S_I := \cap_{i\in I} S_i \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_I$. Then for $I \subset J$, we denote by $j_{IJ} : S_J \hookrightarrow S_I$ the open embedding and $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ the projection, so that $p_{IJ} \circ i_J = i_I \circ j_{IJ}$. This gives the diagram of algebraic varieties $(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{Var}(k))$ which gives the diagram of sites $(\tilde{S}_I) := \mathrm{Ouv}(\tilde{S}_I) \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{Cat})$. It also gives the diagram of sites $(\tilde{S}_I)^{op} := \mathrm{Ouv}(\tilde{S}_I)^{op} \in \mathrm{Fun}(\mathcal{P}(\mathbb{N}), \mathrm{Cat})$. For $I \subset J$, we denote by $m : \tilde{S}_I \backslash (S_I \backslash S_J) \hookrightarrow \tilde{S}_I$ the open embedding.

Definition 41. *Let $S \in \mathrm{Var}(k)$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then, $\mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}(2)fil}((\tilde{S}_I))$ is the full subcategory*

- *whose objects are $(M,F) = ((M_I,F)_{I\subset[1,\cdots l]}, s_{IJ})$, with*
  - *$(M_I,F) \in \mathrm{PSh}_{\mathcal{D}(2)fil}(\tilde{S}_I)$ such that $\mathcal{I}_{S_I} M_I = 0$, in particular $(M_I,F) \in \mathrm{PSh}_{\mathcal{D}(2)fil,S_I}(\tilde{S}_I)$*
  - *$s_{IJ} : m^*(M_I,F) \xrightarrow{\sim} m^* p_{IJ*}(M_J,F)[d_{\tilde{S}_I} - d_{\tilde{S}_J}]$ for $I \subset J$, are isomorphisms, $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ being the projection, satisfying for $I \subset J \subset K$, $p_{IJ*}s_{JK} \circ s_{IJ} = s_{IK}$;*
- *the morphisms $m : (M,F) \to (N,F)$ between $(M,F) = ((M_I,F)_{I\subset[1,\cdots l]}, s_{IJ})$ and $(N,F) = ((N_I,F)_{I\subset[1,\cdots l]}, r_{IJ})$ are by definition a family of morphisms of complexes,*

$$m = (m_I : (M_I,F) \to (N_I,F))_{I\subset[1,\cdots l]}$$

such that $r_{IJ} \circ m_J = p_{IJ*} m_J \circ s_{IJ}$ in $C_{\mathcal{D},S_J}(\tilde{S}_J)$.

We denote by

$$\mathrm{PSh}_{\mathcal{D}(2)fil,rh}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}(2)fil,h}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}(2)fil,c}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

the full subcategory consisting of $((M_I,F),s_{IJ}) \in \mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ *such that for all* $I \subset [1,\ldots,l]$, $(M_I,F) \in \mathrm{PSh}_{\mathcal{D}(2)fil,c}(\tilde{S}_I)$ *resp.* $(M_I,F) \in \mathrm{PSh}_{\mathcal{D}(2)fil,h}(\tilde{S}_I)$, *resp.* $(M_I,F) \in \mathrm{PSh}_{\mathcal{D}(2)fil,h}(\tilde{S}_I)$ *and* $M_I \in \mathrm{PSh}_{\mathcal{D},rh}(\tilde{S}_I)$ *(see definition 39) We have the full subcategories*

$$\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}2fil,rh}(S/(\tilde{S}_I)),\ \mathrm{PSh}_{\mathcal{D}(1,0)fil,h}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}2fil,h}(S/(\tilde{S}_I)),$$
$$\mathrm{PSh}_{\mathcal{D}(1,0)fil,h}(S/(\tilde{S}_I)) \subset \mathrm{PSh}_{\mathcal{D}2fil,h}(S/(\tilde{S}_I)),$$

consisting of $((M_I,F,W),s_{IJ})$ *such that* W^pM_I *are* $D_{\tilde{S}_I}$ *submodules.*

We recall from [10] the following

- A morphism $m=(m_I):((M_I),s_{IJ}) \to ((N_I),r_{IJ})$ in $C(\mathrm{PSh}_{\mathcal{D}}(S/(\tilde{S}_I)))$ is a Zariski, resp. usu, local equivalence if and only if all the m_I are Zariski local equivalences.
- A morphism $m=(m_I):((M_I,F),s_{IJ} \to ((N_I,F),r_{IJ}))$ in $C(\mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$ is a filtered Zariski local equivalence if and only if all the m_I are filtered Zariski local equivalence.
- By definition, a morphism $m=(m_I):((M_I,F),s_{IJ}) \to ((N_I,F),r_{IJ})$ in $C(\mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$ is an r-filtered Zariski local equivalence if there exist $m_i:((C_{iI},F),s_{iIJ}) \to ((C_{(i+1)I},F),s_{(i+1)IJ})$, $0 \leq i \leq s$, with $((C_{iI},F),s_{iIJ}) \in C(\mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$, $((C_{0I},F),s_{iIJ}) = ((M_I,F),s_{IJ})$, $((C_{sI},F),s_{sIJ}) = ((N_I,F),r_{IJ})$ such that

  $$m = m_s \circ \cdots \circ m_i \circ \cdots \circ m_0 : ((M_I,F),s_{IJ} \to ((N_I,F),r_{IJ}))$$

  with $m_i:((C_{iI},F),s_{iIJ}) \to ((C_{(i+1)I},F),s_{(i+1)IJ})$ either filtered Zariski local equivalence or r-filtered homotopy equivalence.

Definition-Proposition 4. *Let* $S \in \mathrm{Var}(k)$ *and let* $S=\cup_i S_i$ *an open cover such that there exist closed embeddings* $i_iS_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{SmVar}(k)$. *Then* $\mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ *does not depend on the open covering of* S *and the closed embeddings and we set*

$$\mathrm{PSh}_{\mathcal{D}(2)fil}(S) := \mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

We denote by $C^0_{\mathcal{D}(2)fil}(S) := C(\mathrm{PSh}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$ *and by* $D^0_{\mathcal{D}(2)fil,r}(S) := K^0_{\mathcal{D}(2)fil,r}(S)([E_1]^{-1})$ *the localization of the* r*-filtered homotopy category with respect to the classes of filtered Zariski local equivalences.*

Proof. Similar to the complex case : see [10]. □

We now give the definition of our category :

Definition 42. *Let* $S \in \mathrm{Var}(k)$ *and let* $S=\cup_i S_i$ *an open cover such that there exist closed embeddings* $i_i:S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{SmVar}(k)$. *Then,* $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil}((\tilde{S}_I))$ *is the full subcategory*

- *whose objects are* $(M,F)=((M_I,F)_{I\subset[1,\cdots l]},u_{IJ})$, *with*
  - $(M_I,F) \in C_{\mathcal{D}(2)fil,S_I}(\tilde{S}_I)$ *that is* $(M_I,F) \in C_{\mathcal{D}(2)fil}(\tilde{S}_I)$ *satisfy* $n_I^* \mathrm{Gr}^p_F M_I \in C_O(\tilde{S}_I)$ *is acyclic for all* $p \in \mathbb{Z}$, *where* $n_I:\tilde{S}_I \backslash S_I \hookrightarrow \tilde{S}_I$ *is the open embedding,*
  - $u_{IJ}: m^*(M_I,F) \to m^*p_{IJ*}(M_J,F)[d_{\tilde{S}_I}-d_{\tilde{S}_J}]$ *for* $J \subset I$, *are morphisms,* $p_{IJ}:\tilde{S}_J \to \tilde{S}_I$ *being the projection, satisfying for* $I \subset J \subset K$, $p_{IJ} * u_{JK} \circ u_{IJ} = u_{IK}$ *in* $C_{\mathcal{D}fil}(\tilde{S}_I)$ *;*
- *the morphisms* $m:((M_I,F),u_{IJ}) \to ((N_I,F),v_{IJ})$ *between* $(M,F)=((M_I,F)_{I\subset[1,\cdots l]},u_{IJ})$ *and* $(N,F)=((N_I,F)_{I\subset[1,\cdots l]},v_{IJ})$ *being a family of morphisms of complexes,*

  $$m = (m_I:(M_I,F) \to (N_I,F))_{I\subset[1,\cdots l]}$$

  such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ *in* $C_{\mathcal{D}fil}(\tilde{S}_I)$.

We denote by $C^{\sim}_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ *the full subcategory consisting of objects* $((M_I, F), u_{IJ})$ *such that the* u_{IJ} *are filtered Zariski local equivalences.*

Let $S \in \mathrm{Var}(k)$ and let $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then, We denote by

$$C_{\mathcal{D}(2)fil,rh}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil,h}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil,c}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

the full subcategories consisting of those $((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ such that for all $I \subset [1, \ldots, l]$, $(M_I, F) \in C_{\mathcal{D}(2)fil,S_I,c}(\tilde{S}_I)$, that is such that $a_\tau H^n(M_I, F) \in \mathrm{PSh}_{\mathcal{D}fil,c}(\tilde{S}_I)$ are coherent endowed with a good filtration for all $n \in \mathbb{Z}$, resp. $(M_I, F) \in C_{\mathcal{D}(2)fil,S_I,h}(\tilde{S}_I)$, that is such that $a_\tau H^n(M_I, F) \in \mathrm{PSh}_{\mathcal{D}fil,h}(\tilde{S}_I)$ are filtered holonomic for all $n \in \mathbb{Z}$, resp. such that $(M_I, F) \in C_{\mathcal{D}(2)fil,S_I,rh}(\tilde{S}_I)$, that is such that $a_\tau H^n(M_I, F) \in \mathrm{PSh}_{\mathcal{D}fil,h}(\tilde{S}_I)$ are filtered holonomic for all $n \in \mathbb{Z}$ and $a_\tau H^n M_I \in \mathrm{PSh}_{\mathcal{D},rh}(\tilde{S}_I)$(see definition 39).

We denote by

$$C_{\mathcal{D}(1,0)fil,h}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}2fil,h}(S/(\tilde{S}_I)),\ C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}2fil,rh}(S/(\tilde{S}_I)),$$
$$C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \subset C_{\mathcal{D}2fil}(S/(\tilde{S}_I)),$$

the full subcategories consisting of those $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}2fil}(S/(\tilde{S}_I))$ such that $W^p M_I$ are $D_{\tilde{S}_I}$ submodules.

We recall from [10] the following

- A morphism $m = (m_I) : (M_I, u_{IJ}) \to (N_I, v_{IJ})$ in $C_{\mathcal{D}}(S/(\tilde{S}_I))$ is a Zariski local equivalence if and only if all the m_I are Zariski local equivalences.
- A morphism $m = (m_I) : ((M_I, F), u_{IJ} \to ((N_I, F), v_{IJ}))$ in $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ is a filtered Zariski local equivalence if and only if all the m_I are filtered Zariski local equivalence.
- Let $r = 1, \cdots, \infty$. By definition, a morphism $m = (m_I) : ((M_I, F), u_{IJ}) \to ((N_I, F), v_{IJ})$ in $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ is an r-filtered Zariski local equivalence if there exist $m_i : (C_{iI}, F), u_{iIJ}) \to (C_{(i+1)I}, F), u_{(i+1)IJ})$, $0 \leq i \leq s$, with $(C_{iI}, F), u_{iIJ}) \in C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$, $(C_{0I}, F), u_{iIJ}) = (M_I, F), u_{IJ})$, $(C_{sI}, F), u_{sIJ}) = (N_I, F), v_{IJ})$ such that
$$m = m_s \circ \cdots \circ m_i \circ \cdots \circ m_0 : ((M_I, F), u_{IJ} \to ((N_I, F), v_{IJ}))$$
with $m_i : (C_{iI}, F), u_{iIJ}) \to (C_{(i+1)I}, F), u_{(i+1)IJ})$ either filtered Zariski local equivalence or r-filtered homotopy equivalence (i.e. r-filtered for the first filtration and filtered for the second filtration).

Definition 43. *Let* $S \in \mathrm{Var}(k)$ *and let* $S = \cup_i S_i$ *an open cover such that there exist closed embeddings* $i_i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{SmVar}(k)$.

(i) We have the derived category
$$D_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) := C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))^{\sim}([E_1]^{-1})$$
the localization with respect to the classes of filtered Zariski local equivalences, together with the localization functor
$$D(zar) : C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))^{\sim} \to K_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \to D_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)).$$

(ii) We have the full subcategories
$$D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}(1,0)fil,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}2fil}(S/(\tilde{S}_I))$$
which are the image of $C_{\mathcal{D}(1,0)fil,h}(S/(\tilde{S}_I))$*, resp. of* $C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))$*, by the localization functor* $D(zar) : C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \to D_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$.

(iii) *We have, for $r = 1, \ldots, \infty$, the r-filtered homotopy category*

$$K_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I)) := \mathrm{Ho}_r(C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)))$$

whose objects are those of $C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ and whose morphism are r-filtered homotopy classes of morphisms (r-filtered for the first filtration and filtered for the second), and

$$D_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I)) := K_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I)))([E_1]^{-1})$$

the localization with respect to the classes of filtered Zariski local equivalences, together with the localization functor

$$D(zar) : C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \to K_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I)) \to D_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I)).$$

(iv) *We have*

$$D_{\mathcal{D}(1,0)fil,\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}2fil,\infty,h}(S/(\tilde{S}_I)) \subset D_{\mathcal{D}2fil,\infty}(S/(\tilde{S}_I))$$

the full subcategories which are the image of $C_{\mathcal{D}2fil,h}(S/(\tilde{S}_I))$, resp. of $C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))$, by the localization functor $D(zar) : C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \to D_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I))$.

Let $S \in \mathrm{Var}(k)$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$.

- We denote by

$$C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))^0 \subset C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \text{ and } D_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))^0 \subset D_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

  the full subcategories consisting of $((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$ such that

$$H^n((M_I, F), u_{IJ}) = (H^n(M_I, F), H^n u_{IJ}) \in \mathrm{PSh}^0_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))$$

  that is such that the $H^n u_{IJ}$ are isomorphism.

- We have the full embedding functor

$$\iota^0_{S/(\tilde{S}_I)} : C^0_{\mathcal{D}(2)fil}(S) := C^0_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)) \hookrightarrow C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))^0 \hookrightarrow C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)),$$
$$((M_I, F), s_{IJ}) \mapsto ((M_I, F), s_{IJ})$$

  This full embedding induces in the derived category the functors

$$\iota^0_{S/(\tilde{S}_I)} : D^0_{\mathcal{D}(2)fil,r}(S) := \mathrm{Ho}_{zar}(C^0_{\mathcal{D}(2)fil,\infty}(S/(\tilde{S}_I))) \to D_{\mathcal{D}(2)fil}(S/(\tilde{S}_I))^0 \hookrightarrow D_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I)),$$
$$((M_I, F), s_{IJ}) \mapsto ((M_I, F), s_{IJ}).$$

  We can show that this functor is a full embedding.

4.2.2 Duality in the singular case

The definition of Saito's category comes with a dual functor :

Definition 44. *Let $S \in \mathrm{Var}(k)$ and let $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We have the dual functor :*

$$\mathbb{D}^K_S : C^0_{\mathcal{D}fil}(S/(\tilde{S}_I)) \to C^0_{\mathcal{D}fil}(S/(\tilde{S}_I)),\ ((M_I, F), s_{IJ}) \mapsto (\mathbb{D}^K_{\tilde{S}_I}(M_I, F), s^d_{IJ}),$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$,

$$u^q_{IJ} : \mathbb{D}^K_{\tilde{S}_I}(M_I, F) \xrightarrow{\mathbb{D}^K(s^{-1}_{IJ})} \mathbb{D}^K_{\tilde{S}_I} p_{IJ*}(M_J, F)[d_{IJ}] \xrightarrow{T_*(p_{IJ}, D)(-)} p_{IJ*}\mathbb{D}^K_{\tilde{S}_J}(M_J, F)[d_{IJ}]$$

It induces in the derived category the functor

$$L\mathbb{D}^K_S : D^0_{\mathcal{D}fil}(S/(\tilde{S}_I)) \to D^0_{\mathcal{D}fil}(S/(\tilde{S}_I)),\ ((M_I, F), s_{IJ}) \mapsto \mathbb{D}^K_S Q((M_I, F), s_{IJ}),$$

with $q : Q((M_I, F), s_{IJ}) \to ((M_I, F), s_{IJ})$ a projective resolution.

For our definition, we have

Definition 45. *Let $S\in\mathrm{Var}(\mathbb{C})$ and let $S=\cup S_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$. We have the dual functors :*

$$\mathbb{D}_S:C_{\mathcal{D}fil}(S/(\tilde{S}_I))\to C_{\mathcal{D}fil}(S/(\tilde{S}_I)^{op}),\ ((M_I,F),s_{IJ})\mapsto(\mathbb{D}_{\tilde{S}_I}(M_I,F),s^d_{IJ}),$$

with, denoting for short $d_{IJ}:=d_{\tilde{S}_J}-d_{\tilde{S}_I}$,

$$u^q_{IJ}:p_{IJ*}\mathbb{D}_{\tilde{S}_J}(M_J,F)[d_{IJ}]$$

and

$$\mathbb{D}_S:C_{\mathcal{D}fil}(S/(\tilde{S}_I)^{op})\to C_{\mathcal{D}fil}(S/(\tilde{S}_I)),\ ((M_I,F),s_{IJ})\mapsto(\mathbb{D}_{\tilde{S}_I}(M_I,F),s^d_{IJ}),$$

with, denoting for short $d_{IJ}:=d_{\tilde{S}_J}-d_{\tilde{S}_I}$,

$$u^q_{IJ}:\mathbb{D}_{\tilde{S}_I}(M_I,F)$$

It induces in the derived category the functors

$$L\mathbb{D}_S:D_{\mathcal{D}fil,r}(S/(\tilde{S}_I))\to D_{\mathcal{D}fil,r}(S/(\tilde{S}_I)^{op}),\ ((M_I,F),s_{IJ})\mapsto\mathbb{D}_SQ((M_I,F),s_{IJ}),$$

with $q:Q((M_I,F),s_{IJ})\to((M_I,F),s_{IJ})$ a projective resolution, and

$$L\mathbb{D}_S:D_{\mathcal{D}fil,r}(S/(\tilde{S}_I)^{op})\to D_{\mathcal{D}fil,r}(S/(\tilde{S}_I)),\ ((M_I,F),s_{IJ})\mapsto\mathbb{D}_SQ((M_I,F),s_{IJ}),$$

with $q:Q((M_I,F),s_{IJ})\to((M_I,F),s_{IJ})$ a projective resolution.

4.2.3 Inverse image in the singular case

We give in this subsection the inverse image functors between our categories.

Let $n:S^o\hookrightarrow S$ be an open embedding with $S\in\mathrm{Var}(k)$ and let $S=\cup_iS_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(k)$. Denote $S^o_I:=n^{-1}(S_I)=S_I\cap S^o$ and $n_I:=n_{|S^o_I}:S^o_I\hookrightarrow S^o$ the open embeddings. Consider open embeddings $\tilde{n}_I:\tilde{S}^o_I\hookrightarrow\tilde{S}_I$ such that $\tilde{S}^o_I\cap S_I=S^o_I$, that is which are lift of n_I. We have the functor

$$\begin{aligned}&n^*:C_{\mathcal{D}fil}(S/(\tilde{S}_I))\to C_{\mathcal{D}fil}(S^o/(\tilde{S}^o_I)),\\(M,F)=((M_I,F),u_{IJ})\mapsto n^*(M,F):=(\tilde{n}_I)^*(M,F)&:=(\tilde{n}^*_I(M_I,F),n^*u_{IJ})\end{aligned}$$

which derive trivially.

Let $f:X\to S$ be a morphism, with $X,S\in\mathrm{Var}(k)$, such that there exist a factorization $f;X\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ with $Y\in\mathrm{SmVar}(k)$, l a closed embedding and p_S the projection, and consider $S=\cup_{i=1}^lS_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$, with $\tilde{S}_i\in\mathrm{SmVar}(k)$; The (graph) inverse image functors is

$$\begin{aligned}&f^{*mod[-],\Gamma}:C_{\mathcal{D}fil}(S/(\tilde{S}_I))\to C_{\mathcal{D}fil}(X/(Y\times\tilde{S}_I)),\\(M,F)=((M_I,F),u_{IJ})\mapsto f^{*mod[-],\Gamma}(M,F)&:=(\Gamma_{X_I}E(p^{*mod[-]}_{\tilde{S}_I}(M_I,F)),\tilde{f}^{*mod[-]}_Ju_{IJ})\end{aligned}$$

with $\tilde{f}^{*mod[-]}_Ju_{IJ}$ as in [10], It induces in the derived categories the functor

$$\begin{aligned}&Rf^{*mod[-],\Gamma}:D_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I))\to D_{\mathcal{D}(2)fil,r}(X/(Y\times\tilde{S}_I)),\\(M,F)=((M_I,F),u_{IJ})\mapsto f^{*mod[-],\Gamma}(M,F)&:=(\Gamma_{X_I}E(p^{*mod[-]}_{\tilde{S}_I}(M_I,F)),\tilde{f}^{*mod[-]}_Ju_{IJ}).\end{aligned}$$

It gives by duality the functor

$$\begin{aligned}&Lf^{\hat{*}mod[-],\Gamma}:D_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I))\to D_{\mathcal{D}(2)fil,r}(X/(Y\times\tilde{S}_I)),\\(M,F)=((M_I,F),u_{IJ})\mapsto Lf^{\hat{*}mod[-],\Gamma}(M,F)&:=L\mathbb{D}_SRf^{*mod[-],\Gamma}L\mathbb{D}_S(M,F).\end{aligned}$$

The following proposition is easy :

Proposition 26. *Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \mathrm{Var}(k)$. Assume there exist factorizations $f_1 : X \xrightarrow{l_1} Y' \times Y \xrightarrow{p_Y} Y$ and $f_2 : Y \xrightarrow{l_2} Y'' \times S \xrightarrow{p_S} S$ with $Y', Y'' \in \mathrm{SmVar}(k)$, l_1, l_2 closed embeddings and p_S, p_Y the projections. We have then the factorization*

$$f_2 \circ f_1 : X \xrightarrow{(l_2 \circ I_{Y'}) \circ l_1} Y' \times Y'' \times S \xrightarrow{p_S} S.$$

We have, for $(M, F) \in C_{\mathcal{D}(2)fil}^{\sim}(S/(\tilde{S}_I))$, $R(f_2 \circ f_1)^{*mod[-],\Gamma}(M, F) = Rf_2^{*mod[-],\Gamma} \circ Rf_1^{*mod[-],\Gamma}(M, F)$.

Proof. Similar to the complex case : see [10]. □

4.2.4 Direct image functor in the singular case

We define the direct image functors between our category.

Let $f : X \to S$ be a morphism with $X, S \in \mathrm{Var}(k)$, and assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S a the projection ; Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i \in I} S_i$ and $X_I = \cap_{i \in I} X_i$. For $I \subset [1, \cdots l]$, denote by $\tilde{S}_I = \Pi_{i \in I} \tilde{S}_i$, We define the direct image functor on our category by

$$f_{*mod}^{FDR} : C_{\mathcal{D}(2)fil}(X/(Y \times \tilde{S}_I)) \to C_{\mathcal{D}(2)fil}(S/(\tilde{S}_I)),$$
$$((M_I, F), u_{IJ}) \mapsto (\tilde{f}_{I*mod}^{FDR}(M_I, F), f^k(u_{IJ})) := (p_{\tilde{S}_I *} E((\Omega^{\bullet}_{Y \times \tilde{S}_I/\tilde{S}_I}, F_b) \otimes_{O_{Y \times \tilde{S}_I}} (M_I, F)[d_Y]), f^k(u_{IJ}))$$

with $f^k(u_{IJ})$ as in [10]. It induces in the derived categories the functor

$$\int_f^{FDR} : D_{\mathcal{D}(2)fil,r}(X/(Y \times \tilde{S}_I)) \to D_{\mathcal{D}(2)fil,r}(S/(\tilde{S}_I)), \ ((M_I, F), u_{IJ}) \mapsto (\tilde{f}_{I*mod}^{FDR}(M_I, F), f^k(u_{IJ})).$$

In the algebraic case, we have the followings:

Proposition 27. *Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \mathrm{QPVar}(\mathbb{C})$ quasi-projective. Then there exist factorizations $f_1 : X \xrightarrow{l_1} Y' \times Y \xrightarrow{p_Y} Y$ and $f_2 : Y \xrightarrow{l_2} Y'' \times S \xrightarrow{p_S} S$ with $Y' = \mathbb{P}^{N,o} \subset \mathbb{P}^N$,$Y'' = \mathbb{P}^{N',o} \subset \mathbb{P}^{N'}$ open subsets, l_1, l_2 closed embeddings and p_S, p_Y the projections. We have then the factorization $f_2 \circ f_1 : X \xrightarrow{(l_2 \circ I_{Y'}) \circ l_1} Y' \times Y'' \times S \xrightarrow{p_S} S$. Let $i : S \hookrightarrow \tilde{S}$ a closed embedding with $\tilde{S} = \mathbb{P}^{n,o} \subset \mathbb{P}^n$ an open subset.*

(i) Let $M \in C_{\mathcal{D}}(X/(Y' \times Y'' \times \tilde{S}))$. Then, we have $\int_{f_2 \circ f_1}^{FDR}(M) = \int_{f_2}^{FDR}(\int_{f_1}^{FDR}(M))$ in $D_{\mathcal{D}}(S/(\tilde{S}_I))$.

(ii) Let $M \in C_{\mathcal{D}(2)fil,h}(X/(Y' \times Y'' \times \tilde{S}))$. Then, we have $\int_{(f_2 \circ f_1)!}^{FDR}(M) = \int_{f_2 !}^{FDR}(\int_{f_1 !}^{FDR}(M))$ in $D_{\mathcal{D},h}(S/(\tilde{S}_I))$.

Proof. Similar to the complex case : see [10]. □

4.2.5 Tensor product in the singular case

Let $S \in \mathrm{Var}(k)$. Let $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We have, as in the complex case, the tensor product functors

$$(-) \otimes_{O_S}^{[-]} (-) : C^2_{\mathcal{D}fil}(S/(\tilde{S}_I)) \to C_{\mathcal{D}fil}(S/(\tilde{S}_I)), \ (((M_I, F), u_{IJ}), ((N_I, F), v_{IJ})) \mapsto$$
$$((M_I, F), u_{IJ}) \otimes_{O_S}^{[-]} ((N_I, F), v_{IJ}) := ((M_I, F) \otimes_{O_{\tilde{S}_I}} (N_I, F)[d_{\tilde{S}_I}], u_{IJ} \otimes v_{IJ}),$$

with, denoting for short $d_{IJ} := d_{\tilde{S}_J} - d_{\tilde{S}_I}$ and $d_I := d_{\tilde{S}_I}$,

$$\begin{aligned}u_{IJ} \otimes v_{IJ} : (M_I, F) \otimes_{O_{\tilde{S}_I}} (N_I, F)[d_I] &\xrightarrow{T(p_{IJ}^{*mod}, p_{IJ})(-)[d_I]} p_{IJ*}p_{IJ}^{*mod}((M_I, F) \otimes_{O_{\tilde{S}_I}} (N_I, F))[d_I] \\ &\xrightarrow{=} p_{IJ*}(p_{IJ}^{*mod}(M_I, F) \otimes_{O_{\tilde{S}_J}} p_{IJ}^{*mod}(N_I, F))[d_I] \\ &\xrightarrow{I(p_{IJ}^{*mod}, p_{IJ})(-,-)(u_{IJ}) \otimes I(p_{IJ}^{*mod}, p_{IJ})(-,-)(v_{IJ})[d_I]} p_{IJ*}((M_J, F) \otimes_{O_{\tilde{S}_J}} (N_J, F))[d_J + d_{IJ}].\end{aligned}$$

It induces in the derived category, for $1 \leq r \leq \infty$, the functors

$$\begin{aligned}(-) \otimes^L_{O_S} (-) : D^2_{\mathcal{D}fil,r}(S/(\tilde{S}_I)) \to D_{\mathcal{D}fil,r}(S/(\tilde{S}_I)),\ (((M_I, F), u_{IJ}), ((N_I, F), v_{IJ})) \mapsto \\ ((M_I, F), u_{IJ}) \otimes^L_{O_S} ((N_I, F), v_{IJ}) := (L_D(M_I, F) \otimes_{O_{\tilde{S}_I}} L_D(N_I, F)[d_{\tilde{S}_I}], u^q_{IJ} \otimes v^q_{IJ}).\end{aligned}$$

We have the following easy proposition :

Proposition 28. *Let $S \in \mathrm{Var}(k)$. Denote $\Delta_S : S \hookrightarrow S \times S$ the diagonal embedding. Denote $p_1 : S \times S \to S$ and $p_2 : S \times S \to S$ the projections. Let $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We have, for $(M_I, u_{IJ}), (N_I, v_{IJ}) \in C_{\mathcal{D}}(S/(\tilde{S}_I))$,*

$$(M_I, u_{IJ}) \otimes^{[-]}_{O_S} (N_I, v_{IJ}) = \Delta_S^{*mod,\Gamma}((p_{1I}^{*mod}M_I, p_{1I}^{*mod}u_{IJ}) \otimes_{O_{S\times S}} (p_{2I}^{*mod}N_I, p_{2I}^{*mod}v_{IJ}))$$

and

$$((M_I, F), u_{IJ}) \otimes^L_{O_S} ((N_I, F), v_{IJ}) = R\Delta_S^{*mod,\Gamma}((p_{1I}^{*mod}M_I, p_{1I}^{*mod}u_{IJ}) \otimes_{O_{S\times S}} (p_{2I}^{*mod}N_I, p_{2I}^{*mod}v_{IJ}))$$

Proof. Follows from proposition 16 and theorem 24. □

4.2.6 The 2 functors of D modules on the category of algebraic varieties over a field k of characteristic zero and the transformation maps

Definition 46. *Consider a commutative diagram in $\mathrm{Var}(k)$ which is cartesian :*

$$D = \begin{array}{ccc} X_T & \xrightarrow{f'} & T \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array}.$$

Assume there exist factorizations $f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$, with $Y_1, Y_2 \in \mathrm{SmVar}(k)$, l_1, l_2 closed embeddings and p_S, p_S the projections. Let $S = \cup S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We then have as in the complex case, for $(M, F) = ((M_I, F), u_{IJ}) \in C_{\mathcal{D}(2)fil}(X/(Y_1 \times \tilde{S}_I))$, the following canonical transformation map in $D_{\mathcal{D}(2)fil,r}(T/(Y_2 \times \tilde{S}_I))$,

$$\begin{aligned}&T^{\mathcal{D}mod}(f, g)(M, F) : \\ Rg^{*mod,\Gamma}\int_f^{FDR}(M, F) &:= (\Gamma_{T_I}E(\tilde{g}_I^{*mod}p_{\tilde{S}_I*}E((\Omega^\bullet_{Y_1\times\tilde{S}_I/\tilde{S}_I}, F_b) \otimes_{O_{Y_1\times\tilde{S}_I}} (M_I, F))), \tilde{g}_J^{*mod}f^k(u_{IJ})) \\ &\xrightarrow{(T^O_\omega(p_{\tilde{S}_I}, \tilde{g}_I)(M_I, F))} \\ &(\Gamma_{T_I}E(p_{Y_2\times\tilde{S}_I*}E((\Omega^\bullet_{Y_1\times Y_2\times\tilde{S}_I/Y_2\times\tilde{S}_I}, F_b) \otimes_{O_{Y_1\times Y_2\times\tilde{S}_I}} p_{Y_1\times\tilde{S}_I}^{*mod}(M_I, F))), f'^k(p_{Y_1\times\tilde{S}_J}^{*mod}(u_{IJ}))) \\ &\xrightarrow{(T^O_\omega(\gamma, \otimes)(p_{Y_1\times\tilde{S}_I}^{*mod}(M_I, F)))^{-1}} \\ &(p_{Y_2\times\tilde{S}_I*}E((\Omega^\bullet_{Y_1\times Y_2\times\tilde{S}_I/Y_2\times\tilde{S}_I}, F_b) \otimes_{O_{Y_1\times Y_2\times\tilde{S}_I}} \Gamma_{Y_1\times T_I}E(p_{Y_1\times\tilde{S}_I}^{*mod}((M_I, F)))), f'^k(\tilde{g}_J''^{*mod}(u^q_{IJ}))) \\ &=: \int_{f'}^{FDR} Rg'^{*mod,\Gamma}(M, F).\end{aligned}$$

Proposition 29. *Consider a commutative diagram in* $\mathrm{Var}(k)$

$$D = (f,g) = \begin{tikzcd} X_T \arrow[r, "f'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd} .$$

which is cartesian. Assume there exist factorizations $f : X \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $g : T \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$, *with* $Y_1, Y_2 \in \mathrm{SmVar}(k)$, l_1, l_2 *closed embeddings and* p_S, p_S *the projections. Let* $S = \cup S_i$ *an open cover such that there exist closed embeddings* $i_i : S_i \hookrightarrow \tilde{S}_i$ *closed embedding with* $\tilde{S}_i \in \mathrm{SmVar}(k)$*. For* $(M,F) = ((M_I,F),u_{IJ}) \in C_{\mathcal{D}(2)fil,c}(X/(Y \times \tilde{S}_I))$,

$$T^{\mathcal{D}mod}(f,g) : Rg^{*mod,\Gamma} \int_f^{FDR} (M,F) \to \int_{f'}^{FDR} Rg'^{*mod,\Gamma}(M,F)$$

is an isomorphism in $D_{\mathcal{D}(2)fil,r}(T/(Y_2 \times \tilde{S}_I))$.

Proof. Similar to the complex case. $\square$

Theorem 29. *Let* $f : X \to S$ *a morphism with* $X, S \in \mathrm{Var}(k)$*. Assume there exists a factorization* $f : X \xrightarrow{l} Y \times S \xrightarrow{p} S$ *with* $Y \in \mathrm{SmVar}(k)$*,* l *a closed embedding and* p *the projection. Let* $S = \cup S_i$ *an open cover such that there exist closed embeddings* $i_i : S_i \hookrightarrow \tilde{S}_i$ *closed embedding with* $\tilde{S}_i \in \mathrm{SmVar}(k)$*. Then,*

(i) For $(M,F) \in C_{\mathcal{D}(2)fil,rh}(S/(\tilde{S}_I)^{op})$*, we have* $L\mathbb{D}_S(M,F) \in D_{\mathcal{D}(2)fil,rh}(S/(\tilde{S}_I))$*.*

(ii) For $M \in C_{\mathcal{D},rh}(S/(\tilde{S}_I))$*,* $Rf^{*mod,\Gamma}(M) \in D_{\mathcal{D},rh}(X/(Y \times \tilde{S}_I))$ *and* $Lf^{\hat{*}mod,\Gamma}M \in D_{\mathcal{D},rh}(X/(Y \times \tilde{S}_I))$*.*

(iii) For $M \in C_{\mathcal{D},rh}(X/(Y \times \tilde{S}_I))$*,* $\int_f M \in D_{\mathcal{D},rh}(S/(\tilde{S}_I))$ *and* $\int_{f!} M := L\mathbb{D}_S \int_f L\mathbb{D}_X \in D_{\mathcal{D},rh}(S/(\tilde{S}_I))$*.*

(iv) If f *is proper, for* $(M,F) \in C_{\mathcal{D}(2)fil,rh}(X/(Y \times \tilde{S}_I)))$*, we have* $\int_f (M,F) \in D_{\mathcal{D}(2)fil,rh}(S/(\tilde{S}_I))$*.*

(v) For $(M,F),(N,F) \in C_{\mathcal{D}(2)fil,rh}(S/(\tilde{S}_I))$*,* $(M,F) \otimes^L_{O_S} (N,F) \in D_{\mathcal{D}(2)fil,rh}(S/(\tilde{S}_I))$

Proof. Follows from theorem 28. $\square$

4.3 The category of complexes of quasi-coherent sheaves on an algebraic variety whose cohomology sheaves has a structure of D-modules

4.3.1 Definition on a smooth algebraic variety and the functorialities

Definition 47. *Let* $S \in \mathrm{SmVar}(k)$*. Let* $Z \subset S$ *a closed subset. Denote by* $j : S \backslash Z \hookrightarrow S$ *the open complementary embedding.*

(i) We denote by $C_{O_S,\mathcal{D},Z}(S) \subset C_{O_S,\mathcal{D}}(S)$ *the full subcategory consisting of* $M \in C_{O_S,\mathcal{D}}(S)$ *such that such that* $j^* H^n M = 0$ *for all* $n \in \mathbb{Z}$*.*

(ii) We denote by $C_{O_S fil,\mathcal{D},Z}(S) \subset C_{O_S fil,\mathcal{D}}(S)$ *the full subcategory consisting of* $(M,F) \in C_{O_S fil,\mathcal{D}}(S)$ *such that there exist* $r \in \mathbb{N}$ *and an* r*-filtered homotopy equivalence* $m : (M,F) \to (M',F)$ *with* $(M',F) \in C_{O_S fil,\mathcal{D}}(S)$ *such that* $j^* H^n \mathrm{Gr}^p_F(M',F) = 0$ *for all* $n, p \in \mathbb{Z}$*.*

Definition 48. *Let* $S \in \mathrm{SmVar}(k)$*. We have then (see section 2), for* $r = 1, \cdots, \infty$*, the homotopy category* $K_{O_S fil,\mathcal{D},r}(S) = \mathrm{Ho}_r(C_{O_S fil,\mathcal{D}}(S))$ *whose objects are those of* $C_{O_S fil,\mathcal{D}}(S)$ *and whose morphisms are* r*-filtered homotopy classes of morphism, and its localization* $D_{O_S fil,\mathcal{D},r}(S) = K_{O_S fil,\mathcal{D},r}(S)([E_1]^{-1})$ *with respect to filtered zariski, resp. usu local equivalence. Note that the classes of filtered* τ *local equivalence constitute a right multiplicative system.*

- Let $S \in \mathrm{SmVar}(k)$. Let $(M,F) \in C_{O_S fil,\mathcal{D}}(S)$. Then, the canonical morphism $q : L_O(M,F) \to (M,F)$ in $C_{O_S fil}(S)$ being a quasi-isomorphism of O_S modules, we get in a unique way $L_O(M,F) \in C_{O_S fil,\mathcal{D}}(S)$ such that $q : L_O(M,F) \to (M,F)$ is a morphism in $C_{O_S fil,\mathcal{D}}(S)$

- Let $f : X \to S$ be a morphism with $X,S \in \mathrm{SmVar}(k)$. Let $(M,F) \in C_{O_S fil,\mathcal{D}}(S)$. Then, $f^{*mod}H^n(M,F) := (O_X,F_b) \otimes_{f^*O_S} f^*H^n(M,F)$ is canonical a filtered D_X module (see section 4.1 or 4.2). Consider the canonical surjective map $q(f) : H^n f^{*mod}(M,F) \to f^{*mod}H^n(M,F)$. Then, $q(f)$ is an isomorphism if f is smooth. Let $h : U \to S$ be a smooth morphism with $U,S \in \mathrm{SmVar}(k)$. We get the functor

$$h^{*mod} : C_{O_S fil,\mathcal{D}}(S) \to C_{O_U fil,\mathcal{D}}(U), (M,F) \mapsto h^{*mod}(M,F),$$

- Let $S \in \mathrm{SmVar}(k)$, and let $i : Z \hookrightarrow S$ a closed embedding and denote by $j : S\backslash Z \hookrightarrow S$ the open complementary. For $M \in C_{O_S,\mathcal{D}}(S)$, the cohomology presheaves of

$$\Gamma_Z M := \mathrm{Cone}(\mathrm{ad}(j^*,j_*)(M) : M \to j_*j^*M)[-1]$$

has a canonical D_S-module structure (as j^*H^nM is a j^*D_S module, $H^nj_*j^*M = j_*j^*H^nM$ has an induced structure of D_S module), and $\gamma_Z(M) : \Gamma_Z M \to M$ is a map in $C_{O_S,\mathcal{D}}(S)$. For $Z_2 \subset Z$ a closed subset and $M \in C_{O_S,\mathcal{D}}(S)$, $T(Z_2/Z,\gamma)(M) : \Gamma_{Z_2}M \to \Gamma_Z M$ is a map in $C_{O_S,\mathcal{D}}(S)$. We get the functor

$$\Gamma_Z : C_{O_S fil,\mathcal{D}}(S) \to C_{O_S fil,\mathcal{D}}(S),$$
$$(M,F) \mapsto \Gamma_Z(M,F) := \mathrm{Cone}(\mathrm{ad}(j^*,j_*)((M,F)) : (M,F) \to j_*j^*(M,F))[-1],$$

together we the canonical map $\gamma_Z(M,F) : \Gamma_Z(M,F) \to (M,F)$

More generally, let $h : Y \to S$ a morphism with $Y,S \in \mathrm{Var}(k)$, S smooth, and let $i : X \hookrightarrow Y$ a closed embedding and denote by $j : Y\backslash X \hookrightarrow Y$ the open complementary. For $M \in C_{h^*O_S,h^*\mathcal{D}}(Y)$,

$$\Gamma_X M := \mathrm{Cone}(\mathrm{ad}(j^*,j_*)(M) : M \to j_*j^*M)[-1]$$

has a canonical h^*D_S-module structure, (as j^*H^nM is a $j^*h^*D_S$ module, $H^nj_*j^*M = j_*j^*H^nM$ has an induced structure of $j^*h^*D_S$ module), and $\gamma_X(M) : \Gamma_X M \to M$ is a map in $C_{h^*O_S,h^*\mathcal{D}}(Y)$. For $X_2 \subset X$ a closed subset and $M \in C_{h^*O_S,h^*\mathcal{D}}(Y)$, $T(Z_2/Z,\gamma)(M) : \Gamma_{X_2}M \to \Gamma_X M$ is a map in $C_{h^*O_S,h^*\mathcal{D}}(Y)$. We get the functor

$$\Gamma_X : C_{h^*O_S fil,h^*\mathcal{D}}(Y) \to C_{h^*O_S fil,h^*\mathcal{D}}(Y),$$
$$(M,F) \mapsto \Gamma_X(M,F) := \mathrm{Cone}(\mathrm{ad}(j^*,j_*)((M,F)) : (M,F) \to j_*j^*(M,F))[-1],$$

together we the canonical map $\gamma_X(M,F) : \Gamma_X(M,F) \to (M,F)$

- Let $f : X \to S$ be a morphism with $X,S \in \mathrm{SmVar}(k)$. Consider the factorization $f : X \xrightarrow{l} X \times S \xrightarrow{p} S$, where l is the graph embedding and p the projection. We get from the two preceding points the functor

$$f^{*mod,\Gamma} : C_{O_S fil,\mathcal{D}}(S) \to C_{O_X fil,\mathcal{D}}(X \times S), (M,F) \mapsto f^{*mod,\Gamma}(M,F) := \Gamma_X p^{*mod}(M,F),$$

and

$$f^{*mod[-],\Gamma} : C_{O_S fil,\mathcal{D}}(S) \to C_{O_X fil,\mathcal{D}}(X \times S),$$
$$(M,F) \mapsto f^{*mod[-],\Gamma}(M,F) := \Gamma_X E(p^{*mod}(M,F))[-d_X],$$

which induces in the derived categories the functor

$$Rf^{*mod[-],\Gamma} : D_{O_S fil,\mathcal{D}}(S) \to D_{O_X fil,\mathcal{D}}(X \times S),$$
$$(M,F) \mapsto Rf^{*mod[-],\Gamma}(M,F) := \Gamma_X E(p^{*mod[-]}(M,F)).$$

4.3.2 Definition on a singular algebraic variety and the functorialities

Definition 49. *Let $S \in \mathrm{Var}(k)$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then, $C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$ is the category*

- *whose objects are $(M,F) = ((M_I,F)_{I\subset[1,\cdots l]}, u_{IJ})$, with*
  - $(M_I,F) \in C_{O_{\tilde{S}_I}fil\mathcal{D},S_I}(\tilde{S}_I)$,
  - $u_{IJ} : m^*(M_I,F) \to m^* p_{IJ*}(M_J,F)[d_{\tilde{S}_J} - d_{\tilde{S}_I}]$ *for $J \subset I$, are morphisms, $p_{IJ} : \tilde{S}_J \to \tilde{S}_I$ being the projection, satisfying for $I \subset J \subset K$, $p_{IJ*}u_{JK} \circ u_{IJ} = u_{IK}$ in $C_{O_{\tilde{S}_I}fil,\mathcal{D}}(\tilde{S}_I)$;*
- *whose morphisms $m : ((M_I,F),u_{IJ}) \to ((N_I,F),v_{IJ})$ between $(M,F) = ((M_I,F)_{I\subset[1,\cdots l]},u_{IJ})$ and $(N,F) = ((N_I,F)_{I\subset[1,\cdots l]},v_{IJ})$ are a family of morphisms of complexes,*
  $$m = (m_I : (M_I,F) \to (N_I,F))_{I\subset[1,\cdots l]}$$
  such that $v_{IJ} \circ m_I = p_{IJ*}m_J \circ u_{IJ}$ in $C_{O_{\tilde{S}_I}fil,\mathcal{D}}(\tilde{S}_I)$.

We denote by $C^{\sim}_{Ofil,\mathcal{D}}(S/(\tilde{S}_I)) \subset C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$ the full subcategory consisting of objects $((M_I,F),u_{IJ})$ such that the u_{IJ} are ∞-filtered Zariski local equivalences.

Definition 50. *Let $S \in \mathrm{Var}(k)$ and let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We have then (see [10]), for $r = 1,\cdots,\infty$, the homotopy category*

$$K_{Ofil,\mathcal{D},r}(S/(\tilde{S}_I)) := \mathrm{Ho}_r(C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I)))$$

whose objects are those of $C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$ and whose morphisms are r-filtered homotopy classes of morphism, and its localization

$$D_{fil,\mathcal{D},r}(S/(\tilde{S}_I)) := K_{Ofil,\mathcal{D},r}(S/(\tilde{S}_I))([E_1]^{-1})$$

with respect to the classes of filtered zariski local equivalence. Note that the classes of filtered τ local equivalence constitute a right multiplicative system.

Let $f : X \to S$ be a morphism, with $X, S \in \mathrm{Var}(k)$, such that there exist a factorization $f; X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection, and consider $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$, with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then, $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. We then have the filtered De Rham the inverse image functor :

$$\begin{aligned} f^{*mod[-],\Gamma} : C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I)) \to C_{Ofil,\mathcal{D}}(X/(Y\times\tilde{S}_I)),\ \ (M,F) = ((M_I,F),u_{IJ}) \mapsto \\ f^{*mod[-],\Gamma}(M,F) := (\Gamma_{X_I}E(p_{\tilde{S}_I}^{*mod[-]}(M_I,F))), \tilde{f}_J^{*mod[-]}u_{IJ}) \end{aligned}$$

with $\tilde{f}_J^{*mod[-]}u_{IJ}$ as in the complex case It induces in the derived categories, the functor

$$\begin{aligned} Rf^{*mod[-],\Gamma} : D_{Ofil,\mathcal{D},r}(S/(\tilde{S}_I) \to D_{Ofil,\mathcal{D},r}(X/(Y\times\tilde{S}_I)), \\ (M,F) = ((M_I,F),u_{IJ}) \mapsto \\ Rf^{*mod[-],\Gamma} := f^{*mod[-],\Gamma}(M,F) := (\Gamma_{X_I}E(p_{\tilde{S}_I}^{*mod[-]}(M_I,F)), \tilde{f}_J^{*mod[-]}u_{IJ}). \end{aligned}$$

4.4 The (filtered) De Rahm functor over a field k of characteristic zero and Riemann Hilbert for holonomic D-modules on smooth algebraic varieties over a subfield $k \subset \mathbb{C}$

Let $j : S^o \hookrightarrow S$ an open embedding with $S = (S, O_S) \in \mathrm{RTop}$. Denote by $Z := S\backslash S^o$ the closed complementary subset. Recall that we have, see [10], for $(M,F) \in C_{\mathcal{D}fil}(S^o)$ the canonical maps in

$C_{fil}(S)$

$$T^w(j,\otimes)(M,F) : DR(S)(j_*(M,F)) := (\Omega^{\bullet}_S, F_b)\otimes_{O_S} j_*(M,F) \xrightarrow{\mathrm{ad}(j^*,j_*)(\Omega^{\bullet}_S)\otimes I}$$
$$j_*j^*(\Omega^{\bullet}_S,F_b)\otimes_{O_S} j_*(M,F) \xrightarrow{=} j_*((\Omega^{\bullet}_{S^o},F_b)\otimes_{O_{S^o}}(M,F)) =: j_*DR(S^o)(M,F)$$

and

$$T^w(\gamma_Z,\otimes)(M,F) : DR(S)(\Gamma_Z(M,F)) := (\Omega^{\bullet}_S,F_b)\otimes_{O_S}\Gamma_Z(M,F)$$
$$\xrightarrow{(I,T^w(j,\otimes)(M,F))} \Gamma_Z((\Omega^{\bullet}_{S^o},F_b)\otimes_{O_{S^o}}(M,F)) =: \Gamma_Z DR(S^o)(M,F).$$

Let k a field of characteristic zero.

Proposition 30. *Let $Y,S\in\mathrm{SmVar}(k)$. Let $p : Y\times S\to S$ the projection. For $(M,F)\in C_{\mathcal{D}fil}(Y\times S)$,*

$$DR(Y\times S/S)(M,F) := (\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}(M,F)\in C_{p^*O_Sfil}(Y\times S)$$

is a naturally a complex of filtered p^*D_S modules, that is

$$DR(Y\times S/S)(M,F) := (\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}(M,F)\in C_{p^*\mathcal{D}fil}(Y\times S),$$

where the p^*D_S module structure on $\Omega^p_{Y\times S/S}\otimes_{O_{Y\times S}}M^n$ is given by for $(Y\times S)^o\subset Y\times S$ an open subset,

$$(\gamma\in\Gamma((Y\times S)^o,T_{Y\times S}),\hat{\omega}\otimes m\in\Gamma((Y\times S)^o,\Omega^p_{Y\times S/S}\otimes_{O_{Y\times S}}M^n))\mapsto\gamma.(\hat{\omega}\otimes m) := (\hat{\omega}\otimes(\gamma.m).$$

Moreover, if $\phi : (M_1,F)\to(M_2,F)$ a morphism with $(M_1,F),(M_2,F)\in C_{\mathcal{D}fil}(Y\times S)$,

$$DR(Y\times S/S)(\phi) := (I\otimes\phi) : (\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}(M_1,F)\to(\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}(M_2,F)$$

is a morphism in $C_{p^*\mathcal{D}fil}(Y\times S)$.

Proof. Follows imediately by definition : see [10]. □

Proposition 31. *Consider a commutative diagram in $\mathrm{SmVar}(k)$:*

$$D = \begin{tikzcd} Y\times S \arrow[r, "p"] & S \\ Y'\times T \arrow[u, "g''=(g''_0\times g)"] \arrow[r, "p'"] & T \arrow[u, "g"'] \end{tikzcd}$$

with p and p' the projections. For $(M,F)\in C_{\mathcal{D}fil}(Y\times S)$ the map in $C_{g''*p^*O_Sfil}(Y'\times T)$

$$\Omega_{(Y'\times T/Y\times S)/(T/S)}(M,F) : g''^*((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}(M,F))\to(\Omega^{\bullet}_{Y'\times T/T},F_b)\otimes_{O_{Y'\times T}}g''^{*mod}(M,F)$$

given in [10] section 4.1 is a map in $C_{g''*p^*\mathcal{D}fil}(Y'\times T)$. Hence, for $(M,F)\in C_{\mathcal{D}fil}(Y\times S)$, the map in $C_{O_Tfil}(T)$ (with L_D instead of L_O)

$$T^O_{\omega}(D)(M) : g^{*mod}L_D(p_*E((\Omega^{\bullet}_{Y\times S/S},F_b)\otimes_{O_{Y\times S}}(M,F)))\to p'_*E((\Omega^{\bullet}_{Y'\times T/T},F_b)\otimes_{O_{Y'\times T}}g''^{*mod}(M,F)),$$

is a map in $C_{\mathcal{D}fil}(T)$.

Proof. Follows imediately by definition. □

Proposition 32. *Let $S\in\mathrm{SmVar}(k)$.*

- *We have the filtered resolutions of K_S by the following complex of locally free right D_S modules: $\omega(S) : \omega(K_S) := (\Omega^{\bullet}_S,F_b)[d_S]\otimes_{O_S}(D_S,F_b)\to(K_S,F_b)$ and $\omega(S) : \omega(K_S,F^{ord}) := (\Omega^{\bullet}_S,F_b)[d_S]\otimes_{O_S}(D_S,F^{ord})\to(K_S,F^{ord})$*

- *Dually, we have the filtered resolution of O_S by the following complex of locally free (left) D_S modules: $\omega^{\vee}(S):\omega(O_S):=(\wedge^{\bullet}T_S,F_b)[d_S]\otimes_{O_S}(D_S,F_b)\to(O_S,F_b)$ and $\omega^{\vee}(S):\omega(O_S,F^{ord}):=(\wedge^{\bullet}T_S,F_b)[d_S]\otimes_{O_S}(D_S,F^{ord})\to(O_S,F^{ord})$.*

Let $S_1,S_2\in\mathrm{SmVar}(k)$. Consider the projection $p=p_1:S_1\times S_2\to S_1$.

- *We have the filtered resolution of $D_{S_1\times S_2\to S_1}$ by the following complexes of (left) ($p^*D_{S_1}$ and right $D_{S_1\times S_2}$) modules :*

$$\omega(S_1\times S_2/S_1):(\Omega^{\bullet}_{S_1\times S_2/S_1}[d_{S_2}],F_b)\otimes_{O_{S_1\times S_2}}(D_{S_1\times S_2},F^{ord})\to(D_{S_1\times S_2\leftarrow S_1},F^{ord}).$$

- *Dually, we have the filtered resolution of $D_{S_1\times S_2\to S_1}$ by the following complexes of (left) ($p^*D_{S_1},D_{S_1\times S_2}$) modules :*

$$\omega^{\vee}(S_1\times S_2/S_1):(\wedge^{\bullet}T_{S_1\times S_2/S_1}[d_{S_2}],F_b)\otimes_{O_{S_1\times S_2}}(D_{S_1\times S_2},F^{ord})\to(D_{S_1\times S_2\to S_1},F^{ord}),$$

Proof. Similar to the complex case: see [18]. □

Definition 51. *(i) Let $i:Z\hookrightarrow S$ be a closed embedding, with $Z,S\in\mathrm{SmVar}(k)$. Then, for $(M,F)\in C_{\mathcal{D}fil}(Z)$, we set*

$$i_{*mod}(M,F):=i^0_{*mod}(M,F):=i_*((M,F)\otimes_{D_Z}(D_{Z\leftarrow S},F^{ord}))\in C_{\mathcal{D}fil}(S)$$

(ii) Let $S_1,S_2\in\mathrm{SmVar}(k)$ and $p:S_1\times S_2\to S_1$ be the projection. Then, for $(M,F)\in C_{\mathcal{D}fil}(S_1\times S_2)$, we set

- $p^0_{*mod}(M,F):=p_*(DR(S_1\times S_2/S_1)(M,F)):=p_*((\Omega^{\bullet}_{S_1\times S_2/S_1},F_b)\otimes_{O_{S_1\times S_2}}(M,F))[d_{S_2}]\in C_{\mathcal{D}fil}(S_1)$,
- $p_{*mod}(M,F):=p_*E(DR(S_1\times S_2/S_1)(M,F)):=p_*E((\Omega^{\bullet}_{S_1\times S_2/S_1},F_b)\otimes_{O_{S_1\times S_2}}(M,F))[d_{S_2}]\in C_{\mathcal{D}fil}(S_1)$.

(iii) Let $f:X\to S$ be a morphism, with $X,S\in\mathrm{SmVar}(k)$. Consider the factorization $f:X\xrightarrow{i}X\times S\xrightarrow{p_S}S$, where i is the graph embedding and $p_S:X\times S\to S$ is the projection. Then, for $(M,F)\in C_{\mathcal{D}fil}(X)$ we set

- $f^{FDR}_{*mod}(M,F):=p_{S*mod}i_{*mod}(M,F)\in C_{\mathcal{D}fil}(S)$,
- $\int_f^{FDR}(M,F):=f^{FDR}_{*mod}(M,F):=p_{S*mod}i_{*mod}(M,F)\in D_{\mathcal{D}fil}(S)$.

By proposition 33 below, we have $\int_f^{FDR}M=\int_f M\in D_{\mathcal{D}}(X)$.

(iii) Let $f:X\to S$ be a morphism, with $X,S\in\mathrm{SmVar}(k)$. Consider the factorization $f:X\xrightarrow{i}X\times S\xrightarrow{p_S}S$, where i is the graph embedding and $p_S:X\times S\to S$ is the projection. Then, for $(M,F)\in C_{\mathcal{D}fil}(X)$ we set

- $f^{FDR}_{!mod}(M,F):=\mathbb{D}^K_S L_D f^{FDR}_{*mod}\mathbb{D}^K_S L_D(M,F):=\mathbb{D}^K_S L_D p_{S*mod}i_{*mod}\mathbb{D}^K_{X\times S}L_D(M,F)\in C_{\mathcal{D}fil}(S)$,
- $\int_{f!}^{FDR}(M,F):=f^{FDR}_{!mod}(M,F):=\mathbb{D}^K_S L_D p_{S*mod}i_{*mod}\mathbb{D}^K_{X\times S}L_D(M,F)\in D_{\mathcal{D}fil}(S)$.

Proposition 33. *(i) Let $i:Z\hookrightarrow S$ a closed embedding with $S,Z\in\mathrm{SmVar}(k)$. Then for $(M,F)\in C_{\mathcal{D}fil}(Z)$, we have*

$$\int_i(M,F):=Ri_*((M,F)\otimes^L_{D_Z}(D_{Z\leftarrow S},F^{ord})=i_*((M,F)\otimes_{D_Z}(D_{Z\leftarrow S},F^{ord}))=i_{*mod}(M,F).$$

(ii) Let $S_1, S_2 \in \mathrm{SmVar}(k)$ and $p : S_{12} := S_1 \times S_2 \to S_1$ be the projection. Then, for $(M,F) \in C_{\mathcal{D}fil}(S_1 \times S_2)$ we have

$$\begin{aligned}\int_p (M,F) : &= Rp_*((M,F) \otimes^L_{D_{S_1\times S_2}} (D_{S_1\times S_2 \leftarrow S_1}, F^{ord})) \\ &= p_*E((\Omega^\bullet_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (D_{S_1\times S_2}, F^{ord}) \otimes_{D_{S_1\times S_2}} (M,F))[d_{S_2}] \\ &= p_*E((\Omega^\bullet_{S_1\times S_2/S_1}, F_b) \otimes_{O_{S_1\times S_2}} (M,F))[d_{S_2}] =: p_{*mod}(M,F).\end{aligned}$$

where the second equality follows from Griffitz transversality (the canonical isomorphism map respect by definition the filtration).

(iii) Let $f : X \to S$ be a morphism with $X, S \in \mathrm{SmVar}(k)$. Then for $M \in C_{\mathcal{D}}(X)$, we have $\int_f^{FDR} M = \int_f M$.

Proof. (i):Follows from the fact that $D_{Z\leftarrow S}$ is a locally free D_Z module and that i_* is an exact functor. (ii): Since $\Omega^\bullet_{S_{12}/S_1}[d_{S_2}], F_b) \otimes_{O_{S_{12}}} D_{S_{12}}$ is a complex of locally free $D_{S_1\times S_2}$ modules, we have in $D_{fil}(S_1 \times S_2)$, using proposition 32,

$$(D_{S_1\times S_2\leftarrow S_1}, F^{ord}) \otimes^L_{D_{S_1\times S_2}} (M,F) = (\Omega^\bullet_{S_{12}/S_1}[d_{S_2}], F_b) \otimes_{O_{S_{12}}} (D_{S_{12}}, F^{ord}) \otimes_{D_{S_{12}}} (M,F).$$

(iii): Follows from (i) and (ii) by proposition 15(ii). □

Let k a field of characteristic zero. Let $S \in \mathrm{SmVar}(k)$ connected. We use to shift the De Rham functor in order to have compatibility with perverse sheaves in the complex or p-adic ananlytic case by setting for $(M,F) \in C_{\mathcal{D}fil}(S)$, $DR(S)^{[-]}(M,F) := DR(S)(M,F)[-d_S] \in C_{fil}(S)$.

- Let $f : X \to S$ a morphism with $S, X \in \mathrm{SmVar}(k)$. Recall that we have for $(M,F) \in C_{\mathcal{D}fil}(X)$ the canonical map in $D_{fil}(S)$

$$\begin{aligned}T_*(f, DR)(M,F) : DR(S)(\int_f (M,F)) &:= (\Omega^\bullet_S, F_b) \otimes_{O_S} Rf_*((D_{X\leftarrow S}, F^{ord}) \otimes^L_{D_X} (M,F)) \\ &\xrightarrow{\iota(S)\otimes I} Rf_*((D_{X\leftarrow S}, F^{ord}) \otimes^L_{D_X} (M,F)) \otimes^L_{D_S} (K_S, F^{ord}) \\ &\xrightarrow{T(f,\otimes)((D_{X\leftarrow S}, F^{ord})\otimes^L_{D_X}(M,F),(K_S,F^{ord}))}\end{aligned}$$

$$\begin{aligned}Rf_*(f^*(K_S, F^{ord}) \otimes^L_{f^*D_S} (D_{X\leftarrow S}, F^{ord}) \otimes^L_{D_X} (M,F)) &\xrightarrow{=} Rf_*((K_X, F^{ord}) \otimes^L_{D_X} (M,F)) \\ &\xrightarrow{\iota(X)\otimes I} Rf_*((\Omega^\bullet_X, F_b) \otimes_{O_X} (M,F)) =: Rf_*DR(X)(M,F)\end{aligned}$$

  which is an isomorphism by the projection formula for quasi-coherent sheaves for a morphism of ringed topos and proposition 32. In particular, for $S \in \mathrm{SmVar}(k)$ and $(M,F) \in C_{\mathcal{D}fil,c}(S^o)$,

$$\begin{aligned}T^w(j,\otimes)(M,F) : DR(S)(j_*(M,F)) &:= (\Omega^\bullet_S, F_b) \otimes_{O_S} j_*(M,F) \\ &\to j_*((\Omega^\bullet_{S^o}, F_b) \otimes_{O_{S^o}} (M,F)) =: j_*DR(S^o)(M,F)\end{aligned}$$

  is a filtered quasi-isomorphism.

- Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(k)$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p the projection. Let $S = \cup_i S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We have for

$((M_I,F),u_{IJ})\in C_{\mathcal{D}fil}(X/(Y\times\tilde{S}_I))$ the canonical map in $D_{fil}(S/(\tilde{S}_I))$

$$T_*(f,DR)((M_I,F),u_{IJ}):$$

$$DR(S)(\int_f((M_I,F),u_{IJ})):=((\Omega^{\bullet}_{\tilde{S}_I},F_b)\otimes_{O_{\tilde{S}_I}}p_{\tilde{S}_I*}E((\Omega^{\bullet}_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F)),DR(fu_{IJ}))$$
$$\xrightarrow{(k\circ T(p_{\tilde{S}_I},\otimes)(-,-))}(p_{\tilde{S}_I*}E((p^*_{\tilde{S}_I}\Omega^{\bullet}_{\tilde{S}_I},F_b)\otimes_{p^*_{\tilde{S}_I}O_{\tilde{S}_I}}(\Omega^{\bullet}_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F)),DR(fu_{IJ}))$$
$$\xrightarrow{w(Y\times\tilde{S}_I)}p_{\tilde{S}_I*}E((\Omega^{\bullet}_{Y\times\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F)),DR(fu_{IJ}))=:Rf_*DR(X)((M_I,F),u_{IJ})$$

$w(Y\times\tilde{S}_I)$ being the wedge product, which is an isomorphism by by the projection formula for quasi-coherent sheaves for a morphism of ringed topos.

Let k a field of characteristic zero. Let $S\in\mathrm{SmVar}(k)$. Recall we denote for $M,N\in C_{\mathcal{D}}(S)$,

$$m(M,N):\mathcal{H}om_{D_S}(M,D_S)\otimes_{D_S}N\to\mathcal{H}om_{D_S}(M,N),\ (\phi\otimes n)\mapsto(m\mapsto\phi(m)n)$$

the multiplication map in $C(S)$. It induces in the derived category for $M,N\in C_{\mathcal{D}}(S)$ the map in $D(S)$

$$\begin{aligned}m(L_DM,N):R\mathcal{H}om_{D_S}(M,D_S)\otimes^L_{D_S}N&=\mathcal{H}om_{D_S}(L_DM,D_S)\otimes_{D_S}N\\&\to\mathcal{H}om_{D_S}(L_DM,N)=R\mathcal{H}om_{D_S}(M,N).\end{aligned}$$

We use for the proof of theorem 31 in the next subsection the following :

Proposition 34. *Let k a field of characteristic zero. Let $S\in\mathrm{SmVar}(k)$.*

(i) Let $M,N\in C_{\mathcal{D}}(S)$. If $N\in C_{\mathcal{D},c}(S)$,

$$\begin{aligned}m(L_DM,N):R\mathcal{H}om_{D_S}(M,D_S)\otimes^L_{D_S}N&=\mathcal{H}om_{D_S}(L_DM,D_S)\otimes_{D_S}N\\&\to\mathcal{H}om_{D_S}(L_DM,N)=R\mathcal{H}om_{D_S}(M,N).\end{aligned}$$

is an isomorphism in $D(S)$.

(ii) Let $M,N\in C_{\mathcal{D}}(S)$. If $N\in C_{\mathcal{D},c}(S)$, we have using (i) a canonical isomorphism in $D(S)$

$$D(M,N):R\mathcal{H}om_{D_S}(M,N)\xrightarrow{m(L_DM,N)^{-1}}R\mathcal{H}om_{D_S}(M,D_S)\otimes^L_{D_S}N$$
$$\xrightarrow{=:}K_S\otimes^L_{O_S}L\mathbb{D}_SM[-d_S]\otimes^L_{D_S}N\xrightarrow{=}K_S\otimes^L_{D_S}\mathbb{D}_SM\otimes^L_{O_S}N[-d_S]=:DR(S)^{[-]}(L\mathbb{D}_SM\otimes^L_{O_S}N)$$

Proof. (i):Standard.
(ii):Follows from (i). □

4.4.1 Some complements on the (filtered)De Rahm functor for D modules on smooth algebraic varieties over a subfield $k\subset\mathbb{C}$

In this section, for $S\in\mathrm{AnSm}(\mathbb{C})$, we write for short $DR(S):=DR(S)^{[-]}$, where we recall for S connected $DR(S)^{[-]}:=DR(S)[-d_S]$.

For $S\in\mathrm{AnSp}(\mathbb{C})$, we denote by

$$\alpha(S):\mathbb{C}_S\hookrightarrow DR(S)(O_S)$$

the inclusion map in $C(S)$. In particular, we get for $S\in\mathrm{Var}(\mathbb{C})$, the inclusion map

$$\alpha(S):\mathbb{C}_S\hookrightarrow DR(S)(O_{S^{an}})$$

in $C(S^{an})$.

For $S\in\mathrm{AnSp}(\mathbb{C})$, we denote by

$$\iota(S)::DR(S)(O_S)\to K_S,\ h\otimes w\mapsto hw$$

the canonical map in $C(S)$. In particular, we get for $S\in\mathrm{Var}(\mathbb{C})$, the canonical map

$$\iota(S):DR(S)(O_{S^{an}})\to K_S$$

in $C(S^{an})$.

- Let $f:X\to S$ a morphism with $S,X\in\mathrm{AnSm}(\mathbb{C})$. Recall that we have for $(M,F)\in C_{\mathcal{D}fil}(X)$ the canonical map in $D_{fil}(S)$

$$\begin{aligned}T_*(f,DR)(M,F):DR(S)(\int_f(M,F))&:=(\Omega^\bullet_S,F_b)\otimes_{O_S}Rf_*((D_{X\leftarrow S},F^{ord})\otimes^L_{D_X}(M,F))\\&\xrightarrow{\iota(S)\otimes I}Rf_*((D_{X\leftarrow S},F^{ord})\otimes^L_{D_X}(M,F))\otimes^L_{D_S}(K_S,F^{ord})\\&\xrightarrow{T(f,\otimes)((D_{X\leftarrow S},F^{ord})\otimes^L_{D_X}(M,F),(K_S,F^{ord}))}\end{aligned}$$

$$\begin{aligned}Rf_*(f^*(K_S,F^{ord})\otimes^L_{f^*D_S}(D_{X\leftarrow S},F^{ord})\otimes^L_{D_X}(M,F))&\xrightarrow{=}Rf_*(K_X\otimes_{O_X}((D_X,F^{ord})\otimes^L_{D_X}(M,F))\\&\xrightarrow{\iota(X)\otimes I}Rf_*((\Omega^\bullet_X,F_b)\otimes_{O_X}(M,F))=:Rf_*DR(X)(M,F)\end{aligned}$$

  which is an isomorphism by the projection formula for quasi-coherent sheaves for a morphism of ringed topos. In particular, for $S\in\mathrm{AnSm}(\mathbb{C})$ and $(M,F)\in C_{\mathcal{D}fil,c}(S^o)$,

$$\begin{aligned}T^w(j,\otimes)(M,F):DR(S)(j_*(M,F))&:=(\Omega^\bullet_S,F_b)\otimes_{O_S}j_*(M,F)\\&\to j_*((\Omega^\bullet_{S^o},F_b)\otimes_{O_{S^o}}(M,F))=:j_*DR(S^o)(M,F)\end{aligned}$$

  is a filtered quasi-isomorphism.

- Let $f:X\to S$ a morphism with $S,X\in\mathrm{AnSp}(\mathbb{C})$. Assume there exist a factorization $f:X\xrightarrow{l}Y\times S\xrightarrow{p}S$ with $Y\in\mathrm{AnSm}(\mathbb{C})$, l a closed embedding and p the projection. Let $S=\cup_iS_i$ an open cover such that there exists closed embedding $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{AnSm}(\mathbb{C})$. We have for $((M_I,F),u_{IJ})\in C_{\mathcal{D}fil}(X/(Y\times\tilde{S}_I))$ the canonical map in $D_{fil}(S/(\tilde{S}_I))$

$$T_*(f,DR)((M_I,F),u_{IJ}):$$

$$\begin{aligned}DR(S)(\int_f((M_I,F),u_{IJ}))&:=((\Omega^\bullet_{\tilde{S}_I},F_b)\otimes_{O_{\tilde{S}_I}}p_{\tilde{S}_I*}E((\Omega^\bullet_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F)),DR(fu_{IJ}))\\&\xrightarrow{(k\circ T(p_{\tilde{S}_I},\otimes)(-,-))}(p_{\tilde{S}_I*}E((p^*_{\tilde{S}_I}\Omega^\bullet_{\tilde{S}_I},F_b)\otimes_{p^*_{\tilde{S}_I}O_{\tilde{S}_I}}(\Omega^\bullet_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F)),DR(fu_{IJ}))\\&\xrightarrow{w(Y\times\tilde{S}_I)}p_{\tilde{S}_I*}E((\Omega^\bullet_{Y\times\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F)),DR(fu_{IJ}))=:Rf_*DR(X)((M_I,F),u_{IJ})\end{aligned}$$

  $w(Y\times\tilde{S}_I)$ being the wedge product, which is an isomorphism by the projection formula for quasi-coherent sheaves for a morphism of ringed topos.

- Let $S\in\mathrm{AnSm}(\mathbb{C})$. Recall that we have for $(M,F)\in C_{\mathcal{D}fil}(S)$ the canonical map in $D_{fil}(S)$

$$T(D,DR)(M,F):DR(S)(L\mathbb{D}_S(M,F))\to\mathbb{D}^v_S(DR(S)(M,F))$$

- Let $S\in\mathrm{AnSp}(\mathbb{C})$. Let $S=\cup_iS_i$ an open cover such that there exists closed embedding $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{AnSm}(\mathbb{C})$. Recall that we have for $((M_I,F),u_{IJ})\in C_{\mathcal{D}fil}(S/(\tilde{S}_I))$ the canonical map in $D_{fil}(S/(\tilde{S}_I))$

$$T(D,DR)(((M_I,F),u_{IJ})):DR(S)(L\mathbb{D}_S((M_I,F),u_{IJ}))\to\mathbb{D}^v_S(DR(S)((M_I,F),u_{IJ}))$$

- Let $f : X \to S$ a morphism with $S, X \in \mathrm{AnSm}(\mathbb{C})$. Recall that we have for $(N,F) \in C_{\mathcal{D}fil}(S)$ the canonical map in $D_{fil}(X)$

$$\begin{aligned}T^*(f,DR)(M,F) : f^*DR(S)(N,F) &\xrightarrow{\iota_S} f^*\mathcal{H}om_{D_S}(O_S, L_D(N,F)) \\ &\xrightarrow{T(f,hom)(-,-)} \mathcal{H}om_{f^*D_S}(f^*O_S, f^*L_D(N,F)) \\ \xrightarrow{Tr(-,-)} \mathcal{H}om_{D_X}(f^{*mod}O_S, f^{*mod}L_D(N,F)) &= \mathcal{H}om_{D_X}(O_X, f^{*mod}L_D(N,F)) \\ &\xrightarrow{(\iota(X)\otimes I)^{-1}} \Omega^{\bullet}_X \otimes_{O_X} f^{*mod}L_D(N,F) =: DR(X)(Lf^{*mod}(N,F))\end{aligned}$$

Let $k \subset \mathbb{C}$ a subfield.

- Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(k)$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p the projection. Let $S = \cup_i S_i$ an affine open cover so that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We have for $((M_I,F),u_{IJ}) \in C_{\mathcal{D}fil}(S/(\tilde{S}_I))$ the canonical map in $D_{fil}(X^{an}_{\mathbb{C}}/(Y \times \tilde{S}_I)^{an}_{\mathbb{C}})$

$$\begin{aligned}f^!DR(S)((M_I,F),u_{IJ}) &= \Gamma_X \mathbb{D}^v p^! \mathbb{D}DR(S)(((M_I,F),u_{IJ})^{an}) \\ &\xrightarrow{\mathbb{D}T^*(p,DR)(-)} \Gamma_X DR(Y \times S)((p^{*mod}_{\tilde{S}_I}((M_I,F),u_{IJ})^{an}) \\ \xrightarrow{T^w(j,\otimes)(-)^{-1}} DR(Y \times S)(\Gamma_X(p^{*mod}_{\tilde{S}_I}(M_I,F),u_{IJ})^{an}) &\xrightarrow{DR(Y\times S)(T(\gamma,an)(-):=(I,T(j,an)(-)))} \\ DR(Y \times S)((\Gamma_X p^{*mod}_{\tilde{S}_I}((M_I,F),u_{IJ}))^{an}) &=: DR(X)(f^{*mod,\Gamma}((M_I,F),u_{IJ})).\end{aligned}$$

- Let $S \in \mathrm{Var}(k)$. Denote by $\Delta_S : S \hookrightarrow S \times S$ the graph embedding and $p_1 : S \times S \to S$ and $p_2 : S \times S \to S$ the projections. Let $S = \cup_i S_i$ an affine open cover so that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We have for $((M_I,F),u_{IJ}),((N_I,F),u_{IJ}) \in C_{\mathcal{D}fil}(S/(\tilde{S}_I))$ the canonical map in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$

$$\begin{aligned}T(\otimes,DR)((M,F)(M_I,F),u_{IJ}),((N_I,F),u_{IJ})) &: DR(S)((M,F)^{an}) \otimes_{\mathbb{C}_S} DR(S)(((N_I,F),u_{IJ})^{an}) \\ &\xrightarrow{=} R\Delta^!_S((p_1^*DR(S)((M_I,F),u_{IJ})^{an}) \otimes_{\mathbb{C}_S} p_2^*DR(S)(((N_I,F),u_{IJ})^{an})) \\ &\xrightarrow{=} R\Delta^!_S((p_1^!DR(S)((M_I,F),u_{IJ})^{an}) \otimes_{\mathbb{C}_S} p_2^!DR(S)(((N_I,F),u_{IJ})^{an}))[2d_S] \\ &\xrightarrow{T^!(p_1,DR)(-)\otimes T^!(p_2,DR)(-)} \Delta^!_S DR(S \times S)(((p_1^{*mod}(M_I,F),u_{IJ}) \otimes_{O_{S\times S}} p_2^{*mod}((N_I,F),u_{IJ}))^{an}) \\ &\xrightarrow{T^!(\Delta_S,DR)(-)} DR(S)((L\Delta^{*mod}_S((p_1^{*mod}(M_I,F),u_{IJ}) \otimes_{O_{S\times S}} p_2^{*mod}((N_I,F),u_{IJ})))^{an}) \\ &\xrightarrow{=} DR(S)((((M_I,F),u_{IJ}) \otimes^L_{O_S} ((N_I,F),u_{IJ}))^{an}).\end{aligned}$$

Theorem 30. *Let $k \subset \mathbb{C}$ a subfield.*

(i) Let $j : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{SmVar}(k)$. Then, for $M \in C_{\mathcal{D},rh}(S^o)$, the map in $C(S^{an}_{\mathbb{C}})$

$$DR(S)(T(j,an)(M)) : DR(S)((j_*M)^{an}) \to DR(S)(j_*(M^{an}))$$

is a quasi-isomorphism.

(ii) Let $S \in \mathrm{Var}(k)$. Let $S = \cup_i S_i$ an affine open cover so that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. For $((M_I,F),u_{IJ}) \in C_{\mathcal{D}fil}(S/(\tilde{S}_I))$ the canonical map in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I\mathbb{C}}))$

$$T(D,DR)(((M_I,F),u_{IJ})) : DR(S)(L\mathbb{D}_S((M_I,F),u_{IJ})) \to \mathbb{D}^v_S(DR(S)((M_I,F),u_{IJ}))$$

is an isomorphism.

(iii) *Let $f : X \to S$ a morphism with $S, X \in \mathrm{SmVar}(k)$. Then, for $(M, F) \in C_{\mathcal{D}fil,rh}(X)$, the map in $D_{fil}(S^{an}_{\mathbb{C}})$*

$$T_*(f, DR)(M, F) : DR(S)((\int_f (M, F))^{an}) \xrightarrow{DR(S)(T(f,an)(-))} DR(S)(\int_f (M, F)^{an})$$
$$\xrightarrow{T_*(f,DR)((M,F)^{an})} Rf_* DR(X)((M, F)^{an})$$

is an isomorphism if f is proper and $o_{fil}T(f, DR)(M, F) =: T(f, DR)(M)$ is an isomorphism in $D(S^{an}_{\mathbb{C}})$.

(iii)' *Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(k)$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p the projection. Let $S = \cup_i S_i$ an affine open cover so that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. For $((M_I, F), u_{IJ}) \in C_{\mathcal{D}fil}(X/(Y \times \tilde{S}_I))$ the canonical map in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$*

$$T_*(f, DR)((M_I, F), u_{IJ}) : DR(S)(\int_f ((M_I, F), u_{IJ})^{an}) \xrightarrow{(T(p_{\tilde{S}_I}, an))}$$
$$DR(S)(\int_f (((M_I, F), u_{IJ})^{an})) \xrightarrow{T_*(f,DR)(((M_I,F),u_{IJ})^{an})} Rf_* DR(X)(((M_I, F), u_{IJ})^{an})$$

is an isomorphism if f is proper, and $o_{fil}T(f, DR)((M_I, F), u_{IJ}) =: T(f, DR)((M_I, u_{IJ}))$ is an isomorphism in $D(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$.

(iv) *Let $S \in \mathrm{SmVar}(k)$. Then, for $M, N \in C_{\mathcal{D},rh}(S)$, the map in $D(S^{an}_{\mathbb{C}})$*

$$T(\otimes, DR)(M, N) : DR(S)(M^{an}) \otimes_{\mathbb{C}_{S^{an}_{\mathbb{C}}}} DR(S)(N^{an})$$
$$\xrightarrow{T(\otimes,DR)(M^{an},N^{an})} DR(S)(M^{an} \otimes_{O_S} N^{an}) = DR(S)((M \otimes_{O_S} N)^{an})$$

is an isomorphism.

Proof. (i): Follows from the complex case : see [18].
(ii): Follows from the complex case which is standard.
(iii) and (iii)': Follows from GAGA for proper morphisms of complex algebraic varieties.
(iv): Follows from (i). □

Theorem 31. *Let $k \subset \mathbb{C}$ a subfield. Let $S \in \mathrm{SmVar}(k)$.*

(i) *For $M, N \in D_{\mathcal{D},rh}(S)$, we have using proposition 34 and theorem 30(iv) and (ii) the following canonical isomorphism in $D(\mathbb{C})$*

$$DR(M, N) : R\,\mathrm{Hom}_{D_S}(M, N) \otimes_k \mathbb{C} = Ra_{S*} R\mathcal{H}om_{D_S}(M, N) \otimes_k \mathbb{C}$$
$$\xrightarrow{Ra_{S*}D(M,N)} \int_{a_S} (L\mathbb{D}_S M \otimes^L_{O_S} N) \otimes_k \mathbb{C} = DR(pt) \int_{a_S} (L\mathbb{D}_S M \otimes^L_{O_S} N) \otimes_k \mathbb{C}$$
$$\xrightarrow{(T(\otimes,DR)(-,-)\circ T_*(a_S,DR)(-)^{-1}} Ra_{S*}(\mathbb{D}^v_S DR(S)(M^{an}) \otimes DR(S)(N^{an}))$$
$$\xrightarrow{Ra_{S*}m(DR(S)(M^{an}),DR(S)(N^{an}))} R\,\mathrm{Hom}_{\mathbb{C}_{S^{an}_{\mathbb{C}}}}(DR(S)(M^{an}), DR(S)(N^{an}))$$

is an isomorphism.

(i)' *For $M, N \in D_{\mathcal{D},rh}(S)$, the canonical map in $D(\mathbb{C})$*

$$DR(S)^{L_D M,N} : R\,\mathrm{Hom}_{D_S}(M, N) \otimes_k \mathbb{C} \xrightarrow{\sim} R\,\mathrm{Hom}_{\mathbb{C}_{S^{an}_{\mathbb{C}}}}(DR(S)(M^{an}), DR(S)(N^{an}))$$

is equal to $DR(M, N)$, hence is an isomorphism.

(ii1) *We have $DR(S)(D_{\mathcal{D},rh}(S)) \subset D_{\mathbb{C}_S,c}(S^{an}_{\mathbb{C}})$, that is the image of the class of a complex of D_S module with regular holonomic cohomology sheaves is a complex of presheaves on $S^{an}_{\mathbb{C}}$ whose cohomology sheaves are constructible for a Zariski stratification of S defined over k.*

(ii2) *For $M \in \mathrm{PSh}_{\mathcal{D},rh}(S)$, $DR(S)(M) \in P(S^{an}_{\mathbb{C}})$ that is is a perverse sheaf for a Zariski stratification of S defined over k.*

Proof. (i): Follows from proposition 34 and theorem 30(iv) and (ii).
(i)': Follows from the following commutative diagram in $D(\mathbb{C})$

$$\begin{tikzcd} R\,\mathrm{Hom}_{D_S}(M,N)\otimes_k\mathbb{C} = Ra_{S*}R\mathcal{H}om_{D_S}(M,N)\otimes_k\mathbb{C} \arrow[r, "DR(S)^{L_D M,N}"] \arrow[d, "Ra_{S*}D(M,N)"] & R\,\mathrm{Hom}_{\mathbb{C}_{S^{an}_{\mathbb{C}}}}(DR(S)(M^{an}), DR(S)(N^{an})) \\ \int_{a_S}(L\mathbb{D}_S M\otimes^L_{O_S} N)\otimes_k\mathbb{C} = DR(pt)\int_{a_S}(L\mathbb{D}_S M\otimes^L_{O_S} N)\otimes_k\mathbb{C} \arrow[r, "(T(\otimes{,}DR)(-{,}-)\circ T(a_S{,}DR)(-))^{-1}"] & Ra_{S*}(\mathbb{D}^v_S DR(S)(M^{an})\otimes DR(S)(N^{an})) \arrow[u, "m(-{,}-)"'] \end{tikzcd}$$

(ii):Similar to the proof of the complex case in [18]: follows by definition from the locally free case by theorem 25. □

4.4.2 On the De Rahm functor for D modules on smooth algebraic varieties over a p-adic field $K \subset \mathbb{C}_p$

For $S \in \mathrm{AnSp}(K)$, we denote by

$$\alpha(S) : \mathbb{B}_{dr,S} \hookrightarrow DR(S)(O\mathbb{B}_{dr,S})$$

the inclusion map in $C_{\mathbb{B}_{dr,S}}(S^{pet})$. In particular, we get for $S \in \mathrm{Var}(K)$, the inclusion map

$$\alpha(S) : \mathbb{B}_{dr,S} \hookrightarrow DR(S)(O\mathbb{B}_{dr,S})$$

in $C_{\mathbb{B}_{dr,S}}(S^{an,pet})$.

For $S \in \mathrm{AnSp}(K)$, we denote by

$$\iota(S) :: DR(S)(O\mathbb{B}_{dr,S}) \to \mathbb{B}_{dr,S}\otimes_{O_S} K_S,\ h\otimes k\otimes w \mapsto k\otimes(hw)$$

the canonical map in $C_{\mathbb{B}_{dr,S}}(S^{pet})$. In particular, we get for $S \in \mathrm{Var}(K)$, the canonical map

$$\iota(S) : DR(S)(O\mathbb{B}_{dr,S}) \to \mathbb{B}_{dr,S}\otimes_{O_S} K_S$$

in $C_{\mathbb{B}_{dr,S}}(S^{an,pet})$.

We have the following theorem

Theorem 32. *Let $K \subset \mathbb{C}_p$ a p adic field.*

(i) *Let $S \in \mathrm{AnSm}(K)$. The inclusion map in $C_{\mathbb{B}_{dr,S}}(S^{an,pet})$*

$$\alpha(S) : \mathbb{B}_{dr,S} \hookrightarrow DR(S)(O\mathbb{B}_{dr,S}).$$

is a quasi-isomorphism.

(i)' *Let $S \in \mathrm{AnSm}(K)$.the canonical map in $C_{\mathbb{B}_{dr,S}}(S^{an,pet})$*

$$\iota(S) : DR(S)(O\mathbb{B}_{dr,S}) \to \mathbb{B}_{dr,S}\otimes K_S$$

is a quasi-isomorphism.

(ii) *Let $S \in \mathrm{Var}(K)$. Let $D = \cup D_i \subset S$ a normal crossing divisor and denote $j : S^o := S \backslash D \hookrightarrow S$ the open embedding. The inclusion map in $C_{\mathbb{B}_{dr,S}}(S^{an,pet})$*

$$\alpha(S) : \mathbb{B}_{dr,S}(\log D) \hookrightarrow \Omega^{\bullet}_S(\log D) \otimes_{O_S} O\mathbb{B}_{dr,S}(\log D) = F^0 DR(S)(j_{*Hdg}(O_{S^o}, F_b) \otimes_{O_S} (O\mathbb{B}_{dr,S}, F)).$$

where $j_{*Hdg}(O_S, F_b) = (j_* O_{S^o}, V_D)$ with V_D the V-filtration (see section 5) that is the filtration by order of the pole in this case, is a quasi-isomorphism.

(iii) *Let $S \in \mathrm{AnSm}(K)$. The functor*

$$\mathrm{Vect}_{\mathcal{D}}(S) \to D_{\mathbb{B}_{dr,S}}(S^{an,pet}),\ (M,F) \mapsto F^0 DR(S)((M,F)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S}, F))$$

is fully faithful whose inverse on the image is given by

$$N \in \mathrm{Shv}_{\mathbb{B}_{dr,S}}(S^{an,pet}) \mapsto Re_*((N,F) \otimes_{\mathbb{B}_{dr,S}} (O\mathbb{B}_{dr,S}, F)).$$

where $e : S^{pet} \to S^{et}$ is the morphism of site given by the inclusion functor.

Proof. (i):See [27].
(i)': Follows from (i) by duality.
(i):See [21].
(ii):See [21]. □

Definition 52. *Let $K \subset \mathbb{C}_p$ a p adic field.*

(i) *Let $f : X \to S$ a morphism with $S, X \in \mathrm{AnSm}(K)$. We have for $(M,F) \in C_{\mathcal{D}fil}(X)$ the canonical map in $D_{\mathbb{B}_{dr}fil}(S)$*

$$\begin{aligned}
T^{B_{dr}}(f, DR)(M,F) : DR(S)(\int_f (M,F) \otimes_{O_S} (OB_{dr,S}, F)) \\
\xrightarrow{:=} (\Omega^{\bullet}_S, F_b) \otimes_{O_S} (OB_{dr,S}, F) \otimes_{O_S} Rf_*((D_{X \leftarrow S}, F^{ord}) \otimes^L_{D_X} (M,F)) \\
\xrightarrow{\iota(S) \otimes I} Rf_*((D_{X \leftarrow S}, F^{ord}) \otimes^L_{D_X} (M,F)) \otimes_{D_S} \mathbb{B}_{dr,S} \otimes_{O_S} K_S \\
\xrightarrow{T(f,\otimes)((D_{X \leftarrow S}, F^{ord}) \otimes^L_{D_X} (M,F), \mathbb{B}_{dr,S} \otimes_{O_S} K_S)} \\
Rf_*(f^* K_S \otimes^L_{f^* D_S} (D_{X \leftarrow S}, F^{ord}) \otimes^L_{D_X} (M,F) \otimes_{f^* O_S} f^* \mathbb{B}_{dr,S}) \\
\xrightarrow{=} Rf_*(K_X \otimes_{O_X} \mathbb{B}_{dr,X} \otimes^L_{D_X} (M,F)) \xrightarrow{\iota(X)} Rf_*((\Omega^{\bullet}_X, F_b) \otimes_{O_X} (M,F) \otimes_{O_X} (O\mathbb{B}_{dr,X}, F)) \\
=: Rf_* DR(X)((M,F) \otimes_{O_X} (O\mathbb{B}_{dr,X}, F))
\end{aligned}$$

which is an isomorphism by the projection formula for quasi-coherent modules for morphisms of ringed topos (see [10]).

(i)' *Let $f : X \to S$ a morphism with $S, X \in \mathrm{AnSp}(K)$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p} S$ with $Y \in \mathrm{AnSm}(K)$, l a closed embedding and p the projection. Let $S = \cup_i S_i$ an affine open cover so that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{AnSm}(K)$. We have for*

$((M_I,F),u_{IJ}) \in C_{\mathcal{D}fil}(X/(Y\times\tilde{S}_I))$ *the canonical map in* $D_{\mathbb{B}_{dr}fil}(S/(\tilde{S}_I))$

$$T^{B_{dr}}(f,DR)((M_I,F),u_{IJ}) : DR(S)(\int_f((M_I,F),u_{IJ})\otimes_{O_S}(OB_{dr,(\tilde{S}_I)},F)) :=$$
$$((\Omega^{\bullet}_{\tilde{S}_I},F_b)\otimes_{O_{\tilde{S}_I}}(OB_{dr,(\tilde{S}_I)},F)\otimes_{O_{\tilde{S}_I}}p_{\tilde{S}_I*}E((\Omega^{\bullet}_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F)),DR(fu_{IJ}))$$
$$\xrightarrow{(k\circ T(p_{\tilde{S}_I},\otimes)(-,-))}$$
$$(p_{\tilde{S}_I*}E((p^*_{\tilde{S}_I}\Omega^{\bullet}_{\tilde{S}_I},F_b)\otimes_{p^*_{\tilde{S}_I}O_{\tilde{S}_I}}p^*_{\tilde{S}_I}(OB_{dr,(\tilde{S}_I)},F)\otimes_{p^*_{\tilde{S}_I}O_{\tilde{S}_I}}(\Omega^{\bullet}_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F)),DR(fu_{IJ}))$$
$$\xrightarrow{=}$$
$$(p_{\tilde{S}_I*}E((p^*_{\tilde{S}_I}\Omega^{\bullet}_{\tilde{S}_I},F_b)\otimes_{p^*_{\tilde{S}_I}O_{\tilde{S}_I}}(OB_{dr,(Y\times\tilde{S}_I)},F)\otimes_{O_{Y\times\tilde{S}_I}}(\Omega^{\bullet}_{Y\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F)),DR(fu_{IJ}))$$
$$\xrightarrow{w(Y\times\tilde{S}_I)}(p_{\tilde{S}_I*}E((\Omega^{\bullet}_{Y\times\tilde{S}_I},F_b)\otimes_{O_{Y\times\tilde{S}_I}}(OB_{dr,(Y\times\tilde{S}_I)},F)\otimes_{O_{Y\times\tilde{S}_I}}(M_I,F)),DR(u_{IJ}))$$
$$=: Rf_*DR(X)(((M_I,F),u_{IJ})\otimes_{O_X}(OB_{dr,(Y\times\tilde{S}_I)},F))$$

where $w(Y\times\tilde{S}_I)$ *is the wedge product, which is an isomorphism by the projection formula for quasi-coherent modules for morphisms of ringed topos (see [10]).*

(ii) Let $S\in\mathrm{AnSm}(K)$. *We have for* $(M,F)\in C_{\mathcal{D}fil}(S)$ *the canonical map in* $D_{fil}(S)$

$$T(D,DR)(M,F) : DR(S)(\mathbb{D}_S(M,F)) \to \mathbb{D}^v_S(DR(S)(M,F))$$

(ii)' Let $S\in\mathrm{AnSp}(K)$. *Let* $S=\cup_i S_i$ *an affine open cover so that there exists closed embedding* $i_i : S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{AnSm}(K)$. *We have for* $((M_I,F),u_{IJ})\in C_{\mathcal{D}fil}(S/(\tilde{S}_I))$ *the canonical map in* $D_{fil}(S/(\tilde{S}_I))$

$$T(D,DR)((M_I,F),u_{IJ}) : DR(S)(L\mathbb{D}_S((M_I,F),u_{IJ})) \to \mathbb{D}^v_S(DR(S)((M_I,F),u_{IJ}))$$

- *Let* $f : X\to S$ *a morphism with* $S,X\in\mathrm{Var}(K)$. *Assume there exist a factorization* $f : X\xrightarrow{l} Y\times S\xrightarrow{p} S$ *with* $Y\in\mathrm{SmVar}(K)$, l *a closed embedding and* p *the projection. Let* $S=\cup_i S_i$ *an affine open cover so that there exists closed embedding* $i_i : S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{SmVar}(K)$. *We have for* $((M_I,F),u_{IJ})\in C_{\mathcal{D}fil}(S/(\tilde{S}_I))$ *the canonical map in* $D_{fil}(X^{an}/(Y\times\tilde{S}_I)^{an})$

$$f^!DR(S)((M_I,F),u_{IJ}) = \Gamma_X\mathbb{D}^v p^!\mathbb{D}DR(S)(((M_I,F),u_{IJ})^{an})$$
$$\xrightarrow{\mathbb{D}T^*(p,DR)(-)}\Gamma_X DR(Y\times S)((p^{*mod}_{\tilde{S}_I}((M_I,F),u_{IJ})^{an})$$
$$\xrightarrow{T^w(j,\otimes)(-)^{-1}} DR(Y\times S)(\Gamma_X(p^{*mod}_{\tilde{S}_I}(M_I,F),u_{IJ})^{an})\xrightarrow{DR(Y\times S)(T(\gamma,an)(-):=(I,T(j,an)(-)))}$$
$$DR(Y\times S)((\Gamma_X p^{*mod}_{\tilde{S}_I}((M_I,F),u_{IJ}))^{an}) =: DR(X)(f^{*mod,\Gamma}((M_I,F),u_{IJ})).$$

- *Let* $S\in\mathrm{Var}(K)$. *Denote by* $\Delta_S : S\hookrightarrow S\times S$ *the graph embedding and* $p_1 : S\times S\to S$ *and* $p_2 : S\times S\to S$ *the projections. Let* $S=\cup_i S_i$ *an affine open cover so that there exists closed embedding* $i_i : S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{SmVar}(K)$. *We have for* $((M_I,F),u_{IJ}),((N_I,F),u_{IJ})\in C_{\mathcal{D}fil}(S/(\tilde{S}_I))$ *the canonical map in* $D_{fil}(S^{an}/(\tilde{S}^{an}_I))$

$$T(\otimes,DR)((M_I,F),u_{IJ}),((N_I,F),u_{IJ})) : DR(S)(((M_I,F),u_{IJ})^{an})\otimes_{\mathbb{C}_S}DR(S)(((N_I,F),u_{IJ})^{an})$$
$$\xrightarrow{=} R\Delta^!_S((p_1^*DR(S)((M_I,F),u_{IJ})^{an})\otimes_{\mathbb{C}_S}p_2^*DR(S)(((N_I,F),u_{IJ})^{an}))$$
$$\xrightarrow{=} R\Delta^!_S((p_1^!DR(S)((M_I,F),u_{IJ})^{an})\otimes_{\mathbb{C}_S}p_2^!DR(S)(((N_I,F),u_{IJ})^{an}))[2d_S]$$
$$\xrightarrow{T^!(p_1,DR)(-)\otimes T^!(p_2,DR)(-)}\Delta^!_S DR(S\times S)(((p_1^{*mod}(M_I,F),u_{IJ})\otimes_{O_{S\times S}}p_2^{*mod}((N_I,F),u_{IJ}))^{an})$$
$$\xrightarrow{T^!(\Delta_S,DR)(-)} DR(S)((L\Delta^{*mod}_S((p_1^{*mod}(M_I,F),u_{IJ})\otimes_{O_{S\times S}}p_2^{*mod}((N_I,F),u_{IJ})))^{an})$$
$$\xrightarrow{=} DR(S)((((M_I,F),u_{IJ})\otimes^L_{O_S}((N_I,F),u_{IJ}))^{an}).$$

Theorem 33. *(i) Let $j : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{SmVar}(K)$. Then, for $M \in C_{\mathcal{D},rh}(S^o)$, the map in $C(S^{an})$*

$$DR(S)(T(j,an)(M)) : DR(S)((j_*M)^{an}) \to DR(S)(j_*(M^{an}))$$

is a quasi-isomorphism.

(ii) Let $S \in \mathrm{Var}(K)$. Then, for $((M_I,F),u_{IJ}) \in C_{\mathcal{D}fil,rh}(S)$, the map in $D_{\mathbb{B}_{dr}fil}(S^{an})$

$$T(D,DR)(M,F) : DR(S)(L\mathbb{D}_S((M_I,F),u_{IJ})^{an}) \to \mathbb{D}^v_S(DR(S)(((M_I,F),u_{IJ})^{an}))$$

is an isomorphism.

(iii) Let $f : X \to S$ a morphism with $S,X \in \mathrm{SmVar}(K)$. Then, for $(M,F) \in C_{\mathcal{D}fil,rh}(X)$, the map in $D_{\mathbb{B}_{dr}fil}(S^{an})$

$$T^{B_{dr}}(f,DR)(M,F) : DR(S)((\int_f (M,F))^{an} \otimes_{O_S} (OB_{dr,S},F)) \xrightarrow{DR(S)(T(\int_f,an)(M,F))}$$
$$DR(S)(\int_f ((M,F)^{an}) \otimes_{O_S} (OB_{dr,S},F)) \xrightarrow{T^{B_{dr}}(f,DR)((M,F)^{an})} Rf_*DR(X)((M,F)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X},F))$$

is an isomorphism if f is proper.

(iii)' Let $f : X \to S$ a morphism with $S,X \in \mathrm{Var}(K)$. Then, for $((M_I,F),u_{IJ}) \in C_{\mathcal{D}fil}(X/(Y \times \tilde{S}_I))$ the map in $D_{\mathbb{B}_{dr}fil}(S^{an}/(\tilde{S}_I^{an}))$

$$T^{B_{dr}}(f,DR)((M_I,F),u_{IJ}) : DR(S)(\int_f ((M_I,F),u_{IJ})^{an} \otimes_{O_S} (OB_{dr,(\tilde{S}_I)},F)) \xrightarrow{DR(S)((T(p_{\tilde{S}_I},an)(-)))}$$
$$DR(S)(\int_f (((M_I,F),u_{IJ})^{an}) \otimes_{O_S} (OB_{dr,(\tilde{S}_I)},F)) \xrightarrow{T^{B_{dr}}(f,DR)(((M_I,F),u_{IJ})^{an})}$$
$$Rf_*DR(X)(((M_I,F),u_{IJ})^{an} \otimes_{O_S} (OB_{dr,(Y\times\tilde{S}_I)},F))$$

is an isomorphism if f is proper.

Proof. Similar to the proof of theorem 30. For (iii) and (iii)', we use GAGA for proper morphism of algebraic varieties over a p-adic field. □

5 The De Rham modules over a field k of characteristic 0 : the Kashiwara Malgrange V-filtration and the Hodge filtration in the geometric case

5.1 The Kashiwara Malgrange V filtration for geometric D modules on smooth algebraic varieties over a field of characteristic zero and the nearby and vanishing cycle functors.

Let $k \subset \mathbb{C}$ a subfield. For $S \in \mathrm{Var}(k)$, consider $\pi := \pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}} := S \otimes_k \mathbb{C} \to S$ the projection so that we have the injective map in $\mathrm{PSh}_{O_S}(S_{\mathbb{C}})$

$$n_{k/\mathbb{C}}(O_S) : \pi^*O_S \hookrightarrow O_{S_{\mathbb{C}}} := \pi^{*mod}O_S := \pi^*O_S \otimes_k \mathbb{C}, \;\; h \mapsto n_{k/\mathbb{C}}(O_S)(h) := h \otimes 1.$$

For $M \in \mathrm{PSh}_{O_S}(S)$, we have (see section 2) the canonical morphism in $\mathrm{PSh}_{O_S}(S_{\mathbb{C}})$

$$n_{O_S/O_{S_{\mathbb{C}}}}(M) : \pi^*M \to \pi^{*mod}M := \pi^*M \otimes_{\pi^*O_S} O_{S_{\mathbb{C}}}, \;\; m \mapsto n_{O_S/O_{S_{\mathbb{C}}}}(M)(m) := m \otimes 1$$

For $S \in \mathrm{SmVar}(k)$ and $M \in \mathrm{PSh}_{\mathcal{D}}(S)$

$$n_{O_S/O_{S_{\mathbb{C}}}}(M) : \pi^*M \to \pi^{*mod}M = \pi^*M \otimes_{\pi^*O_S} O_{S_{\mathbb{C}}}, \;\; m \mapsto n_{O_S/O_{S_{\mathbb{C}}}}(M)(m) := m \otimes 1$$

is a morphism in $\mathrm{PSh}_{D_S}(S_{\mathbb{C}})$, that is is a morphism of π^*D_S modules.

Definition 53. *Let k a field of characteristic zero. Let $S \in \mathrm{SmVar}(k)$.*

(i) Let $D = V(s) \subset S$ be a smooth (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D. Let $M \in \mathrm{PSh}_{\mathcal{D}}(S)$. A V_D-filtration (M, V_D) for M (see [10]) is called a Kashiwara-Malgrange V_D-filtration for M if

- *$V_{D,k}M$ are coherent $V_{D,0}D_S$ modules for all $k \in \mathbb{Z}$, that is (M, V_D) is a good filtration, in particular $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$ is coherent*
- *$sV_{D,k}M = V_{D,k-1}M$ for $k << 0$,*
- *all eigenvalues of $s\partial_s : \mathrm{Gr}_{V_D,k} := V_{D,k}M/V_{D,k-1}M \to \mathrm{Gr}_{V_D,k} M := V_{D,k}M/V_{D,k-1}M$ have real part between $k-1$ and k.*

Almost by definition, a Kashiwara-Malgrange V_D-filtration for M if it exists is unique, so that we denote it by $(M, V_D) \in \mathrm{PSh}_{O_S fil}(S)$ and (M, V_D) is strict. In particular if $m : (M_1, F) \to (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in \mathrm{PSh}_{\mathcal{D}(2)fil}(S)$ such that M_1 and M_2 admit the Kashiwara-Malgrange filtration for $D \subset S$, we have $m(V_{D,q}F^pM_1) \subset V_{D,q}F^pM_2$, that is we get $m : (M_1, F, V_D) \to (M_2, F, V_D)$ a filtered morphism, and if $0 \to M' \to M \to M'' \to 0$ is an exact sequence, $0 \to (M', V_D) \to (M, V_D) \to (M'', V_D) \to 0$ is an exact sequence (strictness).

(ii) More generally, let $Z = V(s_1, \ldots, s_r) = D_1 \cap \cdots \cap D_r \subset S$ be a smooth Zariski closed subset, where $s_i \in \Gamma(S, L_i)$ is a section of the line bundle $L = L_{D_i}$ associated to D_i. Let $M \in \mathrm{PSh}_{\mathcal{D}}(S)$. A V_Z-filtration (M, V_Z) for M (see [10]) is called a Kashiwara-Malgrange V_Z-filtration for M if

- *$V_{Z,k}M$ are coherent $V_{Z,0}O_S$ modules for all $k \in \mathbb{Z}$,*
- *$\sum_{i=1}^r s_iV_{Z,k}M = V_{Z,k-1}M$ for $k << 0$,*
- *all eigenvalues of $\sum_{i=1}^r s_i\partial_{s_i} : \mathrm{Gr}_{V_Z,k} M := V_{Z,k}M/V_{Z,k-1}M \to \mathrm{Gr}^V_{Z,k} M := V_{Z,k}M/V_{Z,k-1}M$ have real part between $k-1$ and k.*

Almost by definition, a Kashiwara-Malgrange V_Z-filtration for M if it exists is unique (see [26]), so that we denote it by $(M, V_Z) \in \mathrm{PSh}_{O_S fil}(S)$ and (M, V_Z) is strict. In particular if $m : (M_1, F) \to (M_2, F)$ a morphism with $(M_1, F), (M_2, F) \in \mathrm{PSh}_{\mathcal{D}(2)fil}(S)$ such that M_1 and M_2 admit the Kashiwara-Malgrange filtration for $D \subset S$, we have $m(V_{Z,q}F^pM_1) \subset V_{Z,q}F^pM_2$, that is we get $m : (M_1, F, V_Z) \to (M_2, F, V_Z)$ a filtered morphism, and if $0 \to M' \to M \to M'' \to 0$ is an exact sequence, $0 \to (M', V_Z) \to (M, V_Z) \to (M'', V_Z) \to 0$ is an exact sequence (strictness).

Definition 54. *Let k a field of characteristic zero. Let $S \in \mathrm{SmVar}(k)$. Let $D = V(s) \subset S$ a (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D. We then have the graph embedding $i : S \hookrightarrow L$, $i(x) := (x, s(x))$. Let $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$. If the Kashiwara-Malgrange V_S-filtration exist on $M' := i_{*mod}M \in \mathrm{PSh}_{\mathcal{D},c}(L)$ (see definition 53) and the eigeinvalues of $s\partial_s$ are rational numbers, we refine the V_S-filtration to all rational numbers as follows : for $\alpha = k-1+r/q \in \mathbb{Q}$, $k, q, r \in \mathbb{Z}$, $q \leq 0$, $0 \leq r \leq q-1$, we set*

$$V_{S,\alpha}M' := q_{V,k}^{-1}(\oplus_{k-1<\beta\leq\alpha} \mathrm{Gr}^{V_S}_{k,\beta} M \subset V_{S,k}M'$$

with $\mathrm{Gr}^{V_S}_{k,\beta} M' := \ker(\partial_s s - \beta I) \subset \mathrm{Gr}^{V_S}_k M'$ and $q_{V,k} : V_{S,k}M' \to \mathrm{Gr}^{V_S}_k M'$ is the projection. We set similarly

$$V_{S,<\alpha}M := q_{V,k}^{-1}(\oplus_{k-1<\beta<\alpha} \mathrm{Gr}^{V_S}_{k,\beta} M') \subset V_{S,k}M'$$

The Hodge filtration induced on $\mathrm{Gr}^V_\alpha M$ *is*

$$F^p \mathrm{Gr}^{V_S}_\alpha M' := (F^p M \cap V_{S,\alpha}M')/(F^p M \cap V_{S,<\alpha}M')$$

We call it the rational Kashiwara-Malgrange V_S*-filtration of* $M' = i_{*mod}M \in \mathrm{PSh}_{\mathcal{D},c}(L)$*. Using theorem 23, we set for* $\alpha \in \mathbb{Q}$,

$$V_{D,\alpha}M := i^\sharp V_{S,\alpha} i_{*mod}M \subset M = i^\sharp i_{*mod}M$$

and (M, V_D) *is the rational Kashiwara-Malgrange* V_D*-filtration of* $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$.

Definition 55. *Let* k *a field of characteristic zero. Let* $S \in \mathrm{SmVar}(k)$*. Let* $D = V(s) \subset S$ *a divisor with* $s \in \Gamma(S, L)$ *and* L *a line bundle (*S *being smooth,* D *is Cartier). For* $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$ *such that the rational Kashiwara-Malgrange* V_D *filtration exists, that is the* V_S*-filtration on* $i_{*mod}M$ *exists, we define, using definition*

- *the nearby cycle functor*

$$\psi_D M := i^\sharp(\oplus_{-1\leq\alpha<0} \mathrm{Gr}_{V_S,\alpha}\, i_{*mod}M) = \oplus_{-1\leq\alpha<0} \mathrm{Gr}_{V_D,\alpha} M \in \mathrm{PSh}_{\mathcal{D},D}(S),$$

- *the vanishing cycle functor*

$$\phi_D M := i^\sharp(\oplus_{-1<\alpha\leq 0} \mathrm{Gr}_{V_S,\alpha}\, i_{*mod}M) = \oplus_{-1<\alpha\leq 0} \mathrm{Gr}_{V_D,\alpha} M \in \mathrm{PSh}_{\mathcal{D},D}(S),$$

- *the canonical maps in* $\mathrm{PSh}_{\mathcal{D},D}(S)$

$$can(M) := (\partial_s, I) : \psi_D M \to \phi_D M\ ,\ var(M) := (I, s) : \phi_D M \to \psi_D M.$$

By the complex case (see [20]), after considering a subfield $k_0 \subset k$ *and an embedding* $\sigma : k_0 \hookrightarrow \mathbb{C}$ *we have*

- *for* $M \in \mathrm{PSh}_{\mathcal{D},h}(S)$, $\psi_D M, \phi_D M \in \mathrm{PSh}_{\mathcal{D},h}(S)$.
- *for* $M \in \mathrm{PSh}_{\mathcal{D},rh}(S)$, $\psi_D M, \phi_D M \in \mathrm{PSh}_{\mathcal{D},rh}(S)$.

Theorem 34. *Let* $k \subset \mathbb{C}$ *a subfield. Let* $S \in \mathrm{SmVar}(k)$*. Let* $D = V(s) \subset S$ *a divisor with* $s \in \Gamma(S, L)$ *and* $p : L \to S$ *a line bundle (*S *being smooth,* D *is Cartier), so that we have the closed embedding* $i : S \hookrightarrow L$, $i(x) = (x, s(x))$ *and* $D = i^{-1}(s_0)$, s_0 *being the zero section. Denote by* $l : L^o := L\backslash S \hookrightarrow L$ *the open embedding which induces the open embedding* $l := l \times_L S : S^o := S\backslash D \hookrightarrow S$*. Denote by* $\pi : \tilde{L}^o \to L^o$ *the universal covering which induces the universal covering* $\pi := \pi \times_{L^o} S^o : \tilde{S}^o \to S^o$*. For* $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$ *with quasi-unipotent monodromy such that the rational Kashiwara-Malgrange* V_D*-filtration on* M *exists, we have the canonical isomorphism in* $D_{c,k}(S^{an}_{\mathbb{C}})$

$$T(\psi_D, DR)(M) := B(M) \circ A(M)^{-1} : DR(S)(\psi_D M^{an}) \xrightarrow{\sim} \psi_D DR(S)(M^{an})[-1]$$

with, for $S = \cup_{i=1}^s S_i$ *an open affine cover such that* $D \cap S_i = V(f_i) \subset S_i$ *is given by* $f_i \in \Gamma(S_i, O_{S_i})$, *denoting* $q : L_i := p^{-1}(S_i) \to \mathbb{A}^1_k$ *the projection and* $j_i : S_i \hookrightarrow S$ *the open embeddings,*

- *the isomorphism in* $D_{c,k}(S^{an}_{\mathbb{C}})$

$$A(M) : (\oplus_{i=1}^s \oplus_{-1\leq\alpha<0} \mathrm{Cone}(\partial_s : DR(L_i/\mathbb{A}^1_k)((V_{D\alpha}M)^{an}) \otimes_{O_S} s^{\alpha+1} O_{S^{an}_{\mathbb{C}}}[\log s]$$
$$\to DR(L_i/\mathbb{A}^1_k)((V_{D\alpha}M)^{an}) \otimes_{O_S} s^\alpha O_{S^{an}_{\mathbb{C}}}[\log s]) \xrightarrow{(j_I^*)} \cdots)[-1]$$
$$\to (\oplus_{i=1}^s DR(S_i)(\psi_D M^{an}) \xrightarrow{j_I^*} \cdots) \xrightarrow{((j_i^*),0)^{-1}} (DR(S)(\psi_D M^{an})),\ (\sum_j m_j \otimes (\log s)^j, m') \mapsto [m_0],$$

- *and the isomorphism in* $D_{c,k}(S_{\mathbb{C}}^{an})$

$$B(M) : (\oplus_{i=1}^{s} \oplus_{-1\leq\alpha<0} \mathrm{Cone}(\partial_s : V_{D\alpha}DR(L_i/\mathbb{A}_k^1)(M^{an}) \otimes_{O_S} s^{\alpha+1}O_{S_{\mathbb{C}}^{an}}[\log s]$$
$$\to V_{D\alpha}DR(L_i/\mathbb{A}_k^1)(M^{an}) \otimes_{O_S} s^{\alpha}O_{S_{\mathbb{C}}^{an}}[\log s]) \xrightarrow{(j_I^*)} \cdots)[-1]$$
$$\to (\oplus_{i=1}^{s} DR(p^*O_{\mathbb{A}_1^k})(i^*(l\circ\pi)_*(l\circ\pi)^*DR(L_i/\mathbb{A}_k^1)(M^{an})) \xrightarrow{(j_I^*)} \cdots)[-1]$$
$$\xrightarrow{=} (\oplus_{i=1}^{s} \psi_D DR(p^*O_{\mathbb{A}_1^k})(DR(L_i/\mathbb{A}_k^1)(M^{an})) \xrightarrow{(j_I^*)} \cdots)[-1] \xrightarrow{((j_i^*),0)^{-1}} \psi_D DR(S)(M^{an})[-1],$$
$$(\sum_j m_j \otimes (\log s)^j, m') \mapsto \sum_j (\log s)^j m_j,$$

so that $T(\psi_D, DR)(M) \circ DR(S)(s\partial_s) = N \circ T(\psi_D, DR)(M)$ *where*

$$N := \log T_u \in \mathrm{Hom}(\psi_D DR(S)(M^{an}), \psi_D DR(S)(M^{an})),\ T = T_u T_s,\ T_u \text{ unipotent},\ T_s \text{ nilpotent}$$

is induced by the monodromy automorphism $T : \tilde{S}^o \xrightarrow{\sim} \tilde{S}^o$ *of the universal covering* $\pi : \tilde{S}^o \to S^o := S\backslash D$. *As is the rest of this paper, we denote for short* $M^{an} := \mathrm{an}_S^{*mod} M = (\pi_{k/\mathbb{C}}(S)^{*mod}M)^{an} \in \mathrm{PSh}_{\mathcal{D},c}(S_{\mathbb{C}}^{an})$ *with* $\mathrm{an}_S : S_{\mathbb{C}}^{an} \xrightarrow{\mathrm{an}_S} S_{\mathbb{C}} \xrightarrow{\pi_{k/\mathbb{C}}(S)} S$.

Proof. See [26]. □

Definition-Proposition 5. *Let k a field of characteristic zero. Let $S \in \mathrm{SmVar}(k)$. Let $D = V(s) \subset S$ a divisor with $s \in \Gamma(S, L)$ and L a line bundle (S being smooth, D is Cartier), so that we have the closed embedding $i : S \hookrightarrow L$, $i(x) = (x, s(x))$ and $D = i^{-1}(s_0)$, s_0 being the zero section. Denote by $j : S^o := S\backslash D \hookrightarrow S$ the open embedding. For $K \in D(S)$, we set (see [4])*

$$\psi_D K := e(S)_* i^* e(S)_* R(j\circ\pi)_* \pi^* \mathcal{H}om(\mathcal{A}_{S^o}, e(S)^*K)$$

together with the monodromy morphism $T : \psi_D K \to \psi_D K$ *where*

$$\pi : \tilde{S}^o := \varinjlim_{n\in\mathbb{N}} \mathrm{Spec}(L[t]/(t^n - s)) \to S^o$$

is the universal cover and $\mathcal{A}_{S^o} := (S^o \times_{S^o\times S^o} S^o) \in \mathrm{Fun}(\Delta^\bullet, \mathrm{Var}(k)^{sm}/S^o)$ the diagram of lattices. Denote by $l : L^o := L\backslash S \hookrightarrow L$ the open embedding which induces the open embedding $l := l \times_L S : S^o := S\backslash D \hookrightarrow S$. Denote again by $\pi : \tilde{L}^o \to L^o$ the universal covering which induces the universal covering $\pi := \pi \times_{L^o} S^o : \tilde{S}^o \to S^o$. For $M \in \mathrm{PSh}_{\mathcal{D},c}(S)$ with quasi-unipotent monodromy such that the rational Kashiwara-Malgrange V_D-filtration on M exists, we have the canonical morphism in $D(S)$

$$T_{alg}(\psi_D, DR)(M) := B_{alg}(M) \circ A_{alg}(M)^{-1} : DR(S)(\psi_D M) \to \psi_D DR(S)(M)[-1]$$

with, for $S = \cup_{i=1}^{s} S_i$ an open affine cover such that $D \cap S_i = V(f_i) \subset S_i$ is given by $f_i \in \Gamma(S_i, O_{S_i})$, denoting $q : L_i := p^{-1}(S_i) \to \mathbb{A}_k^1$ the projection and $j_i : S_i \hookrightarrow S$ the open embeddings,

- *the isomorphism in* $D_c(S)$

$$A_{alg}(M) : (\oplus_{i=1}^{s} \oplus_{-1\leq\alpha<0} \mathrm{Cone}(\partial_s : DR(L_i/\mathbb{A}_k^1)(V_{D\alpha}M) \otimes_{O_S} s^{\alpha+1}O_S[\log s]$$
$$\to DR(L_i/\mathbb{A}_k^1)(V_{D\alpha}M) \otimes_{O_S} s^{\alpha}O_S[\log s]) \xrightarrow{j_I^*} \cdots)[-1]$$
$$\to (\oplus_{i=1}^{s} DR(S_i)(\psi_D M) \xrightarrow{j_I^*} \cdots) \xrightarrow{((j_i^*),0)^{-1}} (DR(S)(\psi_D M)),\ (\sum_j m_j \otimes (\log s)^j, m') \mapsto [m_0]$$

- *and morphism in $D_c(S)$*

$$\begin{aligned}B_{alg}(M) : (\oplus_{i=1}^{s}\oplus_{-1\leq\alpha<0}\mathrm{Cone}(\partial_s : V_{D\alpha}DR(L_i/\mathbb{A}^1_k)(M)\otimes_{O_S} s^{\alpha+1}O_S[\log s] \\ \to V_{D\alpha}DR(L_i/\mathbb{A}^1_k)(M)\otimes_{O_S} s^{\alpha}O_S[\log s]) \xrightarrow{(j_I^*)} \cdots)[-1] \\ \to (\oplus_{i=1}^{s}DR(p^*O_{\mathbb{A}^1_k})(i^*l_*\pi_*\pi^{*mod}\mathcal{H}om(\mathcal{A}_{S^o}, l^*DR(L_i/\mathbb{A}^1_k)(M))) \xrightarrow{(j_I^*)} \cdots)[-1] \\ \xrightarrow{=} (\oplus_{i=1}^{s}\psi_D DR(p^*O_{\mathbb{A}^1_k})(DR(L_i/\mathbb{A}^1_k)(M)) \xrightarrow{(j_I^*)} \cdots)[-1] \xrightarrow{((j_i^*),0)^{-1}} \psi_D DR(S)(M)[-1], \\ (\sum_j m_j\otimes(\log s)^j, m') \mapsto \sum_j (\log s)^j m_j,\end{aligned}$$

with $\log s\in\mathcal{H}om(\mathcal{A}_{S^o}, p^*O_{\mathbb{A}^1_k})$, so that $T_{alg}(\psi_D, DR)(M)\circ DR(S)(s\partial_s) = N\circ T_{alg}(\psi_D, DR)(M)$ where

$$N := \log T_u \in \mathrm{Hom}(\psi_D DR(S)(M), \psi_D DR(S)(M)), T = T_uT_s,\ T_u \text{ unipotent},\ T_s \text{ nilpotent}$$

is induced by the the monodromy morphism $T : \psi_D DR(S)(M)\to\psi_D DR(S)(M)$. By definition we have for $k\subset\mathbb{C}$ a subfield, the following commutaive diagram in $D(S^{an}_{\mathbb{C}})$

$$\begin{tikzcd}
(DR(S)(\psi_D M))^{an} \arrow[r, "T^w(\mathrm{an}_S,\otimes)(M)"] \arrow[d, "T_{alg}(\psi_D,DR)(M)"'] & DR(S)((\psi_D M)^{an}) \arrow[d, "T(\psi_D,DR)(M)"] \\
(\psi_D DR(S)(M))^{an} \arrow[r, "T(\psi_D,an)(M)"] & \psi_D(DR(S)(M)^{an})
\end{tikzcd}$$

where $T(\psi_D, an)(M)$ is the isomorphism given in [1].

Proof. The proof that $A_{alg}(M)$ is an isomorphism is similar to the proof that $A(M)$ is an isomorphism in theorem 34. $\square$

Definition 56. *Let k a field of caracteristic 0. Let $S\in\mathrm{SmVar}(k)$ and $D = V(s)\subset S$ a (Cartier) divisor. Denote by $j : S^o := S\backslash D\hookrightarrow S$ the open embedding. Let $M\in\mathrm{PSh}_{\mathcal{D},rh}(S)$ which admits the rational Kashiwara-Malgrange filtration. There exist a subfield $k_0\subset k$ of finite transcendence degree over $\mathbb{Q}$, $S_0\in\mathrm{SmVar}(k_0)$ and $M_0\in\mathrm{PSh}_{\mathcal{D},rh}(S_0)$ such that $S = S_0\otimes_{k_0}k$ and $M = \pi_{k_0/k}(S_0)^{*mod}M_0$. Denote again by $j : S^o := S\backslash D\hookrightarrow S$ the open embedding. Consider then an embedding $\sigma : k_0\hookrightarrow\mathbb{C}$. Denote by $\pi : \tilde{S}_0^{an(\sigma)}\to S_0^{o,an(\sigma)} := S_0^{an(\sigma)}\backslash D^{an(\sigma)}$ the universal covering and by $j : S_0\hookrightarrow S_0$ the open embedding. Denote*

$$\begin{aligned}A_\pi := \mathrm{ad}(i^*, i_*)(-)\circ\mathrm{ad}(\pi^*,\pi_*)(DR(S_0^o)(M_0^{an}))\in\mathrm{Hom}_{D(S^{an}_{0\mathbb{C}})}(j_*DR(S_0^o)(M_0^{an}), \psi_D DR(S_0^o)(M_0^{an})) \\ = \mathrm{Hom}_{D(S^{an}_{0\mathbb{C}})}(DR(S_0)(j_*M_0^{an}), DR(S_0)(\psi_D M_0^{an}))\end{aligned}$$

where the equality follows from the isomorphisms

- *$T(j, DR)(M_0) := T^w(j,\otimes)(M_0)\circ DR(S)(T(j, an)(M_0)) : DR(S_0)(j_*M_0^{an})\xrightarrow{\sim} j_*DR(S_0^o)(M_0^{an})$,*
- *$T(\psi_D, DR)(M_0) : \psi_D DR(S_0)(M_0^{an})\xrightarrow{\sim}\psi_D DR(S_0)(M_0^{an})$ of theorem 34.*

We have by definition the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{D_{\mathcal{D},rh}(S_0)}(j_*M_0,\psi_D M_0)\otimes_k\mathbb{C} \arrow[r, "\theta_3:=DR(S)^{j_*M_0,\psi_D M_0}"] \arrow[d, "DR(S)^{j_*M_0,\psi_D M_0}"] & \mathrm{Hom}_{D(S^{an}_{0\mathbb{C}})}(DR(S_0)(j_*M_0^{an}), DR(S_0)(\psi_D M_0^{an})) \\
\mathrm{Hom}_{D(S_0)}(DR(S_0)(j_*M_0), DR(S_0)(\psi_D M_0))\otimes_k\mathbb{C} \arrow[r, "\mathrm{Hom}(T^w(j,\otimes)(M_0),T_{alg}(\psi_D,DR)(M_0))"] & \mathrm{Hom}_{D(S_0)}(j_*DR(S_0^o)(M_0),\psi_D DR(S_0^o)(M_0))\otimes_k\mathbb{C} \arrow[u, "\theta_2:=\mathcal{H}om(T^w(an,\otimes)(-),T^w(an,\otimes)(-))\circ\mathrm{an}_S^{*-,-}"']
\end{tikzcd} \tag{15}$$

where $T_{alg}(\psi_D, DR)(M_0)$ is given in definition-proposition 5. By theorem 31, θ_3 is an isomorphism, hence for $m = \theta_2(m_k \otimes 1)$, $\theta_3^{-1}(m) = m_k \otimes 1$ by the diagram (15). In particular for $A_\pi = \theta_2(A_{\pi,k} \otimes 1)$ with

$$A_{\pi,k} := \mathrm{ad}(i^*, i_*)(-) \circ n_{\log s}(-) \circ \mathrm{ad}(\pi^*, \pi_*)(DR(S_0^o)(M_0)) \in \mathrm{Hom}_{D(S_0)}(j_* DR(S_0^o)(M_0), \psi_D DR(S_0^o)(M_0))$$

and $n_{\log s}(K) = K \to \mathcal{H}om(\mathcal{A}_{S^o}, K)$, we get in $\mathrm{PSh}_{\mathcal{D},rh}(S_0)$

$$\rho_{DR,D}(M_0) := (DR(S)^{j_* M_0, \psi_D M_0})^{-1}(A_\pi) = A_{\pi,k} : j_* M_0 \to \psi_D(M_0).$$

If $\sigma' : k_0 \hookrightarrow \mathbb{C}$ is an other embedding, there exists $\theta : \mathbb{C} \to \mathbb{C}$ an algebraic automorphism of $\mathbb{C}$ such that $\theta \circ \sigma = \sigma'$ and $\sigma'(\rho_{DR,D}(M_0)) = \theta(\sigma(\rho_{DR,D}(M_0)))$ by the diagram (15). This map gives in particular the map in $\mathrm{PSh}_{\mathcal{D},rh}(S)$

$$\rho_{DR,D}(M) := \rho_{DR,D}(M_0) \otimes_{O_{S_0}} O_S : j_*(M) \to \psi_D(M).$$

We now show, using the complex case, the existence of the rational Kashiwara-Malgrange filtration in the regular holonomic case.

Lemma 5. *Let $S \in \mathrm{SmVar}(k)$ a smooth affine variety with a closed embedding $l : S \hookrightarrow \mathbb{A}_k^N$. Let $D = V(f) \subset S$ a (Cartier) Divisor which is given by a $f \in \Gamma(S, O_S)$. Then $D = \tilde{D} \cap S$ with $\tilde{D} = V(\tilde{f}) \subset \mathbb{A}_k^N$, where the polynomial $\tilde{f} \in \Gamma(\mathbb{A}_k^N, O_{\mathbb{A}_k^N})$ is a lift of f ; denote by $j : S \backslash D \hookrightarrow S$ and $\tilde{j} : \mathbb{A}_k^N \backslash \tilde{D} \hookrightarrow \mathbb{A}_k^N$ the open embeddings. We then have the graph embedding $i : S \hookrightarrow S \times \mathbb{A}_k^1$, $i(x) = (x, f(x))$ and the zero section embedding $i_0 : S \hookrightarrow S \times \mathbb{A}_k^1$, $i_0(x) = (x, 0)$. Denote $(x,t) \in S \times \mathbb{A}_k^1$ the coordinates and $s = t\partial_t$.*

(i) Let $M \in \mathrm{PSh}_{\mathcal{D}}(S)$ such that the multiplication map $m_f : M \to M$, $m_f(m) = fm$, is an isomorphism. Denote by $\delta = 1/(f-t) \in O_S(*D)$. Consider for $i \in \mathbb{N}$ the polynomials $Q_i = \pi_{j=0}^{i-1}(x+j) \in \mathbb{Z}[x]$ for $i > 0$ and $Q_0 = 1$. We have then an isomorphism of $D_S < t, t^{-1}, s >$ modules

$$A_f(M) : M[s]f^s := i_*(M \otimes_{O_S} O_S[s]) \xrightarrow{\sim} i_{*mod} M = i_* M \otimes_k k[\partial_t] = i_* M \otimes_{O_S} i_{*mod} O_S(*D), \quad m s^j f^s \mapsto m \otimes (t\partial_t)^j \delta$$

whose inverse is

$$B_f(M) : i_{*mod} M \xrightarrow{\sim} M[s]f^s, \ m \otimes \partial_t^j \delta \mapsto m/f^j Q_j(-s) f^s$$

where the structure of $D_S < t, t^{-1}, s >$ module on $M[s]f^s$ is given by

- *$s.(ms^j f^s) = ms^{j+1} f^s$, $t.(ms^j f^s) = m(s+1)^j f^s$,*
- *$P.(ms^j f^s) = P(f)/f m s^{j+1} f^s + P(m)(s+1)^j f^s$, for $P \in \Gamma(S, D_S)$.*

(ii) Consider by (i) the $D_S < t, t^{-1}, s >$ submodule

$$N_f := D_S[s]f^s \subset O_S(*D)[s]f^s \xrightarrow{A_f(O_S(*D))\sim} i_{*mod} O_S(*D).$$

Then $N_f / t N_f$ an holonomic D_S module.

(iii) The endomorphism

$$s : N_f / t N_f \to N_f / t N_f$$

has a minimal polynomial which is equal to the Berstein-Sato Polynomial $b_f \in \mathbb{Q}[x]$ of f.

Proof. (i): See [24].
(ii): By proposition 21, $O_S(*D) = j_* O_{S \backslash D}$ is an holonomic D_S module. Hence, since $N_f / t N_f \subset O_S(*D)$ is a D_S submodule, $N_f / t N_f$ is an holonomic D_S module by proposition 19.
(iii): Follows from (ii) by [14] theorem 3.3.

□

For an arbitrary field of caracteristic zero, we have the following key proposition :

Proposition 35. *Let k a field of characteristic zero.*

- *(i1) Let $S \in \mathrm{SmVar}(k)$. Let $D = V(s) \subset S$ a (Cartier) divisor, where $s \in \Gamma(S, L)$ is a section of the line bundle $L = L_D$ associated to D. We then have the graph embedding $i : S \hookrightarrow L$, $i(x) = (x, s(x))$ and the zero section embedding $i_0 : S \hookrightarrow L$, $i_0(x) = (x, 0)$ and $L_0 = i_0(S)$. Denote $j : S^o := S\backslash D \hookrightarrow S$ and $j : L^o := L\backslash L_0 \hookrightarrow L$ the open complementary subsets. Then $j_*O_{S^o} = O_S(*D) \in \mathrm{PSh}_{\mathcal{D},rh}(S)$ admits the rational Kashiwara-Malgrange V_D-filtration, that is $j_*i_{*mod}O_{S^o} = i_{*mod}O_S(*D) \in \mathrm{PSh}_{\mathcal{D},rh}(L)$ admits the rational Kashiwara-Malgrange V_S-filtration.*
- *(i2) Let $S \in \mathrm{SmVar}(k)$. Let $D \subset S$ a (Cartier) divisor. Denote $j : S^o := S\backslash D \hookrightarrow S$ the open complementary subset. Then for $E \in \mathrm{Vect}_{\mathcal{D}}(S^o)$ regular in the strong sense (see definition 39), $j_*E \in \mathrm{PSh}_{\mathcal{D},rh}(S)$ admits the rational Kashiwara-Malgrange V_E-filtration for all (Cartier) divisor $E = V(s') \subset S$.*
- *(ii) Let $f : X \to S$ a proper morphism with $X, S \in \mathrm{SmVar}(k)$. Let $D \subset S$ a (Cartier) divisor. If $M \in \mathrm{PSh}_{\mathcal{D},rh}(X)$ admits the rational Kashiwara-Malgrange $V_{f^{-1}(D)}$ filtration, then $H^n\int_f M$ admits the rational Kashiwara-Malgrange V_D filtration for all $n \in \mathbb{Z}$.*

Proof. (i1): Let $S = \cup_{i=1}^r S_i$ an open affine covering such that for each i $D \cap S_i = V(f_i) \subset S_i$ is given by one equation $f_i \in \Gamma(S_i, O_{S_i})$. For each i, lemma 5 (iii) applied to S_i and $D \cap S_i$, gives the rational Kashiwara-Malgrange V_{S_i}-filtration for $i_{*mod}O_{S_i}(*D)$. By unicity of the rational Kashiwara-Malgrange V-filtration, we get the rational Kashiwara-Malgrange V_S-filtration on $i_{*mod}O_S(*D)$ since for all $k \in \mathbb{Q}$

$$V_{S_i,k}i_{*mod}O_S(*D)_{|S_i\cap S_j} = V_{S_j,k}i_{*mod}O_S(*D)_{|S_i\cap S_j}.$$

(i2): Let $E \subset S$ a (Cartier) divisor. Consider a subfield $k_0 \subset k$ of finite transcendence degree over $\mathbb{Q}$ such that $S = S_{0k} := S \otimes_{k_0} k$ with $S_0 \in \mathrm{SmVar}(k_0)$, $D = D_{0k} := D \otimes_{k_0} k$ with $D_0 \subset S_0$, $E = E_{0k} := E \otimes_{k_0} k$ with $E_0 \subset S_0$, $E = \pi_{k_0/k}(S_0^o)^{*mod}E_0$ with $E \in \mathrm{Vect}_{\mathcal{D}}(S_0^o)$, $S_0^o := S_0\backslash D_0$, and an embedding $\sigma : k_0 \hookrightarrow \mathbb{C}$. For simplicity, we denote again $S = S_0$, $E = E_0$, $D = D_0$, $j : S^o := S\backslash D \hookrightarrow S$ the open complementary subset and $E = E_0 \in \mathrm{Vect}_{\mathcal{D}}(S^o)$, and denote for short

$$\pi := \pi_{k_0/\mathbb{C}}(S) : S_{\mathbb{C}} := S \otimes_{k_0} \mathbb{C} \to S$$

the projection. Since $E \in \mathrm{Vect}_{\mathcal{D}}(S^o)$ is a locally free O_{S^o} module

$$n_{O_S/O_{S_{\mathbb{C}}}}(E) : E = \pi^*E \to \pi^{*mod}E,\ m \mapsto n_{O_S/O_{S_{\mathbb{C}}}}(E)(m) := m \otimes 1$$

is injective. Hence we get a canonical embedding

$$j_*n_{O_S/O_{S_{\mathbb{C}}}}(E) : j_*\pi^*E = \pi^*j_*E \hookrightarrow j_*(\pi^{*mod}E),\ m \mapsto m \otimes 1$$

By the complex case, see [20], $j_*(\pi^{*mod}E) \in \mathrm{PSh}_{\mathcal{D},rh}(S_{\mathbb{C}})$ admits the rational Kashiwara-Malgrange $V_{E'}$-filtration for all divisor $E' \subset S_{\mathbb{C}}$. We then set for $k \in \mathbb{Q}$

$$V_{E,k}j_*E := \pi_*(V_{E_{\mathbb{C}},k}j_*(\pi^{*mod}E) \cap \pi^*j_*E)$$

so that we get a strict monomorphism

$$j_*n_{O_S/O_{S_{\mathbb{C}}}}(E) : (j_*E, V_E) \hookrightarrow \pi_*(j_*(\pi^{*mod}E), V_{E_{\mathbb{C}}})$$

and so that this filtration satisfies the property of the Kashiwara-Malgrange V_E-filtration. Taking back the initial notations, this means that we get the V_{E_0} filtration on $j_*E_0 \subset \mathrm{PSh}_{\mathcal{D},h}(S_0)$. Then,

$$V_{E,k}j_*E := \pi_{k_0/k}(S)^{*mod}V_{E_0,k}j_*E_0 \subset \pi_{k_0/k}(S)^{*mod}j_*E_0 = j_*E$$

gives the V_E filtration on j_*E.
(ii): By definition $H^n\int_f M = H^n f_*E(D_{X\leftarrow S}\otimes_{D_X} M)$. We then see immediately that $H^n f_*E(D_{X\leftarrow S}\otimes_{D_X} (M, V_{f^{-1}(D)}))$ satisfy the hypothesis of the V_D filtration : since f is proper, the $V_{D,0}O_S$ modules

$$V_k H^n f_*E(((D_{X\leftarrow S}, V_{f^{-1}(D)})\otimes_{D_X} (M, V_{f^{-1}(D)})) := \\ \mathrm{Im}(H^n f_*(I\otimes \iota_{V_{f^{-1}(D)}}(M)) : H^n f_*E(V_{f^{-1}(D),k}(D_{X\leftarrow S}\otimes_{D_X} M) \to H^n f_*E(D_{X\leftarrow S}\otimes_{D_X} M))$$

are coherent. □

Theorem 35. *Let k a field of characteristic zero. Let $S\in \mathrm{SmVar}(k)$. Every $M\in \mathrm{PSh}_{\mathcal{D},rh}(S)$ holonomic, regular in the strong sense (see definition 39), admits the rationnal Kashiwara-Malgrange V_E filtration for each (Cartier) divisor $E\subset S$ (see definition 54).*

Proof. We may assume without loss of generality that S is connected. We argue by induction on $\dim\mathrm{supp}(M)$. If $\dim\mathrm{supp}(M) = 0$, there is noting to prove. Denote $i : Z := \mathrm{supp}\, M \hookrightarrow S$ the closed embedding. There exist by proposition 20 an open subset $j : S^o \hookrightarrow S$ with $D := S\backslash S^o \subset S$ a (Cartier) divisor such that $Z^o := Z\cap S^o$ is smooth and $i^{*mod}j^*M \in \mathrm{Vect}_{\mathcal{D}}(Z^o)$ is an integral connexion. Then $\dim(D\cap Z_i) = \dim(Z_i) - 1$ where $Z_i \subset Z$ are the irreducible component of Z since an holonomic D_S module is generically an integral connexion on its support by proposition 20. Take a desingularization $\epsilon : \tilde{Z} \to Z$ of the pair $(Z, D\cap Z)$ and denote by $l : Z^o \hookrightarrow \tilde{Z}$ the open embedding. By proposition 35 (i0) and (i2), $l_*i^{*mod}j^*M \in \mathrm{PSh}_{\mathcal{D},rh}(\tilde{Z})$ admits the rational Kashiwara Malgrange $V_{\epsilon^{-1}i^{-1}(E)}$ filtration, hence by proposition 35 (ii)

$$j_*j^*M = i_{*mod}\epsilon_{*mod}l_*i^{*mod}j^*M \in \mathrm{PSh}_{\mathcal{D},rh,Z}(S)$$

admits the rational Kashiwara Malgrange V_E filtration for all divisor $E\subset S$. We then consider

- the nearby cycle functor for M

$$\psi_D(M) := \psi_D(j_*j^*M) := \oplus_{-1\leq\alpha<0} V_{D,k}j_*j^*M \in \mathrm{PSh}_{\mathcal{D},rh,D\cap Z}(S),$$

  and we have then, by theorem 34, the canonical isomorphism

$$T(\psi_D, DR)(j_*j^*M) : DR(S)(\psi_D M) \xrightarrow{\sim} \psi_D DR(S)(M)$$

  (note that $\psi_D K = \psi_D(j_*j^*K)$ for $K\in P(S^{an}_{\mathbb{C}})$),

- the vanishing cycle functor for M

$$\phi^{\rho}_D(M) := H^0\,\mathrm{Cone}(\theta_{DR,D}(M) : \Gamma^{\vee,h}_D M \to \psi_D(M)) \in \mathrm{PSh}_{\mathcal{D},rh,D\cap Z}(S)$$

  which is regular holonomic by proposition 25, where $\theta_{DR,D}$ is the factorization in $D_{\mathcal{D},rh}(S)$

$$\rho_{DR,D}(j_*j^*M)\circ\mathrm{ad}(j^*,j_*)(M)) : M \xrightarrow{\gamma^{\vee,h}(M)} \Gamma^{\vee,h}_D M := L\mathbb{D}_S R\Gamma_D L\mathbb{D}_S M \xrightarrow{\theta_{DR,D}(M)} \psi_D(M)$$

  of the map in $\mathrm{PSh}_{\mathcal{D},rh}(S)$ of definition 56

$$\rho_{DR,D}(j_*j^*M)\circ\mathrm{ad}(j^*,j_*)(M)) := M \xrightarrow{\mathrm{ad}(j^*,j_*)(M)} j_*j^*M \\ \xrightarrow{\rho_{DR,D}(j_*j^*M)} \psi_D(j_*j^*M) =: \psi_D(M),$$

  we have then by theorem 31 the canonical isomorphism

$$(T(\gamma^{\vee}_D, DR)(M), T(\psi_D, DR)(j_*j^*M)) : DR(S)(\phi^{\rho}_D M) \xrightarrow{\sim} \phi_D DR(S)(M)$$

- the canonical map $can^{\rho}(M) := H^0c(\phi^{\rho}_D(M)) : \psi_D(M) \to \phi^{\rho}_D M$ in $\mathrm{PSh}_{\mathcal{D},rh,D\cap Z}(S)$,

- the variation map $var^{\rho}(M) := (0, s\partial s) : \phi^{\rho}_D M \to \psi_D M$ in $\mathrm{PSh}_{\mathcal{D},rh,D\cap Z}(S)$.

Consider then the following canonical map in $C_{\mathcal{D},rh}(S)$

$$Is(M) := (0, (\mathrm{ad}(j^*, j_*)(M), \rho_{DR,D}(j_* j^* M) \circ \mathrm{ad}(j^*, j_*)(M)), 0) :$$
$$M \to (\psi_D M \xrightarrow{(c(x_{S^o/S}(M)), can^{\rho}(M))} x_{S^o/S}(M) \oplus \phi^{\rho}_D M \xrightarrow{(0,s\partial s), var^{\rho}(M))} \psi_D M).$$

with

$$x_{S^o/S}(M) := \mathrm{Cone}(\rho_{DR,D}(j_* j^* M) : j_* j^* M \to \psi_D K) \in C_{\mathcal{D},rh}(S)$$

which is a quasi-isomorphism by theorem 13 and theorem 31. By induction hypothesis, $\phi^{\rho}_D M \in \mathrm{PSh}_{\mathcal{D},rh,D\cap Z}(S)$ admits the Kashiwara-Malgrange rational V_E-filtration for all divisor $E \subset S$. Let $E \subset S$ a Cartier divisor. We then set for $k \in \mathbb{Q}$

$$V_{E,k} M := Is(M)^{-1}(V_{E,k} H^1((\psi_D M, V_E) \xrightarrow{(c(x_{S^o/S}(M)), can^{\rho}(M))}$$
$$\mathrm{Cone}(\rho_{DR,D}(j_* j^* M) : (j_* j^* M, V_E) \to (\psi_D K, V_E) \oplus (\phi^{\rho}_D M, V_E) \xrightarrow{((0,s\partial s), var^{\rho}(M))} (\psi_D M, V_E)))).$$

□

Theorem 36. *Let $k \subset \mathbb{C}$ a subfield. Let $S \in \mathrm{SmVar}(k)$. Let $D = V(s) \subset S$ a divisor with $s \in \Gamma(S, L)$ and L a line bundle (S being smooth, D is Cartier). For $M \in \mathrm{PSh}_{\mathcal{D},rh}(S)$ with quasi-unipotent monodromy, so that the rational Kashiwara-Malgrange V_D-filtration on M exists by theorem 35,*

(i) we have the canonical isomorphism in $D_{c,k}(S^{an}_{\mathbb{C}})$ given in theorem 34

$$T(\psi_D, DR)(M) : DR(S)(\psi_D M^{an}) \xrightarrow{\sim} \psi_D DR(S)(M^{an})[-1]$$

so that $T(\psi_D, DR)(M) \circ DR(S)(s\partial_s) = N \circ T(\psi_D, DR)(M)$ where

$$N := \log T_u \in \mathrm{Hom}(\psi_D DR(S)(M^{an}), \psi_D DR(S)(M^{an})),\ T = T_u T_s,\ T_u \text{ unipotent},\ T_s \text{ nilpotent}$$

is induced by the monodromy automorphism $T : \tilde{S}^o \xrightarrow{\sim} \tilde{S}^o$ of the universal covering $\pi : \tilde{S}^o \to S^o := S \backslash D$.

(ii) we have the following canonical isomorphism in $D_{c,k}(S^{an}_{\mathbb{C}})$

$$T(\phi_D, DR)(M) : DR(S)(\phi_D M^{an}) \xrightarrow{DR(S)((0,var(M)))} DR(S)(\phi^{\rho}_D M^{an}) \xrightarrow{(I,T(\psi_D,DR)(M))} \phi_D DR(S)(M^{an})[-1].$$

where

$$\phi^{\rho}_D M := H^0 \,\mathrm{Cone}(\theta_{DR,D}(M) : \Gamma^{\vee,h}_D M \to \psi_D M) \in \mathrm{PSh}_{\mathcal{D},rh,D}(S)$$

with $\theta_{DR,D}$ the factorization in $D_{\mathcal{D},rh}(S)$

$$\rho_{DR,D}(M) \circ \mathrm{ad}(j^*, j_*)(M)) : M \xrightarrow{\gamma^{\vee,h}_D(M)} \Gamma^{\vee,h}_D M := L\mathbb{D}_S R\Gamma_D L\mathbb{D}_S M \xrightarrow{\theta_{DR,D}(M)} \psi_D(M).$$

of the map in $\mathrm{PSh}_{\mathcal{D},rh}(S)$ given in definition 56. We get

- *$T(\phi_D, DR)(M) \circ DR(S)(can(M)) = can(DR(S)(M)) \circ T(\psi_D, DR)(M)$*
- *$T(\psi_D, DR)(M) \circ DR(S)(var(M)) = var(DR(S)(M)) \circ T(\phi_D, DR)(M)$.*

Proof. (i):Follows from theorem 34.
(ii):Follows from (i). □

Theorem 37. *Let $S \in \mathrm{SmVar}(k)$.*

(i) *Let $M\in\mathrm{PSh}_{\mathcal{D},rh}(S)$. Let $S^o\subset S$ an open subset such that $D:=S\backslash S^o=V(s)\subset S$ is a (Cartier) divisor. Denote $i:D\hookrightarrow S$ the closed embedding and $j:S^o\hookrightarrow S$ the open embedding. Take an embedding $\sigma:k\hookrightarrow\mathbb{C}$. We have, using definition 56 and theorem 35, the canonical quasi-isomorphism in $C_{\mathcal{D},rh}(S)$:*

$$\begin{gathered}Is(M):=(0,(\mathrm{ad}(j^*,j_*)(M),\rho_{DR,D}(M)\circ\mathrm{ad}(j^*,j_*)(M)),0):\\ M\to(\psi_DM\xrightarrow{(c(x_{S^o/S}(M)),can(M))}x_{S^o/S}(M)\oplus\phi_DM\xrightarrow{((0,s\partial s),var(M))}\psi_DM).\end{gathered}$$

with

$$x_{S^o/S}(M):=\mathrm{Cone}(\rho_{DR,D}(M):j_*j^*M\to\psi_DK)\in C_{\mathcal{D},rh}(S)$$

(ii) *Let $D=V(s)\subset S$ a Cartier divisor. Denote $i:D\hookrightarrow S$ the closed embedding and $j:S^o:=S\backslash D\hookrightarrow S$ the open embedding. Then the functor*

$$(j^*,\phi_D,can,var):\mathrm{PSh}_{\mathcal{D},rh}(S)\to\mathrm{PSh}_{\mathcal{D},rh}(S^o)\times_J\mathrm{PSh}_{\mathcal{D},rh,D}(S)$$

is an equivalence of category whose inverse is

$$\begin{gathered}\mathrm{PSh}_{\mathcal{D},rh}(S^o)\times_J\mathrm{PSh}_{\mathcal{D},rh}(S)\to\mathrm{PSh}_{\mathcal{D},rh,D}(S),\\ (M',M'',u,v)\mapsto H^1((\psi_DM')\xrightarrow{(c(x_{S^o/S}(M')),u)}x_{S^o/S}(M')\oplus M''\xrightarrow{((0,s\partial s),v)}(\psi_DM')).\end{gathered}$$

Proof. (i):By theorem 31, theorem 36 and definition 60 we have $Is(M)=DR(S)^{-,-1}(Is(DR(S)(M)))$. The result then follows from theorem 13.
(ii):Follows from (i). □

We now give the p-adic version of theorem 36 :

Theorem 38. *Let $k\subset K\subset\mathbb{C}_p$ a subfield with p a prime number and K a p adic field. Let $S\in\mathrm{SmVar}(k)$. Let $D=V(s)\subset S$ a divisor with $s\in\Gamma(S,L)$ and L a line bundle (S being smooth, D is Cartier). so that we have the closed embedding $i:S\hookrightarrow L$, $i(x)=(x,s(x))$ and $D=i^{-1}(s_0)$, s_0 being the zero section. For $M\in\mathrm{PSh}_{\mathcal{D},rh}(S)$ with quasi-unipotent monodromy, so that the rational Kashiwara-Malgrange V_D-filtration on M exists by theorem 35,*

- *we have the canonical isomorphism in $D_{\mathbb{B}_{dr}}(S_K^{an,pet})$*

$$\begin{gathered}T(\psi_D,DR)^{B_{dr}}(M):=B^{B_{dr}}(M)\circ A^{B_{dr}}(M)^{-1}:\\ DR(S)(\psi_DM^{an}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K})\xrightarrow{\sim}\psi_DDR(S)(M^{an}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K})[-1]\end{gathered}$$

  with, for $S=\cup_{i=1}^sS_i$ an open affine cover such that $D\cap S_i=V(f_i)\subset S_i$ is given by $f_i\in\Gamma(S_i,O_{S_i})$, denoting $q:L_i:=p^{-1}(S_i)\to\mathbb{A}^1_k$ the projection and $j_i:S_i\hookrightarrow S$ the open embeddings,

  - *the isomorphism in $D_{\mathbb{B}_{dr}}(S_K^{an,pet})$*

$$\begin{aligned}A^{B_{dr}}(M):(\oplus_{i=1}^s\oplus_{-1\leq\alpha<0}\mathrm{Cone}(\partial_s:DR(L_i/\mathbb{A}^1_k)((V_{D\alpha}M)^{an})\otimes_{O_S}s^{\alpha+1}O\mathbb{B}_{dr,S_K}\\ \to DR(L_i/\mathbb{A}^1_k)((V_{D\alpha}M)^{an})\otimes_{O_S}s^{\alpha}O\mathbb{B}_{dr,S_K})\xrightarrow{(j_I^*)}\cdots)[-1]\\ \to(\oplus_{i=1}^sDR(S_i)(\psi_DM^{an}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K})\xrightarrow{j_I^*}\cdots)[-1]\\ \xrightarrow{((j_i^*),0)^{-1}}(DR(S)(\psi_DM^{an})\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K})[-1],\ (\sum_jm_j\otimes(\log s)^j,m')\mapsto[m_0],\end{aligned}$$

– and the isomorphism in $D_{\mathbb{B}_{dr}}(S_K^{an,pet})$

$$B^{B_{dr}}(M) : (\oplus_{i=1}^{s}\oplus_{-1\leq\alpha<0}\mathrm{Cone}(\partial_s : V_{D\alpha}DR(L_i/\mathbb{A}_k^1)(M^{an})\otimes_{O_S}s^{\alpha+1}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K}$$
$$\to V_{D\alpha}DR(L_i/\mathbb{A}_k^1)(M^{an})\otimes_{O_S}s^{\alpha}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K})\xrightarrow{(j_I^*)}\cdots)[-1]$$
$$\to(\oplus_{i=1}^{s}DR(p^*O_{\mathbb{A}_1^k})(i^*\pi_*\pi^{*mod}DR(L_i/\mathbb{A}_k^1)(M^{an})\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K})\xrightarrow{(j_I^*)}\cdots)[-1]$$
$$\xrightarrow{=:}(\oplus_{i=1}^{s}\psi_D DR(p^*O_{\mathbb{A}_1^k})(DR(L_i/\mathbb{A}_k^1)(M^{an})\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K})\xrightarrow{(j_I^*)}\cdots)[-1]$$
$$\xrightarrow{((j_i^*),0)^{-1}}\psi_D DR(S)(M^{an}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K})[-1],\ (\sum_j m_j\otimes(\log s)^j,m')\mapsto\sum_j(\log s)^j m_j,$$

so that $T_{B_{dr}}(\psi_D,DR)(M)\circ DR(S)((s\partial_s)\otimes I)=N\circ T_{B_{dr}}(\psi_D,DR)(M)$ *where*

$$N:=\log T_u\in\mathrm{Hom}(\psi_D DR(S)(M^{an}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K}),\psi_D DR(S)(M^{an}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K})),$$
$$T=T_uT_s,\ T_u \text{ unipotent},\ T_s \text{ nilpotent}$$

is induced by the monodromy automorphism $T:\tilde{S}^o\xrightarrow{\sim}\tilde{S}^o$ *of the perfectoid universal covering* $\pi:\tilde{S}^o\to S^o:=S\backslash D$ *(see [27]).*

- *there is a canonical isomorphism in* $D_{\mathbb{B}_{dr}}(S_K^{an,pet})$

$$T^{B_{dr}}(\phi_D,DR)(M):DR(S)(\phi_D M^{an}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K})\xrightarrow{DR(S)((0,var(M)\otimes I))}$$
$$DR(S)(\phi_D^{\rho}M^{an}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K})\xrightarrow{(I,T^{B_{dr}}(\psi_D,DR)(M))}\phi_D DR(S)(M^{an}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K})[-1].$$

where

$$\phi_D^{\rho}M:=H^0\,\mathrm{Cone}(\theta_{DR,D}(M):\Gamma_D^{\vee,h}M\to\psi_D M)\in\mathrm{PSh}_{\mathcal{D},rh,D}(S)$$

with $\theta_{DR,D}$ *the factorization in* $D_{\mathcal{D},rh}(S)$

$$\rho_{DR,D}(M)\circ\mathrm{ad}(j^*,j_*)(M)):M\xrightarrow{\gamma_D^{\vee,h}(M)}\Gamma_D^{\vee,h}M:=L\mathbb{D}_S R\Gamma_D L\mathbb{D}_S M\xrightarrow{\theta_{DR,D}(M)}\psi_D(M).$$

of the map given in definition 56. In particular

– $T^{B_{dr}}(\phi_D,DR)(M)\circ DR(S)(can(M)\otimes I)=can(DR(S)(M^{an}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K}))\circ T^{B_{dr}}(\psi_D,DR)(M)$

– $T^{B_{dr}}(\psi_D,DR)(M)\circ DR(S)(var(M)\otimes I)=var(DR(S)(M^{an}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K}))\circ T^{B_{dr}}(\phi_D,DR)(M)$.

Proof. Similar to the proof of theorem 36. □

We now look at the particular case of algebraic integral connexions. The following follows from the work of Bhatwaderkar and Rao ([7]).

Theorem 39. *Let R a local finite type algebra over a field of characteristic zero k. Let $f\in R$ a non zero divisor and non invertible element. If $M\in\mathrm{Mod}(R_f)$ is a projective $R_f:=R[1/f]$ module, then there exist a projective (hence free) module $M'\in\mathrm{Mod}(R)$ such that $M'_f:=M'\otimes_R R_f=M$*

Proof. See [7]. □

Corollary 3. *Let $S\in\mathrm{SmVar}(k)$. Let $S^o\subset S$ an open subset such that $D:=S\backslash S^o=V(s)\subset S$ is a Cartier divisor. Denote $j:S^o\hookrightarrow S$ the open embedding. Let $E=(E,\nabla)\in\mathrm{Vect}_{\mathcal{D}}(S^o)$ an integrable connexion regular along D (see definition 39).*

(i) For all $s\in D$, there exist an open affine neighborhood $W_s\subset S$ of s in S and a free O_{W_s}-submodule $E'\subset(j_*E)_{|W_s}$ such that $E'_{|W_s\cap S^o}=E_{|W_s\cap S^o}$, $D_SE'=(j_*E)_{|W_s}$, $s\nabla_sE'\subset E'$ and such that all the eigeinvalue of the residue matrix $\nabla_s:E'/sE'\to E'/sE'$ are rational numbers $k\in\mathbb{Q}$, $0\leq k<1$.

(i)' For $s \in D$, such an $E' \subset (j_*E)_{|W_s}$ is unique.

(ii) There exist a unique locally free O_S-submodule $E' \subset j_*E$ such that $E'_{|S^o} = E$, $D_SE' = j_*E$ and $s\nabla_sE' \subset E'$ and such that all the eigeinvalue of the residue matrix $\nabla_s : E'/sE' \to E'/sE'$ are rational numbers $k \in \mathbb{Q}$, $0 \leq k < 1$.

Proof. (i):Consider an affine neighborhood $W'_s \subset S$ of s. Then, $E_{|W'_s\cap S^o} \in \mathrm{Vect}(W'_s \cap S^o)$ is projective since it is locally free and $W'_s \cap S^o$ is affine. By the complex case and regularity, there exist an integral lattice $L' \subset j_*(E \otimes_k \mathbb{C})$ such that $D_SL = E$, $s\partial_sL \subset L$ and such that all the eigeinvalue of the residue matrix $\nabla_s : L'/sL' \to L'/sL'$ are rational numbers $k \in \mathbb{Q}$, $0 \leq k < 1$. Then set $L := L' \cap \subset j_*E$. we have then $D_SL = E$, $s\partial_sL \subset L$ and all the eigeinvalue of the residue matrix $\nabla_s : L/sL \to L/sL$ are rational numbers $k \in \mathbb{Q}$, $0 \leq k < 1$. In particular $L_{|S^o} = E$ and L is a coherent j_*O_S module. Denote $R = O_{W'_s,s}$. Then by theorem 39 the projective R_f module $L_{s|S^o} \in \mathrm{Mod}(R_f)$ extend to a free module $\tilde{L} \in \mathrm{Mod}(R)$, that is $\tilde{L} \subset j_*j^*L$ with $\tilde{L} \otimes_R R_f = L_{s|S^o}$. Then take a neighborhood $W_s \subset W'_s$ of s in W'_s and $E' \in \mathrm{Vect}(W_s)$, $E' \subset (j_*E)_{|W_s}$, such that $E'_s = \tilde{L}$. Then, $E'_{|W_s\cap S^o} = E_{|W_s}$, $D_SE' = (j_*E)_{|W_s}$, $s\nabla_sE' \subset E'$ and all the eigeinvalue of the residue matrix $\nabla_s : E'/sE' \to E'/sE'$ are rational numbers $k \in \mathbb{Q}$, $0 \leq k < 1$.
(i)':Follows the unicity of the V_D filtration on j_*E.
(ii):Follows from (i) and (i)'. □

5.2 The De Rham modules on algebraic varieties over a field of caracteristic zero

We recall the theorem of [25]

Theorem 40. *Let $f : X \to S$ a projective morphism with $X, S \in \mathrm{Var}(\mathbb{C})$, where projective means that there exist a factorization $f : X \xrightarrow{l} \mathbb{P}^N \times S \xrightarrow{p_S} S$ with l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exits closed embeddings $i_I : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1, \ldots, s]$, recall that we denote $S_I := \cap_{i\in I}S_i$ and $X_I := f^{-1}(S_I)$. We have then the following commutative diagram*

$$\begin{array}{ccccc} X_I & \xrightarrow{i_I\circ l_I} & \mathbb{P}^N \times \tilde{S}_I & \xrightarrow{p_{\tilde{S}_I}} & \tilde{S}_I \\ \uparrow{\scriptstyle j'_{IJ}} & & \uparrow{\scriptstyle p'_{IJ}} & & \uparrow{\scriptstyle p_{IJ}} \\ X_J & \xrightarrow{i_J\circ l_J} & \mathbb{P}^N \times \tilde{S}_J & \xrightarrow{p_{\tilde{S}_J}} & \tilde{S}_J \end{array}$$

whose right square is cartesian (see section 5).

(i) For

$$(((M_I, F), u_{IJ}), K, \alpha) \in HM(X),$$

where $((M_I, F), u_{IJ}) \in \mathrm{PSh}_{\mathcal{D}fil}(X_I/(\mathbb{P}^N \times \tilde{S}_I))$, $K \in P(X^{an})$, we have

$$H^n(\int_f^{FDR}((M_I, F), u_{IJ}), Rf_*W, f_*(\alpha)) \in HM(S)$$

for all $n \in \mathbb{Z}$, and for all $p \in \mathbb{Z}$, the differentials of $\mathrm{Gr}_F^p \int_f^{FDR}((M_I, F, W), u_{IJ})$ are strict for the the Hodge filtration F.

(ii) Then, for

$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D(MHM(X)),$$

where $((M_I, F, W), u_{IJ}) \in C_{\mathcal{D}(1,0)fil}(X_I/(\mathbb{P}^N \times \tilde{S}_I))$, $(K, W) \in C_{fil}(X^{an})$, we have

$$H^n(\int_f^{FDR}((M_I, F, W), u_{IJ}), Rf_*(K, W), f_*(\alpha)) \in MHM(S)$$

for all $n \in \mathbb{Z}$, and for all $p \in \mathbb{Z}$, the differentials of $\mathrm{Gr}_F^p \int_f^{FDR}((M_I, F, W), u_{IJ})$ are strict for the the Hodge filtration F.

Proof. (i):See [26].
(ii):Follows from (i) using the spectral sequence of the filtered complex $(\int_f^{FDR}((M_I, F, W), u_{IJ}), Rf_*(K, W), f_*(\alpha))$ associated to the filtration W: see [25]. □

Definition 57. *Let $S \in \mathrm{SmVar}(k)$. Let $D = V(s) \subset S$ a divisor with $s \in \Gamma(S, L)$ and L a line bundle (S being smooth, D is Cartier). For $(M, F, W) \in \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S)$, consider using theorem 35 the Kashiwara-Malgrange V_S-filtration on $i_{*mod}M$ and denote $V_{D,k}M := i^* V_{S,k} i_{*mod} M$. We then define, using definition 55,*

- *the nearby cycle functor*

$$\psi_D(M, F, W) := \oplus_{-1 \leq \alpha < 0}(\mathrm{Gr}_{V_D,\alpha} M, F, W) \in \mathrm{PSh}_{\mathcal{D}(1,0)fil,D}(S)$$

 where $\mathrm{Gr}_{V_D,\alpha} M$ is endowed with the induced filtration $F^p \mathrm{Gr}_{V_D,\alpha} M := F^p V_{D,\alpha} M / F^p V_{D,<\alpha} M$,

- *the vanishing cycle functor*

$$\phi_D(M, F, W) := \oplus_{-1 < \alpha \leq 0}(\mathrm{Gr}_{V_D,\alpha} M, F, W) \in \mathrm{PSh}_{\mathcal{D}(1,0)fil,D}(S)$$

 where $\mathrm{Gr}_{V_D,\alpha} M$ is endowed with the induced filtration $F^p \mathrm{Gr}_{V_D,\alpha} M := F^p V_{D,\alpha} M / F^p V_{D,<\alpha} M$,

- *the canonical maps*

$$can(M, F, W) := (\partial_s, I) : \psi_D(M, F, W) \to \phi_D(M, F, W)(-1),$$
$$var(M, F, W) := (I, s) : \phi_D(M, F, W) \to \psi_D(M, F, W).$$

Proposition 36. *Let $S \in \mathrm{SmVar}(k)$. Let $D = V(s) \subset S$ a (Cartier divisor). Consider a composition of proper morphisms $(f : X = X_r \xrightarrow{f_r} X_{r-1} \xrightarrow{f_1} X_0 = S) \in \mathrm{SmVar}(k)$ and*

$$(M, F) = H^{n_0} \int_{f_1} \cdots H^{n_r} \int_{f_r} (O_X, F_b) \in \mathrm{PSh}_{\mathcal{D}fil,rh}(S).$$

which admits a V_D-filtration (see theorem 35). Then,

$$\psi_D(M, F) = H^{n_0} \int_{f_1} \cdots H^{n_r} \int_{f_r} \psi_{f^{-1}(D)}(O_X, F_b) \in \mathrm{PSh}_{\mathcal{D}fil,rh}(S)$$

and

$$\phi_D(M, F) = H^{n_0} \int_{f_1} \cdots H^{n_r} \int_{f_r} \phi_{f^{-1}(D)}(O_X, F_b) \in \mathrm{PSh}_{\mathcal{D}fil,rh}(S).$$

Proof. Immediate from definition. □

Definition 58. *(i) Let $S \in \mathrm{SmVar}(k)$. We define, see theorem 28(iv), the full subcategories*

$$PDRM^1(S) \subset PDRM^2(S) \subset PDRM(S) = \cup_{i \in \mathbb{N}} PDRM^i(S) \subset \mathrm{PSh}_{\mathcal{D}fil,rh}(S)$$

consisting of pure De Rham modules inductively. For each $S \in \mathrm{SmVar}(k)$, we define

$$PDRM^1(S) :=< H^n \int_f (O_X, F_b)(d), (f : X \to S) \in \mathrm{SmVar}(k) \text{ proper}, n, d \in \mathbb{Z} >\subset \mathrm{PSh}_{\mathcal{D}fil,rh}(S),$$

the full abelian subcategory, where $<,>$ means generated by and $(-)$ is the shift of the filtration. Assume we have defined $PDRM^{k-1}(S) \subset \mathrm{PSh}_{\mathcal{D}fil,rh}(S)$ for all $S \in \mathrm{SmVar}(k)$. For each $S \in \mathrm{SmVar}(k)$, we define

$$PDRM^k(S) :=< H^n \int_f (M,F)(d), (f : X \to S) \in \mathrm{SmVar}(k) \textit{ proper},$$
$$(M,F) \in PDRM^{k-1}(X), n, d \in \mathbb{Z} >\subset \mathrm{PSh}_{\mathcal{D}fil,rh}(S),$$

the full abelian subcategory, where $<,>$ means generated by and $(-)$ is the shift of the filtration. By proposition 36, for $D = V(s) \subset S$ a (Cartier) divisor and $(M,F) \in PDRM(S)$, we have, using theorem 35 or proposition 35(ii), for all $k \in \mathbb{Z}$

$$\mathrm{Gr}_k^W \psi_D(M,F), \mathrm{Gr}_k^W \psi_D(M,F) \in PDRM(S).$$

(i)' *Let $S \in \mathrm{Var}(k)$ non smooth. Take an open cover $S = \cup_i S_i$ such that there are closed embedding $S_I \hookrightarrow \tilde{S}_I$ with $S_I \in \mathrm{SmVar}(k)$. We define as in (i), see theorem 29(iv), the full subcategories*

$$PDRM^1(S) \subset PDRM^2(S) \subset PDRM(S) = \cup_{i\in\mathbb{N}} PDRM^i(S) \subset \mathrm{PSh}^0_{\mathcal{D}fil,rh}(S/(\tilde{S}_I))$$

inductively. For each $S \in \mathrm{Var}(k)$, we define

$$PDRM^1(S) :=< H^n \int_{p_S} (\Gamma_X^{\vee,Hdg}(O_{Y\times\tilde{S}_I}, F_b), x_{IJ})(d),$$
$$(f : X \to S) \in \mathrm{Var}(k) \textit{ proper}, \ X \textit{ smooth}, \ f = p_S \circ i, n, d \in \mathbb{Z} >\subset \mathrm{PSh}^0_{\mathcal{D}fil,rh}(S/(\tilde{S}_I))$$

the full abelian subcategory, where $<,>$ means generated by and $(-)$ is the shift of the filtration, $i : X \hookrightarrow Y \times S$ is a closed embedding with $Y \in \mathrm{PSmVar}(k)$,

- *for $j : X^o \hookrightarrow X$ an open embedding we set using proposition 35(i1) $j_{!Hdg}(O_{X^o}, F_b) := (j_* O_{X^o}, F)$ with $F^p j_* O_{X^o} := \oplus_k \partial_s^k F^{p+k} V_{X\backslash X^o,k} j_* O_{X^o}$, $X\backslash X^o = V(s) \subset X$,*
- *we set*

$$(\Gamma_X^{\vee,Hdg}(O_{Y\times\tilde{S}_I}, F_b), x_{IJ}) \in C_{\mathcal{D}fil,rh}(X/(Y \times \tilde{S}_I))$$

where

$$\Gamma_X^{\vee,Hdg}(O_{Y\times\tilde{S}_I}, F_b) := \mathrm{Cone}(\mathrm{ad}(j_{I!Hdg}, j_I^*)(-) : j_{I!Hdg} j_I^*(O_{Y\times\tilde{S}_I\backslash X}, F_b) \to (O_{Y\times\tilde{S}_I}, F_b)),$$

together with the maps

$$x_{IJ} : \Gamma_X^{\vee,Hdg}(O_{Y\times\tilde{S}_I}, F_b) \xrightarrow{\mathrm{ad}(p_{IJ}^{*mod}, p_{IJ*})(-)}$$
$$p_{IJ*}\Gamma^{\vee,Hdg}_{X\times\tilde{S}_{J\backslash I}}(O_{Y\times\tilde{S}_J}, F_b) \xrightarrow{\mathrm{ad}(j_{IJ!Hdg}, j^{IJ*})(-)} p_{IJ*}\Gamma_X^{\vee,Hdg}(O_{Y\times\tilde{S}_J}, F_b).$$

Assume we have defined $PDRM^{k-1}(S) \subset \mathrm{PSh}^0_{\mathcal{D}fil,rh}(S/(\tilde{S}_I))$ for all $S \in \mathrm{Var}(k)$. For each $S \in \mathrm{Var}(k)$, we define

$$PDRM^k(S) :=< H^n \int_{p_S} (\Gamma_X^{Hdg}((M_I,F), u_{IJ})(d), (f : X \to S) \in \mathrm{Var}(k) \textit{ proper}, \ X \textit{ smooth}, \ f = p_S \circ i,$$
$$((M_I,F), u_{IJ}) \in PDRM^{k-1}(X), n, d \in \mathbb{Z} >\subset \mathrm{PSh}^0_{\mathcal{D}fil}(S/(\tilde{S}_I))$$

the full abelian subcategory, where $<,>$ means generated by and $(-)$ is the shift of the filtration, $i : X \hookrightarrow Y \times S$ is a closed embedding with $Y \in \mathrm{PSmVar}(k)$, $((M_I,F), u_{IJ}) \in DRM^{k-1}(X)$ and

- *for* $j : X^o \hookrightarrow X'$ *an open embedding with* $X' \in \mathrm{SmVar}(k)$ *and* $(M,F) \in \mathrm{PSh}_{\mathcal{D}fil,rh}(X^o)$ *we set* $j_{!Hdg}(M,F) := (j_*M,F)$ *using theorem 35 or proposition 35(ii) with* $F^pj_*M := \oplus_k\partial_s^kF^{p+k}V_{X'\backslash X^o,k}j_*(M,F)$, $X'\backslash X^o = V(s) \subset X'$,
- *we set*

$$\Gamma_X^{\vee,Hdg}((M_I,F),u_{IJ}) := (\Gamma_X^{\vee,Hdg}(M_I,F),u_{IJ}^q) \in C_{\mathcal{D}fil}(X/(Y\times\tilde{S}_I))$$

with $\Gamma_X^{\vee,Hdg}(M_I,F) := \mathrm{ad}(j_{I!Hdg}j_I^*)(-) : \mathrm{Cone}(j_{I!Hdg}j_I^*(M_I,F) \to (M_I,F))$, *together with the maps*

$$u_{IJ}^q : \Gamma_X^{\vee,Hdg}(M_I,F) \xrightarrow{I(p_{IJ}^{*mod},p_{IJ*})(u_{IJ})\circ\mathrm{ad}(p_{IJ}^{*mod},p_{IJ*})(-)}$$
$$p_{IJ*}\Gamma^{\vee,Hdg}_{X\times\tilde{S}_{J\backslash I}}(M_J,F) \xrightarrow{\mathrm{ad}(j_{IJ!Hdg},j^{IJ*})(-)} p_{IJ*}\Gamma_X^{\vee,Hdg}(M_I,F).$$

Note that if S *is smooth then this definition of* $PDRM(S)$ *agree with the one given in (i).*

For $k \subset \mathbb{C}$ *and* $S \in \mathrm{Var}(k)$, *we have by theorem 40* $PDRM(S_{\mathbb{C}}) \subset \pi_S(HM(S_{\mathbb{C}}))$ *which are by definition the De Rham factor of geometric pure Hodge modules.*

(ii) Let $S \in \mathrm{Var}(k)$. *Take an open cover* $S = \cup_i S_i$ *such that there are closed embedding* $S_I \hookrightarrow \tilde{S}_I$ *with* $S_I \in \mathrm{SmVar}(k)$. *We define using the pure case (i) and (i)' the full subcategory of weak mixed De Rham modules*

$$\widetilde{DRM}(S) := \left\{((M_I,F,W),u_{IJ}),\ \mathrm{Gr}_k^W((M_I,F,W),u_{IJ}) \in PDRM(S)\right\} \subset \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))$$

whose object consists of $((M_I,F,W),u_{IJ}) \in \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))$ *such that*

$$\mathrm{Gr}_k^W((M_I,F,W),u_{IJ}) := ((\mathrm{Gr}_k^W M_I,F),\mathrm{Gr}_W^k u_{IJ}) \in PDRM(S).$$

For $S \in \mathrm{SmVar}(k)$ *and* $D = V(s) \subset S$ *a (Cartier) divisor, we have for* $(M,F,W) \in \widetilde{DRM}(S)$, *using theorem 35,*

$$\psi_D(M,F,W),\phi_D(M,F,W) \in \widetilde{DRM}(S),$$

by the pure case (c.f. (i) and proposition 36) and the strictness of the V*-filtration.*

(ii)' Let $S \in \mathrm{Var}(k)$. *Take an open cover* $S = \cup_i S_i$ *such that there are closed embedding* $S_I \hookrightarrow \tilde{S}_I$ *with* $S_I \in \mathrm{SmVar}(k)$. *The category of (mixed) de Rham modules over* S *is the full subcategory*

$$DRM(S) \subset \widetilde{DRM}(S) \subset \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))$$

whose object consists of $((M_I,F,W),u_{IJ}) \in \widetilde{DRM}(S)$ *such that for each* $I \subset [1,\ldots s]$ *and every Cartier divisor* $D \subset \tilde{S}_I$, *the three filtrations* F,W *and* V_D *on* M_I *are compatible, the filtration* W *is admissible for* D *(that is the relative monodromy filtration* $< W,W(N) >$ *exists on* M_I *where* N *is the monodromy of* $\tilde{S}_I\backslash D$ *on* M_I*), and so on inductively on the graded* ψ_DM_I *until* $\mathrm{supp}(\psi_{D_1}\cdots\psi_{D_r}M_I)$ *has dimension zero.*

For $k \subset \mathbb{C}$ *and* $S \in \mathrm{Var}(k)$, *we have by theorem 40* $DRM(S_{\mathbb{C}}) \subset \pi_S(MHM(S_{\mathbb{C}}))$ *which are by definition the De Rham factor of geometric mixed Hodge modules. In particular, for* $S \in \mathrm{Var}(k)$, *a morphism*

$$m : ((M_I,F,W),u_{IJ}) \to ((N_I,F,W),u_{IJ}),\ ((M_I,F,W),u_{IJ}),((N_I,F,W),u_{IJ}) \in DRM(S)$$

is strict for the Hodge filtration F. *For* $S \in \mathrm{Var}(k)$ *we get* $D(DRM(S)) := \mathrm{Ho}_{zar}(C(DRM(S)))$ *after localization with Zariski local equivalence.*

Remark 3. *(i) Let $S \in \mathrm{SmVar}(k)$. By definition,*

$$PDRM(S) :=< H^{n_1}\int_{f_1}\cdots H^{n_r}\int_{f_r}(O_X,F_b)(d),\ (f : X = X_r \xrightarrow{f_r} X_{r-1} \xrightarrow{f_1} X_0 = S) \in \mathrm{SmVar}(k),$$
$$f_i \text{ proper}, 1 \leq i \leq r, n_1,\ldots n_r, d \in \mathbb{Z} >\subset \mathrm{PSh}_{\mathcal{D}fil,rh}(S).$$

(ii) Let $S \in \mathrm{Var}(k)$ non smooth. Take an open cover $S = \cup_i S_i$ such that there are closed embedding $S_I \hookrightarrow \tilde{S}_I$ with $S_I \in \mathrm{SmVar}(k)$. By definition,

$$PDRM(S) :=< H^{n_1}\int_{p_S}(\Gamma^{\vee,Hdg}_{X_1}(\cdots H^{n_r}\int_{p_{X_{r-1}}}\Gamma^{\vee,Hdg}_{X}(O_{Y_r\times\tilde{X}_{r-1}},F_b)))(d),$$
$$(f : X = X_r \xrightarrow{f_r} X_{r-1} \xrightarrow{f_1} X_0 = S) \in \mathrm{Var}(k) \text{ proper}, f_i = p_{S_{i-1}}\circ i_i, 1 \leq i \leq r, n_1,\ldots n_r, d \in \mathbb{Z} >$$
$$\subset \mathrm{PSh}_{\mathcal{D}fil,rh}(S),$$

where $i_i : X_i \hookrightarrow Y_i \times X_{i-1}$ is a closed embedding with $Y_i \in \mathrm{PSmVar}(k)$. Note that if S is smooth then this definition of $PDRM(S)$ agree with the one given in (i).

- Let $S \in \mathrm{SmVar}(k)$. We consider the canonical embedding

$$\iota_S : C(DRM(S)) \hookrightarrow C_{\mathcal{D}(1,0)fil}(S)$$

which induces in the derived category the functor

$$\iota_S : D(DRM(S)) \to D_{\mathcal{D}(1,0)fil}(S) \to D_{\mathcal{D}(1,0)fil,\infty}(S)$$

after localization with respect to filtered Zariski local equivalences and ∞-filtered Zariski local equivalences respectively. Note that if $m : (M,F,W) \to (N,F,W)$ with $(M,F,W),(N,F,W) \in C(DRM(S))$ is a Zariski local equivalence, then it is a filltered Zariski local equivalence by strictness.

- Let $S \in \mathrm{Var}(k)$ non smooth. Take an open cover $S = \cup_i S_i$ such that there are closed embedding $S_I \hookrightarrow \tilde{S}_I$ with $S_I \in \mathrm{SmVar}(k)$. We consider the canonical embedding

$$\iota_S : C(DRM(S)) \hookrightarrow C_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)),$$

which induces in the derived category the functor

$$\iota_S : D(DRM(S)) \to D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \to D_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I))$$

after localization with respect to filtered Zariski local equivalences and ∞-filtered Zariski local equivalences respectively. Note that if $m : (M,F,W) \to (N,F,W)$ with $(M,F,W),(N,F,W) \in C(DRM(S))$ is a Zariski local equivalence, then it is a filltered Zariski local equivalence by strictness.

Definition 59. *(i) Let $S \in \mathrm{SmVar}(k)$. Let $D = V(s) \subset S$ a divisor with $s \in \Gamma(S,L)$ and L a line bundle (S being smooth, D is Cartier). Denote by $j : S^o := S\backslash D \hookrightarrow S$ the open complementary embedding. Let $(M,F,W) \in DRM(S^o))$. By theorem 35, M admits the Kashiwara-Malgrange rational V_D-filtration, that is $i_{*mod}M$ admits the Kashiwara-Malgrange rational V_S-filtration and $V_{D,k}M := i^*V_{S,k}i_{*mod}M$. We then define,*

- *the canonical extension*

$$j_{*Hdg}(M,F,W) := (j_*M,F,W) \in DRM(S),\ \ F^pj_*M = \sum_{k\in\mathbb{N}}\partial_s^kF^{p+k}V_{D,<0}j_*M \subset j_*M,$$
$$W_kj_*M := W_kj_{*w}(M,W) :=< j_*W_kM, W(N)_kM >\subset j_*M$$

and $(j_*M,W) := j_{*w}(M,W)$ is given by monodromy weight filtration similarly as in the complex case in [25], so that $j^*j_{*Hdg}(M,F,W) = (M,F,W)$ and $DR(S)(j_{*Hdg}(M,F,W)) = j_*DR(S^o)(M,W)$,

- *the canonical extension*

$$j_{!Hdg}(M,F,W) := \mathbb{D}^{Hdg}_S j_{*Hdg}\mathbb{D}^{Hdg}_S(M,F,W) \in DRM(S)$$

so that $j^*j_{!Hdg}(M,F,W) = (M,F,W)$ *and* $DR(S)(j_{!Hdg}(M,F,W)) = j_!DR(S^o)(M,W)$.

Moreover for $(M',F,W) \in DRM(S)$,

- *there is a canonical map* $\mathrm{ad}(j^*, j_{*Hdg})(M',F,W) : (M',F,W) \to j_{*Hdg}j^*(M',F,W)$ *in* $DRM(S)$,
- *there is a canonical map* $\mathrm{ad}(j_{!Hdg}, j^*)(M',F,W) : j_{!Hdg}j^*(M',F,W) \to (M',F,W)$ *in* $DRM(S)$.

(ii) *Let* $S \in \mathrm{SmVar}(k)$. *Let* $Z = V(\mathcal{I}) \subset S$ *an arbitrary closed subset,* $\mathcal{I} \subset O_S$ *being an ideal subsheaf. Taking generators* $\mathcal{I} = (s_1, \ldots, s_r)$, *we get* $Z = V(s_1, \ldots, s_r) = \cap_{i=1}^r Z_i \subset S$ *with* $Z_i = V(s_i) \subset S$, $s_i \in \Gamma(S, \mathcal{L}_i)$ *and* L_i *a line bundle. Note that* Z *is an arbitrary closed subset,* $d_Z \geq d_X - r$ *needing not be a complete intersection. Denote by* $j : S^o := S\backslash Z \hookrightarrow S$, $j_I : S^{o,I} := \cap_{i\in I}(S\backslash Z_i) = S\backslash(\cup_{i\in I} Z_i) \xrightarrow{j_I^o} S^o \xrightarrow{j} S$ *the open complementary embeddings, where* $I \subset \{1, \cdots, r\}$. *Denote*

$$\mathcal{D}(Z/S) := \left\{(Z_i)_{i\in[1,\ldots r]}, Z_i \subset S, \cap Z_i = Z\right\}, Z'_i \subset Z_i$$

the flag category. For $(M,F,W) \in DRM(S^o)$, *we define by (i)*

- *the (bi)-filtered complex of* D_S*-modules*

$$j_{*Hdg}(M,F,W) := \varinjlim_{\mathcal{D}(Z/S)} \mathrm{Tot}_{cardI=\bullet}(j^{Hdg}_{I*} j_I^{o*}(M,F,W)) \in C(DRM(S)),$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \mathrm{ad}(j^*_{IJ}, j^{Hdg}_{IJ*})(j_I^{o*}(M,F,W))$, $j_{IJ} : S^{oJ} \hookrightarrow S^{oI}$ *being the open embedding, and* $d_{IJ} = 0$ *if* $I \notin J$,

- *the (bi)-filtered complex of* D_S*-modules*

$$j_{!Hdg}(M,F,W) := \varprojlim_{\mathcal{D}(Z/S)} \mathrm{Tot}_{cardI=-\bullet}(j^{Hdg}_{I!} j_I^{o*}(M,F,W)) = \mathbb{D}^{Hdg}_S j_{*Hdg}\mathbb{D}^{Hdg}_S(M,F,W) \in C(DRM(S)),$$

where the horizontal differential are given by, if $I \subset J$, $d_{IJ} := \mathrm{ad}(j^{Hdg}_{IJ!}, j^*_{IJ})(j_I^{o*}(M,F,W))$, $j_{IJ} : S^{oJ} \hookrightarrow S^{oI}$ *being the open embedding, and* $d_{IJ} = 0$ *if* $I \notin J$.

By definition, we have for $(M,F,W) \in C(DRM(S^o))$, $j^*j_{*Hdg}(M,F,W) = (M,F,W)$ *and* $j^*j_{!Hdg}(M,F,W) = (M,F,W)$. *For* $(M',F,W) \in C(DRM(S))$, *there is, by construction,*

- *a canonical map* $\mathrm{ad}(j^*, j_{*Hdg})(M',F,W) : (M',F,W) \to j_{*Hdg}j^*(M',F,W)$ *in* $C(DRM(S))$,
- *a canonical map* $\mathrm{ad}(j_{!Hdg}, j^*)(M',F,W) : j_{!Hdg}j^*(M',F,W) \to (M',F,W)$ *in* $C(DRM(S))$.

Let $j : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{SmVar}(k)$. For $(M,F,W) \in C(DRM(S^o))$,

- we have the canonical map in $C_{\mathcal{D}(1,0)fil}(S)$

$$T(j_{*Hdg}, j_*)(M,F,W) := k \circ \mathrm{ad}(j^*, j_*)(j_{*Hdg}(M,F,W)) : j_{*Hdg}(M,F,W) \to j_*E(M,F,W),$$

- we have the canonical map in $C_{\mathcal{D}(1,0)fil}(S)$

$$T(j_!, j_{!Hdg})(M,F,W) := \mathbb{D}^K_S L_D(k \circ \mathrm{ad}(j^*, j_*)(-)) :$$
$$j_!(M,F,W) := \mathbb{D}^K_S L_D j_*E(\mathbb{D}^K_S(M,F,W)) \to \mathbb{D}^K_S L_D j_{*Hdg}\mathbb{D}^K_S(M,F,W) = j_{!Hdg}(M,F,W).$$

Remark 4. *Let* $j : S^o \hookrightarrow S$ *an open embedding, with* $S \in \mathrm{SmVar}(k)$. *Then, for* $(M,F,W) \in DRM(S^o)$,

- *the map* $T(j_!, j_{!Hdg})(M,W) : j_{!w}(M,W) \to j_{!Hdg}(M,W)$ *in* $C_{\mathcal{D}0fil}(S)$ *is a filtered quasi-isomorphism (by the acyclicity of the functor* j_* *in the divisor case).*
- *the map* $T(j_{*Hdg}, j_*)(M,W) : j_{*Hdg}(M,W) \to j_{*w}(M,W)$ *in* $C_{\mathcal{D}0fil}(S)$ *is a filtered quasi-isomorphism (by the acyclicity of the functor* j_* *in the divisor case).*

Hence, for $(M,F,W) \in DRM(S^o)$*,*

- *we get, for all* $p, n \in \mathbb{N}$*, monomorphisms*

$$F^pH^nT(j_!, j_{!Hdg})(M,F,W) : F^pH^nj_{!w}(M,F,W) \hookrightarrow F^pH^nj_{!Hdg}(M,F,W)$$

  in $\mathrm{PSh}_{O_S}(S)$*, but* $F^pH^nj_{!w}(M,F,W) \neq F^pH^nj_{!Hdg}(M,F,W)$ *(it leads to different F-filtrations), since* $F^pH^nj_!(M,F) \subset H^nj_!M$ *are sub* D_S *module while the F-filtration on* $H^nj_{!Hdg}(M,F)$ *is given by Kashiwara-Malgrange V-filtrations, hence satisfy a non trivial Griffith transversality property, thus* $H^nj_!(M,F)$ *and* $H^nj_{!Hdg}(M,F)$ *are isomorphic as* D_S*-modules but NOT isomorphic as filtered* D_S*-modules.*

- *we get, for all* $p, n \in \mathbb{N}$*, monomorphisms*

$$T(j_{*Hdg}, j_*)(M,F,W) : F^pH^nj_{*Hdg}(M,F,W) \hookrightarrow F^pH^nj_{*w}(M,F,W)$$

  in $\mathrm{PSh}_{O_S}(S)$*, but* $F^pH^nj_{*Hdg}(M,F,W) \neq F^pH^nj_{*w}(M,F,W)$ *(it leads to different F-filtrations), since* $F^pH^nj_*E(M,F) \subset H^nj_*E(M)$ *are sub* D_S *module while the F-filtration on* $H^nj_{*Hdg}(M,F)$ *is given by Kashiwara-Malgrange V-filtrations, hence satisfy a non trivial Griffith transversality property, thus* $H^nj_*E(M,F)$ *and* $H^nj_{*Hdg}(M,F)$ *are isomorphic as* D_S*-modules but NOT isomorphic as filtered* D_S*-modules.*

Proposition 37. *(i) Let* $S \in \mathrm{SmVar}(k)$*. Let* $D = V(s) \subset S$ *a divisor with* $s \in \Gamma(S,L)$ *and* L *a line bundle (*S *being smooth,* D *is Cartier). Denote by* $j : S^o := S\backslash D \hookrightarrow S$ *the open complementary embedding. Then,*

  - $(j^*, j_{*Hdg}) : DRM(S) \leftrightarrows DRM(S^o)$ *is a pair of adjoint functors*
  - $(j_{!Hdg}, j^*) : DRM(S^o) \leftrightarrows DRM(S)$ *is a pair of adjoint functors.*

(ii) Let $S \in \mathrm{SmVar}(k)$*. Let* $Z = V(\mathcal{I}) \subset S$ *an arbitrary closed subset,* $\mathcal{I} \subset O_S$ *being an ideal subsheaf. Denote by* $j : S^o := S\backslash Z \hookrightarrow S$*. Then,*

  - $(j^*, j_{*Hdg}) : D(DRM(S)) \leftrightarrows D(DRM(S^o))$ *is a pair of adjoint functors*
  - $(j_{!Hdg}, j^*) : D(DRM(S^o)) \leftrightarrows D(DRM(S))$ *is a pair of adjoint functors.*

Proof. (i): Follows from the fact that for $(M,F) \in DRM(S)$, we have $F^pV_{D,<0}M = j_*F^pj^*M \cap V_{D,<0}M$, where $V_{D,p}M := i^\sharp V_{S,p}i_{*mod}M$.
(ii):Follows from (i). □

The map given in definition 56 induces the following:

Definition 60. *Let* k *a field of caracteristic* 0*. Let* $S \in \mathrm{SmVar}(k)$ *and* $D \subset S$ *a (Cartier) divisor. Let* $(M,F,W) \in DRM(S)$*. The map of definition 56 given by theorem 31 and theorem 34 in* $\mathrm{PSh}_{\mathcal{D},rh}(S)$

$$\rho_{DR,D}(M) := DR(S)^{-,-1}(\mathrm{ad}(i^*, i_*)(-) \circ \mathrm{ad}(\pi^*, \pi_*)(DR(S)(M))) : j_*M \to \psi_D(M).$$

induces, similarly to the complex case ([25]) by theorem 31, the unicity of the V-filtration, and the definition of the monomdromy filtration, the following map in $\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S)$

$$\rho_{DR,D}(M,F,W) := \rho_{DR,D}(M) : j_{*Hdg}(M,F,W) \to \psi_D(M,F,W).$$

Proposition 38. *(i) Let $(M,F,W)\in DRM(S)$. Let $S^o\subset S$ an open subset such that $M_{|S^o}\in \mathrm{Vect}_{\mathcal{D}}(S^o)$. Denote $i:D:=S\backslash D\hookrightarrow S$ the closed embedding and $j:S^o\hookrightarrow S$ the open embedding. We have the canonical quasi-isomorphism in $C_{\mathcal{D},rh}(S)$ given in theorem 37:*

$$Is(M):=(0,(\mathrm{ad}(j^*,j_*)(M),\rho_{DR,D}(M)\circ\mathrm{ad}(j^*,j_*)(M)),0):$$
$$M\to(\psi_D M\xrightarrow{(c(x_{S^o/S}(M)),can(M))}x_{S^o/S}(M)\oplus\phi_D M\xrightarrow{(0,s\partial s),var(M))}\psi_D M).$$

gives a filtered quasi-isomorphism in $C_{\mathcal{D}(1,0)fil,rh}(S)$

$$Is(M,F,W):=(0,(\mathrm{ad}(j^*,j_{*Hdg})(M,F,W),\rho_{DR,D}(M,F,W)\circ\mathrm{ad}(j^*,j_{*Hdg})(M,F,W)),0):$$
$$(M,F,W)\to(\psi_D(M,F,W)\xrightarrow{(c(x_{S^o/S}(M,F,W)),can(M,F,W))}x_{S^o/S}(M,F,W)\oplus\phi_D(M,F,W)$$
$$\xrightarrow{(0,s\partial s),var(M,F,W))}\psi_D(M,F,W)).$$

with, see definition 60,

$$x_{S^o/S}(M,F,W):=\mathrm{Cone}(\rho_{DR,D}(M,F,W):j_{*Hdg}(M,F,W)\to\psi_D(M,F,W))\in C_{\mathcal{D}(1,0)fil,rh}(S)$$

(ii) Let $S\in\mathrm{SmVar}(k)$. Let $D=V(s)\subset S$ a (Cartier) divisor, where $s\in\Gamma(S,L)$ is a section of the line bundle $L=L_D$ associated to D. We then have the zero section embedding $i:S\hookrightarrow L$. We denote $L_0=i(S)$ and $j:L^o:=L\backslash L_0\hookrightarrow L$ the open complementary subset. We denote by $DRM(S\backslash D)\times_J DRM(D)$ the category whose set of objects consists of

$$\{(\mathcal{M},\mathcal{N},a,b),\mathcal{M}\in DRM(S\backslash D),\mathcal{N}\in DRM(D),a:\psi_{D1}\mathcal{M}\to N,b:N\to\psi_{D1}M\}$$

The functor (see definition 57)

$$(j^*,\phi_{D1},can,var):DRM(S)\to DRM(S\backslash D)\times_J DRM(D),$$
$$(M,F,W)\mapsto(j^*(M,F,W),\phi_{D1}(M,F,W),can(M,F,W),var(M,F,W))$$

is an equivalence of category.

Proof. (i):Similar to the complex case ([25]) by theorem 31, the unicity of the V-filtration and the definition of the monodromy weight filtration.
(ii): follows from (i). □

We make the following key definition

Definition 61. *Let $S\in\mathrm{SmVar}(k)$. Let $Z\subset S$ a closed subset. Denote by $j:S\backslash Z\hookrightarrow S$ the complementary open embedding.*

(i) We define using definition 59, the filtered Hodge support section functor

$$\Gamma_Z^{Hdg}:C(DRM(S))\to C(DRM(S)),$$
$$(M,F,W)\mapsto\Gamma_Z^{Hdg}(M,F,W):=\mathrm{Cone}(\mathrm{ad}(j^*,j_{*Hdg})(M,F,W):(M,F,W)\to j_{*Hdg}j^*(M,F,W))[-1],$$

together we the canonical map $\gamma_Z^{Hdg}(M,F,W):\Gamma_Z^{Hdg}(M,F,W)\to(M,F,W)$.

(i)' Since $j_{*Hdg}:C(DRM(S^o))\to C(DRM(S))$ is an exact functor, Γ_Z^{Hdg} induces the functor

$$\Gamma_Z^{Hdg}:D(DRM(S))\to D(DRM(S)),\ (M,F,W)\mapsto\Gamma_Z^{Hdg}(M,F,W)$$

(ii) We define using definition 59, the dual filtered Hodge support section functor

$$\Gamma_Z^{\vee,Hdg}:C(DRM(S))\to C(DRM(S)),$$
$$(M,F,W)\mapsto\Gamma_Z^{\vee,Hdg}(M,F,W):=\mathrm{Cone}(\mathrm{ad}(j_{!Hdg},j^*)(M,F,W):j_{!Hdg},j^*(M,F,W)\to(M,F,W)),$$

together we the canonical map $\gamma_Z^{\vee,Hdg}(M,F,W):(M,F,W)\to\Gamma_Z^{\vee,Hdg}(M,F,W)$.

(ii)' Since $j_{!Hdg} : C(DRM(S^o)) \to C(DRM(S))$ *is an exact functor,* $\Gamma_Z^{\vee,Hdg}$ *induces the functor*

$$\Gamma_Z^{\vee,Hdg} : D(DRM(S)) \to D(DRM(S)), \ (M,F,W) \mapsto \Gamma_Z^{\vee,Hdg}(M,F,W)$$

We now give the definition of the filtered Hodge inverse image functor :

Definition 62. *(i) Let* $i : Z \hookrightarrow S$ *be a closed embedding, with* $Z, S \in \mathrm{SmVar}(k)$*. Then, for* $(M,F,W) \in C(DRM(S))$*, we set*

$$i^{*mod}_{Hdg}(M,F,W) := i^* \mathrm{Gr}_{V_Z,0} \Gamma_Z^{Hdg}(M,F,W) \in D(DRM(Z))$$

and

$$i^{\hat{*}mod}_{Hdg}(M,F,W) := i^* \mathrm{Gr}_{V_Z,0} \Gamma_Z^{\vee,Hdg}(M,F,W) \in D(DRM(Z)),$$

noting that $i_{*mod} : D(DRM(Z)) \to D(DRM_Z(S))$ *is an equivalence of category whose inverse is* $i^* \mathrm{Gr}_{V_Z,0} : D(DRM_Z(S)) \to D(DRM(Z))$.

(ii) Let $f : X \to S$ *be a morphism, with* $X, S \in \mathrm{SmVar}(k)$*. Consider the factorization* $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$*, where* i *is the graph embedding and* $p_S : X \times S \to S$ *is the projection.*

- *For* $(M,F,W) \in C(DRM(S))$ *we set*

$$f^{*mod}_{Hdg}(M,F,W) := i^{*mod}_{Hdg} p_S^{*mod[-]}(M,F,W)(d_X)[2d_X] \in D(DRM(X)),$$

- *For* $(M,F,W) \in C(DRM(S))$ *we set*

$$f^{\hat{*}mod}_{Hdg}(M,F,W) := i^{\hat{*}mod}_{Hdg} p_S^{*mod[-]}(M,F,W) \in D(DRM(X)),$$

If $j : S^o \hookrightarrow S$ *is a closed embedding, we have, for* $(M,F,W) \in C(DRM(S))$,

$$j^{*mod}_{Hdg}(M,F,W) = j^{\hat{*}mod}_{Hdg}(M,F,W) = j^*(M,F,W) \in D(DRM(S^o)))$$

(iii) Let $f : X \to S$ *be a morphism, with* $X, S \in \mathrm{SmVar}(k)$*. Consider the factorization* $f : X \xrightarrow{i} X \times S \xrightarrow{p_S} S$*, where* i *is the graph embedding and* $p_S : X \times S \to S$ *is the projection.*

- *For* $(M,F,W) \in C(DRM(S))$ *we set*

$$f^{*mod}_{Hdg}(M,F,W) := \Gamma_X^{Hdg} p_S^{*mod[-]}(M,F,W)(d_X)[2d_X] \in C(DRM(X \times S)),$$

- *For* $(M,F,W) \in C(DRM(S)))$ *we set*

$$f^{\hat{*}mod}_{Hdg}(M,F,W) := \Gamma_X^{\vee,Hdg} p_S^{*mod[-]}(M,F,W) \in C(DRM(X \times S)),$$

Definition-Proposition 6. *(i) Let* $g : S' \to S$ *a morphism with* $S', S \in \mathrm{SmVar}(k)$ *and* $i : Z \hookrightarrow S$ *a closed subset. Then, for* $(M,F,W) \in C(DRM(S)))$*, there is a canonical map in* $C(DRM_{S'}(S' \times S))$

$$T^{Hdg}(g,\gamma)(M,F,W) : g^{*mod,\Gamma}_{Hdg} \Gamma_Z^{Hdg}(M,F,W) \to \Gamma^{Hdg}_{Z \times_S S'} g^{*mod,\Gamma}_{Hdg}(M,F,W)$$

unique up to homotopy such that

$$\gamma^{Hdg}_{Z \times_S S'}(g^{*mod,\Gamma}_{Hdg}(M,F,W)) \circ T^{Hdg}(g,\gamma)(M,F,W) = g^{*mod,\Gamma}_{Hdg} \gamma_Z^{Hdg}(M,F,W).$$

(i)' Let $g : S' \to S$ *a morphism with* $S', S \in \mathrm{SmVar}(k)$ *and* $i : Z \hookrightarrow S$ *a closed subset. Then, for* $(M,F,W) \in C(DRM(S)))$*, there is a canonical isomorphism in* $C(DRM_{S'}(S' \times S))$

$$T^{Hdg}(g,\gamma^{\vee})(M,F,W) : \Gamma^{Hdg}_{Z \times_S S'} g^{\hat{*}mod,\Gamma}_{Hdg}(M,F,W) \xrightarrow{\sim} g^{\hat{*}mod,\Gamma}_{Hdg} \Gamma_Z^{Hdg}(M,F,W)$$

unique up to homotopy such that

$$\gamma^{\vee,Hdg}_{Z \times_S S'}(g^{\hat{*}mod,\Gamma}_{Hdg}(M,F,W)) \circ g^{\hat{*}mod,\Gamma}_{Hdg} \gamma_Z^{\vee,Hdg}(M,F,W) = T^{Hdg}(g,\gamma)(M,F,W).$$

(ii) Let $S \in \mathrm{SmVar}(k)$ and $i_1 : Z_1 \hookrightarrow S$, $i_2 : Z_2 \hookrightarrow Z_1$ be closed embeddings. Then, for $(M,F,W) \in C(DRM(S)))$,

- *there is a canonical map $T(Z_2/Z_1,\gamma^{Hdg})(M,F,W) : \Gamma^{Hdg}_{Z_2}(M,F,W) \to \Gamma^{Hdg}_{Z_1}(M,F,W)$ in $C(DRM(S)))$ unique up to homotopy such that*

$$\gamma^{Hdg}_{Z_1}(M,F,W) \circ T(Z_2/Z_1,\gamma^{Hdg})(M,F,W) = \gamma^{Hdg}_{Z_2}(M,F,W)$$

together with a distinguish triangle in $K(DRM(S))$

$$\Gamma^{Hdg}_{Z_2}(M,F,W) \xrightarrow{T(Z_2/Z_1,\gamma^{Hdg})(M,F,W)} \Gamma^{Hdg}_{Z_1}(M,F,W)$$
$$\xrightarrow{\mathrm{ad}(j_2^*,j_{2*}^{Hdg})(\Gamma^{Hdg}_{Z_1}(M,F,W))} \Gamma^{Hdg}_{Z_1/\backslash Z_2}(G,F) \to \Gamma^{Hdg}_{Z_2}(M,F,W)[1]$$

- *there is a canonical map $T(Z_2/Z_1,\gamma^{\vee,Hdg})(M,F,W) : \Gamma^{\vee,Hdg}_{Z_1}(M,F,W) \to \Gamma^{\vee,Hdg}_{Z_2}(M,F,W)$ in $C(DRM(S)))$ unique up to homotopy such that*

$$\gamma^{\vee,Hdg}_{Z_2}(M,F,W) = T(Z_2/Z_1,\gamma^{\vee,Hdg})(M,F,W) \circ \gamma^{\vee,Hdg}_{Z_1}(M,F,W).$$

together with a distinguish triangle in $K(DRM(S))$

$$\Gamma^{\vee,Hdg}_{Z_1\backslash Z_2}(M,F,W) \xrightarrow{\mathrm{ad}(j_{2!}^{Hdg},j_2^*)(M,F,W)} \Gamma^{\vee,Hdg}_{Z_1}(M,F,W)$$
$$\xrightarrow{T(Z_2/Z_1,\gamma^{\vee,Hdg})(M,F,W))} \Gamma^{\vee,Hdg}_{Z_2}(M,F,W) \to \Gamma^{\vee,Hdg}_{Z_2\backslash Z_1}(M,F,W)[1]$$

Proof. Follows from the projection case and the closed embedding case using the adjonction maps. □

The definitions 61 and 62 immediately extends to the non smooth case :

Definition 63. *Let $S \in \mathrm{Var}(k)$. Let $Z \subset S$ a closed subset. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Denote $Z_I := Z \cap S_I$. Denote by $j : S\backslash Z \hookrightarrow S$ and $\tilde{j}_I : \tilde{S}_I\backslash Z_I \hookrightarrow \tilde{S}_I$ the complementary open embeddings.*

(i) We define using definition 59, the filtered Hodge support section functor

$$\Gamma^{Hdg}_Z : C(DRM(S))) \to C(DRM(S))),\ ((M_I,F,W),u_{IJ}) \mapsto \Gamma^{Hdg}_Z((M_I,F,W),u_{IJ}) :=$$
$$\mathrm{Cone}(\mathrm{ad}(j^*,j_{*Hdg})(M_I,F,W),u_{IJ}) := (\mathrm{ad}(\tilde{j}_I^*,\tilde{j}_{I*Hdg})(M_I,F,W)) :$$
$$((M_I,F,W),u_{IJ}) \to (\tilde{j}_{I*Hdg}\tilde{j}_I^*(M_I,F,W),\tilde{j}_{J*Hdg}u_{IJ}))[-1],$$

together with the canonical map $\gamma^{Hdg}_Z((M_I,F,W),u_{IJ}) : \Gamma^{Hdg}_Z((M_I,F,W),u_{IJ}) \to ((M_I,F,W),u_{IJ})$.

(i)' Since $\tilde{j}_{I*Hdg} : C(DRM(\tilde{S}_I\backslash S_I)) \to C(DRM(\tilde{S}_I))$ are exact functors, Γ^{Hdg}_Z induces the functor

$$\Gamma^{Hdg}_Z : D(DRM(S)) \to D(DRM(S)), ((M_I,F,W),u_{IJ}) \mapsto \Gamma^{Hdg}_Z((M_I,F,W),u_{IJ})$$

(ii) We define using definition 59, the dual filtered Hodge support section functor

$$\Gamma^{\vee,Hdg}_Z : C(DRM(S)) \to C(DRM(S)),$$
$$((M_I,F,W),u_{IJ}) \mapsto \Gamma^{\vee,Hdg}_Z((M_I,F,W),u_{IJ}) := \mathbb{D}^{Hdg}_S\Gamma^{Hdg}_Z\mathbb{D}^{Hdg}_S((M_I,F,W),u_{IJ}) =$$
$$\mathrm{Cone}(\mathrm{ad}(j_{!Hdg},j^*)((M_I,F,W),u_{IJ}) := (\mathrm{ad}(\tilde{j}_{I!Hdg},\tilde{j}_I^*)(M_I,F,W)) :$$
$$(\tilde{j}_{I!Hdg}\tilde{j}_I^*(M_I,F,W),(\tilde{j}_{J*Hdg}u^d_{IJ})^d) \to ((M_I,F,W),u_{IJ})),$$

together we the canonical map $\gamma^{\vee,Hdg}_Z((M_I,F,W),u_{IJ}) : ((M_I,F,W),u_{IJ}) \to \Gamma^{\vee,Hdg}_Z((M_I,F,W),u_{IJ})$.

(ii)' Since $\tilde{j}_{I!Hdg} : C(DRM(\tilde{S}_I\backslash S_I)) \to C(DRM(\tilde{S}_I))$ *are exact functors,* $\Gamma_Z^{Hdg,\vee}$ *induces the functor*

$$\Gamma_Z^{\vee,Hdg} : D(DRM(S)) \to D(DRM(S)),\ ((M_I,F,W),u_{IJ}) \mapsto \Gamma_Z^{\vee,Hdg}((M_I,F,W),u_{IJ})$$

Definition 64. *Let* $f : X \to S$ *a morphism with* $X, S \in \mathrm{Var}(k)$. *Assume there exist a factorization* $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ *with* $Y \in \mathrm{SmVar}(k)$, l *a closed embedding and* p_S *the projection. Let* $S = \cup_{i\in I}$ *an open cover such that there exist closed embeddings* $i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{SmVar}(k)$. *Denote* $X_I := f^{-1}(S_I)$. *We have then* $X = \cup_{i\in I} X_i$ *and the commutative diagrams*

$$\begin{tikzcd} f : X_I \arrow[r, "l_I"] \arrow[dr] & Y \times S_I \arrow[r, "p_{S_I}"] \arrow[d, "i'_I:=(I\times i_I)"] & S_I \arrow[d, "i_I"] \\ & Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I}=:\tilde{f}_I"] & \tilde{S}_I \end{tikzcd}$$

(i) For $((M_I,F,W),u_{IJ}) \in C(DRM(S))$ *we set (see definition 63 for l)*

$$f_{Hdg}^{*mod}((M_I,F,W),u_{IJ}) := \Gamma_X^{Hdg}(p_{\tilde{S}_I}^{*mod[-]}(M_I,F,W),u_{IJ})(d_Y)[2d_Y] \in C(DRM(X)),$$

(ii) For $((M_I,F,W),u_{IJ}) \in C(DRM(S)))$ *we set (see definition 63 for l)*

$$f_{Hdg}^{\hat{*}mod}(M,F,W) := \Gamma_X^{\vee,Hdg}(p_{\tilde{S}_I}^{*mod[-]}(M_I,F,W),p_{\tilde{S}_I}^{*mod[-]}u_{IJ}) \in C(DRM(X)),$$

Let $j : S^o \hookrightarrow S$ *an open embedding with* $S \in \mathrm{Var}(k)$. *Let* $S = \cup_{i\in I}$ *an open cover such that there exist closed embeddings* $i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{SmVar}(k)$. *We have then, for* $((M_I,F,W),u_{IJ}) \in C(DRM(S))$, *quasi-isomorphisms in* $C(DRM(S))$

$$I(j^*, j_{Hdg}^{*mod})(-) : j^*((M_I,F,W),u_{IJ}) := (\tilde{j}_I^*(M_I,F,W),\tilde{j}_I^* u_{IJ}) \to j_{Hdg}^{*mod}((M_I,F,W),u_{IJ})$$

and

$$I(j^*, j_{Hdg}^{\hat{*}mod})(-) : j_{Hdg}^{\hat{*}mod}((M_I,F,W),u_{IJ}) \to j^*((M_I,F,W),u_{IJ}) := (\tilde{j}_I^*(M_I,F,W),\tilde{j}_I^* u_{IJ}).$$

Definition 65. *Let* $S \in \mathrm{Var}(k)$. *Let* $S = \cup_{i\in I}$ *an open cover such that there exist closed embeddings* $i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_i \in \mathrm{SmVar}(k)$. *We have the following functor*

$$\begin{aligned} (-) \otimes^{Hdg} (-) : D(DRM(S))^2 \to D(DRM(S)), \\ (((M,F,W),u_{IJ}),((N,F,W),v_{IJ})) \mapsto ((M,F,W),u_{IJ}) \otimes_{O_S}^{Hdg} ((N,F,W),v_{IJ}) := \\ \Delta_{S,Hdg}^{*mod}(p_{1I}^{*mod}(M_I,F,W) \otimes_{O_{\tilde{S}_I\times\tilde{S}_I}} p_{2I}^{*mod}(N_I,F,W), p_{1I}^{*mod}u_{IJ} \otimes p_{2I}^{*mod}v_{IJ}) := \\ \Delta_S^* \mathrm{Gr}_{V_{\Delta_S},0} \Gamma_{\Delta_S}^{\vee,Hdg}(p_{1I}^{*mod}(M_I,F,W) \otimes_{O_{\tilde{S}_I\times\tilde{S}_I}} p_{2I}^{*mod}(N_I,F,W), p_{1I}^{*mod}u_{IJ} \otimes p_{2I}^{*mod}v_{IJ}) \end{aligned}$$

using the definition 64 for the diagonal closed embedding $\Delta_S : S \hookrightarrow S \times S$.

Proposition 39. *Let* $f_1 : X \to Y$ *and* $f_2 : Y \to S$ *two morphism with* $X, Y, S \in \mathrm{QPVar}(k)$.

(i) Let $(M,F,W) \in C(DRM(S)))$. *Then,*

$$(f_2 \circ f_1)_{Hdg}^{*mod}(M,F,W) = f_{1Hdg}^{*mod} f_{2Hdg}^{*mod}(M,F,W) \in D(DRM(X)).$$

(ii) Let $(M,F,W) \in C(DRM(S)))$. *Then,*

$$(f_2 \circ f_1)_{Hdg}^{\hat{*}mod}(M,F,W) = f_{1Hdg}^{\hat{*}mod} f_{2Hdg}^{\hat{*}mod}(M,F,W) \in D(DRM(X))$$

Proof. Immediate from definition. □

Theorem 41. *(i) Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i\in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then the full embedding*

$$\iota_S : DRM(S) \hookrightarrow \mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))$$

induces a full embedding

$$\iota_S : D(DRM(S) \hookrightarrow D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))$$

whose image consists of $((M_I, F, W), u_{IJ}) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))$ such that $(H^n(M_I, F, W), H^n(u_{IJ})) \in DRM(S)$ for all $n \in \mathbb{Z}$ and such that for all $p \in \mathbb{Z}$, the differentials of $\mathrm{Gr}^p_W(M_I, F)$ are strict for the filtrations F.

(i)' Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i\in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We have then

$$\begin{aligned} D(DRM(S) =&< \int_{p_S} (n \times I)_{!Hdg}(\Gamma^{\vee,Hdg}_X(O_{\mathbb{P}^{N,o}\times\tilde{S}_I}, F_b), x_{IJ}),\ (f : X \xrightarrow{l} \mathbb{P}^{N,o} \times S \xrightarrow{p_S} S) \in \mathrm{QPVar}(k) > \\ =&< \int_{p_S} (\Gamma^{\vee,Hdg}_X(O_{\mathbb{P}^{N,o}\times\tilde{S}_I}, F_b), x_{IJ})(f : X \xrightarrow{l} \mathbb{P}^N \times S \xrightarrow{p_S} S) \in \mathrm{QPVar}(k),\ \textit{proper},\ X \textit{smooth} > \\ & \subset D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \end{aligned}$$

where $n : \mathbb{P}^{N,o} \hookrightarrow \mathbb{P}^N$ are open embeddings, l are closed embedding and $<,>$ means the full triangulated category generated by.

(ii) Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i\in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then the full embedding

$$\iota_S : DRM(S) \hookrightarrow \mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))$$

induces a full embedding

$$\iota_S : D(DRM(S)) \hookrightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I))$$

whose image consists of $((M_I, F, W), u_{IJ}) \in D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I))$ such that $(H^n(M_I, F, W), H^n(u_{IJ})) \in DRM(S)$ for all $n \in \mathbb{Z}$ and such that there exist $r \in \mathbb{Z}$ and an r-filtered homotopy equivalence $((M_I, F, W), u_{IJ}) \to ((M'_I, F, W), u_{IJ})$ such that for all $p \in \mathbb{Z}$ the differentials of $\mathrm{Gr}^p_W(M'_I, F)$ are strict for the filtrations F.

Proof. (i):We first show that ι_S is fully faithfull, that is for all $\mathcal{M} = ((M_I, F, W), u_{IJ}), \mathcal{M}' = ((M'_I, F, W), u_{IJ}) \in DRM(S)$ and all $n \in \mathbb{Z}$,

$$\begin{aligned} \iota_S : \mathrm{Ext}^n_{D(DRM(S))}(\mathcal{M}, \mathcal{M}') &:= \mathrm{Hom}_{D(DRM(S))}(\mathcal{M}, \mathcal{M}'[n]) \\ \to \mathrm{Ext}^n_{\mathcal{D}(S)_0}(\mathcal{M}, \mathcal{M}') &:= \mathrm{Hom}_{\mathcal{D}(S)_0 := D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))}(\mathcal{M}, \mathcal{M}'[n]) \end{aligned}$$

For this it is enough to assume S smooth. We then proceed by induction on $max(\dim \mathrm{supp}(M), \dim \mathrm{supp}(M'))$.

- For $\mathrm{supp}(M) = \mathrm{supp}(M') = \{s\}$, it is obvious. If $\mathrm{supp}(M) = \{s\}$ and $\mathrm{supp}(M') = \{s'\}$ wit $s' \neq s$, then by the localization exact sequence

$$\mathrm{Ext}^n_{D(MHM(S))}(\mathcal{M}, \mathcal{M}') = 0 = \mathrm{Ext}^n_{\mathcal{D}(S)}(\mathcal{M}, \mathcal{M}')$$

- Denote $\mathrm{supp}(M) = Z \subset S$ and $\mathrm{supp}(M') = Z' \subset S$. There exist an open subset $S^o \subset S$ such that $Z^o := Z \cap S^o$ and $Z'^o := Z' \cap S^o$ are smooth, and $\mathcal{M}_{|Z^o} := (i^* \mathrm{Gr}_{V_{Z^o},0} M_{|S^o}, F, W) \in DRM(Z^o)$ and $\mathcal{M}'_{|Z'^o} := (i'^* \mathrm{Gr}_{V_{Z'^o},0} M'_{|S^o}, F, W) \in DRM(Z'^o)$ are filtered vector bundles, where $j : S^o \hookrightarrow S$ is the open embedding, and $i : Z^o \hookrightarrow S^o$, $i : Z'^o \hookrightarrow S^o$ the closed embeddings. Considering the connected components of Z^o and Z'^o, we way assume that Z^o and Z'^o are connected. Shrinking S^o if necessary, we may assume that either $Z^o = Z'^o$ or $Z^o \cap Z'^o = \emptyset$, We denote $D = S \backslash S^o$. Shrinking S^o if necessary, we may assume that D is a divisor and denote by $l : S \hookrightarrow L_D$ the zero section embedding.
  - If $Z^o = Z'^o$, denote $i : Z^o \hookrightarrow S^o$ the closed embedding. We have then the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^n_{D(DRM(S^o))}(\mathcal{M}_{|S^o}, \mathcal{M}'_{|S^o}) \arrow[r, "\iota_{S^o}"] \arrow[d, shift right, "i^* \mathrm{Gr}_{V_{Z^o},0}"'] & \mathrm{Ext}^n_{\mathcal{D}(S^o)_0}(\mathcal{M}_{|S^o}, \mathcal{M}'_{|S^o}) \arrow[d, shift left, "i^* \mathrm{Gr}_{V_{Z^o},0}"] \\
\mathrm{Ext}^n_{D(DRM(Z^o))}(\mathcal{M}_{|Z^o}, \mathcal{M}'_{|Z^o}) \arrow[r, "\iota_{Z^o}"] \arrow[u, shift right, "i_{*mod}"'] & \mathrm{Ext}^n_{\mathcal{D}(Z^o)_0}(\mathcal{M}_{|Z^o}, \mathcal{M}'_{|Z^o}) \arrow[u, shift left, "i_{*mod}"]
\end{tikzcd}$$

    Now we prove that ι_{Z^o} is an isomorphism similarly to the proof the the generic case of [6]. On the other hand the left and right colummn are isomorphisms. Hence ι_{S^o} is an isomorphism by the diagram.
  - If $Z^o \cap Z'^o = \emptyset$, we consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^n_{D(DRM(S^o))}(\mathcal{M}_{|S^o}, \mathcal{M}'_{|S^o}) \arrow[r, "\iota_{S^o}"] \arrow[d, shift right, "i^* \mathrm{Gr}_{V_{Z^o},0}"'] & \mathrm{Ext}^n_{\mathcal{D}(S^o)_0}(\mathcal{M}_{|S^o}, \mathcal{M}'_{|S^o}) \arrow[d, shift left, "i^* \mathrm{Gr}_{V_{Z^o},0}"] \\
\mathrm{Ext}^n_{D(DRM(Z^o))}(\mathcal{M}_{|Z^o}, 0) = 0 \arrow[r, "\iota_{Z^o}"] \arrow[u, shift right, "i_{*mod}"'] & \mathrm{Ext}^n_{\mathcal{D}(Z^o)_0}(\mathcal{M}_{|Z^o}, 0) = 0 \arrow[u, shift left, "i_{*mod}"]
\end{tikzcd}$$

    where the left and right column are isomorphism by strictness of the V_{Z^o} filtration (use a bi-filtered injective resolution with respect to F and V_{Z^o} for the right column).

- We consider now the following commutative diagram in $C(\mathbb{Z})$ where we denote for short $H_0 := D(DRM(S))$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}^\bullet_{H_0}(\Gamma^{\vee,Hdg}_D \mathcal{M}, \Gamma^{Hdg}_D \mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\gamma^{Hdg}_D(\mathcal{M}'))"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_{H_0}(\Gamma^{\vee,Hdg}_D \mathcal{M}, \mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\mathrm{ad}(j^*,j_{*Hdg})(\mathcal{M}'))"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_{H_0}(\Gamma^{\vee,Hdg}_D \mathcal{M}, j_{*Hdg} j^* \mathcal{M}') \arrow[r] \arrow[d, "\iota_S"] & {} \\
0 \arrow[r] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)_0}(\Gamma^{\vee,Hdg}_D \mathcal{M}, \Gamma^{Hdg}_D \mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\gamma^{Hdg}_D(\mathcal{M}'))"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)_0}(\Gamma^{\vee,Hdg}_D \mathcal{M}, \mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\mathrm{ad}(j^*,j_{*Hdg})(\mathcal{M}'))"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)_0}(\Gamma^{\vee,Hdg}_D \mathcal{M}, j_{*Hdg} j^* \mathcal{M}') \arrow[r] & {}
\end{tikzcd}$$

  whose lines are exact sequence. We have on the one hand,

$$\mathrm{Hom}^\bullet_{D(DRM(S))}(\Gamma^{\vee,Hdg}_D \mathcal{M}, j_{*Hdg} j^* \mathcal{M}') = 0 = \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D \mathcal{M}, j_{*Hdg} j^* \mathcal{M}')$$

  On the other hand by induction hypothesis

$$\iota_S : \mathrm{Hom}^\bullet_{D(DRM(S))}(\Gamma^{\vee,Hdg}_D \mathcal{M}, \Gamma^{Hdg}_D \mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)_0}(\Gamma^{\vee,Hdg}_D \mathcal{M}, \Gamma^{Hdg}_D \mathcal{M}')$$

  is a quasi-isomorphism. Hence, by the diagram

$$\iota_S : \mathrm{Hom}^\bullet_{D(DRM(S))}(\Gamma^{\vee,Hdg}_D \mathcal{M}, \mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)_0}(\Gamma^{\vee,Hdg}_D \mathcal{M}, \mathcal{M}')$$

  is a quasi-isomorphism.

- We consider now the following commutative diagram in $C(\mathbb{Z})$ where we denote for short $H_0 := D(DRM(S))$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}^{\bullet}_{H_0}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(\gamma^{\vee,Hdg}_D(\mathcal{M}),-)"] \arrow[d, "\iota_S"] & \mathrm{Hom}^{\bullet}_{H_0}(\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(\mathrm{ad}(j_{!Hdg},j^*)(\mathcal{M}'),-)"] \arrow[d, "\iota_S"] & \mathrm{Hom}^{\bullet}_{H_0}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \arrow[r] \arrow[d, "\iota_S"] & 0 \\
0 \arrow[r] & \mathrm{Hom}^{\bullet}_{\mathcal{D}(S)_0}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(\gamma^{\vee,Hdg}_D(\mathcal{M}),-)"] & \mathrm{Hom}^{\bullet}_{\mathcal{D}(S)_0}(\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(\mathrm{ad}(j_{!Hdg},j^*)(\mathcal{M}),-)"] & \mathrm{Hom}^{\bullet}_{\mathcal{D}(S)_0}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \arrow[r] & 0
\end{tikzcd}$$

whose lines are exact sequence. On the one hand, the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^{\bullet}_{D(DRM(S))}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \arrow[r, "j^*"] \arrow[d, "\iota_S"] & \mathrm{Hom}^{\bullet}_{D(DRM(S^o))}(j^*\mathcal{M},j^*\mathcal{M}') \arrow[d, "\iota_{S^o}"] \\
\mathrm{Hom}^{\bullet}_{\mathcal{D}(S)_0}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \arrow[r, "j^*"] & \mathrm{Hom}^{\bullet}_{\mathcal{D}(S^o)_0}(j^*\mathcal{M},j^*\mathcal{M}')
\end{tikzcd}$$

together with the fact that the horizontal arrows j^* are quasi-isomorphism by the functoriality given the uniqueness of the V_S filtration for the embedding $l : S \hookrightarrow L_D$, (use a bi-filtered injective resolution with respect to F and V_S for the lower arrow) and the fact that ι_{S^o} is a quasi-isomorphism by the first two point, show that

$$\iota_S : \mathrm{Hom}^{\bullet}_{D(DRM(S))}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \to \mathrm{Hom}^{\bullet}_{\mathcal{D}(S)_0}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}')$$

is a quasi-isomorphism. On the other hand, by the third point

$$\iota_S : \mathrm{Hom}^{\bullet}_{D(DRM(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \to \mathrm{Hom}^{\bullet}_{\mathcal{D}(S)_0}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}')$$

is a quasi-isomorphism. Hence, by the diagram

$$\iota_S : \mathrm{Hom}^{\bullet}_{D(DRM(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \to \mathrm{Hom}^{\bullet}_{\mathcal{D}(S)_0}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}')$$

is a quasi-isomorphism.

This shows the fully faithfulness. We now prove the essential surjectivity : let

$$((M_I,F,W),u_{IJ}) \in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))$$

such that the cohomology are mixed hodge modules and such that the differential are strict. We proceed by induction on $card\{n \in \mathbb{Z}\}$, s.t.$H^n(M_I,F,W) \neq 0$ by taking the cohomological troncation and using the fact that the differential are strict for the filtration F and the fully faithfullness.
(i)':Follows from (i).
(ii):Follows from (i). Indeed, in the composition of functor

$$\iota_S : D(DRM(S)) \xrightarrow{\iota_S} D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \to D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I))$$

the second functor which is the localization functor is an isomorphism on the full subcategory $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))^{st} \subset D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))$ constisting of complex such that the differentials are strict for F, and the first functor ι_S is a full embedding by (i) and $\iota_S(D(DRM(S))) \subset D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))^{st}$. □

Definition 66. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(k)$. Consider a compactification $f : X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} S$ of f, in particular j is an open embedding and $\bar{f}$ is proper.*

(i) *For* $(M,F,W)\in C(DRM(X))$*, we define, using definition 59,*

$$\int_f^{Hdg}(M,F,W):=\int_{\bar{f}}^{FDR}j_{*Hdg}(M,F,W)\in D_{\mathcal{D}(1,0)fil}(S)$$

It does not depends on the choice of the compactification as in the complex case: for two compactification $\bar{f}:\bar{X}\to S$*,* $\bar{f}':X'\to S$*, there exist a compactification* $\bar{f}'':\bar{X}''\to S$ *together with morphisms* $e:\bar{X}''\to\bar{X}$ *and* $e':\bar{X}''\to\bar{X}'$ *such that* $\bar{f}\circ e=\bar{f}\circ e'=\bar{f}''$*. Let* $(M,F,W)\in C(DRM(X)))$*, then*

- *by definition* $H^i\int_{\bar{f}}^{FDR}\mathrm{Gr}_W^k j_{*Hdg}(M,F,W)\in PDRM(S)$ *for all* $i,k\in\mathbb{Z}$*, hence by the spectral sequence for the filtered complex* $\int_{\bar{f}}^{FDR}j_{*Hdg}(M,W)$

$$\mathrm{Gr}_W^k(H^i\int_f^{Hdg}(M,F,W))=\mathrm{Gr}_W^k(H^i\int_{\bar{f}}^{FDR}j_{*Hdg}(M,F,W))\in PDRM(S)$$

  since it is a sub-quotient of $H^i\int_{\bar{f}}^{FDR}\mathrm{Gr}_W^k j_{*Hdg}(M,F,W)$*, this gives by definition* $H^i\int_f^{Hdg}(M,F,W))\in DRM(S)$ *for all* $i\in\mathbb{Z}$*.*
- $\int_f^{Hdg}(M,F,W)$ *is the class of a complex such that the differential are strict for* F *by theorem 40 in the complex case*

We then set using theorem 41

$$Rf_*^{Hdg}(M,F,W):=\iota_S^{-1}\int_f^{Hdg}(M,F,W)\in D(DRM(S))$$

(ii) *For* $(M,F,W)\in\in C(DRM(X))$*, we define, using definition 59,*

$$\int_{f!}^{Hdg}(M,F,W):=\int_{\bar{f}}^{FDR}j_{!Hdg}(M,F,W)\in D_{\mathcal{D}(1,0)fil}(S)$$

It does not depends on the choice of the compactification as in the complex case: for two compactification $\bar{f}:\bar{X}\to S$*,* $\bar{f}':X'\to S$*, there exist a compactification* $\bar{f}'':\bar{X}''\to S$ *together with morphisms* $e:\bar{X}''\to\bar{X}$ *and* $e':\bar{X}''\to\bar{X}'$ *such that* $\bar{f}\circ e=\bar{f}\circ e'=\bar{f}''$*. Let* $(M,F,W)\in C(DRM(X)))$*, then*

- *by definition* $H^i\int_{\bar{f}}^{FDR}\mathrm{Gr}_W^k j_{!Hdg}(M,F,W)\in PDRM(S)$ *for all* $i,k\in\mathbb{Z}$*, hence by the spectral sequence for the filtered complex* $\int_{\bar{f}}^{FDR}j_{!Hdg}(M,W)$

$$\mathrm{Gr}_W^k(H^i\int_{f!}^{Hdg}(M,F,W))==\mathrm{Gr}_W^k(H^i\int_{\bar{f}}^{FDR}j_{!Hdg}(M,F,W))\in PDRM(S)$$

  since it is a sub-quotient of $H^i\int_{\bar{f}}^{FDR}\mathrm{Gr}_W^k j_{!Hdg}(M,F,W)$*, this gives by definition* $H^i\int_{f!}^{Hdg}(M,F,W))\in DRM(S)$ *for all* $i\in\mathbb{Z}$*.*
- $\int_{f!}^{Hdg}(M,F,W)$ *is the class of a complex such that the differential are strict for* F *by theorem 40 in the complex case.*

We then set using theorem 41

$$Rf_!^{Hdg}(M,F,W):=\iota_S^{-1}\int_{f!}^{Hdg}(M,F,W)\in D(DRM(S))$$

In the singular case, we set the following

Definition 67. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. Let $\bar{Y} \in \mathrm{PSmVar}(k)$ a compactification of Y and denote by $n : Y \hookrightarrow \bar{Y}$ the open embedding. Denote again $p_S : \bar{Y} \times S \to S$ the projection. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We have then the open cover $X = \cup_i X_i$ with $X_i := f^{-1}(S_i)$ together with closed embeddings $i'_I : X_I :\hookrightarrow Y \times \tilde{S}_I$.*

(i) For $((M_I, F, W), u_{IJ}) \in C(DRM(X))$, we define, using definition 59 for $n \times I : Y \times S \hookrightarrow \bar{Y} \times S$,

$$\int_{p_S}^{Hdg} ((M_I, F, W), u_{IJ}) := \int_{p_S}^{FDR} (n \times I)_{*Hdg}((M_I, F, W), u_{IJ}) \in D_{\mathcal{D}(1,0)fil}(S/\tilde{S}_I)$$

with

$$(n \times I)_{*Hdg}((M_I, F, W), u_{IJ}) := ((n \times I)_{*Hdg}(M_I, F, W), (n \times I)_{*Hdg} u_{IJ}) \in C(\bar{X}/(\bar{Y} \times \tilde{S}_I))$$

We then set using theorem 41 and theorem 40

$$Rf_*^{Hdg}((M_I, F, W), u_{IJ}) := \iota_S^{-1} \int_{p_S}^{Hdg} ((M_I, F, W), u_{IJ}) \in D(DRM(S)).$$

(ii) For $((M_I, F, W), u_{IJ}) \in C(DRM(X))$, we define, using definition 59 for $n \times I : Y \times S \hookrightarrow \bar{Y} \times S$,

$$\int_{p_S!}^{Hdg} ((M_I, F, W), u_{IJ}) := \int_{p_S}^{FDR} (n \times I)_{!Hdg}((M_I, F, W), u_{IJ}) \in D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$$

with

$$(n \times I)_{!Hdg}((M_I, F, W), u_{IJ}) := ((n \times I)_{!Hdg}(M_I, F, W), ((n \times I)_{*Hdg} u_{IJ}^d)^d) \in C(\bar{X}/(\bar{Y} \times \tilde{S}_I))$$

We then set using theorem 41 and theorem 40

$$Rf_!^{Hdg}((M_I, F, W), u_{IJ}) := \iota_S^{-1} \int_{p_S!}^{Hdg} ((M_I, F, W), u_{IJ}) \in D(DRM(S)).$$

Proposition 40. *Let $f_1 : X \to Y$ and $f_2 : Y \to S$ two morphism with $X, Y, S \in \mathrm{QPVar}(k)$ or with $X, Y, S \in \mathrm{SmVar}(k)$.*

(i) Let $(M, F, W) \in C(DRM(X))$. Then,

$$R(f_2 \circ f_1)_*^{Hdg}(M, F, W) = Rf_{2*}^{Hdg} Rf_{1*}^{Hdg}(M, F, W) \in D(DRM(S)).$$

(ii) Let $(M, F, W) \in C(DRM(X))$. Then,

$$R(f_2 \circ f_1)_!^{Hdg}(M, F, W) = Rf_{2!}^{Hdg} Rf_{1!}^{Hdg}(M, F, W) \in D(DRM(S))$$

Proof. Immediate from definition. □

Proposition 41. *Let $f : X \to S$ with $S, X \in \mathrm{SmVar}(k)$ or with $S, X \in \mathrm{QPVar}(k)$. Then*

(i) $(f_{Hdg}^{\hat{*}mod}, Rf_*^{Hdg}) : D(DRM(S)) \to D(DRM(X))$ is a pair of adjoint functors. For $(M, F, W) \in C(DRM(S))$ we denote by

$$\mathrm{ad}(f_{Hdg}^{\hat{*}mod}, Rf_*^{Hdg})(M, F, W) : (M, F, W) \to Rf_*^{Hdg} f_{Hdg}^{\hat{*}mod}(M, F, W)$$

the adjonction map in $D(DRM(S))$. For $(N, F, W) \in C(DRM(X))$, we denote by

$$\mathrm{ad}(f_{Hdg}^{\hat{*}mod}, Rf_*^{Hdg})(N, F, W) : f_{Hdg}^{\hat{*}mod} Rf_*^{Hdg}(N, F, W) \to (N, F, W)$$

the adjonction map in $D(DRM(X))$.

(ii) $(Rf_!^{Hdg}, f^{*mod}_{Hdg}) : D(DRM(X)) \to D(DRM(S))$ *is a pair of adjoint functors. For* $(M, F, W) \in C(DRM(S))$ *we denote by*

$$\mathrm{ad}(Rf_!^{Hdg}, f^{*mod}_{Hdg})(M, F, W) : Rf_!^{Hdg} f^{*mod}_{Hdg}(M, F, W) \to (M, F, W)$$

the adjonction map in $D(DRM(S))$*. For* $(N, F, W) \in C(DRM(X))$*, we denote by*

$$\mathrm{ad}(Rf_!^{Hdg}, f^{*mod}_{Hdg})(N, F, W) : (N, F, W) \to f^{*mod}_{Hdg} Rf_!^{Hdg}(N, F, W)$$

the adjonction map in $D(DRM(X))$*.*

Proof. Follows from proposition 37 after considering the graph factorization $f : X \hookrightarrow \bar{X} \times S \xrightarrow{p_S} S$ with $\bar{X} \in \mathrm{PSmVar}(k)$ a compactification of X. □

We have by proposition 39 and proposition 40 the 2 functors on $\mathrm{QPVar}(k)$:

- $D(DRM(-)) : \mathrm{QPVar}(k) \to D(DRM(-)),\ S \mapsto D(DRM(S)),\ (f : T \to S) \mapsto Rf_*^{Hdg}$,
- $D(DRM(-)) : \mathrm{QPVar}(k) \to D(DRM(-)),\ S \mapsto D(DRM(S)),\ (f : T \to S) \mapsto Rf_!^{Hdg}$,
- $D(DRM(-)) : \mathrm{QPVar}(k) \to D(DRM(-)),\ S \mapsto D(DRM(S)),\ (f : T \to S) \mapsto f^{*mod}_{Hdg}$,
- $D(DRM(-)) : \mathrm{QPVar}(k) \to D(DRM(-)),\ S \mapsto D(DRM(S)),\ (f : T \to S) \mapsto f^{\hat{*}mod}_{Hdg}$.

Proposition 42. *Let* $f : X \to S$ *with* $S, X \in \mathrm{SmVar}(k)$ *or with* $S, X \in \mathrm{QPVar}(k)$*. Then*

(i) $(f^{\hat{*}mod}_{Hdg}, Rf_*^{Hdg}) : D(DRM(S)) \to D(DRM(X))$ *is a pair of adjoint functors. For* $((M_I, F, W), u_{IJ}) \in C(DRM(S))$ *we denote by*

$$\mathrm{ad}(f^{\hat{*}mod}_{Hdg}, Rf_*^{Hdg})((M_I, F, W), u_{IJ}) : ((M_I, F, W),) \to Rf_*^{Hdg} f^{\hat{*}mod}_{Hdg}((M_I, F, W), u_{IJ})$$

the adjonction map in $D(DRM(S))$*. For* $((N_I, F, W), u_{IJ}) \in C(DRM(X))$*, we denote by*

$$\mathrm{ad}(f^{\hat{*}mod}_{Hdg}, Rf_*^{Hdg})((N_I, F, W), u_{IJ}) : f^{\hat{*}mod}_{Hdg} Rf_*^{Hdg}((N_I, F, W), u_{IJ}) \to ((N_I, F, W), u_{IJ})$$

the adjonction map in $D(DRM(X))$

(ii) $(Rf_!^{Hdg}, f^{*mod}_{Hdg}) : D(DRM(X)) \to D(DRM(S))$ *is a pair of adjoint functors. For* $((M_I, F, W), u_{IJ}) \in C(DRM(S))$ *we denote by*

$$\mathrm{ad}(Rf_!^{Hdg}, f^{*mod}_{Hdg})((M_I, F, W), u_{IJ}) : Rf_!^{Hdg} f^{*mod}_{Hdg}((M_I, F, W), u_{IJ}) \to ((M_I, F, W), u_{IJ})$$

the adjonction map in $D(DRM(S))$*. For* $((N_I, F, W), u_{IJ}) \in C(DRM(X))$*, we denote by*

$$\mathrm{ad}(Rf_!^{Hdg}, f^{*mod}_{Hdg})((N_I, F, W), u_{IJ}) : ((N_I, F, W), u_{IJ}) \to f^{*mod}_{Hdg} Rf_!^{Hdg}((N_I, F, W), u_{IJ})$$

the adjonction map in $D(DRM(X))$*.*

Proof. Follows from proposition 37 after considering a factorization $f : X \hookrightarrow \bar{Y} \times S \xrightarrow{p_S} S$ with $\bar{Y} \in \mathrm{PSmVar}(k)$. □

Theorem 42. *Let* k *a field of characteristic zero.*

(i) We have the six functor formalism on $D(DRM(-)) : \mathrm{SmVar}(k) \to \mathrm{TriCat}$.

(ii) We have the six functor formalism on $D(DRM(-)) : \mathrm{QPVar}(k) \to \mathrm{TriCat}$.

Proof. Follows from proposition 42. □

Theorem 43. *Let $k\subset K\subset\mathbb{C}_p$ a subfield with p a prime number and K a p adic field. Let $S\in\mathrm{SmVar}(k)$. Let $D=V(s)\subset S$ a divisor with $s\in\Gamma(S,L)$ and L a line bundle (S being smooth, D is Cartier). so that we have the closed embedding $i:S\hookrightarrow L$, $i(x)=(x,s(x))$ and $D=i^{-1}(s_0)$, s_0 being the zero section. For $(M,F,W)\in DRM(S)$,*

- *we have the canonical isomorphism in $D_{\mathbb{B}_{dr}}(S_K^{an,pet})$*

$$T^{B_{dr}}(\psi_D,DR)(M,F,W):=B^{B_{dr}}(M,F,W)\circ A^{B_{dr}}(M,F,W)^{-1}: DR(S)(\psi_D(M,F,W)^{an}\otimes_{O_{S_K}}(O\mathbb{B}_{dr,S_K},F))\xrightarrow{\sim}\psi_D DR(S)((M,F,W)^{an}\otimes_{O_{S_K}}(O\mathbb{B}_{dr,S_K},F))[-1]$$

  with, for $S=\cup_{i=1}^s S_i$ an open affine cover such that $D\cap S_i=V(f_i)\subset S_i$ is given by $f_i\in\Gamma(S_i,O_{S_i})$, denoting $q:L_i:=p^{-1}(S_i)\to\mathbb{A}_k^1$ the projection and $j_i:S_i\hookrightarrow S$ the open embeddings,

  - *the isomorphism in $D_{\mathbb{B}_{dr}2fil}(S_K^{an,pet})$*

$$\begin{aligned}A^{B_{dr}}(M,F,W):(\oplus_{i=1}^s\oplus_{-1\leq\alpha<0}\mathrm{Cone}(\partial_s:DR(L_i/\mathbb{A}_k^1)((V_{D\alpha}(M,F^*,W))^{an})\otimes_{O_S}s^{\alpha+1}O\mathbb{B}_{dr,S_K}\\ \to DR(L_i/\mathbb{A}_k^1)((V_{D\alpha}(M,F^{*-1},W)^{an})\otimes_{O_S}s^{\alpha}O\mathbb{B}_{dr,S_K}))\xrightarrow{(j_I^*)}\cdots)[-1]\\ \to(\oplus_{i=1}^s DR(S_i)(\psi_D(M,F,W)^{an}\otimes_{O_{S_K}}(O\mathbb{B}_{dr,S_K},F))\xrightarrow{j_I^*}\cdots)[-1]\\ \xrightarrow{((j_i^*),0)^{-1}}(DR(S)(\psi_D(M,F,W)^{an})\otimes_{O_{S_K}}(O\mathbb{B}_{dr,S_K},F)),\ (\sum_j m_j\otimes(\log s)^j,m')\mapsto[m_0],\end{aligned}$$

  - *and the isomorphism in $D_{\mathbb{B}_{dr}2fil}(S_K^{an,pet})$*

$$\begin{aligned}B^{B_{dr}}(M,F,W):(\oplus_{i=1}^s\oplus_{-1\leq\alpha<0}\mathrm{Cone}(\partial_s:V_{D\alpha}DR(L_i/\mathbb{A}_k^1)((M,F^*,W)^{an})\otimes_{O_{S_K}}s^{\alpha+1}O\mathbb{B}_{dr,S_K}\\ \to V_{D\alpha}DR(L_i/\mathbb{A}_k^1)((M,F^{*-1},W)^{an})\otimes_{O_{S_K}}s^{\alpha}O\mathbb{B}_{dr,S_K})\xrightarrow{(j_I^*)}\cdots)[-1]\\ \to(\oplus_{i=1}^s DR(p^*O_{\mathbb{A}_1^k})(i^*\pi_*\pi^{*mod}DR(L_i/\mathbb{A}_k^1)((M,F,W)^{an})\otimes_{O_{S_K}}(O\mathbb{B}_{dr,S_K},F))\xrightarrow{(j_I^*)}\cdots)[-1]\\ \xrightarrow{=}(\oplus_{i=1}^s\psi_D DR(p^*O_{\mathbb{A}_1^k})(DR(L_i/\mathbb{A}_k^1)((M,F,W)^{an})\otimes_{O_{S_K}}(O\mathbb{B}_{dr,S_K},F))\xrightarrow{(j_I^*)}\cdots)[-1]\\ \xrightarrow{((j_i^*),0)^{-1}}\psi_D DR(S)((M,F,W)^{an}\otimes_{O_{S_K}}(O\mathbb{B}_{dr,S_K},F))[-1],\\ (\sum_j m_j\otimes(\log s)^j,m')\mapsto\sum_j(\log s)^j m_j,\end{aligned}$$

  so that $T_{B_{dr}}(\psi_D,DR)(M)\circ DR(S)((s\partial_s)\otimes I)=N\circ T_{B_{dr}}(\psi_D,DR)(M)$ where

$$N:=\log T_u\in\mathrm{Hom}(\psi_D DR(S)(M^{an}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K}),\psi_D DR(S)(M^{an}\otimes_{O_{S_K}}O\mathbb{B}_{dr,S_K})),$$

  is induced by the monodromy automorphism $T:\tilde{S}^o\xrightarrow{\sim}\tilde{S}^o$ of the perfectoid universal covering $\pi:\tilde{S}^o\to S^o:=S\backslash D$ (see [27]).

- *there is a canonical isomorphism in $D_{\mathbb{B}_{dr}2fil}(S_K^{an,pet})$*

$$\begin{aligned}T^{B_{dr}}(\phi_D,DR)(M,F,W):DR(S)(\phi_D(M,F,W)^{an}\otimes_{O_{S_K}}(O\mathbb{B}_{dr,S_K},F))\\ \xrightarrow{DR(S)(0,(var(M,F,W)\otimes I))}DR(S)(\phi_D^{\rho}(M,F,W)^{an}\otimes_{O_{S_K}}(O\mathbb{B}_{dr,S_K},F))\\ \xrightarrow{(I,T^{B_{dr}}(M,F,W))}\phi_D DR(S)((M,F,W)^{an}\otimes_{O_{S_K}}(O\mathbb{B}_{dr,S_K},F))[-1].\end{aligned}$$

  where

$$\phi_D^{\rho}(M,F,W):=\mathrm{Cone}(\theta_{DR,D}(M,F,W):\Gamma_D^{\vee,Hdg}(M,F,W)\to\psi_D(M,F,W))$$

with $\theta_{DR,D}(M,F,W)$ the factorization in $C_{\mathcal{D}(1,0)fil,rh}(S)$

$$\rho_{DR,D}(M,F,W)\circ\mathrm{ad}(j^*,j_{*Hdg})(M,F,W)):$$
$$(M,F,W)\xrightarrow{\gamma_D^{\vee,Hdg}(M,F,W)}\Gamma_D^{\vee,Hdg}(M,F,W)\xrightarrow{\theta_{DR,D}(M)}\psi_D(M,F,W).$$

of the map given in definition 60.

Proof. Follows from the proof of theorem 38: $j_i^*A^{B_{dr}}(M,F,W)$ are filtered quasi-isomorphism, hence

$$\mathrm{Cone}(\partial_s:V_{D\alpha}DR(L_i/\mathbb{A}^1_k)((M,F^*,W)^{an})\otimes_{O_{S_K}}s^{\alpha+1}O\mathbb{B}_{dr,S_K}$$
$$\to V_{D\alpha}DR(L_i/\mathbb{A}^1_k)((M,F^{*-1},W)^{an})\otimes_{O_{S_K}}s^{\alpha}O\mathbb{B}_{dr,S_K})$$

is strict for the F-filtration (i.e. the spectral sequence for the F-filtration is E_1-degenerate, hence the fact that $j_i^*B^{B_{dr}}(M,F,W)$ is a quasi-isomorphism implies that $j_i^*B^{B_{dr}}(M,F,W)$ is a filtered quasi-isomorphism. □

6 The geometric Mixed Hodge Modules over a field k of characteristic 0

6.1 The complex case where $k\subset\mathbb{C}$

Let $k\subset\mathbb{C}$ a subfield. For $S\in\mathrm{Var}(k)$, we denote by $\mathrm{an}_S:S^{an}:=S^{an}_{\mathbb{C}}\xrightarrow{\mathrm{an}_S}S_{\mathbb{C}}\xrightarrow{\pi_{k/\mathbb{C}}(S)}S$ the morphism of ringed spaces given by the analytical functor.

- For $(M,F)\in C_{O_Sfil}(S)$, we denote by $(M,F)^{an}:=\mathrm{an}_S^{*mod}(M,F)\in C_{O_Sfil}(S^{an}_{\mathbb{C}})$.
- For $(M,F)\in C_{\mathcal{D}fil}(S)$, we denote by $(M,F)^{an}:=\mathrm{an}_S^{*mod}(M,F)\in C_{\mathcal{D}fil}(S^{an}_{\mathbb{C}})$.

We denote for short

$$DR(S):=DR(S^{an}_{\mathbb{C}})\circ\mathrm{an}_S^{*mod}:C_{\mathcal{D}fil}(S)\to C_{fil}(S^{an}_{\mathbb{C}}),\ M\mapsto DR(S)(M^{an})$$

the De Rham functor.

- Let $S\in\mathrm{SmVar}(k)$. The category $C_{\mathcal{D}(1,0)fil,rh}(S)\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$ is the category
  - whose set of objects is the set of triples $\{((M,F,W),(K,W),\alpha)\}$ with

    $(M,F,W)\in C_{\mathcal{D}(1,0)fil,rh}(S),\ (K,W)\in D_{fil,c,k}(S^{an}_{\mathbb{C}}),\ \alpha:(K,W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}}\to DR(S)^{[-]}((M,W)^{an})$

    where $DR(S)^{[-]}:=DR(S)^{[-]}(S^{an}_{\mathbb{C}}):C_{\mathcal{D}(1,0)fil,rh}(S^{an}_{\mathbb{C}})\to C_{fil}(S^{an}_{\mathbb{C}})$ is the De Rahm functor (recall for $S'\subset S$ a connected component of S of dimension d, $DR(S)^{[-]}_{|S'}:=DR(S)_{|S'}[d]$) and α is an morphism in $D_{fil}(S^{an}_{\mathbb{C}})$,
  - and whose set of morphisms are

    $$\phi=(\phi_D,\phi_C,[\theta]):((M_1,F,W),(K_1,W),\alpha_1)\to((M_2,F,W),(K_2,W),\alpha_2)$$

    where $\phi_D:(M_1,F,W)\to(M_2,F,W)$ and $\phi_C:(K_1,W)\to(K_2,W)$ are morphisms, and

    $\theta=(\theta^\bullet,I(DR(S)(\phi_D^{an}))\circ I(\alpha_1),I(\alpha_2)\circ I(\phi_C\otimes I)):I(K_1,W)\otimes\mathbb{C}_{S^{an}}[1]\to I(DR(S)(M_2^{an},W))$

    is an homotopy, i.e. for all $i\in\mathbb{Z}$,

    $$\theta^i\circ\partial^i-\partial^{i+1}\circ\theta^i=(I(DR(S)(\phi_D^{an}))\circ I(\alpha_1))^i-(I(\alpha_2)\circ I(\phi_C\otimes I))^i,$$

$I : C_{fil}(S^{an}_{\mathbb{C}}) \to K_{fil}(S^{an}_{\mathbb{C}})$ being the injective resolution functor : for $(K,W) \in C_{fil}(S^{an}_{\mathbb{C}})$, we take an injective resolution $k : (K,W) \to I(K,W)$ with $I(K,W) \in C_{fil}(S^{an}_{\mathbb{C}})$ which is unique modulo homotopy, and the class $[\theta]$ of θ does NOT depend of the injective resolution ; in particular, we have

$$DR(S)^{[-]}(\phi_D^{an}) \circ \alpha_1 = \alpha_2 \circ (\phi_C \otimes I)$$

in $D_{fil}(S^{an}_{\mathbb{C}})$; and for

* $\phi = (\phi_D, \phi_C, [\theta]) : ((M_1,F,W),(K_1,W),\alpha_1) \to ((M_2,F,W),(K_2,W),\alpha_2)$
* $\phi' = (\phi'_D, \phi'_C, [\theta']) : ((M_2,F,W),(K_2,W),\alpha_2) \to ((M_3,F,W),(K_3,W),\alpha_3)$

the composition law is given by

$$\phi' \circ \phi := (\phi'_D \circ \phi_D, \phi'_C \circ \phi_C, I(DR(S)(\phi_D^{'an})) \circ [\theta] + [\theta'] \circ I(\phi_C \otimes I)[1]) : \\ ((M_1,F,W),(K_1,W),\alpha_1) \to ((M_3,F,W),(K_3,W),\alpha_3),$$

in particular for $((M,F,W),(K,W),\alpha) \in C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$,

$$I_{((M,F,W),(K,W),\alpha)} = (I_M, I_K, 0).$$

We have then the full embedding

$$\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil,k}(S^{an}_{\mathbb{C}}) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$$

where $\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{fil,k}(S^{an}_{\mathbb{C}})$ is the category

- whose set of objects is the set of triples $\{((M,F,W),(K,W),\alpha)\}$ with

  $(M,F,W) \in \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S),\ (K,W) \in P_{fil,k}(S^{an}_{\mathbb{C}}),\ \alpha : (K,W) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}} \to DR(S)^{[-]}((M,W)^{an})$

  where $DR(S)^{[-]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S^{an}_{\mathbb{C}})$,
- and whose set of morphisms are

  $$\phi = (\phi_D, \phi_C) = (\phi_D, \phi_C, 0) : ((M_1,F,W),(K_1,W),\alpha_1) \to ((M_2,F,W),(K_2,W),\alpha_2)$$

  where $\phi_D : (M_1,F,W) \to (M_2,F,W)$ and $\phi_C : (K_1,W) \to (K_2,W)$ are morphisms (of filtered sheaves) and $DR(S)^{[-]}(\phi_D^{an}) \circ \alpha_1 = \alpha_2 \circ (\phi_C \otimes I)$ in $P_{fil,k}(S^{an}_{\mathbb{C}})$.

- Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I \in \mathrm{SmVar}(k)$. The category $C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$ is the category
  - whose set of objects is the set of triples $\{(((M_I,F,W),u_{IJ}),(K,W),\alpha)\}$ with

    $$((M_I,F,W),u_{IJ}) \in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)),\ (K,W) \in D_{fil,c,k}(S^{an}_{\mathbb{C}}), \\ \alpha : T(S/(\tilde{S}_I))((K,W) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}}) \to DR(S)^{[-]}(((M_I,W),u_{IJ})^{an})$$

    where

    $$DR(S)^{[-]} := DR(S^{an}_{\mathbb{C}})^{[-]} : C_{\mathcal{D}(1,0)fil,rh}(S^{an}/(\tilde{S}^{an}_{I,\mathbb{C}})) \to C_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$$

    is the De Rahm functor and α is a morphism in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$,
  - and whose set of morphisms consists of

    $$\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1) \to (((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2)$$

    where $\phi_D : ((M_1,F,W),u_{IJ}) \to ((M_2,F,W),u_{IJ})$ and $\phi_C : (K_1,W) \to (K_2,W)$ are morphisms, and

    $$\theta = (\theta^{\bullet}, I(DR(S)(\phi_D^{an})) \circ I(\alpha_1), I(\alpha_2) \circ I(\phi_C \otimes I)) : \\ I(T(S/(\tilde{S}_I))((K_1,W) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}}))[1] \to I(DR(S)(((M_{2I},W),u_{IJ})^{an}))$$

is an homotopy, $I : C_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}})) \to K_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$ being the injective resolution functor : for $((K_I, W), t_{IJ}) \in C_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$, we take an injective resolution

$$k : ((K_I, W), t_{IJ}) \to I((K_I, W), t_{IJ})$$

with $I((K, W), t_{IJ}) \in C_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$ which is unique modulo homotopy, and the class $[\theta]$ of θ does NOT depend of the injective resolution ; in particular we have

$$DR(S)^{[-]}(\phi^{an}_D) \circ \alpha_1 = \alpha_2 \circ (\phi_C \otimes I)$$

in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$; and for

- $\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$
- $\phi' = (\phi'_D, \phi'_C, [\theta']) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \to (((M_{3I}, F, W), u_{IJ}), (K_3, W), \alpha_3)$

the composition law is given by

$$\phi' \circ \phi := (\phi'_D \circ \phi_D, \phi'_C \circ \phi_C, I(DR(S)(\phi'^{an}_D)) \circ [\theta] + [\theta'] \circ I(\phi_C \otimes I)[1]) :$$
$$(((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{3I}, F, W), u_{IJ}), (K_3, W), \alpha_3)$$

in particular for $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$,

$$I_{(((M_I,F,W),u_{IJ}),(K,W),\alpha)} = ((I_{M_I}), I_K, 0).$$

We have then full embeddings

$$\mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I P_{fil,k}(S^{an}_{\mathbb{C}}) \hookrightarrow C^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$$
$$\xrightarrow{\iota^0_{S/\tilde{S}_I}} C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))^0 \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}}) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$$

where $\mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I P_{fil,k}(S^{an}_{\mathbb{C}})$ is the category

- whose set of objects is the set of triples $\{(((M_I, F, W), u_{IJ}), (K, W), \alpha)\}$ with

$$((M_I, F, W), u_{IJ}) \in \mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)),\ (K, W) \in P_{fil,k}(S^{an}_{\mathbb{C}}),$$
$$\alpha : T(S/(\tilde{S}_I))((K, W) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}}) \to DR(S)^{[-]}(((M_I, W), u_{IJ})^{an})$$

  where $DR(S)^{[-]}$ is the De Rahm functor and α is an isomorphism in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$,
- and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) = (\phi_D, \phi_C, 0) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

  where $\phi_D : ((M_1, F, W), u_{IJ}) \to ((M_2, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \to (K_2, W)$ are morphisms (of filtered sheaves) such that $\phi^{an}_D \circ \alpha_1 = \alpha_2 \circ (\phi_C \otimes I)$ in $P_{fil,k}(S^{an}_{\mathbb{C}})$.

Moreover,

- For $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$, we set

$$(((M_I, F, W), u_{IJ}), (K, W), \alpha)[1] := (((M_I, F, W), u_{IJ})[1], (K, W)[1], \alpha[1]).$$

- For

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

  a morphism in $C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$, we set (see [11] definition 3.12)

$$\mathrm{Cone}(\phi) := (\mathrm{Cone}(\phi_D), \mathrm{Cone}(\phi_C), ((\alpha_1, \theta), (\alpha_2, 0))) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}}),$$

  $((\alpha_1, \theta), (\alpha_2, 0))$ being the matrix given by the composition law, together with the canonical maps

- $c_1(-) = (c_1(\phi_D), c_1(\phi_C), 0) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \to \mathrm{Cone}(\phi)$
- $c_2(-) = (c_2(\phi_D), c_2(\phi_C), 0) : \mathrm{Cone}(\phi) \to (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1)[1]$.

Remark 5. *By [11] theorem 3.25, if*

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_1, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_2, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

is a morphism in $C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$ such that ϕ_D is a Zariski local equivalence and ϕ_C is an isomorphism then ϕ is an isomorphism.

We get from [10] the following definition :

Definition 68. *(i) Let $f : X \to S$ a morphism with $S, X \in \mathrm{SmVar}(k)$. Let $f : X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} S$ a compactification of f with $\bar{X} \in \mathrm{SmVar}(k)$ and j the open embedding. Denote $Z := \bar{X} \backslash X = \cap_i Z_i$ with $Z_i \subset \bar{X}$ (Cartier) divisor. Let*

$$\alpha : (K, W) \otimes \mathbb{C}_{X^{an}_{\mathbb{C}}} \to DR(X)((M, W)^{an})$$

a morphism in $D_{fil}(X^{an}_{\mathbb{C}})$, with

$$(M, F, W) \in C(DRM(X)),\ (K, W) \in D_{fil,c,k}(X^{an}_{\mathbb{C}})^{ad,(Z_i)}.$$

We then consider the maps in $D_{fil}(S^{an}_{\mathbb{C}})$

$$\begin{aligned} f_*\alpha : Rf_{*w}(K, W) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}} &:= R\bar{f}_*Rj_{*w}(K, W) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}} \\ \xrightarrow{R\bar{f}_*j_*\alpha} R\bar{f}_*Rj_{*w}DR(X)((M, W)^{an}) &\xrightarrow{T^w(j,\otimes)(-)^{-1}} R\bar{f}_*DR(\bar{X})(j_{*Hdg}(M, W)^{an}) \\ \xrightarrow{T(\bar{f},DR)(-)^{-1}} DR(S)((\int_{\bar{f}}(j_{*Hdg}(M, W))^{an}) &= DR(S)((\int_f^{Hdg}(M, W))^{an}) \end{aligned}$$

and

$$\begin{aligned} f_!\alpha : Rf_{!w}(K, W) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}} &:= R\bar{f}_*Rj_{!w}(K, W) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}} \\ \xrightarrow{R\bar{f}_*j_!\alpha} R\bar{f}_*Rj_{!w}DR(X)((M, W)^{an}) &\xrightarrow{\mathbb{D}T^w(j,\otimes)(-)} R\bar{f}_*DR(\bar{X})(j_{!Hdg}(M, W)^{an}) \\ \xrightarrow{T(\bar{f},DR)(-)^{-1}} DR(S)((\int_{\bar{f}}(j_{!Hdg}(M, W))^{an}) &= DR(S)((\int_{f!}^{Hdg}(M, W))^{an}), \end{aligned}$$

see definition 6 and definition 66 .

(ii) Let $f : X \to S$ a morphism with $S, X \in \mathrm{QPVar}(k)$. Consider a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. Let $\bar{Y} \in \mathrm{PSmVar}(k)$ a smooth compactification of Y with $j : Y \hookrightarrow \bar{Y}$ the open embedding. Then $\bar{f} : \bar{X} \xrightarrow{\bar{l}} \bar{Y} \times_S \xrightarrow{\bar{p}} S$ is a compactification of f, with $\bar{X} \subset \bar{Y} \times S$ the closure of X and $\bar{l}$ the closed embedding. Denote $Z := \bar{X} \backslash X = \cap_i Z_i$ with $Z_i \subset \bar{X}$ Cartier divisors. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Let

$$\alpha : T(X/(Y \times \tilde{S}_I))((K, W) \otimes \mathbb{C}_{X^{an}_{\mathbb{C}}}) \to DR(X)((M_I, W)^{an}, u^{an}_{IJ})$$

a morphism in $D_{fil}(X^{an}_{\mathbb{C}}/(Y \times \tilde{S}_{I\mathbb{C}}^{an}))$, with

$$((M_I, W), u_{IJ}) \in C(DRM(X)) \subset C_{\mathcal{D}0fil,rh}(X/(Y \times \tilde{S}_I)),\ (K, W) \in D_{fil,c,k}(X^{an}_{\mathbb{C}})^{ad,(Z_i)}.$$

We then consider the maps in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$

$$\begin{aligned}
&f_*\alpha = f_*(\alpha) : T(S/\tilde{S}_I)(Rf_{*w}(K,W)) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}} \\
&\xrightarrow{:=} T(S/\tilde{S}_I)(R\bar{p}_*(I\times j)_{*w}(K,W)) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}} \xrightarrow{=} R\bar{p}_*(I\times j)_{*w}T(X/(Y\times\tilde{S}_I))((K,W)\otimes\mathbb{C}_{X^{an}_{\mathbb{C}}}) \\
&\xrightarrow{Rp_*\alpha} R\bar{p}_*(I\times j)_{*w}DR(X)(((M_I,W),u_{IJ})^{an}) \\
&\xrightarrow{(T^w(I\times j,\otimes)(-))} R\bar{p}_*DR(X)((I\times j)_{*Hdg}((M_I,W),u_{IJ})^{an}) \\
&\xrightarrow{T(\bar{f},DR)(-)} DR(S)((\int_{\bar{f}}(I\times j)_{*Hdg}((M_I,W),u_{IJ}))^{an}) = DR(S)((\int_f^{Hdg}((M_I,W),u_{IJ}))^{an})
\end{aligned}$$

and

$$\begin{aligned}
&f_!\alpha = f_!(\alpha) : T(S/\tilde{S}_I)(Rf_{!w}(K,W)) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}} \\
&\xrightarrow{:=} T(S/\tilde{S}_I)(R\bar{p}_*(I\times j)_{*w}(K,W)) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}} \xrightarrow{=} R\bar{p}_*(I\times j)_{!w}T(X/(Y\times\tilde{S}_I))((K,W)\otimes\mathbb{C}_{X^{an}_{\mathbb{C}}}) \\
&\xrightarrow{R\bar{p}_*\mathbb{D}^vR(I\times j)_*\mathbb{D}^v\alpha} R\bar{p}_*(I\times j)_{!w}DR(X)(((M_I,W),u_{IJ})^{an}) \\
&\xrightarrow{T(D,DR)(-)\circ(\mathbb{D}T^w(I\times j,\otimes)(-))\circ T(D,DR)(-)} R\bar{p}_*DR(X)((I\times j)_{!Hdg}((M_I,W),u_{IJ})^{an}) \\
&\xrightarrow{T(\bar{f},DR)(-)} DR(S)((\int_{\bar{f}}(I\times j)_{!Hdg}((M_I,W),u_{IJ}))^{an}) = DR(S)((\int_{f!}^{Hdg}((M_I,W),u_{IJ}))^{an}),
\end{aligned}$$

see definition 6 and definition 67 .

(iii) *Let* $l : S^o \hookrightarrow S$ *an open embedding with* $S \in \mathrm{Var}(k)$ *and denote* $Z = S\backslash S^o = \cap_i Z_i$. *with* $Z_i \subset S$ *Cartier divisors. Let* $S = \cup_i S_i$ *an open affine cover and* $i_i : S_i \hookrightarrow \tilde{S}_i$ *closed embedding with* $\tilde{S}_i \in \mathrm{SmVar}(k)$. *Let* $l_I : \tilde{S}^o_I \hookrightarrow \tilde{S}_I$ *open embeddings such that* $\tilde{S}^o_I \cap S = S^o \cap S_I$. *Let*

$$\alpha : T(S/(\tilde{S}_I))((K,W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}}) \to DR(S)(((M_I,W),u_{IJ})^{an})$$

a morphism in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$, *with*

$$((M_I,W),u_{IJ}) \in C(DRM(S)) \subset C_{\mathcal{D}0fil,rh}(S/(\tilde{S}_I)),\ (K,W) \in D_{fil,c,k}(S^{an}_{\mathbb{C}}).$$

We then consider the maps in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$

$$\begin{aligned}
&\Gamma_Z(\alpha) : T(S/(\tilde{S}_I))(\Gamma^w_Z(K,W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}}) \xrightarrow{=} \Gamma^w_Z T(S/(\tilde{S}_I))((K,W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}}) \\
&\xrightarrow{R\Gamma_Z\alpha} \Gamma^w_Z DR(S)(((M_I,W),u_{IJ})^{an}) \\
&\xrightarrow{(T(\gamma_Z,DR)((M_I,W),u_{IJ})^{-1}:=((I,T^w(l_I,\otimes)(M_I,W))\circ(T^w(an,\otimes)(M_I,W)))^{-1}} DR(S)((\Gamma^{Hdg}_Z((M_I,W),u_{IJ}))^{an})
\end{aligned}$$

and

$$\begin{aligned}
&\Gamma^\vee_Z(\alpha) : T(S/(\tilde{S}_I))(\Gamma^{\vee,w}_Z(K,W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}}) \xrightarrow{=} \Gamma^\vee_Z T(S/(\tilde{S}_I))((K,W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}}) \\
&\xrightarrow{\Gamma^\vee_Z\alpha} \Gamma^{\vee,w}_Z DR(S)(((M_I,W),u_{IJ})^{an}) \\
&\xrightarrow{T(\gamma^\vee_Z,DR)((M_I,W),u_{IJ}):=(\mathbb{D}(I,T^w(l_I,\otimes)(\mathbb{D}(M_I,W)))\circ(\mathbb{D}T^w(an,\otimes)(\mathbb{D}(M_I,W))))} DR(S)((\Gamma^{\vee,w}_Z((M_I,W),u_{IJ}))^{an}),
\end{aligned}$$

see definition 5 and definition 61.

(iv) *Let* $f : X \to S$ *a morphism with* $S, X \in \mathrm{QPVar}(k)$. *Consider a factorization* $f : X \hookrightarrow Y\times S \xrightarrow{p} S$ *with* $Y \in \mathrm{SmVar}(k)$. *Let* $S = \cup_i S_i$ *an open affine cover and* $i_i : S_i \hookrightarrow \tilde{S}_i$ *closed embedding with* $\tilde{S}_i \in \mathrm{SmVar}(k)$. *Let*

$$\alpha : T(S/(\tilde{S}_I))((K,W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}}) \to DR(S)(((M_I,W),u_{IJ})^{an})$$

a morphism in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$*, with*

$$((M_I,W),u_{IJ})\in C(DRM(S))\subset C_{\mathcal{D}0fil,rh}(S/(\tilde{S}_I)),\ (K,W)\in D_{fil,c,k}(S^{an}_{\mathbb{C}})^{ad,(\Gamma_{f,i})}.$$

We then consider, see (iii), the maps in $D_{fil}(X^{an}_{\mathbb{C}}/(Y\times\tilde{S}_{I,\mathbb{C}})^{an})$

$$\begin{aligned}f^!\alpha=f^!(\alpha):T(X/(Y\times\tilde{S}_I))(f^{!w}(K,W)\otimes\mathbb{C}_{X^{an}_{\mathbb{C}}})&\xrightarrow{:=}T(X/(Y\times\tilde{S}_I))(\Gamma^w_Xp^*(K,W))\otimes\mathbb{C}_{X^{an}_{\mathbb{C}}})\\&\xrightarrow{=}(\Gamma^w_Xp^*_{\tilde{S}_I}T(S/\tilde{S}_I)((K,W)_I\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}}),\Gamma^w_Xp^*T(D_{IJ})(-))\\&\xrightarrow{R\Gamma_Xp^*\alpha}(\Gamma^w_Xp^*_{\tilde{S}_I}DR(S)((M_I,W),u_{IJ})_I,\Gamma^w_Xp^*DR(u_{IJ}))\\&\xrightarrow{T^!(f,DR)(-):=T(\gamma_X,DR)(-)^{-1}\circ T^!(p,DR)(-)}DR(X)(f^{*mod}_{Hdg}((M_I,W),u_{IJ})^{an})\end{aligned}$$

and

$$\begin{aligned}f^*\alpha=f^*(\alpha):T(X/(Y\times\tilde{S}_I))(f^{*w}(K,W)\otimes\mathbb{C}_{X^{an}_{\mathbb{C}}})&\xrightarrow{:=}T(X/(Y\times\tilde{S}_I))(\Gamma^{\vee,w}_Xp^*(K,W)\otimes\mathbb{C}_{X^{an}_{\mathbb{C}}})\\&\xrightarrow{=}(\Gamma^{\vee,w}_Xp^*_{\tilde{S}_I}T(S/(\tilde{S}_I))((K,W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}})_I,\Gamma^{\vee,w}_Xp^*T(D_{IJ})(-))\\&\xrightarrow{\Gamma^{\vee}_Xp^*\alpha}(\Gamma^{\vee,w}_Xp^*_{\tilde{S}_I}DR(S)((M_I,W),u_{IJ})_I,\Gamma^{\vee,w}_Xp^*DR(u_{IJ}))\\&\xrightarrow{T^*(f,DR)(-):=T(\gamma^{\vee}_X,DR)(-)\circ T^*(p,DR)(-)}DR(X)(f^{\hat{*}mod}_{Hdg}((M_I,W),u_{IJ})^{an})\end{aligned}$$

(v) *Let* $S\in\mathrm{Var}(k)$*.Let* $S=\cup_iS_i$ *an open affine cover and* $i_i:S_i\hookrightarrow\tilde{S}_i$ *closed embedding with* $\tilde{S}_i\in\mathrm{SmVar}(k)$*. Let*

$$\begin{aligned}\alpha&:T(S/(\tilde{S}_I))((K,W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}})\to DR(S)(((M_I,W),u_{IJ})^{an}),\\\alpha'&:T(S/(\tilde{S}_I))((K',W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}})\to DR(S)(((M'_I,W),v_{IJ})^{an})\end{aligned}$$

two morphism in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$*, with*

$$((M_I,W),u_{IJ}),((M'_I,W),u_{IJ})\in C(DRM(S))\subset C_{\mathcal{D}0fil,rh}(S/(\tilde{S}_I)),\ (K,W),(K',W)\in D_{fil}(S^{an}_{\mathbb{C}}).$$

We then consider the map in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$

$$\begin{aligned}\alpha\otimes\alpha':T(S/(\tilde{S}_I))((K,W)\otimes^{L,w}(K',W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}})\\\xrightarrow{=}T(S/(\tilde{S}_I))((K,W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}})\otimes T(S/(\tilde{S}_I))((K',W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}})\\\xrightarrow{\alpha\otimes\alpha'}DR(S)(((M_I,W),u_{IJ})^{an})\otimes^{L,w}DR(S)(((M'_I,W),v_{IJ})^{an})\\\xrightarrow{T(\otimes^w,DR)(-,-)}DR(S)((((M_I,W),u_{IJ})\otimes^{L,w}_{O_S}((M'_I,W),v_{IJ}))^{an})\\\xrightarrow{=}DR(S)((((M_I,W),u_{IJ})\otimes^{Hdg}_{O_S}((M'_I,W),v_{IJ}))^{an})\end{aligned}$$

with $T(\otimes^w,DR)(-,-):=T(\gamma_S,DR)(-)\circ T(\otimes,DR)(-,-)\circ(T(p_1,DR)(-)\otimes T(p_2,DR)(-))$.

Definition 69. *Let* $S\in\mathrm{SmVar}(k)$*. Let* $D=V(s)\subset S$ *a divisor with* $s\in\Gamma(S,L)$ *and* L *a line bundle (S being smooth, D is Cartier). For* $\mathcal{M}=((M,F,W),(K,W),\alpha)\in\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S)\times_IP_{fil,k}(S^{an}_{\mathbb{C}})$*, we then define, using definition 57 and theorem 36,*

- *the nearby cycle functor*

  $$\psi_D((M,F,W),(K,W),\alpha):=(\psi_D(M,F,W),\psi_D(K,W)[-1],\psi_D\alpha)\in\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S)\times_IP_{fil,k}(S^{an}_{\mathbb{C}}),$$

  with $\psi_D\alpha:=T(\psi_D,DR)(M)\circ\psi_D(\alpha)$.

- *the vanishing cycle functor*

  $$\phi_D((M,F,W),(K,W),\alpha) := (\phi_D(M,F,W),\phi_D(K,W)[-1],\phi_D\alpha) \in \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S)\times_I P_{fil,k}(S_{\mathbb{C}}^{an}),$$

  with $\phi_D\alpha := T(\phi_D, DR)(M)\circ\phi_D(\alpha)$.

- *the canonical maps in* $\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S)\times_I P_{fil,k}(S_{\mathbb{C}}^{an})$

  $$can(\mathcal{M}) := (can(M,F,W), can(K,W)) : \psi_D((M,F,W),(K,W),\alpha) \to \phi_D((M,F,W),(K,W),\alpha)(-),$$
  $$var(\mathcal{M}) := (var(M,F,W), var(K,W) : \phi_D((M,F,W),(K,W),\alpha) \to \psi_D((M,F,W),(K,W),\alpha).$$

Proposition 43. *Let $S\in\mathrm{SmVar}(k)$. Let $D=V(s)\subset S$ a (Cartier divisor). Consider a composition of proper morphisms $(f : X = X_r \xrightarrow{f_r} X_{r-1} \xrightarrow{f_1} X_0 = S)\in\mathrm{SmVar}(k)$ and*

$$(M,F) = H^{n_0}\int_{f_1}\cdots H^{n_r}\int_{f_r}((O_X,F_b), H^{n_0}Rf_{1*}\cdots H^{n_r}Rf_{r*}\mathbb{Z}_X,$$
$$H^{n_0}f_{1*}\circ\cdots\circ H^{n_r}f_{r*}\alpha(X))\in\mathrm{PSh}_{\mathcal{D}fil,rh}(S)\times_I P_k(S_{\mathbb{C}}^{an}).$$

Then,

$$\psi_D(M,F) = H^{n_0}\int_{f_1}\cdots H^{n_r}\int_{f_r}(\psi_{f^{-1}(D)}(O_X,F_b)), H^{n_0}Rf_{1*}\cdots H^{n_r}Rf_{r*}\psi_{f^{-1}(D)}\mathbb{Z}_X,$$
$$H^{n_0}f_{1*}\circ\cdots\circ H^{n_r}f_{r*}\psi_{f^{-1}(D)}\alpha(X))\in\mathrm{PSh}_{\mathcal{D}fil,rh}(S)\times_I P_k(S_{\mathbb{C}}^{an})$$

and

$$\phi_D(M,F) = H^{n_0}\int_{f_1}\cdots H^{n_r}\int_{f_r}(\psi_{f^{-1}(D)}(O_X,F_b)), H^{n_0}Rf_{1*}\cdots H^{n_r}Rf_{r*}\phi_{f^{-1}(D)}\mathbb{Z}_X,$$
$$H^{n_0}f_{1*}\circ\cdots\circ H^{n_r}f_{r*}\phi_{f^{-1}(D)}\alpha(X))\in\mathrm{PSh}_{\mathcal{D}fil,rh}(S)\times_I P_k(S_{\mathbb{C}}^{an}).$$

Proof. Immediate from definition. □

We now come to the main definition of this section :

Definition 70. *Let $k\subset\mathbb{C}$ a subfield.*

(i0) Let $S\in\mathrm{Var}(k)$. Take an open cover $S=\cup_i S_i$ such that there are closed embedding $S_I\hookrightarrow\tilde{S}_I$ with $S_I\in\mathrm{SmVar}(k)$. The category of mixed hodge modules over k is the full subcategory

$$MHM_{k,\mathbb{C}}(S)\subset DRM(S)\times_I P_{fil,k}(S_{\mathbb{C}}^{an})\subset\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I P_{fil,k}(S_{\mathbb{C}}^{an})$$

whose object consists of $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in DRM(S)\times_I P_{fil,k}(S_{\mathbb{C}}^{an})$ such that

$$((\pi_{k/\mathbb{C}}^{*mod}(M_I,F,W),u_{IJ}),(K,W),\alpha)\in MHM(S_{\mathbb{C}})$$

where

- *$\pi_{k/\mathbb{C}} := \pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}}\to S$ is the projection (see section 2),*
- *$DRM(S)$ is the category of de Rham modules introduced in section 5 definition 58,*
- *$MHM(S_{\mathbb{C}})$ is the category of mixed hodge modules on $S_{\mathbb{C}}$ introduced by Saito ([25]).*

(i) Let $S\in\mathrm{SmVar}(k)$. We denote by

$$HM_{gm,k,\mathbb{C}}(S) :=< (H^{n_1}\int_{f_1}\cdots H^{n_r}\int_{f_r}(O_X,F_b)(d), R^{n_1}f_{1*}\cdots R^{n_r}f_{r*}\mathbb{Z}_{X_{\mathbb{C}}^{an}}, H^{n_1}f_{1*}\cdots H^{n_r}f_{r*}\alpha(X)),$$
$$(f : X = X_r \xrightarrow{f_r} X_{r-1}\to\cdots\xrightarrow{f_1} X_0 = S)\in\mathrm{SmVar}(k),\ proper,\ n_1,\ldots n_r, d\in\mathbb{Z} >$$
$$\subset PDRM(S)\times_I P_{fil,k}(S_{\mathbb{C}}^{an})\subset\mathrm{PSh}_{\mathcal{D}fil,rh}(S)\times_I P_k(S_{\mathbb{C}}^{an})$$

the full abelian subcategory, where $<,>$ *means generated by and* $(-)$ *is the shift of the filtration,*

$$\alpha(X):\mathbb{C}_{X^{an}_{\mathbb{C}}}\hookrightarrow DR(X)(O^{an}_X)$$

is the inclusion quasi-isomorphism in $C(X^{an}_{\mathbb{C}})$*, and we use definition 68. We have by proposition 43 for* $((M,F),K,\alpha)\in HM_{gm,k,\mathbb{C}}(S)$,

$$\mathrm{Gr}^W_k\,\psi_D((M,F),K,\alpha):=\mathrm{Gr}^W_k\,\psi_D(M,F),\mathrm{Gr}^W_k\,\psi_D K,\mathrm{Gr}^W_k\,\psi_D\alpha)\in HM_{gm,k,\mathbb{C}}(S).$$

and

$$\mathrm{Gr}^W_k\,\psi_D((M,F),K,\alpha):=\mathrm{Gr}^W_k\,\psi_D(M,F),\mathrm{Gr}^W_k\,\psi_D K,\mathrm{Gr}^W_k\,\psi_D\alpha)\in HM_{gm,k,\mathbb{C}}(S).$$

for all $k\in\mathbb{Z}$*. We have by theorem 40, for* $S\in\mathrm{SmVar}(k)$, $HM_{gm,k,\mathbb{C}}(S)\subset HM(S_{\mathbb{C}})$ *which consists of geometric Hodge module defined over* k.

(i)' Let $S\in\mathrm{Var}(k)$*. Let* $S=\cup_{i\in I}S_i$ *an open cover such that there exists closed embeddings* $i_i:S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_I\in\mathrm{SmVar}(k)$*. We denote by*

$$\begin{aligned}HM_{gm,k,\mathbb{C}}(S):=<(H^{n_1}\int_{p_1}\cdots H^{n_r}\int_{p_r}(\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{X}_{r-1,I}},F_b),x_{IJ})(d),\\ R^{n_1}p_{1*}\cdots R^{n_r}p_{r*}T(X/(Y_r\times\tilde{X}_{r-1,I}))(\mathbb{Z}_{X^{an}_{\mathbb{C}}}),H^{n_1}p_{1*}\cdots H^{n_r}p_{r*}\alpha(X)),\\ (f:X=X_r\xrightarrow{f_r}X_{r-1}\to\cdots\xrightarrow{f_1}X_0=S)\in\mathrm{Var}(k),>\\ \subset PDRM(S)\times_I P_k(S^{an}_{\mathbb{C}})\subset\mathrm{PSh}_{\mathcal{D}fil,rh}(S/(\tilde{S}_I))\times_I P_k(S^{an}_{\mathbb{C}})\end{aligned}$$

the full abelian subcategory, where $<,>$ *means generated by and* $(-)$ *is the shift of the filtration,* $f_i:X_i\hookrightarrow Y_i\times X_{i-1}\xrightarrow{p_i}X_{i-1}$ *proper,* $Y_i\in\mathrm{PSmVar}(k)$, X_i *smooth,*

$$\begin{aligned}\alpha(X):=(\Gamma^{\vee}_{X_I}\alpha(Y_r\times\tilde{X}_{r-1,I})):T(X/(Y_r\times\tilde{X}_{r-1,I}))(\mathbb{C}_{X^{an}}):=(\Gamma^{\vee}_{X_I}\mathbb{C}_{(Y\times\tilde{X}_{r-1,I})^{an}_{\mathbb{C}}},t_{IJ})\\ \xrightarrow{\sim}DR(X)(o_F(\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{X}_{r-1,I}},F_b),x_{IJ})^{an}).\end{aligned}$$

is the canonical isomorphism in $D(X^{an}_{\mathbb{C}}/(Y_r\times\tilde{X}_{r-1,I})^{an}_{\mathbb{C}})$*, and we use definition 68. Note that if* S *is smooth then this definition of* $HM_{gm,k,\mathbb{C}}(S)$ *agree with the one given in (i). We have by theorem 40, for* $S\in\mathrm{Var}(k)$, $HM_{gm,k,\mathbb{C}}(S)\subset HM(S_{\mathbb{C}})$ *which consists of geometric Hodge module defined over* k.

(ii) Let $S\in\mathrm{Var}(k)$*. Take an open cover* $S=\cup_iS_i$ *such that there are closed embedding* $S_I\hookrightarrow\tilde{S}_I$ *with* $S_I\in\mathrm{SmVar}(k)$*. We define using the pure case (i) and (i)' the full subcategory of geometric mixed Hodge modules defined over* k

$$\begin{aligned}MHM_{gm,k,\mathbb{C}}(S):=\\ \{(((M_I,F,W),u_{IJ}),(K,W),\alpha),\ s.t.\ \mathrm{Gr}^W_k(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in HM_{gm,k,\mathbb{C}}(S)\}\\ \subset DRM(S)\times_I P_{fil,k}(S^{an}_{\mathbb{C}})\subset\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I P_{fil,k}(S^{an}_{\mathbb{C}})\end{aligned}$$

whose object consists of $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in DRM(S)\times_I P_{fil,k}(S^{an}_{\mathbb{C}})$ *such that*

$$\mathrm{Gr}^W_k(((M_I,F,W),u_{IJ}),(K,W),\alpha):=(\mathrm{Gr}^W_k((M_I,F),u_{IJ}),\mathrm{Gr}^W_k K,\mathrm{Gr}^W_k\alpha)\in HM_{gm,k,\mathbb{C}}(S).$$

We set

$$\mathbb{Q}^{Hdg}_S:=((\Gamma^{\vee,Hdg}_{S_I}(O_{\tilde{S}_I},F_b),x_{IJ}),\mathbb{Q}^w_{S^{an}},\alpha(S))\in C(MHM_{gm,k,\mathbb{C}}(S))$$

where $\mathbb{Q}^w_{S^{an}}\in C(P_{fil,k}(S^{an}_{\mathbb{C}}))$ *is such that* $j^*_I\mathbb{Q}^w_{S^{an}}=i^*_I\Gamma^{\vee,w}_{S_I}\mathbb{Q}_{\tilde{S}^{an}_I}$ *and*

$$\alpha(S):=(\Gamma^{\vee}_{S_I}\alpha(\tilde{S}_I)):T(S/(\tilde{S}_I))((\mathbb{Q}^w_{S^{an}})\otimes\mathbb{C}_{S^{an}})\xrightarrow{=}(\Gamma^{\vee,w}_{S_I}(\mathbb{C}_{\tilde{S}^{an}_I}),t_{IJ})\xrightarrow{\sim}DR(S)(o_F(\Gamma^{\vee,Hdg}_{S_I}(O_{\tilde{S}_I},F_b),x_{IJ})).$$

For $S \in \mathrm{SmVar}(k)$ and $D = V(s) \subset S$ a (Cartier) divisor, we have for $((M,F,W),(K,W),\alpha) \in MHM_{gm,k,\mathbb{C}}(S)$, using theorem 35,

$$\psi_D((M,F,W),(K,W),\alpha), \phi_D((M,F,W),(K,W),\alpha) \in MHM_{gm,k,\mathbb{C}}(S),$$

by the pure case (c.f. (i) and proposition 43) and the strictness of the V-filtration.

For $k \subset \mathbb{C}$ and $S \in \mathrm{Var}(k)$, we have by theorem 40

$$MHM_{gm,k,\mathbb{C}}(S) \subset MHM_{k,\mathbb{C}}(S) \subset MHM(S_{\mathbb{C}}).$$

For $S \in \mathrm{Var}(k)$ we get $D(MHM_{gm,k,\mathbb{C}}(S)) := \mathrm{Ho}_{(zar,usu)}(C(MHM_{gm,k,\mathbb{C}}(S)))$ after localization with Zariski local equivalence and usu local equivalence.

We now look at functorialities :

Definition 71. *Let $k \subset \mathbb{C}$ a subfield. Let $S \in \mathrm{SmVar}(k)$. Let $j : S^o \hookrightarrow S$ an open embedding. Let $Z := S \backslash S^o = V(\mathcal{I}) \subset S$ an the closed complementary subset, $\mathcal{I} \subset O_S$ being an ideal subsheaf. Taking generators $\mathcal{I} = (s_1, \ldots, s_r)$, we get $Z = V(s_1, \ldots, s_r) = \cap_{i=1}^r Z_i \subset S$ with $Z_i = V(s_i) \subset S$, $s_i \in \Gamma(S, \mathcal{L}_i)$ and L_i a line bundle. Note that Z is an arbitrary closed subset, $d_Z \geq d_X - r$ needing not be a complete intersection. Denote by $j_I : S^{o,I} := \cap_{i \in I}(S \backslash Z_i) = S \backslash (\cup_{i \in I} Z_i) \xrightarrow{j_I^o} S^o \xrightarrow{j} S$ the open embeddings. Let $(M,F,W) \in MHM_{gm,k,\mathbb{C}}(S^o))$. We then define, using definition 59 and definition 4*

- *the canonical extension*

$$j_{*Hdg}((M,F,W),(K,W),\alpha) := (j_{*Hdg}(M,F,W), j_{*w}(K,W), j_*\alpha)$$
$$:= \mathrm{Tot}((j_{I*Hdg} j_I^*(M,F,W), j_{I*w} j_I^*(K,W), j_{I*}\alpha)) \in MHM_{gm,k,\mathbb{C}}(S),$$

  so that $j^*(j_{*Hdg}((M,F,W),(K,W),\alpha)) = ((M,F,W),(K,W),\alpha)$,

- *the canonical extension*

$$j_{!Hdg}((M,F,W),(K,W),\alpha) := (j_{!Hdg}(M,F,W), j_{!w}(K,W), j_!\alpha)$$
$$:= \mathrm{Tot}((j_{I!Hdg} j_I^*(M,F,W), j_{I!w} j_I^*(K,W), j_{I!}\alpha)) \in MHM_{gm,k,\mathbb{C}}(S),$$

  so that $j^*(j_{!Hdg}((M,F,W),(K,W),\alpha)) = ((M,F,W),(K,W),\alpha)$.

Moreover for $((M',F,W),(K',W),\alpha') \in MHM_{gm,k,\mathbb{C}}(S)$,

- *there is a canonical map in $MHM_{gm,k,\mathbb{C}}(S)$*

  $\mathrm{ad}(j^*, j_{*Hdg})((M',F,W),(K',W),\alpha') : ((M',F,W),(K',W),\alpha') \to j_{*Hdg} j^*((M',F,W),(K',W),\alpha')$,

- *there is a canonical map in $MHM_{gm,k,\mathbb{C}}(S)$*

  $\mathrm{ad}(j_{!Hdg}, j^*)((M',F,W),(K',W),\alpha') : j_{!Hdg} j^*((M',F,W),(K',W),\alpha') \to ((M',F,W),(K',W),\alpha')$.

Let $j : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{SmVar}(k)$. For $(M,F,W) \in C(MHM_{gm,k,\mathbb{C}}(S^o))$,

- we have the canonical map in $C_{\mathcal{D}(1,0)fil}(S) \times_I C_{fil}(S_{\mathbb{C}}^{an})$

$$T(j_{*Hdg}, j_*)((M,F,W),(K,W),\alpha) := (k \circ \mathrm{ad}(j^*, j_*)(-), k \circ \mathrm{ad}(j^*, j_*), 0) :$$
$$j_{*Hdg}((M,F,W),(K,W),\alpha) \to (j_* E(M,F,W), j_* E(K,W), \alpha)$$

- we have the canonical map in $C_{\mathcal{D}(1,0)fil}(S) \times_I C_{fil}(S_{\mathbb{C}}^{an})$

$$T(j_!, j_{!Hdg})((M,F,W),(K,W),\alpha) := (k \circ \mathrm{ad}(j_!, j^*)(-), k \circ \mathrm{ad}(j_!, j^*)(-), 0) :$$
$$(j_!(M,F,W), j_!(K,W), j_!\alpha) \to j_{!Hdg}((M,F,W),(K,W),\alpha)$$

the canonical maps.

Proposition 44. *(i) Let $S \in \mathrm{SmVar}(k)$. Let $D = V(s) \subset S$ a divisor with $s \in \Gamma(S, L)$ and L a line bundle (S being smooth, D is Cartier). Denote by $j : S^o := S \backslash D \hookrightarrow S$ the open complementary embedding. Then,*

- *$(j^*, j_{*Hdg}) : MHM_{gm,k,\mathbb{C}}(S) \leftrightarrows MHM_{gm,k,\mathbb{C}}(S^o)$ is a pair of adjoint functors*
- *$(j_{!Hdg}, j^*) : MHM_{gm,k,\mathbb{C}}(S^o) \leftrightarrows MHM_{gm,k,\mathbb{C}}(S)$ is a pair of adjoint functors.*

(ii) Let $S \in \mathrm{SmVar}(k)$. Let $Z = V(\mathcal{I}) \subset S$ an arbitrary closed subset, $\mathcal{I} \subset O_S$ being an ideal subsheaf. Denote by $j : S^o := S \backslash Z \hookrightarrow S$. Then,

- *$(j^*, j_{*Hdg}) : D(MHM_{gm,k,\mathbb{C}}(S)) \leftrightarrows D(MHM_{gm,k,\mathbb{C}}(S^o))$ is a pair of adjoint functors*
- *$(j_{!Hdg}, j^*) : D(MHM_{gm,k,\mathbb{C}}(S^o)) \leftrightarrows D(MHM_{gm,k,\mathbb{C}}(S))$ is a pair of adjoint functors.*

Proof. (i): Follows from proposition 37.
(ii):Follows from (i) and the exactness of j^*, j_{*Hdg} and $j_{!Hdg}$. □

Definition 72. *Let $S \in \mathrm{SmVar}(k)$. Let $Z \subset S$ a closed subset. Denote by $j : S \backslash Z \hookrightarrow S$ the complementary open embedding.*

(i) We define using definition 61, definition 5 and definition 68(iii), the filtered Hodge support section functor

$$\Gamma_Z^{Hdg} : C(MHM_{gm,k,\mathbb{C}}(S)) \to C(MHM_{gm,k,\mathbb{C}}(S)), \quad ((M, F, W), (K, W), \alpha) \mapsto$$
$$\Gamma_Z^{Hdg}((M, F, W), (K, W), \alpha) := (\Gamma_Z^{Hdg}(M, F, W), \Gamma_Z^w(K, W), \Gamma_Z(\alpha))$$
$$= \mathrm{Cone}(\mathrm{ad}(j^*, j_{*Hdg})(-) : j_{*Hdg}, j^*((M, F, W), (K, W), \alpha) \to ((M, F, W), (K, W), \alpha)[-1]$$

see definition 71 for the last equality, together we the canonical map

$$\gamma_Z^{Hdg}((M, F, W), (K, W), \alpha) : \Gamma_Z^{Hdg}((M, F, W), (K, W), \alpha) \to ((M, F, W), (K, W), \alpha).$$

(i)' Since $j_{*Hdg} : C(MHM_{gm,k,\mathbb{C}}(S^o)) \to C(MHM_{gm,k,\mathbb{C}}(S))$ is an exact functor, Γ_Z^{Hdg} induces the functor

$$\Gamma_Z^{Hdg} : D(MHM_{gm,k,\mathbb{C}}(S)) \to D(MHM_{gm,k,\mathbb{C}}(S)),$$
$$((M, F, W), (K, W), \alpha) \mapsto \Gamma_Z^{Hdg}((M, F, W), (K, W), \alpha)$$

(ii) We define using definition 61, definition 5 and definition 68(iii) the dual filtered Hodge support section functor

$$\Gamma_Z^{\vee,Hdg} : C(MHM_{gm,k,\mathbb{C}}(S)) \to C(MHM_{gm,k,\mathbb{C}}(S)), \quad ((M, F, W), (K, W), \alpha) \mapsto$$
$$\Gamma_Z^{\vee,Hdg}((M, F, W), (K, W), \alpha) := (\Gamma_Z^{\vee,Hdg}(M, F, W), \Gamma_Z^{\vee,w}(K, W), \Gamma_Z^{\vee}(\alpha))$$
$$= \mathrm{Cone}(\mathrm{ad}(j_{!Hdg}, j^*)(-) : j_{!Hdg}, j^*((M, F, W), (K, W), \alpha) \to ((M, F, W), (K, W), \alpha))$$

see definition 71 for the last equality, together we the canonical map

$$\gamma_Z^{\vee,Hdg}((M, F, W), (K, W), \alpha) : ((M, F, W), (K, W), \alpha) \to \Gamma_Z^{\vee,Hdg}((M, F, W), (K, W), \alpha).$$

(ii)' Since $j_{!Hdg} : C(MHM_{gm,k,\mathbb{C}}(S^o)) \to C(MHM_{gm,k,\mathbb{C}}(S))$ is an exact functor, $\Gamma_Z^{Hdg,\vee}$ induces the functor

$$\Gamma_Z^{\vee,Hdg} : D(MHM_{gm,k,\mathbb{C}}(S)) \to D(MHM_{gm,k,\mathbb{C}}(S)),$$
$$((M, F, W), (K, W), \alpha) \mapsto \Gamma_Z^{\vee,Hdg}((M, F, W), (K, W), \alpha)$$

In the singular case it gives :

Definition 73. *Let $S \in \mathrm{Var}(k)$. Let $Z \subset S$ a closed subset. Let $S = \cup_{i=1}^{s} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Denote $Z_I := Z \cap S_I$. Denote by $n : S \backslash Z \hookrightarrow S$ and $\tilde{n}_I : \tilde{S}_I \backslash Z_I \hookrightarrow \tilde{S}_I$ the complementary open embeddings.*

(i) We define using definition 63, definition 5 and definition 68(iii) the filtered Hodge support section functor

$$\Gamma_Z^{Hdg} : C(MHM_{gm,k,\mathbb{C}}(S)) \to C(MHM_{gm,k,\mathbb{C}}(S)),$$
$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto \Gamma_Z^{Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) :=$$
$$:= (\Gamma_Z^{Hdg}((M_I, F, W), u_{IJ}), \Gamma_Z^{w}(K, W), \Gamma_Z(\alpha))$$

together with the canonical map

$$\gamma_Z^{Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) :$$
$$\Gamma_Z^{Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) \to (((M_I, F, W), u_{IJ}), (K, W), \alpha).$$

(i)' By exactness of Γ_Z^{Hdg} and Γ_Z^{w} it induces the functor

$$\Gamma_Z^{Hdg} : D(MHM_{gm,k,\mathbb{C}}(S)) \to D(MHM_{gm,k,\mathbb{C}}(S)),$$
$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto \Gamma_Z^{Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha)$$

(ii) We define using definition 63, definition 5 and definition 68(iii) the dual filtered Hodge support section functor

$$\Gamma_Z^{\vee,Hdg} : C(MHM_{gm,k,\mathbb{C}}(S)) \to C(MHM_{gm,k,\mathbb{C}}(S)), \;\; (((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto$$
$$\Gamma_Z^{\vee,Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) := (\Gamma_Z^{\vee,Hdg}((M_I, F, W), u_{IJ}), \Gamma_Z^{\vee,w}(K, W), \Gamma_Z^{\vee}(\alpha)),$$

together we the canonical map

$$\gamma_Z^{\vee,Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) :$$
$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \to \Gamma_Z^{\vee,Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha).$$

(ii)' By exactness of $\Gamma_Z^{\vee,Hdg}$ and $\Gamma_Z^{\vee,w}$, it induces the functor

$$\Gamma_Z^{\vee,Hdg} : D(MHM_{gm,k,\mathbb{C}}(S)) \to D(MHM_{gm,k,\mathbb{C}}(S)),$$
$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto \Gamma_Z^{\vee,Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha)$$
$$:= (\Gamma_Z^{\vee,Hdg}((M_I, F, W), u_{IJ}), \Gamma_Z^{\vee,w}(K, W), \Gamma_Z^{\vee}(\alpha))$$

This gives the inverse image functor :

Definition 74. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. Let $S = \cup_{i \in I}$ an open cover such that there exist closed embeddings $i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Denote $X_I := f^{-1}(S_I)$. We have then $X = \cup_{i \in I} X_i$ and the commutative diagrams*

$$\begin{tikzcd} f : X_I \arrow[r, "l_I"] \arrow[dr] & Y \times S_I \arrow[r, "p_{S_I}"] \arrow[d, "i'_I := (I \times i_I)"] & S_I \arrow[d, "i_I"] \\ & Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I} =: \tilde{f}_I"] & \tilde{S}_I \end{tikzcd}$$

(i) For $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in C(MHM_{gm,k,\mathbb{C}}(S))$ we set (see definition 73 for l)

$$f^{*Hdg}(((M_I,F,W),u_{IJ}),(K,W),\alpha):=$$
$$\Gamma^{Hdg}_X((p^{*mod[-]}_{\tilde{S}_I}(M_I,F,W),p^{*mod[-]}_{\tilde{S}_I}u_{IJ}),p_S^*(K,W),p_S^*\alpha)(d_Y)[2d_Y]\in C(MHM_{gm,k,\mathbb{C}}(X)),$$

(ii) For $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in C(MHM_{gm,k,\mathbb{C}}(S))$ we set (see definition 73 for l)

$$f^{!Hdg}(((M_I,F,W),u_{IJ}),(K,W),\alpha):=$$
$$\Gamma^{\vee,Hdg}_X((p^{*mod[-]}_{\tilde{S}_I}(M_I,F,W),p^{*mod[-]}_{\tilde{S}_I}u_{IJ}),p_S^*(K,W),p_S^*\alpha)\in C(MHM_{gm,k,\mathbb{C}}(X)),$$

Let $j:S^o\hookrightarrow S$ an open embedding with $S\in\mathrm{Var}(k)$. Let $S=\cup_{i\in I}$ an open cover such that there exist closed embeddings $i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(k)$. We have then, for $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in C(MHM_{gm,k,\mathbb{C}}(S))$, quasi-isomorphisms in $C(MHM_{gm,k,\mathbb{C}}(S))$

$$(I(j^*,j^{*mod}_{Hdg})(-),I):j_{*Hdg}(((M_I,F,W),u_{IJ}),(K,W),\alpha)\to$$
$$j^*(((M_I,F,W),u_{IJ}),(K,W),\alpha):=(j^*((M_I,F,W),u_{IJ}),j^*(K,W),\alpha)$$

and

$$(I(j^*,j^{\hat{*}mod}_{Hdg})(-),I):j^{!Hdg}(((M_I,F,W),u_{IJ}),(K,W),\alpha)\to$$
$$j^*(((M_I,F,W),u_{IJ}),(K,W),\alpha):=(j^*((M_I,F,W),u_{IJ}),j^*(K,W),\alpha)$$

Definition 75. *Let $S\in\mathrm{Var}(k)$. Let $S=\cup_{i\in I}$ an open cover such that there exist closed embeddings $i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(k)$. We have, using definition 65 the following bi-functor*

$$(-)\otimes^{Hdg}_{O_S}(-):D(MHM_{gm,k,\mathbb{C}}(S))^2\to D(MHM_{gm,k,\mathbb{C}}(S)),$$
$$(((M_I,F,W),u_{IJ}),(K,W),\alpha),(((M'_I,F,W),v_{IJ}),(K',W),\alpha')\mapsto$$
$$(((M_I,F,W),u_{IJ}),(K,W),\alpha)\otimes^{Hdg}_{O_S}(((M'_I,F,W),v_{IJ}),(K',W),\alpha'):=$$
$$((M_I,F,W),u_{IJ})\otimes^{Hdg}_{O_S}((M'_I,F,W),v_{IJ}),(K,W)\otimes^{L,w}(K',W),\alpha\otimes\alpha')$$

where the map $\alpha\otimes\alpha'$ is given in definition 68(v).

Proposition 45. *Let $f_1:X\to Y$ and $f_2:Y\to S$ two morphism with $X,Y,S\in\mathrm{QPVar}(k)$.*

(i) *Let $\mathcal{M}\in C(MHM_{gm,k,\mathbb{C}}(S))$. Then,*

$$(f_2\circ f_1)^{*Hdg}(\mathcal{M})=f_1^{*Hdg}f_2^{*Hdg}(\mathcal{M})\in D(MHM_{gm,k,\mathbb{C}}(X)).$$

(ii) *Let $(M,F,W)\in C(MHM_{gm,k,\mathbb{C}}(S))$. Then,*

$$(f_2\circ f_1)^{!Hdg}(\mathcal{M})=f_1^{!Hdg}f_2^{!Hdg}(\mathcal{M})\in D(MHM_{gm,k,\mathbb{C}}(X))$$

Proof. Immediate from definition. $\square$

Proposition 46. *Let $S\in\mathrm{SmVar}(k)$. Let $D=V(s)\subset S$ a (Cartier) divisor, where $s\in\Gamma(S,L)$. Denote $i:D:=S\backslash D\hookrightarrow S$ the closed embedding and $j:S^o\hookrightarrow S$ the open embedding.*

(i) *Let $((M,F,W),(K,W),\alpha)\in MHM_{gm,k,\mathbb{C}}(S)$. We have, using proposition 38, the canonical quasi-isomorphism in $C(MHM_{gm,k,\mathbb{C}}(S))$:*

$$Is(M):=(Is(M),Is(K),0):$$
$$((M,F,W),(K,W),\alpha)\to(\psi_D((M,F,W),(K,W),\alpha)\xrightarrow{((c(x_{S^o/S}(M)),can(M)),(c(x_{S^o/S}(K)),can(K)),0)}$$
$$x_{S^o/S}((M,F,W),(K,W),\alpha)\oplus\phi_D((M,F,W),(K,W),\alpha)\xrightarrow{:=}$$
$$(x_{S^o/S}(M,F,W),x_{S^o/S}(K,W),x_{S^o/S}(\alpha))\oplus(\phi_D(M,F,W),\phi_D(K,W),\phi_D\alpha)$$
$$\xrightarrow{((0,s\partial s),var(M)),((0,T-I),var(K)),0)}\psi_D((M,F,W),(K,W),\alpha)).$$

(ii) We denote by $MHM_{gm,k,\mathbb{C}}(S\backslash D)\times_J MHM_{gm,k,\mathbb{C}}(D)$ the category whose set of objects consists of

$$\{(\mathcal{M},\mathcal{N},a,b),\mathcal{M}\in MHM_{gm,k,\mathbb{C}}(S\backslash D),\mathcal{N}\in MHM_{gm,k,\mathbb{C}}(D),a:\psi_{D1}\mathcal{M}\to N,b:N\to\psi_{D1}M\}$$

The functor (see definition 69)

$$(j^*,\phi_{D1},c,v):MHM_{gm,k,\mathbb{C}}(S)\to MHM_{gm,k,\mathbb{C}}(S\backslash D)\times_J MHM_{gm,k,\mathbb{C}}(D),$$
$$((M,F,W),(K,W),\alpha)\mapsto((j^*(M,F,W),j^*(K,W),j^*\alpha),\phi_D((M,F,W),(K,W),\alpha),can(-),var(-))$$

is an equivalence of category.

Proof. (i):Follows from proposition 38.
(ii):Follows from (i). □

Let $S\in\mathrm{Var}(k)$. Let $S=\cup_{i\in I}S_i$ an open cover such that there exists closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_I\in\mathrm{SmVar}(k)$. We have the category $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$

- whose set of objects is the set of triples $\{(((M_I,F,W),u_{IJ}),(K,W),\alpha)\}$ with

$$((M_I,F,W),u_{IJ})\in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)),\ (K,W)\in D_{fil,c,k}(S^{an}_{\mathbb{C}}),$$
$$\alpha:T(S/(\tilde{S}_I))(K,W)\otimes\mathbb{C}_{S^{an}_{\mathbb{C}}}\to DR(S)^{[-]}(((M_I,W),u_{IJ})^{an})$$

  where α is an morphism in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$,

- and whose set of morphisms consists of

$$\phi=(\phi_D,\phi_C,[\theta]):(((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1)\to(((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2)$$

  where $\phi_D:((M_1,F,W),u_{IJ})\to((M_2,F,W),u_{IJ})$ and $\phi_C:(K_1,W)\to(K_2,W)$ are morphisms and

$$\theta=(\theta^\bullet,I(DR(S)(\phi_D^{an}))\circ I(\alpha_1),I(\alpha_2)\circ I(\phi_C\otimes I)):$$
$$I(T(S/(\tilde{S}_I))(K_1,W))\otimes\mathbb{C}_{S^{an}}[1]\to I(DR(S)(((M_{2I},W),u_{IJ})^{an}))$$

  is an homotopy, $I:D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))\to K_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$ being the injective resolution functor, and for

  - $\phi=(\phi_D,\phi_C,[\theta]):(((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1)\to(((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2)$
  - $\phi'=(\phi'_D,\phi'_C,[\theta']):(((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2)\to(((M_{3I},F,W),u_{IJ}),(K_3,W),\alpha_3)$

  the composition law is given by

$$\phi'\circ\phi:=(\phi'_D\circ\phi_D,\phi'_C\circ\phi_C,I(DR(S)(\phi_D^{'an}))\circ[\theta]+[\theta']\circ I(\phi_C\otimes I)[1]):$$
$$(((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1)\to(((M_{3I},F,W),u_{IJ}),(K_3,W),\alpha_3),$$

  in particular for $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$,

$$I_{(((M_I,F,W),u_{IJ}),(K,W),\alpha)}=((I_{M_I}),I_K,0),$$

and also the category $D_{\mathcal{D}(1,0)fil,rh,\infty}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$ defined in the same way, together with the localization functor

$$(D(zar),I):C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})\to D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$$
$$\to D_{\mathcal{D}(1,0)fil,rh,\infty}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}}).$$

Note that if $\phi=(\phi_D,\phi_C,[\theta]):(((M_1,F,W),u_{IJ}),(K_1,W),\alpha_1)\to(((M_2,F,W),u_{IJ}),(K_2,W),\alpha_2)$ is a morphism in $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$ such that ϕ_D and ϕ_C are isomorphism then ϕ is an isomorphism (see remark 5). Moreover,

- For $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$, we set
$$(((M_I,F,W),u_{IJ}),(K,W),\alpha)[1]:=(((M_I,F,W),u_{IJ})[1],(K,W)[1],\alpha[1]).$$

- For
$$\phi=(\phi_D,\phi_C,[\theta]):(((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1)\to(((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2)$$
a morphism in $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$, we set (see [11] definition 3.12)
$$\mathrm{Cone}(\phi):=(\mathrm{Cone}(\phi_D),\mathrm{Cone}(\phi_C),((\alpha_1,\theta),(\alpha_2,0)))\in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}}),$$
$((\alpha_1,\theta),(\alpha_2,0))$ being the matrix given by the composition law, together with the canonical maps
  - $c_1(-)=(c_1(\phi_D),c_1(\phi_C),0):(((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2)\to\mathrm{Cone}(\phi)$
  - $c_2(-)=(c_2(\phi_D),c_2(\phi_C),0):\mathrm{Cone}(\phi)\to(((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1)[1]$.

We have then the following :

Theorem 44. *(i) Let $S\in\mathrm{Var}(k)$. Let $S=\cup_{i\in I}S_i$ an open cover such that there exists closed embedding $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(k)$. Then the full embedding*
$$\iota_S:MHM_{gm,k,\mathbb{C}}(S)\hookrightarrow\mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I P_{fil,k}(S^{an}_{\mathbb{C}})\hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$$
induces a full embedding
$$\iota_S:D(MHM_{gm,k,\mathbb{C}}(S))\hookrightarrow D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$$
whose image consists of $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$ such that
$$((H^n(M_I,F,W),H^n(u_{IJ})),H^n(K,W),H^n\alpha)\in MHM_{gm,k,\mathbb{C}}(S)$$
for all $n\in\mathbb{Z}$ and such that for all $p\in\mathbb{Z}$, the differentials of $\mathrm{Gr}^p_W(M_I,F)$ are strict for the filtrations F.

(i)' Let $S\in\mathrm{Var}(k)$. Let $S=\cup_{i\in I}S_i$ an open cover such that there exists closed embedding $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(k)$. Then,
$$\begin{aligned}D(MHM_{gm,k,\mathbb{C}}(S))=&<(\int_f^{FDR}(n\times I)_{!Hdg}(\Gamma^{\vee,Hdg}_X(O_{\mathbb{P}^{N,o}\times\tilde{S}_I},F_b),x_{IJ})(d),Rf_*\mathbb{Q}^w_X,f_*\alpha(X)),\\&(f:X\xrightarrow{l}\mathbb{P}^{N,o}\times S\xrightarrow{p}S)\in\mathrm{QPVar}(k),\ d\in\mathbb{Z}>\\=&<(\int_f^{FDR}(\Gamma^{\vee,Hdg}_X(O_{\mathbb{P}^{N,o}\times\tilde{S}_I},F_b),x_{IJ})(d),Rf_*\mathbb{Q}_X,f_*\alpha(X)),\\&(f:X\xrightarrow{l}\mathbb{P}^{N,o}\times S\xrightarrow{p}S)\in\mathrm{QPVar}(k),\ \text{proper},\ X\text{smooth},\ d\in\mathbb{Z}>\\&\subset D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})\end{aligned}$$
where $n:\mathbb{P}^{N,o}\hookrightarrow\mathbb{P}^N$ are open embeddings, l are closed embedding and $<,>$ means the full triangulated category generated by and $(-)$ the shift of the F-filtration.

(ii) Let $S\in\mathrm{Var}(k)$. Let $S=\cup_{i\in I}S_i$ an open cover such that there exists closed embedding $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(k)$. Then the full embedding
$$\iota_S:MHM_{gm,k,\mathbb{C}}(S)\hookrightarrow\mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I P_{fil,k}(S^{an}_{\mathbb{C}})\hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$$

induces a full embedding

$$\iota_S : D(MHM_{gm,k,\mathbb{C}}(S)) \hookrightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S_{\mathbb{C}}^{an})$$

whose image consists of $(((M_I,F,W),u_{IJ}),(K,W),\alpha) \in D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{fil}(S_{\mathbb{C}}^{an})$ *such that*

$$((H^n(M_I,F,W),H^n(u_{IJ})),H^n(K,W),H^n\alpha) \in MHM_{gm,k,\mathbb{C}}(S)$$

for all $n \in \mathbb{Z}$ *and such that there exist* $r \in \mathbb{Z}$ *and an* r*-filtered homotopy equivalence* $((M_I,F,W),u_{IJ}) \to ((M'_I,F,W),u_{IJ})$ *such that for all* $p \in \mathbb{Z}$ *the differentials of* $\mathrm{Gr}_W^p(M'_I,F)$ *are strict for the filtrations* F.

Proof. (i): We first show that ι_S is fully faithfull, that is for all $\mathcal{M} = (((M_I,F,W),u_{IJ}),(K,W),\alpha), \mathcal{M}' = (((M'_I,F,W),u_{IJ}),(K',W),\alpha') \in MHM_{gm,k,\mathbb{C}}(S)$ and all $n \in \mathbb{Z}$,

$$\iota_S : \mathrm{Ext}^n_{D(MHM_{gm,k,\mathbb{C}}(S))}(\mathcal{M},\mathcal{M}') := \mathrm{Hom}_{D(MHM_{gm,k,\mathbb{C}}(S))}(\mathcal{M},\mathcal{M}'[n])$$
$$\to \mathrm{Ext}^n_{\mathcal{D}(S)}(\mathcal{M},\mathcal{M}') := \mathrm{Hom}_{\mathcal{D}(S):=D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil}(S_{\mathbb{C}}^{an})}(\mathcal{M},\mathcal{M}'[n])$$

For this it is enough to assume S smooth. We then proceed by induction on $max(\dim \mathrm{supp}(M), \dim \mathrm{supp}(M'))$.

- For $\mathrm{supp}(M) = \mathrm{supp}(M') = \{s\}$, it is the theorem for mixed hodge complexes or absolute Hodge complexes, see [11]. If $\mathrm{supp}(M) = \{s\}$ and $\mathrm{supp}(M') = \{s'\}$ and $s' \neq s$, then by the localization exact sequence
$$\mathrm{Ext}^n_{D(MHM_{gm,k,\mathbb{C}}(S))}(\mathcal{M},\mathcal{M}') = 0 = \mathrm{Ext}^n_{\mathcal{D}(S)}(\mathcal{M},\mathcal{M}')$$
- Denote $\mathrm{supp}(M) = Z \subset S$ and $\mathrm{supp}(M') = Z' \subset S$. There exist an open subset $S^o \subset S$ such that $Z^o := Z \cap S^o$ and $Z'^o := Z' \cap S^o$ are smooth, and $\mathcal{M}_{|Z^o} := ((i^* \mathrm{Gr}_{V_{Z^o},0} M_{|S^o},F,W), i^*j^*(K,W), \alpha^*(i)) \in MHM_{gm,k}(Z^o)$ and $\mathcal{M}'_{|Z'^o} := ((i'^* \mathrm{Gr}_{V_{Z'^o},0} M'_{|S^o},F,W), i'^*j^*K, \alpha^*(i')) \in MHM_{gm,k}(Z'^o)$ are variation of geometric mixed Hodge structure over $k \subset \mathbb{C}$, where $j : S^o \hookrightarrow S$ is the open embedding, and $i : Z^o \hookrightarrow S^o$, $i' : Z'^o \hookrightarrow S^o$ the closed embeddings. Considering the connected components of Z^o and Z'^o, we way assume that Z^o and Z'^o are connected. Shrinking S^o if necessary, we may assume that either $Z^o = Z'^o$ or $Z^o \cap Z'^o = \emptyset$, We denote $D = S \backslash S^o$. Shrinking S^o if necessary, we may assume that D is a divisor and denote by $l : S \hookrightarrow L_D$ the zero section embedding.
  - If $Z^o = Z'^o$, denote $i : Z^o \hookrightarrow S^o$ the closed embedding. We have then the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^n_{D(MHM_{gm,k,\mathbb{C}}(S^o))}(\mathcal{M}_{|S^o},\mathcal{M}'_{|S^o}) \arrow[r, "\iota_{S^o}"] \arrow[d, shift left, "(i_{*mod},i_*,\alpha_*(i))"] & \mathrm{Ext}^n_{\mathcal{D}(S^o)}(\mathcal{M}_{|S^o},\mathcal{M}'_{|S^o}) \arrow[d, shift left, "(i^* \mathrm{Gr}_{V_{Z^o},0},i^*,\alpha^*(i))"] \\
\mathrm{Ext}^n_{D(MHM_{gm,k,\mathbb{C}}(Z^o))}(\mathcal{M}_{|Z^o},\mathcal{M}'_{|Z^o}) \arrow[r, "\iota_{Z^o}"] \arrow[u, shift left, "(i^* \mathrm{Gr}_{V_{Z^o},0},i^*,\alpha^*(i))"] & \mathrm{Ext}^n_{\mathcal{D}(Z^o)}(\mathcal{M}_{|Z^o},\mathcal{M}'_{|Z^o}) \arrow[u, shift left, "(i_{*mod},i_*,\alpha_*(i))"]
\end{tikzcd}$$

    Now we prove that ι_{Z^o} is an isomorphism similarly to the proof the the generic case of [6]. On the other hand the left and right colummn are isomorphisms. Hence ι_{S^o} is an isomorphism by the diagram.
  - If $Z^o \cap Z'^o = \emptyset$, we consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^n_{D(MHM_{gm,k,\mathbb{C}}(S^o))}(\mathcal{M}_{|S^o},\mathcal{M}'_{|S^o}) \arrow[r, "\iota_{S^o}"] \arrow[d, shift left, "(i_{*mod},i_*,\alpha_*(i))"] & \mathrm{Ext}^n_{\mathcal{D}(S^o)}(\mathcal{M}_{|S^o},\mathcal{M}'_{|S^o}) \arrow[d, shift left, "(i^* \mathrm{Gr}_{V_{Z^o},0},i^*,\alpha^*(i))"] \\
\mathrm{Ext}^n_{D(MHM_{gm,k,\mathbb{C}}(Z^o))}(\mathcal{M}_{|Z^o},0) = 0 \arrow[r, "\iota_{Z^o}"] \arrow[u, shift left, "(i^* \mathrm{Gr}_{V_{Z^o},0},i^*,\alpha^*(i))"] & \mathrm{Ext}^n_{\mathcal{D}(Z^o)}(\mathcal{M}_{|Z^o},0) = 0 \arrow[u, shift left, "(i_{*mod},i_*,\alpha_*(i))"]
\end{tikzcd}$$

    where the left and right column are isomorphism by strictness of the V_{Z^o} filtration (use a bi-filtered injective resolution with respect to F and V_{Z^o} for the right column).

- We consider now the following commutative diagram in $C(\mathbb{Z})$ where we denote for short $H := D(MHM_{gm,k,\mathbb{C}}(S))$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}^\bullet_H(\Gamma^{\vee,Hdg}_D\mathcal{M},\Gamma^{Hdg}_D\mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\gamma^{Hdg}_D(\mathcal{M}'))"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_H(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\mathrm{ad}(j^*,j_{*Hdg})(\mathcal{M}'))"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_H(\Gamma^{\vee,Hdg}_D\mathcal{M},j_{*Hdg}j^*\mathcal{M}') \arrow[r] \arrow[d, "\iota_S"] & 0 \\
0 \arrow[r] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\Gamma^{Hdg}_D\mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\gamma^{Hdg}_D(\mathcal{M}'))"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\mathrm{ad}(j^*,j_{*Hdg})(\mathcal{M}'))"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},j_{*Hdg}j^*\mathcal{M}') \arrow[r] & 0
\end{tikzcd}$$

whose lines are exact sequence. We have on the one hand,

$$\mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}}(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M},j_{*Hdg}j^*\mathcal{M}') = 0 = \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},j_{*Hdg}j^*\mathcal{M}')$$

On the other hand by induction hypothesis

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}}(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M},\Gamma^{Hdg}_D\mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\Gamma^{Hdg}_D\mathcal{M}')$$

is a quasi-isomorphism. Hence, by the diagram

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}}(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}')$$

is a quasi-isomorphism.

- We consider now the following commutative diagram in $C(\mathbb{Z})$ where we denote for short $H := D(MHM_{gm,k,\mathbb{C}}(S))$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}^\bullet_H(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(\gamma^{\vee,Hdg}_D(\mathcal{M}),-)"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_H(\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(\mathrm{ad}(j_{!Hdg},j^*)(\mathcal{M}'),-)"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_H(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \arrow[r] \arrow[d, "\iota_S"] & 0 \\
0 \arrow[r] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(\gamma^{\vee,Hdg}_D(\mathcal{M}),-)"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(\mathrm{ad}(j_{!Hdg},j^*)(\mathcal{M}),-)"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \arrow[r] & 0
\end{tikzcd}$$

whose lines are exact sequence. On the one hand, the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}}(S))}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \arrow[r, "j^*"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}}(S^o))}(j^*\mathcal{M},j^*\mathcal{M}') \arrow[d, "\iota_{S^o}"] \\
\mathrm{Hom}^\bullet_{\mathcal{D}(S)}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \arrow[r, "j^*"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S^o)}(j^*\mathcal{M},j^*\mathcal{M}')
\end{tikzcd}$$

together with the fact that the horizontal arrows j^* are quasi-isomorphism by the functoriality given the uniqueness of the V_S filtration for the embedding $l : S \hookrightarrow L_D$, (use a bi-filtered injective resolution with respect to F and V_S for the lower arrow) and the fact that ι_{S^o} is a quasi-isomorphism by the first two point, show that

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}}(S))}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(j_{!Hdg}j^*\mathcal{M},\mathcal{M}')$$

is a quasi-isomorphism. On the other hand, by the third point

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}}(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}')$$

is a quasi-isomorphism. Hence, by the diagram

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}}(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}')$$

is a quasi-isomorphism.

This shows the fully faithfulness. We now prove the essential surjectivity : let

$$(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I C_{fil}(S^{an}_{\mathbb{C}})$$

such that the cohomology are mixed hodge modules and such that the differential are strict. We proceed by induction on $card\{n\in\mathbb{Z}\}$, s.t.$H^n(M_I,F,W)\neq 0$ by taking for the cohomological troncation

$$\tau^{\leq n}(((M_I,F,W),u_{IJ}),(K,W),\alpha):=((\tau^{\leq n}(M_I,F,W),\tau^{\leq n}u_{IJ}),\tau^{\leq n}(K,W),\tau^{\leq n}\alpha)$$

and using the fact that the differential are strict for the filtration F and the fully faithfullness.
(i)':Follows from (i).
(ii):Follows from (i).Indeed, in the composition of functor

$$\begin{aligned}\iota_S: D(MHM_{gm,k,\mathbb{C}}(S)) &\xrightarrow{\iota_S} D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil}(S^{an}_{\mathbb{C}})\\ &\to D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I))\times_I D_{fil}(S^{an}_{\mathbb{C}})\end{aligned}$$

the second functor which is the localization functor is an isomorphism on the full subcategory

$$D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))^{st}\times_I D_{fil}(S^{an}_{\mathbb{C}})\subset D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil}(S^{an}_{\mathbb{C}})$$

constisting of complex such that the differentials are strict for F, and the first functor ι_S is a full embedding by (i) and $\iota_S(D(MHM_{gm,k,\mathbb{C}}(S)))\subset D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))^{st}\times_I D_{fil}(S^{an}_{\mathbb{C}})$. □

Definition 76. *Let $f:X\to S$ a morphism with $X,S\in\mathrm{Var}(k)$. Assume there exist a factorization $f:X\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ with $Y\in\mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. Let $\bar{Y}\in\mathrm{PSmVar}(k)$ a smooth compactification of Y with $n:Y\hookrightarrow\bar{Y}$ the open embedding. Then $\bar{f}:\bar{X}\xrightarrow{\bar{l}}\bar{Y}\times_S\xrightarrow{\bar{p}_S}S$ is a compactification of f, with $\bar{X}\subset\bar{Y}\times S$ the closure of X and $\bar{l}$ the closed embedding, and we denote by $n':X\hookrightarrow\bar{X}$ the open embedding so that $f=\bar{f}\circ n'$.*

(i) For $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in C(MHM_{gm,k,\mathbb{C}}(X))$, we define, using definition 67 and theorem 44

$$\begin{aligned}Rf_{*Hdg}(((M_I,F,W),u_{IJ}),(K,W),\alpha):=\iota_S^{-1}(\int_{\bar{f}}^{FDR}(n\times I)_{*Hdg}((M_I,F,W),u_{IJ}),Rf_{*w}(K,W),f_*(\alpha))\\ \in D(MHM_{gm,k,\mathbb{C}}(S))\end{aligned}$$

where $f_*(\alpha)$ is given in definition 68, and since

- *by definition*

$$H^i(\int_{\bar{f}}^{FDR}\mathrm{Gr}^k_W(I\times n)_{Hdg*}((M_I,F,W),u_{IJ}),R\bar{f}_*\,\mathrm{Gr}^k_W n'_{*w}(K,W),\bar{f}_*\,\mathrm{Gr}^k_W n'_*\alpha)\in HM_{gm,k,\mathbb{C}}(S)$$

for all $i,k\in\mathbb{Z}$, hence by the spectral sequence for the filtered complexes $\int_{\bar{f}}^{FDR}(I\times n)_{Hdg*}((M_I,F,W),u_{IJ})$ and $R\bar{f}_*((I\times n)_{*w}(K,W))$

$$\begin{aligned}\mathrm{Gr}^k_W H^i(\int_f^{Hdg}((M_I,F,W),u_{IJ}),Rf_{*w}(K,W),f_*\alpha)):=\\ (\mathrm{Gr}^k_W H^i\int_{\bar{f}}^{FDR}(I\times n)_{Hdg*}((M_I,F,W),u_{IJ}),\mathrm{Gr}^k_W H^iR\bar{f}_*n'_{*w}(K,W),\mathrm{Gr}^k_W H^if_*\alpha)\in HM_{gm,k,\mathbb{C}}(S)\end{aligned}$$

this gives by definition $H^i(\int_f^{Hdg}((M_I,F,W),u_{IJ}),Rf_{*w}(K,W),f_*(\alpha))\in MHM_{gm,k,\mathbb{C}}(S)$ for all $i\in\mathbb{Z}$.

– $\int_f^{Hdg}((M_I,F,W),u_{IJ})$ *is the class of a complex such that the differential are strict for* F *by theorem 40 in the complex case.*

(ii) For $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in C(MHM_{gm,k,\mathbb{C}}(X))$*, we define, using definition 67 and theorem 44,*

$$Rf_{!Hdg}(((M_I,F,W),u_{IJ}),(K,W),\alpha):=\iota_S^{-1}(\int_{\bar{f}}^{FDR}(n\times I)_{!Hdg}((M_I,F,W),u_{IJ}),Rf_{!w}(K,W),f_!(\alpha))$$
$$\in D(MHM_{gm,k,\mathbb{C}}(S))$$

where $f_!(\alpha)$ *is given in definition 68, and since*

– *by definition*

$$H^i(\int_{\bar{f}}^{FDR}\mathrm{Gr}_W^k(n\times I)_{!Hdg}((M_I,F,W),u_{IJ}),R\bar{f}_*\,\mathrm{Gr}_W^k\,n'_{!w}(K,W),\mathrm{Gr}_W^k\,f_!\alpha)\in HM_{gm,k,\mathbb{C}}(S)$$

for all $i,k\in\mathbb{Z}$*, hence by the spectral sequence for the filtered complexes* $\int_{\bar{f}}^{FDR}(n\times I)_{!Hdg}((M_I,F,W),u_{IJ})$ *and* $R\bar{f}_*(n\times I)_{!w}(K,W)$

$$\mathrm{Gr}_W^k\,H^i(\int_{f!}^{Hdg}((M_I,F,W),u_{IJ}),Rf_{!w}(K,W),f_!\alpha):=$$
$$(\mathrm{Gr}_W^k\,H^i\int_{\bar{f}}^{FDR}(n\times I)_{!Hdg}((M_I,F,W),u_{IJ}),\mathrm{Gr}_W^k\,H^iR\bar{f}_*n'_{!w}(K,W),\mathrm{Gr}_W^k\,H^i f_!\alpha)\in HM_{gm,k,\mathbb{C}}(S)$$

this gives by definition $H^i(\int_{f!}^{Hdg}((M_I,F,W),u_{IJ}),Rf_{!w}(K,W),f_!(\alpha))\in MHM_{gm,k,\mathbb{C}}(S)$ *for all* $i\in\mathbb{Z}$.

– $\int_{f!}^{Hdg}((M_I,F,W),u_{IJ})$ *is the class of a complex such that the differential are strict for* F *by theorem 40 in the complex case.*

Proposition 47. *Let* $f_1:X\to Y$ *and* $f_2:Y\to S$ *two morphism with* $X,Y,S\in\mathrm{QPVar}(k)$.

(i) Let $\mathcal{M}\in C(MHM_{gm,k,\mathbb{C}}(X))$*. Then,*

$$R(f_2\circ f_1)_*^{Hdg}\mathcal{M}=Rf_{2*}^{Hdg}Rf_{1*}^{Hdg}\mathcal{M}\in D(MHM_{gm,k,\mathbb{C}}(S)).$$

(ii) Let $(M,F,W)\in C(MHM_{gm,k,\mathbb{C}}(X))$*. Then,*

$$R(f_2\circ f_1)_!^{Hdg}\mathcal{M}=Rf_{2!}^{Hdg}Rf_{1!}^{Hdg}\mathcal{M}\in D(MHM_{gm,k,\mathbb{C}}(S))$$

Proof. Immediate from definition. □

Let $k\subset\mathbb{C}$ a subfield. Definition 74, definition 76 and gives by proposition 45 and proposition 47 respectively, the following 2 functors :

- We have the following 2 functor on the category of algebraic varieties over $k\subset\mathbb{C}$

$$D(MHM_{gm,k,\mathbb{C}}(\cdot)):\mathrm{QPVar}(k)\to\mathrm{TriCat},\ S\mapsto D(MHM_{gm,k,\mathbb{C}}(S)),$$
$$(f:T\to S)\longmapsto(f^{*Hdg}:(((M_I,F,W),u_{IJ}),(K,W),\alpha)\mapsto$$
$$f^{!Hdg}(((M_I,F,W),u_{IJ}),(K,W),\alpha):=(f_{Hdg}^{*mod}(((M_I,F,W),u_{IJ})),f^{!w}(K,W),f^!\alpha)).$$

see definition 64 and definition 68 for the equality.

- We have the following 2 functor on the category of quasi-projective algebraic varieties over $k \subset \mathbb{C}$

$$D(MHM_{gm,k,\mathbb{C}}(\cdot)) : \mathrm{QPVar}(k) \to \mathrm{TriCat}, \ S \mapsto D(MHM_{gm,k,\mathbb{C}}(S)),$$
$$(f : T \to S) \longmapsto (f_{*Hdg} : (((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto Rf_{*Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha)).$$

- We have the following 2 functor on the category of quasi-projective algebraic varieties over $k \subset \mathbb{C}$

$$D(MHM_{gm,k,\mathbb{C}}(\cdot)) : \mathrm{QPVar}(k) \to \mathrm{TriCat}, \ S \mapsto D(MHM_{gm,k,\mathbb{C}}(S)),$$
$$(f : T \to S) \longmapsto (f_{!Hdg} : (((M_I, F, W)), (K, W), \alpha) \mapsto f_{!Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha)).$$

- We have the following 2 functor on the category of algebraic varieties over $k \subset \mathbb{C}$

$$D(MHM_{gm,k,\mathbb{C}}(\cdot)) : \mathrm{QPVar}(k) \to \mathrm{TriCat}, \ S \mapsto D(MHM_{gm,k,\mathbb{C}}(S)),$$
$$(f : T \to S) \longmapsto (f^{!Hdg} : (((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto$$
$$f^{*Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) := (f_{Hdg}^{\hat{*}mod}(((M_I, F, W), u_{IJ})), f^{*w}(K, W), f^*\alpha)).$$

see definition 64 and definition 68 for the equality.

Proposition 48. *Let $f : X \to S$ with $S, X \in \mathrm{QPVar}(k)$. Then*

(i) $(f^{*Hdg}, Rf_*^{Hdg}) : D(MHM_{gm,k,\mathbb{C}}(S)) \to D(MHM_{gm,k,\mathbb{C}}(X))$ *is a pair of adjoint functors.*

- *For* $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C(MHM_{gm,k,\mathbb{C}}(S))$,

$$\mathrm{ad}(f^{*Hdg}, Rf_*^{Hdg})(((M_I, F, W), u_{IJ}), (K, W), \alpha) :=$$
$$(\mathrm{ad}(f_{Hdg}^{\hat{*}mod}, Rf_*^{Hdg})((M_I, F, W), u_{IJ}), \mathrm{ad}(f^{*w}, Rf_{*w})(K, W)) :$$
$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \to Rf_*^{Hdg} f^{*Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha)$$

is the adjonction map in $D(MHM_{gm,k,\mathbb{C}}(S))$.

- *For* $(((N_I, F, W), u_{IJ}), (P, W), \beta) \in C(MHM_{gm,k,\mathbb{C}}(X))$,

$$\mathrm{ad}(f^{*Hdg}, Rf_*^{Hdg})(((N_I, F, W), u_{IJ}), (P, W), \beta) :=$$
$$(\mathrm{ad}(f_{Hdg}^{\hat{*}mod}, Rf_*^{Hdg})((N_I, F, W), u_{IJ}), \mathrm{ad}(f^{*w}, Rf_{*w})(P, W)) :$$
$$f^{*Hdg} Rf_*^{Hdg}(((N_I, F, W), u_{IJ}), (P, W), \beta) \to (((N_I, F, W), u_{IJ}), (P, W), \beta)$$

is the adjonction map in $D(MHM_{gm,k,\mathbb{C}}(X))$

(ii) $(Rf_!^{Hdg}, f^{!Hdg}) : D(MHM_{gm,k,\mathbb{C}}(X)) \to D(MHM_{gm,k,\mathbb{C}}(S))$ *is a pair of adjoint functors.*

- *For* $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C(MHM_{gm,k,\mathbb{C}}(S))$,

$$\mathrm{ad}(Rf_!^{Hdg}, f^{!Hdg})(((M_I, F, W), u_{IJ}), (K, W), \alpha) :=$$
$$(\mathrm{ad}(f_{Hdg}^{*mod}, Rf_!^{Hdg})((M_I, F, W), u_{IJ}), \mathrm{ad}(f^{!w}, Rf_{!w})(K, W)) :$$
$$Rf_!^{Hdg} f^{!Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) \to (((M_I, F, W), u_{IJ}), (K, W), \alpha)$$

is the adjonction map in $D(MHM_{gm,k,\mathbb{C}}(S))$.

- *For* $(((N_I, F, W), u_{IJ}), (P, W), \beta) \in C(MHM_{gm,k,\mathbb{C}}(X))$,

$$\mathrm{ad}(Rf_!^{Hdg}, f^{!Hdg})(((N_I, F, W), u_{IJ}), (P, W), \beta) :=$$
$$(\mathrm{ad}(f_{Hdg}^{*mod}, Rf_!^{Hdg})((N_I, F, W), u_{IJ}), \mathrm{ad}(f^{!w}, Rf_{!w})(P, W)) :$$
$$(((N_I, F, W), u_{IJ}), (P, W), \beta) \to f^{!Hdg} Rf_!^{Hdg}(((N_I, F, W), u_{IJ}), (P, W), \beta)$$

is the adjonction map in $D(MHM_{gm,k,\mathbb{C}}(X))$.

Proof. Follows from proposition 44 after considering a factorization $f : X \hookrightarrow \bar{Y} \times S \xrightarrow{p_S} S$ with $\bar{Y} \in \mathrm{PSmVar}(k)$. □

Theorem 45. *Let $k \subset \mathbb{C}$ a subfield.*

(i) We have the six functor formalism on $D(MHM_{gm,k,\mathbb{C}}(-)) : \mathrm{SmVar}(k) \to \mathrm{TriCat}$.

(ii) We have the six functor formalism on $D(MHM_{gm,k,\mathbb{C}}(-)) : \mathrm{QPVar}(k) \to \mathrm{TriCat}$.

Proof. Follows from proposition 48. □

We recall the definition of the Deligne complex of a complex manifold and the Deligne cohomology class of an algebraic cycle of a complex algebraic variety.

Definition 77. *(i) Let $X \in \mathrm{AnSm}(\mathbb{C})$. We have for $d \in \mathbb{Z}$ the Deligne complex*

$$\mathbb{Z}_{\mathcal{D},X}(d) := (\mathbb{Z}_X(d) \hookrightarrow \tau^{\leq d} DR(X)) = (\mathbb{Z}_X(d) \hookrightarrow (O_X \to \cdots \to \Omega_X^{d-1}) \in C(X)$$

Let $D \subset X$ a normal crossing divisor. We have for $d \in \mathbb{Z}$ the Deligne complexes

$$\mathbb{Z}_{\mathcal{D},(X,D)}(d) := (\mathbb{Z}_X(d) \hookrightarrow (O_X \to \cdots \to \Omega_X^{d-1}(\log D))) \in C(X)$$

and

$$\mathbb{Z}_{\mathcal{D},(X,D)}(d)^\vee := (\mathbb{Z}_X(d) \hookrightarrow (O_X \to \cdots \to \Omega_X^{d-1}(\mathrm{nul}\, D))) \in C(X).$$

Moreover we have (see [15]) canonical products

- $(-) \cdot (-) : \mathbb{Z}_{\mathcal{D},(X,D)}(d) \otimes \mathbb{Z}_{\mathcal{D},(X,D)}(d') \to \mathbb{Z}_{\mathcal{D},(X,D)}(d+d')$
- $(-) \cdot (-) : \mathbb{Z}_{\mathcal{D},(X,D)}(d)^\vee \otimes \mathbb{Z}_{\mathcal{D},(X,D)}(d')^\vee \to \mathbb{Z}_{\mathcal{D},(X,D)}(d+d')^\vee$

(ii) Let $X \in \mathrm{AnSm}(\mathbb{C})$. We have for $d \in \mathbb{Z}$ the Deligne (homology) complex

$$C^\bullet_{\mathcal{D}}(X, \mathbb{Z}(d)) := \mathrm{Cone}(\mathbb{Z}\,\mathrm{Hom}_{Diff(\mathbb{R})}(\Delta^\bullet, X) \oplus \Gamma(X, F^d\mathcal{D}_X^\bullet) \hookrightarrow \Gamma(X, \mathcal{D}_X^\bullet)) \in C(\mathbb{Z})$$

Let $D \subset X$ a normal crossing divisor. Denote $U := X \backslash D$. We have for $d \in \mathbb{Z}$ the Deligne (homology) complexes

$$C^\bullet_{\mathcal{D}}((X,D), \mathbb{Z}(d)) := \mathrm{Cone}(\mathbb{Z}\,\mathrm{Hom}_{Diff(\mathbb{R})}(\Delta^\bullet, U) \oplus \Gamma(X, F^d\mathcal{D}_X^\bullet(\log D)) \hookrightarrow \Gamma(X, \mathcal{D}_X^\bullet(\log D))) \in C(\mathbb{Z})$$

and

$$C^\bullet_{\mathcal{D}}(X, D, \mathbb{Z}(d)) := \mathrm{Cone}(\mathbb{Z}\,\mathrm{Hom}_{Diff(\mathbb{R})}(\Delta^\bullet, (X,D)) \oplus \Gamma(X, F^d\mathcal{D}_X^\bullet(\mathrm{nul}\, D)) \hookrightarrow \Gamma(X, \mathcal{D}_X^\bullet(\mathrm{nul}\, D))) \in C(\mathbb{Z}).$$

(iii) Let $k \subset \mathbb{C}_p$ a subfield. Let $X \in \mathrm{PSmVar}(k)$. We have, for $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$, the Deligne cohomology

$$H^k_{\mathcal{D}}(X^{an}_{\mathbb{C}}, \mathbb{Z}(d)) := \mathbb{H}^k(X^{an}_{\mathbb{C}}, \mathbb{Z}_{X,\mathcal{D}}(d)) = H^k C^\bullet_{\mathcal{D}}(X^{an}_{\mathbb{C}}, D, \mathbb{Z}(d))^\vee$$

Let $U \in \mathrm{SmVar}(k)$. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X\backslash U$ a normal crossing divisor. We have, for $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$, the Deligne cohomology

$$H^k_{\mathcal{D}}(U^{an}_{\mathbb{C}}, \mathbb{Z}(d)) := \mathbb{H}^k(X, \mathbb{Z}_{(X^{an}_{\mathbb{C}}, D^{an}_{\mathbb{C}}),\mathcal{D}}(d)) = H^k C^\bullet_{\mathcal{D}}((X^{an}_{\mathbb{C}}, D^{an}_{\mathbb{C}}), \mathbb{Z}(d))^\vee$$

and

$$H^k_{\mathcal{D}}(X, D, \mathbb{Z}(d)) := \mathbb{H}^k(X^{an}_{\mathbb{C}}, \mathbb{Z}_{(X^{an}_{\mathbb{C}}, D^{an}_{\mathbb{C}}),\mathcal{D}}(d)^\vee) = H^k C^\bullet_{\mathcal{D}}(X^{an}_{\mathbb{C}}, D^{an}_{\mathbb{C}}, \mathbb{Z}(d))^\vee.$$

(iv) Let $k \subset \mathbb{C}_p$ a subfield. Let $U \in \mathrm{SmVar}(k)$. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X \backslash U$ a normal crossing divisor. We define the Deligne cohomology of a (higher) cycle $Z \in \mathcal{Z}^d(U,n)^{\partial=0}$ by

$$[Z]_{\mathcal{D}} := \mathrm{Im}(H^{2d-n}(\gamma_{\mathrm{supp}(Z)})([Z])),$$
$$H^k(\gamma_{\mathrm{supp}(Z)}) : \mathbb{H}^{2d-n}_{\mathcal{D},\mathrm{supp}(Z)}(X^{an}_{\mathbb{C}}, \mathbb{Z}_{X^{an}_{\mathbb{C}},D^{an}_{\mathbb{C}}}(d)) \to \mathbb{H}^{2d-n}_{\mathcal{D}}(X^{an}_{\mathbb{C}}, \mathbb{Z}_{X^{an}_{\mathbb{C}},D^{an}_{\mathbb{C}}}(d))$$

with $\mathrm{supp}(Z) := p_X(\mathrm{supp}(Z)) \subset X$, where $\mathrm{supp}(Z) \subset X \times \square^n$ is the support of Z.

(v) Let $k \subset \mathbb{C}_p$ a subfield. Let $U \in \mathrm{SmVar}(k)$. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X \backslash U$ a normal crossing divisor. We have for $d \in \mathbb{Z}$ the morphism of complexes

$$\mathcal{R}^d_U : \mathcal{Z}^d(U,\bullet) \to C^{\bullet}_{\mathcal{D}}(X^{an}_{\mathbb{C}}, D^{an}_{\mathbb{C}}, \mathbb{Z}(d)),\ Z \mapsto \mathcal{R}^d_U(Z) := (T_{\bar{Z}}, \Omega_{\bar{Z}}, R_{\bar{Z}})$$

which gives for $Z \in \mathcal{Z}^d(U,n)^{\partial=0}$,

$$[\mathcal{R}^d_U(Z)] = [Z]_{\mathcal{D}} \in H^{2d-n}_{\mathcal{D}}(U^{an}_{\mathbb{C}}, \mathbb{Z}(d))$$

Let $f : X \to S$ a morphism with $S, X \in \mathrm{AnSm}(\mathbb{C})$. We have for $d \in \mathbb{Z}$ the canonical morphism of Deligne complexes

$$(\mathrm{ad}(f^*, f_*)(\mathbb{Z}_S), \Omega^{\leq d}_{X/S}) : \mathbb{Z}_{\mathcal{D},S}(d) \to f_*\mathbb{Z}_{\mathcal{D},X}(d)$$

which induces after taking the resolution of the Deligne complexes by differential forms the morphism in $C(\mathbb{Z})$

$$f^* := (f^*, f^*, \theta(f)^t) : \mathrm{Cone}(\mathbb{Z}\,\mathrm{Hom}_{Diff(\mathbb{R})}(\Delta^{\bullet}, S)^{\vee} \oplus \Gamma(S, F^d\mathcal{A}^{\bullet}_S) \hookrightarrow \Gamma(S, \mathcal{A}^{\bullet}_S))$$
$$\to \mathrm{Cone}(\mathbb{Z}\,\mathrm{Hom}_{Diff(\mathbb{R})}(\Delta^{\bullet}, X)^{\vee} \oplus \Gamma(X, F^d\mathcal{A}^{\bullet}_X) \hookrightarrow \Gamma(X, \mathcal{A}^{\bullet}_X))$$

where $\theta(f)^t$ is the homotopy in the morphism in $D_{fil}(k) \otimes_I D(\mathbb{Z})$

$$(f^*, f^*, \theta(f)^t) : (\Gamma(S, (\Omega^{\bullet}_S, F_b)), \mathbb{Z}\,\mathrm{Hom}_{Diff(\mathbb{R})}(\Delta^{\bullet}, S)^{\vee}, a_{S*}\alpha(S))$$
$$\to (\Gamma(X, (\Omega^{\bullet}_X, F_b)), \mathbb{Z}\,\mathrm{Hom}_{Diff(\mathbb{R})}(\Delta^{\bullet}, X)^{\vee}, a_{X*}\alpha(X)),$$

which induces in cohomology for $n \in \mathbb{Z}$, the morphisms of abelian groups

$$f^* : H^n_{\mathcal{D}}(S, \mathbb{Z}(d)) \to H^n_{\mathcal{D}}(X, \mathbb{Z}(d));$$

we get dually, after taking the resolution of the Deligne complexes by currents the morphism in $C(\mathbb{Z})$

$$f_* := (f_*, f_*, \theta(f)) : C^{\bullet}_{\mathcal{D}}(X, \mathbb{Z}(d)) := \mathrm{Cone}(\mathbb{Z}\,\mathrm{Hom}_{Diff(\mathbb{R})}(\Delta^{\bullet}, X) \oplus \Gamma(X, F^d\mathcal{D}^{\bullet}_X) \hookrightarrow \Gamma(X, \mathcal{D}^{\bullet}_X))$$
$$\to C^{\bullet}_{\mathcal{D}}(S, \mathbb{Z}(d)) := \mathrm{Cone}(\mathbb{Z}\,\mathrm{Hom}_{Diff(\mathbb{R})}(\Delta^{\bullet}, S) \oplus \Gamma(S, F^d\mathcal{D}^{\bullet}_S) \hookrightarrow \Gamma(S, \mathcal{D}^{\bullet}_S))$$

where $\theta(f)$ is the homotopy in the morphism in $D_{fil}(k) \otimes_I D(\mathbb{Z})$

$$(f_*, f_*, \theta(f)) : (\Gamma(X, (\Omega^{\bullet}_X, F_b)), \mathbb{Z}\,\mathrm{Hom}_{Diff(\mathbb{R})}(\Delta^{\bullet}, X), a_{X!}\alpha(X))$$
$$\to (\Gamma(S, (\Omega^{\bullet}_S, F_b)), \mathbb{Z}\,\mathrm{Hom}_{Diff(\mathbb{R})}(\Delta^{\bullet}, S), a_{S!}\alpha(S)),$$

which induces in homology for $n \in \mathbb{Z}$, the morphisms of abelian groups

$$f_* : H_{n,\mathcal{D}}(X, \mathbb{Z}(d)) \to H_{n,\mathcal{D}}(S, \mathbb{Z}(d)).$$

Theorem 46. *Let $k \subset \mathbb{C}$ a subfield.*

(i) Let $U \in \mathrm{SmVar}(k)$. Denote by $a_U : U \to \mathrm{pt}$ the terminal map. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X\backslash U$ a normal crossing divisor. The embedding (see theorem 44)

$$\iota : D(MHM_{gm,k,\mathbb{C}}(\{\mathrm{pt}\})) \to D_{fil}(k) \times_I D(\mathbb{Z})$$

induces for $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$, canonical isomorphisms

$$\iota(a_{U!Hdg}\mathbb{Z}_U^{Hdg}) : H^k(a_{U!Hdg}\mathbb{Z}_U^{Hdg}) \xrightarrow{\sim} H^k_{\mathcal{D}}(X^{an}_{\mathbb{C}}, D^{an}_{\mathbb{C}}, \mathbb{Z}(d)), \text{ and}$$
$$\iota(a_{U*Hdg}\mathbb{Z}_U^{Hdg}) : H^k(a_{U*Hdg}\mathbb{Z}_U^{Hdg}) \xrightarrow{\sim} H^k_{\mathcal{D}}(U^{an}_{\mathbb{C}}, \mathbb{Z}(d)).$$

(ii) Let $h : U \to S$ and $h' : U' \to S$ two morphism with $S, U, U' \in \mathrm{SmVar}(k)$. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X\backslash U$ a normal crossing divisor such that $h : U \to S$ extend to $f : X \to \bar{S}$. Let $X' \in \mathrm{PSmVar}(k)$ a compactification of U' with $D' := X'\backslash U'$ a normal crossing divisor such that $h' : U' \to S$ extend to $f' : X' \to \bar{S}$. The embedding $\iota : D(MHM_{gm,k,\mathbb{C}}(\mathrm{pt})) \to D_{fil}(k) \times_I D(\mathbb{Z})$ (see theorem 44) induces for $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$ a canonical isomorphism

$$\begin{aligned}
&\iota(a_{U'\times_S U!Hdg}\mathbb{Z}^{Hdg}_{U'\times_S U}) : \mathrm{Hom}_{D(MHM_{gm,k,\mathbb{C}}(S))}(h_{U'!Hdg}\mathbb{Z}^{Hdg}_{U'}, h_{U!Hdg}\mathbb{Z}^{Hdg}_U(d)[k]) \\
&\xrightarrow{RI(-,-)} \mathrm{Hom}_{D(MHM_{gm,k,\mathbb{C}}(\mathrm{pt}))}(\mathbb{Z}^{Hdg}_{\mathrm{pt}}, a_{U'\times_S U!Hdg}\mathbb{Z}^{Hdg}_{U'\times_S U}(d)[k]) = H^k(a_{U'\times_S U!Hdg}\mathbb{Z}^{Hdg}_{U'\times_S U}(d)) \\
&\xrightarrow{\sim} H^k_{\mathcal{D}}((X'\times_S X)^{an}_{\mathbb{C}}, ((X'\times_S U) \cup (U'\times_S X))^{an}_{\mathbb{C}}, \mathbb{Z}(d)).
\end{aligned}$$

(iii) Let $U \in \mathrm{SmVar}(k)$. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X\backslash U$ a normal crossing divisor. For $[Z] \in \mathrm{CH}^d(U, n)$ and $[Z'] \in \mathrm{CH}^{d'}(U, n')$, we have

$$([Z] \cdot [Z'])_{\mathcal{D}} = [Z]_{\mathcal{D}} \cdot [Z']_{\mathcal{D}} \in H^{2d+2d'-n-n'}(U^{an}_{\mathbb{C}}, \mathbb{Z}(d + d'))$$

where the product on the left is the intersection of higher Chow cycle which is well defined modulo boundary (they intersect properly modulo boundary) while the right product of Deligne cohomology classes is induced by the product of Deligne complexes $(-)\cdot(-) : \mathbb{Z}_{\mathcal{D},(X,D)}(d) \otimes \mathbb{Z}_{\mathcal{D},(X,D)}(d') \to \mathbb{Z}_{\mathcal{D},(X,D)}(d + d')$.

(iv) Let $h : U \to S$,$h' : U' \to S$, $h'' : U'' \to S$ three morphism with $S, U, U', U'' \in \mathrm{SmVar}(k)$. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X\backslash U$ a normal crossing divisor such that $h : U \to S$ extend to $f : X \to \bar{S}$. Let $X' \in \mathrm{PSmVar}(k)$ a compactification of U' with $D' := X'\backslash U'$ a normal crossing divisor such that $h' : U' \to S$ extend to $f' : X' \to \bar{S}$. Let $X' \in \mathrm{PSmVar}(k)$ a compactification of U' with $D' := X'\backslash U'$ a normal crossing divisor such that $h' : U' \to S$ extend to $f' : X' \to \bar{S}$. For $[Z] \in \mathrm{CH}^d(U \times_S U', n)$ and $[Z'] \in \mathrm{CH}^{d'}(U' \times_S U'', n')$, we have

$$([Z] \circ [Z'])_{\mathcal{D}} = [Z]_{\mathcal{D}} \circ [Z']_{\mathcal{D}} \in H^{d''-n''}((U \times_S U'')^{an}_{\mathbb{C}}, \mathbb{Z}(d'' - n''))$$

where the composition on the left is the composition of higher correspondence modulo boundary while the composition on the right is given by (ii).

Proof. (i):Standard.
(ii):Follows on the one hand from (i) and on the other hand the six functor formalism on the 2-functor $D(MHM_{gm,k,\mathbb{C}}(-)) : \mathrm{SmVar}(k) \to \mathrm{TriCat}$ (theorem 45) gives the isomorphism $RI(-,-)$.
(iii):Standard.
(iv):Follows from (iii).

□

6.2 The p-adic case where $k \subset K \subset \mathbb{C}_p$

Let p a prime integer. Let $k \subset K \subset \mathbb{C}_p$ a subfield of a p adic field K. Denote by $\bar{k} \subset \mathbb{C}_p$ its algebraic closure.

- We denote by $G := \mathrm{Gal}(\bar{K}/K) \subset \mathrm{Gal}(\bar{k}/k)$ the Galois group of K.
- For $S \in \mathrm{Var}(k)$,
  - we will consider $\mathbb{B}_{dr,S_K} := \mathbb{B}_{dr,R_K(S_K^{an})}$ and $O\mathbb{B}_{dr,S_K} := O\mathbb{B}_{dr,R_K(S_K^{an})}$ where $R_K : \mathrm{AnSp}(K) \to \mathrm{AdSp}/(K, O_K)$ the canonical functor
  - we will consider $\mathbb{B}_{dr,S_{\mathbb{C}_p}} := \mathbb{B}_{dr,R_{\mathbb{C}_p}(S_{\mathbb{C}_p}^{an})}$ and $O\mathbb{B}_{dr,S_{\mathbb{C}_p}} := O\mathbb{B}_{dr,R_{\mathbb{C}_p}(S_{\mathbb{C}_p}^{an})}$ where $R_{\mathbb{C}_p} : \mathrm{AnSp}(\mathbb{C}_p) \to \mathrm{AdSp}/(\mathbb{C}_p, O_{\mathbb{C}_p})$ the canonical functor.
- Recall (see section 2) that for a prime number l, a $\mathbb{Z}_l$ module $K = (K_n)_{n\in\mathbb{N}} \in \mathrm{Fun}(\mathbb{N}, \mathrm{Ab})$ is a projective system with K_n a l^n torsion group such that $K_n \to K_{n+1}/l^n K_{n+1}$ is an isomorphism. For $S \in \mathrm{Var}(k)$ and l a prime integer, we have (see section 2)
  - the full subcategory $C_{\mathbb{Z}_l fil}(S^{et}) \subset \mathrm{PSh}(S^{et}, \mathrm{Fun}(\mathbb{N}, C(\mathbb{Z})))$ and the full subcategory
  $$D_{\mathbb{Z}_l fil,c,k}(S^{et}) \subset D_{\mathbb{Z}_l fil}(S^{et}) := \mathrm{Ho}_{et}\, C_{\mathbb{Z}_l fil}(S^{et}),$$
  whose cohomology sheaves of the graded piece are constructible with respect to a Zariski stratification of S, the full subcategory $C_{\mathbb{Z}_l fil}(S_K^{an,pet}) \subset \mathrm{PSh}(S_K^{an,pet}, \mathrm{Fun}(\mathbb{N}, C(\mathbb{Z})))$ and the full subcategory
  $$D_{\mathbb{Z}_l fil,c,k}(S_K^{an,pet}) \subset D_{\mathbb{Z}_l fil}(S_K^{an,pet}) := \mathrm{Ho}_{pet}\, C_{\mathbb{Z}_l fil}(S_K^{an,pet})$$
  whose cohomology sheaves of the graded piece are constructible with respect to a Zariski stratification of S,
  - $P_{\mathbb{Z}_l,fil}(S^{et}) \subset D_{\mathbb{Z}_l fil,c}(S^{et})$ and $P_{\mathbb{Z}_l,fil,k}(S_K^{an,pet}) \subset D_{\mathbb{Z}_l fil,c,k}(S_K^{an,pet})$ the full subcategories of filtered perverse sheaves.
- Let $S \in \mathrm{Var}(k)$ and $D \subset S$ a Cartier divisor. For $(K, W) \in D_{\mathbb{Z}_l fil}(S^{et})$, we denote for short
  $$\psi_D(K, W) := \psi_D(K, W)[-1] \in D_{\mathbb{Z}_l fil}(S^{et}), \phi_D(K, W) := \phi_D(K, W)[-1] \in D_{\mathbb{Z}_l fil}(S^{et}),$$
  $$x_{S\backslash D/S}(K, W) := x_{S\backslash D/S}(K, W)[-1] \in D_{\mathbb{Z}_l fil}(S^{et})$$
  so that it sends (filtered) perverse sheaves to (filtered) perverse sheaves.
- For $S \in \mathrm{Var}(k)$, we denote by $\mathrm{an}_S : S^{an} := S_{\mathbb{C}_p}^{an} \xrightarrow{\mathrm{an}_S} S_{\mathbb{C}_p} \xrightarrow{\pi_{k/\mathbb{C}_p}(S)} S$ the morphism of ringed spaces given by the analytical functor.
  - For $(M, F) \in C_{O_S fil}(S)$, we denote by $(M, F)^{an} := \mathrm{an}_S^{*mod}(M, F) \in C_{O_S fil}(S_{\mathbb{C}_p}^{an})$.
  - For $(M, F) \in C_{\mathcal{D} fil}(S)$, we denote by $(M, F)^{an} := \mathrm{an}_S^{*mod}(M, F) \in C_{\mathcal{D} fil}(S_{\mathbb{C}_p}^{an})$.

  We denote for short
  $$DR(S) := DR(S_{\mathbb{C}_p}^{an}) \circ \mathrm{an}_S^{*mod} : C_{\mathcal{D} fil}(S) \to C_{fil}(S_{\mathbb{C}_p}^{an,pet}),\ M \mapsto DR(S)(M^{an})$$
  the De Rham functor.
- Let $S \in \mathrm{Var}(k)$.
  - For $K_1, K_2 \in D_{\mathbb{Z}_p}(S_K^{an,pet})$, we denote for short $K_1 \otimes_{\mathbb{Q}_p} K_2 := K_1 \otimes_{\mathbb{Q}_p}^L K_2 \in D_{\mathbb{Z}_p}(S_K^{an,pet})$ the derived tensor product.

– For $(K_1,W),(K_2,W)\in D_{\mathbb{Z}_pfil}(S_K^{an,pet})$, we denote for short $(K_1,W)\otimes_{\mathbb{Q}_p}(K_2,W):=(K_1,W)\otimes^L_{\mathbb{Q}_p}(K_2,W)\in D_{\mathbb{Z}_pfil}(S_K^{an,pet})$ the derived tensor product.
– For $M,N\in D_{\mathbb{B}_{dr,S_K}}(S_K^{an,pet})$, we denote for short $M\otimes_{\mathbb{B}_{dr,S}}N:=M\otimes^L_{\mathbb{B}_{dr,S_K}}N\in D_{\mathbb{B}_{dr,S_K}}(S_K^{an,pet})$ the derived tensor product.
– For $(M,W),(N,W)\in D_{\mathbb{B}_{dr,S_K}fil}(S_K^{an,pet})$, we denote for short $(M,W)\otimes_{\mathbb{B}_{dr,S}}(N,W):=(M,W)\otimes^L_{\mathbb{B}_{dr,S_K}}(N,W)\in D_{\mathbb{B}_{dr,S_K}}(S_K^{an,pet})$ the derived tensor product.

- Let $S\in\mathrm{SmVar}(k)$ and $D\subset S$ a (Cartier) divisor. Denote by $j:S^o:=S\backslash D\hookrightarrow S$ the open embedding. Then $j_*:C(S_K^{o,an,pet})\to C(S_K^{an,pet})$ is preserve pro-etale equivalence, that is $Rj_*=j_*$.

6.2.1 The $\mathbb{B}_{dr}$ functor

Motivated by theorem 32 and theorem 15, we make the following definition:

Definition 78. *Let $S\in\mathrm{SmVar}(k)$. Let $D=V(s)\subset S$ a (Cartier) divisor, where $s\in\Gamma(S,L)$ is a section of the line bundle $L=L_D$ associated to D. We denote by $j:S^o:=S\backslash D\hookrightarrow S$ the open complementary subset. Let $\pi:\tilde{S}_K^{o,an}\to S_K^{o,an}$ the perfectoid universal covering.*

(i) We define, using definition 59,

$$\mathbb{B}_{dr,S^o/S,K}:=F^0DR(S)((j_{*Hdg}(O_{S^o},F_b))^{an}\otimes_{O_{S_K^{an}}}(O\mathbb{B}_{dr,S_K},F))\in C_{\mathbb{B}_{dr,S_K}}(S_K^{an,pet})$$

together with the canonical map in $C_{\mathbb{B}_{dr,S_K}}(S_K^{an,pet})$

$$a_S(\mathbb{B}_{dr,S^o/S,K}):=F^0DR(S)(\mathrm{ad}(j^*,j_{*Hdg})(O_S,F_b)^{an}\otimes I):\mathbb{B}_{dr,S_K}\to\mathbb{B}_{dr,S^o/S,K}$$

(ii) We define, using definition 57, the nearby cycle module

$$\mathbb{B}_{dr,\psi_D,K}:=F^0DR(S)((\psi_D(O_{S^o},F_b))^{an}\otimes_{O_{S_K^{an}}}(O\mathbb{B}_{dr,S_K},F))\in C_{\mathbb{B}_{dr,S}}(S_K^{an,pet})$$

together with the canonical maps in $C_{\mathbb{B}_{dr,S_K}}(S_K^{an,pet})$

$$\rho_{\mathbb{B}_{dr},D}(O_S):=F^0DR(S)(\rho_{DR,D}(O_{S^o},F_b)^{an}\otimes I):\mathbb{B}_{dr,S^o/S,K}\to\mathbb{B}_{dr,\psi_D,K}$$

and

$$a_S(\mathbb{B}_{dr,\psi_D}):=F^0DR(S)((\rho_{DR,D}(O_{S^o},F_b)\circ\mathrm{ad}(j^*,j_{*Hdg})(O_S,F_b))^{an}\otimes I):$$
$$\mathbb{B}_{dr,S_K}\xrightarrow{a_S(\mathbb{B}_{dr,S^o/S,K})}\mathbb{B}_{dr,S^o/S,K}\xrightarrow{\rho_{\mathbb{B}_{dr},D}(O_S)}\mathbb{B}_{dr,\psi_D,K}$$

where $\rho_{DR,D}(O_{S^o},F_b):j_{*Hdg}(O_{S^o},F_b)\to\psi_D(O_{S^o},F_b)$ is given in definition 60.

(iii) We define, using definition 57 and (ii), the vanishing cycle module

$$\mathbb{B}_{dr,\phi_D,K}:=F^0DR(S)((\phi_D(O_{S^o},F_b))^{an}\otimes_{O_{S_K^{an}}}(O\mathbb{B}_{dr,S_K},F))\in C_{\mathbb{B}_{dr,S_K}}(S_K^{an,pet}).$$

together with the canonical maps in $C_{\mathbb{B}_{dr,S_K}}(S_K^{an,pet})$ We have using definition 57 the following maps

$$can_{\mathbb{B}_{dr},D}(O_S):=F^0DR(S)(can(O_{S^o},F_b)^{an}\otimes I):\mathbb{B}_{dr,\psi_D,K}\to\mathbb{B}_{dr,\phi_D,K},$$

and

$$var_{\mathbb{B}_{dr},D}(O_S):=F^0DR(S)(var(O_{S^o},F_b)^{an}\otimes I):\mathbb{B}_{dr,\phi_D,K}\to\mathbb{B}_{dr,\psi_D,K}$$

and

$$a_S(\mathbb{B}_{dr,\phi_D,K}):=F^0DR(S)((can(O_{S^o},F_b)\circ\rho_{DR,D}(O_{S^o},F_b)\circ\mathrm{ad}(j^*,j_{*Hdg})(O_S,F_b))^{an}\otimes I):$$
$$\mathbb{B}_{dr,S_K}\xrightarrow{a_S(\mathbb{B}_{dr,\psi_D,K})}\mathbb{B}_{dr,\psi_D,K}\xrightarrow{can_{\mathbb{B}_{dr},D}(O_S)}\mathbb{B}_{dr,\phi_D,K}$$

(iv) Using (ii), we set

$$\mathbb{B}_{dr,x_{S^o/S},K} := \mathrm{Cone}(\rho_{\mathbb{B}_{dr},D}(O_S) : \mathbb{B}_{dr,S^o/S,K} \to \mathbb{B}_{dr,\psi_D,K}) \in C_{\mathbb{B}_{dr,S_K}}(S_K^{an,pet})$$

together with the canonical map in $C_{\mathbb{B}_{dr,S_K}}(S_K^{an,pet})$

$$a_S(\mathbb{B}_{dr,x_{S^o/S},K}) := (a_S(\mathbb{B}_{dr,S^o/S,K}), a_S(\mathbb{B}_{dr,\psi_D,K})) : \mathbb{B}_{dr,S_K} \to \mathbb{B}_{dr,x_{S^o/S},K}.$$

(v) For $L \in \mathrm{Shv}_{\mathbb{Z}_p}(S^{o,et})$ *a local system, we set using theorem 35 for* $j_*(L \otimes O_{S_K^o})$

$$V_{D0}j_*(L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K^o}) := V_{D0}j_*(L \otimes_{\mathbb{Q}_p} O_{S_K^o}) \otimes_{O_{S_K}} \mathbb{B}_{dr,S_K} \in C_{\mathbb{B}_{dr}}(S_K^{an,pet})$$

so that we have the isomorphism in $D_{\mathbb{B}_{dr}}(S_K^{an,pet})$

$$\begin{aligned} m(L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K^o}) : V_{D0}j_*(L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K^o}) \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,S^o/S,K} \\ \xrightarrow{:=} V_{D0}j_*(L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K^o}) \otimes_{\mathbb{B}_{dr,S_K}} F^0DR(S)(j_{*Hdg}(O_{S^o}, F_b) \otimes_{O_S} (O\mathbb{B}_{dr,S}, F)) \\ \to j_*L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S^o/S,K}, \xrightarrow{:=} j_*L \otimes_{\mathbb{Q}_p} F^0DR(S)(j_{*Hdg}(O_{S^o}, F_b) \otimes_{O_S} (O\mathbb{B}_{dr,S}, F)), \\ s \otimes h \otimes w \mapsto s \otimes hw. \end{aligned}$$

More generally, for $Z \subset S$ *a closed subset and* $L \in \mathrm{Shv}_{\mathbb{Z}_p}(Z^{o,et})$ *a local system with* $Z^o := Z \cap S^o$, *we set using theorem 35 for* $j_*(L \otimes O_{Z_K^o})$

$$V_{D0}j_*(L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K^o}) := V_{D0}j_*(L \otimes_{\mathbb{Q}_p} O_{Z_K^o}) \otimes_{O_{S_K}} \mathbb{B}_{dr,S_K} \in C_{\mathbb{B}_{dr}}(S_K^{an,pet})$$

so that we have the isomorphism in $D_{\mathbb{B}_{dr}}(S_K^{an,pet})$

$$\begin{aligned} m(L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K^o}) : V_{D0}j_*(L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K^o}) \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,S^o/S,K} \to j_*L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S^o/S,K}, \\ s \otimes h \otimes w \mapsto s \otimes hw. \end{aligned}$$

(vi) For $L \in \mathrm{Shv}_{\mathbb{Z}_p}(S^{o,et})$ *a local system, we set using theorem 35 for* $j_*(L \otimes O_{S_K^o})$

$$\psi_D(L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K^o}) := \mathrm{Gr}^{V_D}_{-1 \leq \alpha < 0} j_*(L \otimes_{\mathbb{Q}_p} O_{S_K^o}) \otimes_{O_{S_K}} \mathbb{B}_{dr,S_K} \in C_{\mathbb{B}_{dr}}(S_K^{an,pet})$$

so that we have the isomorphism in $D_{\mathbb{B}_{dr}}(S_K^{an,pet})$

$$\begin{aligned} m(L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K^o}) : \psi_D(L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K^o}) \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,\psi_D,K} \to \psi_D L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,\psi_D,K} \\ \xrightarrow{:=} \psi_D L \otimes_{\mathbb{Q}_p} F^0DR(S)(\mathrm{Gr}^{V_D}_{-1 \leq \alpha < 0}(O_{S^o}, F_b) \otimes_{O_S} (O\mathbb{B}_{dr,S}, F)), \; s \otimes h \otimes w \mapsto s \otimes hw \end{aligned}$$

More generally, for $Z \subset S$ *a closed subset and* $L \in \mathrm{Shv}_{\mathbb{Z}_p}(Z^{o,et})$ *a local system with* $Z^o := Z \cap S^o$, *we set using theorem 35 for* $j_*(L \otimes O_{Z_K^o})$

$$\psi_D(L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K^o}) := \mathrm{Gr}^{V_D}_{-1 \leq \alpha < 0} j_*(L \otimes_{\mathbb{Q}_p} O_{Z_K^o}) \otimes_{O_{S_K}} \mathbb{B}_{dr,S_K} \in C_{\mathbb{B}_{dr}}(S_K^{an,pet})$$

so that we have the isomorphism in $D_{\mathbb{B}_{dr}}(S_K^{an,pet})$

$$\begin{aligned} m(L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K^o}) : \psi_D(L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K^o}) \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,\psi_D,K} \to \psi_D L \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,\psi_D,K} \\ \xrightarrow{:=} \psi_D L \otimes_{\mathbb{Q}_p} F^0DR(S)(\mathrm{Gr}^{V_D}_{-1 \leq \alpha < 0}(O_{S^o}, F_b) \otimes_{O_S} (O\mathbb{B}_{dr,S}, F)), \; s \otimes h \otimes w \mapsto s \otimes hw \end{aligned}$$

We then give using definition 78 and the local system case an inverse functor to the De Rham functor for De Rham modules

Definition 79. *(i0) Let $S \in \mathrm{SmVar}(k)$ irreducible. We then have the morphism of site $\mathrm{an}_S : S_K^{an,pet} \to S^{et}$ given by the analytical functor. Let $K \in P_{\mathbb{Z}_p,k}(S^{et})$ a perverse sheaf, in particular there exist an open subset $S^o \subset S$ with $D := S\backslash S^o$ a (Cartier) divisor such that $K_{|S^o} := j^*K \in C(S^{o,et})$ is a local system for the etale topology, where we denote $j : S_0 \hookrightarrow S$ the open embedding and $i : D \hookrightarrow S$ the closed embedding of the Cartier divisor. Assume first that $K_{|D} := i^*K$ is a local system. Then, $\psi_D K, \phi_D K \in C_{\mathbb{Z}_p}(D^{et})$ are local systems. We denote again $K := \mathrm{an}_S^* K \in C(S_K^{an,pet})$ and $K := j^*K \in C(S_K^{o,an,pet})$. Denote by $\pi : \tilde{S}_K^{o,an} \to S_K^{o,an}$ the perfectoid universal covering. We then have by theorem 15 a canonical isomorphism in $D_{\mathbb{Z}_p,c}(S_K^{an,pet})$*

$$Is(K) : K \xrightarrow{\sim} (\psi_D K \xrightarrow{(c(x_{S^o/S}(K)),can(K))} x_{S^o/S}(K) \oplus \phi_D K \xrightarrow{((0,T-I),var(K))} \psi_D K)$$

We then set using definition 78

$$\begin{gathered}\mathbb{B}_{dr,S}(x_{S^o/S}(K)) := x_{S^o/S}(K) \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,x_{S^o/S},K} := \\ \mathrm{Cone}((\mathrm{ad}(i^*,i_*)(-) \circ \mathrm{ad}(\pi^*,\pi_*)(K)) \otimes \rho_{\mathbb{B}_{dr,D}}(O_S) : \\ j_*K \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S^o/S,K} \to \psi_D(K) \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,\psi_D,K}) \in D_{\mathbb{B}_{dr}}(S_K^{an,pet})\end{gathered}$$

and

$$\begin{aligned}\mathbb{B}_{dr,S}(K) := (\mathbb{B}_{dr,S}(\psi_D K) := \psi_D K \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,\psi_D,K} &\xrightarrow{(c(x_{S^o/S}(K)) \otimes c(\mathbb{B}_{dr,x_{S^o/S},K}),can(K) \otimes can_{\mathbb{B}_{dr},D}(O_S))} \\ \mathbb{B}_{dr,S}(x_{S^o/S}(K)) \oplus \mathbb{B}_{dr,S}(\phi_D K) &:= (x_{S^o/S}(K) \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,x_{S^o/S},K}) \oplus \phi_D K \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,\phi_D,K}) \\ \xrightarrow{((0,T-I) \otimes (0,s\partial s),var(K) \otimes var_{\mathbb{B}_{dr},D}(O_S))} &\mathbb{B}_{dr,S}(\psi_D K) := \psi_D K \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,\psi_D,K}) \in D_{\mathbb{B}_{dr}}(S_K^{an,pet}).\end{aligned}$$

(i) Let $S \in \mathrm{SmVar}(k)$ irreducible. We then have the morphism of site $\mathrm{an}_S : S^{an,pet} \to S^{et}$ given by the analytical functor. We define the functor

$$\mathbb{B}_{dr,S} : D_{\mathbb{Z}_p fil,c,k}(S^{et}) \to D_{\mathbb{B}_{dr,S} fil}(S_K^{an,pet})$$

using the nearby and vanishing cycle functors. For $(K,W) \in P_{pfil}(S^{et})$ a filtered perverse sheaf and $(D_1,\ldots,D_d) \in \mathcal{S}(K)$ a stratification by (Cartier) divisor $D_i \subset S$, $1 \leq i \leq d$ such that $K_{|D(r)\backslash D(r+1)} := l_r^* K \in D_{\mathbb{Z}_p,c}(D(r)\backslash D(r+1)^{et})$ are local systems for all $1 \leq r \leq d$, where $D(r) := \cap_{1 \leq i \leq r} D_i$ and $l_r : D(r)\backslash D(r+1) \hookrightarrow S$ is the locally closed embedding, we have by theorem 15 a canonical isomorphism in $D_{\mathbb{Z}_p,c}(S_K^{an,pet})$

$$Is(K) : K \xrightarrow{\sim} (\cdots \to \bigoplus_{1<i_1 \cdots < i_d \leq d} x_{S\backslash D_{i_1}/S} \cdots x_{S\backslash D_{i_r}/S} \phi_{D_{i_{r+1}}} \cdots \phi_{D_{i_s}} \psi_{D_{i_{s+1}}} \cdots \psi_{D_{i_d}}(K) \to \cdots),$$

we then define by (i0)

$$\begin{aligned}\mathbb{B}_{dr,S}(K,W) := \lim_{(D_1,\ldots,D_d) \in \mathcal{S}(K)} (\cdots \to \\ \bigoplus_{1 \leq i_1 < \cdots < i_d \leq d} \phi_{D_{i_{r+1}}} \cdots \phi_{D_{i_s}} \psi_{D_{i_{s+1}}} \cdots \psi_{D_{i_d}} x_{S\backslash D_{i_1}/S} \cdots x_{S\backslash D_{i_r}/S}(K,W) \\ \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,x_{S\backslash D_{i_1}/S},K} \otimes_{\mathbb{B}_{dr,S_K}} \cdots \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,x_{S\backslash D_{i_r}/S},K} \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,\phi_{D_{i_{r+1}}},K} \otimes_{\mathbb{B}_{dr,S_K}} \cdots \\ \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,\phi_{D_{i_s}},K} \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,\psi_{D_{i_{s+1}}},K} \otimes_{\mathbb{B}_{dr,S_K}} \cdots \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,\psi_{D_{i_d}},K} \to \cdots).\end{aligned}$$

For $m : (K_1,W) \to (K_2,W)$ a morphism with $(K_1,W),(K_2,W) \in P_{pfil}(S^{et})$, considering a stratification $(D_1,\ldots,D_d) \in \mathcal{S}(K_1) \cap \mathcal{S}(K_2)$ by (Cartier) divisor $D_i \subset S$, $1 \leq i \leq d$ such that

$K_{1|D(r)\backslash D(r+1)}, K_{2|D(r)\backslash D(r+1)} \in D_{\mathbb{Z}_p,c}(D(r)\backslash D(r+1)^{et})$ are local systems for all $1 \leq r \leq d$, we get

$$\mathbb{B}_{dr,S}(m):$$

$$\mathbb{B}_{dr,S}(K_1,W) := (\cdots \to \bigoplus_{1\leq i_1<\cdots<i_d\leq d} \phi_{D_{i_{r+1}}}\cdots\phi_{D_{i_s}}\psi_{D_{i_{s+1}}}\cdots\psi_{D_{i_d}} x_{S\backslash D_{i_1}/S}\cdots x_{S\backslash D_{i_r}/S}(K_1,W)$$
$$\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,x_{S\backslash D_{i_1}/S},K}\otimes_{\mathbb{B}_{dr,S_K}}\cdots\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,x_{S\backslash D_{i_r}/S},K}\otimes_{\mathbb{B}_{dr,S_K}}$$
$$\mathbb{B}_{dr,\phi_{D_{i_{r+1}}},K}\otimes_{\mathbb{B}_{dr,S_K}}\cdots\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,\phi_{D_{i_s}},K}$$
$$\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,\psi_{D_{i_{s+1}}},K}\otimes_{\mathbb{B}_{dr,S_K}}\cdots\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,\psi_{D_{i_d}},K}\to\cdots)$$
$$\xrightarrow{(x_{S\backslash D_{i_1}/S}\cdots x_{S\backslash D_{i_r}/S}\phi_{D_{i_{r+1}}}\cdots\phi_{D_{i_s}}\psi_{D_{i_{s+1}}}\cdots\psi_{D_{i_d}}(m)\otimes I)}$$
$$\mathbb{B}_{dr,S}(K_2,W) := (\cdots \to \bigoplus_{1\leq i_1<\cdots<i_d\leq d} \phi_{D_{i_{r+1}}}\cdots\phi_{D_{i_s}}\psi_{D_{i_{s+1}}}\cdots\psi_{D_{i_d}} x_{S\backslash D_{i_1}/S}\cdots x_{S\backslash D_{i_r}/S}(K_2,W)$$
$$\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,x_{S\backslash D_{i_1}/S},K}\otimes_{\mathbb{B}_{dr,S_K}}\cdots\otimes\mathbb{B}_{dr,x_{S\backslash D_{i_r}/S},K}\otimes_{\mathbb{B}_{dr,S_K}}$$
$$\mathbb{B}_{dr,\phi_{D_{i_{r+1}}},K}\otimes_{\mathbb{B}_{dr,S_K}}\cdots\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,\phi_{D_{i_s}},K}$$
$$\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,\psi_{D_{i_{s+1}}},K}\otimes_{\mathbb{B}_{dr,S_K}}\cdots\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,\psi_{D_{i_d}},K}\to\cdots).$$

Note that if L is a local system on S then $\mathbb{B}_{dr,S}(L) = L\otimes\mathbb{B}_{dr,S,K}$, that is it does NOT depend on the choice of a stratification (see remark 6). This gives the functor

$$\mathbb{B}_{dr,S} : D_{\mathbb{Z}_p fil,c,k}(S^{et}) = D(P_{pfil}(S^{et})) \to D_{\mathbb{B}_{dr,S}fil}(S_K^{an,pet}).$$

(ii) *Let $S\in\mathrm{Var}(k)$. Let $S=\cup_{i\in I}S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I\in\mathrm{SmVar}(k)$. We define as in (i) the functor*

$$\mathbb{B}_{dr,(\tilde{S}_I)} : D_{\mathbb{Z}_p fil,c,k}(S^{et}) \xrightarrow{T(S/(\tilde{S}_I))} D_{\mathbb{Z}_p fil,c,k}(S^{et}/(\tilde{S}_I^{et})) \to D_{\mathbb{B}_{dr}fil}(S_K^{an,pet}/(\tilde{S}_{I,K}^{an,pet})),$$
$$(K,W)\mapsto\mathbb{B}_{dr,(\tilde{S}_I)}(K,W) := \mathbb{B}_{dr,(\tilde{S}_I)}(i_{I*}j_I^*(K,W),t_{IJ})$$

with for $(K,W)\in P_{pfil}(S^{et})$,

$$\mathbb{B}_{dr,(\tilde{S}_I)}(K,W) := \lim_{(D_1,\ldots,D_d)\in\mathcal{S}(K)}$$
$$((\bigoplus_{1\leq i_1<\cdots<i_d\leq d}\phi_{\tilde{D}_{i_{r+1},I}}\cdots\phi_{\tilde{D}_{i_s,I}}\psi_{\tilde{D}_{i_{s+1},I}}\cdots\psi_{\tilde{D}_{i_d,I}}x_{\tilde{S}_I\backslash\tilde{D}_{i_1,I}/\tilde{S}_I}\cdots x_{\tilde{S}_I\backslash\tilde{D}_{i_r,I}/\tilde{S}_I}(i_{I*}j_I^*(K,W))$$
$$\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{D}_{i_1,I}/S},K}\otimes_{\mathbb{B}_{dr,\tilde{S}_{I,K}}}\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_{I,K}}}\mathbb{B}_{dr,x_{\tilde{S}_I\backslash D_{i_r,I}/\tilde{S}_I},K}\otimes_{\mathbb{B}_{dr,\tilde{S}_{I,K}}}$$
$$\mathbb{B}_{dr,\phi_{\tilde{D}_{i_{r+1},I}},K}\otimes_{\mathbb{B}_{dr,\tilde{S}_{I,K}}}\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_{I,K}}}\mathbb{B}_{dr,\phi_{\tilde{D}_{i_s,I}},K}\otimes_{\mathbb{B}_{dr,\tilde{S}_{I,K}}}$$
$$\mathbb{B}_{dr,\psi_{\tilde{D}_{i_{s+1},I}},K}\otimes_{\mathbb{B}_{dr,\tilde{S}_{I,K}}}\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_{I,K}}}\mathbb{B}_{dr,\psi_{\tilde{D}_{i_d,I}},K}\to\cdots),\mathbb{B}_{dr}(t_{IJ}))$$

with $(D_1,\ldots,D_d)\in\mathcal{S}(K)$ stratifications by Cartier divisor $D_i\subset S$, $1\leq i\leq d$ such that

$$K_{|D(r)\backslash D(r+1)} := l_r^*K\in D_{\mathbb{Z}_p,c}(D(r)\backslash D(r+1)^{et})$$

are local systems for all $1\leq r\leq d$, and $\tilde{D}_{s,I}\subset\tilde{S}_I$ are (Cartier) divisor such that $D_s\cap S_I\subset\tilde{D}_{s,I}\cap S$ (that is $D_s\cap S_I$ is a union of irreducible components of $\tilde{D}_{S,I}\cap S$ which are (Cartier) divisors), having by theorem 15 the canonical isomorphism in $D_{\mathbb{Z}_p,c,k}(S_K^{an,pet})$

$$Is(K) : K \xrightarrow{\sim} (\cdots\to\bigoplus_{1<i_1\cdots<i_d\leq d}x_{S\backslash D_{i_1}/S}\cdots x_{S\backslash D_{i_r}/S}\phi_{D_{i_{r+1}}}\cdots\phi_{D_{i_s}}\psi_{D_{i_{s+1}}}\cdots\psi_{D_{i_d}}(K)\to\cdots).$$

Remark 6. *Let $S \in \mathrm{SmVar}(k)$. Let $K \in P_{\mathbb{Z}_p,k}(S^{et})$. $(D_1,\ldots,D_d) \in \mathcal{S}(K)$ stratifications by Cartier divisor $D_i \subset S$, $1 \leq i \leq d$ such that $K_{|D(r)\backslash D(r+1)} := l_r^*K \in D_{\mathbb{Z}_p,c}(D(r)\backslash D(r+1)^{et})$ are local systems for all $1 \leq r \leq d$. We then have the canonical map in $D_{\mathbb{B}_{dr}}(S_K^{an,pet})$*

$$a_S(K,\mathbb{B}_{dr}) := (I \otimes a_S(\mathbb{B}_{dr,x_{S\backslash D_i/S}}), I \otimes a_S(\mathbb{B}_{dr,\phi_{D_i}}), I \otimes a_S(\mathbb{B}_{dr,\psi_{D_i}})) \circ (I(K) \otimes I) :$$
$$K \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K} \to \mathbb{B}_{dr,S}(K) := \bigoplus_{1 \leq i_1 < \cdots < i_d \leq d} x_{S\backslash D_{i_1}/S} \cdots x_{S\backslash D_{i_r}/S} \phi_{D_{i_{r+1}}} \cdots \phi_{D_{i_s}} \psi_{D_{i_{s+1}}} \cdots \psi_{D_{i_d}} K$$
$$\otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,x_{S\backslash D_{i_1}/S},K} \otimes_{\mathbb{B}_{dr,S_K}} \cdots \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,x_{S\backslash D_{i_r}/S},K} \otimes_{\mathbb{B}_{dr,S_K}}$$
$$\mathbb{B}_{dr,\phi_{D_{i_{r+1}}}} \otimes_{\mathbb{B}_{dr,S_K}} \cdots \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,\phi_{D_{i_s}},K} \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,\psi_{D_{i_{s+1}}},K} \otimes_{\mathbb{B}_{dr,S_K}} \cdots \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,\psi_{D_{i_d}},K} \to \cdots).$$

On the other hand we have by theorem 32,

$$\alpha(S) : K \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S_K} \xrightarrow{\sim} F^0DR(S)((K \otimes O_{S_K^{an}}, F_b) \otimes_{O_{S_K^{an}}} (O\mathbb{B}_{dr,S_K}, F))$$

in $D_{\mathbb{B}_{dr}}(S_K^{an,pet})$.

- *(i) If $K \in P_{\mathbb{Z}_p,k}(S^{et})$ is a local system then the map $a_S(K,\mathbb{B}_{dr})$ is an isomorphism since by proposition 46*

$$(O_S,F_b) \xrightarrow{\sim} \bigoplus_{1 \leq i_1 < \cdots < i_d \leq d} x_{S\backslash D_{i_1}/S} \cdots x_{S\backslash D_{i_r}/S} \phi_{D_{i_{r+1}}} \cdots \phi_{D_{i_s}} \psi_{D_{i_{s+1}}} \cdots \psi_{D_{i_d}} (O_S,F_b)$$

in $D(DRM(S))$ and since the functor

$$K \otimes_{\mathbb{Q}_p} (-) : C_{\mathbb{Z}_p}(S_K^{an,pet}) \to C_{\mathbb{Z}_p}(S_K^{an,pet}),\ N \mapsto K \otimes_{\mathbb{Q}_p} N$$

respect etale hence pro-etale equivalences.

- *(ii) If $K \in P_{\mathbb{Z}_p,k}(S^{et})$ is NOT a local system then the map $a_S(K,\mathbb{B}_{dr})$ is NOT an isomorphism is general. For example, for $j : S^o \hookrightarrow S$ an open embedding with $D := S\backslash S^o$ a Cartier divisor, we have in $D_{\mathbb{B}_{dr}}(S_K^{an,pet})$, by proposition 46,*

$$T(j,\mathbb{B}_{dr})(\mathbb{Z}_{p,S^o}) := (0, c(\mathbb{B}_{dr,x_{S\backslash D/S},K}), 0) : \mathbb{B}_{dr,S}(j_*\mathbb{Z}_{p,S^o}) \xrightarrow{:=}$$
$$(\phi_D(\mathbb{Z}_{p,S_K}) \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,\psi_D,K} \to (x_{S\backslash D/S}(\mathbb{Z}_{p,S_K}) \otimes \mathbb{B}_{dr,x_{S\backslash D/S},K}) \oplus (\phi_D(\mathbb{Z}_{p,S_K}) \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,\phi_D,K}) \to$$
$$\psi_D(\mathbb{Z}_{p,S_K}) \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,\psi_D,K})$$
$$\xrightarrow{\sim} \mathbb{B}_{dr,S^o/S,K} := F^0DR(S)(j_{*Hdg}(O_{S^o},F_b)^{an} \otimes_{O_{S_K^{an}}} (O\mathbb{B}_{dr,S_K},F))$$

which is, by theorem 32 (in the case of D a normal crossing divisor, $\mathbb{B}_{dr,S^o/S,K} = \mathbb{B}_{dr,S_K}(\log D_K)$), NOT isomorphic in $D_{\mathbb{B}_{dr}}(S_K^{an,pet})$ to

$$j_*\alpha(S^o) : j_*\mathbb{B}_{dr,S_K^o} \xrightarrow{\sim} F^0DR(S)(j_*(O_S,F_b)^{an} \otimes_{O_{S_K^{an}}} (O\mathbb{B}_{dr,S_K},F)),$$

and also NOT isomorphic to

$$\mathbb{D}_S^v T_!(j,\otimes)(-,-) : (j_*\mathbb{Z}_{p,S_K^o}) \otimes \mathbb{B}_{dr,S_K} \xrightarrow{\sim} \mathbb{D}_S^v(j_!\mathbb{B}_{dr,S_K})$$

see also remark 1. If $K \in P_{\mathbb{Z}_p,k}(S^{et})$ is NOT a local system, the functor

$$K \otimes_{\mathbb{Q}_p} (-) : C_{\mathbb{Z}_p}(S_K^{an,pet}) \to C_{\mathbb{Z}_p}(S_K^{an,pet}),\ N \mapsto K \otimes_{\mathbb{Q}_p} N$$

does NOT preserve etale or pro-etale equivalence. Recall also that the filtered De Rham functor does NOT commutes with filtered tensor product in general (it may leads to different F-filtration).

Let $k \subset K \subset \mathbb{C}_p$ a subfield of a p-adic field. Let $S \in \mathrm{SmVar}(k)$ and $D \subset S$ a Cartier divisor. Denote $S^o := S \backslash D$. We write for simplicity,

- $\mathbb{B}_{dr,S} := \mathbb{B}_{dr,S_K}$, $O\mathbb{B}_{dr,S} := O\mathbb{B}_{dr,S_K}$,
- $\mathbb{B}_{dr,S^o/S} := \mathbb{B}_{dr,S^o/S,K}$, $\mathbb{B}_{dr,\psi_D} := \mathbb{B}_{dr,\psi_D,K}$ and $\mathbb{B}_{dr,\phi_D} := \mathbb{B}_{dr,\phi_D,K}$
- $\mathbb{B}_{dr,x_{S^o/S}} := \mathbb{B}_{dr,x_{S^o/S},K}$.

We now look at the functorialities with respect to the proper morphisms and with respect to the open embeddings :

Let $f : X \to S$ a morphism with $X, S \in \mathrm{SmVar}(k)$. Let $D \subset S$ a (Cartier) divisor and denote $S^o := S \backslash D$ and $X^o := X \backslash f^{-1}(D)$. Denote $j : S^o \hookrightarrow S$, $j' : X^o \hookrightarrow X$ the open embeddings.

- We have the following quasi-isomorphism in $C_{\mathbb{B}_{dr,S}}(X_K^{an,pet})$

$$\begin{aligned}m_f(\mathbb{B}_{dr,S^o/S}) : f^*\mathbb{B}_{dr,S^o/S} \otimes_{f^*\mathbb{B}_{dr,S}} \mathbb{B}_{dr,X} \xrightarrow{I\otimes\alpha(X_K)} \\ f^*F^0DR(S)(j_{*Hdg}(O_{S^o}, F_b)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S}, F)) \otimes_{f^*\mathbb{B}_{dr,S}} F^0DR(X)((O_X, F_b)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X}), F) \\ \xrightarrow{=} F^0DR(f^*O_S)(f^*j_{*Hdg}(O_{S^o}, F_b)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S}, F)) \otimes_{f^*\mathbb{B}_{dr,S}} \\ F^0DR(X)((O_X, F_b)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X}), F) \\ \xrightarrow{w_X\circ\Omega_{f^*O_S/O_X}(-)} F^0DR(X)(f^{*mod}j_{*Hdg}(O_{S^o}, F_b)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X}, F)) \\ \xrightarrow{=} F^0DR(X)(j'_{*Hdg}(O_{X^o}, F_b)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X}), F) =: \mathbb{B}_{dr,X^o/X}.\end{aligned}$$

- We have the following quasi-isomorphism in $C_{\mathbb{B}_{dr,S}}(X_K^{an,pet})$

$$\begin{aligned}m_f(\mathbb{B}_{dr,\psi_D}) : f^*\mathbb{B}_{dr,\psi_D} \otimes_{f^*\mathbb{B}_{dr,S}} \mathbb{B}_{dr,X} \xrightarrow{I\otimes\alpha(X_K)} \\ f^*F^0DR(S)(\psi_D(O_{S^o}, F_b)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S}, F)) \otimes_{f^*\mathbb{B}_{dr,S}} F^0DR(X)((O_X, F_b)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X}), F) \\ \xrightarrow{=} F^0DR(f^*O_S)(f^*\psi_D(O_{S^o}, F_b)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S}, F)) \otimes_{f^*\mathbb{B}_{dr,S}} \\ F^0DR(X)((O_X, F_b)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X}), F) \\ \xrightarrow{w_X\circ\Omega_{f^*O_S/O_X}(-)} F^0DR(X)(f^{*mod}\psi_D(O_{S^o}, F_b)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X}, F)) \\ \xrightarrow{=} F^0DR(X)(\psi_{f^{-1}(D)}(O_{X^o}, F_b)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X}, F)) =: \mathbb{B}_{dr,\psi_{f^{-1}(D)}}.\end{aligned}$$

- We have the following quasi-isomorphism in $C_{\mathbb{B}_{dr,S}}(X_K^{an,pet})$

$$\begin{aligned}m_f(\mathbb{B}_{dr,\psi_D}) : f^*\mathbb{B}_{dr,\psi_D} \otimes_{f^*\mathbb{B}_{dr,S}} \mathbb{B}_{dr,X} \xrightarrow{I\otimes\alpha(X_K)} \\ f^*F^0DR(S)(\phi_D(O_{S^o}, F_b)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S}, F)) \otimes_{f^*\mathbb{B}_{dr,S}} F^0DR(X)((O_X, F_b)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X}), F) \\ \xrightarrow{=} F^0DR(f^*O_S)(f^*\phi_D(O_{S^o}, F_b)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S}, F)) \otimes_{f^*\mathbb{B}_{dr,S}} \\ F^0DR(X)((O_X, F_b)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X}), F) \\ \xrightarrow{w_X\circ\Omega_{f^*O_S/O_X}(-)} F^0DR(X)(f^{*mod}\phi_D(O_{S^o}, F_b)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X}, F)) \\ \xrightarrow{=} F^0DR(X)(\phi_{f^{-1}(D)}(O_{X^o}, F_b)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X}, F)) =: \mathbb{B}_{dr,\phi_{f^{-1}(D)}}.\end{aligned}$$

- We have the following map in $C_{\mathbb{B}_{dr,S}}(S_K^{an,pet})$

$$m_f(\mathbb{B}_{dr,x_{S^o/S}}) := (m_f(\mathbb{B}_{dr,S^o/S}), m_f(\mathbb{B}_{dr,\psi_D})) : f^*\mathbb{B}_{dr,x_{S^o/S}} \otimes_{f^*\mathbb{B}_{dr,S}} \mathbb{B}_{dr,X} \to \mathbb{B}_{dr,x_{X^o/X}}$$

Definition 80. *(i) Let $f : X \to S$ be a proper morphism with $X, S \in \mathrm{SmVar}(k)$. Let $(K, W) \in P_{\mathbb{Z}_p fil,k}(X^{et}) \cap D_{\mathbb{Z}_p fil,c,k,gm}(X^{et})$ be a filtered perverse sheaf of geometric origin, i.e. $\mathrm{Gr}^n_W K \in D_{\mathbb{Z}_p,c,k,gm}(X^{et})$ for all $n \in \mathbb{Z}$. We have then, by the perverse hard Lefchetz thorem, a canonical isomorphism in $D_{\mathbb{Z}_p fil,c,k}(S^{et})$*

$$l(K, W) : (K, W) \to \oplus_{k \in \mathbb{Z}} {}^p R^k f_*(K, W).$$

Consider a stratification $(E_1, \ldots, E_d) \in \mathcal{S}(K)$ by (Cartier) divisor $E_i \subset X$, $1 \leq i \leq d$, such that

$$K_{|E(r)\backslash E(r+1)} := l_r^* K \in D_{\mathbb{Z}_p,c}((E(r)\backslash E(r+1))^{et})$$

are local systems for all $1 \leq r \leq d$, $l_r : E(r) \hookrightarrow X$ being the locally closed embeddings. Let $k \in \mathbb{Z}$. Take a stratification $(D_1, \ldots, D_e) \in \mathcal{S}(K)$ by (Cartier) divisor $D_i \subset S$, $1 \leq i \leq e$, such that

$$({}^p R^k f_* x_{i_1} \cdots x_{i_r} \phi_{E_{i_{r+1}}} \cdots \phi_{E_{i_s}} \psi_{E_{i_{s+1}}} \cdots \psi_{E_{i_d}} K)_{|D(r')\backslash D(r'+1)} :=$$
$$m_{r'}^* \, {}^p R^k f_* x_{i_1} \cdots x_{i_r} \phi_{E_{i_{r+1}}} \cdots \phi_{E_{i_s}} \psi_{E_{i_{s+1}}} \in D_{\mathbb{Z}_p,c}((D(r')\backslash D(r'+1))^{et})$$

are local systems for all $1 \leq i_1 < \cdots < i_{r+1} < \cdots < i_{s+1} < \cdots < i_d \leq d$ and all $1 \leq r' \leq e$, $m_{r'} : D(r') \hookrightarrow S$ being the locally closed embeddings. This implies that

$${}^p R^k f_* K_{|D(r')\backslash D(r'+1)} := m_{r'}^* \, {}^p R^k f_* K \in D_{\mathbb{Z}_p,c}((D(r')\backslash D(r'+1))^{et})$$

are local systems for all $1 \leq r' \leq e$. We then define the canonical maps in $D_{\mathbb{B}_{dr,S}fil}(S_K^{an,pet})$

$$T^k(f,\mathbb{B}_{dr})(K,W) : \mathbb{B}_{dr,S}({}^pR^kf_*(K,W)) \xrightarrow{B_{dr,S}({}^pR^kf_*Is(K,W))}$$

$$(\cdots \to \bigoplus_{1\leq i_1\cdots<i_d\leq d} \mathbb{B}_{dr,S}({}^pR^kf_*x_{X\backslash E_{i_1}/S}\cdots x_{X\backslash E_{i_r}/S}\phi_{E_{i_{r+1}}}\cdots\phi_{E_{i_s}}\psi_{E_{i_{s+1}}}\cdots\psi_{E_{i_d}}(K,W)) \to \cdots)$$

$$\xrightarrow{:=} (\cdots \to \bigoplus_{1\leq i_1<\cdots<i_d\leq d, 1\leq j_1<\cdots<j_e\leq e} \phi_{D_{j_{r'+1}}}\cdots\phi_{D_{j_{s'}}}\psi_{D_{j_{s'+1}}}\cdots\psi_{D_{j_e}}$$

$$x_{S\backslash D_{j_1}/S}\cdots x_{S\backslash D_{i_{r'}}/S}({}^pR^kf_*x_{X\backslash E_{i_1}/X}\cdots x_{X\backslash E_{i_r}/X}\phi_{E_{i_{r+1}}}\cdots\phi_{E_{i_s}}\psi_{E_{i_{s+1}}}\cdots\psi_{E_{i_d}}(K,W))$$

$$\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,x_{S\backslash D_{j_1}/S}}\otimes\cdots\otimes\mathbb{B}_{dr,x_{S\backslash D_{j_{r'}}/S}}\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\phi_{D_{j_{r'+1}}}}$$

$$\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{D_{j_{s'}}}}\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\psi_{D_{j_{s'+1}}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{D_{j_e}}}\to\cdots)$$

$$\xrightarrow{((T(f,\psi)(-)\circ\cdots\circ T(f,\psi)(-)\circ T(f,\phi)(-)\circ\cdots\circ T(f,\phi)(-)\circ T(f,x)(-)\circ\cdots\circ T(f,x)(-))\circ l_k(-))\otimes I}$$

$$(\cdots \to \bigoplus_{1\leq i_1<\cdots<i_d\leq d, 1\leq j_1<\cdots<j_e\leq e} Rf_*(x_{X\backslash f^{-1}(D_{j_1})/S}x_{X\backslash E_{i_1}/S}\cdots x_{X\backslash f^{-1}(D_{j_{r'}})/X}x_{X\backslash E_{i_r}/X}$$

$$\phi_{f^{-1}(D_{j_{r'+1}})}\phi_{E_{i_{r+1}}}\cdots\phi_{f^{-1}(D_{j_{s'}})}\phi_{E_{i_s}}\psi_{f^{-1}(D_{j_{s'+1}})}\psi_{E_{i_{s+1}}}\cdots\psi_{f^{-1}(D_{j_d})}\psi_{E_{i_d}}(K,W))$$

$$\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,x_{S\backslash D_{j_1}/S}}\otimes\cdots\otimes\mathbb{B}_{dr,x_{S\backslash D_{j_{r'}}/S}}\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\phi_{D_{j_{r'+1}}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{D_{j_{s'}}}}$$

$$\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\psi_{D_{j_{s'+1}}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{D_{j_e}}}\to\cdots)$$

$$\xrightarrow{T(f,\otimes)(-,-)}$$

$$(\cdots \to \bigoplus_{1\leq i_1<\cdots<i_d\leq d, 1\leq j_1<\cdots<j_e\leq e} Rf_*(x_{X\backslash f^{-1}(D_{j_1})/S}x_{X\backslash E_{i_1}/S}\cdots x_{X\backslash f^{-1}(D_{j_{r'}})/X}x_{X\backslash E_{i_r}/X}$$

$$\phi_{f^{-1}(D_{j_{r'+1}})}\phi_{E_{i_{r+1}}}\cdots\phi_{f^{-1}(D_{j_{s'}})}\phi_{E_{i_s}}\psi_{f^{-1}(D_{j_{s'+1}})}\psi_{E_{i_{s+1}}}\cdots\psi_{f^{-1}(D_{j_d})}\psi_{E_{i_d}}(K,W)$$

$$\otimes_{\mathbb{Q}_p}f^*\mathbb{B}_{dr,x_{S\backslash D_{j_1}/S}}\otimes\cdots\otimes f^*\mathbb{B}_{dr,x_{S\backslash D_{j_{r'}}/S}}\otimes_{f^*\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\phi_{D_{j_{r'+1}}}}\otimes\cdots\otimes f^*\mathbb{B}_{dr,\phi_{D_{j_{s'}}}}$$

$$\otimes_{f^*\mathbb{B}_{dr,S}}f^*\mathbb{B}_{dr,\psi_{D_{j_{s'+1}}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{D_{j_e}}})\to\cdots)$$

$$\xrightarrow{Rf_*(I\otimes a_X(\mathbb{B}_{dr,x_{X\backslash E_i/X}})\otimes a_X(\mathbb{B}_{dr,\phi_{E_i}})\otimes a_X(\mathbb{B}_{dr,\psi_{E_i}}))}$$

$$(\cdots \to \bigoplus_{1\leq i_1<\cdots<i_d\leq d, 1\leq j_1<\cdots<j_e\leq e} Rf_*(x_{X\backslash f^{-1}(D_{j_1})/S}x_{X\backslash E_{i_1}/S}\cdots x_{X\backslash f^{-1}(D_{j_{r'}})/X}x_{X\backslash E_{i_r}/X}$$

$$\phi_{f^{-1}(D_{j_{r'+1}})}\phi_{E_{i_{r+1}}}\cdots\phi_{f^{-1}(D_{j_{s'}})}\phi_{E_{i_s}}\psi_{f^{-1}(D_{j_{s'+1}})}\psi_{E_{i_{s+1}}}\cdots\psi_{f^{-1}(D_{j_d})}\psi_{E_{i_d}}(K,W)$$

$$\otimes_{\mathbb{Q}_p}f^*\mathbb{B}_{dr,x_{S\backslash D_{j_1}/S}}\otimes\cdots\otimes f^*\mathbb{B}_{dr,x_{S\backslash D_{j_{r'}}/S}}\otimes_{f^*\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\phi_{D_{j_{r'+1}}}}\otimes\cdots\otimes f^*\mathbb{B}_{dr,\phi_{D_{j_{s'}}}}$$

$$\otimes_{f^*\mathbb{B}_{dr,S}}f^*\mathbb{B}_{dr,\psi_{D_{j_{s'+1}}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{D_{j_e}}}\otimes_{f^*\mathbb{B}_{dr,S}}$$

$$\mathbb{B}_{dr,x_{X\backslash E_{i_1}/X}}\otimes_{\mathbb{B}_{dr,X}}\otimes\cdots\otimes\mathbb{B}_{dr,x_{X\backslash E_{i_r}/X}}$$

$$\otimes_{\mathbb{B}_{dr,X}}\mathbb{B}_{dr,\phi_{E_{i_{r+1}}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{E_{i_s}}}\otimes_{\mathbb{B}_{dr,X}}\mathbb{B}_{dr,\psi_{E_{i_{s+1}}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{E_{i_d}}})\to\cdots)$$

$$\xrightarrow{Rf_*(I\otimes m_f(\mathbb{B}_{dr,x_{S\backslash D_j/S}})\otimes m_f(\mathbb{B}_{dr\phi_{D_j}})\otimes m_f(\mathbb{B}_{dr,\psi_{D_j}}))}$$

$$(\cdots \to \bigoplus_{1\leq i_1<\cdots<i_d\leq d, 1\leq j_1<\cdots<j_e\leq e} Rf_*(x_{X\backslash f^{-1}(D_{j_1})/X}x_{X\backslash E_{i_1}/X}\cdots x_{X\backslash f^{-1}(D_{j_{r'}})/X}x_{X\backslash E_{i_r}/X}$$

$$\phi_{f^{-1}(D_{j_{r'+1}})}\phi_{E_{i_{r+1}}}\cdots\phi_{f^{-1}(D_{j_{s'}})}\phi_{E_{i_s}}\psi_{f^{-1}(D_{j_{s'+1}})}\psi_{E_{i_{s+1}}}\cdots\psi_{f^{-1}(D_{j_d})}\psi_{E_{i_d}}(K,W)$$

$$\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,x_{X\backslash E_{i_1}/X}}\otimes_{\mathbb{B}_{dr,X}}\otimes\cdots\otimes\mathbb{B}_{dr,x_{X\backslash E_{i_r}/X}}$$

$$\otimes_{\mathbb{B}_{dr,X}}\mathbb{B}_{dr,\phi_{E_{i_{r+1}}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{E_{i_s}}}\otimes_{\mathbb{B}_{dr,X}}\mathbb{B}_{dr,\psi_{E_{i_{s+1}}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{E_{i_d}}}$$

$$\otimes_{\mathbb{B}_{dr,X}}\mathbb{B}_{dr,x_{X\backslash f^{-1}(D_{j_1})/X}}\otimes\cdots\otimes\mathbb{B}_{dr,x_{X\backslash f^{-1}(D_{j_{r'}})/X}}$$

$$\otimes_{\mathbb{B}_{dr,X}}\mathbb{B}_{dr,\phi_{f^{-1}(D_{j_{r'+1}})}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{f^{-1}(D_{j_{s'}})}}\otimes_{\mathbb{B}_{dr,X}}\mathbb{B}_{dr,\psi_{f^{-1}(D_{j_{s'+1}})}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{f^{-1}(D_{j_e})}})\to\cdots)$$

$$\xrightarrow{=} Rf_*\mathbb{B}_{dr,X}(K,W),$$

with $l_k(K,W): {}^pR^kf_*(K,W)\hookrightarrow Rf_*(K,W)$*, which gives the canonical map in* $D_{\mathbb{B}_{dr,S}}(S_K^{an,pet})$

$$T(f,\mathbb{B}_{dr})(K,W):\mathbb{B}_{dr,S}(Rf_*(K,W))\xrightarrow{\mathbb{B}_{dr,S}(l(K,W))}\oplus_{k\in\mathbb{Z}}\mathbb{B}_{dr,S}({}^pR^kf_*(K,W)) \xrightarrow{(T^k(f,\mathbb{B}_{dr})(K,W))} Rf_*\mathbb{B}_{dr,X}(K,W).$$

It gives, by functoriality, for $(K,W)\in D_{\mathbb{Z}_pfil,c,k,gm}(S^{et})$*, the canonical map in* $D_{\mathbb{B}_{dr,S}fil}(S_K^{an,pet})$

$$T(f,\mathbb{B}_{dr})(K,W):\mathbb{B}_{dr,S}(Rf_*(K,W))\to Rf_*\mathbb{B}_{dr,X}(K,W).$$

(ii) *Let* $f:X\to S$ *a morphism with* $S,X\in\mathrm{QPVar}(k)$*. Consider a factorization* $f:X\hookrightarrow Y\times S\xrightarrow{p}S$ *with* $Y\in\mathrm{SmVar}(k)$*. Let* $S=\cup_iS_i$ *an open affine cover and* $i_i:S_i\hookrightarrow\tilde{S}_i$ *closed embedding with* $\tilde{S}_i\in\mathrm{SmVar}(k)$*. Denote by* $i'_I:X_I\hookrightarrow Y\times\tilde{S}_I$ *the closed embeddings. For* $(K,W)\in P_{\mathbb{Z}_pfil,k}(X^{et})\cap D_{\mathbb{Z}_pfil,c,k,gm}(X^{et})$*, we have as in (i) the following map in* $D_{\mathbb{B}_{dr}fil}(S_K^{an,pet}/(\tilde{S}_{I,K})^{an,pet})$

$$\begin{aligned}T^k(f,\mathbb{B}_{dr})(K,W):\mathbb{B}_{dr,(\tilde{S}_I)}({}^pRf_*(K,W))\xrightarrow{=}(H^k(\cdots\to\bigoplus_{1\le i_1<\cdots<i_d\le d,1\le j_1<\cdots<j_e\le e}\\
x_{\tilde{S}_I\backslash D_{j_1,I}/\tilde{S}_I}\cdots x_{\tilde{S}_I\backslash\tilde{D}_{i_{r'},I}/\tilde{S}_I}\phi_{\tilde{D}_{j_{r'+1},I}}\cdots\phi_{\tilde{D}_{j_{s'},I}}\psi_{\tilde{D}_{j_{s'+1},I}}\cdots\psi_{\tilde{D}_{je,I}}(p_{\tilde{S}_I*}E(x_{(Y\times\tilde{S}_I)\backslash\tilde{E}_{i_1,I}/Y\times\tilde{S}_I}\cdots\\
x_{(Y\times\tilde{S}_I)\backslash\tilde{E}_{i_r,I}/Y\times\tilde{S}_I}\phi_{\tilde{E}_{i_{r+1}}}\cdots\phi_{\tilde{E}_{i_s}}\psi_{\tilde{E}_{i_{s+1}}}\cdots\psi_{\tilde{E}_{i_d}}(K,W)))\otimes_{\mathbb{Q}_p}\\
\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{D}_{j_1}/\tilde{S}}}\otimes\cdots\otimes\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{D}_{j_{r'}}/\tilde{S}_I}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\phi_{\tilde{D}_{j_{r'+1},I}}}\otimes\cdots\otimes\\
\mathbb{B}_{dr,\phi_{\tilde{D}_{j_{s'},I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{D}_{j_{s'+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{\tilde{D}_{j_e,I}}}\to\cdots),\mathbb{B}_{dr}(t_{IJ}))\\
\xrightarrow{((I\otimes m_f(-))\circ(I\otimes a_-(-))\circ T(p,\otimes)(-,-)\circ(T(p,-)(-)\circ\cdots\circ T(p,-)(-)\circ l_k(-)\otimes I))}\\
((\cdots\to\bigoplus_{1\le i_1<\cdots<i_d\le d,1\le j_1<\cdots<j_e\le e}p_{\tilde{S}_I*}E(x_{(Y\times\tilde{S}_I)\backslash p^{-1}(\tilde{D}_{j_1,I})/Y\times\tilde{S}_I}x_{(Y\times\tilde{S}_I)\backslash\tilde{E}_{i_1,I}/Y\times\tilde{S}_I}\\
\cdots x_{(Y\times\tilde{S}_I)\backslash p^{-1}(\tilde{D}_{j_{r'},I})/Y\times\tilde{S}_I}x_{(Y\times\tilde{S}_I)\backslash\tilde{E}_{i_r}/Y\times\tilde{S}_I}\\
\phi_{p^{-1}(\tilde{D}_{j_{r'+1},I})}\phi_{\tilde{E}_{i_{r+1}}}\cdots\phi_{p^{-1}(\tilde{D}_{j_{s'},I})}\phi_{\tilde{E}_{i_s}}\psi_{p^{-1}(\tilde{D}_{j_{s'+1}})}\psi_{\tilde{E}_{i_{s+1}}}\cdots\psi_{p^{-1}(\tilde{D}_{j_d,I})}\psi_{\tilde{E}_{i_d,I}}(K,W)\otimes_{\mathbb{Q}_p}\\
\mathbb{B}_{dr,x_{(Y\times\tilde{S}_I)\backslash\tilde{E}_{i_1}/Y\times\tilde{S}_I}}\otimes_{\mathbb{B}_{dr,Y\times\tilde{S}_I}}\otimes\cdots\otimes\mathbb{B}_{dr,x_{(Y\times\tilde{S}_I)\backslash\tilde{E}_{i_r,I}/Y\times\tilde{S}_I}}\otimes_{\mathbb{B}_{dr,Y\times\tilde{S}_I}}\\
\mathbb{B}_{dr,\phi_{\tilde{E}_{i_{r+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{\tilde{E}_{i_s,I}}}\otimes_{\mathbb{B}_{dr,Y\times\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{E}_{i_{s+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{\tilde{E}_{i_d,I}}}\\
\otimes_{\mathbb{B}_{dr,Y\times\tilde{S}_I}}\mathbb{B}_{dr,x_{(Y\times\tilde{S}_I)\backslash p^{-1}(\tilde{D}_{j_1,I})/X}}\otimes\cdots\otimes\mathbb{B}_{dr,x_{(Y\times\tilde{S}_I)\backslash p^{-1}(\tilde{D}_{j_{r'}})/X}}\\
\otimes_{\mathbb{B}_{dr,Y\times\tilde{S}_I}}\mathbb{B}_{dr,\phi_{p^{-1}(\tilde{D}_{j_{r'+1},I})}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{p^{-1}(\tilde{D}_{j_{s'},I})}}\\
\otimes_{\mathbb{B}_{dr,Y\times\tilde{S}_I}}\mathbb{B}_{dr,\psi_{p^{-1}(\tilde{D}_{j_{s'+1},I})}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{p^{-1}(\tilde{D}_{j_e,I})}})\to\cdots),\mathbb{B}_{dr}(t_{IJ}))\xrightarrow{=:}Rp_*\mathbb{B}_{dr,(Y\times\tilde{S}_I)}(K,W)\end{aligned}$$

where $(E_1,\ldots,E_d)\in\mathcal{S}(K)$ *is a stratification by Cartier divisor* $E_i\subset X$*,* $1\le i\le d$*, such that*

$$K_{|E(r)\backslash E(r+1)}:=l_r^*K\in D_{\mathbb{Z}_p,c}((E(r)\backslash E(r+1))^{et})$$

are local systems for all $1\le r\le d$*,* $l_r:E(r)\hookrightarrow X$ *being the locally closed embeddings, and* $(D_1,\ldots,D_e)\in\mathcal{S}(K)$ *is a stratification by Cartier divisor* $D_i\subset S$*,* $1\le i\le e$*, such that*

$$\begin{aligned}({}^pR^kf_*x_{i_1}\cdots x_{i_r}\phi_{E_{i_{r+1}}}\cdots\phi_{E_{i_s}}\psi_{E_{i_{s+1}}}\cdots\psi_{E_{i_d}}K)_{|D(r')\backslash D(r'+1)}:=\\
m_{r'}^*\,{}^pR^kf_*x_{i_1}\cdots x_{i_r}\phi_{E_{i_{r+1}}}\cdots\phi_{E_{i_s}}\psi_{E_{i_{s+1}}}\in D_{\mathbb{Z}_p,c}((D(r')\backslash D(r'+1))^{et})\end{aligned}$$

are local systems for all $1\le i_1<\cdots<i_{r+1}<\cdots<i_{s+1}<\cdots<i_d\le d$*, all* $1\le r'\le e$*,* $k\in\mathbb{Z}$*,* $m_{r'}:D(r')\hookrightarrow S$ *being the locally closed embeddings,* $\tilde{D}_{s,I}\subset\tilde{S}_I$ *(Cartier) divisor such that* $D_s\cap S_I\subset\tilde{D}_{s,I}\cap S$*, and* $\tilde{E}_{s,I}\subset Y\times\tilde{S}_I$ *(Cartier) divisor such that* $E_s\cap X_I\subset\tilde{E}_{s,I}\cap X$*. It gives, by functoriality, for* $(K,W)\in D_{\mathbb{Z}_pfil,c,k,gm}(S^{et})$*, the canonical map in* $D_{\mathbb{B}_{dr}fil}(S_K^{an,pet}/(\tilde{S}_{I,K}^{an,pet}))$

$$T(f,\mathbb{B}_{dr})(K,W):\mathbb{B}_{dr,(\tilde{S}_I)}(Rf_*(K,W))\to Rf_*\mathbb{B}_{dr,(Y\times\tilde{S}_I)}(K,W).$$

Lemma 6. *Let $f : X \to S$ a proper morphism with $X, S \in \mathrm{SmVar}(k)$. Let $E \subset X$ a (Cartier) divisor. Denote $j : U := X \backslash E \hookrightarrow X$ the open embedding. Let $K \in P_{\mathbb{Z}_p,k}(X^{et})$ such that $K_{|U}$ and $K_{|E}$ are local systems and such that ${}^pR^kf_*K = R^kf_*K$ are local systems for all $k \in \mathbb{Z}$. Then,*

(i) The map

$$Rf_*(I \otimes a_X(\mathbb{B}_{dr,U/X,\mathbb{C}_p})) : Rf_*Rj_*\pi^*_{k/\mathbb{C}_p}K \otimes \mathbb{B}_{dr,S_{\mathbb{C}_p}} \to Rf_*(Rj_*\pi^*_{k/\mathbb{C}_p}K \otimes \mathbb{B}_{dr,U/X,\mathbb{C}_p})$$

is an isomorphism.

(ii) The map

$$Rf_*(I \otimes a_X(\mathbb{B}_{dr,\psi_E})) : Rf_*\psi_E K \otimes \mathbb{B}_{dr,S} \to Rf_*(\psi_E K \otimes \mathbb{B}_{dr,\psi_E})$$

is an isomorphism.

(iii) The map

$$Rf_*(I \otimes a_X(\mathbb{B}_{dr,\phi_E})) : Rf_*\phi_E K \otimes \mathbb{B}_{dr,S} \to Rf_*(\phi_E K \otimes \mathbb{B}_{dr,\phi_E})$$

is an isomorphism.

(iv) The map

$$Rf_*(I \otimes a_X(\mathbb{B}_{dr,x_{U/X},\mathbb{C}_p})) : Rf_*x_{U/X}(\pi^*_{k/\mathbb{C}_p}K) \otimes \mathbb{B}_{dr,S_{\mathbb{C}_p}} \to Rf_*(x_{U/X}(K) \otimes \mathbb{B}_{dr,x_{U/X,\mathbb{C}_p}})$$

is an isomorphism.

Proof. (i):Consider a desingularization of the pair (X, E). Then the E_2 degenerescence of the perverse Leray spectral sequence, and the a normal crossing divisor case (see [21]). shows that $H^i(\mathbb{B}_{dr,U/X,\mathbb{C}_p}) = 0$ for all $i \in \mathbb{Z}$, $i \neq 0$. Hence, (i) follows from theorem 6 and theorem 8.
(ii):Follows from theorem 43 and on the other hand theorem 6 and theorem 8.
(iii):Follows from theorem 43 and on the other hand theorem 6 and theorem 8.
(iv):Follows from (i),(ii). ☐

Theorem 47. *(i) Let $X \in \mathrm{PSmVar}(k)$. Let $Z \subset X$ a closed subset. Denote by $j : U := X\backslash Z \hookrightarrow X$ the open complementary embedding. Take (Cartier) divisor $D_1, \ldots, D_r \subset X$ such that $Z = \cap_{i=1}^r D_i$. The map in $D(\mathbb{B}_{dr,\mathbb{C}_p}, G)$*

$$\begin{aligned} R\Gamma(X_{\bar{k}}, I \otimes a_X(\mathbb{B}_{dr,D_i/X})) : R\Gamma(X_{\bar{k}}, Rj_*\mathbb{Z}_{U^{et},p}) \otimes \mathbb{B}_{dr,\mathbb{C}_p} &= R\Gamma(U_{\bar{k}}, \mathbb{Z}_{p,U^{et}}) \otimes \mathbb{B}_{dr,\mathbb{C}_p} \\ &\to R\Gamma(X_{\bar{k}}, j_*\mathbb{Z}_{p,U^{et}} \otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,X\backslash D_1/X} \otimes_{\mathbb{B}_{dr,X}} \cdots \otimes_{\mathbb{B}_{dr,X}} \mathbb{B}_{dr,X\backslash D_1/X}) \\ &\xrightarrow{=} R\Gamma(X_{\bar{k}}, F^0DR(X)(j_{*Hdg}(O_U, F_b)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X}, F))) \end{aligned}$$

is an isomorphism.

(ii) Let $f : X \to S$ be a proper morphism with $X, S \in \mathrm{SmVar}(k)$. For $K \in D_{\mathbb{Z}_p,c,k}(X^{et})$, the map in $D_{\mathbb{B}_{dr},G}(S^{an,pet}_{\mathbb{C}_p})$ (where the a G module structure is a continuous action of the Galois group)

$$T(f, B_{dr})(K) : \mathbb{B}_{dr,S}(Rf_*K) \to Rf_*\mathbb{B}_{dr,X}(K)$$

given in definition 80 is an isomorphism.

(ii)' Let $f : X \to S$ a morphism with $S, X \in \mathrm{QPVar}(k)$. Consider a factorization $f : X \hookrightarrow Y \times S \xrightarrow{p} S$ with $Y \in \mathrm{SmVar}(k)$. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Denote by $i'_I : X_I \hookrightarrow Y \times \tilde{S}_I$ the closed embeddings. For $K \in D_{\mathbb{Z}_p,c,k}(X^{et})$, the map in $D_{\mathbb{B}_{dr},G}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}_{I,\mathbb{C}_p})^{an,pet})$

$$T(f, \mathbb{B}_{dr})(K) : \mathbb{B}_{dr,(\tilde{S}_I)}(Rf_*K) \to Rp_*\mathbb{B}_{dr,(Y\times\tilde{S}_I)}(K)$$

given in definition 80 is an isomorphism.

Proof. (i):Follows from lemma 6(i).
(ii):Follows from lemma 6 and on the other hand theorem 5 together with theorem 8.
(ii)':Follows from lemma 6 and on the other hand theorem 5 together with theorem 8 as for (ii). $\square$

Remark 7. *Let $f : X \to S$ a proper morphism with $S, X \in \mathrm{Var}(k)$. Then for $K \in C_{\mathbb{Z}_p}(X^{et})$, the map in $D_{\mathbb{Z}_p}(S^{an,pet})$*

$$Rf_*K \otimes \mathbb{B}_{dr,S} \xrightarrow{\mathrm{ad}(Lf^{*mod}, Rf_*)(\mathbb{B}_{dr,S})} Rf_*K \otimes Rf_*\mathbb{B}_{dr,X} \xrightarrow{T(f,f,\otimes)(K,\mathbb{B}_{dr,X})} Rf_*(K \otimes \mathbb{B}_{dr,X})$$

is an isomorphism by theorem 6 and theorem 8. In the analytic case ([27]), for $f : X \to S$ a smooth proper morphism with $X, S \in \mathrm{AnSm}(K)$ and $L \in Loc_{\mathbb{Z}_p}(X^{et})$ an analytic local system, the map in $D_{\mathbb{Z}_p}(S^{pet})$

$$Rf_*L \otimes \mathbb{B}_{dr,S} \xrightarrow{\mathrm{ad}(Lf^{*mod}, Rf_*)(\mathbb{B}_{dr,S})} Rf_*L \otimes Rf_*\mathbb{B}_{dr,X} \xrightarrow{T(f,f,\otimes)(L,\mathbb{B}_{dr,X})} Rf_*(L \otimes \mathbb{B}_{dr,X})$$

is an isomorphism.

Definition 81. *(i) Let $j : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{SmVar}(k)$ and $D := S \backslash S^o$ a (Cartier) divisor. We will consider, using definition 78(vi) for $(K, W) \in P_{\mathbb{Z}_p fil,k}(S^{o,et})$, the canonical isomorphism in $D_{\mathbb{B}_{dr,S}fil}(S_K^{an,pet})$*

$$\begin{aligned}
&T(j, \mathbb{B}_{dr})(K, W) : \mathbb{B}_{dr,S}(j_{*w}(K, W)) \xrightarrow{:=} \\
&((\cdots \to \bigoplus_{1 \leq i_1 \cdots < i_d \leq d} \psi_D x_{S\backslash \bar{E}_{i_1}/S} \cdots x_{S\backslash \bar{E}_{i_r}/S} \phi_{\bar{E}_{i_{r+1}}} \cdots \phi_{\bar{E}_{i_s}} \psi_{\bar{E}_{i_{s+1}}} \cdots \psi_{\bar{E}_{i_d}} j_{*w}(K, W) \\
&\otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,\psi_D} \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,x_{S\backslash \bar{E}_{i_1}/S}} \otimes_{\mathbb{B}_{dr,S}} \cdots \otimes \mathbb{B}_{dr,x_{S\backslash \bar{E}_{i_r}/S}} \\
&\otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,\phi_{\bar{E}_{i_{r+1}}}} \otimes \cdots \otimes \mathbb{B}_{dr,\phi_{\bar{E}_{i_s}}} \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,\psi_{\bar{E}_{i_{s+1}}}} \otimes \cdots \otimes \mathbb{B}_{dr,\psi_{\bar{E}_{i_d}}} \to \cdots) \to \\
&(\cdots \to \bigoplus_{1 \leq i_1 \cdots < i_d \leq d} x_{S^o/S} x_{S\backslash \bar{E}_{i_1}/S} \cdots x_{S\backslash \bar{E}_{i_r}/S} \phi_{\bar{E}_{i_{r+1}}} \cdots \phi_{\bar{E}_{i_s}} \psi_{\bar{E}_{i_{s+1}}} \cdots \psi_{\bar{E}_{i_d}} j_{*w}(K, W) \oplus \\
&\phi_D x_{S\backslash \bar{E}_{i_1}/S} \cdots x_{S\backslash \bar{E}_{i_r}/S} \phi_{\bar{E}_{i_{r+1}}} \cdots \phi_{\bar{E}_{i_s}} \psi_{\bar{E}_{i_{s+1}}} \cdots \psi_{\bar{E}_{i_d}} j_{*w}(K, W) \\
&\otimes_{\mathbb{Q}_p} (\mathbb{B}_{dr,x_{S^o/S}} \oplus \mathbb{B}_{dr,\phi_D}) \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,x_{S\backslash \bar{E}_{i_1}/S}} \otimes_{\mathbb{B}_{dr,S}} \cdots \otimes \mathbb{B}_{dr,x_{S\backslash \bar{E}_{i_r}/S}} \\
&\otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,\phi_{\bar{E}_{i_{r+1}}}} \otimes \cdots \otimes \mathbb{B}_{dr,\phi_{\bar{E}_{i_s}}} \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,\psi_{\bar{E}_{i_{s+1}}}} \otimes \cdots \otimes \mathbb{B}_{dr,\psi_{\bar{E}_{i_d}}} \to \cdots) \to \\
&(\cdots \to \bigoplus_{1 \leq i_1 \cdots < i_d \leq d} \psi_D x_{S\backslash \bar{E}_{i_1}/S} \cdots x_{S\backslash \bar{E}_{i_r}/S} \phi_{\bar{E}_{i_{r+1}}} \cdots \phi_{\bar{E}_{i_s}} \psi_{\bar{E}_{i_{s+1}}} \cdots \psi_{\bar{E}_{i_d}} j_{*w}(K, W) \\
&\otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,\psi_D} \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,x_{S\backslash \bar{E}_{i_1}/S}} \otimes_{\mathbb{B}_{dr,S}} \cdots \otimes \mathbb{B}_{dr,x_{S\backslash \bar{E}_{i_r}/S}} \\
&\otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,\phi_{\bar{E}_{i_{r+1}}}} \otimes \cdots \otimes \mathbb{B}_{dr,\phi_{\bar{E}_{i_s}}} \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,\psi_{\bar{E}_{i_{s+1}}}} \otimes \cdots \otimes \mathbb{B}_{dr,\psi_{\bar{E}_{i_d}}} \to \cdots)) \\
&\xrightarrow{(0,(c(x_{S^o/S}(-),0)) \otimes c(\mathbb{B}_{dr,S^o/S}) \otimes I, 0)} \\
&(\cdots \to \bigoplus_{1 \leq i_1 \cdots < i_d \leq d} j_{*w} x_{S\backslash \bar{E}_{i_1}/S} \cdots x_{S\backslash \bar{E}_{i_r}/S} \phi_{\bar{E}_{i_{r+1}}} \cdots \phi_{\bar{E}_{i_s}} \psi_{\bar{E}_{i_{s+1}}} \cdots \psi_{\bar{E}_{i_d}} (K, W) \\
&\otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,S^o/S} \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,x_{S\backslash \bar{E}_{i_1}/S}} \otimes_{\mathbb{B}_{dr,S}} \cdots \otimes \mathbb{B}_{dr,x_{S\backslash \bar{E}_{i_r}/S}} \\
&\otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,\phi_{\bar{E}_{i_{r+1}}}} \otimes \cdots \otimes \mathbb{B}_{dr,\phi_{\bar{E}_{i_s}}} \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,\psi_{\bar{E}_{i_{s+1}}}} \otimes \cdots \otimes \mathbb{B}_{dr,\psi_{\bar{E}_{i_d}}} \to \cdots) \\
&\xrightarrow{m(-)^{-1}} \\
&(\cdots \to \bigoplus_{1 \leq i_1 \cdots < i_d \leq d} V_{D0} j_{*w}(x_{S^o\backslash E_{i_1}/S} \cdots x_{S^o\backslash E_{i_r}/S} \phi_{E_{i_{r+1}}} \cdots \phi_{E_{i_s}} \psi_{E_{i_{s+1}}} \cdots \psi_{E_{i_d}} (K, W) \\
&\otimes_{\mathbb{Q}_p} \mathbb{B}_{dr,x_{S^o\backslash E_{i_1}/S}} \otimes_{\mathbb{B}_{dr,S}} \cdots \mathbb{B}_{dr,x_{S^o\backslash E_{i_r}/S}} \\
&\otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,\phi_{E_{i_{r+1}}}} \otimes \cdots \otimes \mathbb{B}_{dr,\phi_{E_{i_s}}} \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,\psi_{E_{i_{s+1}}}} \otimes \cdots \otimes \mathbb{B}_{dr,\psi_{E_{i_d}}}) \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,S^o/S}) \to \cdots) \\
&\xrightarrow{=:} V_{D0} j_{*w} \mathbb{B}_{dr,S^o}(K, W) \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,S^o/S}.
\end{aligned}$$

where $(E_1,\ldots,E_d)\in\mathcal{S}(K)$ *is a stratification by (Cartier) divisor* $E_i\subset S^o$, $1\leq i\leq d$, *such that*

$$K_{|E(r)\backslash E(r+1)}:=l_r^*K\in D_{\mathbb{Z}_p,c}((E(r)\backslash E(r+1))^{et})$$

are local systems for all $1\leq r\leq d$, $l_r:E(r)\hookrightarrow S^o$ *being the locally closed embeddings.*

(ii) *Let* $l:S^o\hookrightarrow S$ *an open embedding with* $S\in\mathrm{Var}(k)$ *such that* $D=S\backslash S^o\subset S$ *is a Cartier divisor. Let* $S=\cup_i S_i$ *an open affine cover and* $i_i:S_i\hookrightarrow\tilde{S}_i$ *closed embedding with* $\tilde{S}_i\in\mathrm{SmVar}(k)$. *Let* $l_I:\tilde{S}_I^o\hookrightarrow\tilde{S}_I$ *open embeddings such that* $\tilde{S}_I^o\cap S=S^o\cap S_I$ *and* $\tilde{D}_I\subset\tilde{S}_I$ *a Cartier divisor such that* $D\cap S_I\subset\tilde{D}_I\cap S$. *We will consider, using definition 78(vi), for* $(K,W)\in P_{\mathbb{Z}_pfil,k}(S^{o,et})$, *the canonical isomorphism in* $D_{\mathbb{B}_{dr}fil}(S_K^{an,pet}/(\tilde{S}_{I,K})^{an,pet})$

$$\begin{aligned}
&T(l,\mathbb{B}_{dr})(K,W):\mathbb{B}_{dr,(\tilde{S}_I)}(l_{*w}(K,W))\xrightarrow{:=}\\
&(((\cdots\to\bigoplus_{1\leq i_1\cdots<i_d\leq d}\psi_{\tilde{D}_I}x_{\tilde{S}_I\backslash\tilde{D}_{i_1,I}/\tilde{S}_I}\cdots x_{\tilde{S}_I\backslash\tilde{D}_{I,i_r}/S}\phi_{\tilde{E}_{i_{r+1},I}}\cdots\phi_{\tilde{E}_{i_s,I}}\psi_{\tilde{E}_{i_{s+1},I}}\cdots\psi_{\tilde{E}_{i_d,I}}i_{I*}j_I^*l_{*w}(K,W)\\
&\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,\psi_{\tilde{D}_I}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{E}_{i_1,I}/S}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{E}_{i_r,I}/S}}\\
&\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\phi_{\tilde{E}_{i_{r+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{\tilde{E}_{i_s,I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{E}_{i_{s+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{\tilde{E}_{i_d,I}}}\to\cdots)\to\\
&(\cdots\to\bigoplus_{1\leq i_1\cdots<i_d\leq d}x_{\tilde{S}_I^o/\tilde{S}_I}x_{S\backslash\tilde{E}_{i_1,I}/S}\cdots x_{\tilde{S}_I\backslash\tilde{E}_{i_r,I}/S}\phi_{\tilde{E}_{i_{r+1},I}}\cdots\phi_{\tilde{E}_{i_s,I}}\psi_{\tilde{E}_{i_{s+1},I}}\cdots\psi_{\tilde{E}_{i_d,I}}i_{I*}j_I^*l_{*w}(K,W)\\
&\oplus\phi_{\tilde{D}_I}x_{S\backslash\tilde{E}_{i_1,I}/S}\cdots x_{\tilde{S}_I\backslash\tilde{E}_{i_r,I}/S}\phi_{\tilde{E}_{i_{r+1},I}}\cdots\phi_{\tilde{E}_{i_s,I}}\psi_{\tilde{E}_{i_{s+1},I}}\cdots\psi_{\tilde{E}_{i_d,I}}i_{I*}j_I^*l_{*w}(K,W)\\
&\otimes_{\mathbb{Q}_p}(\mathbb{B}_{dr,x_{\tilde{S}_I^o/\tilde{S}_I}}\oplus\mathbb{B}_{dr,\phi_D})\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{E}_{i_1,I}/S}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{E}_{i_r,I}/S}}\\
&\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\phi_{\tilde{E}_{i_{r+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{\tilde{E}_{i_s,I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{E}_{i_{s+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{\tilde{E}_{i_d,I}}}\to\cdots)\to\\
&(\cdots\to\bigoplus_{1\leq i_1\cdots<i_d\leq d}\psi_{\tilde{D}_I}x_{S\backslash\tilde{E}_{i_1,I}/S}\cdots x_{S\backslash\tilde{E}_{i_r,I}/S}\phi_{\tilde{E}_{i_{r+1},I}}\cdots\phi_{\tilde{E}_{i_s,I}}\psi_{\tilde{E}_{i_{s+1},I}}\cdots\psi_{\tilde{E}_{i_d,I}}i_{I*}j_I^*l_{*w}(K,W)\\
&\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,\psi_D}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{E}_{i_1,I}/S}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{E}_{i_r,I}/\tilde{S}_I}}\\
&\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\phi_{\tilde{E}_{i_{r+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{\tilde{E}_{i_s,I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{E}_{i_{s+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{\tilde{E}_{i_d,I}}}\to\cdots)),\mathbb{B}_{dr}(t_{IJ}))\\
&\xrightarrow{((0,(c(x_{\tilde{S}_I^o/\tilde{S}_I}(-),0))\otimes c(\mathbb{B}_{dr,\tilde{S}_I^o/\tilde{S}_I})\otimes I,0))}\\
&(\cdots\to\bigoplus_{1\leq i_1\cdots<i_d\leq d}l_{I*w}x_{S\backslash\tilde{E}_{i_1,I}/S}\cdots x_{S\backslash\tilde{E}_{i_r,I}/S}\phi_{\tilde{E}_{i_{r+1},I}}\cdots\phi_{\tilde{E}_{i_s,I}}\psi_{\tilde{E}_{i_{s+1},I}}\cdots\psi_{\tilde{E}_{i_d,I}}i_{I*}j_I^*(K,W)\\
&\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{E}_{i_1,I}/S}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{E}_{i_r,I}/\tilde{S}_I}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\phi_{\tilde{E}_{i_{r+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{\tilde{E}_{i_s,I}}}\\
&\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{E}_{i_{s+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{\tilde{E}_{i_d,I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\tilde{S}_I^o/\tilde{S}_I}\to\cdots),\mathbb{B}_{dr}(t_{IJ}))\\
&\xrightarrow{(m(-)^{-1})}\\
&((\cdots\to\bigoplus_{1\leq i_1\cdots<i_d\leq d}V_{\tilde{D}_I0}l_{I*w}(x_{\tilde{S}_I^o\backslash\tilde{E}_{i_1,I}/\tilde{S}_I}\cdots x_{\tilde{S}_I^o\backslash\tilde{E}_{i_r,I}/S}\phi_{\tilde{E}_{i_{r+1},I}}\cdots\phi_{\tilde{E}_{i_s,I}}\psi_{\tilde{E}_{i_{s+1}}}\cdots\psi_{\tilde{E}_{i_d,I}}i_{I*}j_I^*(K,W)\\
&\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,x_{\tilde{S}_I^o\backslash\tilde{E}_{i_1,I}/S}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes\mathbb{B}_{dr,x_{\tilde{S}_I^o\backslash\tilde{E}_{i_r,I}/S}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\phi_{\tilde{E}_{i_{r+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{\tilde{E}_{i_s,I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\\
&\mathbb{B}_{dr,\psi_{\tilde{E}_{i_{s+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{\tilde{E}_{i_d,I}}})\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\tilde{S}_I^o/\tilde{S}_I}\to\cdots),\mathbb{B}_{dr}(t_{IJ}))\\
&\xrightarrow{=:}V_{D0}l_{*w}\mathbb{B}_{dr,(\tilde{S}_I^o)}(K,W)\otimes_{\mathbb{B}_{dr,S}}(\mathbb{B}_{dr,\tilde{S}_I^o/\tilde{S}_I},t_{IJ}).
\end{aligned}$$

where $(E_1,\ldots,E_d)\in\mathcal{S}(K)$ *is a stratification by Cartier divisor* $E_i\subset S^o$, $1\leq i\leq d$, *such that*

$$K_{|E(r)\backslash E(r+1)}:=l_r^*K\in D_{\mathbb{Z}_p,c}((E(r)\backslash E(r+1))^{et})$$

are local systems for all $1\leq r\leq d$, $l_r:E(r)\hookrightarrow S^o$ *being the locally closed embeddings, and* $\tilde{E}_{s,I}\subset\tilde{S}_I$, $\tilde{D}_I\subset\tilde{S}_I$ *are (Cartier) divisor such that* $\bar{E}_s\cap S_I\subset\tilde{E}_{s,I}\cap S_I$ *and* $D\cap S_I\subset\tilde{D}_I\cap S_I$. *It* *gives for* $(K,W)\in D_{\mathbb{Z}_pfil,c,k}(S^{o,et})$*, the canonical isomorphism in* $D_{\mathbb{B}_{dr}fil}(S_K^{an,pet}/(\tilde{S}_{I,K})^{an,pet})$

$$T(l,\mathbb{B}_{dr})(K,W):\mathbb{B}_{dr,(\tilde{S}_I)}(l_{*w}(K,W))\xrightarrow{\sim}V_{D0}l_{*w}\mathbb{B}_{dr,(\tilde{S}_I^o)}(K,W)\otimes_{\mathbb{B}_{dr,S}}(\mathbb{B}_{dr,\tilde{S}_I^o/\tilde{S}_I},t_{IJ}).$$

Let $S\in\mathrm{SmVar}(k)$. Let $j:S^o\hookrightarrow S$ an open embedding such that $D=S\backslash S^o\subset S$ is a Cartier divisor. Denote by $\Delta_S:S\hookrightarrow S\times S$ the diagonal closed embedding and $p_1:S\times S\to S$ and $p_2:S\times S\to S$ the projections.

- We have the isomorphism in $C_{\mathbb{B}_{dr}}(S_K^{an,pet})$

$$\begin{aligned}
&m(\mathbb{B}_{dr,S^o/S,K}):\mathbb{B}_{dr,S^o/S,K}\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,S^o/S,K}\xrightarrow{:=}\\
&F^0DR(S)(j_{*Hdg}(O_{S^o},F_b)\otimes_{O_S}(O\mathbb{B}_{dr,S_K},F))\otimes_{\mathbb{B}_{dr,S_K}}F^0DR(S)(j_{*Hdg}(O_{S^o},F_b)\otimes_{O_S}(O\mathbb{B}_{dr,S_K},F))\\
&\xrightarrow{F^0w_S}\Delta_S^{*mod}F^0DR(S\times S)((p_1^{*mod}j_{*Hdg}(O_{S^o},F_b)\otimes_{O_{S\times S}}\\
&p_2^{*mod}j_{*Hdg}(O_{S^o},F_b)\otimes_{O_{S\times S}}(O\mathbb{B}_{dr,(S\times S)_K},F)))\\
&\xrightarrow{DR(S\times S)(\mathrm{ad}(\Delta_{S,Hdg}^{*mod},\Delta_{S*mod})(-))}\\
&\Delta_S^{*mod}F^0DR(S\times S)(\Delta_{S*mod}\Delta_{S,Hdg}^{*mod}(p_1^{*mod}j_{*Hdg}(O_{S^o},F_b)\otimes_{O_{S\times S}}p_2^{*mod}j_{*Hdg}(O_{S^o},F_b))\\
&\otimes_{O_{S\times S}}(O\mathbb{B}_{dr,(S\times S)_K},F))\xrightarrow{T^{B_{dr}}(\Delta_S,DR)(-)}\\
&\Delta_S^{*mod}\Delta_{S*}F^0DR(S)(\Delta_{S,Hdg}^{*mod}(p_1^{*mod}j_{*Hdg}(O_{S^o},F_b)\otimes_{O_{S\times S}}p_2^{*mod}j_{*Hdg}(O_{S^o},F_b))\otimes_{O_S}(O\mathbb{B}_{dr,S_K},F))\\
&\xrightarrow{=}F^0DR(S)(j_{*Hdg}(O_{S^o},F_b)\otimes_{O_S}^{Hdg}j_{*Hdg}(O_{S^o},F_b)\otimes_{O_S}(O\mathbb{B}_{dr,S_K},F))\\
&\xrightarrow{F^0DR(S)(m)}F^0DR(S)(j_{*Hdg}(O_{S^o},F_b)\otimes_{O_S}(O\mathbb{B}_{dr,S_K},F)):=\mathbb{B}_{dr,S^o/S,K}
\end{aligned}$$

  where

$$m:(j_*O_{S^o},V_D)\otimes_{O_S}(j_*O_{S^o},V_D)\xrightarrow{\sim}(j_*O_{S^o},V_D),\ m(b_1\otimes b_2)=b_1b_2$$

  is the multiplication map whose inverse is

$$n:(j_*O_{S^o},V_D)\xrightarrow{\sim}(j_*O_{S^o},V_D)\otimes_{O_S}(j_*O_{S^o},V_D),\ n(b)=b\otimes 1.$$

- We have the isomorphism

$$\begin{aligned}
&m(\mathbb{B}_{dr,\psi_D,K}):\mathbb{B}_{dr,\psi_D,K}\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,\psi_D,K}\xrightarrow{:=}\\
&F^0DR(S)(\psi_D(O_{S^o},F_b)\otimes_{O_S}(O\mathbb{B}_{dr,S_K},F))\otimes_{\mathbb{B}_{dr,S_K}}F^0DR(S)(\psi_D(O_{S^o},F_b)\otimes_{O_S}(O\mathbb{B}_{dr,S_K},F))\\
&\xrightarrow{F^0w_S}\Delta_S^{*mod}F^0DR(S\times S)(p_1^{*mod}\psi_D(O_{S^o},F_b)\otimes_{O_{S\times S}}p_2^{*mod}\psi_D(O_{S^o},F_b)\otimes_{O_{S\times S}}(O\mathbb{B}_{dr,(S\times S)_K},F))\\
&\xrightarrow{DR(S\times S)(\mathrm{ad}(\Delta_{S,Hdg}^{*mod},\Delta_{S*mod})(-))}\\
&\Delta_S^{*mod}F^0DR(S\times S)(\Delta_{S*mod}\Delta_{S,Hdg}^{*mod}(p_1^{*mod}\psi_D(O_{S^o},F_b)\otimes_{O_{S\times S}}p_2^{*mod}\psi_D(O_{S^o},F_b))\\
&\otimes_{O_{S\times S}}(O\mathbb{B}_{dr,(S\times S)_K},F))\xrightarrow{T^{B_{dr}}(\Delta_S,DR)(-)}\\
&\Delta_S^{*mod}\Delta_{S*}F^0DR(S)(\Delta_{S,Hdg}^{*mod}(p_1^{*mod}\psi_D(O_{S^o},F_b)\otimes_{O_{S\times S}}p_2^{*mod}\psi_D(O_{S^o},F_b))\otimes_{O_S}(O\mathbb{B}_{dr,S_K},F))\\
&\xrightarrow{=}F^0DR(S)(\psi_D(O_{S^o},F_b)\otimes_{O_S}^{Hdg}\psi_D(O_{S^o},F_b)\otimes_{O_S}(O\mathbb{B}_{dr,S_K},F))\\
&\xrightarrow{F^0DR(S)(m)}F^0DR(S)(\psi_D(O_{S^o},F_b)\otimes_{O_S}(O\mathbb{B}_{dr,S_K},F)):=\mathbb{B}_{dr,\psi_D,K}
\end{aligned}$$

  where

$$m:\psi_D(O_{S^o})\otimes_{O_S}\psi_D(O_{S^o})\xrightarrow{\sim}\psi_DO_{S^o},\ m(b_1\otimes b_2)=b_1b_2$$

  is the multiplication map whose inverse is

$$n:\psi_D(O_{S^o})\xrightarrow{\sim}\psi_D(O_{S^o})\otimes_{O_S}\psi_D(O_{S^o}),\ n(b)=b\otimes 1.$$

- We have similarly to $\mathbb{B}_{dr,\psi_D,K}$ the isomorphism

$$m(\mathbb{B}_{dr,\phi_D,K}):\mathbb{B}_{dr,\phi_D,K}\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,\phi_D,K}\xrightarrow{\sim}\mathbb{B}_{dr,\phi_D,K}$$

with

$$m:\phi_D(O_{S^o})\otimes_{O_S}\phi_D(O_{S^o})\xrightarrow{\sim}\phi_D O_{S^o},\ m(b_1\otimes b_2)=b_1b_2$$

is the multiplication map whose inverse is

$$n:\phi_D(O_{S^o})\xrightarrow{\sim}\phi_D(O_{S^o})\otimes_{O_S}\phi_D(O_{S^o}),\ n(b)=b\otimes 1.$$

- We have similarly the isomorphism

$$m(\mathbb{B}_{dr,x_{S^o/S},K}):\mathbb{B}_{dr,x_{S^o/S},K}\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,x_{S^o/S},K}\xrightarrow{\sim}\mathbb{B}_{dr,x_{S^o/S},K}$$

with

$$m:=(m,0,m):\mathrm{Cone}(j_*O_{S^o}\to\psi_D(O_{S^o}))\otimes_{O_S}\mathrm{Cone}(j_*O_{S^o}\to\psi_D(O_{S^o}))\xrightarrow{\sim}\mathrm{Cone}(j_*O_{S^o}\to\psi_D(O_{S^o})).$$

Definition 82. *(i) Let $S\in\mathrm{SmVar}(k)$. For $(K_1,W),(K_2,W)\in P_{\mathbb{Z}pfil,k}(S^{et})$ filtered perverse sheaves, we have the isomorphism in $D_{\mathbb{B}_{dr}fil}(S_K^{an,pet})$*

$$\begin{aligned}
&T(\otimes,\mathbb{B}_{dr})((K_1,W),(K_2,W)):\mathbb{B}_{dr,S}(K_1,W)\otimes_{B_{dr,S}}\mathbb{B}_{dr,S}(K_2,W)\\
&\xrightarrow{:=}(\cdots\to\bigoplus_{1\le i_1<\cdots<i_d\le d}x_{S\backslash D_{i_1}/S}\cdots x_{S\backslash D_{i_r}/S}\phi_{D_{i_{r+1}}}\cdots\phi_{D_{i_s}}\psi_{D_{i_{s+1}}}\cdots\psi_{D_{i_d}}(K_1,W)\otimes_{\mathbb{Q}_p}\\
&\mathbb{B}_{dr,x_{S\backslash D_{i_1}/S}}\otimes_{\mathbb{B}_{dr,S}}\cdots\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,x_{S\backslash D_{i_r}/S}}\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,\phi_{D_{i_{r+1}}}}\otimes_{\mathbb{B}_{dr,S}}\cdots\\
&\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\phi_{D_{i_s}}}\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\psi_{D_{i_{s+1}}}}\otimes_{\mathbb{B}_{dr,S}}\cdots\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\psi_{D_{i_d}}}\to\cdots)\otimes_{B_{dr,S}}\\
&(\cdots\to\bigoplus_{1\le i_1<\cdots<i_d\le d}x_{S\backslash D_{i_1}/S}\cdots x_{S\backslash D_{i_r}/S}\phi_{D_{i_{r+1}}}\cdots\phi_{D_{i_s}}\psi_{D_{i_{s+1}}}\cdots\psi_{D_{i_d}}(K_2,W)\otimes_{\mathbb{Q}_p}\\
&\mathbb{B}_{dr,x_{S\backslash D_{i_1}/S}}\otimes_{\mathbb{B}_{dr,S}}\cdots\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,x_{S\backslash D_{i_r}/S}}\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\phi_{D_{i_{r+1}}}}\otimes_{\mathbb{B}_{dr,S}}\cdots\\
&\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\phi_{D_{i_s}}}\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\psi_{D_{i_{s+1}}}}\otimes_{\mathbb{B}_{dr,S}}\cdots\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\psi_{D_{i_d}}}\to\cdots)\\
&\xrightarrow{=}(\cdots\to\bigoplus_{1\le i_1<\cdots<i_d\le d}x_{S\backslash D_{i_1}/S}\cdots x_{S\backslash D_{i_r}/S}\phi_{D_{i_{r+1}}}\cdots\phi_{D_{i_s}}\psi_{D_{i_{s+1}}}\cdots\psi_{D_{i_d}}((K_1,W)\otimes(K_2,W))\\
&\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,x_{S\backslash D_{i_1}/S}}\otimes_{\mathbb{B}_{dr,S}}\cdots\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,x_{S\backslash D_{i_r}/S}}\otimes_{\mathbb{B}_{dr,S_K}}\\
&\mathbb{B}_{dr,\phi_{D_{i_{r+1}}}}\otimes_{\mathbb{B}_{dr,S_K}}\cdots\otimes_{\mathbb{B}_{dr,S_K}}\mathbb{B}_{dr,\phi_{D_{i_s}},K}\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\psi_{D_{i_{s+1}}}}\otimes_{\mathbb{B}_{dr,S_K}}\cdots\\
&\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\psi_{D_{i_d}},K}\otimes_{B_{dr,S}}\mathbb{B}_{dr,x_{S\backslash D_{i_1}/S}}\otimes_{\mathbb{B}_{dr,S}}\cdots\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,x_{S\backslash D_{i_r}/S}}\otimes_{\mathbb{B}_{dr,S}}\\
&\mathbb{B}_{dr,\phi_{D_{i_{r+1}}}}\otimes_{\mathbb{B}_{dr,S}}\cdots\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\phi_{D_{i_s}}}\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\psi_{D_{i_{s+1}}}}\otimes_{\mathbb{B}_{dr,S}}\cdots\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\psi_{D_{i_d}}}\to\cdots)\\
&\xrightarrow{(m(\mathbb{B}_{dr,x_{S\backslash D_{i_1}/S}}),\cdots,m(\mathbb{B}_{dr,\phi_{D_{i_{r+1}}}}),\cdots,m(\mathbb{B}_{dr,\psi_{D_{i_d}},K}))}\\
&(\cdots\to\bigoplus_{1\le i_1<\cdots<i_d\le d}x_{S\backslash D_{i_1}/S}\cdots x_{S\backslash D_{i_r}/S}\phi_{D_{i_{r+1}}}\cdots\phi_{D_{i_s}}\psi_{D_{i_{s+1}}}\cdots\psi_{D_{i_d}}((K_1,W)\otimes(K_2,W))\otimes_{\mathbb{Q}_p}\\
&\mathbb{B}_{dr,x_{S\backslash D_{i_1}/S}}\otimes_{\mathbb{B}_{dr,S}}\cdots\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,x_{S\backslash D_{i_r}/S}}\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\phi_{D_{i_{r+1}}},K}\otimes_{\mathbb{B}_{dr,S}}\cdots\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\phi_{D_{i_s}}})\\
&\xrightarrow{=:}\mathbb{B}_{dr,S}((K_1,W)\otimes^{L,w}(K_2,W))
\end{aligned}$$

with $(D_1,\ldots,D_d)\in\mathcal{S}(K)$ a stratification by (Cartier) divisor $D_i\subset S$, $1\le i\le d$ such that

$$K_{1|D(r)\backslash D(r+1)},K_{2|D(r)\backslash D(r+1)}\in D_{\mathbb{Z}_p,c}(D(r)\backslash D(r+1)^{et})$$

are local systems for all $1\le r\le d$.

(ii) Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i\in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I \in \mathrm{SmVar}(k)$. For $(K_1, W), (K_2, W) \in P_{\mathbb{Z}_p fil,k}(S^{et})$ filtered perverse sheaves, we have the isomorphism in $D_{\mathbb{B}_{dr} fil}(S_K^{an,pet}/(\tilde{S}_{I,K}^{an,pet}))$ given as in (i)

$$\begin{aligned}
T(\otimes, \mathbb{B}_{dr})((K_1, W), (K_2, W)) : \mathbb{B}_{dr,(\tilde{S}_I)}(K_1, W) \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,(\tilde{S}_I)}(K_2, W) \xrightarrow{:=} \\
((\bigoplus_{1\leq i_1<\cdots<i_d\leq d} x_{\tilde{S}_I\backslash\tilde{D}_{i_1,I}/\tilde{S}_I}\cdots x_{\tilde{S}_I\backslash\tilde{D}_{i_r,I}/\tilde{S}_I}\phi_{\tilde{D}_{i_{r+1},I}}\cdots\phi_{\tilde{D}_{i_s,I}}\psi_{\tilde{D}_{i_{s+1},I}}\cdots\psi_{\tilde{D}_{i_d,I}}(i_{I*}j_I^*(K_1, W)) \\
\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{D}_{i_1,I}/S}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,x_{\tilde{S}_I\backslash D_{i_r,I}/\tilde{S}_I}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\phi_{\tilde{D}_{i_{r+1},I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}} \\
\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\phi_{\tilde{D}_{i_s,I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{D}_{i_{s+1},I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{D}_{i_d,I}}}\to\cdots),\mathbb{B}_{dr}(t_{IJ}))\otimes_{\mathbb{B}_{dr,S}} \\
((\bigoplus_{1\leq i_1<\cdots<i_d\leq d} x_{\tilde{S}_I\backslash\tilde{D}_{i_1,I}/\tilde{S}_I}\cdots x_{\tilde{S}_I\backslash\tilde{D}_{i_r,I}/\tilde{S}_I}\phi_{\tilde{D}_{i_{r+1},I}}\cdots\phi_{\tilde{D}_{i_s,I}}\psi_{\tilde{D}_{i_{s+1},I}}\cdots\psi_{\tilde{D}_{i_d,I}}(i_{I*}j_I^*(K_2, W)) \\
\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{D}_{i_1,I}/S}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,x_{\tilde{S}_I\backslash D_{i_r,I}/\tilde{S}_I}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\phi_{\tilde{D}_{i_{r+1},I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}} \\
\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\phi_{\tilde{D}_{i_s,I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{D}_{i_{s+1},I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{D}_{i_d,I}}}\to\cdots),\mathbb{B}_{dr}(t_{IJ}))\xrightarrow{=} \\
((\bigoplus_{1\leq i_1<\cdots<i_d\leq d} x_{\tilde{S}_I\backslash\tilde{D}_{i_1,I}/\tilde{S}_I}\cdots x_{\tilde{S}_I\backslash\tilde{D}_{i_r,I}/\tilde{S}_I} \\
\phi_{\tilde{D}_{i_{r+1},I}}\cdots\phi_{\tilde{D}_{i_s,I}}\psi_{\tilde{D}_{i_{s+1},I}}\cdots\psi_{\tilde{D}_{i_d,I}}(i_{I*}j_I^*((K_1, W)\otimes(K_2, W)))\otimes_{\mathbb{Q}_p} \\
\mathbb{B}^2_{dr,x_{\tilde{S}_I\backslash\tilde{D}_{i_1,I}/S}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}^2_{dr,x_{\tilde{S}_I\backslash D_{i_r,I}/\tilde{S}_I}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}^2_{dr,\phi_{\tilde{D}_{i_{r+1},I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}} \\
\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}^2_{dr,\phi_{D_{i_s,I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}^2_{dr,\psi_{\tilde{D}_{i_{s+1},I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}^2_{dr,\psi_{\tilde{D}_{i_d,I}}}\to\cdots),\mathbb{B}_{dr}(t_{IJ})) \\
\xrightarrow{(m(\mathbb{B}_{dr,x_{\tilde{S}_I\backslash D_{i_1,I}/S},K}),\cdots,m(\mathbb{B}_{dr,\phi_{\tilde{D}_{i_{r+1},I}}}),\cdots,m(\mathbb{B}_{dr,\psi_{\tilde{D}_{i_d,I}}}))} \\
((\bigoplus_{1\leq i_1<\cdots<i_d\leq d} x_{\tilde{S}_I\backslash\tilde{D}_{i_1,I}/\tilde{S}_I}\cdots x_{\tilde{S}_I\backslash\tilde{D}_{i_r,I}/\tilde{S}_I} \\
\phi_{\tilde{D}_{i_{r+1},I}}\cdots\phi_{\tilde{D}_{i_s,I}}\psi_{\tilde{D}_{i_{s+1},I}}\cdots\psi_{\tilde{D}_{i_d,I}}(i_{I*}j_I^*((K_1, W)\otimes(K_2, W))) \\
\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{D}_{i_1,I}/S}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,x_{\tilde{S}_I\backslash D_{i_r,I}/\tilde{S}_I}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\phi_{\tilde{D}_{i_{r+1},I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_I}} \\
\mathbb{B}_{dr,\phi_{\tilde{D}_{i_s,I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{D}_{i_{s+1},I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{D}_{i_d,I}}}\to\cdots),\mathbb{B}_{dr}(t_{IJ})) \\
\xrightarrow{=:}\mathbb{B}_{dr,(\tilde{S}_I)}((K_1, W)\otimes^{L,w}(K_2, W))
\end{aligned}$$

with $(D_1, \ldots, D_d) \in \mathcal{S}(K)$ stratifications by Cartier divisor $D_i \subset S$, $1 \leq i \leq d$ such that

$$K_{1|D(r)\backslash D(r+1)}, K_{2|D(r)\backslash D(r+1)} \in D_{\mathbb{Z}_p,c}(D(r)\backslash D(r+1)^{et})$$

are local systems for all $1 \leq r \leq d$, and $\tilde{D}_{s,I} \subset \tilde{S}_I$ (Cartier) divisor such that $D_s \cap S_I \subset \tilde{D}_{s,I} \cap S$. This gives, for $(K_1, W), (K_2, W) \in D_{\mathbb{Z}_p fil,c,k}(S^{et})$, isomorphism in $D_{\mathbb{B}_{dr} fil}(S_K^{an,pet}/(\tilde{S}_{I,K}^{an,pet})$,

$$\begin{aligned}
T(\otimes, \mathbb{B}_{dr})((K_1, W), (K_2, W)) : \mathbb{B}_{dr,(\tilde{S}_I)}(K_1, W) \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,(\tilde{S}_I)}(K_2, W) \\
\xrightarrow{\sim} \mathbb{B}_{dr,(\tilde{S}_I)}((K_1, W) \otimes^{L,w} (K_2, W)).
\end{aligned}$$

Definition 83. *(i) Let $S \in \mathrm{SmVar}(k)$. For $(K, W) \in D_{\mathbb{Z}_p fil,c,k}(S^{et})$, we have the isomorphism*

$$T(D, \mathbb{B}_{dr})(K, W) : \mathbb{B}_{dr,S}(\mathbb{D}_S^v(K, W)) \xrightarrow{B((K,W),\mathbb{D}_S^v(K,W))} \mathbb{D}_S\mathbb{B}_{dr,S}(K, W)$$

given by the pairing

$$\begin{aligned}
B((K, W), \mathbb{D}_S^v(K, W)) : \mathbb{B}_{dr,S}(\mathbb{D}_S^v K) \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,S}(K, W) \xrightarrow{T(\otimes,\mathbb{B}_{dr})((K,W),\mathbb{D}_S^v(K,W))} \\
\mathbb{B}_{dr,S}(\mathbb{D}_S^v(K, W) \otimes (K, W)) \xrightarrow{\mathbb{B}_{dr,S}(ev_K)} \mathbb{B}_{dr,S}(\mathbb{Z}_{p,S^{et}}) \xrightarrow{\alpha(S_K)^{-1}} \mathbb{B}_{dr,S_K}
\end{aligned}$$

using definition 82.

(ii) Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i\in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I \in \mathrm{SmVar}(k)$. For $(K,W) \in D_{\mathbb{Z}_p fil,c,k}(S^{et})$, we have the isomorphism in $D_{\mathbb{B}_{dr} fil}(S_K^{an,pet}/(\tilde{S}_{I,K}^{an,pet})$

$$T(D,\mathbb{B}_{dr})(K,W) : \mathbb{B}_{dr,(\tilde{S}_I)}(\mathbb{D}_S^v(K,W)) \xrightarrow{B((K,W),\mathbb{D}_S^v(K,W))} \mathbb{D}_S\mathbb{B}_{dr,(\tilde{S}_I)}(K,W)$$

given by the pairing

$$\begin{aligned} B((K,W),\mathbb{D}_S^v(K,W)) : \mathbb{B}_{dr,(\tilde{S}_I)}(\mathbb{D}_S^v(K,W)) \otimes_{\mathbb{B}_{dr,S_K}} \mathbb{B}_{dr,(\tilde{S}_I)}(K,W) \\ \xrightarrow{T(\otimes,\mathbb{B}_{dr})((K,W),\mathbb{D}_S^v(K,W))} \mathbb{B}_{dr,(\tilde{S}_I)}(\mathbb{D}_S^v(K,W) \otimes (K,W)) \\ \xrightarrow{\mathbb{B}_{dr,(\tilde{S}_I)}(ev_K)} \mathbb{B}_{dr,(\tilde{S}_I)}(\mathbb{Z}_{p,S^{et}}) \xrightarrow{(\alpha(\tilde{S}_{I,K}))^{-1}} (\mathbb{B}_{dr,\tilde{S}_{I,K}}, t_{IJ}) \end{aligned}$$

using definition 82.

Definition 84. *(i) Let $j : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{SmVar}(k)$ and $D := S\backslash S^o$ a (Cartier) divisor. We will consider, for $(K,W) \in D_{\mathbb{Z}_p fil,c,k}(S^{o,et})$, the canonical isomorphism in $D_{\mathbb{B}_{dr} fil}(S_K^{an,pet})$*

$$\begin{aligned} T_!(j,\mathbb{B}_{dr})(K,W) : \mathbb{B}_{dr,S}(j_{!w}(K,W)) := \mathbb{B}_{dr,S}(\mathbb{D}_S^v j_{*w}\mathbb{D}_S^v(K,W)) \\ \xrightarrow{T(D,\mathbb{B}_{dr})(j_{*w}\mathbb{D}_S^v(K,W))} \mathbb{D}_S\mathbb{B}_{dr,S}(j_{*w}\mathbb{D}_{S^o}^v(K,W)) \\ \xrightarrow{\mathbb{D}_S T(j,\mathbb{B}_{dr})(\mathbb{D}_{S^o}^v(K,W))} \mathbb{D}_S(V_{D0}j_{*w}\mathbb{B}_{dr,S^o}(\mathbb{D}_{S^o}^v(K,W)) \otimes \mathbb{B}_{dr,S^o/S}) \\ \xrightarrow{T(D,\mathbb{B}_{dr})(K,W)} \mathbb{D}_S(V_{D0}j_{*w}\mathbb{D}_{S^o}\mathbb{B}_{dr,S^o}(K,W) \otimes \mathbb{B}_{dr,S^o/S}) \\ \xrightarrow{=} V_{D0}j_{!w}\mathbb{B}_{dr,S^o}(K,W) \otimes_{\mathbb{B}_{dr,S}} \mathbb{D}_S\mathbb{B}_{dr,S^o/S}, \end{aligned}$$

using definition 81 and definition 83.

(ii) Let $l : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{Var}(k)$ such that $D = S\backslash S^o$ is a Cartier divisor. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Let $l_I : \tilde{S}_I^o \hookrightarrow \tilde{S}_I$ open embeddings such that $\tilde{S}_I^o \cap S = S^o \cap S_I$ and $\tilde{D}_I \subset \tilde{S}_I$ a Cartier divisor such that $D \cap S_I \subset \tilde{D}_I \cap S$. We will consider, for $(K,W) \in D_{\mathbb{Z}_p fil,c,k}(S^{o,et})$, the canonical isomorphism in $D_{\mathbb{B}_{dr} fil}(S_K^{an,pet}/(\tilde{S}_{I,K})^{an,pet})$

$$\begin{aligned} T_!(l,\mathbb{B}_{dr})(K,W) : \mathbb{B}_{dr,(\tilde{S}_I)}(l_{!w}(K,W)) := \mathbb{B}_{dr,(\tilde{S}_I)}(\mathbb{D}_S^v l_{*w}\mathbb{D}_S^v(K,W)) \\ \xrightarrow{T(D,\mathbb{B}_{dr})(l_{*w}\mathbb{D}_S^v(K,W))} \mathbb{D}_S\mathbb{B}_{dr,(\tilde{S}_I)}(l_{*w}\mathbb{D}_{S^o}^v(K,W)) \\ \xrightarrow{\mathbb{D}_S T(l,\mathbb{B}_{dr})(\mathbb{D}_{S^o}^v(K,W))} \mathbb{D}_S(V_{D0}l_{*w}\mathbb{B}_{dr,(\tilde{S}_I^o)}(\mathbb{D}_{S^o}^v(K,W)) \otimes_{\mathbb{B}_{dr,S}} (\mathbb{B}_{dr,\tilde{S}_I^o/\tilde{S}_I}, t_{IJ})) \\ \xrightarrow{T(D,\mathbb{B}_{dr})(K,W)} \mathbb{D}_S(V_{D0}l_{*w}\mathbb{D}_{S^o}\mathbb{B}_{dr,(\tilde{S}_I^o)}(K,W) \otimes_{\mathbb{B}_{dr,S}} (\mathbb{B}_{dr,\tilde{S}_I^o/\tilde{S}_I}, t_{IJ})) \\ \xrightarrow{=} V_{D0}l_{!w}\mathbb{B}_{dr,(\tilde{S}_I^o}(K,W) \otimes_{\mathbb{B}_{dr,S}} \mathbb{D}_S(\mathbb{B}_{dr,\tilde{S}_I^o/\tilde{S}_I}) \end{aligned}$$

using definition 81 and definition 83.

As a consequence of this formalism we have :

Theorem 48. *(i) Let $S \in \mathrm{SmVar}(k)$ irreducible. Let $\bar{S} \in \mathrm{PSmVar}(k)$ a compactification of S, with $D := \bar{S}\backslash S \subset S$ a (Cartier) divisor. Denote by $j : S^o \hookrightarrow S$ the open embedding. We have the* *canonical isomorphisms, given by, using definition 80, definition 81, definition 82 and definition 83, for* $K, K' \in D_{\mathbb{Z}_p,c,k}(S^{et}_{\bar{k}})$,

$$\begin{aligned}\mathbb{B}_{dr,S}(K_1,K_2) : R\operatorname{Hom}(K_1,K_2)\otimes\mathbb{B}_{dr,\bar{k}} &\xrightarrow{=} \mathbb{B}_{dr,\bar{k}}(Ra_{\bar{S}*}Rj_*\mathcal{H}om(K_1,K_2)) \\ &\xrightarrow{T(a_{\bar{S}},\mathbb{B}_{dr})(-)} Ra_{\bar{S}*}\mathbb{B}_{dr,\bar{S}}(Rj_*\mathcal{H}om(K_1,K_2)) \\ &\xrightarrow{\mathbb{B}_{dr,\bar{S}}(T(j,\mathbb{B}_{dr})(-))} Ra_{\bar{S}*}(V_{D,0}Rj_*\mathbb{B}_{dr,S}(\mathcal{H}om(K_1,K_2))\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,S/\bar{S}}) \\ &\xrightarrow{\mathbb{B}_{dr,S}(m(K_1,K_2)^{-1})} Ra_{\bar{S}*}(V_{D,0}Rj_*\mathbb{B}_{dr,S}(\mathbb{D}^v_S K_1\otimes K_2)\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,S/\bar{S}}) \\ &\xrightarrow{(T(D,\mathbb{B}_{dr})(K_1)\otimes I)\circ T(\otimes,\mathbb{B}_{dr})(\mathbb{D}_S K_1,K_2)} Ra_{\bar{S}*}(V_{D,0}Rj_*(\mathbb{D}_S\mathbb{B}_{dr,S}(K_1)\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,S}(K_2))\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,S/\bar{S}}) \\ &\xrightarrow{m(\mathbb{B}_{dr,S}(K_1),\mathbb{B}_{dr,S}(K_2))} Ra_{\bar{S}*}(V_{D,0}Rj_*\mathcal{H}om(\mathbb{B}_{dr,S}(K_1),\mathbb{B}_{dr,S}(K_2))\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,S/\bar{S}}) \\ &\xrightarrow{=} R\operatorname{Hom}(\mathbb{B}_{dr,S}(K_1),\mathbb{B}_{dr,S}(K_2)).\end{aligned}$$

(ii) *Let* $S \in \mathrm{Var}(k)$. *Let* $\bar{S} \in \mathrm{PVar}(k)$ *a compactification of* S, *with* $D := \bar{S}\backslash S \subset S$ *a Cartier divisor. Denote by* $j : S^o \hookrightarrow S$ *the open embedding. Let* $S = \cup_{i\in I} S_i$ *an open cover such that there exists closed embeddings* $i_i : S_i \hookrightarrow \tilde{S}_i$ *with* $\tilde{S}_I \in \mathrm{SmVar}(k)$. *We have the canonical isomorphisms, given by, using definition 80, definition 81, definition 82 and definition 83, for* $K, K' \in D_{\mathbb{Z}_p,c,k}(S^{et}_{\bar{k}})$,

$$\begin{aligned}\mathbb{B}_{dr,(\tilde{S}_I)}(K_1,K_2) : R\operatorname{Hom}(K_1,K_2)\otimes\mathbb{B}_{dr,\bar{k}} &\xrightarrow{=} \mathbb{B}_{dr,\bar{k}}(Ra_{\bar{S}*}Rj_*\mathcal{H}om(K_1,K_2)) \\ &\xrightarrow{T(a_{\bar{S}},\mathbb{B}_{dr})(-)} Ra_{\bar{S}*}\mathbb{B}_{dr,(\bar{S}_I)}(Rj_*\mathcal{H}om(K_1,K_2)) \\ &\xrightarrow{\mathbb{B}_{dr,(\bar{S}_I)}(T(j,\mathbb{B}_{dr})(-))} Ra_{\bar{S}*}(V_{D,0}j_*\mathbb{B}_{dr,(\tilde{S}_I)}(\mathcal{H}om(K_1,K_2))\otimes_{\mathbb{B}_{dr,S}}(\mathbb{B}_{dr,\tilde{S}_I/\bar{S}_I},t_{IJ})) \\ &\xrightarrow{\mathbb{B}_{dr,S}(m(K_1,K_2)^{-1})} Ra_{\bar{S}*}(V_{D,0}j_*\mathbb{B}_{dr,(\tilde{S}_I)}(\mathbb{D}^v_S K_1\otimes K_2)\otimes_{\mathbb{B}_{dr,S}}(\mathbb{B}_{dr,\tilde{S}_I/\bar{S}_I},t_{IJ})) \\ &\xrightarrow{(T(D,\mathbb{B}_{dr})(K_1)\otimes I)\circ T(\otimes,\mathbb{B}_{dr})(\mathbb{D}_S K_1,K_2)} \\ Ra_{\bar{S}*}(V_{D,0}j_*(\mathbb{D}_S\mathbb{B}_{dr,(\tilde{S}_I)}(K_1)&\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,(\tilde{S}_I)}(K_2))\otimes_{\mathbb{B}_{dr,S}}(\mathbb{B}_{dr,\tilde{S}_I/\bar{S}_I},t_{IJ})) \\ &\xrightarrow{m(\mathbb{B}_{dr,(\tilde{S}_I)}(K_1),\mathbb{B}_{dr,(\tilde{S}_I)}(K_2))} Ra_{\bar{S}*}(V_{D,0}j_*\mathcal{H}om(\mathbb{B}_{dr,(\tilde{S}_I)}(K_1),\mathbb{B}_{dr,(\tilde{S}_I)}(K_2))\otimes_{\mathbb{B}_{dr,S}}(\mathbb{B}_{dr,\tilde{S}_I/\bar{S}_I},t_{IJ})) \\ &\xrightarrow{=} R\operatorname{Hom}(\mathbb{B}_{dr,S}(K_1),\mathbb{B}_{dr,S}(K_2)).\end{aligned}$$

Proof. Follows from theorem 47. □

Let $S \in \mathrm{SmVar}(k)$ and $D \subset S$ a (Cartier) divisor. We have by theorem 43 the following isomorphisms

$$\begin{aligned}F^0T^{B_{dr}}(O_S,F_b) : \mathbb{B}_{dr,\psi_D} &:= F^0DR(S)(\psi_D(O_S,F_b)^{an}\otimes_{O_S}(O\mathbb{B}_{dr,S},F)) \\ &\xrightarrow{\sim} F^0\psi_D DR(S)(O\mathbb{B}_{dr,S},F) \xrightarrow{=} \psi_D F^0DR(S)(O\mathbb{B}_{dr,S},F) =: \psi_D\mathbb{B}_{dr,S}\end{aligned}$$

and

$$\begin{aligned}F^0T'^{B_{dr}}(O_S,F_b) : \mathbb{B}_{dr,\phi_D} &:= F^0DR(S)(\phi_D(O_S,F_b)^{an}\otimes_{O_S}(O\mathbb{B}_{dr,S},F)) \\ &\xrightarrow{\sim} F^0\phi_D DR(S)(O\mathbb{B}_{dr,S},F) \xrightarrow{=} \phi_D F^0DR(S)(O\mathbb{B}_{dr,S},F) =: \phi_D\mathbb{B}_{dr,S}.\end{aligned}$$

Definition 85. (i) *Let* $j : S^o \hookrightarrow S$ *an open embedding with* $S \in \mathrm{SmVar}(k)$ *and* $D := S\backslash S^o$ *a (Cartier) divisor. We will consider, for* $(K,W) \in P_{\mathbb{Z}_p fil,k}(S^{et})$, *the canonical isomorphism in*

$D_{\mathbb{B}_{dr,S}fil}(S_K^{an,pet})$

$$T(\psi_D,\mathbb{B}_{dr})(K,W):\mathbb{B}_{dr,S}(\psi_D(K,W))\xrightarrow{:=}$$
$$(\cdots\to\bigoplus_{1\leq i_1\cdots<i_d\leq d}x_{S\backslash\bar{E}_{i_1}/S}\cdots x_{S\backslash\bar{E}_{i_r}/S}\phi_{\bar{E}_{i_{r+1}}}\cdots\phi_{\bar{E}_{i_s}}\psi_{\bar{E}_{i_{s+1}}}\cdots\psi_{\bar{E}_{i_d}}\psi_D(K,W)$$
$$\otimes\mathbb{B}_{dr,\psi_D}\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,x_{S\backslash\bar{E}_{i_1}/S}}\otimes_{\mathbb{B}_{dr,S}}\cdots\otimes\mathbb{B}_{dr,x_{S\backslash\bar{E}_{i_r}/S}}$$
$$\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\phi_{\bar{E}_{i_{r+1}}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{\bar{E}_{i_s}}}\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\psi_{\bar{E}_{i_{s+1}}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{\bar{E}_{i_d}}}\to\cdots)$$
$$\xrightarrow{m(-)\otimes I}(\cdots\to\bigoplus_{1\leq i_1\cdots<i_d\leq d}\psi_D(x_{S\backslash\bar{E}_{i_1}/S}\cdots x_{S\backslash\bar{E}_{i_r}/S}\phi_{\bar{E}_{i_{r+1}}}\cdots\phi_{\bar{E}_{i_s}}\psi_{\bar{E}_{i_{s+1}}}\cdots\psi_{\bar{E}_{i_d}}(K,W)$$
$$\otimes\mathbb{B}_{dr,x_{S\backslash\bar{E}_{i_1}/S}}\otimes_{\mathbb{B}_{dr,S}}\cdots\otimes\mathbb{B}_{dr,x_{S\backslash\bar{E}_{i_r}/S}}$$
$$\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\phi_{\bar{E}_{i_{r+1}}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{\bar{E}_{i_s}}}\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,\psi_{\bar{E}_{i_{s+1}}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{\bar{E}_{i_d}}}))\to\cdots)\xrightarrow{=:}\psi_D\mathbb{B}_{dr,S}((K,W))$$

using definition 78(vi). We will also consider, for $(K,W)\in D_{\mathbb{Z}_pfil,c,k}(S^{o,et})$*, the canonical isomorphism in* $D_{\mathbb{B}_{dr}fil}(S_K^{an,pet})$

$$T(\phi_D,\mathbb{B}_{dr})(K,W):\mathbb{B}_{dr,S}(\phi_D(K,W))=\mathbb{B}_{dr,S}(\mathbb{D}_S^v\psi_D\mathbb{D}_S^v(K,W))$$
$$\xrightarrow{T(D,\mathbb{B}_{dr})(\psi_D\mathbb{D}_S^v(K,W))}\mathbb{D}_S\mathbb{B}_{dr,S}(\psi_D\mathbb{D}_{S^o}^v(K,W))\xrightarrow{\mathbb{D}_ST(\psi_D,\mathbb{B}_{dr})(\mathbb{D}_S^v(K,W))}$$
$$\mathbb{D}_S(\psi_D\mathbb{B}_{dr,S}(\mathbb{D}_S^v(K,W)))\xrightarrow{T(D,\mathbb{B}_{dr})(K,W)}\mathbb{D}_S\psi_D\mathbb{D}_S\mathbb{B}_{dr,S}(K,W)\xrightarrow{=}\phi_D\mathbb{B}_{dr,S}(K,W),$$

using definition 83.

(ii) *Let* $l:S^o\hookrightarrow S$ *an open embedding with* $S\in\mathrm{Var}(k)$ *such that* $D=S\backslash S^o\subset S$ *is a Cartier divisor. Let* $S=\cup_iS_i$ *an open affine cover and* $i_i:S_i\hookrightarrow\tilde{S}_i$ *closed embedding with* $\tilde{S}_i\in\mathrm{SmVar}(k)$*. Let* $l_I:\tilde{S}_I^o\hookrightarrow\tilde{S}_I$ *open embeddings such that* $\tilde{S}_I^o\cap S=S^o\cap S_I$ *and* $\tilde{D}_I\subset\tilde{S}_I$ *a Cartier divisor such that* $D\cap S_I\subset\tilde{D}_I\cap S$*. We will consider, for* $(K,W)\in P_{\mathbb{Z}_pfil,k}(S^{o,et})$*, the canonical isomorphism in* $D_{\mathbb{B}_{dr}fil}(S_K^{an,pet}/(\tilde{S}_{I,K})^{an,pet})$

$$T(\psi_D,\mathbb{B}_{dr})(K,W):\mathbb{B}_{dr,(\tilde{S}_I)}(\psi_D(K,W))\xrightarrow{:=}$$
$$((\cdots\to\bigoplus_{1\leq i_1\cdots<i_d\leq d}x_{\tilde{S}_I\backslash\tilde{D}_{i_1,I}/\tilde{S}_I}\cdots x_{\tilde{S}_I\backslash\tilde{D}_{I,i_r}/S}\phi_{\tilde{E}_{i_{r+1},I}}\cdots\phi_{\tilde{E}_{i_s,I}}\psi_{\tilde{E}_{i_{s+1},I}}\cdots\psi_{\tilde{E}_{i_d,I}}i_{I*}j_I^*\psi_D(K,W)$$
$$\otimes\mathbb{B}_{dr,\psi_{\tilde{D}_I}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{E}_{i_1,I}/S}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{E}_{i_r,I}/S}}$$
$$\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\phi_{\tilde{E}_{i_{r+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{\tilde{E}_{i_s,I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{E}_{i_{s+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{\tilde{E}_{i_d,I}}}\to\cdots),\mathbb{B}_{dr}(t_{IJ}))$$
$$\xrightarrow{m(-)\otimes I}$$
$$((\cdots\to\bigoplus_{1\leq i_1\cdots<i_d\leq d}\psi_{\tilde{D}_I}(x_{\tilde{S}_I\backslash\tilde{D}_{i_1,I}/\tilde{S}_I}\cdots x_{\tilde{S}_I\backslash\tilde{D}_{I,i_r}/S}\phi_{\tilde{E}_{i_{r+1},I}}\cdots\phi_{\tilde{E}_{i_s,I}}\psi_{\tilde{E}_{i_{s+1},I}}\cdots\psi_{\tilde{E}_{i_d,I}}i_{I*}j_I^*(K,W)$$
$$\otimes\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{E}_{i_1,I}/S}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\cdots\otimes\mathbb{B}_{dr,x_{\tilde{S}_I\backslash\tilde{E}_{i_r,I}/S}}$$
$$\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\phi_{\tilde{E}_{i_{r+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\phi_{\tilde{E}_{i_s,I}}}\otimes_{\mathbb{B}_{dr,\tilde{S}_I}}\mathbb{B}_{dr,\psi_{\tilde{E}_{i_{s+1},I}}}\otimes\cdots\otimes\mathbb{B}_{dr,\psi_{\tilde{E}_{i_d,I}}}))\to\cdots),\mathbb{B}_{dr}(t_{IJ}))$$
$$\xrightarrow{=:}\psi_D\mathbb{B}_{dr,(\tilde{S}_I)}(K,W)$$

using definition 78(vi), where $(E_1,\ldots,E_d)\in\mathcal{S}(K)$ *is a stratification by Cartier divisor* $E_i\subset S^o$*,* $1\leq i\leq d$*, such that*

$$K_{|E(r)\backslash E(r+1)}:=l_r^*K\in D_{\mathbb{Z}_p,c}((E(r)\backslash E(r+1))^{et})$$

are local systems for all $1\leq r\leq d$*,* $l_r:E(r)\hookrightarrow S^o$ *being the locally closed embeddings, and* $\tilde{E}_{s,I}\subset\tilde{S}_I$*,* $\tilde{D}_I\subset\tilde{S}_I$ *are (Cartier) divisor such that* $\bar{E}_s\cap S_I\subset\tilde{E}_{s,I}\cap S_I$ *and* $D\cap S_I\subset\tilde{D}_I\cap S_I$*. We will also* *consider, for* $(K,W)\in D_{\mathbb{Z}_pfil,c,k}(S^{et})$*, the canonical isomorphism in* $D_{\mathbb{B}_{dr}fil}(S_K^{an,pet}/(\tilde{S}_{I,K})^{an,pet})$

$$T(\phi_D,\mathbb{B}_{dr})(K,W):\mathbb{B}_{dr,(\tilde{S}_I)}(\phi_D(K,W))=\mathbb{B}_{dr,(\tilde{S}_I)}(\mathbb{D}_S^v\psi_D\mathbb{D}_S^v(K,W))$$
$$\xrightarrow{T(D,\mathbb{B}_{dr})(\psi_D\mathbb{D}_S^v(K,W))}\mathbb{D}_S\mathbb{B}_{dr,(\tilde{S}_I)}(\psi_D\mathbb{D}_S^v(K,W))\xrightarrow{\mathbb{D}_ST(\psi_D,\mathbb{B}_{dr})(\mathbb{D}_S^v(K,W))}$$
$$\mathbb{D}_S\psi_D\mathbb{B}_{dr,(\tilde{S}_I)}(\mathbb{D}_S^v(K,W))\xrightarrow{T(D,\mathbb{B}_{dr})(K,W)}\mathbb{D}_S\psi_D\mathbb{D}_S\mathbb{B}_{dr,(\tilde{S}_I)}(K,W)\xrightarrow{=}\phi_D\mathbb{B}_{dr,(\tilde{S}_I)}(K,W),$$

using definition 83. It gives for $(K,W)\in D_{\mathbb{Z}_pfil,c,k}(S^{o,et})$*, the canonical isomorphisms*

$$T(\psi_D,\mathbb{B}_{dr})(K,W):\mathbb{B}_{dr,(\tilde{S}_I)}(\psi_D(K,W))\xrightarrow{\sim}\psi_D\mathbb{B}_{dr,(\tilde{S}_I)}(K,W).$$

and

$$T(\phi_D,\mathbb{B}_{dr})(K,W):\mathbb{B}_{dr,(\tilde{S}_I)}(\phi_D(K,W))\xrightarrow{\sim}\phi_D\mathbb{B}_{dr,(\tilde{S}_I)}(K,W).$$

in $D_{\mathbb{B}_{dr}fil}(S_K^{an,pet}/(\tilde{S}_{I,K})^{an,pet})$*.*

6.2.2 The geometric p-adic Mixed Hodge Modules

Let p a prime integer. Let $k\subset K\subset\mathbb{C}_p$ a subfield of a p-adic field. Denote by $\bar{k}\subset\mathbb{C}_p$ its algebraic closure. Recall $G=\mathrm{Gal}(\bar{K},K)\subset\mathrm{Gal}(\bar{k},k)$ denotes the Galois group of K.

For $S\in\mathrm{Var}(k)$, we denote for short $O_S:=O_{S^{an}_{\mathbb{C}_p}}$, $\mathbb{B}_{dr,S}:=\mathbb{B}_{dr,S_{\mathbb{C}_p}}:=\mathbb{B}_{dr,R_{\mathbb{C}_p}(S^{an}_{\mathbb{C}_p})}$ and $O\mathbb{B}_{dr,S}:=O\mathbb{B}_{dr,S_{\mathbb{C}_p}}:=O\mathbb{B}_{dr,R_{\mathbb{C}_p}(S^{an}_{\mathbb{C}_p})}$. where $R_{\mathbb{C}_p}:\mathrm{AnSp}(\mathbb{C}_p)\to\mathrm{AdSp}/(\mathbb{C}_p,O_{\mathbb{C}_p})$ is the canonical functor (see section 2).

Let $S\in\mathrm{Var}(k)$. Recall that $S^{et}\subset\mathrm{Var}(k)^{sm}/S$ denote the small etale site. We then have the morphism of site $\mathrm{an}_S:S^{an,pet}:=S^{an,pet}_{\mathbb{C}_p}\to S^{et}$ given by the analytical functor where $S^{an,pet}_{\mathbb{C}_p}\subset(\mathrm{AnSp}(\mathbb{C}_p)^{sm}/S)^{pro}$ is the small pro-etale site. Then, $\mathrm{PSh}_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p})$ is the category whose objects are $\pi^{*mod}_{K/\mathbb{C}_p}(N,F)$ with $N\in\mathrm{PSh}_{\mathbb{B}_{dr}fil}(S^{an,pet})$ together with a continuous action of G compatible with the $\mathbb{B}_{dr,S}$ module structure.

- Let $S\in\mathrm{SmVar}(k)$. The category $C_{\mathcal{D}(1,0)fil,rh}(S)\times_I D_{\mathbb{Z}_pfil,c,k}(S^{et})$ is the category
  - whose set of objects is the set of triples $\{((M,F,W),(K,W),\alpha)\}$ with

    $$(M,F,W)\in C_{\mathcal{D}(1,0)fil,rh}(S),\ (K,W)\in D_{\mathbb{Z}_pfil,c,k}(S^{et}),$$
    $$\alpha:\mathbb{B}_{dr,S}(K,W)\to F^0DR(S)^{[-]}((M,F,W)^{an}\otimes_{O_S}(O\mathbb{B}_{dr,S},F))$$

    where
    - we recall that

      $$DR(S)^{[-]}=DR(S^{an}_{\mathbb{C}_p})^{[-]}:C_{\mathcal{D}(1,0)fil,rh}(S^{an}_{\mathbb{C}_p})\to C_{\mathbb{B}_{dr}2fil}(S^{an,pet}_{\mathbb{C}_p})$$

      is the De Rahm functor (for $S'\subset S$ a connected component of dimension d, $DR(S)^{[-]}_{|S'}=DR(S)_{|S'}[d]$),
    - the functor

      $$\mathbb{B}_{dr,S}:D_{\mathbb{Z}_pfil,c,k}(S^{et})\to D_{\mathbb{B}_{dr}fil}(S^{an,pet}_{\mathbb{C}_p})$$

      is the functor from complexes of presheaves with constructible etale cohomology to complexes of $\mathbb{B}_{dr,S}$ modules given in definition 79 (recall that for L a local system, it is given by $\mathbb{B}_{dr,S}(L):=\mathrm{an}_S^*L\otimes_{\mathbb{Q}_p}\mathbb{B}_{dr,S_{\mathbb{C}_p}}$),
    - α is a morphism in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p})$, that is a morphism in $D_{\mathbb{B}_{dr},fil}(S^{an,pet}_{\mathbb{C}_p})$ compatible with the action of the galois group $G=\mathrm{Gal}(\bar{K},K)\subset\mathrm{Gal}(\bar{k},k)$,

– and whose set of morphisms are

$$\phi = (\phi_D, \phi_C, [\theta]) : ((M_1, F, W), (K_1, W), \alpha_1) \to ((M_2, F, W), (K_2, W), \alpha_2)$$

where $\phi_D : (M_1, F, W) \to (M_2, F, W)$ and $\phi_C : (K_1, W) \to (K_2, W)$ are morphisms and

$$\theta = (\theta^\bullet, I(F^0 DR(S)(\phi_D^{an} \otimes I)) \circ I(\alpha_1), I(\alpha_2) \circ I(\mathbb{B}_{dr,S}(\phi_C))) : \\ I(\mathbb{B}_{dr,S}(K_1, W))[1] \to I(F^0 DR(S)((M, F, W)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S}, F))$$

is an homotopy, $I : C_{\mathbb{B}_{dr,S},G,fil}(S^{an,pet}_{\mathbb{C}_p}) \to K_{\mathbb{B}_{dr,S},G,fil}(S^{an,pet}_{\mathbb{C}_p})$ being the injective resolution functor : for $(N, W) \in C_{\mathbb{B}_{dr,S},G,fil}(S^{an,pet}_{\mathbb{C}_p})$, $k : (N, W) \to I(N, W)$ with $I(N, W) \in C_{\mathbb{B}_{dr,S},G,fil}(S^{an,pet}_{\mathbb{C}_p})$ is an injective resolution, and the class $[\theta]$ of θ does NOT depend of the injective resolution ; in particular

$$F^0 DR(S)^{[-]}(\phi_D^{an} \otimes I) \circ \alpha_1 = \alpha_2 \circ \mathbb{B}_{dr,S}(\phi_C)$$

in $D_{\mathbb{B}_{dr,S},G,fil}(S^{an,pet}_{\mathbb{C}_p})$, and for

- $\phi = (\phi_D, \phi_C, [\theta]) : ((M_1, F, W), (K_1, W), \alpha_1) \to ((M_2, F, W), (K_2, W), \alpha_2)$
- $\phi' = (\phi'_D, \phi'_C, [\theta']) : ((M_2, F, W), (K_2, W), \alpha_2) \to ((M_2, F, W), (K_3, W), \alpha_3)$

the composition law is given by

$$\phi' \circ \phi := (\phi'_D \circ \phi_D, \phi'_C \circ \phi_C, I(F^0 DR(S)(\phi_C^{'an} \otimes I)) \circ [\theta] + [\theta'] \circ I(\mathbb{B}_{dr,S}(\phi_C))[1]) : \\ ((M_1, F, W), (K_1, W), \alpha_1) \to ((M_3, F, W), (K_3, W), \alpha_3),$$

in particular for $((M, F, W), (K, W), \alpha) \in C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$,

$$I_{((M,F,W),(K,W),\alpha)} = (I_M, I_K, 0).$$

We have then the full embedding

$$\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{\mathbb{Z}_p fil,k}(S^{et}) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$$

where the category $\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S) \times_I P_{\mathbb{Z}_p,fil}(S^{et})$ is the category

– whose set of objects is the set of triples $\{((M, F, W), (K, W), \alpha)\}$ with

$$(M, F, W) \in \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S),\ (K, W) \in P_{\mathbb{Z}_p fil,k}(S^{et}), \\ \alpha : \mathbb{B}_{dr,S}(K, W) \to F^0 DR(S)^{[-]}((M, F, W)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S}, F))$$

where α is an isomorphism,

– and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) = (\phi_D, \phi_C, 0) : ((M_1, F, W), (K_1, W), \alpha_1) \to ((M_2, F, W), (K_2, W), \alpha_2)$$

where $\phi_D : (M_1, F, W) \to (M_2, F, W)$ and $\phi_C : (K_1, W) \to (K_2, W)$ are morphisms such that

$$F^0 DR(S)^{[-]}(\phi_D^{an} \otimes I) \circ \alpha_1 = \alpha_2 \circ \mathbb{B}_{dr,S}(\phi_C)$$

in $P_{\mathbb{B}_{dr,S},G,fil}(S^{an,pet}_{\mathbb{C}_p})$.

- Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I \in \mathrm{SmVar}(k)$. The category $C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$ is the category

– whose set of objects is the set of triples $\{(((M_I,F,W),u_{IJ}),(K,W),\alpha)\}$ with

$$((M_I,F,W),u_{IJ})\in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)),\ (K,W)\in D_{\mathbb{Z}_pfil,c,k}(S^{et}),$$
$$\alpha:\mathbb{B}_{dr,(\tilde{S}_I)}(K,W)\to F^0DR(S)^{[-]}(((M_I,F,W),u_{IJ})^{an}\otimes_{O_S}((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$

where

* the functor

$$DR(S)^{[-]}=DR(S^{an}_{\mathbb{C}_p})^{[-]}:C_{\mathcal{D}(1,0)fil,rh}(S^{an}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))\to C_{\mathbb{B}_{dr}2fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$$

  is the De Rahm functor,

* the functor
$$\mathbb{B}_{dr,(\tilde{S}_I)}:D_{\mathbb{Z}_pfil,c,k}(S^{et})\to D_{\mathbb{B}_{dr}fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$$
  is the functor from complexes of presheaves with constructible etale cohomology to complexes of $\mathbb{B}_{dr}$ modules given in definition 79,

* α is a morphism in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$, that is a morphism in $D_{\mathbb{B}_{dr},fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$ compatible with the action of the galois group $G=\mathrm{Gal}(\bar{\hat{k}},\hat{k})\subset\mathrm{Gal}(\bar{k},k)$

– and whose set of morphisms are

$$\phi=(\phi_D,\phi_C,[\theta]):(((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1)\to(((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2)$$

where $\phi_D:((M_1,F,W),u_{IJ})\to((M_2,F,W),u_{IJ})$ and $\phi_C:(K_1,W)\to(K_2,W)$ are morphisms (of filtered complexes) and

$$\theta=(\theta^\bullet,I(F^0DR(S)(\phi_D\otimes I))\circ I(\alpha_1),I(\alpha_2)\circ I(\mathbb{B}_{dr,S}(\phi_C))):$$
$$I(\mathbb{B}_{dr,(\tilde{S}_I)}(K_1,W))[1]\to I(DR(S)(((M_{2I},F,W),u_{IJ})^{an}\otimes_{O_S}((O\mathbb{B}_{dr,(\tilde{S}_I)},F),t_{IJ})))$$

is an homotopy, $I:C_{\mathbb{B}_{dr},G,fil}(S^{an}_{\mathbb{C}_p}/(\tilde{S}^{an}_{I,\mathbb{C}_p}))\to K_{\mathbb{B}_{dr},G,fil}(S^{an}_{\mathbb{C}_p}/(\tilde{S}^{an}_{I,\mathbb{C}_p}))$ being the injective resolution functor : for $((N_I,W),t_{IJ})\in C_{\mathbb{B}_{dr},G,fil}(S^{an}_{\mathbb{C}_p}/(\tilde{S}^{an}_{I,\mathbb{C}_p}))$,

$$k:((N_I,W),t_{IJ})\to I((N_I,W),t_{IJ})$$

with $I((N_I,W),t_{IJ})\in C_{\mathbb{B}_{dr},G,fil}(S^{an}_{\mathbb{C}_p}/(\tilde{S}^{an}_{I,\mathbb{C}_p}))$ is an injective resolution, and the class $[\theta]$ of θ does NOT depend of the injective resolution ; in particular we have

$$F^0DR(S)^{[-]}(\phi^{an}_D\otimes I)\circ\alpha_1=\alpha_2\circ\mathbb{B}_{dr,(\tilde{S}_I)}(\phi_C)$$

in $D_{\mathbb{B}_{dr},G,fil}(S_{\mathbb{C}_p}/\tilde{S}^{an,pet}_{I,\mathbb{C}_p})$, and for

* $\phi=(\phi_D,\phi_C,[\theta]):(((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1)\to(((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2)$
* $\phi'=(\phi'_D,\phi'_C,[\theta']):(((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2)\to(((M_{3I},F,W),u_{IJ}),(K_3,W),\alpha_3)$

the composition law is given by

$$\phi'\circ\phi:=(\phi'_D\circ\phi_D,\phi'_C\circ\phi_C,I(F^0DR(S)(\phi'^{an}_D\otimes I))\circ[\theta]+[\theta']\circ I(\mathbb{B}_{dr,(\tilde{S}_I)}(\phi_C))[1]):$$
$$(((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1)\to(((M_{3I},F,W),u_{IJ}),(K_3,W),\alpha_3).$$

in particular for $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{\mathbb{Z}_pfil,c,k}(S^{et})$,

$$I_{(((M_I,F,W),u_{IJ}),(K,W),\alpha)}=((I_{M_I}),I_K,0).$$

We have then full embeddings

$$\mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I P_{\mathbb{Z}_p fil,k}(S^{et}) \hookrightarrow C^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$$

$$\xrightarrow{\iota^0_{S/\tilde{S}_I}} C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))^0 \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et}) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$$

where the category $\mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I P_{\mathbb{Z}_p fil,k}(S^{et})$ is the category

- whose set of objects is the set of triples $\{(((M_I, F, W), u_{IJ}), (K, W), \alpha)\}$ with

$$((M_I, F, W), u_{IJ}) \in \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)),\ (K, W) \in P_{\mathbb{Z}_p fil,k}(S^{et}),$$
$$\alpha : \mathbb{B}_{dr,(\tilde{S}_I)}(K, W) \to F^0 DR(S)^{[-]}(((M_I, F, W), u_{IJ})^{an} \otimes_{O_S} (O\mathbb{B}_{dr,(\tilde{S}_I)}, F))$$

  where α is an isomorphism,
- and whose set of morphisms are

$$\phi = (\phi_D, \phi_C) = (\phi_D, \phi_C, 0) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

  where $\phi_D : ((M_{1I}, F, W), u_{IJ}) \to ((M_{2I}, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \to (K_2, W)$ are morphisms such that

$$F^0 DR(S)^{[-]}(\phi_D^{an} \otimes I) \circ \alpha_1 = \alpha_2 \circ \mathbb{B}_{dr,(\tilde{S}_I)}(\phi_C)$$

  in $P_{\mathbb{B}_{dr,S},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$.

Moreover,

- For $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$, we set

$$(((M_I, F, W), u_{IJ}), (K, W), \alpha)[1] := (((M_I, F, W), u_{IJ})[1], (K, W)[1], \alpha[1]).$$

- For

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

  a morphism in $C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$, we set (see [11] definition 3.12)

$$\mathrm{Cone}(\phi) := (\mathrm{Cone}(\phi_D), \mathrm{Cone}(\phi_C), ((\alpha_1, \theta), (\alpha_2, 0))) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et}),$$

  $((\alpha_1, \theta), (\alpha_2, 0))$ being the matrix given by the composition law, together with the canonical maps
  - $c_1(-) = (c_1(\phi_D), c_1(\phi_C), 0) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \to \mathrm{Cone}(\phi)$
  - $c_2(-) = (c_2(\phi_D), c_2(\phi_C), 0) : \mathrm{Cone}(\phi) \to (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1)[1]$.

Remark 8. *By [11] theorem 3.25, if*

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_1, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_2, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

is a morphism in $C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$ such that ϕ_D is a Zariski local equivalence and ϕ_C is an isomorphism then ϕ is an isomorphism.

Definition 86. *Let $k \subset \mathbb{C}_p$ a subfield.*

(i1) *Let* $f : X \to S$ *a proper morphism with* $S, X \in \mathrm{SmVar}(k)$. *Let*

$$\alpha : \mathbb{B}_{dr,X}(K,W) \to F^0DR(X)((M,F,W)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X},F))$$

a morphism in $D_{\mathbb{B}_{dr},G,fil}(X^{an,pet}_{\mathbb{C}_p})$, *with*

$$(M,F,W) \in C(DRM(X)),\ (K,W) \in D_{\mathbb{Z}_pfil,c,k,gm}(X^{et}).$$

We then consider, using definition 80 and definition 52, the map in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p})$

$$\begin{aligned} f_*\alpha = f_*(\alpha) : \mathbb{B}_{dr,S}(Rf_*(K,W)) &\xrightarrow{T(f,\mathbb{B}_{dr})(K,W)} Rf_*\mathbb{B}_{dr,X}(K,W) \\ &\xrightarrow{Rf_*\alpha} Rf_*F^0DR(X)((M,F,W)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X},F)) \\ &\xrightarrow{Rf_*\iota_{F^0}(-)} F^0Rf_*DR(X)((M,F,W)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X},F)) \\ &\xrightarrow{F^0T^{B_{dr}}(f,DR)(M,F,W)^{-1}} F^0DR(S)(\int_f((M,F,W)^{an}) \otimes_{O_S} (O\mathbb{B}_{dr,S},F)) \\ &\xrightarrow{F^0DR(S)(T(an,\int_f)(M,F,W)^{-1})} F^0DR(S)((\int_f(M,F,W))^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F)) \\ &\xrightarrow{=} F^0DR(S)((Rf_{*Hdg}(M,F,W))^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F)) \end{aligned}$$

where $\iota_{F^0}(A) = D_{fil}(\iota_{F^0}(A))$ *is the image of the embedding* $\iota_{F^0}(A) : F^0A \hookrightarrow A$ *by the localization functor.*

(i2) *Let* $j : S^o \hookrightarrow S$ *an open embedding with* $S \in \mathrm{SmVar}(k)$ *and* $D = S\backslash S^o$ *a (Cartier) divisor. Let*

$$\alpha : \mathbb{B}_{dr,S^o}(K,W) \to F^0DR(S^o)((M,F,W)^{an} \otimes_{O_{S^o}} (O\mathbb{B}_{dr,S^o},F))$$

a morphism in $D_{\mathbb{B}_{dr},G,fil}(S^{o,an,pet}_{\mathbb{C}_p})$, *with*

$$(M,F,W) \in C(DRM(S^o)),\ (K,W) \in D_{\mathbb{Z}_pfil,c,k}(S^{o,et})^{ad,D}.$$

We then consider, using definition 81 and the strictness of the V*-filtration, the maps in* $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p})$

$$\begin{aligned} j_*\alpha = j_*(\alpha) : \mathbb{B}_{dr,S}(j_{*w}(K,W)) &\xrightarrow{T(j,\mathbb{B}_{dr})(K,W)} V_{D0}j_{*w}\mathbb{B}_{dr,S^o}(K,W) \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,S^o/S} \\ &\xrightarrow{V_{D0}j_*\alpha\otimes I} V_{D0}j_{*w}(F^0DR(S^o)((M,F,W)^{an} \otimes_{O_{S^o}} (O\mathbb{B}_{dr,S^o},F))) \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,S^o/S} \\ &\xrightarrow{:=} (V_{D0}j_{*w}F^0DR(S^o)((M,F,W)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F))) \otimes_{\mathbb{B}_{dr,S}} \\ &\qquad (F^0DR(S)(j_{*Hdg}(O_{S^o},F_b)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F))) \\ &\xrightarrow{w_S\otimes m(M)^{an}} F^0DR(S)(j_{*Hdg}(M,F,W)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F)) \end{aligned}$$

where $m(M) : O_{S^o} \otimes_{O_{S^o}} M \xrightarrow{\sim} M$, $m \otimes f \mapsto fm$ *is the multiplication map structure of the module* M *and* w_S *is the wedge product, and*

$$\begin{aligned} j_!\alpha = j_!(\alpha) : \mathbb{B}_{dr,S}(j_{!w}(K,W)) &\xrightarrow{T_!(j,\mathbb{B}_{dr})(K,W)} V_{D0}\mathbb{D}j_*\mathbb{D}\mathbb{B}_{dr,S^o}(K,W) \otimes_{\mathbb{B}_{dr,S}} \mathbb{D}_S\mathbb{B}_{dr,S^o/S}) \\ &\xrightarrow{(V_{D0}\mathbb{D}j_*\mathbb{D}\alpha)\otimes I} V_{D0}\mathbb{D}j_{*w}\mathbb{D}(F^0DR(S^o)(\mathbb{D}(M,F,W)^{an} \otimes_{O_{S^o}} (O\mathbb{B}_{dr,S^o},F))) \otimes_{\mathbb{B}_{dr,S}} \mathbb{D}_S\mathbb{B}_{dr,S^o/S} \\ &\xrightarrow{:=} V_{D0}\mathbb{D}j_{*w}\mathbb{D}(F^0DR(S^o)(\mathbb{D}(M,F,W)^{an} \otimes_{O_{S^o}} (O\mathbb{B}_{dr,S^o},F))) \\ &\qquad \otimes_{\mathbb{B}_{dr,S}}(F^0DR(S)(j_{!Hdg}(O_{S^o},F_b)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F))) \\ &\xrightarrow{w_S\otimes m(M)^{an}} F^0DR(S)(j_{!Hdg}(M,F,W)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F)) \end{aligned}$$

where $m(M) : O_{S^o} \otimes_{O_{S^o}} M \xrightarrow{\sim} M$, $h \otimes m \mapsto hm$ *is the multiplication map and* w_S *is the wedge product.*

(i2)' Let $l : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{Var}(k)$ and $D = S\backslash S^o$ a Cartier divisor. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Let $l_I : \tilde{S}^o_I \hookrightarrow \tilde{S}_I$ closed embeddings such that $\tilde{S}^o_I \cap S = S^o \cap S_I$. Let

$$\alpha : \mathbb{B}_{dr,(\tilde{S}^o_I)}(K,W) \to F^0DR(S^o)(((M_I,F,W),u_{IJ})^{an} \otimes_{O_{S^o}} ((O\mathbb{B}_{dr,(\tilde{S}^o_I)},F),t_{IJ}))$$

a morphism in $D_{\mathbb{B}_{dr},G,fil}(S^{o,an,pet}_{\mathbb{C}_p}/(\tilde{S}^{o,an,pet}_{I,\mathbb{C}_p}))$, with

$$((M_I,F,W),u_{IJ}) \in C(DRM(S^o)) \subset C_{\mathcal{D}(1,0)fil,rh}(S^o/(\tilde{S}^o_I)),\ (K,W) \in D_{\mathbb{Z}_pfil,c,k}(S^{o,et})^{ad,D}.$$

We then consider as in (i2) the maps in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I\mathbb{C}_p}))$

$$\begin{aligned}
l_*\alpha = l_*(\alpha) : \mathbb{B}_{dr,(\tilde{S}_I)}(l_{*w}(K,W)) \\
\xrightarrow{T(l,\mathbb{B}_{dr})(K,W)} V_{D0}l_{*w}\mathbb{B}_{dr,(\tilde{S}^o_I)}(K,W) \otimes_{\mathbb{B}_{dr,(\tilde{S}_I)}} (\mathbb{B}_{dr,\tilde{S}^o_I/\tilde{S}_I},t_{IJ}) \\
\xrightarrow{V_{D0}l_*\alpha\otimes I} V_{D0}l_{*w}F^0DR(S^o)(((M_I,F,W),u_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}^o_I},F),t_{IJ})) \otimes_{\mathbb{B}_{dr,(\tilde{S}_I)}} (\mathbb{B}_{dr,\tilde{S}^o_I/\tilde{S}_I},t_{IJ}) \\
\xrightarrow{:=} \\
V_{D0}l_{*w}F^0DR(S^o)(((M_I,F,W),u_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}^o_I},F),t_{IJ})) \otimes_{\mathbb{B}_{dr,(\tilde{S}_I)}} \\
(F^0DR((\tilde{S}_I))((l_{I*Hdg}(O_{\tilde{S}^o_I},F_b),x_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))) \\
\xrightarrow{w_{\tilde{S}_I}\otimes m(M_I)^{an}} F^0DR(S)((l_{*Hdg}((M_I,F,W),u_{IJ})^{an}) \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))
\end{aligned}$$

and

$$\begin{aligned}
l_!\alpha = l_!(\alpha) : \mathbb{B}_{dr,(\tilde{S}_I)}(l_{!w}(K,W)) \\
\xrightarrow{T_!(l,\mathbb{B}_{dr})(K,W)} V_{D0}\mathbb{D}l_*\mathbb{D}(\mathbb{B}_{dr,(\tilde{S}^o_I)}(K,W)) \otimes_{\mathbb{B}_{dr,(\tilde{S}_I)}} \mathbb{D}(\mathbb{B}_{dr,\tilde{S}^o_I/\tilde{S}_I},t_{IJ}) \xrightarrow{V_{D0}\mathbb{D}l_*\mathbb{D}(\alpha\otimes I)} \\
V_{D0}\mathbb{D}l_{*w}\mathbb{D}(F^0DR(S^o)(((M_I,F,W),u_{IJ})^{an} \otimes_{O_{S^o}} ((O\mathbb{B}_{dr,\tilde{S}^o_I},F),t_{IJ}))) \otimes_{\mathbb{B}_{dr,(\tilde{S}_I)}} \mathbb{D}(\mathbb{B}_{dr,\tilde{S}^o_I/\tilde{S}_I},t_{IJ}) \\
\xrightarrow{:=} V_{D0}\mathbb{D}l_{*w}\mathbb{D}(F^0DR(S^o)(((M_I,F,W),u_{IJ})^{an} \otimes_{O_{S^o}} ((O\mathbb{B}_{dr,\tilde{S}^o_I},F),t_{IJ}))) \otimes_{\mathbb{B}_{dr,(\tilde{S}_I)}} \\
(F^0DR((\tilde{S}_I))((l_{I!Hdg}(O_{\tilde{S}^o_I},F_b),x_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))) \\
\xrightarrow{w_{\tilde{S}_I}\otimes m(M_I)^{an}} F^0DR(S)(l_{!Hdg}((M_I,F,W),u_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))
\end{aligned}$$

(ii0) Let $f : X \to S$ a morphism with $S, X \in \mathrm{SmVar}(k)$. Take a compactification $f : X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} S$ of f with $\bar{X} \in \mathrm{SmVar}(k)$, j an open embedding and $D = \bar{X}\backslash X$ a divisor (see section 2, we can take D a normal crossing divisor but it is unnecessary). Let

$$\alpha : \mathbb{B}_{dr,X}(K,W) \to F^0DR(X)((M,F,W)^{an} \otimes_{O_X} O\mathbb{B}_{dr,X})$$

a morphism in $D_{\mathbb{B}_{dr},G,fil}(X^{an,pet}_{\mathbb{C}_p})$, with

$$(M,F,W) \in C(DRM(X)),\ (K,W) \in D_{\mathbb{Z}_pfil,c,k,gm}(X^{et})^{ad,D}.$$

We then consider, using (i1) and (i2) the maps in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p})$

$$\begin{aligned}
f_*\alpha = f_*(\alpha) : \mathbb{B}_{dr,S}(Rf_{*w}(K,W)) = \mathbb{B}_{dr,S}(R\bar{f}_*j_{*w}(K,W)) \\
\xrightarrow{T(\bar{f},B_{dr})(-)} R\bar{f}_*\mathbb{B}_{dr,\bar{X}}(j_{*w}(K,W)) \xrightarrow{j_*(\alpha)} R\bar{f}_*F^0DR(X)(j_{*Hdg}(M,F,W)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X},F)) \\
\xrightarrow{F^0DR(S)(T(an,\int_f)(-)^{-1})\circ F^0T^{B_{dr}}(\bar{f},DR)(-)^{-1}\circ R\bar{f}_*\iota_{F^0}} \\
F^0DR(S)(R\bar{f}_{*Hdg}j_{*Hdg}(M,F,W)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F)) \\
\xrightarrow{=:} F^0DR(S)(Rf_{*Hdg}(M,F,W)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F))
\end{aligned}$$

and

$$f_!\alpha = f_!(\alpha) : \mathbb{B}_{dr,S}(Rf_{!w}(K,W)) = \mathbb{B}_{dr,S}(R\bar{f}_*j_{!w}(K,W))$$
$$\xrightarrow{T(\bar{f},B_{dr})(-)} R\bar{f}_*\mathbb{B}_{dr,\bar{X}}(j_{!w}(K,W)) \xrightarrow{j_!(\alpha)} R\bar{f}_*F^0DR(X)(j_{!Hdg}(M,F,W)^{an} \otimes_{O_X} (O\mathbb{B}_{dr,X},F))$$
$$\xrightarrow{F^0DR(S)(T(an,\int_f)(-)^{-1})\circ F^0T^{B_{dr}}(\bar{f},DR)(-)^{-1}\circ R\bar{f}_*\iota_{F^0}}$$
$$F^0DR(S)(R\bar{f}_{*Hdg}j_{!Hdg}(M,F,W)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F))$$
$$\xrightarrow{=:} F^0DR(S)(Rf_{!Hdg}(M,F,W)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F)).$$

(ii) *Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(k)$. Consider a factorization $f : X \hookrightarrow Y \times S \xrightarrow{p} S$ with $Y \in \mathrm{SmVar}(k)$, and let $f : X \xrightarrow{j} \bar{X} \hookrightarrow \bar{Y} \times S \xrightarrow{\bar{p}} S$ be a compactification of f, with $\bar{Y} \in \mathrm{PSmVar}(k)$ and $D = \bar{Y}\backslash Y$ a (Cartier) divisor (e.g. a normal crossing divisor). Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Let*

$$\alpha : \mathbb{B}_{dr,(Y\times\tilde{S}_I)}(K,W) \to F^0DR(X)(((M_I,F,W),u_{IJ})^{an} \otimes_{O_{Y\times\tilde{S}_I}} (O\mathbb{B}_{dr,Y\times\tilde{S}_I},F))$$

a morphism in $D_{\mathbb{B}_{dr},G,fil}(X^{an,pet}_{\mathbb{C}_p}/(Y\times \tilde{S}_{I\mathbb{C}_p}^{an,pet}))$, with

$$((M_I,F,W),u_{IJ}) \in C(DRM(X)) \subset C_{\mathcal{D}(1,0)fil,rh}(X/(Y\times\tilde{S}_I)),\ (K,W) \in D_{\mathbb{Z}_pfil,c,k,gm}(X^{et})^{ad,D}.$$

We then consider, using definitions 80 and (i2)', the maps in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$

$$f_*\alpha = f_*(\alpha) : \mathbb{B}_{dr,(\tilde{S}_I)}(Rf_{*w}(K,W)) = \mathbb{B}_{dr,(\tilde{S}_I)}((R\bar{f}_*j_{*w}(K,W)))$$
$$\xrightarrow{T(\bar{f},\mathbb{B}_{dr})(-)} R\bar{p}_*\mathbb{B}_{dr,(\bar{Y}\times\tilde{S}_I)}(j_{*w}(K,W))$$
$$\xrightarrow{j_*\alpha} R\bar{p}_*F^0DR(X)((j_{*Hdg}((M_I,F,W),u_{IJ})^{an}) \otimes_{O_X} ((O\mathbb{B}_{dr,\bar{Y}\times\tilde{S}_I},F),t_{IJ}))$$
$$\xrightarrow{Rp_*\iota_{F^0}} F^0R\bar{p}_*DR(X)((j_{*Hdg}((M_I,F,W),u_{IJ})^{an}) \otimes_{O_X} ((O\mathbb{B}_{dr,\bar{Y}\times\tilde{S}_I},F),t_{IJ}))$$
$$\xrightarrow{F^0T^{B_{dr}}(f,DR)(j_{*Hdg}((M_I,F,W),u_{IJ}))} F^0DR(S)(\int_{\bar{f}}(j_{*Hdg}((M_I,F,W),u_{IJ}))^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$
$$\xrightarrow{F^0DR(S)(T(an,\int_f)(-)\otimes I)} F^0DR(S)((\int_{\bar{f}} j_{*Hdg}((M_I,F,W),u_{IJ}))^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$
$$\xrightarrow{=} F^0DR(S)((Rf_{*Hdg}((M_I,F,W),u_{IJ}))^{an} \otimes_{O_S} (O\mathbb{B}_{dr,\tilde{S}_I},F)),$$

and

$$f_!\alpha = f_!(\alpha) : \mathbb{B}_{dr,(\tilde{S}_I)}(R\bar{f}_*j_{!w}(K,W))) \xrightarrow{=} \mathbb{B}_{dr,(\tilde{S}_I)}(R\bar{f}_*j_{!w}(K,W))$$
$$\xrightarrow{\mathbb{D}(T(\bar{f},\mathbb{B}_{dr})(-))^{-1}} R\bar{p}_*\mathbb{B}_{dr,(\bar{Y}\times\tilde{S}_I)}(j_{!w}(K,W))$$
$$\xrightarrow{j_!\alpha} R\bar{p}_*F^0DR(X)((j_{!Hdg}((M_I,F,W),u_{IJ})^{an}) \otimes_{O_X} ((O\mathbb{B}_{dr,\bar{Y}\times\tilde{S}_I},F),t_{IJ}))$$
$$\xrightarrow{Rp_*\iota_{F^0}} F^0R\bar{p}_*DR(X)((j_{!Hdg}((M_I,F,W),u_{IJ})^{an}) \otimes_{O_X} ((O\mathbb{B}_{dr,\bar{Y}\times\tilde{S}_I},F),t_{IJ}))$$
$$\xrightarrow{F^0T^{B_{dr}}(f,DR)(j_{!Hdg}((M_I,F,W),u_{IJ}))} F^0DR(S)(\int_{\bar{f}}(j_{!Hdg}((M_I,F,W),u_{IJ}))^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$
$$\xrightarrow{F^0DR(S)(T(an,\int_f)(-)\otimes I)} F^0DR(S)((\int_{\bar{f}} j_{!Hdg}((M_I,F,W),u_{IJ}))^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$
$$\xrightarrow{=} F^0DR(S)((Rf_{!Hdg}((M_I,F,W),u_{IJ}))^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,(\tilde{S}_I)},F),t_{IJ}).$$

(iii) *Let $l : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{Var}(k)$ and denote $Z = S \backslash S^o$. Let $D_1, \cdots, D_d \subset S$ Cartier divisor such that $Z = \cap_{s=1}^d D_s$. Denote $l_s : D_s \hookrightarrow S$ the closed embeddings. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Let $l_{I,s} : \tilde{D}_{I,s} \hookrightarrow \tilde{S}_I$ closed embeddings such that $\tilde{D}_{I,s} \cap S = S \cap D_{I,s}$. Let*

$$\alpha : \mathbb{B}_{dr,(\tilde{S}_I)}(K,W) \to F^0DR(S)(((M_I,F,W),u_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$

a morphism in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$, with

$$((M_I,F,W),u_{IJ}) \in C(DRM(S)) \subset C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)),\ (K,W) \in D_{\mathbb{Z}_pfil,c,k}(S^{et})^{ad,(D_i)}.$$

We then have by (i2), the maps in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$

$$\Gamma_Z(\alpha) : \mathbb{B}_{dr,(\tilde{S}_I)}(\Gamma^w_Z(K,W)) \xrightarrow{:=} \mathbb{B}_{dr,(\tilde{S}_I)}(\Gamma^w_{D_1}\cdots\Gamma^w_{D_s}(K,W))$$
$$\xrightarrow{(I,(l_{1*}\cdots l_{s*}(\alpha)))} F^0DR(S)((\Gamma^{Hdg}_Z((M_I,F,W),u_{IJ}))^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$

and

$$\Gamma^\vee_Z(\alpha) : \mathbb{B}_{dr,(\tilde{S}_I)}(\Gamma^{\vee,w}_Z(K,W)) \xrightarrow{:=} \mathbb{B}_{dr,(\tilde{S}_I)}(\Gamma^{\vee,w}_{D_1}\cdots\Gamma^{\vee,w}_{D_s}(K,W))$$
$$\xrightarrow{(I,(l_{1!}\cdots l_{s!}(\alpha)))} F^0DR(S)((\Gamma^{\vee,Hdg}_Z((M_I,F,W),u_{IJ}))^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$

(iv) *Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(k)$. Consider a factorization $f : X \hookrightarrow Y \times S \xrightarrow{p} S$ with $Y \in \mathrm{SmVar}(k)$. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Let*

$$\alpha : \mathbb{B}_{dr,(\tilde{S}_I)}(K,W) \to F^0DR(S)(((M_I,F,W),u_{IJ})^{an} \otimes_{O_{\tilde{S}_I}} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$

a morphism in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I\mathbb{C}_p}))$, with

$$((M_I,F,W),u_{IJ}) \in C(DRM(S)) \subset C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)),\ (K,W) \in D_{\mathbb{Z}_pfil,c,k}(S^{et})^{ad,(\Gamma_{f,i})}.$$

We then have by (iii), the maps in $D_{\mathbb{B}_{dr},G,fil}(X^{an,pet}_{\mathbb{C}_p}/(Y \times \tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$

$$f^!\alpha = f^!(\alpha) : \mathbb{B}_{dr,(Y\times\tilde{S}_I)}(f^{!w}(K,W)) \xrightarrow{=} \mathbb{B}_{dr,(Y\times\tilde{S}_I)}(\Gamma^w_X p^*(K,W))$$
$$\xrightarrow{\Gamma_X(p^*\alpha)} F^0DR(X)((\Gamma^{Hdg}_X p^{*mod}((M_I,F,W),u_{IJ}))^{an} \otimes_{O_X} ((O\mathbb{B}_{dr,Y\times\tilde{S}_I},F),t_{IJ}))$$
$$\xrightarrow{=} DR(X)(f^{*mod}_{Hdg}((M_I,F,W),u_{IJ})^{an} \otimes_{O_X} ((O\mathbb{B}_{dr,Y\times\tilde{S}_I},F),t_{IJ}))$$

and

$$f^*\alpha = f^*(\alpha) : \mathbb{B}_{dr,(Y\times\tilde{S}_I)}(f^{*w}(K,W)) \xrightarrow{=} \mathbb{B}_{dr,(Y\times\tilde{S}_I)}(\Gamma^{\vee,w}_X p^*(K,W))$$
$$\xrightarrow{\Gamma^\vee_X(p^*\alpha)} F^0DR(Y\times S)((\Gamma^{\vee,Hdg}_X p^{\hat{*}mod}((M_I,F,W),u_{IJ}))^{an} \otimes_{O_X} ((O\mathbb{B}_{dr,Y\times\tilde{S}_I},F),t_{IJ}))$$
$$\xrightarrow{=} F^0DR(X)(f^{\hat{*}mod}_{Hdg}((M_I,F,W),u_{IJ})^{an} \otimes_{O_X} ((O\mathbb{B}_{dr,Y\times\tilde{S}_I},F),t_{IJ}))$$

with

$$p^*\alpha : \mathbb{B}_{dr,(Y\times\tilde{S}_I)}(p^*(K,W)) \xrightarrow{=} p^{*mod}\mathbb{B}_{dr,(\tilde{S}_I)}(K,W)$$
$$\xrightarrow{p^{*mod}\alpha} p^{*mod}F^0DR(S)(((M_I,F,W),u_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$
$$\xrightarrow{=} F^0DR(Y\times S)(p^{*mod}((M_I,F,W),u_{IJ})^{an} \otimes_{O_{Y\times S}} ((O\mathbb{B}_{dr,Y\times\tilde{S}_I},F),t_{IJ})).$$

(v) *Let $S\in\mathrm{Var}(k)$. Denote by $\Delta_S:S\hookrightarrow S\times S$ the diagonal closed embedding and $p_1:S\times S\to S$, $p_2:S\times S\to S$ the projections. Let $S=\cup_i S_i$ an open affine cover and $i_i:S_i\hookrightarrow\tilde{S}_i$ closed embedding with $\tilde{S}_i\in\mathrm{SmVar}(k)$. Let*

$$\alpha:\mathbb{B}_{dr,(\tilde{S}_I)}(K,W)\to F^0DR(S)(((M_I,F,W),u_{IJ})^{an}\otimes_{O_S}((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ})),$$
$$\alpha':\mathbb{B}_{dr,(\tilde{S}_I)}(K',W)\to F^0DR(S)(((M'_I,F,W),v_{IJ})^{an}\otimes_{O_S}((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$

two morphisms in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I\mathbb{C}_p}))$, with

$$((M_I,F,W),u_{IJ}),((M'_I,F,W),v_{IJ})\in C(DRM(S))\subset C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)),$$
$$(K,W),(K',W)\in D_{\mathbb{Z}_pfil,c,k}(S^{et}).$$

We have then, as in (iv), the following map in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I\mathbb{C}_p}))$

$$\alpha\otimes\alpha':\mathbb{B}_{dr,(\tilde{S}_I)}((K,W)\otimes^{L,w}(K',W))$$
$$\xrightarrow{T(\otimes,\mathbb{B}_{dr})((K,W),(K',W))^{-1}}\mathbb{B}_{dr,(\tilde{S}_I)}((K,W)\otimes_{\mathbb{B}_{dr,S}}\mathbb{B}_{dr,(\tilde{S}_I)}(K',W))$$
$$\xrightarrow{\alpha\otimes\alpha'}F^0DR(S)(((M_I,F,W),u_{IJ})^{an}\otimes_{O_S}((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))\otimes_{\mathbb{B}_{dr,S}}$$
$$F^0DR(S)(((M'_I,F,W),v_{IJ})^{an}\otimes_{O_S}((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$
$$\xrightarrow{w_S}V_{S_1 0}\cdots V_{S_r 0}\Gamma^w_S F^0DR(S\times S)(p_1^{*mod}((M_I,F,W),u_{IJ})^{an}\otimes_{O_{S\times S}}p_2^{*mod}((M'_I,F,W),v_{IJ})^{an}$$
$$\otimes_{O_{S\times S}}((O\mathbb{B}_{dr,\tilde{S}_I\times\tilde{S}_I},F),t_{IJ}))$$
$$\xrightarrow{(I,m(-)\otimes w_S}F^0DR(S)(((M_I,F,W),u_{IJ})^{an}\otimes^{Hdg}_{O_S}((M'_I,F,W),v_{IJ})^{an}\otimes_{O_S}((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$

where $S=\cap_i S_i$ with $S_i\subset S$ Cartier divisor, and (see (ii)) for $j_i:S\backslash S_i\hookrightarrow S$ the open embedding $m(M):V_{S_i0}j_{iw}(M,F,W)\otimes_{O_S}j_{i*Hdg}(O_{S\backslash S_i},F_b)\to j_{i*Hdg}(M,F,W)$ is the multiplication map.

Lemma 7. *(i) Let $j:S^o\hookrightarrow S$ an open embedding with $S\in\mathrm{SmVar}(k)$ such that $D=S\backslash S^o=V(s)\subset S$ is a (Cartier) divisor. For $(M,F,W)\in C(DRM(S^o))$,*

$$m(M):V_{D0}j_{*w}(M,F,W)\otimes_{O_S}j_{*Hdg}(O_{S^o},F_b)\to j_{*Hdg}(M,F,W),m\otimes h\mapsto m(M)(m\otimes h):=hm$$

is an isomorphism in $C(DRM(S))$, whose inverse is given by

$$n(M):j_{*Hdg}(M,F,W)\to V_{D0}j_{*w}(M,F,W)\otimes_{O_S}j_{*Hdg}(O_{S^o},F_b),m\mapsto n(M)(m):=s^rm\otimes 1/s^r.$$

where $r\in\mathbb{N}$ is such that $s^rm\in\Gamma(W,V_{D0}j_*M)$ for $m\in\Gamma(W,M)$ and $W\subset S$ an open subset.

(i)' *Let $l:S^o\hookrightarrow S$ an open embedding with $S\in\mathrm{Var}(k)$ such that $D=S\backslash S^o=V(s)\subset S$ is a Cartier divisor. Let $S=\cup_i S_i$ an open affine cover and $i_i:S_i\hookrightarrow\tilde{S}_i$ closed embedding with $\tilde{S}_i\in\mathrm{SmVar}(k)$. Let $l_I:\tilde{S}^o_I\hookrightarrow\tilde{S}_I$ closed embeddings such that $\tilde{S}^o_I\cap S=S^o\cap S_I$. For $((M_I,F,W),u_{IJ})\in C(DRM(S^o))$*

$$(m(M_I)):V_{D0}l_{*w}((M_I,F,W),u_{IJ})\otimes_{O_S}(l_{I*Hdg}(O_{\tilde{S}^o_I},F_b),x_{IJ})\to(l_{*Hdg}((M_I,F,W),u_{IJ})$$

is an isomorphism in $C(DRM(S))$ whose inverse is given by

$$(n(M_I)):l_{*Hdg}((M_I,F,W),u_{IJ})\to V_{D0}l_{*w}((M_I,F,W),u_{IJ})\otimes_{O_S}(l_{I*Hdg}(O_{\tilde{S}^o_I},F_b),x_{IJ}).$$

(ii) *Let $j:S^o\hookrightarrow S$ an open embedding with $S\in\mathrm{SmVar}(k)$ such that $D=S\backslash S^o=V(s)\subset S$ is a (Cartier) divisor. For $(M,F,W)\in C(DRM(S^o))$,*

$$w_S\otimes m(M)^{an}V_{D0}j_{*w}F^0DR(S^o)(((M,F,W))^{an}\otimes_{O_{S^o}}(O\mathbb{B}_{dr,S^o},F))\otimes_{\mathbb{B}_{dr,S}}$$
$$F^0DR(S)(j_{*Hdg}(O_{S^o},F_b)^{an}\otimes_{O_S}(O\mathbb{B}_{dr,S},F))$$
$$\to F^0DR(S)(j_{*Hdg}(M,F,W)^{an}\otimes_{O_S}(O\mathbb{B}_{dr,S},F)),\ (w_1\otimes m)\otimes(w_2\otimes h)\mapsto(w_1\wedge w_2)\otimes(hm)$$

is an isomorphism in $C(DRM(S))$ whose inverse is

$$w_S^{-1} \otimes n(M)^{an} : F^0DR(S)(j_{*Hdg}(M,F,W)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F)) \\ \to (V_{D0}j_{*w}F^0DR(S^o)(((M,F,W))^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F))) \otimes_{\mathbb{B}_{dr,S}} \\ (F^0DR(S)(j_{*Hdg}(O_{S^o},F_b)^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F))).$$

(ii)' *Let $l : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{Var}(k)$ such that $D = S\backslash S^o = V(s) \subset S$ is a Cartier divisor. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Let $l_I : \tilde{S}_I^o \hookrightarrow \tilde{S}_I$ closed embeddings such that $\tilde{S}_I^o \cap S = S^o \cap S_I$. For $((M_I,F,W),u_{IJ}) \in C(DRM(S^o))$*

$$(w_{\tilde{S}_I} \otimes m(M_I)^{an}) : V_{D0}l_{*w}F^0DR(S^o)(((M_I,F,W),u_{IJ})^{an} \otimes_{O_{S^o}} ((O\mathbb{B}_{dr,\tilde{S}_I^o},F),t_{IJ})) \otimes_{\mathbb{B}_{dr,(\tilde{S}_I)}} \\ (F^0DR((\tilde{S}_I))((l_{I*Hdg}(O_{\tilde{S}_I^o},F_b),x_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))) \\ \to F^0DR(S)((l_{*Hdg}((M_I,F,W),u_{IJ})^{an}) \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$

is an isomorphism in $C(DRM(S))$ whose inverse is

$$(w_{\tilde{S}_I}^{-1} \otimes n(M_I)^{an}) : F^0DR(S)((l_{*Hdg}((M_I,F,W),u_{IJ})^{an}) \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ})) \\ \to V_{D0}l_{*w}F^0DR(S^o)(((M_I,F,W),u_{IJ})^{an} \otimes_{O_{S^o}} ((O\mathbb{B}_{dr,\tilde{S}_I^o},F),t_{IJ})) \otimes_{\mathbb{B}_{dr,(\tilde{S}_I)}} \\ F^0DR((\tilde{S}_I))((l_{I*Hdg}(O_{\tilde{S}_I^o},F_b),x_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ})).$$

Proof. (i): Follows from the definition of the V-filtration and the F-filtration.
(i)':Follows from (i).
(ii): Follows from (i).
(ii)':Follows from (ii). □

Proposition 49. *Let $k \subset \mathbb{C}_p$ a subfield.*

(i0) *Let $j : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{SmVar}(k)$ and $D = S\backslash S^o$ a (Cartier) divisor. If*

$$\alpha : \mathbb{B}_{dr,S^o}(K,W) \to F^0DR(S^o)((M,F,W)^{an} \otimes_{O_{S^o}} (O\mathbb{B}_{dr,S^o},F))$$

is an isomorphism in $D_{\mathbb{B}_{dr},G,fil}(S^{0,an,pet}_{\mathbb{C}_p})$, with

$$(M,F,W) \in C(DRM(S^o)),\ (K,W) \in D_{\mathbb{Z}_pfil,c,k}(S^{o,et}),$$

then the maps in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p})$

$$j_*\alpha = j_*(\alpha) : \mathbb{B}_{dr,S}(j_{*w}(K,W)) \to F^0DR(S)((j_{*Hdg}(M,F,W))^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F))$$

and

$$j_!\alpha = j_!(\alpha) : \mathbb{B}_{dr,S}(j_{!w}(K,W)) \to F^0DR(S)((j_{!Hdg}(M,F,W))^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S},F))$$

given in definition 68 are isomorphism.

(i0)' *Let $l : S^o \hookrightarrow S$ an open embedding with $S \in \mathrm{Var}(k)$ and $D = S\backslash S^o$ a Cartier divisor. Let $S = \cup_i S_i$ an open affine cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embedding with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Let $l_I : \tilde{S}_I^o \hookrightarrow \tilde{S}_I$ closed embeddings such that $\tilde{S}_I^o \cap S = S^o \cap S_I$. If*

$$\alpha : \mathbb{B}_{dr,(\tilde{S}_I^o)}(K,W) \to F^0DR(S^o)(((M_I,F,W),u_{IJ})^{an} \otimes_{O_{\tilde{S}_I^o}} (O\mathbb{B}_{dr,\tilde{S}_I^o}))$$

is an isomorphism in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$*, with*

$$((M_I,F,W),u_{IJ})\in C(DRM(S^o)),\ (K,W)\in D_{\mathbb{Z}_pfil,c,k}(S^{et}),$$

then the maps in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$

$$l_*\alpha=l_*(\alpha):\mathbb{B}_{dr,(\tilde{S}_I)}(l_{*w}(K,W))\to F^0DR(S)(l_{!Hdg}((M_I,F,W),u_{IJ})^{an}\otimes_{O_S}(O\mathbb{B}_{dr,(\tilde{S}_I)},F))$$

and

$$l_!\alpha=l_!(\alpha):\mathbb{B}_{dr,(\tilde{S}_I)}(l_{!w}(K,W))\to F^0DR(S)(l_{!Hdg}((M_I,F,W),u_{IJ})^{an}\otimes_{O_S}(O\mathbb{B}_{dr,(\tilde{S}_I)},F))$$

are isomorphisms.

(i) Let $f:X\to S$ *a morphism with* $S,X\in\mathrm{SmVar}(k)$*. If*

$$\alpha:\mathbb{B}_{dr,X}(K,W)\to F^0DR(X)((M,F,W)^{an}\otimes_{O_X}(O\mathbb{B}_{dr,X},F))$$

an isomorphism in $D_{\mathbb{B}_{dr},G,fil}(X^{an,pet}_{\mathbb{C}_p})$*, with*

$$(M,F,W)\in C(DRM(X)),\ (K,W)\in D_{\mathbb{Z}_pfil,c,k}(X^{et}),$$

then the morphisms given in definition 86

$$f_*\alpha=f_*(\alpha):\mathbb{B}_{dr,S}(Rf_{*w}(K,W))\to F^0DR(S)((Rf_{*Hdg}(M,F,W))^{an}\otimes_{O_S}(O\mathbb{B}_{dr,S},F))$$

and

$$f_!\alpha=f_!(\alpha):\mathbb{B}_{dr,S}(Rf_{!w}(K,W))\to F^0DR(S)((Rf_{!Hdg}(M,F,W))^{an}\otimes_{O_S}(O\mathbb{B}_{dr,S},F))$$

are isomorphisms.

(i)' Let $f:X\to S$ *a morphism with* $S,X\in\mathrm{QPVar}(k)$*. Consider a factorization* $f:X\hookrightarrow Y\times S\xrightarrow{p}S$ *with* $Y\in\mathrm{SmVar}(k)$*. Let* $S=\cup_iS_i$ *an open affine cover and* $i_i:S_i\hookrightarrow\tilde{S}_i$ *closed embedding with* $\tilde{S}_i\in\mathrm{SmVar}(k)$*. If*

$$\alpha:\mathbb{B}_{dr,(Y\times\tilde{S}_I)}(K,W)\to F^0DR(X)((((M_I,F,W),u_{IJ})^{an}\otimes_{O_{Y\times\tilde{S}_I}}(O\mathbb{B}_{dr,Y\times\tilde{S}_I},F))$$

is an isomorphism in $D_{\mathbb{B}_{dr},G,fil}(X^{an,pet}_{\mathbb{C}_p}/(Y\times\tilde{S}_{I\mathbb{C}_p}^{an,pet}))$*, with*

$$((M_I,F,W),u_{IJ})\in C(DRM(X)),\ (K,W)\in D_{\mathbb{Z}_pfil,c,k}(X^{et}),$$

then, the maps in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$

$$f_*\alpha=f_*(\alpha):\mathbb{B}_{dr,(\tilde{S}_I)}(Rf_{*w}(K,W))\to F^0DR(S)((Rf_{*Hdg}((M_I,F,W),u_{IJ}))^{an}\otimes_{O_S}(O\mathbb{B}_{dr,(\tilde{S}_I)},F)),$$

and

$$f_!\alpha=f_!(\alpha):\mathbb{B}_{dr,(\tilde{S}_I)}((Rf_{!w}(K,W)))\to F^0DR(S)((Rf_{!Hdg}((M_I,F,W),u_{IJ}))^{an}\otimes_{O_S}(O\mathbb{B}_{dr,(\tilde{S}_I)},F)),$$

are isomorphisms.

(ii) Let $f:X\to S$ *a morphism with* $S,X\in\mathrm{QPVar}(k)$*. Consider a factorization* $f:X\hookrightarrow Y\times S\xrightarrow{p}S$ *with* $Y\in\mathrm{SmVar}(k)$*. Let* $S=\cup_iS_i$ *an open affine cover and* $i_i:S_i\hookrightarrow\tilde{S}_i$ *closed embedding with* $\tilde{S}_i\in\mathrm{SmVar}(k)$*. If*

$$\alpha:\mathbb{B}_{dr,(\tilde{S}_I)}(K,W)\to F^0DR(S)((((M_I,F,W),u_{IJ})^{an}\otimes_{O_{\tilde{S}_I}}(O\mathbb{B}_{dr,(\tilde{S}_I)},F))$$

is an isomorphism in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$, *with*

$$((M_I,F,W),u_{IJ})\in C(DRM(S)),\ (K,W)\in D_{\mathbb{Z}_pfil,c,k}(S^{et}),$$

the maps in $D_{\mathbb{B}_{dr},G,fil}(X^{an,pet}_{\mathbb{C}_p}/(Y\times\tilde{S}_I)^{an,pet}_{\mathbb{C}_p})$

$$f^!\alpha=f^!(\alpha):\mathbb{B}_{dr,(Y\times\tilde{S}_I)}(f^{!w}(K,W))\to F^0DR(X)(f^{*mod}_{Hdg}((M_I,F,W),u_{IJ})^{an}\otimes_{O_X}(O\mathbb{B}_{dr,(Y\times\tilde{S}_I)},F))$$

and

$$f^*\alpha=f^*(\alpha):\mathbb{B}_{dr,(Y\times\tilde{S}_I)}(f^!(K,W))\to F^0DR(X)(f^{\hat{*}mod}_{Hdg}((M_I,F,W),u_{IJ})^{an}\otimes_{O_X}(O\mathbb{B}_{dr,(Y\times\tilde{S}_I)},F))$$

given in definition 86 are isomorphisms.

Proof. (i0): Follows from lemma 7(ii).
(i0)':Follows from lemma 7(ii)'.
(i): Follows from (i0) and on the other hand theorem 47 and GAGA for proper morphism of algebraic varieties over a p-adic field.
(i): Follows from (i0)' and on the other hand theorem 47 and GAGA for proper morphism of algebraic varieties over a p-adic field.
(ii): Follows from (i0). □

Definition 87. *Let* $S\in\mathrm{SmVar}(k)$. *Let* $D=V(s)\subset S$ *a divisor with* $s\in\Gamma(S,L)$ *and* L *a line bundle (*S *being smooth,* D *is Cartier). For* $\mathcal{M}=((M,F,W),(K,W),\alpha)\in\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S)\times_I P_{\mathbb{Z}_pfil,k}(S^{et})$, *we then define, using definition 57, theorem 43 and definition TphipsiBdr,*

- *the nearby cycle functor*

$$\psi_D((M,F,W),(K,W),\alpha):=(\psi_D(M,F,W),\psi_D(K,W)[-1],\psi_D\alpha)\in\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S)\times_I P_{\mathbb{Z}_pfil,k}(S^{et}),$$

with

$$\begin{aligned}\psi_D\alpha:\mathbb{B}_{dr,S}(\psi_D(K,W))&\xrightarrow{T(\psi_D,\mathbb{B}_{dr})(K,W)}\psi_D\mathbb{B}_{dr,S}(K,W)\xrightarrow{\psi_D\alpha}\\ \psi_DDR(S)((M,F,W)^{an}\otimes_{O_S}(O\mathbb{B}_{dr,S},F))&\xrightarrow{T^{B_{dr}}(\psi_D,DR)(M,F,W)}\\ &DR(S)(\psi_D(M,F,W)^{an}\otimes_{O_S}(O\mathbb{B}_{dr,S},F)),\end{aligned}$$

- *the vanishing cycle functor*

$$\phi_D((M,F,W),(K,W),\alpha):=(\phi_D(M,F,W),\phi_D(K,W)[-1],\phi_D\alpha)\in\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S)\times_I P_{\mathbb{Z}_pfil,k}(S^{et}),$$

with

$$\begin{aligned}\phi_D\alpha:\mathbb{B}_{dr,S}(\phi_D(K,W))&\xrightarrow{T(\phi_D,\mathbb{B}_{dr})(K,W)}\phi_D\mathbb{B}_{dr,S}(K,W)\xrightarrow{\phi_D\alpha}\\ \phi_DDR(S)((M,F,W)^{an}\otimes_{O_S}(O\mathbb{B}_{dr,S},F))&\xrightarrow{T^{B_{dr}}(\phi_D,DR)(M,F,W)}\\ &DR(S)(\phi_D(M,F,W)^{an}\otimes_{O_S}(O\mathbb{B}_{dr,S},F)),\end{aligned}$$

- *the canonical maps in* $\mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S)\times_I P_{\mathbb{Z}_pfil,k}(S^{et})$

$$\begin{aligned}can(\mathcal{M}):=(can(M,F,W),can(K,W)):\psi_D((M,F,W),(K,W),\alpha)\to\phi_D((M,F,W),(K,W),\alpha)(-1),\\ var(\mathcal{M}):=(var(M,F,W),var(K,W)):\phi_D((M,F,W),(K,W),\alpha)\to\psi_D((M,F,W),(K,W),\alpha).\end{aligned}$$

Proposition 50. *Let $S \in \mathrm{SmVar}(k)$. Let $D = V(s) \subset S$ a (Cartier divisor). Consider a composition of proper morphisms*

$$(f : X = X_r \xrightarrow{f_r} X_{r-1} \xrightarrow{f_1} X_0 = S) \in \mathrm{SmVar}(k), \text{ proper}, 1 \leq i \leq r,$$

and

$$(M, F) = H^{n_0} \int_{f_1} \cdots H^{n_r} \int_{f_r} ((O_X, F_b), H^{n_0} R f_{1*} \cdots H^{n_r} R f_{r*} \mathbb{Z}_{X_{\bar{k}}}, \\ H^{n_0} f_{1*} \circ \cdots \circ H^{n_r} f_{r*} \alpha(X_{\mathbb{C}_p})) \in \mathrm{PSh}_{\mathcal{D} fil, rh}(S) \times_I P_{\mathbb{Z}_p, k}(S^{et}).$$

Then,

$$\psi_D(M, F) = H^{n_0} \int_{f_1} \cdots H^{n_r} \int_{f_r} (\psi_{f^{-1}(D)}(O_X, F_b)), H^{n_0} R f_{1*} \cdots H^{n_r} R f_{r*} \psi_{f^{-1}(D)} \mathbb{Z}_{X_{\bar{k}}}, \\ H^{n_0} f_{1*} \circ \cdots \circ H^{n_r} f_{r*} \psi_{f^{-1}(D)} \alpha(X_{\mathbb{C}_p})) \in \mathrm{PSh}_{\mathcal{D} fil, rh}(S) \times_I P_{\mathbb{Z}_p, k}(S^{et}),$$

and

$$\phi_D(M, F) = H^{n_0} \int_{f_1} \cdots H^{n_r} \int_{f_r} (\psi_{f^{-1}(D)}(O_X, F_b)), H^{n_0} R f_{1*} \cdots H^{n_r} R f_{r*} \phi_{f^{-1}(D)} \mathbb{Z}_{X_{\bar{k}}}, \\ H^{n_0} f_{1*} \circ \cdots \circ H^{n_r} f_{r*} \phi_{f^{-1}(D)} \alpha(X_{\mathbb{C}_p})) \in \mathrm{PSh}_{\mathcal{D} fil, rh}(S) \times_I P_{\mathbb{Z}_p, k}(S^{et}),$$

Proof. Immediate from definition. □

Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I \in \mathrm{SmVar}(k)$. We consider $\mathbb{Z}^w_{p,S^{et}} \in C_{\mathbb{Z}_p fil}(S^{et})$ such that $j_I^* \mathbb{Z}^w_{p,S^{et}} = i_I^* \Gamma^{\vee,w}_{S_I} \mathbb{Z}^{et}_{p,\tilde{S}_I}$ and set

$$\alpha(S) : \mathbb{B}_{dr,(\tilde{S}_I)}(\mathbb{Z}^w_{p,S^{et}}) \xrightarrow{:=} \\ (\Gamma^{\vee,w}_{S_I} \mathbb{Z}_{p,(\tilde{S}_I)} \otimes \mathbb{B}_{dr,\phi_{\tilde{D}_{1,I}}} \otimes_{\mathbb{B}_{dr,\tilde{S}_I}} \cdots \otimes_{\mathbb{B}_{dr,\tilde{S}_I}} \mathbb{B}_{dr,\phi_{\tilde{D}_{d,I}}}, \mathbb{B}_{dr}(t_{IJ})) \\ \xrightarrow{=} (\Gamma^{\vee,w}_{S_I} \mathbb{Z}_{p,\tilde{S}_I} \otimes \mathbb{B}_{dr,\tilde{S}_I} \otimes_{\mathbb{B}_{dr,\tilde{S}_I}} \mathbb{B}_{dr,\phi_{\tilde{D}_{1,I}}} \otimes_{\mathbb{B}_{dr,\tilde{S}_I}} \cdots \otimes_{\mathbb{B}_{dr,\tilde{S}_I}} \mathbb{B}_{dr,\phi_{\tilde{D}_{d,I}}}, \mathbb{B}_{dr}(x_{IJ})) \\ \xrightarrow{\alpha((\tilde{S}_{I,\mathbb{C}_p}) \otimes I)} \\ (V_{\tilde{D}_{1,I}0} \cdots V_{\tilde{D}_{d,I}0} \Gamma^{\vee}_{S_I}(F^0 DR(\tilde{S}_I)((O\mathbb{B}_{dr,\tilde{S}_I}), F)) \otimes_{\mathbb{B}_{dr,\tilde{S}_I}} \mathbb{B}_{dr,\phi_{\tilde{D}_{1,I}}} \otimes_{\mathbb{B}_{dr,\tilde{S}_I}} \cdots \otimes_{\mathbb{B}_{dr,\tilde{S}_I}} \mathbb{B}_{dr,\phi_{\tilde{D}_{d,I}}}, \mathbb{B}_{dr}(t_{IJ})) \\ \xrightarrow{(\mathbb{D}T(\gamma_{S_I}, \otimes)(-)) \otimes (DR(S)(\mathbb{D}\rho_{DR,\tilde{D}_{1,I}}(O_{\tilde{S}_I}, F_b)) \otimes \cdots \otimes \mathbb{D}(\rho_{DR,\tilde{D}_{d,I}}(O_{\tilde{S}_I}, F_b)))} \\ F^0 DR(S)(V_{\tilde{D}_{1,I}0} \cdots V_{\tilde{D}_{d,I}0} \Gamma^{\vee,h}_{S_I}(O_{\tilde{S}_I}, F_b) \otimes_{\mathbb{B}_{dr,\tilde{S}_I}} O\mathbb{B}_{dr,\tilde{S}_I}, x_{IJ}) \otimes_{\mathbb{B}_{dr,(\tilde{S}_I)}} \\ F^0 DR(S)((\Gamma^{\vee,Hdg}_{S_I}(O_{\tilde{S}_I}, F_b))^{an} \otimes_{O_{\tilde{S}_I}} (O\mathbb{B}_{dr,\tilde{S}_I}, F), x_{IJ}) \\ \xrightarrow{T(DR,\otimes)(-,-)} F^0 DR(S)(\Gamma^{\vee,h}_{S_I}(O_{\tilde{S}_I}, F_b) \otimes_{O_{\tilde{S}_I}} (\Gamma^{\vee,Hdg}_{S_I}(O_{\tilde{S}_I}, F_b))^{an} \otimes_{O_{\tilde{S}_I}} (O\mathbb{B}_{dr,\tilde{S}_I}, F), x_{IJ}) \\ \xrightarrow{F^0 DR(S)(m(O_{\tilde{S}_I}))} F^0 DR(S)((\Gamma^{\vee,Hdg}_{S_I}(O_{\tilde{S}_I}, F_b))^{an} \otimes_{O_{\tilde{S}_I}} (O\mathbb{B}_{dr,\tilde{S}_I}, F), x_{IJ}) \\ \xrightarrow{=:} F^0 DR(S)((\Gamma^{\vee,Hdg}_{S_I}(O_{\tilde{S}_I}, F_b), x_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,(\tilde{S}_I)}, F), t_{IJ}))$$

is an isomorphism in $D_{\mathbb{B}_{dr},G}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}_I)^{an,pet}_{\mathbb{C}_p})$, where $D_1, \ldots, D_d \subset S$ are Cartier divisors such that $S = \cap_{s=1}^d D_s$, $\tilde{D}_{s,I} \subset \tilde{S}_I$ are Cartier divisor such that $D_s \cap S_I \subset \tilde{D}_{s,I} \cap S_I$, $i_I : S_I \hookrightarrow \tilde{S}_I$ are the closed embeddings, $m(O) : O \times_O O \xrightarrow{\sim} O, h \otimes f \mapsto hf$ is the multiplication map, and we use definition 86 and proposition 49.

We now give the definition of p adic mixed Hodge modules which is the main definition of this section :

Definition 88. *Let $k \subset \mathbb{C}_p$ a subfield.*

(i) Let $S \in \mathrm{SmVar}(k)$. We denote by

$$HM_{gm,k,\mathbb{C}_p}(S) :=< (H^{n_1}\int_{f_1}\cdots H^{n_r}\int_{f_r}(O_X,F_b)(d), R^{n_1}f_{1*}\cdots R^{n_r}f_{r*}\mathbb{Z}_{p,X^{et}}, H^{n_1}f_{1*}\cdots H^{n_r}f_{r*}\alpha(X)),$$
$$(f : X = X_r \xrightarrow{f_r} X_{r-1} \to \cdots \xrightarrow{f_1} X_0 = S) \in \mathrm{SmVar}(k),\ proper,\ n_1,\ldots,n_r,d \in \mathbb{Z} >$$
$$\subset PDRM(S) \times_I P_{\mathbb{Z}_p,k}(S^{et}) \subset \mathrm{PSh}_{\mathcal{D}fil,rh}(S) \times_I P_{\mathbb{Z}_p,k}(S^{et})$$

the full abelian subcategory, where $<,>$ means generated by and $(-)$ the shift of the filtration,

$$\alpha(X) : \mathbb{B}_{dr,X}(\mathbb{Z}_{p,X^{et}}) := \mathbb{B}_{dr,X_{\mathbb{C}_p}} \hookrightarrow DR(X)(O\mathbb{B}_{dr,X_{\mathbb{C}_p}})$$

is the inclusion quasi-isomorphism in $C_{\mathbb{B}_{dr},G}(X^{pet}_{\mathbb{C}_p})$, and we use definition 86. We have by proposition 50 for $((M,F),K,\alpha) \in HM_{gm,k,\mathbb{C}_p}(S)$,

$$\mathrm{Gr}^W_k\,\psi_D((M,F),K,\alpha) := \mathrm{Gr}^W_k\,\psi_D(M,F), \mathrm{Gr}^W_k\,\psi_D K, \mathrm{Gr}^W_k\,\psi_D\alpha) \in HM_{gm,k,\mathbb{C}_p}(S).$$

and

$$\mathrm{Gr}^W_k\,\psi_D((M,F),K,\alpha) := \mathrm{Gr}^W_k\,\psi_D(M,F), \mathrm{Gr}^W_k\,\psi_D K, \mathrm{Gr}^W_k\,\psi_D\alpha) \in HM_{gm,k,\mathbb{C}_p}(S).$$

for all $k \in \mathbb{Z}$. We set

$$\mathbb{Z}^{Hdg}_{p,S} := ((O_S,F_b),\mathbb{Z}_{p,S^{et}},\alpha(S)) \in HM_{gm,k,\mathbb{C}_p}(S)$$

(i)' Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i\in I}S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I \in \mathrm{SmVar}(k)$.

$$HM_{gm,k,\mathbb{C}_p}(S) :=< (R^{n_1}p_{1*Hdg}\cdots R^{n_r}p_{r*Hdg}(\Gamma^{Hdg}_{X_I}(O_{Y\times\tilde{X}_{r-1,I}},F_b),x_{IJ})(d),$$
$$R^{n_1}p_{1*}\cdots R^{n_r}p_{r*}T(X/(Y_r\times\tilde{X}_{r-1,I}))(\mathbb{Z}_{p,X^{et}}), H^{n_1}p_{1*}\cdots H^{n_r}p_{r*}\alpha(X)),$$
$$(f : X = X_r \xrightarrow{f_r} X_{r-1} \to \cdots \xrightarrow{f_1} X_0 = S) \in \mathrm{Var}(k),\ n_1,\ldots,n_r,d \in \mathbb{Z} >$$
$$\subset PDRM(S) \times_I P_{\mathbb{Z}_p,k}(S^{et}) \subset \mathrm{PSh}_{\mathcal{D}fil,rh}(S/(\tilde{S}_I)) \times_I P_{\mathbb{Z}_p,k}(S^{et})$$

the full abelian subcategory, where $<,>$ means generated by and $(-)$ the shift of the filtration, $f_i : X_i \hookrightarrow Y_i \times X_{i-1} \xrightarrow{p_i} X_{i-1}$ proper, $Y_i \in \mathrm{PSmVar}(k)$, X_i smooth, and $\alpha(X)$ is given above. Note that if S is smooth then this definition of $HM_{gm,k,\mathbb{C}_p}(S)$ agree with the one given in (i).

(ii) Let $S \in \mathrm{Var}(k)$. Take an open cover $S = \cup_i S_i$ such that there are closed embedding $S_I \hookrightarrow \tilde{S}_I$ with $S_I \in \mathrm{SmVar}(k)$. We define using the pure case (i) and (i)' the full subcategory of geometric mixed Hodge modules defined over k

$$MHM_{gm,k,\mathbb{C}_p}(S) :=$$
$$\left\{(((M_I,F,W),u_{IJ}),(K,W),\alpha),\ s.t.\ \mathrm{Gr}^W_k(((M_I,F,W),u_{IJ}),(K,W),\alpha) \in HM_{gm,k,\mathbb{C}_p}(S)\right\}$$
$$\subset DRM(S) \times_I P_{\mathbb{Z}_p fil,k}(S^{et}) \subset \mathrm{PSh}_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I P_{\mathbb{Z}_p fil,k}(S^{et})$$

whose object consists of $(((M_I,F,W),u_{IJ}),(K,W),\alpha) \in DRM(S) \times_I P_{\mathbb{Z}_p fil,k}(S^{et})$ such that

$$\mathrm{Gr}^W_k(((M_I,F,W),u_{IJ})(K,W),\alpha) := (\mathrm{Gr}^W_k((M_I,F),u_{IJ}), \mathrm{Gr}^W_k K, \mathrm{Gr}^W_k\alpha) \in HM_{gm,k,\mathbb{C}_p}(S).$$

where $DRM(S)$ is the category of de Rham modules introduced in section 5 definition 58. The fact that α is an isomorphism implies that the Galois representation of G induced on each k-point of S is a de Rham representation. We set

$$\mathbb{Z}^{Hdg}_{p,S} := ((\Gamma^{\vee,Hdg}_{S_I}(O_{\tilde{S}_I},F_b),x_{IJ}),\mathbb{Z}^w_{p,S^{et}},\alpha(S)) \in C(MHM_{gm,k,\mathbb{C}_p}(S))$$

where $\mathbb{Z}^w_{p,S^{et}} \in C(P_{fil,k}(S^{et}))$ *is such that* $j_I^*\mathbb{Z}^w_{p,S^{et}} = i_I^*\Gamma^{\vee,w}_{S_I}\mathbb{Z}_{p,\tilde{S}_I^{et}}$ *and* $\alpha(S)$ *given above. For* $S \in \mathrm{SmVar}(k)$ *and* $D = V(s) \subset S$ *a (Cartier) divisor, we have for* $((M,F,W),(K,W),\alpha) \in MHM_{gm,k,\mathbb{C}_p}(S)$*, using theorem 35,*

$$\psi_D((M,F,W),(K,W),\alpha), \phi_D((M,F,W),(K,W),\alpha) \in MHM_{gm,k,\mathbb{C}_p}(S),$$

by the pure case (c.f. (i) and proposition 50) and the strictness of the V*-filtration.*

For $S \in \mathrm{Var}(k)$ *we get* $D(MHM_{gm,k,\mathbb{C}_p}(S)) := \mathrm{Ho}_{(zar,et)}(C(MHM_{gm,k,\mathbb{C}_p}(S)))$ *after localization with Zariski local equivalence and etale local equivalence.*

We now look at functorialities :

Definition 89. *Let* $k \subset \mathbb{C}_p$ *a subfield. Let* $S \in \mathrm{SmVar}(k)$*. Let* $j : S^o \hookrightarrow S$ *an open embedding. Let* $Z := S\backslash S^o = V(\mathcal{I}) \subset S$ *an the closed complementary subset,* $\mathcal{I} \subset O_S$ *being an ideal subsheaf. Taking generators* $\mathcal{I} = (s_1, \ldots, s_r)$*, we get* $Z = V(s_1, \ldots, s_r) = \cap_{i=1}^r Z_i \subset S$ *with* $Z_i = V(s_i) \subset S$*,* $s_i \in \Gamma(S, \mathcal{L}_i)$ *and* L_i *a line bundle. Note that* Z *is an arbitrary closed subset,* $d_Z \geq d_X - r$ *needing not be a complete intersection. Denote by* $j_I : S^{o,I} := \cap_{i\in I}(S\backslash Z_i) = S\backslash(\cup_{i\in I} Z_i) \xrightarrow{j_I^o} S^o \xrightarrow{j} S$ *the open embeddings. Let* $(M,F,W) \in MHM_{gm,k,\mathbb{C}_p}(S^o))$*. We then define, using definition 59 and definition 10*

- *the canonical extension*

$$\begin{aligned} j_{*Hdg}((M,F,W),(K,W),\alpha) &:= (j_{*Hdg}(M,F,W), j_{*w}(K,W), j_*\alpha) \\ &:= \in MHM_{gm,k,\mathbb{C}_p}(S), \end{aligned}$$

  so that $j^*(j_{*Hdg}((M,F,W),(K,W),\alpha)) = ((M,F,W),(K,W),\alpha)$*,*

- *the canonical extension*

$$j_{!Hdg}((M,F,W),(K,W),\alpha) := (j_{!Hdg}(M,F,W), j_{!w}(K,W), j_!\alpha) := \in MHM_{gm,k,\mathbb{C}_p}(S),$$

  so that $j^*(j_{!Hdg}((M,F,W),(K,W),\alpha)) = ((M,F,W),(K,W),\alpha)$*.*

Moreover for $((M',F,W),(K',W),\alpha') \in MHM_{gm,k,\mathbb{C}_p}(S)$*,*

- *there is a canonical map in* $MHM_{gm,k,\mathbb{C}_p}(S)$

  $\mathrm{ad}(j^*, j_{*Hdg})((M',F,W),(K',W),\alpha') : ((M',F,W),(K',W),\alpha') \to j_{*Hdg}j^*((M',F,W),(K',W),\alpha')$,

- *there is a canonical map in* $MHM_{gm,k,\mathbb{C}_p}(S)$

  $\mathrm{ad}(j_{!Hdg}, j^*)((M',F,W),(K',W),\alpha') : j_{!Hdg}j^*((M',F,W),(K',W),\alpha') \to ((M',F,W),(K',W),\alpha')$.

For $(M,F,W) \in C(MHM_{gm,k,\mathbb{C}_p}(S^o))$,

- we have the canonical map in $C_{\mathcal{D}(1,0)fil}(S) \times_I C_{fil}(S^{et})$

$$\begin{aligned} T(j_{*Hdg}, j_*)((M,F,W),(K,W),\alpha) &:= (k \circ \mathrm{ad}(j^*, j_*)(-), k \circ \mathrm{ad}(j^*, j_*), 0) : \\ j_{*Hdg}((M,F,W),(K,W),\alpha) &\to (j_*E(M,F,W), j_*E(K,W), \alpha) \end{aligned}$$

- we have the canonical map in $C_{\mathcal{D}(1,0)fil}(S) \times_I C_{fil}(S^{et})$

$$\begin{aligned} T(j_!, j_{!Hdg})((M,F,W),(K,W),\alpha) &:= (k \circ \mathrm{ad}(j_!, j^*)(-), k \circ \mathrm{ad}(j_!, j^*)(-), 0) : \\ (j_!(M,F,W), j_!(K,W), j_!\alpha) &\to j_{!Hdg}((M,F,W),(K,W),\alpha). \end{aligned}$$

Proposition 51. *(i) Let $S \in \mathrm{SmVar}(k)$. Let $D = V(s) \subset S$ a divisor with $s \in \Gamma(S, L)$ and L a line bundle (S being smooth, D is Cartier). Denote by $j : S^o := S \backslash D \hookrightarrow S$ the open complementary embedding. Then,*

- *$(j^*, j_{*Hdg}) : MHM_{gm,k,\mathbb{C}_p}(S) \leftrightarrows MHM_{gm,k,\mathbb{C}_p}(S^o)$ is a pair of adjoint functors*
- *$(j_{!Hdg}, j^*) : MHM_{gm,k,\mathbb{C}_p}(S^o) \leftrightarrows MHM_{gm,k,\mathbb{C}_p}(S)$ is a pair of adjoint functors.*

(ii) Let $S \in \mathrm{SmVar}(k)$. Let $Z = V(\mathcal{I}) \subset S$ an arbitrary closed subset, $\mathcal{I} \subset O_S$ being an ideal subsheaf. Denote by $j : S^o := S \backslash Z \hookrightarrow S$. Then,

- *$(j^*, j_{*Hdg}) : D(MHM_{gm,k,\mathbb{C}_p}(S)) \leftrightarrows D(MHM_{gm,k,\mathbb{C}_p}(S^o))$ is a pair of adjoint functors*
- *$(j_{!Hdg}, j^*) : D(MHM_{gm,k,\mathbb{C}_p}(S^o)) \leftrightarrows D(MHM_{gm,k,\mathbb{C}_p}(S))$ is a pair of adjoint functors.*

Proof. (i): Follows from proposition 37.
(ii):Follows from (i) and the exactness of j^*, j_{*Hdg} and $j_{!Hdg}$. □

Definition 90. *Let $S \in \mathrm{SmVar}(k)$. Let $Z \subset S$ a closed subset. Denote by $j : S \backslash Z \hookrightarrow S$ the complementary open embedding.*

(i) We define using definition 61, definition 11 and definition 86(iii), the filtered Hodge support section functor

$$\Gamma_Z^{Hdg} : C(MHM_{gm,k,\mathbb{C}_p}(S)) \to C(MHM_{gm,k,\mathbb{C}_p}(S)), \quad ((M,F,W),(K,W),\alpha) \mapsto \Gamma_Z^{Hdg}((M,F,W),(K,W),\alpha) := (\Gamma_Z^{Hdg}(M,F,W), \Gamma_Z^w(K,W), \Gamma(\alpha)) = \mathrm{Cone}(\mathrm{ad}(j^*, j_{*Hdg})(-) : j_{*Hdg}, j^*((M,F,W),(K,W),\alpha) \to ((M,F,W),(K,W),\alpha)[-1]$$

see definition 89 for the last equality, together we the canonical map

$$\gamma_Z^{Hdg}((M,F,W),(K,W),\alpha) : \Gamma_Z^{Hdg}((M,F,W),(K,W),\alpha) \to ((M,F,W),(K,W),\alpha).$$

(i)' Since $j_{*Hdg} : C(MHM_{gm,k,\mathbb{C}_p}(S^o)) \to C(MHM_{gm,k,\mathbb{C}_p}(S))$ is an exact functor, Γ_Z^{Hdg} induces the functor

$$\Gamma_Z^{Hdg} : D(MHM_{gm,k,\mathbb{C}_p}(S)) \to D(MHM_{gm,k,\mathbb{C}_p}(S)),$$
$$((M,F,W),(K,W),\alpha) \mapsto \Gamma_Z^{Hdg}((M,F,W),(K,W),\alpha)$$

(ii) We define using definition 61, definition 11 and definition 86(iii) the dual filtered Hodge support section functor

$$\Gamma_Z^{\vee,Hdg} : C(MHM_{gm,k,\mathbb{C}_p}(S)) \to C(MHM_{gm,k,\mathbb{C}_p}(S)), \quad ((M,F,W),(K,W),\alpha) \mapsto \Gamma_Z^{\vee,Hdg}((M,F,W),(K,W),\alpha) := (\Gamma_Z^{\vee,Hdg}(M,F,W), \Gamma_Z^{\vee,w}(K,W), \Gamma^{\vee}(\alpha)) = \mathrm{Cone}(\mathrm{ad}(j_{!Hdg}, j^*)(-) : j_{!Hdg}, j^*((M,F,W),(K,W),\alpha) \to ((M,F,W),(K,W),\alpha))$$

see definition 89 for the last equality, together we the canonical map

$$\gamma_Z^{\vee,Hdg}((M,F,W),(K,W),\alpha) : ((M,F,W),(K,W),\alpha) \to \Gamma_Z^{\vee,Hdg}((M,F,W),(K,W),\alpha).$$

(ii)' Since $j_{!Hdg} : C(MHM_{gm,k,\mathbb{C}_p}(S^o)) \to C(MHM_{gm,k,\mathbb{C}_p}(S))$ is an exact functor, $\Gamma_Z^{Hdg,\vee}$ induces the functor

$$\Gamma_Z^{\vee,Hdg} : D(MHM_{gm,k,\mathbb{C}_p}(S)) \to D(MHM_{gm,k,\mathbb{C}_p}(S)),$$
$$((M,F,W),(K,W),\alpha) \mapsto \Gamma_Z^{\vee,Hdg}((M,F,W),(K,W),\alpha)$$

In the singular case it gives :

Definition 91. *Let $S \in \mathrm{Var}(k)$. Let $Z \subset S$ a closed subset. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Denote $Z_I := Z \cap S_I$. Denote by $n : S \backslash Z \hookrightarrow S$ and $\tilde{n}_I : \tilde{S}_I \backslash Z_I \hookrightarrow \tilde{S}_I$ the complementary open embeddings.*

(i) We define using definition 63, definition 11 and definition 86(iii) the filtered Hodge support section functor

$$\Gamma_Z^{Hdg} : C(MHM_{gm,k,\mathbb{C}_p}(S)) \to C(MHM_{gm,k,\mathbb{C}_p}(S)),$$
$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto \Gamma_Z^{Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) :=$$
$$:= (\Gamma_Z^{Hdg}((M_I, F, W), u_{IJ}), \Gamma_Z^w(K, W), \Gamma(\alpha))$$

together with the canonical map

$$\gamma_Z^{Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) :$$
$$\Gamma_Z^{Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) \to (((M_I, F, W), u_{IJ}), (K, W), \alpha).$$

(i)' By exactness of Γ_Z^{Hdg} and Γ_Z^w it induces the functor

$$\Gamma_Z^{Hdg} : D(MHM_{gm,k,\mathbb{C}_p}(S)) \to D(MHM_{gm,k,\mathbb{C}_p}(S)),$$
$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto \Gamma_Z^{Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha)$$

(ii) We define using definition 63, definition 11 and definition 86(iii) the dual filtered Hodge support section functor

$$\Gamma_Z^{\vee,Hdg} : C(MHM_{gm,k,\mathbb{C}_p}(S)) \to C(MHM_{gm,k,\mathbb{C}_p}(S)), \quad (((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto$$
$$\Gamma_Z^{\vee,Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) := (\Gamma_Z^{\vee,Hdg}((M_I, F, W), u_{IJ}), \Gamma_Z^{\vee,w}(K, W), \Gamma(\alpha)),$$

together we the canonical map

$$\gamma_Z^{\vee,Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) :$$
$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \to \Gamma_Z^{\vee,Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha).$$

(ii)' By exactness of $\Gamma_Z^{\vee,Hdg}$ and $\Gamma_Z^{\vee,w}$, it induces the functor

$$\Gamma_Z^{\vee,Hdg} : D(MHM_{gm,k,\mathbb{C}_p}(S)) \to D(MHM_{gm,k,\mathbb{C}_p}(S)),$$
$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto \Gamma_Z^{\vee,Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha)$$
$$:= (\Gamma_Z^{\vee,Hdg}((M_I, F, W), u_{IJ}), \Gamma_Z^{\vee,w}(K, W), \Gamma(\alpha))$$

This gives the inverse image functor :

Definition 92. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. Let $S = \cup_{i \in I}$ an open cover such that there exist closed embeddings $i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Denote $X_I := f^{-1}(S_I)$. We have then $X = \cup_{i \in I} X_i$ and the commutative diagrams*

$$\begin{tikzcd} f : X_I \arrow[r, "l_I"] \arrow[dr] & Y \times S_I \arrow[r, "p_{S_I}"] \arrow[d, "i'_I := (I \times i_I)"] & S_I \arrow[d, "i_I"] \\ & Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I} =: \tilde{f}_I"] & \tilde{S}_I \end{tikzcd}$$

(i) For $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in C(MHM_{gm,k,\mathbb{C}_p}(S))$ *we set (see definition 91 for l)*

$$f^{*mod}_{Hdg}(((M_I,F,W),u_{IJ}),(K,W),\alpha):=$$
$$\Gamma^{Hdg}_X((p^{*mod[-]}_{\tilde{S}_I}(M_I,F,W),p^{*mod[-]}_{\tilde{S}_I}u_{IJ}),p^*_S(K,W),p^*_S\alpha)(d_Y)[2d_Y]\in C(MHM_{gm,k,\mathbb{C}_p}(X)),$$

(ii) For $(((M_I,F,W),u_{IJ}),(K,W),\alpha)\in C(MHM_{gm,k,\mathbb{C}_p}(S)))$ *we set (see definition 91 for l)*

$$f^{\hat{*}mod}_{Hdg}(((M_I,F,W),u_{IJ}),(K,W),\alpha):=$$
$$\Gamma^{\vee,Hdg}_X((p^{*mod[-]}_{\tilde{S}_I}(M_I,F,W),p^{*mod[-]}_{\tilde{S}_I}u_{IJ}),p^*_S(K,W),p^*_S\alpha)\in C(MHM_{gm,k,\mathbb{C}_p}(X)),$$

Definition 93. *Let* $S\in\mathrm{Var}(k)$*. Let* $S=\cup_{i\in I}$ *an open cover such that there exist closed embeddings* $i:S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{SmVar}(k)$*. We have the following bi-functor*

$$(-)\otimes^{Hdg}_{O_S}(-):D(MHM_{gm,k,\mathbb{C}_p}(S))^2\to D(MHM_{gm,k,\mathbb{C}_p}(S)),$$
$$(((M_I,F,W),u_{IJ}),(K,W),\alpha),(((M'_I,F,W),v_{IJ}),(K',W),\alpha')\mapsto$$
$$(((M_I,F,W),u_{IJ}),(K,W),\alpha)\otimes^{Hdg}_{O_S}(((M'_I,F,W),v_{IJ}),(K',W),\alpha'):=$$
$$((M_I,F,W),u_{IJ})\otimes^{Hdg}_{O_S}((M'_I,F,W),v_{IJ}),(K,W)\otimes^{L,w}(K',W),\alpha\otimes\alpha')$$

where the map $\alpha\otimes\alpha'$ *is given in definition 86.*

Proposition 52. *Let* $f_1:X\to Y$ *and* $f_2:Y\to S$ *two morphism with* $X,Y,S\in\mathrm{QPVar}(k)$*.*

(i) Let $\mathcal{M}\in C(MHM_{gm,k,\mathbb{C}_p}(S)))$*. Then,*

$$(f_2\circ f_1)^{!Hdg}(\mathcal{M})=f_1^{!Hdg}f_2^{!Hdg}(\mathcal{M})\in D(MHM_{gm,k,\mathbb{C}_p}(X)).$$

(ii) Let $(M,F,W)\in C(MHM_{gm,k,\mathbb{C}_p}(S)))$*. Then,*

$$(f_2\circ f_1)^{*Hdg}(\mathcal{M})=f_1^{*Hdg}f_2^{*Hdg}(\mathcal{M})\in D(MHM_{gm,k,\mathbb{C}_p}(X))$$

Proof. Immediate from definition. □

Proposition 53. *Let* $S\in\mathrm{SmVar}(k)$*. Let* $D=V(s)\subset S$ *a (Cartier) divisor, where* $s\in\Gamma(S,L)$ *is a section of the line bundle* $L=L_D$ *associated to* D*.*

(i) Let $((M,F,W),(K,W),\alpha)\in MHM_{gm,k,\mathbb{C}_p}(S)$*. We have, using proposition 38, the canonical quasi-isomorphism in* $C(MHM_{gm,k,\mathbb{C}_p}(S))$ *:*

$$Is(M):=(Is(M),Is(K),0):$$
$$((M,F,W),(K,W),\alpha)\to(\psi_D((M,F,W),(K,W),\alpha)\xrightarrow{((c(x_{S^o/S}(M)),can(M)),(c(x(K)),can(K)),0)}$$
$$x_{S^o/S}((M,F,W),(K,W),\alpha)\oplus\phi_D((M,F,W),(K,W),\alpha)\xrightarrow{:=}$$
$$(x_{S^o/S}(M,F,W),x_{S^o/S}(K,W),x_{S^o/S}(\alpha))\oplus(\phi_D(M,F,W),\phi_D(K,W),\phi_D\alpha)$$
$$\xrightarrow{((\mathbb{D}c(x_{S^o/S}(\mathbb{D}M)),var(M)),(\mathbb{D}c(x_{S^o/S}(\mathbb{D}K)),var(K)),0)}\psi_D((M,F,W),(K,W),\alpha)).$$

(ii) We denote by $MHM_{gm,k,\mathbb{C}_p}(S\backslash D)\times_J MHM_{gm,k,\mathbb{C}_p}(D)$ *the category whose set of objects consists of*

$$\left\{(\mathcal{M},\mathcal{N},a,b),\mathcal{M}\in MHM_{gm,k,\mathbb{C}_p}(S\backslash D),\mathcal{N}\in MHM_{gm,k,\mathbb{C}_p}(D),a:\psi_{D1}\mathcal{M}\to N,b:N\to\psi_{D1}M\right\}$$

The functor (see definition 87)

$$(j^*,\phi_D,c,v):MHM_{gm,k,\mathbb{C}_p}(S)\to MHM_{gm,k,\mathbb{C}_p}(S\backslash D)\times_J MHM_{gm,k,\mathbb{C}_p}(D),$$
$$((M,F,W),(K,W),\alpha)\mapsto((j^*(M,F,W),j^*(K,W),j^*\alpha),\phi_D((M,F,W),(K,W),\alpha),can(-),var(-))$$

is an equivalence of category.

Proof. Follows from proposition 38. $\square$

Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i\in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I \in \mathrm{SmVar}(k)$. We have the category $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$

- whose set of objects is the set of triples $\{(((M_I,F,W),u_{IJ}),(K,W),\alpha)\}$ with

$$((M_I,F,W),u_{IJ}) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)),\ (K,W) \in D_{\mathbb{Z}_p fil,c,k}(S^{et}),$$
$$\alpha : \mathbb{B}_{dr,(\tilde{S}_I)}(K,W) \to F^0 DR(S)^{[-]}(((M_I,F,W),u_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ}))$$

  where α is a morphism in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$,

- and whose set of morphisms consists of

$$\phi = (\phi_D,\phi_C,[\theta]) : (((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1) \to (((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2)$$

  where $\phi_D : ((M_1,F,W),u_{IJ}) \to ((M_2,F,W),u_{IJ})$ and $\phi_C : (K_1,W) \to (K_2,W)$ are morphisms and

$$\theta = (\theta^\bullet, I(F^0DR(S)(\phi_D^{an}) \otimes I) \circ I(\alpha_1), I(\alpha_2) \circ I(\mathbb{B}_{dr,(\tilde{S}_I)}(\phi_C \otimes I))) :$$
$$I(\mathbb{B}_{dr,(\tilde{S}_I)}(K_1,W))[1] \to I(F^0DR(S)(((M_{2I},F,W),u_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I},F),t_{IJ})))$$

  is an homotopy, $I : D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p})) \to K_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$ being the injective resolution functor, and for

  - $\phi = (\phi_D,\phi_C,[\theta]) : (((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1) \to (((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2)$
  - $\phi' = (\phi'_D,\phi'_C,[\theta']) : (((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2) \to (((M_{3I},F,W),u_{IJ}),(K_3,W),\alpha_3)$

  the composition law is given by

$$\phi' \circ \phi := (\phi'_D \circ \phi_D, \phi'_C \circ \phi_C, I(DR(S)(\phi_D^{'an} \otimes I)) \circ [\theta] + [\theta'] \circ I(\mathbb{B}_{dr,(\tilde{S}_I)}(\phi_C))[1]) :$$
$$(((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1) \to (((M_{3I},F,W),u_{IJ}),(K_3,W),\alpha_3),$$

  in particular for $(((M_I,F,W),u_{IJ}),(K,W),\alpha) \in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$,

$$I_{(((M_I,F,W),u_{IJ}),(K,W),\alpha)} = ((I_{M_I}), I_K, 0),$$

and also the category $D_{\mathcal{D}(1,0)fil,rh,\infty}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$ defined in the same way, together with the localization functor

$$(D(zar),I) : C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{et}) \to D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{et})$$
$$\to D_{\mathcal{D}(1,0)fil,rh,\infty}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{et}).$$

Note that if $\phi = (\phi_D,\phi_C,[\theta]) : (((M_1,F,W),u_{IJ}),(K_1,W),\alpha_1) \to (((M_2,F,W),u_{IJ}),(K_2,W),\alpha_2)$ is a morphism in $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$ such that ϕ_D and ϕ_C are isomorphism then ϕ is an isomorphism (see remark 8). Moreover,

- For $(((M_I,F,W),u_{IJ}),(K,W),\alpha) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$, we set

$$(((M_I,F,W),u_{IJ}),(K,W),\alpha)[1] := (((M_I,F,W),u_{IJ})[1],(K,W)[1],\alpha[1]).$$

- For

$$\phi = (\phi_D,\phi_C,[\theta]) : (((M_{1I},F,W),u_{IJ}),(K_1,W),\alpha_1) \to (((M_{2I},F,W),u_{IJ}),(K_2,W),\alpha_2)$$

  a morphism in $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p,fil,c,k}(S^{et})$, we set (see [11] definition 3.12)

$$\mathrm{Cone}(\phi) := (\mathrm{Cone}(\phi_D), \mathrm{Cone}(\phi_C), ((\alpha_1,\theta),(\alpha_2,0))) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et}),$$

  $((\alpha_1,\theta),(\alpha_2,0))$ being the matrix given by the composition law, together with the canonical maps

- $c_1(-) = (c_1(\phi_D), c_1(\phi_C), 0) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \to \mathrm{Cone}(\phi)$
- $c_2(-) = (c_2(\phi_D), c_2(\phi_C), 0) : \mathrm{Cone}(\phi) \to (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1)[1]$.

We have then the following :

Theorem 49. *(i) Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i\in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then the full embedding*

$$\iota_S : MHM_{gm,k,\mathbb{C}_p}(S) \hookrightarrow \mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I P_{\mathbb{Z}_p fil,k}(S^{et}) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$$

induces a full embedding

$$\iota_S : D(MHM_{gm,k,\mathbb{C}_p}(S)) \hookrightarrow D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$$

whose image consists of $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$ such that

$$((H^n(M_I, F, W), H^n(u_{IJ})), H^n(K, W), H^n\alpha) \in MHM_{gm,k,\mathbb{C}_p}(S)$$

for all $n \in \mathbb{Z}$ and such that for all $p \in \mathbb{Z}$, the differentials of $\mathrm{Gr}^p_W(M_I, F)$ are strict for the filtrations F.

(i)' Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i\in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then,

$$\begin{aligned} D(MHM_{gm,k,\mathbb{C}_p}(S)) =< (\int_f^{FDR} (n \times I)_{!Hdg}(\Gamma^{\vee,Hdg}_X(O_{\mathbb{P}^{N,o}\times\tilde{S}_I}, F_b), x_{IJ})(d), Rf_*\mathbb{Z}^w_{p,X^{et}}, f_*\alpha(X)), \\ (f : X \xrightarrow{l} \mathbb{P}^{N,o} \times S \xrightarrow{p} S) \in \mathrm{QPVar}(k),\ d \in \mathbb{Z} > \\ =< (\int_f^{FDR} ((\Gamma^{\vee,Hdg}_X(O_{\mathbb{P}^{N,o}\times\tilde{S}_I}, F_b), x_{IJ})(d), Rf_*\mathbb{Z}_{p,X^{et}}, f_*\alpha(X)), \\ (f : X \xrightarrow{l} \mathbb{P}^{N,o} \times S \xrightarrow{p} S) \in \mathrm{QPVar}(k),\ proper,\ X smooth > \\ \subset D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et}) \end{aligned}$$

where $n : \mathbb{P}^{N,o} \hookrightarrow \mathbb{P}^N$ are open embeddings, l are closed embedding and $<,>$ means the full triangulated category generated by and $(-)$ is the shift of the F-filtration.

(ii) Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i\in I} S_i$ an open cover such that there exists closed embedding $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then the full embedding

$$\iota_S : MHM_{gm,k,\mathbb{C}_p}(S) \hookrightarrow \mathrm{PSh}^0_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I P_{\mathbb{Z}_p fil,k}(S^{et}) \hookrightarrow C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$$

induces a full embedding

$$\iota_S : D(MHM_{gm,k,\mathbb{C}_p}(S)) \hookrightarrow D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$$

whose image consists of $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$ such that

$$((H^n(M_I, F, W), H^n(u_{IJ})), H^n(K, W), H^n\alpha) \in MHM_{gm,k,\mathbb{C}_p}(S)$$

for all $n \in \mathbb{Z}$ and such that there exist $r \in \mathbb{Z}$ and an r-filtered homotopy equivalence $((M_I, F, W), u_{IJ}) \to ((M'_I, F, W), u_{IJ})$ such that for all $p \in \mathbb{Z}$ the differentials of $\mathrm{Gr}^p_W(M'_I, F)$ are strict for the filtrations F.

Proof. (i): We first show that ι_S is fully faithfull, that is for all $\mathcal{M}=(((M_I,F,W),u_{IJ}),(K,W),\alpha),\mathcal{M}'=(((M'_I,F,W),u_{IJ}),(K',W),\alpha')\in MHM_{gm,k,\mathbb{C}_p}(S)$ and all $n\in\mathbb{Z}$,

$$\iota_S:\mathrm{Ext}^n_{D(MHM_{gm,k,\mathbb{C}_p}(S))}(\mathcal{M},\mathcal{M}'):=\mathrm{Hom}_{D(MHM_{gm,k,\mathbb{C}_p}(S))}(\mathcal{M},\mathcal{M}'[n])$$
$$\to\mathrm{Ext}^n_{\mathcal{D}(S)}(\mathcal{M},\mathcal{M}'):=\mathrm{Hom}_{\mathcal{D}(S):=D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))\times_I D_{fil}(S^{et})}(\mathcal{M},\mathcal{M}'[n])$$

For this it is enough to assume S smooth. We then proceed by induction on $max(\dim\mathrm{supp}(M),\dim\mathrm{supp}(M'))$.

- For $\mathrm{supp}(M)=\mathrm{supp}(M')=\{s\}$, it is the theorem for mixed hodge complexes or absolute Hodge complexes, see [11]. If $\mathrm{supp}(M)=\{s\}$ and $\mathrm{supp}(M')=\{s'\}$ and $s'\neq s$, then by the localization exact sequence
$$\mathrm{Ext}^n_{D(MHM_{gm,k,\mathbb{C}_p}(S))}(\mathcal{M},\mathcal{M}')=0=\mathrm{Ext}^n_{\mathcal{D}(S)}(\mathcal{M},\mathcal{M}')$$

- Denote $\mathrm{supp}(M)=Z\subset S$ and $\mathrm{supp}(M')=Z'\subset S$. There exist an open subset $S^o\subset S$ such that $Z^o:=Z\cap S^o$ and $Z'^o:=Z'\cap S^o$ are smooth, and $\mathcal{M}_{|Z^o}:=((i^*\mathrm{Gr}_{V_{Z^o},0}M_{|S^o},F,W),i^*j^*(K,W),\alpha^*(i))\in MHM_{gm,k}(Z^o)$ and $\mathcal{M}'_{|Z'^o}:=((i'^*\mathrm{Gr}_{V_{Z'^o},0}M'_{|S^o},F,W),i'^*j^*K,\alpha^*(i'))\in MHM_{gm,k}(Z'^o)$ are variation of geometric mixed Hodge structure over $k\subset\mathbb{C}$, where $j:S^o\hookrightarrow S$ is the open embedding, and $i:Z^o\hookrightarrow S^o$, $i':Z'^o\hookrightarrow S^o$ the closed embeddings. Considering the connected components of Z^o and Z'^o, we way assume that Z^o and Z'^o are connected. Shrinking S^o if necessary, we may assume that either $Z^o=Z'^o$ or $Z^o\cap Z'^o=\emptyset$, We denote $D=S\backslash S^o$. Shrinking S^o if necessary, we may assume that D is a divisor and denote by $l:S\hookrightarrow L_D$ the zero section embedding.
  - If $Z^o=Z'^o$, denote $i:Z^o\hookrightarrow S^o$ the closed embedding. We have then the following commutative diagram
$$\begin{tikzcd}
\mathrm{Ext}^n_{D(MHM_{gm,k,\mathbb{C}_p}(S^o))}(\mathcal{M}_{|S^o},\mathcal{M}'_{|S^o}) \arrow[r, "\iota_{S^o}"] \arrow[d, shift left, "(i_{*mod},i_*,\alpha_*(i))"'] & \mathrm{Ext}^n_{\mathcal{D}(S^o)}(\mathcal{M}_{|S^o},\mathcal{M}'_{|S^o}) \arrow[d, shift left, "(i^*\mathrm{Gr}_{V_{Z^o},0},i^*,\alpha^*(i))"] \\
\mathrm{Ext}^n_{D(MHM_{gm,k,\mathbb{C}_p}(Z^o))}(\mathcal{M}_{|Z^o},\mathcal{M}'_{|Z^o}) \arrow[r, "\iota_{Z^o}"] \arrow[u, shift left, "(i^*\mathrm{Gr}_{V_{Z^o},0},i^*,\alpha^*(i))"] & \mathrm{Ext}^n_{\mathcal{D}(Z^o)}(\mathcal{M}_{|Z^o},\mathcal{M}'_{|Z^o}) \arrow[u, shift left, "(i_{*mod},i_*,\alpha_*(i))"]
\end{tikzcd}$$
Now we prove that ι_{Z^o} is an isomorphism similarly to the proof the the generic case of [6]. On the other hand the left and right colummn are isomorphisms. Hence ι_{S^o} is an isomorphism by the diagram.
  - If $Z^o\cap Z'^o=\emptyset$, we consider the following commutative diagram
$$\begin{tikzcd}
\mathrm{Ext}^n_{D(MHM_{gm,k,\mathbb{C}_p}(S^o))}(\mathcal{M}_{|S^o},\mathcal{M}'_{|S^o}) \arrow[r, "\iota_{S^o}"] \arrow[d, shift left, "(i_{*mod},i_*,\alpha_*(i))"'] & \mathrm{Ext}^n_{\mathcal{D}(S^o)}(\mathcal{M}_{|S^o},\mathcal{M}'_{|S^o}) \arrow[d, shift left, "(i^*\mathrm{Gr}_{V_{Z^o},0},i^*,\alpha^*(i))"] \\
\mathrm{Ext}^n_{D(MHM_{gm,k,\mathbb{C}_p}(Z^o))}(\mathcal{M}_{|Z^o},0)=0 \arrow[r, "\iota_{Z^o}"] \arrow[u, shift left, "(i^*\mathrm{Gr}_{V_{Z^o},0},i^*,\alpha^*(i))"] & \mathrm{Ext}^n_{\mathcal{D}(Z^o)}(\mathcal{M}_{|Z^o},0)=0 \arrow[u, shift left, "(i_{*mod},i_*,\alpha_*(i))"]
\end{tikzcd}$$
where the left and right column are isomorphism by strictness of the V_{Z^o} filtration (use a bi-filtered injective resolution with respect to F and V_{Z^o} for the right column).

- We consider now the following commutative diagram in $C(\mathbb{Z})$ where we denote for short $H:=D(MHM_{gm,k,\mathbb{C}_p}(S))$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}^\bullet_H(\Gamma^{\vee,Hdg}_D\mathcal{M},\Gamma^{Hdg}_D\mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\gamma^{Hdg}_D(\mathcal{M}'))"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_H(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\mathrm{ad}(j^*,j_{*Hdg})(\mathcal{M}'))"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_H(\Gamma^{\vee,Hdg}_D\mathcal{M},j_{*Hdg}j^*\mathcal{M}') \arrow[r] \arrow[d, "\iota_S"] & 0 \\
0 \arrow[r] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\Gamma^{Hdg}_D\mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\gamma^{Hdg}_D(\mathcal{M}'))"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},\mathcal{M}') \arrow[r, "\mathrm{Hom}(-,\mathrm{ad}(j^*,j_{*Hdg})(\mathcal{M}'))"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M},j_{*Hdg}j^*\mathcal{M}') \arrow[r] & 0
\end{tikzcd}$$

whose lines are exact sequence. We have on the one hand,

$$\mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}_p}(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M}, j_{*Hdg}j^*\mathcal{M}') = 0 = \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M}, j_{*Hdg}j^*\mathcal{M}')$$

On the other hand by induction hypothesis

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}_p}(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M}, \Gamma^{Hdg}_D\mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M}, \Gamma^{Hdg}_D\mathcal{M}')$$

is a quasi-isomorphism. Hence, by the diagram

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}_p}(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M}, \mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M}, \mathcal{M}')$$

is a quasi-isomorphism.

- We consider now the following commutative diagram in $C(\mathbb{Z})$ where we denote for short $H := D(MHM_{gm,k,\mathbb{C}_p}(S))$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}^\bullet_H(\Gamma^{\vee,Hdg}_D\mathcal{M}, \mathcal{M}') \arrow[r, "\mathrm{Hom}(\gamma^{\vee,Hdg}_D(\mathcal{M}),-)"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_H(\mathcal{M}, \mathcal{M}') \arrow[r, "\mathrm{Hom}(\mathrm{ad}(j_{!Hdg},j^*)(\mathcal{M}'),-)"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_H(j_{!Hdg}j^*\mathcal{M}, \mathcal{M}') \arrow[r] \arrow[d, "\iota_S"] & 0 \\
0 \arrow[r] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M}, \mathcal{M}') \arrow[r, "\mathrm{Hom}(\gamma^{\vee,Hdg}_D(\mathcal{M}),-)"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\mathcal{M}, \mathcal{M}') \arrow[r, "\mathrm{Hom}(\mathrm{ad}(j_{!Hdg},j^*)(\mathcal{M}),-)"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(j_{!Hdg}j^*\mathcal{M}, \mathcal{M}') \arrow[r] & 0
\end{tikzcd}$$

whose lines are exact sequence. On the one hand, the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}_p}(S))}(j_{!Hdg}j^*\mathcal{M}, \mathcal{M}') \arrow[r, "j^*"] \arrow[d, "\iota_S"] & \mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}_p}(S^o))}(j^*\mathcal{M}, j^*\mathcal{M}') \arrow[d, "\iota_{S^o}"] \\
\mathrm{Hom}^\bullet_{\mathcal{D}(S)}(j_{!Hdg}j^*\mathcal{M}, \mathcal{M}') \arrow[r, "j^*"] & \mathrm{Hom}^\bullet_{\mathcal{D}(S^o)}(j^*\mathcal{M}, j^*\mathcal{M}')
\end{tikzcd}$$

together with the fact that the horizontal arrows j^* are quasi-isomorphism by the functoriality given the uniqueness of the V_S filtration for the embedding $l : S \hookrightarrow L_D$, (use a bi-filtered injective resolution with respect to F and V_S for the lower arrow) and the fact that ι_{S^o} is a quasi-isomorphism by the first two point, show that

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}_p}(S))}(j_{!Hdg}j^*\mathcal{M}, \mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(j_{!Hdg}j^*\mathcal{M}, \mathcal{M}')$$

is a quasi-isomorphism. On the other hand, by the third point

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}_p}(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M}, \mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M}, \mathcal{M}')$$

is a quasi-isomorphism. Hence, by the diagram

$$\iota_S : \mathrm{Hom}^\bullet_{D(MHM_{gm,k,\mathbb{C}_p}(S))}(\Gamma^{\vee,Hdg}_D\mathcal{M}, \mathcal{M}') \to \mathrm{Hom}^\bullet_{\mathcal{D}(S)}(\Gamma^{\vee,Hdg}_D\mathcal{M}, \mathcal{M}')$$

is a quasi-isomorphism.

This shows the fully faithfulness. We now prove the essential surjectivity : let

$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I C_{fil}(S^{et})$$

such that the cohomology are mixed hodge modules and such that the differential are strict. We proceed by induction on $card\{n \in \mathbb{Z}\}$, s.t.$H^n(M_I, F, W) \neq 0$ by taking for the cohomological troncation

$$\tau^{\leq n}(((M_I, F, W), u_{IJ}), (K, W), \alpha) := ((\tau^{\leq n}(M_I, F, W), \tau^{\leq n}u_{IJ}), \tau^{\leq n}(K, W), \tau^{\leq n}\alpha)$$

and using the fact that the differential are strict for the filtration F and the fully faithfullness.
(i)':Follows from (i).
(ii):Follows from (i).Indeed, in the composition of functor

$$\iota_S : D(MHM_{gm,k,\mathbb{C}_p}(S)) \xrightarrow{\iota_S} D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_pfil,c,k}(S^{et}) \\ \to D_{\mathcal{D}(1,0)fil,\infty,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_pfil,c,k}(S^{et})$$

the second functor which is the localization functor is an isomorphism on the full subcategory

$$D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))^{st} \times_I D_{\mathbb{Z}_pfil,c,k}(S^{et}) \subset D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_pfil,c,k}(S^{et})$$

constisting of complex such that the differentials are strict for F, and the first functor ι_S is a full embedding by (i) and $\iota_S(D(MHM_{gm,k,\mathbb{C}_p}(S))) \subset D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I))^{st} \times_I D_{\mathbb{Z}_pfil,c,k}(S^{et})$. □

Definition 94. *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. Let $\bar{Y} \in \mathrm{PSmVar}(k)$ a smooth compactification of Y with $n : Y \hookrightarrow \bar{Y}$ the open embedding. Then $\bar{f} : \bar{X} \xrightarrow{\bar{l}} \bar{Y} \times_S \xrightarrow{\bar{p}_S} S$ is a compactification of f, with $\bar{X} \subset \bar{Y} \times S$ the closure of X and $\bar{l}$ the closed embedding, we denote by $n' : X \hookrightarrow \bar{X}$ the closed embedding so that $f = \bar{f} \circ n'$.*

- *(i) For $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C(MHM_{gm,k,\mathbb{C}_p}(X))$, we define, using definition 67 and theorem 49,*

$$Rf_{*Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) := \iota_S^{-1}(\int_f^{Hdg}((M_I, F, W), u_{IJ}), Rf_{*w}(K, W), f_*(\alpha)) \\ \in D(MHM_{gm,k,\mathbb{C}_p}(S))$$

  where $f_*(\alpha)$ is given in definition 86, and since

  - *by definition*

  $$H^i(\int_{\bar{f}}^{FDR} \mathrm{Gr}_W^k (n \times I)_{*Hdg}((M_I, F, W), u_{IJ}), R\bar{f}_* \mathrm{Gr}_W^k n'_{*w}(K, W), \mathrm{Gr}_W^k f_*\alpha) \in HM_{gm,k,\mathbb{C}_p}(S)$$

  for all $i, k \in \mathbb{Z}$, hence by the spectral sequence for the filtered complexes $\int_{\bar{f}}^{FDR}(n \times I)_{*Hdg}((M_I, W), u_{IJ})$ and $R\bar{f}_* n'_{*w}(K, W)$

  $$\mathrm{Gr}_W^k H^i(\int_f^{Hdg}((M_I, F, W), u_{IJ}), Rf_{*w}(K, W), f_*\alpha) := \\ (\mathrm{Gr}_W^k H^i \int_{\bar{f}}^{FDR}((M_I, F, W), u_{IJ}), \mathrm{Gr}_W^k H^i R\bar{f}_* n'_{*w}(K, W), \mathrm{Gr}_W^k H^i f_*\alpha) \in HM_{gm,k,\mathbb{C}_p}(S)$$

  this gives by definition $H^i(\int_f^{Hdg}((M_I, F, W), u_{IJ}), Rf_{*w}(K, W), f_*(\alpha)) \in MHM_{gm,k,\mathbb{C}_p}(S)$ for all $i \in \mathbb{Z}$.
  - *$\int_f^{Hdg}((M_I, F, W), u_{IJ})$ is the class of a complex such that the differential are strict for F by theorem 40 in the complex case.*

- *(ii) For $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C(MHM_{gm,k,\mathbb{C}_p}(X))$, we define, using definition 66 and theorem 49,*

$$Rf_{!Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) := \iota_S^{-1}(\int_{f!}^{Hdg}((M_I, F, W), u_{IJ}), Rf_{!w}(K, W), f_!(\alpha)) \\ \in D(MHM_{gm,k,\mathbb{C}_p}(S))$$

  where $f_!(\alpha)$ is given in definition 86, and since

– *by definition*

$$H^i(\int_{\bar{f}}^{FDR}\mathrm{Gr}_W^k(n\times I)_{!Hdg}((M_I,F,W),u_{IJ}),R\bar{f}_*\mathrm{Gr}_W^k n'_{!w}(K,W),\mathrm{Gr}_W^k f_!\alpha)\in HM_{gm,k,\mathbb{C}_p}(S)$$

for all $i,k\in\mathbb{Z}$, *hence by the spectral sequence for the filtered complexes* $\int_{\bar{f}}^{FDR}(n\times I)_{!Hdg}((M_I,W),u_{IJ})$ *and* $R\bar{f}_*n'_{!w}(K,W)$

$$\mathrm{Gr}_W^k H^i(\int_{f!}^{Hdg}((M_I,F,W),u_{IJ}),Rf_{!w}K,f_!\alpha):=$$

$$(\mathrm{Gr}_W^k H^i\int_{\bar{f}}^{FDR}(n\times I)_{!Hdg}((M_I,F,W),u_{IJ}),\mathrm{Gr}_W^k H^iR\bar{f}_*n'_{!w}(K,W),\mathrm{Gr}_W^k H^if_!\alpha)\in HM_{gm,k,\mathbb{C}_p}(S)$$

this gives by definition $H^i(\int_{f!}^{Hdg}((M_I,F,W),u_{IJ}),Rf_{!w}(K,W),f_!(\alpha))\in MHM_{gm,k,\mathbb{C}_p}(S)$ *for all* $i\in\mathbb{Z}$.

– $\int_{f!}^{Hdg}((M_I,F,W),u_{IJ})$ *is the class of a complex such that the differential are strict for* F *by theorem 40 in the complex case.*

Proposition 54. *Let* $f_1:X\to Y$ *and* $f_2:Y\to S$ *two morphism with* $X,Y,S\in\mathrm{QPVar}(k)$.

(i) Let $\mathcal{M}\in C(MHM_{gm,k,\mathbb{C}_p}(X))$. *Then,*

$$R(f_2\circ f_1)_*^{Hdg}(\mathcal{M})=Rf_{2*}^{Hdg}Rf_{1*}^{Hdg}(\mathcal{M})\in D(MHM_{gm,k,\mathbb{C}_p}(S)).$$

(ii) Let $\mathcal{M}\in C(MHM_{gm,k,\mathbb{C}_p}(X))$. *Then,*

$$R(f_2\circ f_1)_!^{Hdg}(\mathcal{M})=Rf_{2!}^{Hdg}Rf_{1!}^{Hdg}(\mathcal{M})\in D(MHM_{gm,k,\mathbb{C}_p}(S))$$

Proof. Immediate from definition. □

Let $k\subset K\subset\mathbb{C}_p$ a subfield of a p-adic field K. Definition 92, definition 94 and gives by proposition 52 and proposition 54 respectively, the following 2 functors :

- We have the following 2 functor on the category of algebraic varieties over $k\subset\mathbb{C}_p$

$$D(MHM_{gm,k,\mathbb{C}_p}(\cdot)):\mathrm{QPVar}(k)\to\mathrm{TriCat},\ S\mapsto D(MHM_{gm,k,\mathbb{C}_p}(S)),$$
$$(f:T\to S)\longmapsto(f^{*Hdg}:(((M_I,F,W),u_{IJ}),(K,W),\alpha)\mapsto$$
$$f^{!Hdg}(((M_I,F,W),u_{IJ}),(K,W),\alpha):=(f_{Hdg}^{*mod}(((M_I,F,W),u_{IJ})),f^{!w}(K,W),f^!\alpha)).$$

  see definition 64 and definition 86 for the equality.

- We have the following 2 functor on the category of quasi-projective algebraic varieties over $k\subset\mathbb{C}_p$

$$D(MHM_{gm,k,\mathbb{C}_p}(\cdot)):\mathrm{QPVar}(k)\to\mathrm{TriCat},\ S\mapsto D(MHM_{gm,k,\mathbb{C}_p}(S)),$$
$$(f:T\to S)\longmapsto(f_{*Hdg}:(((M_I,F,W),u_{IJ}),(K,W),\alpha)\mapsto Rf_{*Hdg}(((M_I,F,W),u_{IJ}),(K,W),\alpha)).$$

- We have the following 2 functor on the category of quasi-projective algebraic varieties over $k\subset\mathbb{C}_p$

$$D(MHM_{gm,k,\mathbb{C}_p}(\cdot)):\mathrm{QPVar}(k)\to\mathrm{TriCat},\ S\mapsto D(MHM_{gm,k,\mathbb{C}_p}(S)),$$
$$(f:T\to S)\longmapsto(f_{!Hdg}:(((M_I,F,W)),(K,W),\alpha)\mapsto f_{!Hdg}(((M_I,F,W),u_{IJ}),(K,W),\alpha)).$$

- We have the following 2 functor on the category of algebraic varieties over $k \subset \mathbb{C}_p$

$$D(MHM_{gm,k,\mathbb{C}_p}(\cdot)) : \mathrm{QPVar}(k) \to \mathrm{TriCat},\ S \mapsto D(MHM_{gm,k,\mathbb{C}_p}(S)),$$
$$(f : T \to S) \longmapsto (f^{!Hdg} : (((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto$$
$$f^{*Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) := (f^{\hat{*}mod}_{Hdg}(((M_I, F, W), u_{IJ})), f^{*w}(K, W), f^*\alpha)).$$

see definition 64 and definition 86 for the equality.

Proposition 55. *Let $f : X \to S$ with $S, X \in \mathrm{QPVar}(k)$. Then*

(i) $(f^{*Hdg}, Rf^{Hdg}_*) : D(MHM_{gm,k,\mathbb{C}_p}(S)) \to D(MHM_{gm,k,\mathbb{C}_p}(X))$ *is a pair of adjoint functors.*

- *For* $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C(MHM_{gm,k,\mathbb{C}_p}(S))$,

$$\mathrm{ad}(f^{*Hdg}, Rf^{Hdg}_*)(((M_I, F, W), u_{IJ}), (K, W), \alpha) :=$$
$$(\mathrm{ad}(f^{\hat{*}mod}_{Hdg}, Rf^{Hdg}_*)((M_I, F, W), u_{IJ}), \mathrm{ad}(f^{*w}, Rf_{*w})(K, W)) :$$
$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \to Rf^{Hdg}_* f^{*Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha)$$

is the adjonction map in $D(MHM_{gm,k,\mathbb{C}_p}(S))$.

- *For* $(((N_I, F, W), u_{IJ}), (P, W), \beta) \in C(MHM_{gm,k,\mathbb{C}_p}(X))$,

$$\mathrm{ad}(f^{*Hdg}, Rf^{Hdg}_*)(((N_I, F, W), u_{IJ}), (P, W), \beta) :=$$
$$(\mathrm{ad}(f^{\hat{*}mod}_{Hdg}, Rf^{Hdg}_*)((N_I, F, W), u_{IJ}), \mathrm{ad}(f^{*w}, Rf_{*w})(P, W)) :$$
$$f^{*Hdg} Rf^{Hdg}_*(((N_I, F, W), u_{IJ}), (P, W), \beta) \to (((N_I, F, W), u_{IJ}), (P, W), \beta)$$

is the adjonction map in $D(MHM_{gm,k,\mathbb{C}_p}(X))$

(ii) $(Rf^{Hdg}_!, f^{!Hdg}) : D(MHM_{gm,k,\mathbb{C}_p}(X)) \to D(MHM_{gm,k,\mathbb{C}_p}(S))$ *is a pair of adjoint functors.*

- *For* $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C(MHM_{gm,k,\mathbb{C}_p}(S))$,

$$\mathrm{ad}(Rf^{Hdg}_!, f^{!Hdg})(((M_I, F, W), u_{IJ}), (K, W), \alpha) :=$$
$$(\mathrm{ad}(f^{*mod}_{Hdg}, Rf^{Hdg}_!)((M_I, F, W), u_{IJ}), \mathrm{ad}(f^{!w}, Rf_{!w})(K, W)) :$$
$$Rf^{Hdg}_! f^{!Hdg}(((M_I, F, W), u_{IJ}), (K, W), \alpha) \to (((M_I, F, W), u_{IJ}), (K, W), \alpha)$$

is the adjonction map in $D(MHM_{gm,k,\mathbb{C}_p}(S))$.

- *For* $(((N_I, F, W), u_{IJ}), (P, W), \beta) \in C(MHM_{gm,k,\mathbb{C}_p}(X))$,

$$\mathrm{ad}(Rf^{Hdg}_!, f^{!Hdg})(((N_I, F, W), u_{IJ}), (P, W), \beta) :=$$
$$(\mathrm{ad}(f^{*mod}_{Hdg}, Rf^{Hdg}_!)((N_I, F, W), u_{IJ}), \mathrm{ad}(f^{!w}, Rf_{!w})(P, W)) :$$
$$(((N_I, F, W), u_{IJ}), (P, W), \beta) \to f^{!Hdg} Rf^{Hdg}_!(((N_I, F, W), u_{IJ}), (P, W), \beta)$$

is the adjonction map in $D(MHM_{gm,k,\mathbb{C}_p}(X))$.

Proof. Follows from proposition 51 after considering a factorization $f : X \hookrightarrow \bar{Y} \times S \xrightarrow{p_S} S$ with $\bar{Y} \in \mathrm{PSmVar}(k)$. □

Theorem 50. *Let $k \subset \mathbb{C}_p$ a subfield.*

(i) We have the six functor formalism on $D(MHM_{gm,k,\mathbb{C}_p}(-)) : \mathrm{SmVar}(k) \to \mathrm{TriCat}$.

(ii) We have the six functor formalism on $D(MHM_{gm,k,\mathbb{C}_p}(-)) : \mathrm{QPVar}(k) \to \mathrm{TriCat}$.

Proof. Follows from proposition 55. □

We give the following version (where the De Rham cohomology is twisted by the p-adic periods) of the syntomic complex of a p adic analytic space and the syntomic cohomology class of an algebraic cycle of a p adic algebraic variety.

Definition 95. *(i) Let K a p adic field. Let $X \in \mathrm{AnSm}(K)$. We have for $d \in \mathbb{Z}$ the syntomic complex*

$$\mathbb{Z}_{syn,X}(d) := (\mathbb{Z}_{p,X}(d) \hookrightarrow DR(X)(O\mathbb{B}_{dr,X})/F_b^d := (\Omega_X^{\bullet,\leq d} \otimes_{O_X} O\mathbb{B}_{dr,X})) \in C(X^{et})$$

Let $D \subset X$ a normal crossing divisor. We have for $d \in \mathbb{Z}$ the Deligne complexes

$$\mathbb{Z}_{syn,(X,D)}(d) := (\mathbb{Z}_X(d) \hookrightarrow DR(X)(O\mathbb{B}_{dr,X}(\log D))/F_b^d) := (\Omega_X^{\bullet,\leq d} \otimes_{O_X} O\mathbb{B}_{dr,X}(\log D)) \in C(X^{et})$$

and

$$\mathbb{Z}_{syn,(X,D)}(d)^\vee := (\mathbb{Z}_X(d) \hookrightarrow DR(X)(O\mathbb{B}_{dr,X}(\mathrm{nul}\, D))/F_b^d) := (\Omega_X^{\bullet,\leq d} \otimes_{O_X} O\mathbb{B}_{dr,X}(\mathrm{nul}\, D)) \in C(X^{et}).$$

Moreover we have as for Deligne complexes canonical products

- $(-) \cdot (-) : \mathbb{Z}_{syn,(X,D)}(d) \otimes \mathbb{Z}_{syn,(X,D)}(d') \to \mathbb{Z}_{\mathcal{D},(X,D)}(d+d')$
- $(-) \cdot (-) : \mathbb{Z}_{syn,(X,D)}(d)^\vee \otimes \mathbb{Z}_{syn,(X,D)}(d')^\vee \to \mathbb{Z}_{syn,(X,D)}(d+d')^\vee$

(ii) Let K a p adic field. Let $X \in \mathrm{AnSm}(K)$. We have for $d \in \mathbb{Z}$ the syntomic (cohomology) complex

$$C^\bullet_{syn}(X, \mathbb{Z}(d)) := \mathrm{Cone}(\Gamma(X, E_{et}(\mathbb{Z}_{p,X})) \oplus \Gamma(X, F^d E_{et}(DR(X)(O\mathbb{B}_{dr,X}), F_b)) \hookrightarrow \Gamma(X, E_{et}(DR(X)(O\mathbb{B}_{dr,X})))) \in C(\mathbb{Z}_p)$$

Let $D \subset X$ a normal crossing divisor. Denote $U := X \backslash D$. We have for $d \in \mathbb{Z}$ the syntomic (cohomology) complexes

$$C^\bullet_{syn}((X,D), \mathbb{Z}(d)) := \mathrm{Cone}(\Gamma(X, E_{et}(\mathbb{Z}_{p,X})) \oplus \Gamma(X, F^d E_{et}(DR(X)(O\mathbb{B}_{dr,X}(\log D)), F_b)) \hookrightarrow \Gamma(X, E_{et}(DR(X)(O\mathbb{B}_{dr,X}(\log D)))))) \in C(\mathbb{Z}_p)$$

and

$$C^\bullet_{syn}(X, D, \mathbb{Z}(d)) := \mathrm{Cone}(\Gamma(X, E_{et}(\mathbb{Z}_{p,X})) \oplus \Gamma(X, F^d E_{et}(DR(X)(O\mathbb{B}_{dr,X}(\mathrm{nul}\, D)), F_b)) \hookrightarrow \Gamma(X, E_{et}(DR(X)(O\mathbb{B}_{dr,X}(\mathrm{nul}\, D))))) \in C(\mathbb{Z}_p).$$

(iii) Let $k \subset K$ an embedding of a field of characteristic zero into a p adic field. Let $X \in \mathrm{PSmVar}(k)$. We have, for $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$, the syntomic cohomology

$$H^k_{syn}(X_K^{an}, \mathbb{Z}(d)) := \mathbb{H}^k(X_K^{an}, \mathbb{Z}_{X,syn}(d)) = H^k C^\bullet_{syn}(X_K^{an}, D, \mathbb{Z}(d))$$

Let $U \in \mathrm{SmVar}(k)$. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X \backslash U$ a normal crossing divisor. We have, for $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$, the syntomic cohomology

$$H^k_{syn}(U_K^{an}, \mathbb{Z}(d)) := \mathbb{H}^k(X, \mathbb{Z}_{(X_K^{an}, D_K^{an}),syn}(d)) = H^k C^\bullet_{syn}((X_K^{an}, D_K^{an}), \mathbb{Z}(d))$$

and

$$H^k_{syn}(X, D, \mathbb{Z}(d)) := \mathbb{H}^k(X_K^{an}, \mathbb{Z}_{(X_K^{an}, D_K^{an}),syn}(d)^\vee) = H^k C^\bullet_{syn}(X_K^{an}, D_K^{an}, \mathbb{Z}(d)).$$

(iv) *Let $k \subset K \subset \mathbb{C}_p$ an embedding of a field of characteristic zero into a p adic field. Let $U \in \mathrm{SmVar}(k)$. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X \backslash U$ a normal crossing divisor. We define the Deligne cohomology of a (higher) cycle $Z \in \mathcal{Z}^d(U,n)^{\partial=0}$ by*

$$[Z]_{syn} := \mathrm{Im}(H^{2d-n}(\gamma_{\mathrm{supp}(Z)})([Z])),$$
$$H^k(\gamma_{\mathrm{supp}(Z)}) : \mathbb{H}^{2d-n}_{syn,\mathrm{supp}(Z)}(X^{an}_{\mathbb{C}_p}, \mathbb{Z}_{X^{an}_{\mathbb{C}_p},D^{an}_{\mathbb{C}_p}}(d)) \to \mathbb{H}^{2d-n}_{syn}(X^{an}_{\mathbb{C}_p}, \mathbb{Z}_{X^{an}_{\mathbb{C}_p},D^{an}_{\mathbb{C}_p}}(d))$$

with $\mathrm{supp}(Z) := p_X(\mathrm{supp}(Z)) \subset X$*, where* $\mathrm{supp}(Z) \subset X \times \square^n$ *is the support of* Z.

(v) *Let $k \subset K$ an embedding of a field of characteristic zero into a p adic field. Let $U \in \mathrm{SmVar}(k)$. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X \backslash U$ a normal crossing divisor. We have for $d \in \mathbb{Z}$ the morphism of complexes*

$$\mathcal{R}^d_U : \mathcal{Z}^d(U, \bullet) \to C^{\bullet}_{syn}(X^{an}_{\mathbb{C}_p}, D^{an}_{\mathbb{C}_p}, \mathbb{Z}(d)),\ Z \mapsto \mathcal{R}^d_U(Z) := (T_{\bar{Z}}, \Omega_{\bar{Z}}, R_{\bar{Z}})$$

which gives for $Z \in \mathcal{Z}^d(U,n)^{\partial=0}$,

$$[\mathcal{R}^d_U(Z)] = [Z]_{syn} \in H^{2d-n}_{syn}(U^{an}_{\mathbb{C}_p}, \mathbb{Z}(d))$$

Let K a p adic field. Let $f : X \to S$ a morphism with $S, X \in \mathrm{AnSm}(K)$. We have for $d \in \mathbb{Z}$ the canonical morphism of Deligne complexes

$$(\mathrm{ad}(f^*, f_*)(\mathbb{Z}_{p,S}), \Omega^{\leq d}_{X/S}) : \mathbb{Z}_{syn,S}(d) \to f_*\mathbb{Z}_{syn,X}(d)$$

which induces after taking the canonical flasque resolution of the syntomic complexes the morphism in $C(\mathbb{Z}_p)$

$$\begin{aligned} f^* := (f^*, f^*, \theta(f)^t) : C^{\bullet}_{syn}(S, \mathbb{Z}(d)) := \\ \mathrm{Cone}(\Gamma(S, E_{et}(\mathbb{Z}_{p,S})) \oplus \Gamma(S, F^d E_{et}(DR(S)(O\mathbb{B}_{dr,S}))) \hookrightarrow \Gamma(S, E_{et}(DR(S)(O\mathbb{B}_{dr,S})))) \\ \to C^{\bullet}_{syn}(X, \mathbb{Z}(d)) := \\ \mathrm{Cone}(\Gamma(X, E_{et}(\mathbb{Z}_{p,X})) \oplus \Gamma(X, F^d E_{et}(DR(X)(O\mathbb{B}_{dr,X}))) \hookrightarrow \Gamma(X, E_{et}(DR(X)(O\mathbb{B}_{dr,X})))) \end{aligned}$$

where $\theta(f)^t$ is the homotopy in the morphism in $D_{fil}(k) \otimes_I D(\mathbb{Z}_p)$ (where here the comparaison morphisms α are in $D_{\mathbb{B}_{dr},G,fil}(K)$ instead of $D_{\mathbb{B}_{dr},G,fil}(\mathbb{C}_p)$)

$$\begin{aligned} (f^*, f^*, \theta(f)^t) : (\Gamma(S, E_{et}(DR(S)(O\mathbb{B}_{dr,S}), F_b)), \Gamma(S, E_{et}(\mathbb{Z}_{p,S})), \alpha(S)) \\ \to (\Gamma(X, E_{et}(DR(X)(O\mathbb{B}_{dr,X}), F_b)), \Gamma(X, E_{et}(\mathbb{Z}_{p,X})), \alpha(X)), \end{aligned}$$

which induces in cohomology for $n \in \mathbb{Z}$, the morphisms of abelian groups

$$f^* : H^n_{syn}(S, \mathbb{Z}(d)) \to H^n_{syn}(X, \mathbb{Z}(d))\ ;$$

we get dually,

$$\begin{aligned} f_* := (f_*, f_*, \theta(f)) : \\ \mathrm{Cone}(\Gamma(X, E_{et}(\mathbb{Z}_{p,X}))^{\vee} \oplus F^d \Gamma(X, E_{et}(DR(X)(O\mathbb{B}_{dr,X}), F_b))^{\vee} \hookrightarrow \Gamma(X, E_{et}(DR(X)(O\mathbb{B}_{dr,X})))^{\vee}) \\ \to \mathrm{Cone}(\Gamma(S, E_{et}(\mathbb{Z}_{p,S}))^{\vee} \oplus F^d \Gamma(S, E_{et}(DR(S)(O\mathbb{B}_{dr,S}), F_b))^{\vee} \hookrightarrow \Gamma(S, E_{et}(DR(S)(O\mathbb{B}_{dr,S})))^{\vee}) \end{aligned}$$

where $\theta(f)$ is the homotopy in the morphism in $D_{fil}(k) \otimes_I D(\mathbb{Z}_p)$ (where here the comparaison morphisms α are in $D_{\mathbb{B}_{dr},G,fil}(K)$ instead of $D_{\mathbb{B}_{dr},G,fil}(\mathbb{C}_p)$)

$$\begin{aligned} (f_*, f_*, \theta(f)) : (\Gamma(X, E_{et}(DR(X)(O\mathbb{B}_{dr,X}), F_b))^{\vee}, \Gamma(X, E_{et}(\mathbb{Z}_{p,X}))^{\vee}, \alpha(X)) \\ \to (\Gamma(S, E_{et}(DR(S)(O\mathbb{B}_{dr,S}), F_b))^{\vee}, \Gamma(S, E_{et}(\mathbb{Z}_{p,S}))^{\vee}, \alpha(S)), \end{aligned}$$

which induces in homology for $n \in \mathbb{Z}$, the morphisms of abelian groups

$$f_* : H_{n,syn}(X, \mathbb{Z}(d)) \to H_{n,syn}(S, \mathbb{Z}(d)).$$

Theorem 51. *Let $k \subset \mathbb{C}_p$ a subfield.*

(i) Let $U \in \mathrm{SmVar}(k)$. Denote by $a_U : U \to \mathrm{pt}$ the terminal map. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X\backslash U$ a normal crossing divisor. The embedding (see theorem 49)

$$\iota : D(MHM_{gm,k,\mathbb{C}_p}(\{\mathrm{pt}\})) \to D_{fil}(k) \times_I D(\mathbb{Z}_p)$$

induces for $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$, canonical isomorphisms

$$\iota(a_{U!Hdg}\mathbb{Z}_U^{Hdg}) : H^k(a_{U!Hdg}\mathbb{Z}_U^{Hdg}) \xrightarrow{\sim} H^k_{syn}(X^{an}_{\mathbb{C}_p}, D^{an}_{\mathbb{C}_p}, \mathbb{Z}(d)), \text{ and}$$
$$\iota(a_{U*Hdg}\mathbb{Z}_U^{Hdg}) : H^k(a_{U*Hdg}\mathbb{Z}_U^{Hdg}) \xrightarrow{\sim} H^k_{syn}(U^{an}_{\mathbb{C}_p}, \mathbb{Z}(d)).$$

(ii) Let $h : U \to S$ and $h' : U' \to S$ two morphism with $S, U, U' \in \mathrm{SmVar}(k)$. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X\backslash U$ a normal crossing divisor such that $h : U \to S$ extend to $f : X \to \bar{S}$. Let $X' \in \mathrm{PSmVar}(k)$ a compactification of U' with $D' := X'\backslash U'$ a normal crossing divisor such that $h' : U' \to S$ extend to $f' : X' \to \bar{S}$. The embedding $\iota : D(MHM_{gm,k,\mathbb{C}_p}(\mathrm{pt})) \to D_{fil}(k) \times_I D(\mathbb{Z}_p)$ (see theorem 49) induces for $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$ a canonical isomorphism

$$\begin{aligned}
&\iota(a_{U'\times_S U!Hdg}\mathbb{Z}^{Hdg}_{U'\times_S U}) : \mathrm{Hom}_{D(MHM_{gm,k,\mathbb{C}_p}(S))}(h_{U'!Hdg}\mathbb{Z}^{Hdg}_{U'}, h_{U!Hdg}\mathbb{Z}^{Hdg}_U(d)[k]) \\
\xrightarrow{RI(-,-)} &\mathrm{Hom}_{D(MHM_{gm,k,\mathbb{C}_p}(\mathrm{pt}))}(\mathbb{Z}^{Hdg}_{\mathrm{pt}}, a_{U'\times_S U!Hdg}\mathbb{Z}^{Hdg}_{U'\times_S U}(d)[k]) = H^k(a_{U'\times_S U!Hdg}\mathbb{Z}^{Hdg}_{U'\times_S U}(d)) \\
&\xrightarrow{\sim} H^k_{\mathcal{D}}((X' \times_S X)^{an}_{\mathbb{C}_p}, ((X' \times_S U) \cup (U' \times_S X))^{an}_{\mathbb{C}_p}, \mathbb{Z}(d)).
\end{aligned}$$

(iii) Let $U \in \mathrm{SmVar}(k)$. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X\backslash U$ a normal crossing divisor. For $[Z] \in \mathrm{CH}^d(U, n)$ and $[Z'] \in \mathrm{CH}^{d'}(U, n')$, we have

$$([Z] \cdot [Z'])_{syn} = [Z]_{syn} \cdot [Z']_{syn} \in H^{2d+2d'-n-n'}(U^{an}_{\mathbb{C}_p}, \mathbb{Z}(d + d'))$$

where the product on the left is the intersection of higher Chow cycle which is well defined modulo boundary (they intersect properly modulo boundary) while the right product of Deligne cohomology classes is induced by the product of Deligne complexes $(-) \cdot (-) : \mathbb{Z}_{syn,(X,D)}(d) \otimes \mathbb{Z}_{syn,(X,D)}(d') \to \mathbb{Z}_{syn,(X,D)}(d + d')$.

(iv) Let $h : U \to S$,$h' : U' \to S$, $h'' : U'' \to S$ three morphism with $S, U, U', U'' \in \mathrm{SmVar}(k)$. Let $X \in \mathrm{PSmVar}(k)$ a compactification of U with $D := X\backslash U$ a normal crossing divisor such that $h : U \to S$ extend to $f : X \to \bar{S}$. Let $X' \in \mathrm{PSmVar}(k)$ a compactification of U' with $D' := X'\backslash U'$ a normal crossing divisor such that $h' : U' \to S$ extend to $f' : X' \to \bar{S}$. Let $X' \in \mathrm{PSmVar}(k)$ a compactification of U' with $D' := X'\backslash U'$ a normal crossing divisor such that $h' : U' \to S$ extend to $f' : X' \to \bar{S}$. For $[Z] \in \mathrm{CH}^d(U \times_S U', n)$ and $[Z'] \in \mathrm{CH}^{d'}(U' \times_S U'', n')$, we have

$$([Z] \circ [Z'])_{syn} = [Z]_{syn} \circ [Z']_{syn} \in H^{d''-n''}((U \times_S U'')^{an}_{\mathbb{C}_p}, \mathbb{Z}(d'' - n''))$$

where the composition on the left is the composition of higher correspondence modulo boundary while the composition on the right is given by (ii).

Proof. (i):Standard.
(ii):Follows on the one hand from (i) and on the other hand the six functor formalism on the 2-functor $D(MHM_{gm,k,\mathbb{C}_p}(-)) : \mathrm{SmVar}(k) \to \mathrm{TriCat}$ (theorem 45) gives the isomorphism $RI(-,-)$.
(iii):Standard.
(iv):Follows from (iii).

□

7 The algebraic filtered De Rham realizations for Voevodsky relative motives over a field k of characteristic 0

7.1 The algebraic Gauss-Manin filtered De Rham realization functor

Let k a field of characteristic zero. Consider, for $S \in \mathrm{Var}(k)$, the following composition of morphism in RCat (see section 2)

$$\tilde{e}(S) : (\mathrm{Var}(k)/S, O_{\mathrm{Var}(k)/S}) \xrightarrow{\rho_S} (\mathrm{Var}(k)^{sm}/S, O_{\mathrm{Var}(k)^{sm}/S}) \xrightarrow{e(S)} (S, O_S)$$

with, for $X/S = (X, h) \in \mathrm{Var}(k)/S$,

- $O_{\mathrm{Var}(k)/S}(X/S) := O_X(X)$,
- $(\tilde{e}(S)^* O_S(X/S) \to O_{\mathrm{Var}(k)/S}(X/S)) := (h^* O_S \to O_X)$.

and $O_{\mathrm{Var}(k)^{sm}/S} := \rho_{S*} O_{\mathrm{Var}(k)/S}$, that is, for $U/S = (U, h) \in \mathrm{Var}(k)^{sm}/S$, $O_{\mathrm{Var}(k)^{sm}/S}(U/S) := O_{\mathrm{Var}(k)/S}(U/S) := O_U(U)$

Definition 96. *(i) For $S \in \mathrm{Var}(k)$, we consider the complexes of presheaves*

$$\Omega^\bullet_{/S} := \mathrm{coker}(\Omega_{O_{\mathrm{Var}(k)/S}/\tilde{e}(S)^* O_S} : \Omega^\bullet_{\tilde{e}(S)^* O_S} \to \Omega^\bullet_{O_{\mathrm{Var}(k)/S}}) \in C_{O_S}(\mathrm{Var}(k)/S)$$

which is by definition given by

- *for X/S a morphism $\Omega^\bullet_{/S}(X/S) = \Omega^\bullet_{X/S}(X)$*
- *for $g : X'/S \to X/S$ a morphism,*

$$\Omega^\bullet_{/S}(g) := \Omega_{(X'/X)/(S/S)}(X') : \Omega^\bullet_{X/S}(X) \to g^* \Omega_{X/S}(X') \to \Omega^\bullet_{X'/S}(X')$$
$$\omega \mapsto \Omega_{(X'/X)/(S/S)}(X')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{X'}(X') \mapsto \omega(dg(\alpha)))$$

(ii) For $S \in \mathrm{Var}(k)$, we consider the complexes of presheaves

$$\Omega^\bullet_{/S} := \rho_{S*} \tilde{\Omega}^\bullet_{/S} = \mathrm{coker}(\Omega_{O_{\mathrm{Var}(k)^{sm}/S}/e(S)^* O_S} : \Omega^\bullet_{e(S)^* O_S} \to \Omega^\bullet_{O_{\mathrm{Var}(k)^{sm}/S}}) \in C_{O_S}(\mathrm{Var}(k)^{sm}/S)$$

which is by definition given by

- *for U/S a smooth morphism $\Omega^\bullet_{/S}(U/S) = \Omega^\bullet_{U/S}(U)$*
- *for $g : U'/S \to U/S$ a morphism,*

$$\Omega^\bullet_{/S}(g) := \Omega_{(U'/U)/(S/S)}(U') : \Omega^\bullet_{U/S}(U) \to g^* \Omega_{U/S}(U') \to \Omega^\bullet_{U'/S}(U')$$
$$\omega \mapsto \Omega_{(U'/U)/(S/S)}(U')(\omega) := g^*(\omega) : (\alpha \in \wedge^k T_{U'}(U') \mapsto \omega(dg(\alpha)))$$

Remark 9. *For $S \in \mathrm{Var}(k)$, $\Omega^\bullet_{/S} \in C(\mathrm{Var}(k)/S)$ is by definition a natural extension of $\Omega^\bullet_{/S} \in C(\mathrm{Var}(k)^{sm}/S)$. However $\Omega^\bullet_{/S} \in C(\mathrm{Var}(k)/S)$ does NOT satisfy cdh descent.*

For a smooth morphism $h : U \to S$ with $S, U \in \mathrm{SmVar}(\mathbb{C})$, the cohomology presheaves $H^n \Omega^\bullet_{U/S}$ of the relative De Rham complex

$$DR(U/S) := \Omega^\bullet_{U/S} := \mathrm{coker}(h^* \Omega_S \to \Omega_U) \in C_{h^* O_S}(U)$$

for all $n \in \mathbb{Z}$, have a canonical structure of a complex of $h^* D_S$ modules given by the Gauss Manin connexion : for $S^o \subset S$ an open subset, $U^o = h^{-1}(S^o)$, $\gamma \in \Gamma(S^o, T_S)$ a vector field and $\hat{\omega} \in \Omega^p_{U/S}(U^o)^c$ a closed form, the action is given by

$$\gamma \cdot [\hat{\omega}] = [\widehat{\iota(\tilde{\gamma}) \partial \omega}],$$

$\omega \in \Omega^p_U(U^o)$ being a representative of $\hat{\omega}$ and $\tilde{\gamma} \in \Gamma(U^o, T_U)$ a relevement of γ (h is a smooth morphism), so that

$$DR(U/S) := \Omega^\bullet_{U/S} := \mathrm{coker}(h^*\Omega_S \to \Omega_U) \in C_{h^*O_S, h^*\mathcal{D}}(U)$$

with this h^*D_S structure. Hence we get $h_*\Omega^\bullet_{U/S} \in C_{O_S,\mathcal{D}}(S)$ considering this structure. Since h is a smooth morphism, $\Omega^p_{U/S}$ are locally free O_U modules.

The point (ii) of the definition 105 above gives the object in $\mathrm{DA}(S)$ which will, for S smooth, represent the algebraic Gauss-Manin De Rham realisation. It is the class of an explicit complex of presheaves on $\mathrm{Var}(k)^{sm}/S$.

Proposition 56. *Let $S \in \mathrm{Var}(k)$.*

(i) For $U/S = (U, h) \in \mathrm{Var}(k)^{sm}/S$, we have $e(U)_* h^*\Omega^\bullet_{/S} = \Omega^\bullet_{U/S}$.

(ii) The complex of presheaves $\Omega^\bullet_{/S} \in C_{O_S}(\mathrm{Var}(k)^{sm}/S)$ is $\mathbb{A}^1$ homotopic, in particular $\mathbb{A}^1$ invariant. Note that however, for $p > 0$, the complexes of presheaves $\Omega^{\bullet \geq p}$ are NOT $\mathbb{A}^1$ local. On the other hand, $(\Omega^\bullet_{/S}, F_b)$ admits transferts (recall that means $\mathrm{Tr}(S)_* \mathrm{Tr}(S)^*\Omega^p_{/S} = \Omega^p_{/S}$).

(iii) If S is smooth, we get $(\Omega^\bullet_{/S}, F_b) \in C_{O_S fil, D_S}(\mathrm{Var}(k)^{sm}/S)$ with the structure given by the Gauss Manin connexion. Note that however the D_S structure on the cohomology groups given by Gauss Main connexion does NOT comes from a structure of D_S module structure on the filtered complex of O_S module. The D_S structure on the cohomology groups satisfy a non trivial Griffitz transversality (in the non projection cases), whereas the filtration on the complex is the trivial one.

Proof. Similar to the proof of [10] proposition. □

We have the following canonical transformation map given by the pullback of (relative) differential forms:

Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(k)$. Consider the following commutative diagram in RCat :

$$\begin{tikzcd} D(g,e): (\mathrm{Var}(k)^{sm}/T, O_{\mathrm{Var}(k)^{sm}/T}) \arrow[r, "P(g)"] \arrow[d, "e(T)"] & (\mathrm{Var}(k)^{sm}/S, O_{\mathrm{Var}(k)^{sm}/S}) \arrow[d, "e(S)"] \\ (T, O_T) \arrow[r, "P(g)"] & (S, O_S) \end{tikzcd}$$

It gives (see section 2) the canonical morphism in $C_{g^*O_S fil}(\mathrm{Var}(k)^{sm}/T)$

$$\Omega_{/(T/S)} := \Omega_{(O_{\mathrm{Var}(k)^{sm}/T}/g^*O_{\mathrm{Var}(k)^{sm}/S})/(O_T/g^*O_S)} :$$
$$g^*(\Omega^\bullet_{/S}, F_b) = \Omega^\bullet_{g^*O_{\mathrm{Var}(k)^{sm}/S}/g^*e(S)^*O_S} \to (\Omega^\bullet_{/T}, F_b) = \Omega^\bullet_{O_{\mathrm{Var}(k)^{sm}/T}/e(T)^*O_T}$$

which is by definition given by the pullback on differential forms : for $(V/T) = (V, h) \in \mathrm{Var}(k)^{sm}/T$,

$$\Omega_{/(T/S)}(V/T) : g^*(\Omega^\bullet_{/S})(V/T) := \lim_{(h':U \to S\mathrm{sm}, g':V \to U, h, g)} \Omega^\bullet_{U/S}(U) \xrightarrow{\Omega_{(V/U)/(T/S)}(V/T)} \Omega^\bullet_{V/T}(V) =: \Omega^\bullet_{/T}(V/T)$$
$$\hat{\omega} \mapsto \Omega_{(V/U)/(T/S)}(V/T)(\omega) := \hat{g'^*\omega}.$$

If S and T are smooth, $\Omega_{/(T/S)} : g^*(\Omega^\bullet_{/S}, F_b) \to (\Omega^\bullet_{/T}, F_b)$ is a map in $C_{g^*O_S fil, g^*D_S}(\mathrm{Var}(k)^{sm}/T)$ It induces the canonical morphisms in $C_{g^*O_S fil, g^*D_S}(\mathrm{Var}(k)^{sm}/T)$:

$$E\Omega_{/(T/S)} : g^*E_{et}(\Omega^\bullet_{/S}, F_b) \xrightarrow{T(g, E_{et})(\Omega^\bullet_{/S}, F_b)} E_{et}(g^*(\Omega^\bullet_{/S}, F_b)) \xrightarrow{E_{et}(\Omega_{/(T/S)})} E_{et}(\Omega^\bullet_{/T}, F_b).$$

and

$$E\Omega_{/(T/S)} : g^*E_{zar}(\Omega^\bullet_{/S}, F_b) \xrightarrow{T(g, E_{zar})(\Omega^\bullet_{/S}, F_b)} E_{zar}(g^*(\Omega^\bullet_{/S}, F_b)) \xrightarrow{E_{zar}(\Omega_{/(T/S)})} E_{zar}(\Omega^\bullet_{/T}, F_b).$$

Definition 97. *(i) Let $g:T\to S$ a morphism with $T,S\in\mathrm{Var}(k)$. We have, for $F\in C(\mathrm{Var}(k)^{sm}/S)$, the canonical transformation in $C_{O_T fil}(T)$:*

$$\begin{aligned}
T^O(g,\Omega_{/.})(F) : g^{*mod}L_Oe(S)_*\mathcal{H}om^\bullet(F,E_{et}(\Omega^\bullet_{/S},F_b)) \\
\xrightarrow{:=} (g^*L_Oe(S)_*\mathcal{H}om^\bullet(F,E_{et}(\Omega^\bullet_{/S},F_b)))\otimes_{g^*O_S}O_T \\
\xrightarrow{T(e,g)(-)\circ T(g,L_O)(-)} L_O(e(T)_*g^*\mathcal{H}om^\bullet(F,E_{et}(\Omega^\bullet_{/S},F))\otimes_{g^*O_S}O_T) \\
\xrightarrow{T(g,hom)(F,E_{et}(\Omega^\bullet_{/S}))\otimes I} L_O(e(T)_*\mathcal{H}om^\bullet(g^*F,g^*E_{et}(\Omega^\bullet_{/S},F_b))\otimes_{g^*O_S}O_T) \\
\xrightarrow{ev(hom,\otimes)(-,-,-)} L_Oe(T)_*\mathcal{H}om^\bullet(g^*F,g^*E_{et}(\Omega^\bullet_{/S},F_b)\otimes_{g^*e(S)^*O_S}e(T)^*O_T) \\
\xrightarrow{\mathcal{H}om^\bullet(g^*F,E\Omega_{/(T/S)}\otimes I)} L_Oe(T)_*\mathcal{H}om^\bullet(g^*F,E_{et}(\Omega^\bullet_{/T},F_b)\otimes_{g^*e(S)^*O_S}e(T)^*O_T) \\
\xrightarrow{m} L_Oe(T)_*\mathcal{H}om^\bullet(g^*F,E_{et}(\Omega^\bullet_{/T},F_b)
\end{aligned}$$

where $m(\alpha\otimes h):=h.\alpha$ is the multiplication map.

(ii) Let $g:T\to S$ a morphism with $T,S\in\mathrm{Var}(k)$, S smooth. Assume there is a factorization $g: T\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ with $Y\in\mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have, for $F\in C(\mathrm{Var}(k)^{sm}/S)$, the canonical transformation in $C_{O_T fil}(Y\times S)$:

$$\begin{aligned}
T(g,\Omega_{/.})(F) : g^{*mod,\Gamma}e(S)_*\mathcal{H}om^\bullet(F,E_{et}(\Omega^\bullet_{/S},F_b)) \\
\xrightarrow{:=} \Gamma_T E_{zar}(p_S^{*mod}e(S)_*\mathcal{H}om^\bullet(F,E_{et}(\Omega^\bullet_{/S},F_b))) \\
\xrightarrow{T^O(p_S,\Omega_{/.})(F)} \Gamma_T E_{zar}(e(T\times S)_*\mathcal{H}om^\bullet(p_S^*F,E_{et}(\Omega^\bullet_{/Y\times S},F_b))) \\
\xrightarrow{=} e(T\times S)_*\Gamma_T(\mathcal{H}om^\bullet(p_S^*F,E_{et}(\Omega^\bullet_{/Y\times S},F_b))) \\
\xrightarrow{I(\gamma,\mathrm{hom})(-,-)} e(T\times S)_*\mathcal{H}om^\bullet(\Gamma^\vee_T p_S^*F,E_{et}(\Omega^\bullet_{/Y\times S},F_b)).
\end{aligned}$$

For $Q\in Proj\,\mathrm{PSh}(\mathrm{Var}(k)^{sm}/S)$,

$$T(g,\Omega_{/.})(Q):g^{*mod,\Gamma}e(S)_*\mathcal{H}om^\bullet(Q,E_{et}(\Omega^\bullet_{/S},F_b))\to e(T\times S)_*\mathcal{H}om^\bullet(\Gamma^\vee_T p_S^*Q,E_{et}(\Omega^\bullet_{/Y\times S},F_b))$$

is a map in $C_{O_T fil,\mathcal{D}}(Y\times S)$.

Let $S\in\mathrm{Var}(k)$. We have the canonical map in $C_{O_S fil}(\mathrm{Var}(k)^{sm}/S)$

$$w_S:(\Omega^\bullet_{/S},F_b)\otimes_{O_S}(\Omega^\bullet_{/S},F_b)\to(\Omega^\bullet_{/S},F_b)$$

given by for $h:U\to S\in\mathrm{Var}(k)^{sm}/S$ by the wedge product

$$w_S(U/S):(\Omega^\bullet_{U/S},F_b)\otimes_{h^*O_S}(\Omega^\bullet_{U/S},F_b)(U)\xrightarrow{w_{U/S}(U)}(\Omega^\bullet_{U/S},F_b)(U)$$

It gives the map

$$Ew_S:E_{et}(\Omega^\bullet_{/S},F_b)\otimes_{O_S}E_{et}(\Omega^\bullet_{/S},F_b)\xrightarrow{=}E_{et}((\Omega^\bullet_{/S},F_b)\otimes_{O_S}(\Omega^\bullet_{/S},F_b))\xrightarrow{E_{et}(w_S)}E_{et}(\Omega^\bullet_{/S},F_b)$$

If $S\in\mathrm{SmVar}(\mathbb{C})$,

$$w_S:(\Omega^\bullet_{/S},F_b)\otimes_{O_S}(\Omega^\bullet_{/S},F_b)\to(\Omega^\bullet_{/S},F_b)$$

is a map in $C_{O_S fil,D_S}(\mathrm{Var}(k)^{sm}/S)$.

Definition 98. *Let $S \in \mathrm{Var}(k)$. We have, for $F, G \in C(\mathrm{Var}(k)^{sm}/S)$, the canonical transformation in $C_{O_S fil}(S)$:*

$$\begin{aligned} T(\otimes,\Omega)(F,G) : e(S)_*\mathcal{H}om(F, E_{et}(\Omega^\bullet_{/S}, F_b)) \otimes_{O_S} e(S)_*\mathcal{H}om(G, E_{et}(\Omega^\bullet_{/S}, F_b)) \\ \xrightarrow{=} e(S)_*(\mathcal{H}om(F, E_{et}(\Omega^\bullet_{/S}, F_b)) \otimes_{O_S} \mathcal{H}om(G, E_{et}(\Omega^\bullet_{/S}, F_b))) \\ \xrightarrow{e(S)_*T(\mathcal{H}om,\otimes)(-)} e(S)_*\mathcal{H}om(F \otimes G, E_{et}(\Omega^\bullet_{/S}, F_b) \otimes_{O_S} E_{et}(\Omega^\bullet_{/S}, F_b)) \\ \xrightarrow{\mathcal{H}om(F\otimes G, Ew_S)} e(S)_*\mathcal{H}om(F \otimes G, E_{et}(\Omega^\bullet_{/S}, F_b)) \end{aligned}$$

If $S \in \mathrm{SmVar}(\mathbb{C})$, $T(\otimes,\Omega)(F,G)$ is a map in $C_{O_S fil,\mathcal{D}}(S)$.

Definition 99. *(i) Let $S \in \mathrm{SmVar}(\mathbb{C})$. We have the functor*

$$C(\mathrm{Var}(k)^{sm}/S)^{op} \to C_{Ofil,\mathcal{D}}(S), \quad F \mapsto e(S)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega^\bullet_{/S}, F_b))[-d_S].$$

(ii) Let $S \in \mathrm{Var}(k)$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1, \cdots l]$, denote by $S_I := \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. We have the functor

$$C(\mathrm{Var}(k)^{sm}/S)^{op} \to C_{Ofil,\mathcal{D}}(S/(\tilde{S}_I)), \quad F \mapsto (e(\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u^q_{IJ}(F))$$

where

$$\begin{aligned} u^q_{IJ}(F)[d_{\tilde{S}_J}] : e(\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)) \\ \xrightarrow{\mathrm{ad}(p_{IJ}^{*mod}, p_{IJ*})(-)} p_{IJ*}p_{IJ}^{*mod}e(\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)) \\ \xrightarrow{p_{IJ*}T(p_{IJ},\Omega.)(L(i_{I*}j_I^*F))} p_{IJ*}e(\tilde{S}_J)_*\mathcal{H}om^\bullet(p_{IJ}^*L(i_{I*}j_I^*F), E_{et}(\Omega^\bullet_{/\tilde{S}_J}, F_b)) \\ \xrightarrow{p_{IJ*}e(\tilde{S}_J)_*\mathcal{H}om(S^q(D_{IJ})(F), E_{et}(\Omega^{\bullet,\Gamma}_{/\tilde{S}_J}, F_b))} p_{IJ*}e(\tilde{S}_J)_*\mathcal{H}om^\bullet(L(i_{J*}j_J^*F), E_{et}(\Omega^\bullet_{/\tilde{S}_J}, F_b)). \end{aligned}$$

For $I \subset J \subset K$, we have obviously $p_{IJ*}u_{JK}(F) \circ u_{IJ}(F) = u_{IK}(F)$.

We then have the following key proposition

Proposition 57. *(i) Let $S \in \mathrm{Var}(k)$. Let $m : Q_1 \to Q_2$ be an equivalence $(\mathbb{A}^1, et)$ local in $C(\mathrm{Var}(k)^{sm}/S)$ with Q_1, Q_2 complexes of projective presheaves. Then,*

$$e(S)_*\mathcal{H}om(m, E_{et}(\Omega^\bullet_{/S}, F_b)) : e(S)_*\mathcal{H}om^\bullet(Q_2, E_{et}(\Omega^\bullet_{/S}, F_b)) \to e(S)_*\mathcal{H}om^\bullet(Q_1, E_{et}(\Omega^\bullet_{/S}, F_b))$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{O_S fil,\mathcal{D},\infty}(S)$ if S is smooth.

(ii) Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Let $m = (m_I) : (Q_{1I}, s^1_{IJ}) \to (Q_{2I}, s^2_{IJ})$ be an equivalence $(\mathbb{A}^1, et)$ local in $C(\mathrm{Var}(k)^{sm}/(\tilde{S}_I)^{op})$ with Q_{1I}, Q_{2I} complexes of projective presheaves. Then,

$$\begin{aligned} (e(\tilde{S}_I)_*\mathcal{H}om(m_I, E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))) : \\ (e(\tilde{S}_I)_*\mathcal{H}om^\bullet(Q_{2I}, E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)), u_{IJ}(Q_{2I}, s^2_{IJ})) \to (e(\tilde{S}_I)_*\mathcal{H}om^\bullet(Q_{1I}, E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)), u_{IJ}(Q_{1I}, s^1_{IJ})) \end{aligned}$$

is an 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{O_S fil,\mathcal{D},\infty}((\tilde{S}_I))$.

Proof. Similar to the proof of [10] proposition. □

Definition 100. *(i) We define, using definition 99, by proposition 57, the filtered algebraic Gauss-Manin realization functor defined as*

$$\mathcal{F}_S^{GM} : \mathrm{DA}_c(S)^{op} \to D_{O_S fil,\mathcal{D},\infty}(S), \;\; M \mapsto \mathcal{F}_S^{GM}(M) := e(S)_*\mathcal{H}om^\bullet(L(F), E_{et}(\Omega^\bullet_{/S}, F_b))[-d_S]$$

where $F \in C(\mathrm{Var}(k)^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$,

(ii) Let $S \in \mathrm{Var}(k)$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. For $I \subset [1, \cdots l]$, denote by $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. We define, using definition 99 and proposition 57 the filtered algebraic Gauss-Manin realization functor defined as

$$\mathcal{F}_S^{GM} : \mathrm{DA}_c(S)^{op} \to D_{Ofil,\mathcal{D},\infty}(S/(\tilde{S}_I)), \; M \mapsto$$
$$\mathcal{F}_S^{GM}(M) := ((e(\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega^\bullet_{/\tilde{S}_I}), F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F))$$

where $F \in C(\mathrm{Var}(k)^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

Proposition 58. *Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Then $X = \cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i\in I} S_i$ and $X_I = \cap_{i\in I} X_i$. Assume there exist a factorization*

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We then have, for $I \subset [1, \cdots l]$, the following commutative diagrams which are cartesian

$$\begin{tikzcd} f_I = f_{|X_I} : X_I \arrow[r, "l_I"] \arrow[dr] & Y \times S_I \arrow[r, "p_{S_I}"] \arrow[d, "i'_I"] & S_I \arrow[d, "i_I"] \\ & Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I \end{tikzcd} \quad , \quad \begin{tikzcd} Y \times \tilde{S}_J \arrow[r, "p_{\tilde{S}_J}"] \arrow[d, "p'_{IJ}"'] & \tilde{S}_J \arrow[d, "p_{IJ}"] \\ Y \times \tilde{S}_I \arrow[r, "p_{\tilde{S}_I}"] & \tilde{S}_I \end{tikzcd}$$

Let $F(X/S) := p_{S,\sharp}\Gamma_X^\vee \mathbb{Z}(Y \times S/Y \times S)$. The transformations maps $(N_I(X/S) : Q(X_I/\tilde{S}_I) \to i_{I*}j_I^*F(X/S))$ and $(k \circ I(\gamma, \hom)(-,-))$, for $I \subset [1, \cdots, l]$, induce an isomorphism in $D_{Ofil,\mathcal{D},\infty}(S/(\tilde{S}_I))$

$$I^{GM}(X/S) :$$
$$\mathcal{F}_S^{GM}(M(X/S)) := (e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*F(X/S)), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F(X/S)))$$
$$\xrightarrow{(e(\tilde{S}_I)_*\mathcal{H}om(LN_I(X/S), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)))} (e(\tilde{S}_I)_*\mathcal{H}om(Q(X_I/\tilde{S}_I), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], v_{IJ}^q(F(X/S)))$$
$$\xrightarrow{(k\circ I(\gamma,\hom)(-,-))^{-1}} (p_{\tilde{S}_I*}\Gamma_{X_I}E_{zar}(\Omega^\bullet_{Y\times\tilde{S}_I/\tilde{S}_I}, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)).$$

Proof. Similar to the proof of [10] proposition. □

Definition 101. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{SmVar}(\mathbb{C})$. Consider the factorization $g : T \xrightarrow{l} T \times S \xrightarrow{p_S} S$ where l is the graph embedding and p_S the projection. Let $M \in \mathrm{DA}_c(S)$ and $F \in C(\mathrm{Var}(k)^{sm}/S)$ such that $M = D(\mathbb{A}^1_S, et)(F)$. Then, $D(\mathbb{A}^1_T, et)(g^*F) = g^*M$.*

(i) We have then the canonical transformation in $D_{Ofil,\mathcal{D},\infty}(T \times S)$ (see definition 97) :

$$T(g, \mathcal{F}^{GM})(M) : Rg^{*mod[-],\Gamma}\mathcal{F}_S^{GM}(M) := g^{*mod,\Gamma}e(S)_*\mathcal{H}om^\bullet(LF, E_{et}(\Omega^\bullet_{/S}, F_b)))[-d_T] \xrightarrow{T(g,\Omega_{/.})(LF)}$$
$$e(T \times S)_*\mathcal{H}om^\bullet(\Gamma_T^\vee p_S^* LF, E_{et}(\Omega^\bullet_{/T\times S}, F_b))[-d_T] =: \mathcal{F}_{T\times S}^{GM}(l_*g^*(M, W)).$$

(ii) *We have then the canonical transformation in $D_{Ofil,\infty}(T)$(see definition 97) :*

$$T^O(g,\mathcal{F}^{GM})(M,W) : Lg^{*mod[-]}\mathcal{F}_S^{GM}(M) := g^{*mod}e(S)_*\mathcal{H}om^\bullet(LF, E_{et}(\Omega^\bullet_{/S}, F_b)))[-d_T] \xrightarrow{T^O(g,\Omega_{/\cdot})(LF)} e(T)_*\mathcal{H}om^\bullet(g^*LF, E_{et}(\Omega^\bullet_{/T}, F_b))[-d_T] =: \mathcal{F}_T^{GM}(g^*M).$$

Definition 102. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(k)$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{|T_I} : T_I \to S_I$. Denote for short $d_{YI} := d_Y + d_{\tilde{S}_I}$. Let $M \in \mathrm{DA}_c(S)$ and $F \in C(\mathrm{Var}(k)^{sm}/S)$ such that $M = D(\mathbb{A}^1_S, et)(F)$. Then, $D(\mathbb{A}^1_T, et)(g^*F) = g^*M$. We have the canonical transformation in $D_{Ofil,\mathcal{D},\infty}(T/(Y \times \tilde{S}_I))$*

$$\begin{aligned}
T(g,\mathcal{F}^{GM})(M) : Rg^{*mod[-],\Gamma}\mathcal{F}_S^{GM}(M) := \\
(\Gamma_{T_I}E_{zar}(\tilde{g}_I^{*mod}e(\tilde{S}_I)_*\mathcal{H}om^\bullet(L(i_{I*}j_I^*F), E_{et}(\Omega^\bullet_{/\tilde{S}_I}, F_b)))[-d_Y - d_{\tilde{S}_I}], \tilde{g}_J^{*mod}u_{IJ}^q(F)) \\
\xrightarrow{(\Gamma_{T_I}E(T(\tilde{g}_I,\Omega_{/\cdot})(L(i_{I*}j_I^*(F,W)))))} \\
(\Gamma_{T_I}e(Y \times \tilde{S}_I)_*\mathcal{H}om^\bullet(\tilde{g}_I^*L(i_{I*}j_I^*F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I}, F_b))[-d_Y - d_{\tilde{S}_I}], \tilde{g}_J^*u_{IJ}^q(F)_1) \\
\xrightarrow{(I(\gamma,\hom(-,-)))} \\
(e(Y \times \tilde{S}_I)_*\mathcal{H}om^\bullet(\Gamma^\vee_{T_I}\tilde{g}_I^*L(i_{I*}j_I^*F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I}, F_b))[-d_Y - d_{\tilde{S}_I}], \tilde{g}_J^*u_{IJ}^q(F)_2) \\
\xrightarrow{(e(Y\times\tilde{S}_I)_*\mathcal{H}om(T^{q,\gamma}(D_{gI})(j_I^*F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I}, F_b)))^{-1}} \\
(e(Y \times \tilde{S}_I)_*\mathcal{H}om^\bullet(L(i'_{I*}j_I^{'*}g^*F), E_{et}(\Omega^\bullet_{/Y\times\tilde{S}_I}, F_b))[-d_Y - d_{\tilde{S}_I}], u_{IJ}^q(g^*F)) =: \mathcal{F}_T^{GM}(g^*M).
\end{aligned}$$

Proposition 59. (i) *Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(k)$. Assume we have a factorization $g : T \xrightarrow{l} Y_2 \times S \xrightarrow{p_S} S$ with $Y_2 \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$ Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y_2 \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{|T_I} : T_I \to S_I$. Let $f : X \to S$ a morphism with $X \in \mathrm{Var}(k)$. Assume that there is a factorization $f : X \xrightarrow{l} Y_1 \times S \xrightarrow{p_S} S$, with $Y_1 \in \mathrm{SmVar}(\mathbb{C})$, l a closed embedding and p_S the projection. We have then the following commutative diagram whose squares are cartesians*

$$\begin{tikzcd}
f' : X_T \arrow[r] \arrow[d] & Y_1 \times T \arrow[r] \arrow[d] & T \arrow[d] \\
f'' = f \times I : Y_2 \times X \arrow[r] \arrow[d] & Y_1 \times Y_2 \times S \arrow[r] \arrow[d] & Y_2 \times S \arrow[d] \\
f : X \arrow[r] & Y_1 \times S \arrow[r] & S
\end{tikzcd}$$

Consider $F(X/S) := p_{S,\sharp}\Gamma^\vee_X\mathbb{Z}(Y_1 \times S/Y_1 \times S)$ and the isomorphism in $C(\mathrm{Var}(k)^{sm}/S)$

$$T(f,g,F(X/S)) : g^*F(X/S) := g^*p_{S,\sharp}\Gamma^\vee_X\mathbb{Z}(Y_1 \times S/Y_1 \times S) \xrightarrow{\sim} p_{T,\sharp}\Gamma^\vee_{X_T}\mathbb{Z}(Y_1 \times T/Y_1 \times T) =: F(X_T/T).$$

which gives in $\mathrm{DA}(S)$ *the isomorphism* $T(f,g,F(X/S)) : g^*M(X/S) \xrightarrow{\sim} M(X_T/T)$*. Then, the following diagram in* $D_{Ofil,\mathcal{D},\infty}(T/(Y_2\times\tilde{S}_I))$ *commutes*

$$\begin{array}{ccc}
Rg^{*mod,\Gamma}\mathcal{F}_S^{GM}(M(X/S)) & \xrightarrow{T(g,\mathcal{F}^{GM})(M(X/S))} & \mathcal{F}_T^{GM}(M(X_T/T)) \\
\downarrow{\scriptstyle I^{GM}(X/S)} & & \downarrow{\scriptstyle I^{GM}(X_T/T)} \\
g^{*mod[-],\Gamma}(p_{\tilde{S}_I*}\Gamma_{X_I}E_{zar}(\Omega^{\bullet}_{Y_1\times\tilde{S}_I/\tilde{S}_I},F_b)[-d_{\tilde{S}_I}], w_{IJ}(X/S)) & \xrightarrow{(T(\tilde{g}_I\times I,\gamma)(-)\circ T^O_w(\tilde{g}_I,p_{\tilde{S}_I}))} & (p_{Y_2\times\tilde{S}_I*}\Gamma_{X_{T_I}}E_{zar}(\Omega^{\bullet}_{Y_2\times Y_1\times\tilde{S}_I/Y_2\times\tilde{S}_I},F_b)[-d_{Y_2}-\ldots], w_{IJ}(X_T/T))
\end{array}$$

(ii) *Let* $g:T\to S$ *a morphism with* $T,S\in\mathrm{SmVar}(\mathbb{C})$*. Let* $f:X\to S$ *a morphism with* $X\in\mathrm{Var}(k)$*. Assume that there is a factorization* $f:X\xrightarrow{l}Y\times S\xrightarrow{p_S}S$*, with* $Y\in\mathrm{SmVar}(\mathbb{C})$*,* l *a closed embedding and* p_S *the projection. Consider* $F(X/S):=p_{S,\sharp}\Gamma^{\vee}_X\mathbb{Z}(Y\times S/Y\times S)$ *and the isomorphism in* $C(\mathrm{Var}(k)^{sm}/S)$

$$T(f,g,F(X/S)) : g^*F(X/S) := g^*p_{S,\sharp}\Gamma^{\vee}_X\mathbb{Z}(Y\times S/Y\times S)\xrightarrow{\sim} p_{T,\sharp}\Gamma^{\vee}_{X_T}\mathbb{Z}(Y\times T/Y\times T) =: F(X_T/T).$$

which gives in $\mathrm{DA}(S)$ *the isomorphism* $T(f,g,F(X/S)) : g^*M(X/S)\xrightarrow{\sim} M(X_T/T)$*. Then, the following diagram in* $D_{Ofil,\infty}(T)$ *commutes*

$$\begin{array}{ccc}
Lg^{*mod[-]}\mathcal{F}_S^{GM}(M(X/S)) & \xrightarrow{T^O(g,\mathcal{F}^{GM})(M(X/S))} & \mathcal{F}_T^{GM}(M(X_T/T)) \\
\downarrow{\scriptstyle I^{GM}(X/S)} & & \downarrow{\scriptstyle I^{GM}(X_T/T)} \\
g^{*mod}L_O(p_{S*}\Gamma_X E_{zar}(\Omega^{\bullet}_{Y\times S/S},F_b)[-d_T] & \xrightarrow{(T(g\times I,\gamma)(-)\circ T^O_w(g,p_S))} & p_{Y\times T*}\Gamma_{X_T}E_{zar}(\Omega^{\bullet}_{Y\times T/T},F_b)[-d_T] \\
\downarrow{\scriptstyle T_w(\otimes,\gamma)(O_{Y\times S})} & & \downarrow{\scriptstyle T_w(\otimes,\gamma)(O_{Y\times T})} \\
Lg^{*mod}\int^{FDR}_{p_S}\Gamma_X E(O_{Y\times S},F_b)[-d_Y-d_T] & \xrightarrow{T^{\mathcal{D}mod}(g,f)(\Gamma_X E(O_{Y\times S},F_b))} & \int^{FDR}_{p_T}\Gamma_{X_T}E(O_{Y\times T},F_b)[-d_Y-d_T].
\end{array}$$

Proof. Follows immediately from definition. □

We have the following theorem:

Theorem 52. (i) *Let* $g:T\to S$ *is a morphism with* $T,S\in\mathrm{Var}(k)$*. Assume there exist a factorization* $g:T\xrightarrow{l}Y\times S\xrightarrow{p_S}$ *with* $Y\in\mathrm{SmVar}(k)$*,* l *a closed embedding and* p_S *the projection. Let* $S=\cup_{i=1}^l S_i$ *be an open cover such that there exists closed embeddings* $i_i:S_i\hookrightarrow\tilde{S}_i$ *with* $\tilde{S}_i\in\mathrm{SmVar}(\mathbb{C})$*. Then, for* $M\in\mathrm{DA}_c(S)$

$$T(g,\mathcal{F}^{GM})(M) : Rg^{*mod[-],\Gamma}\mathcal{F}_S^{GM}(M)\xrightarrow{\sim}\mathcal{F}_T^{GM}(g^*M)$$

is an isomorphism in $D_{O_T fil,\mathcal{D},\infty}(T/(Y\times\tilde{S}_I))$*.*

(ii) *Let* $g:T\to S$ *is a morphism with* $T,S\in\mathrm{SmVar}(\mathbb{C})$*. Then, for* $M\in\mathrm{DA}_c(S)$

$$T^O(g,\mathcal{F}^{GM})(M) : Lg^{*mod[-]}\mathcal{F}_S^{GM}(M)\xrightarrow{\sim}\mathcal{F}_T^{GM}(g^*M)$$

is an isomorphism in $D_{O_T}(T)$*.*

(iii) *A base change theorem for algebraic De Rham cohomology : Let* $g:T\to S$ *is a morphism with* $T,S\in\mathrm{SmVar}(k)$*. Let* $h:U\to S$ *a smooth morphism with* $U\in\mathrm{Var}(k)$*. Then the map (see definition [10] section 2)*

$$T^O_w(g,h) : Lg^{*mod}Rh_*(\Omega^{\bullet}_{U/S},F_b)\xrightarrow{\sim} Rh'_*(\Omega^{\bullet}_{U_T/T},F_b)$$

is an isomorphism in $D_{O_T}(T)$*.*

Proof. Similar to the proof of [10] theorem. □

Definition 103. *Let $S \in \mathrm{Var}(k)$ and $S = \cup_{i=1}^{l} S_i$ an open affine covering and denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have, for $M, N \in \mathrm{DA}(S)$ and $F, G \in C(\mathrm{Var}(k)^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$*

$$\begin{aligned}
T(\mathcal{F}_S^{GM}, \otimes)(M, N) : \mathcal{F}_S^{GM}(M) \otimes_{O_S}^{L[-]} \mathcal{F}_S^{GM}(N) :=\\
(e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*F), E_{et}(\Omega^{\bullet}_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{IJ}(F)) \otimes_{O_S}^{[-]}\\
(e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*G), E_{et}(\Omega^{\bullet}_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{IJ}(G))\\
\xrightarrow{=} ((e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*F), E_{et}(\Omega^{\bullet}_{/\tilde{S}_I}, F_b)) \otimes_{O_{\tilde{S}_I}}\\
e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*G), E_{et}(\Omega^{\bullet}_{/\tilde{S}_I}, F_b)))[-d_{\tilde{S}_I}], u_{IJ}(F) \otimes u_{IJ}(G))\\
\xrightarrow{(T(\otimes,\Omega_{/\tilde{S}_I})(L(i_{I*}j_I^*F),L(i_{I*}j_I^*G)))}\\
(e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*F) \otimes L(i_{I*}j_I^*G), E_{et}(\Omega^{\bullet}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], v_{IJ}(F \otimes G))\\
\xrightarrow{=} (e(\tilde{S}_I)_*\mathcal{H}om(L(i_{I*}j_I^*(F \otimes G), E_{et}(\Omega^{\bullet}_{/\tilde{S}_I}, F_b)))[-d_{\tilde{S}_I}], u_{IJ}(F \otimes G)) =: \mathcal{F}_S^{GM}(M \otimes N)
\end{aligned}$$

Proposition 60. *Let $f_1 : X_1 \to S$, $f_2 : X_2 \to S$ two morphism with $X_1, X_2, S \in \mathrm{Var}(k)$. Assume that there exist factorizations $f_1 : X_1 \xrightarrow{l_1} Y_1 \times S \xrightarrow{p_S} S$, $f_2 : X_2 \xrightarrow{l_2} Y_2 \times S \xrightarrow{p_S} S$ with $Y_1, Y_2 \in \mathrm{SmVar}(\mathbb{C})$, l_1, l_2 closed embeddings and p_S the projections. We have then the factorization*

$$f_1 \times f_2 : X_{12} := X_1 \times_S X_2 \xrightarrow{l_1 \times l_2} Y_1 \times Y_2 \times S \xrightarrow{p_S} S$$

Let $S = \cup_{i=1}^{l} S_i$ an open affine covering and denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. We have, for $M, N \in \mathrm{DA}(S)$ and $F, G \in C(\mathrm{Var}(k)^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following commutative diagram in $D_{Ofil,\mathcal{D}}(S/(\tilde{S}_I))$

$$\begin{tikzcd}
\mathcal{F}_S^{GM}(M(X_1/S)) \otimes_{O_S}^{L} \mathcal{F}_S^{GM}(M(X_2/S)) \arrow[r, "T(\mathcal{F}_S^{GM},\otimes)(M(X_1/S),M(X_2/S))"] \arrow[d, "I^{GM}(X_1/S)\otimes I^{GM}(X_2/S)"] & \begin{array}{c}\mathcal{F}_S^{GM}(M(X_1/S) \otimes M(X_2/S)) \\ = \mathcal{F}_S^{GM}(M(X_1 \times_S X_2/S))\end{array} \arrow[d, "I^{GM}(X_{12}/S)"] \\
\begin{array}{c}(p_{\tilde{S}_I*}\Gamma_{X_{1I}} E_{zar}(\Omega^{\bullet}_{Y_1\times\tilde{S}_I/\tilde{S}_I}, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X_1/S)) \otimes_{O_S} \\ (p_{\tilde{S}_I*}\Gamma_{X_{2I}} E_{zar}(\Omega^{\bullet}_{Y_2\times\tilde{S}_I/\tilde{S}_I}, F_b)[-d_{\tilde{S}_I}], w_{IJ}(X_2/S))\end{array} \arrow[r, "(Ew_{(Y_1\times\tilde{S}_I, Y_2\times\tilde{S}_I)/\tilde{S}_I})"] & \begin{array}{c}(p_{\tilde{S}_I*}\Gamma_{X_{12I}} E_{zar}(\Omega^{\bullet}_{Y_1\times Y_2\times\tilde{S}_I/\tilde{S}_I}, F_b)[-d_{\tilde{S}_I}], \\ w_{IJ}(X_{12}/S))\end{array}
\end{tikzcd}.$$

Proof. Immediate from definition. □

7.2 The algebraic filtered De Rham realization functor

Let k a field of caracteristic zero. We recall (see section 2), for $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(k)$, the commutative diagrams of sites (3) and (4)

$$\begin{tikzcd}
\mathrm{Var}(k)^2/T \arrow[rr, "\mu_T"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"'] & & \mathrm{Var}(k)^{2,pr}/T \arrow[dr, "\rho_T"] \arrow[dd, "P(f)", near start] & \\
& \mathrm{Var}(k)^{2,sm}/T \arrow[rr, "\mu_T", near start] \arrow[dd, "P(f)", near start] & & \mathrm{Var}(k)^{2,smpr}/T \arrow[dd, "P(f)"] \\
\mathrm{Var}(k)^2/S \arrow[rr, "\mu_S", near start] \arrow[dr, "\rho_S"] & & \mathrm{Var}(k)^{2,pr}/S \arrow[dr, "\rho_S"] & \\
& \mathrm{Var}(k)^{2,sm}/S \arrow[rr, "\mu_S"] & & \mathrm{Var}(k)^{2,smpr}/S
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Var}(k)^{2,pr}/T \arrow[rr, "\mathrm{Gr}_T^{12}"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"'] & & \mathrm{Var}(k)/T \arrow[dr, "\rho_T"] \arrow[dd, "P(f)", near start] & \\
& \mathrm{Var}(k)^{2,smpr}/T \arrow[rr, "\mathrm{Gr}_T^{12}", near start] \arrow[dd, "P(f)", near start] & & \mathrm{Var}(k)^{sm}/T \arrow[dd, "P(f)"] \\
\mathrm{Var}(k)^{2,pr}/S \arrow[rr, "\mathrm{Gr}_S^{12}", near start] \arrow[dr, "\rho_S"] & & \mathrm{Var}(k)/S \arrow[dr, "\rho_S"] & \\
& \mathrm{Var}(k)^{2,sm}/S \arrow[rr, "\mathrm{Gr}_S^{12}"] & & \mathrm{Var}(k)^{sm}/S
\end{tikzcd} .$$

For $s : \mathcal{I} \to \mathcal{J}$ a functor, with $\mathcal{I}, \mathcal{J} \in \mathrm{Cat}$, and $f_\bullet : T_\bullet \to S_{s(\bullet)}$ a morphism with $T_\bullet \in \mathrm{Fun}(\mathcal{J}, \mathrm{Var}(k))$ and $S_\bullet \in \mathrm{Fun}(\mathcal{I}, \mathrm{Var}(k))$, we have then the commutative diagrams of sites (5) and (6)

$$\begin{tikzcd}
\mathrm{Var}(k)^2/T_\bullet \arrow[rr, "\mu_{T_\bullet}"] \arrow[dr, "\rho_{T_\bullet}"] \arrow[dd, "P(f_\bullet)"'] & & \mathrm{Var}(k)^{2,pr}/T_\bullet \arrow[dr, "\rho_{T_\bullet}"] \arrow[dd, "P(f_\bullet)", near start] & \\
& \mathrm{Var}(k)^{2,sm}/T_\bullet \arrow[rr, "\mu_{T_\bullet}", near start] \arrow[dd, "P(f_\bullet)", near start] & & \mathrm{Var}(k)^{2,smpr}/T_\bullet \arrow[dd, "P(f_\bullet)"] \\
\mathrm{Var}(k)^2/S_\bullet \arrow[rr, "\mu_{S_\bullet}", near start] \arrow[dr, "\rho_{S_\bullet}"] & & \mathrm{Var}(k)^{2,pr}/S_\bullet \arrow[dr, "\rho_{S_\bullet}"] & \\
& \mathrm{Var}(k)^{2,sm}/S_\bullet \arrow[rr, "\mu_{S_\bullet}"] & & \mathrm{Var}(k)^{2,smpr}/S_\bullet
\end{tikzcd} .$$

and

$$\begin{tikzcd}
\mathrm{Var}(k)^{2,pr}/T_\bullet \arrow[rr, "\mathrm{Gr}^{12}_{T_\bullet}"] \arrow[dr, "\rho_{T_\bullet}"] \arrow[dd, "P(f_\bullet)"'] & & \mathrm{Var}(k)/T \arrow[dr, "\rho_{T_\bullet}"] \arrow[dd, "P(f_\bullet)"] & \\
& \mathrm{Var}(k)^{2,smpr}/T_\bullet \arrow[rr, "\mathrm{Gr}^{12}_{T_\bullet}"] \arrow[dd, "P(f_\bullet)"] & & \mathrm{Var}(k)^{sm}/T_\bullet \arrow[dd, "P(f_\bullet)"] \\
\mathrm{Var}(k)^{2,pr}/S_\bullet \arrow[rr, "\mathrm{Gr}^{12}_{S_\bullet}"] \arrow[dr, "\rho_{S_\bullet}"] & & \mathrm{Var}(k)/S_\bullet \arrow[dr, "\rho_{S_\bullet}"] & \\
& \mathrm{Var}(k)^{2,sm}/S_\bullet \arrow[rr, "\mathrm{Gr}^{12}_{S_\bullet}"] & & \mathrm{Var}(k)^{sm}/S_\bullet
\end{tikzcd}.$$

We will use the following map from the property of De Rham modules (see section 5) together with the specialization map of a filtered D module for a closed embedding (see [10] section 4.1) :

Definition-Proposition 7. *(i) Let $l : Z \hookrightarrow S$ a closed embedding with $S, Z \in \mathrm{SmVar}(k)$. Consider an open embedding $j : S^o \hookrightarrow S$. We then have the cartesian square*

$$\begin{tikzcd}
S^o \arrow[r, "j"] & S \\
Z^o := Z \times_S S^o \arrow[u, "l'"] \arrow[r, "j'"] & Z \arrow[u, "l"']
\end{tikzcd}$$

where j' is the open embedding given by base change. Using proposition 35(ii) or theorem 35, the morphisms $Q^{p,0}_{V_Z,V_D}(O_S, F_b)$ for $D \subset S$ a closed subset of definition-proposition of [10] induces a canonical morphism in $C_{l^*O_Sfil}(Z)$

$$Q(Z, j_!)(O_S, F_b) : l^*Q_{V_Z,0}j_{!Hdg}(O_{S^o}, F_b) \to j'_{!Hdg}(O_{Z^o}, F_b),$$

where V_Z is the Kashiwara-Malgrange V_Z-filtration and V_D is the Kashiwara-Malgrange V_D-filtration, which commutes with the action of T_Z.

(ii) Let $l : Z \hookrightarrow S$ and $k : Z' \hookrightarrow Z$ be closed embeddings with $S, Z, Z' \in \mathrm{SmVar}(k)$. Consider an open embedding $j : S^o \hookrightarrow S$. We then have the commutative diagram whose squares are cartesian.

$$\begin{tikzcd}
S^o \arrow[r, "j"] & S \\
Z^o := Z \times_S S^o \arrow[u, "l'"] \arrow[r, "j'"] & Z \arrow[u, "l"'] \\
Z'^o := Z' \times_S S^o \arrow[u, "k'"] \arrow[r, "j''"] & Z' \arrow[u, "k"']
\end{tikzcd}$$

where j' is the open embedding given by base change. Then,

$$Q(Z', j_!)(O_S, F_b) = Q(Z', j'_!)(O_Z, F_b) \circ (k^*Q_{V_{Z'},0}Q(Z, j_!)(O_S, F_b)) :$$
$$k^*Q_{V_{Z'},0}l^*Q_{V_Z,0}j_{!Hdg}(O_{S^o}, F_b) \xrightarrow{k^*Q_{V_{Z'},0}Q(Z,j_!)(O_S,F_b)} k^*Q_{V_{Z'},0}j'_{!Hdg}(O_{Z^o}, F_b)$$
$$\xrightarrow{Q(Z',j'_!)(O_Z,F_b)} j''_{!Hdg}(O_{Z'^o}, F_b)$$

in $C_{k^*l^*O_Sfil}(Z')$ which commutes with the action of $T_{Z'}$.

(iii) Consider a commutative diagram whose squares are cartesian

$$\begin{tikzcd}
S^{oo} \arrow[r, "j_2"] & S^o \arrow[r, "j_1"] & S \\
Z^{oo} := Z \times_S S^{oo} \arrow[u, "l''"] \arrow[r, "j'_2"] & Z^o := Z \times_S S^o \arrow[u, "l'"] \arrow[r, "j'_1"] & Z \arrow[u, "l"]
\end{tikzcd}$$

where j_1, j_2, and hence j'_1,j'_2 are open embeddings. We have then the following commutative diagram

$$\begin{tikzcd}
l^*Q_{V_Z,0}j_{1!Hdg}(O_{S^o},F_b) \arrow[r, "\mathrm{ad}(j_{2!Hdg},j_2^*)(O_{S^o},F_b)"] \arrow[d, "Q(Z,j_!)(O_S,F_b)"] & l^*Q_{V_Z,0}(j_1 \circ j_2)_!^{Hdg}(O_{S^{oo}},F_b) \arrow[d, "Q(Z,(j_1\circ j_2)_!)(O_S,F_b)"] \\
j'_{1!Hdg}(O_{Z^o},F_b) \arrow[r, "\mathrm{ad}(j'_{2!Hdg},j_2'^*)(O_{Z^o},F_b)"] & (j'_1 \circ j'_2)_{!Hdg}(O_{Z^{oo}},F_b)
\end{tikzcd}$$

in $C_{l^*O_Sfil}(Z)$ which commutes with the action of T_Z.

Proof. (i): By definition of $j_{!Hdg} : \pi_{S^o}(MHM(S^o)) \to C(DRM(S)))$, we have to construct the isomorphism for each complement of a (Cartier) divisor $j = j_D : S^o = S \backslash D \hookrightarrow S$. In this case, we have the closed embedding $i : S \hookrightarrow L$ given by the zero section of the line bundle $L = L_D$ associated to D. We have then, using definition-proposition of [10] section 4.1, the canonical morphism in $PSh_{l^*O_Sfil}(Z)$ which commutes with the action of T_Z

$$Q(Z,j_!)(O_S,F_b) : l^*Q_{V_Z,0}j_{!Hdg}(O_{S^o},F_b) \xrightarrow{T_!(l,j)(-)^{-1}} l^*j_{!Hdg}Q_{V_{Z^o},0}(O_{S^o},F_b) = j'_{!Hdg}(O_{Z^o},F_b).$$

and $V_Z^pT_!(l,j)(-)^{-1} = Q^{p,0}_{V_Z,V_S}(i_{*mod}(O_S,F_b))$. Now for $j : S^o = S\backslash R \hookrightarrow S$ an arbitrary open embedding, we set

$$Q(Z,j_!)(O_S,F_b) := \varprojlim_{(D_i),R\subset D_i\subset S} (Q(Z,j_{D_J!})(j^*_{D_I}(O_S,F_b)) : l^*Q_{V_Z,0}j_{!Hdg}(O_{S^o},F_b) \xrightarrow{\sim} j'_{!Hdg}(O_{Z^o},F_b)$$

(ii): Follows from [10] section 4.1.
(iii): Follows from [10] section 4.1. □

Using definition-proposition 6 in the projection case, and the specialization map given in [10] section 4 and the isomorphism of definition-proposition 7, in the closed embedding case, we have the following canonical map :

Definition 104. *Consider a commutative diagram in* SmVar(k) *whose square are cartesian*

$$\begin{tikzcd}
Z_T \arrow[rr, "i'"] \arrow[dr, "l'"] \arrow[dd, "g'"] & & T \arrow[dr, "l"] \arrow[dd] & & T\backslash Z_T \arrow[ll, "j'"'] \arrow[dr, "l"] \arrow[dd] & \\
& T \times_g Z \arrow[rr, "I\times i"] \arrow[dl, "p_Z"] & & T \times_g S \arrow[dl, "p_S"] & & T \times S\backslash(T\times Z) \arrow[ll, "I\times j"'] \arrow[dl, "p_S"] \\
Z \arrow[rr, "i"] & & S & & S\backslash Z \arrow[ll, "j"'] &
\end{tikzcd}$$

where i and hence $I \times i$ and i', are closed embeddings, j, $I \times j$, j' are the complementary open embeddings and $g : T \xrightarrow{l} T \times S \xrightarrow{p_S} S$ is the graph factorization, where l is the graph embedding and p_S the projection. Then, the map in $C_{l^*O_{T\times S}fil}(T)$

$$sp_{V_T}(\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b)) : l^*\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b) \xrightarrow{q_{V_T,0}} l^*Q_{V_T,0}(\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b))$$
$$\xrightarrow{Q(T,(I\times j)_!)(O_{T\times S},F_b):=T_!(l,(I\times j))(-)} \Gamma^{\vee,Hdg}_{Z_T}(O_T,F_b)$$

which commutes with the action of T_T, where the first map is given in [10] section 4.1 and the last map is studied definition-proposition 7, factors through

$$sp_{V_T}(\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b)) : l^*\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b) \xrightarrow{n} l^{*mod}\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b) \xrightarrow{\bar{sp}_{V_T}(\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b))} \Gamma^{\vee,Hdg}_{Z_T}(O_T,F_b),$$

with for $U\subset T\times S$ an open subset, $m\in\Gamma(U,O_{T\times S})$ and $h\in\Gamma(U_T,O_T)$, $n(m):=n\otimes 1$ and $\bar{sp}_{V_T}(-)(m\otimes h)=h\cdot sp_{V_T}(m)$; see definition-proposition 6 and theorem 35. Then,

$$\bar{sp}_{V_T}(\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b)) : l^{*mod}\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b)\to\Gamma^{\vee,Hdg}_{Z_T}(O_T,F_b),$$

is a map in $C_{\mathcal{D}(1,0)fil}(T)$, i.e. is D_T linear. We then consider the canonical map in $C_{\mathcal{D}(1,0)fil}(T)$

$$a(g,Z)(O_S,F_b) : g^{*mod}\Gamma^{\vee,Hdg}_Z(O_S,F_b) = l^{*mod}p_S^{*mod}\Gamma^{\vee,Hdg}_Z(O_S,F_b) \xrightarrow{l^{*mod}T^{Hdg}(p,\gamma^\vee)(O_S,F_b)^{-1}} l^{*mod}\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b) \xrightarrow{\bar{sp}_{V_T}(\Gamma^{\vee,Hdg}_{T\times Z}(O_{T\times S},F_b))} \Gamma^{\vee,Hdg}_{Z_T}(O_T,F_b).$$

Lemma 8. *(i) For $g:T\to S$ and $g:T'\to T$ two morphism with $S,T,T'\in\mathrm{SmVar}(k)$, considering the commutative diagram whose squares are cartesian*

$$\begin{tikzcd}
Z_{T'} \arrow[r, "i''"] \arrow[d, "g'"] & T' \arrow[d, "g'"] & T'\backslash Z_{T'} \arrow[l, "j''"'] \arrow[d, "g'"] \\
Z_T \arrow[r, "i'"] \arrow[d, "g"] & T \arrow[d, "g"] & T\backslash Z_T \arrow[l, "j'"'] \arrow[d, "g"] \\
Z \arrow[r, "i"] & S & S\backslash Z \arrow[l, "j"']
\end{tikzcd}$$

we have then

$$a(g\circ g',Z)(O_S,F_b) = a(g',Z_T)(O_T,F_b)\circ(g^{'*mod}a(g,Z)(O_S,F_b)) :$$
$$(g\circ g')^{*mod}\Gamma^{\vee,Hdg}_Z(O_S,F_b) = g^{'*mod}g^{*mod}\Gamma^{\vee,Hdg}_Z(O_S,F_b) \xrightarrow{g^{'*mod}a(g,Z)(O_S,F_b)} g^{'*mod}\Gamma^{\vee,Hdg}_{Z_T}(O_T,F_b) \xrightarrow{a(g',Z_T)(O_T,F_b)} \Gamma^{\vee,Hdg}_{Z_{T'}}(O_{T'},F_b).$$

(ii) For $g:T\to S$ a morphism with $S,T\in\mathrm{SmVar}(k)$, considering the commutative diagram whose squares are cartesian

$$\begin{tikzcd}
Z'_T \arrow[r, "k'"] \arrow[d, "g"] & Z_T \arrow[r, "i'"] \arrow[d, "g"] & T \arrow[d, "g"] \\
Z' \arrow[r, "k"] & Z \arrow[r, "i"] & S
\end{tikzcd}$$

we have then the following commutative diagram

$$\begin{tikzcd}
g^{*mod}\Gamma^{\vee,Hdg}_Z(O_S,F_b) \arrow[r, "g^{*mod}T(Z'/Z,\gamma^{\vee,Hdg})(O_S,F_b)"] \arrow[d, "a(g,Z)(O_S,F_b)"] & g^{*mod}\Gamma^{\vee,Hdg}_{Z'}(O_S,F_b) \arrow[d, "a(g,Z')(O_S,F_b)"] \\
\Gamma^{\vee,Hdg}_{Z_T}(O_T,F_b) \arrow[r, "T(Z'_T/Z_T,\gamma^{\vee,Hdg})(O_T,F_b)"] & \Gamma^{\vee,Hdg}_{Z'_T}(O_T,F_b)
\end{tikzcd}$$

Proof. (i):Follows from definition-proposition 7 (ii)
(ii):Follows from definition-proposition 7 (iii) □

We can now define the main object :

Definition 105. *(i) For $S \in \mathrm{SmVar}(k)$, we consider the filtered complexes of presheaves*

$$(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}) \in C_{D_S fil}(\mathrm{Var}(k)^{2,smpr}/S)$$

given by,

- *for $(Y \times S, Z)/S = ((Y \times S, Z), p) \in \mathrm{Var}(k)^{2,smpr}/S$,*

$$(\Omega^{\bullet,\Gamma,pr}_{/S}((Y \times S, Z)/S), F_{DR}) := ((\Omega^{\bullet}_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} \Gamma^{\vee,Hdg}_Z(O_{Y\times S}, F_b))(Y \times S)$$

with the structure of p^*D_S module given by proposition 30,

- *for $g : (Y_1 \times S, Z_1)/S = ((Y_1 \times S, Z_1), p_1) \to (Y \times S, Z)/S = ((Y \times S, Z), p)$ a morphism in $\mathrm{Var}(k)^{2,smpr}/S$, denoting for short $\hat{Z} := Z \times_{Y\times S} (Y_1 \times S)$,*

$$\begin{aligned}\Omega^{\bullet,\Gamma,pr}_{/S}(g) : &((\Omega^{\bullet}_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} \Gamma^{\vee,Hdg}_Z(O_{Y\times S}, F_b))(Y \times S)\\ &\xrightarrow{i_-} g^*((\Omega^{\bullet}_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} \Gamma^{\vee,Hdg}_Z(O_{Y\times S}, F_b))(Y_1 \times S)\\ &\xrightarrow{\Omega_{(Y_1\times S/Y\times S)/(S/S)}(\Gamma^{\vee,Hdg}_Z(O_{Y\times S},F_b))(Y_1\times S)} (\Omega^{\bullet}_{Y_1\times S/S}, F_b) \otimes_{O_{Y_1\times S}} g^{*mod}\Gamma^{\vee,Hdg}_Z(O_{Y\times S}, F_b))(Y_1 \times S)\\ &\xrightarrow{DR(Y_1\times S/S)(a(g,Z)(O_{Y\times S},F_b))(Y_1\times S)} (\Omega^{\bullet}_{Y_1\times S/S}, F_b) \otimes_{O_{Y_1\times S}} \Gamma^{\vee,Hdg}_{\hat{Z}}(O_{Y_1\times S}, F_b))(Y_1 \times S)\\ &\xrightarrow{DR(Y_1\times S/S)(T(Z_1/\hat{Z},\gamma^{\vee,Hdg})(O_{Y_1\times S},F_b))(Y_1\times S)} (\Omega^{\bullet}_{Y_1\times S/S}, F_b) \otimes_{O_{Y_1\times S}} \Gamma^{\vee,Hdg}_{Z_1}(O_{Y_1\times S}, F_b))(Y_1 \times S),\end{aligned}$$

where

  - *i_- is the arrow of the inductive limit,*
  - *we recall that*

$$\begin{aligned}\Omega_{(Y_1\times S/Y\times S)/(S/S)}(\Gamma^{\vee,Hdg}_Z(O_{Y\times S}, F_b)) : &g^*((\Omega^{\bullet}_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} \Gamma^{\vee,Hdg}_Z(O_{Y\times S}, F_b))\\ &\to (\Omega^{\bullet}_{Y_1\times S/S}, F_b) \otimes_{O_{Y_1\times S}} g^{*mod}\Gamma^{\vee,Hdg}_Z(O_{Y\times S}, F_b))\end{aligned}$$

    is the map given in [10] section 4.1, which is $p_1^*D_S$ linear by proposition 31,
  - *the map*

$$a(g, Z)(O_{Y\times S}, F_b) : g^{*mod}\Gamma^{\vee,Hdg}_Z(O_{Y\times S}, F_b) \to \Gamma^{\vee,Hdg}_{\hat{Z}}(O_{Y_1\times S}, F_b)$$

    is the map given in definition 104
  - *the map*

$$T(Z_1/\hat{Z}, \gamma^{\vee,Hdg})(O_{Y_1\times S}, F_b) : \Gamma^{\vee,Hdg}_{\hat{Z}}(O_{Y_1\times S}, F_b) \to \Gamma^{\vee,Hdg}_{Z_1}(O_{Y_1\times S}, F_b)$$

    is given in definition-proposition 6.

For $g : ((Y_1 \times S, Z_1), p_1) \to ((Y \times S, Z), p)$ and $g' : ((Y_1' \times S, Z_1'), p_1) \to ((Y_1 \times S, Z_1), p)$ two morphisms in $\mathrm{Var}(k)^{2,smpr}/S$, we have

$$\begin{aligned}\Omega^{\bullet,\Gamma,pr}_{/S}(g \circ g') = \Omega^{\bullet,\Gamma,pr}_{/S}(g') \circ \Omega^{\bullet,\Gamma,pr}_{/S}(g) : &((\Omega^{\bullet}_{Y\times S/S}, F_b) \otimes_{O_{Y\times S}} \Gamma^{\vee,Hdg}_Z(O_{Y\times S}, F_b))(Y \times S)\\ &\xrightarrow{\Omega^{\bullet,\Gamma,pr}_{/S}(g)} (\Omega^{\bullet}_{Y_1\times S/S}, F_b) \otimes_{O_{Y_1\times S}} \Gamma^{\vee,Hdg}_{Z_1}(O_{Y_1\times S}, F_b))(Y_1 \times S)\\ &\xrightarrow{\Omega^{\bullet,\Gamma,pr}_{/S}(g')} (\Omega^{\bullet}_{Y_1'\times S/S}, F_b) \otimes_{O_{Y_1'\times S}} \Gamma^{\vee,Hdg}_{Z_1'}(O_{Y_1'\times S}, F_b))(Y_1' \times S),\end{aligned}$$

since, denoting for short $\hat{Z} := Z \times_{Y\times S} (Y_1 \times S)$ and $\hat{Z}' := Z \times_{Y\times S} (Y_1' \times S)$

– *we have by lemma 8(i)*

$$a(g\circ g',\hat{Z}')(O_{Y\times S},F_b)=a(g',\hat{Z})(O_{Y_1\times S},F_b)\circ g'^{*mod}a(g,Z)(O_{Y\times S},F_b),$$

– *we have by lemma 8(ii)*

$$T(Z_1'/\hat{Z}',\gamma^{\vee,Hdg})(O_{Y_1'\times S},F_b)\circ a(g',\hat{Z})(O_{Y_1\times S},F_b)$$
$$=a(g',Z_1)(O_{Y_1\times S},F_b)\circ g'^{*mod}T(Z_1/\hat{Z},\gamma^{\vee,Hdg})(O_{Y_1\times S},F_b).$$

(ii) *For $S\in\mathrm{SmVar}(k)$, we have the canonical map $C_{O_Sfil,D_S}(\mathrm{Var}(k)^{sm}/S)$*

$$\mathrm{Gr}(\Omega_{/S}):\mathrm{Gr}^{12}_{S*}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\to(\Omega^{\bullet}_{/S},F_b)$$

given by, for $U/S=(U,h)\in\mathrm{Var}(k)^{sm}/S$

$$\begin{aligned}\mathrm{Gr}(\Omega_{/S})(U/S):\mathrm{Gr}^{12}_{S*}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})(U/S)&:=((\Omega^{\bullet}_{U\times S/S},F_b)\otimes_{O_{U\times S}}\Gamma^{\vee,Hdg}_U(O_{U\times S},F_b))(U\times S)\\ &\xrightarrow{\mathrm{ad}(i_U^*,i_{U*})(-)(U\times S)} i^*((\Omega^{\bullet}_{U\times S/S},F_b)\otimes_{O_{U\times S}}\Gamma^{\vee,Hdg}_U(O_{U\times S},F_b))(U)\\ &\xrightarrow{\Omega_{(U/U\times S)/(S/S)}(-)(U)}((\Omega^{\bullet}_{U/S},F_b)\otimes_{O_U}i_U^{*mod}\Gamma^{\vee,Hdg}_U(O_{U\times S},F_b))(U)\\ &\xrightarrow{DR(U/S)(a(i_U,U))(U)}(\Omega^{\bullet}_{U/S},F_b)(U)=:(\Omega^{\bullet}_{/S},F_b)(U/S)\end{aligned}$$

where $h:U\xrightarrow{i_U}U\times S\xrightarrow{p_S}S$ is the graph factorization with i_U the graph embedding and p_S the projection, note that $a(i_U,U)$ is an isomorphism since for $j_U:U\times S\backslash U\hookrightarrow U\times S$ the open complementary $i_U^{*mod}j^{Hdg}_{U!}(M,F,W)=0$.

Let $g:T\to S$ a morphism with $T,S\in\mathrm{SmVar}(k)$. We have the canonical morphism in $C_{g^*D_Sfil}(\mathrm{Var}(k)^{2,smpr}/T)$

$$\Omega^{\Gamma,pr}_{/(T/S)}:g^*(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\to(\Omega^{\bullet,\Gamma,pr}_{/T},F_{DR})$$

induced by the pullback of differential forms : for $((Y_1\times T,Z_1)/T)=((Y_1\times T,Z_1),p)\in\mathrm{Var}(k)^{2,smpr}/T$,

$$\begin{aligned}&\Omega^{\Gamma,pr}_{/(T/S)}((Y_1\times T,Z_1)/T):\\ g^*\Omega^{\bullet,\Gamma,pr}_{/S}((Y_1\times T,Z_1)/T):=&\lim_{(h:(Y\times S,Z)\to S,\,g_1:(Y_1\times T,Z_1)\to(Y\times T,Z_T),h,g)}\Omega^{\bullet,\Gamma,pr}_{/S}((Y\times T,Z)/S)\\ &\xrightarrow{\Omega^{\bullet,\Gamma,pr}_{/S}(g'\circ g_1)}\Omega^{\bullet,\Gamma,pr}_{/S}((Y_1\times T,Z_1)/S)\xrightarrow{q(-)(Y_1\times T)}\Omega^{\bullet,\Gamma,pr}_{/T}((Y_1\times T,Z_1)/T),\end{aligned}$$

where $g'=(I_Y\times g):Y\times T\to Y\times S$ is the base change map and $q(M):\Omega_{Y_1\times T/S}\otimes_{O_{Y_1\times T}}(M,F)\to\Omega_{Y_1\times T/T}\otimes_{O_{Y_1\times T}}(M,F)$ is the quotient map. It induces the canonical morphisms in $C_{g^*D_Sfil}(\mathrm{Var}(k)^{2,smpr}/T)$:

$$E\Omega^{\Gamma,pr}_{/(T/S)}:g^*E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\xrightarrow{T(g,E)(-)}E_{et}(g^*(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\xrightarrow{E_{et}(\Omega^{\Gamma,pr}_{/(T/S)})}E_{et}(\Omega^{\bullet,\Gamma,pr}_{/T},F_{DR})$$

and

$$E\Omega^{\Gamma,pr}_{/(T/S)}:g^*E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\xrightarrow{T(g,E)(-)}E_{zar}(g^*(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\xrightarrow{E_{zar}(\Omega^{\Gamma,pr}_{/(T/S)})}E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/T},F_{DR}).$$

Definition 106. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{SmVar}(k)$. We have, for $F \in C(\mathrm{Var}(k)^{2,smpr}/S)$, the canonical transformation in $C_{\mathcal{D}fil}(T)$:*

$$\begin{aligned} T(g, \Omega^{\Gamma,pr}_{/\cdot})(F) : g^{*mod} L_D e(S)_* \mathrm{Gr}^{12}_{S*} \mathcal{H}om^\bullet(F, E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) \\ \xrightarrow{=} (g^* L_D e(S)_* \mathcal{H}om^\bullet(F, E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))) \otimes_{g^* O_S} O_T \\ \xrightarrow{T(g, \mathrm{Gr}^{12})(-) \circ T(e,g)(-) \circ q} e(T)_* \mathrm{Gr}^{12}_{T*} g^* \mathcal{H}om^\bullet(F, E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) \otimes_{g^* O_S} O_T \\ \xrightarrow{(T(g,hom)(-,-) \otimes I)} e(T)_* \mathrm{Gr}^{12}_{T*} \mathcal{H}om^\bullet(g^* F, g^* E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) \otimes_{g^* O_S} O_T \\ \xrightarrow{ev(hom, \otimes)(-,-,-)} e(T)_* \mathrm{Gr}^{12}_{T*} \mathcal{H}om^\bullet(g^* F, g^* E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) \otimes_{g^* e(S)^* O_S} e(T)^* O_T \\ \xrightarrow{\mathcal{H}om^\bullet(g^* F, (E\Omega^{\Gamma,pr}_{/(T/S)} \otimes m))} e(T)_* \mathrm{Gr}^{12}_{T*} \mathcal{H}om^\bullet(g^* F, E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/T}, F_{DR})) \end{aligned}$$

The complex of presheaves $(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}) \in C_{D_S fil}(\mathrm{Var}(k)^{2,smpr}/S)$ have a monoidal structure given by the wedge product of differential forms: for $p : (Y \times S, Z) \to S \in \mathrm{Var}(k)^{2,smpr}/S$, the map

$$\begin{aligned} DR(-)(\gamma^{\vee,Hdg}_Z(-)) \circ w_{Y \times S/S} : (\Omega^\bullet_{Y \times S/S} \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \otimes_{p^* O_S} (\Omega^\bullet_{Y \times S/S} \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \\ \to \Omega^\bullet_{Y \times S/S} \otimes_{O_{Y \times S}} \Gamma^{\vee,Hdg}_Z (O_{Y \times S}, F_b) \end{aligned}$$

factors trough

$$\begin{aligned} DR(-)(\gamma^{\vee,Hdg}_Z(-)) \circ w_{Y \times S/S} : \\ (\Omega^\bullet_{Y \times S/S} \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \otimes_{p^* O_S} (\Omega^\bullet_{Y \times S/S} \otimes_{O_{Y \times S}} (O_{Y \times S}, F_b)) \\ \xrightarrow{DR(-)(\gamma^{\vee,Hdg}_Z(-)) \otimes DR(-)(\gamma^{\vee,Hdg}_Z(-))} \\ (\Omega^\bullet_{Y \times S/S} \otimes_{O_{Y \times S}} \Gamma^{\vee,Hdg}_Z)(O_{Y \times S}, F_b) \otimes_{p^* O_S} \Omega^\bullet_{Y \times S/S} \otimes_{O_{Y \times S}} \Gamma^{\vee,Hdg}_Z (O_{Y \times S}, F_b) \\ \xrightarrow{(DR(-)(\gamma^{\vee,Hdg}_Z(-)) \circ w_{Y \times S/S})^\gamma} \Omega^\bullet_{Y \times S/S} \otimes_{O_{Y \times S}} \Gamma^{\vee,Hdg}_Z (O_{Y \times S}, F_b) \end{aligned}$$

unique up to homotopy, giving the map in $C_{D_S fil}(\mathrm{Var}(k)^{2,smpr}/S)$:

$$w_S : (\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}) \otimes_{O_S} (\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}) \to (\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})$$

given by for $p : (Y \times S, Z) \to S \in \mathrm{Var}(k)^{2,smpr}/S$,

$$\begin{aligned} w_S((Y \times S, Z)/S) : \\ (((\Omega^\bullet_{Y \times S/S} \otimes_{O_{Y \times S}} \Gamma^{\vee,Hdg}_Z)(O_{Y \times S}, F_b)) \otimes_{p^* O_S} (\Omega^\bullet_{Y \times S/S} \otimes_{O_{Y \times S}} \Gamma^{\vee,Hdg}_Z (O_{Y \times S}, F_b)))(Y \times S) \\ \xrightarrow{(DR(-)(\gamma^{\vee,Hdg}_Z(-)) \circ w_{Y \times S/S})^\gamma (Y \times S)} (\Omega^\bullet_{Y \times S/S} \otimes_{O_{Y \times S}} \Gamma^{\vee,Hdg}_Z (O_{Y \times S}, F_b))(Y \times S) \end{aligned}$$

which induces the map in $C_{D_S fil}(\mathrm{Var}(k)^{2,smpr}/S)$

$$\begin{aligned} Ew_S : E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}) \otimes_{O_S} E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}) \xrightarrow{=} \\ E_{zar}((\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}) \otimes_{O_S} (\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR})) \xrightarrow{E_{zar}(w_S)} E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}) \end{aligned}$$

by the functoriality of the Godement resolution (see section 2).

Definition 107. *Let $S\in\mathrm{SmVar}(k)$. We have, for $F,G\in C(\mathrm{Var}(k)^{2,smpr}/S)$, the canonical transformation in $C_{\mathcal{D}fil}(S)$:*

$$\begin{aligned}&T(\otimes,\Omega)(F,G):\\ &e(S)_*\operatorname{Gr}^{12}_{S*}\mathcal{H}om(F,E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\otimes_{O_S}e(S)_*\operatorname{Gr}^{12}_{S*}\mathcal{H}om(G,E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\\ &\xrightarrow{=}e(S)_*\operatorname{Gr}^{12}_{S*}(\mathcal{H}om(F,E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))\otimes_{O_S}\mathcal{H}om(G,E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})))\\ &\xrightarrow{T(\mathcal{H}om,\otimes)(-)}e(S)_*\operatorname{Gr}^{12}_{S*}\mathcal{H}om(F\otimes G,E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\otimes_{O_S}E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})))\\ &\xrightarrow{=}e(S)_*\operatorname{Gr}^{12}_{S*}\mathcal{H}om(F\otimes G,(E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})\otimes_{O_S}E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})))\\ &\xrightarrow{\mathcal{H}om(F\otimes G,Ew_S)}e(S)_*\operatorname{Gr}^{12}_{S*}\mathcal{H}om(F\otimes G,E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR})).\end{aligned}$$

Let $S\in\mathrm{Var}(k)$. Let $S=\cup_{i=1}^{l}S_i$ an open affine cover and denote by $S_I=\cap_{i\in I}S_i$. Let $i_i:S_i\hookrightarrow\tilde{S}_i$ closed embeddings, with $\tilde{S}_i\in\mathrm{Var}(k)$. For $I\subset[1,\cdots l]$, denote by $\tilde{S}_I=\Pi_{i\in I}\tilde{S}_i$. We then have closed embeddings $i_I:S_I\hookrightarrow\tilde{S}_I$ and for $J\subset I$ the following commutative diagram

$$D_{IJ}=\quad\begin{array}{ccc}S_I&\xrightarrow{i_I}&\tilde{S}_I\\ {\scriptstyle j_{IJ}}\uparrow&&\uparrow{\scriptstyle p_{IJ}}\\ S_J&\xrightarrow{i_J}&\tilde{S}_J\end{array}$$

where $p_{IJ}:\tilde{S}_J\to\tilde{S}_I$ is the projection and $j_{IJ}:S_J\hookrightarrow S_I$ is the open embedding so that $j_I\circ j_{IJ}=j_J$. This gives the diagram of algebraic varieties $(\tilde{S}_I)\in\mathrm{Fun}(\mathcal{P}(\mathbb{N}),\mathrm{Var}(k))$ which the diagram of sites $\mathrm{Var}(k)^{2,smpr}/(\tilde{S}_I)\in\mathrm{Fun}(\mathcal{P}(\mathbb{N}),\mathrm{Cat})$. This gives also the diagram of algebraic varieties $(\tilde{S}_I)^{op}\in\mathrm{Fun}(\mathcal{P}(\mathbb{N})^{op},\mathrm{Var}(k))$ which the diagram of sites $\mathrm{Var}(k)^{2,smpr}/(\tilde{S}_I)^{op}\in\mathrm{Fun}(\mathcal{P}(\mathbb{N})^{op},\mathrm{Cat})$. We then get

$$((\Omega^{\bullet,\Gamma,pr}_{/(\tilde{S}_I)},F_{DR})[-d_{\tilde{S}_I}],T_{IJ})\in C_{D_{(\tilde{S}_I)}fil}(\mathrm{Var}(k)^{2,smpr}/(\tilde{S}_I))$$

with

$$\begin{aligned}T_{IJ}:(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR})[-d_{\tilde{S}_I}]&\xrightarrow{\mathrm{ad}(p_{IJ}^{*mod[-]},p_{IJ*}(-))}p_{IJ*}p_{IJ}^*(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR})\otimes_{p_{IJ}^*O_{\tilde{S}_I}}O_{\tilde{S}_J}[-d_{\tilde{S}_J}]\\ &\xrightarrow{m\circ p_{IJ*}\Omega^{\Gamma,pr}_{/(\tilde{S}_J/\tilde{S}_I)}[-d_{\tilde{S}_J}]}p_{IJ*}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J},F_{DR})[-d_{\tilde{S}_J}].\end{aligned}$$

For $(G_I,K_{IJ})\in C(\mathrm{Var}(k)^{2,smpr}/(\tilde{S}_I)^{op})$, we denote (see section 2)

$$\begin{aligned}&e'((\tilde{S}_I))_*\mathcal{H}om((G_I,K_{IJ}),(E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/(\tilde{S}_I)},F_{DR})[-d_{\tilde{S}_I}],T_{IJ})):=\\ &(e'(\tilde{S}_I)_*\mathcal{H}om(G_I,E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}],u_{IJ}((G_I,K_{IJ})))\in C_{\mathcal{D}fil}((\tilde{S}_I))\end{aligned}$$

with

$$\begin{aligned}&u_{IJ}((G_I,K_{IJ})):e'(\tilde{S}_I)_*\mathcal{H}om(G_I,E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))[-d_{\tilde{S}_I}]\\ &\xrightarrow{\mathrm{ad}(p_{IJ}^{*mod[-]},p_{IJ*})(-)\circ T(p_{IJ},e)(-)}p_{IJ*}e'(\tilde{S}_J)_*p_{IJ}^*\mathcal{H}om(G_I,E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))\otimes_{p_{IJ}^*O_{\tilde{S}_I}}O_{\tilde{S}_J}[-d_{\tilde{S}_J}]\\ &\xrightarrow{T(p_{IJ},hom)(-,-)}p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om(p_{IJ}^*G_I,p_{IJ}^*E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR}))\otimes_{p_{IJ}^*O_{\tilde{S}_I}}O_{\tilde{S}_J}[-d_{\tilde{S}_J}]\\ &\xrightarrow{m\circ\mathcal{H}om(p_{IJ}^*G_I,T_{IJ})}p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om(p_{IJ}^*G_I,E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J},F_{DR}))[-d_{\tilde{S}_J}]\\ &\xrightarrow{\mathcal{H}om(K_{IJ},E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J},F_{DR}))}p_{IJ*}e'(\tilde{S}_J)_*\mathcal{H}om(G_J,E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J},F_{DR}))[-d_{\tilde{S}_J}].\end{aligned}$$

This gives in particular

$$(\Omega^{\bullet,\Gamma,pr}_{/(\tilde{S}_I)}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}) \in C_{D_{(\tilde{S}_I)}fil}(\mathrm{Var}(k)^{2,(sm)pr}/(\tilde{S}_I)^{op}).$$

We now define the filtered De Rahm realization functor.

Definition 108. *(i) Let $S \in \mathrm{SmVar}(k)$. We have, using definition 105 and definition 27, the functor*

$$\mathcal{F}_S^{FDR} : C(\mathrm{Var}(k)^{sm}/S) \to C_{\mathcal{D}fil}(S), \; F \mapsto$$
$$\mathcal{F}_S^{FDR}(F) := e(S)_* \mathrm{Gr}^{12}_{S*} \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_S^* L(F)), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))[-d_S]$$

Moreover, the differentials of $\mathcal{F}_S^{FDR}$ are strict for the filtration by theorem 40.

(ii) Let $S \in \mathrm{Var}(k)$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. For $I \subset [1, \cdots l]$, denote by $S_I := \cap_{i\in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. We then have closed embeddings $i_I : S_I \hookrightarrow \tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. Consider, for $I \subset J$, the following commutative diagram

$$D_{IJ} = \begin{array}{ccc} S_I & \xrightarrow{i_I} & \tilde{S}_I \\ {\scriptstyle j_{IJ}}\uparrow & & \uparrow{\scriptstyle p_{IJ}} \\ S_J & \xrightarrow{i_J} & \tilde{S}_J \end{array}$$

and $j_{IJ} : S_J \hookrightarrow S_I$ is the open embedding so that $j_I \circ j_{IJ} = j_J$. We have, using definition 105 and definition 27, the functor

$$\mathcal{F}_S^{FDR} : C(\mathrm{Var}(k)^{sm}/S) \to C_{\mathcal{D}fil}(S/(\tilde{S}_I)), \; F \mapsto$$
$$\mathcal{F}_S^{FDR}(F) := e'((\tilde{S}_I))_* \mathcal{H}om^\bullet((\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*F)), \hat{R}^{CH}_{\tilde{S}_J}(T^q(D_{IJ})(j_I^*F))),$$
$$(E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/(\tilde{S}_I)}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))$$
$$:= (e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*F)), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^q_{IJ}(F))$$

where we have denoted for short $e'(\tilde{S}_I) = e(\tilde{S}_I) \circ \mathrm{Gr}^{12}_{\tilde{S}_I}$, and

$$u^q_{IJ}(F)[d_{\tilde{S}_J}] : e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*F)), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))$$
$$\xrightarrow{\mathrm{ad}(p_{IJ}^{*mod}, p_{IJ})(-)} p_{IJ*}p_{IJ}^{*mod} e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*F)), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))$$
$$\xrightarrow{p_{IJ*}T(p_{IJ},\Omega^{\gamma,pr}_{\cdot})(-)} p_{IJ*}e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(p_{IJ}^*\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*F)), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J}, F_{DR}))$$
$$\xrightarrow{\mathcal{H}om(T(p_{IJ},\hat{R}^{CH})(Li_{I*}j_I^*F)^{-1}, E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J}, F_{DR}))}$$
$$p_{IJ*}e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_J}^* p_{IJ}^* L(i_{I*}j_I^*F)), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J}, F_{DR}))$$
$$\xrightarrow{\mathcal{H}om(\hat{R}^{CH}_{\tilde{S}_J}(T^q(D_{IJ})(j_I^*F)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J}, F_{DR}))}$$
$$p_{IJ*}e'(\tilde{S}_J)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_J}^* L(i_{J*}j_J^*F)), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_J}, F_{DR})).$$

For $I \subset J \subset K$, we have obviously $p_{IJ*}u_{JK}(F) \circ u_{IJ}(F) = u_{IK}(F)$. Moreover, the differentials of $\mathcal{F}_S^{FDR}$ are strict for the filtration by theorem 40.

Recall, see section 2, that we have the projection morphisms of sites $p_a : \mathrm{Var}(k)^{2,smpr}/(\tilde{S}_I)^{op} \to \mathrm{Var}(k)^{2,smpr}/(\tilde{S}_I)^{op}$ given by the functor

$$p_a : \mathrm{Var}(k)^{2,smpr}/(\tilde{S}_I)^{op} \to \mathrm{Var}(k)^{2,smpr}/(\tilde{S}_I)^{op},$$
$$p_a((Y_I \times \tilde{S}_I, Z_I)/\tilde{S}_I, s_{IJ}) := ((Y_I \times \mathbb{A}^1 \times \tilde{S}_I, Z_I \times \mathbb{A}^1)/\tilde{S}_I, s_{IJ} \times I),$$
$$p_a((g_I) : ((Y'_I \times \tilde{S}_I, Z'_I)/\tilde{S}_I, s'_{IJ}) \to ((Y_I \times \tilde{S}_I, Z_I)/\tilde{S}_I, s_{IJ})) =$$
$$(g_I \times I) : ((Y'_I \times \mathbb{A}^1 \times \tilde{S}_I, Z'_I \times \mathbb{A}^1)/\tilde{S}_I, s'_{IJ} \times I) \to ((Y_I \times \mathbb{A}^1 \times \tilde{S}_I, Z_I \times \mathbb{A}^1)/\tilde{S}_I), s_{IJ} \times I)).$$

We have the following key proposition :

Proposition 61. *(i1) Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. The complex of presheaves $(\Omega^{\bullet,\Gamma,pr}_{/(\tilde{S}_I)}, F_{DR}) \in C_{D_{(\tilde{S}_I)}fil}(\mathrm{Var}(k)^{2,smpr}/(\tilde{S}_I)^{op})$ is 2-filtered $\mathbb{A}^1$ homotopic, that is*

$$\mathrm{ad}(p_a^*, p_{a*})(\Omega^{\bullet,\Gamma,pr}_{/(\tilde{S}_I)}, F_{DR}) : (\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}) \to p_{a*}p_a^*(\Omega^{\bullet,\Gamma,pr}_{/(\tilde{S}_I)}, F_{DR})$$

is a 2-filtered homotopy.

(i2) Let $S \in \mathrm{SmVar}(k)$. The complex of presheaves $(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}) \in C_{D_S fil}(\mathrm{Var}(k)^{2,smpr}/S)$ admits transferts, i.e.

$$\mathrm{Tr}(S)_* \mathrm{Tr}(S)^*(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}) = (\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}).$$

(iii) Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Let $m = (m_I) : (Q_{1I}, K^1_{IJ}) \to (Q_{2I}, K^2_{IJ})$ be an equivalence $(\mathbb{A}^1, et)$ local with $(Q_{1I}, K_{IJ}) \to (Q_{2I}, K_{IJ}) \in C(\mathrm{Var}(k)^{smpr}/(\tilde{S}_I)^{op})$ complexes of representable presheaves. Then, the map in $C_{\mathcal{D}fil}((\tilde{S}_I))$

$$\begin{aligned} M := (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(m_I, E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR})[-d_{\tilde{S}_I}])) :& \\ e'((\tilde{S}_I))_* \mathcal{H}om^\bullet((Q_{2I}, K^1_{IJ}), (E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/(\tilde{S}_I)}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))& \\ \to e'((\tilde{S}_I))_* \mathcal{H}om^\bullet((Q_{1I}, K^1_{IJ}), (E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/(\tilde{S}_I)}, F_{DR})[-d_{\tilde{S}_I}], T_{IJ}))& \end{aligned}$$

is a 2-filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil,\infty}((\tilde{S}_I))$.

Proof. (i1): Similar to the proof of [10], proposition (ii1)
(i2): Similar to the proof of [10], proposition (ii2) : Let $\alpha \in \mathrm{Cor}(\mathrm{Var}(\mathbb{C})^{2,smpr}/S)((Y_1 \times S, Z_1)/S, (Y_2 \times S, Z_2)/S)$ irreducible. Denote by $i : \alpha \hookrightarrow Y_1 \times Y_2 \times S$ the closed embedding, and $p_1 : Y_1 \times Y_2 \times S \to Y_1 \times S$, $p_2 : Y_1 \times Y_2 \times S \to Y_2 \times S$ the projections. The morphism $p_1 \circ i : \alpha \to Y_1 \times S$ is then finite surjective and $(Z_1 \times Y_2) \cap \alpha \subset Y_1 \times Z_2$ (i.e. $p_2(p_1^{-1}(Z_2) \cap \alpha) \subset Z_2$). Then, the transfert map is given by

$$\begin{aligned} \Omega^{\bullet,\Gamma,pr}_{/S}(\alpha) : ((\Omega^\bullet_{Y_2 \times S/S}, F_b) \otimes_{O_{Y_2 \times S}} \Gamma^{\vee,Hdg}_{Z_2}(O_{Y_2 \times S}, F_b))(Y_2 \times S) \\ \xrightarrow{i_-} p_2^*((\Omega^\bullet_{Y_2 \times S/S}, F_b) \otimes_{O_{Y_2 \times S}} \Gamma^{\vee,Hdg}_{Z_2}(O_{Y_2 \times S}, F_b))(Y_1 \times Y_2 \times S) \\ \xrightarrow{\Omega_{(Y_1 \times Y_2 \times S/Y_2 \times S)/(S/S)}(-)(-)} ((\Omega^\bullet_{Y_1 \times Y_2 \times S/S}, F_b) \otimes_{O_{Y_1 \times Y_2 \times S}} \Gamma^{\vee,Hdg}_{Y_1 \times Z_2}(O_{Y_1 \times Y_2 \times S}, F_b))(Y_1 \times Y_2 \times S) \\ \xrightarrow{DR(-)(T((Z_1 \times Y_2) \cap \alpha/Y_1 \times Z_2, \gamma^{\vee,Hdg})(-)(-)} \\ ((\Omega^\bullet_{Y_1 \times Y_2 \times S/S}, F_b) \otimes_{O_{Y_1 \times Y_2 \times S}} \Gamma^{\vee,Hdg}_{(Z_1 \times Y_2) \cap \alpha}(O_{Y_1 \times Y_2 \times S}, F_b))(Y_1 \times Y_2 \times S) \\ \xrightarrow{i_-} i^*((\Omega^\bullet_{Y_1 \times Y_2 \times S/S}, F_b) \otimes_{O_{Y_1 \times Y_2 \times S}} \Gamma^{\vee,Hdg}_{(Z_1 \times Y_2) \cap \alpha}(O_{Y_1 \times Y_2 \times S}, F_b))(\alpha) \\ \xrightarrow{\Omega_{(\alpha/Y_1 \times Y_2 \times S)/(S/S)}(-)(-)} ((\Omega^\bullet_{\alpha/S}, F_b) \otimes_{O_\alpha} i^{*mod}\Gamma^{\vee,Hdg}_{(Z_1 \times Y_2) \cap \alpha}(O_{Y_1 \times Y_2 \times S}, F_b))(\alpha) \\ \xrightarrow{\Omega_{(\alpha/Y_1 \times S)(S/S)}(-)(-)^{tr}} ((\Omega^\bullet_{Y_1 \times S/S}, F_b) \otimes_{O_{Y_1 \times S}} \Gamma^{\vee,Hdg}_{Z_1}(O_{Y_1 \times S}, F_b))(Y_1 \times S). \end{aligned}$$

(ii):Follows from (i) and theorem 18. □

Proposition 62. *Let $S \in \mathrm{Var}(k)$.Let $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$.*

(i) *Let* $m=(m_I):(Q_{1I},K^1_{IJ})\to(Q_{2I},K^2_{IJ})$ *be an etale local equivalence local with* $(Q_{1I},K^1_{IJ}),(Q_{2I},K^2_{IJ})\in C(\mathrm{Var}(k)^{sm}/(\tilde S_I))$ *complexes of projective presheaves. Then,*

$$(e'(\tilde S_I)_*\mathcal{H}om^\bullet(\hat R^{CH}_S(m_I),E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde S_I},F_{DR}))[-d_{\tilde S_I}]):$$
$$e'(\tilde S_I)_*\mathcal{H}om^\bullet((\hat R^{CH}(\rho_S^*Q_{1I}),\hat R^{CH}(K^1_{IJ})),(E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde S_I},F_{DR})[-d_{\tilde S_I}],T_{IJ}))$$
$$\to e'(\tilde S_I)_*\mathcal{H}om^\bullet((\hat R^{CH}(\rho_S^*Q_{2I}),\hat R^{CH}(K^2_{IJ})),(E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde S_I},F_{DR})[-d_{\tilde S_I}],T_{IJ}))$$

is a filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil}((\tilde S_I))$.

(ii) *Let* $m=(m_I):(Q_{1I},K^1_{IJ})\to(Q_{2I},K^2_{IJ})$ *be an equivalence* $(\mathbb{A}^1,et)$ *local equivalence local with* $(Q_{1I},K^1_{IJ}),(Q_{2I},K^2_{IJ})\in C(\mathrm{Var}(k)^{sm}/(\tilde S_I))$ *complexes of projective presheaves. Then,*

$$(e'(\tilde S_I)_*\mathcal{H}om^\bullet(\hat R^{CH}_S(m_I),E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde S_I},F_{DR}))[-d_{\tilde S_I}]):$$
$$e'(\tilde S_I)_*\mathcal{H}om^\bullet((\hat R^{CH}(\rho_S^*Q_{1I}),\hat R^{CH}(K^1_{IJ})),(E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde S_I},F_{DR})[-d_{\tilde S_I}],T_{IJ}))$$
$$\to e'(\tilde S_I)_*\mathcal{H}om^\bullet((\hat R^{CH}(\rho_S^*Q_{2I}),\hat R^{CH}(K^2_{IJ})),(E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde S_I},F_{DR})[-d_{\tilde S_I}],T_{IJ}))$$

is a filtered quasi-isomorphism. It is thus an isomorphism in $D_{\mathcal{D}fil}((\tilde S_I))$.

Proof. Follows from proposition 61 (see the proof the complex case in [10] section 6) and the fact that the differential of the complexes involved are strict for the F-fitration. □

Definition 109. (i) *Let* $S\in\mathrm{SmVar}(k)$. *We define using definition 108(i) and proposition 62(ii) the filtered algebraic De Rahm realization functor defined as*

$$\mathcal{F}^{FDR}_S:\mathrm{DA}_c(S)\to D_{\mathcal{D}fil}(S),\ M\mapsto$$
$$\mathcal{F}^{FDR}_S(M):=e(S)_*\operatorname{Gr}^{12}_{S*}\mathcal{H}om^\bullet(\hat R^{CH}(\rho_S^*L(F)),E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))[-d_S]$$

where $F\in C(\mathrm{Var}(k)^{sm}/S)$ *is such that* $M=D(\mathbb{A}^1,et)(F)$.

(i)' *For the Corti-Hanamura weight structure* W *on* $\mathrm{DA}_c(S)^-$, *we define using definition 108(i) and proposition 62(ii)*

$$\mathcal{F}^{FDR}_S:\mathrm{DA}^-_c(S)\to D^-_{\mathcal{D}(1,0)fil}(S),\ M\mapsto$$
$$\mathcal{F}^{FDR}_S((M,W)):=e(S)_*\operatorname{Gr}^{12}_{S*}\mathcal{H}om^\bullet(\hat R^{CH}(\rho_S^*L(F,W)),E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S},F_{DR}))[-d_S]$$

where $(F,W)\in C_{fil}(\mathrm{Var}(k)^{sm}/S)$ *is such that* $M=D(\mathbb{A}^1,et)((F,W))$ *using corollary 2. Note that the filtration induced by* W *is a filtration by sub* D_S *module, which is a stronger property then Griffitz transversality. Of course, the filtration induced by* F *satisfy only Griffitz transversality in general.*

(ii) *Let* $S\in\mathrm{Var}(k)$ *and* $S=\cup_{i=1}^l S_i$ *an open cover such that there exist closed embeddings* $i_i:S_i\hookrightarrow\tilde S_i$ *with* $\tilde S_i\in\mathrm{SmVar}(k)$. *For* $I\subset[1,\cdots l]$, *denote by* $S_I=\cap_{i\in I}S_i$ *and* $j_I:S_I\hookrightarrow S$ *the open embedding. We then have closed embeddings* $i_I:S_I\hookrightarrow\tilde S_I:=\Pi_{i\in I}\tilde S_i$. *We define, using definition 108(ii) and proposition 62(ii), the filtered algebraic De Rahm realization functor defined as*

$$\mathcal{F}^{FDR}_S:\mathrm{DA}_c(S)\to D_{\mathcal{D}fil}(S/(\tilde S_I)),\ M\mapsto$$
$$\mathcal{F}^{FDR}_S(M):=(e'(\tilde S_I)_*\mathcal{H}om^\bullet(\hat R^{CH}(\rho_{\tilde S_I}^*L(i_{I*}j_I^*F)),E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde S_I},F_{DR}))[-d_{\tilde S_I}],u^q_{IJ}(F))$$

where $F\in C(\mathrm{Var}(k)^{sm}/S)$ *is such that* $M=D(\mathbb{A}^1,et)(F)$, *see definition 108.*

(ii)' For the Corti-Hanamura weight structure W on $\mathrm{DA}_c^-(S)$, using definition 108(ii) and proposition 62(ii),

$$\mathcal{F}_S^{FDR} : \mathrm{DA}_c^-(S) \to D^-_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)), \; M \mapsto \mathcal{F}_S^{FDR}((M,W)) :=$$
$$(e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F,W))), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^q_{IJ}(F,W))$$

where $(F,W) \in C_{fil}(\mathrm{Var}(k)^{sm}/S)$ is such that $(M,W) = D(\mathbb{A}^1, et)(F,W)$ using corollary 2. Note that the filtration induced by W is a filtration by sub $D_{\tilde{S}_I}$-modules, which is a stronger property then Griffitz transversality. Of course, the filtration induced by F satisfy only Griffitz transversality in general.

Proposition 63. *For $S \in \mathrm{Var}(k)$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$, the functor $\mathcal{F}_S^{FDR}$ is well defined.*

Proof. Similar to the proof of [10] proposition : follows from proposition 62. □

Remark 10. *(i) Let $S \in \mathrm{SmVar}(k)$. We have, by proposition 61, for $M \in \mathrm{DA}_c(S)$ the isomorphism in $D^-_{\mathcal{D}(1,0)fil,\infty}(S)$*

$$\mathcal{H}om(-,k) \circ \mathcal{H}om(T(\hat{R}^{CH}, R^{CH})(\rho_S^* L(F,W)), -)^{-1} :$$
$$\mathcal{F}_S^{FDR}((M,W)) := e'(S)_*\mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_S^* L(F,W)), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))[-d_S]$$
$$\xrightarrow{\sim} e'(S)_*\mathcal{H}om^\bullet(L\mu_{S*}\rho_{S*}R^{CH}(\rho_S^* L(F,W)), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/S}, F_{DR}))[-d_S]$$

as it was defined in [10].

(ii) Let $S \in \mathrm{Var}(k)$ and $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We have, by proposition 61, for $M \in \mathrm{DA}_c(S)$ the isomorphism in $D^-_{\mathcal{D}(1,0)fil,\infty}(S/(\tilde{S}_I))$

$$(\mathcal{H}om(-,k)) \circ (\mathcal{H}om(T(\hat{R}^{CH}, R^{CH})(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F,W))), -))^{-1} :$$
$$\mathcal{F}_S^{FDR}((M,W)) := (e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F,W))), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^q_{IJ}(F,W))$$
$$\xrightarrow{\sim} (e'(\tilde{S}_I)_*\mathcal{H}om^\bullet(L\mu_{\tilde{S}_I*}\rho_{\tilde{S}_I*}R^{CH}(\rho_{\tilde{S}_I}^* L(i_{I*}j_I^*(F,W))), E_{et}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^q_{IJ}(F,W))$$

as it was defined in [10].

Proposition 64. *Let $f : X \to S$ a morphism with $S, X \in \mathrm{Var}(k)$. Assume there exist a factorization*

$$f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$$

of f with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. Let $\bar{Y} \in \mathrm{PSmVar}(k)$ a compactification of Y with $\bar{Y}\backslash Y = D$ a normal crossing divisor, denote $k : D \hookrightarrow \bar{Y}$ the closed embedding and $n : Y \hookrightarrow \bar{Y}$ the open embedding. Denote $\bar{X} \subset \bar{Y} \times S$ the closure of $X \subset \bar{Y} \times S$. We have then the following commutative diagram in $\mathrm{Var}(k)$

$$\begin{tikzcd}
X \arrow[r, "l"] \arrow[d] & Y\times S \arrow[d, "(n\times I)"] \arrow[dr, "p_S"] & \\
\bar{X} \arrow[r, "l"] & \bar{Y}\times S \arrow[r, "\bar{p}_S"] & S \\
Z:=\bar{X}\backslash X \arrow[u] \arrow[ur, "l_Z"] \arrow[r] & D\times S \arrow[u, "(k\times I)"] \arrow[ur] &
\end{tikzcd} .$$

Let $S=\cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then $X=\cup_{i=1}^l X_i$ with $X_i := f^{-1}(S_i)$. Denote, for $I\subset [1,\cdots l]$, $S_I=\cap_{i\in I}S_i$ and $X_I=\cap_{i\in I}X_i$. Denote $\bar{X}_I := \bar{X}\cap(\bar{Y}\times S_I)\subset \bar{Y}\times\tilde{S}_I$ the closure of $X_I\subset\bar{Y}\times\tilde{S}_I$, and $Z_I := Z\cap(\bar{Y}\times S_I)=\bar{X}_I\backslash X_I\subset\bar{Y}\times\tilde{S}_I$. We have then for $I\subset[1,\cdots l]$, the following commutative diagram in $\mathrm{Var}(k)$

$$\begin{tikzcd}
X_I \arrow[r, "l_I"] \arrow[d] & Y\times\tilde{S}_I \arrow[d, "(n\times I)"] \arrow[dr, "p_{\tilde{S}_I}"] & \\
\bar{X}_I \arrow[r, "l_I"] & \bar{Y}\times\tilde{S}_I \arrow[r, "\bar{p}_{\tilde{S}_I}"] & \tilde{S}_I \\
Z_I=\bar{X}_I\backslash X_I \arrow[u] \arrow[ur, "l_{Z_I}"] \arrow[r] & D\times\tilde{S}_I \arrow[u, "(k\times I)"] \arrow[ur] &
\end{tikzcd}.$$

Let $F(X/S) := p_{S,\sharp}\Gamma^{\vee}_X\mathbb{Z}(Y\times S/Y\times S)\in C(\mathrm{Var}(k)^{sm}/S)$. We have then the following isomorphism in $D_{\mathcal{D}fil}(S/(\tilde{S}_I))$

$$\begin{aligned}
&I(X/S) : \mathcal{F}_S^{FDR}(M(X/S)) \xrightarrow{:=} \\
&(e'(\tilde{S}_I)_*\mathcal{H}om(\hat{R}^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*F(X/S))), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u^q_{IJ}(F(X/S))) \\
&\xrightarrow{(\mathcal{H}om(\hat{R}^{CH}_{\tilde{S}_I}(N_I(X/S)),E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR})))} \\
&(e'(\tilde{S}_I)_*\mathcal{H}om(\hat{R}^{CH}(\rho^*_{\tilde{S}_I}Q(X_I/\tilde{S}_I)), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], v^q_{IJ}(F(X/S))) \\
&\xrightarrow{(\mathcal{H}om(\rho_{\tilde{S}_I*}I_\delta((\bar{X}_I,Z_I)/\tilde{S}_I),-)[-d_{\tilde{S}_I}])^{-1}} \\
&(p_{\tilde{S}_I*}E_{zar}((\Omega^{\bullet}_{\bar{Y}\times\tilde{S}_I/\tilde{S}_I},F_b)\otimes_{O_{\bar{Y}\times\tilde{S}_I}}(n\times I)^{Hdg}_!\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I},F_b))(d_Y+d_{\tilde{S}_I})[2d_Y+d_{\tilde{S}_I}], w_{IJ}(X/S)) \\
&\xrightarrow{=:} \iota_S Rf^{Hdg}_!(\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I},F_b)(d_Y)[2d_Y], x_{IJ}(X/S)). \xrightarrow{=:} \iota_S Rf^{Hdg}_! f^{*mod}_{Hdg}\mathbb{Z}^{Hdg}_S.
\end{aligned}$$

Proof. Similar to the proof of [10], proposition □

Corollary 4. *Let $S\in\mathrm{Var}(k)$ and $S=\cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(k)$. Then for $M\in\mathrm{DA}_c(S)$, $\mathcal{F}_S^{FDR}\in\iota_S(D(DRM(S)))$, where $\iota_S : D(DRM(S))\hookrightarrow D_{\mathcal{D}fil}(S/(\tilde{S}_I))$ is a full embedding by theorem 41.*

Proof. There exist by definition of constructible motives an isomorphism $\mathrm{DA}(S)$

$$w(M) : M \xrightarrow{\sim} \mathrm{Cone}(M(X_1/S)\to\cdots\to M(X_r/S)).$$

Hence we have the isomorphism in $D_{\mathcal{D}fil}(S/(\tilde{S}_I))$

$$\mathcal{F}_S^{FDR}(w(M)) : \mathcal{F}_S^{FDR}(M) \xrightarrow{\sim} \mathrm{Cone}(\mathcal{F}_S^{FDR}(M(X_1/S))\to\cdots\to\mathcal{F}_S^{FDR}(M(X_r/S))).$$

The result then follows from proposition 64. □

Proposition 65. *For $S\in\mathrm{Var}(k)$ not smooth, the functor (see corollary 4)*

$$\iota_S^{-1}\mathcal{F}_S^{FDR} : \mathrm{DA}_c^-(S)^{op}\to D(DRM(S))$$

does not depend on the choice of the open cover $S=\cup_i S_i$ and the closed embeddings $i_i : S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(k)$.

Proof. Similar to the proof of [10] proposition □

We have the canonical transformation map between the filtered De Rham realization functor and the Gauss-Manin realization functor :

Definition 110. *Let $S \in \mathrm{Var}(k)$ and $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Let $M \in \mathrm{DA}_c(S)$ and $F \in C(\mathrm{Var}(k)^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$. We have, using definition 105(ii), definition 26, proposition 1 and proposition 61, the canonical map in $D_{O_S fil,\mathcal{D},\infty}(S/(\tilde{S}_I))$*

$$\begin{aligned}
&T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : \\
&\mathcal{F}_S^{GM}(L\mathbb{D}_S M) = (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L(i_{I*} j_I^* \mathbb{D}_S LF), E_{zar}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\
&\xrightarrow{\sim} (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\mathbb{D}^0_{\tilde{S}_I} L(i_{I*} j_I^* F), E_{zar}(\Omega^\bullet_{/\tilde{S}_I}, F_b))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\
&\xrightarrow{\mathcal{H}om(-,\mathrm{Gr}(\Omega_{\tilde{S}_I}))^{-1}} (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(L\mathbb{D}^0_{\tilde{S}_I} L(i_{I*} j_I^* F), \mathrm{Gr}^{12}_{\tilde{S}_I *} E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\
&\xrightarrow{I(\mathrm{Gr}^{12*}_{\tilde{S}_I}, \mathrm{Gr}^{12}_{\tilde{S}_I *})(-,-)} (e(\tilde{S}_I)_* \mathcal{H}om^\bullet(\mathrm{Gr}^{12*}_{\tilde{S}_I} L\mathbb{D}^0_{\tilde{S}_I} L(i_{I*} j_I^* F), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) \\
&\xrightarrow{(\mathcal{H}om^\bullet(r^{CH}(L(i_{I*} j_I^* F) \circ T(\hat{R}^{CH}, R^{CH})(L(i_{I*} j_I^* F)), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}])} \\
&(e'(\tilde{S}_I)_* \mathcal{H}om^\bullet(\hat{R}^{CH}(\rho^*_{\tilde{S}_I} L(i_{I*} j_I^* F)), E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}))[-d_{\tilde{S}_I}], u_{IJ}^q(F)) =: \mathcal{F}_S^{FDR}(M)
\end{aligned}$$

Proposition 66. *Let $S \in \mathrm{Var}(k)$ and $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$.*

(i) For $M \in \mathrm{DA}_c(S)$ the map in $D_{O_S,\mathcal{D}}(S/(\tilde{S}_I)) = D_{O_S,\mathcal{D}}(S)$

$$o_{fil} T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : o_{fil} \mathcal{F}_S^{GM}(L\mathbb{D}_S M) \xrightarrow{\sim} o_{fil} \mathcal{F}_S^{FDR}(M)$$

given in definition 110 is an isomorphism if we forgot the Hodge filtration F.

(ii) For $M \in \mathrm{DA}_c(S)$ and all $n, p \in \mathbb{Z}$, the map in $\mathrm{PSh}_{O_S,\mathcal{D}}(S/(\tilde{S}_I))$

$$F^p H^n T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M) : F^p H^n \mathcal{F}_S^{GM}(L\mathbb{D}_S M) \hookrightarrow F^p H^n \mathcal{F}_S^{FDR}(M)$$

given in definition 110 is a monomorphism. Note that $F^p H^n T(\mathcal{F}_S^{GM}, \mathcal{F}_S^{FDR})(M)$ is NOT an isomorphism in general : take for example $M(S^o/S)^\vee = D(\mathbb{A}^1, et)(j_* E_{et}(\mathbb{Z}(S^o/S)))$ for an open embedding $j : S^o \hookrightarrow S$, then

$$H^n \mathcal{F}_S^{GM}(L\mathbb{D}_S M(S^o/S)^\vee) = \mathcal{F}_S^{GM}(\mathbb{Z}(S^o/S)) = j_* E(O_{S^o}, F_b) \notin \pi_S(MHM(S))$$

and hence is NOT isomorphic to $H^n \mathcal{F}_S^{FDR}(L\mathbb{D}_S M(S^o/S)^\vee) \in \pi_S(MHM(S))$ as filtered D_S-modules (see remark 4). It is an isomorphism in the very particular cases where $M = D(\mathbb{A}^1, et)(\mathbb{Z}(X/S))$ or $M = D(\mathbb{A}^1, et)(\mathbb{Z}(X^o/S))$ for $f : X \to S$ is a smooth proper morphism and $n : X^o \hookrightarrow X$ is an open subset such that $X \backslash X^o = \cup D_i$ is a normal crossing divisor and such that $f_{|D_i} = f \circ i_i : D_i \to X$ are SMOOTH morphism with $i_i : D_i \hookrightarrow X$ the closed embedding and considering $f_{|X^o} = f \circ n : X^o \to S$ (see [10] section 6.1 in the complex case).

Proof. (i):Follows from the computation for a Borel-Moore motive.
(ii):Follows from (i). □

We now define the functorialities of $\mathcal{F}_S^{FDR}$ with respect to S which makes $\mathcal{F}_{FDR}^-$ a morphism of 2 functor.

Definition 111. *Let $S \in \mathrm{Var}(k)$. Let $Z \subset S$ a closed subset. Let $S = \cup_{i=1}^{l} S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Denote $Z_I := Z \cap S_I$. We then have closed embeddings $Z_I \hookrightarrow S_I \hookrightarrow \tilde{S}_I$.*

(i) For $F\in C(\mathrm{Var}(k)^{sm}/S)$, we will consider the following canonical map in $D(DRM(S))\subset D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$

$$\begin{aligned}T(\Gamma_Z^{\vee,Hdg},\Omega_{/S}^{\Gamma,pr})(F,W):\\ \Gamma_Z^{\vee,Hdg}\iota_S^{-1}(e'_*\mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^*L(i_{I*}j_I^*(F,W))),E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr},F_{DR}))[-d_{\tilde{S}_I}],u_{IJ}^q(F,W))\\ \xrightarrow{\mathcal{H}om^\bullet(\hat{R}_{\tilde{S}_I}^{CH}(\gamma^{\vee,Z_I}(L(i_{I*}j_I^*(F,W)))),E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr},F_{DR}))}\\ \Gamma_Z^{\vee,Hdg}\iota_S^{-1}(e'_*\mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^*\Gamma_{Z_I}^{\vee}L(i_{I*}j_I^*(F,W))),E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr},F_{DR}))[-d_{\tilde{S}_I}],u_{IJ}^{q,Z}(F,W))\\ \xrightarrow{=}\iota_S^{-1}(e'_*\mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^*\Gamma_{Z_I}^{\vee}L(i_{I*}j_I^*(F,W))),E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr},F_{DR}))[-d_{\tilde{S}_I}],u_{IJ}^{q,Z}(F,W)).\end{aligned}$$

with $u_{IJ}^{q,Z}(F)$ as in [10].

(ii) For $F\in C(\mathrm{Var}(k)^{sm}/S)$, we have also the following canonical map in $D(DRM(S))\subset D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$

$$\begin{aligned}T(\Gamma_Z^{Hdg},\Omega_{/S}^{\Gamma,pr})(F,W):\\ \iota_S^{-1}(e'_*\mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^*L\Gamma_{Z_I}E(i_{I*}j_I^*\mathbb{D}_S(F,W))),E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr},F_{DR}))[-d_{\tilde{S}_I}],u_{IJ}^{q,Z,d}(F,W))\xrightarrow{=}\\ \Gamma_Z^{Hdg}\iota_S^{-1}(e'_*\mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^*L\Gamma_{Z_I}E(i_{I*}j_I^*\mathbb{D}_S(F,W))),E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr},F_{DR}))[-d_{\tilde{S}_I}],u_{IJ}^{q,Z,d}(F,W))\\ \xrightarrow{\mathcal{H}om^\bullet(\hat{R}_{\tilde{S}_I}^{CH}(\gamma^{Z_I}(-)),E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr},F_{DR}))}\\ \Gamma_Z^{Hdg}\iota_S^{-1}(e'_*\mathcal{H}om^\bullet(\hat{R}^{CH}(\rho_{\tilde{S}_I}^*L(i_{I*}j_I^*\mathbb{D}_S(F,W))),E_{zar}(\Omega_{/\tilde{S}_I}^{\bullet,\Gamma,pr},F_{DR}))[-d_{\tilde{S}_I}],u_{IJ}^{q}(F,W))\end{aligned}$$

with $u_{IJ}^{q,Z}(F)$ as in [10].

Definition 112. *Let $g:T\to S$ a morphism with $T,S\in\mathrm{Var}(k)$. Assume we have a factorization $g:T\xrightarrow{l}Y\times S\xrightarrow{p_S}S$ with $Y\in\mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. Let $S=\cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(k)$ Then, $T=\cup_{i=1}^l T_i$ with $T_i:=g^{-1}(S_i)$ and we have closed embeddings $i'_i:=i_i\circ l:T_i\hookrightarrow Y\times\tilde{S}_i$, Moreover $\tilde{g}_I:=p_{\tilde{S}_I}:Y\times\tilde{S}_I\to\tilde{S}_I$ is a lift of $g_I:=g_{|T_I}:T_I\to S_I$. Let $M\in\mathrm{DA}_c(S)^-$ and $(F,W)\in C_{fil}(\mathrm{Var}(k)^{sm}/S)$ such that $(M,W)=D(\mathbb{A}_S^1,et)(F,W)$. Then, $D(\mathbb{A}_T^1,et)(g^*F)=g^*M$ and there exist $(F',W)\in C_{fil}(\mathrm{Var}(k)^{sm}/S)$ and an equivalence $(\mathbb{A}^1,et)$ local $e:g^*(F,W)\to(F',W)$ such that $D(\mathbb{A}_T^1,et)(F',W)=(g^*M,W)$.Denote for short $d_{YI}:=-d_Y-d_{\tilde{S}_I}$. We have, using definition 106 and definition 111(i), the canonical map in*

$D(DRM(T)) \subset D_{\mathcal{D}(1,0)fil}(T/(Y \times \tilde{S}_I))$

$$T(g,\mathcal{F}^{FDR})(M) : g^{\hat{*}mod}_{Hdg}\iota_S^{-1}\mathcal{F}^{FDR}_S(M) :=$$
$$\Gamma^{\vee,Hdg}_T\iota_T^{-1}(\tilde{g}_I^{*mod}(e'_*\mathcal{H}om^{\bullet}(\hat{R}^{CH}(\rho^*_{\tilde{S}_I}(L(i_{I*}j_I^*(F,W))),E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I},F_{DR})))[d_{YI}],\tilde{g}_J^{*mod}u^q_{IJ}(F,W)))$$
$$\xrightarrow{(T(\tilde{g}_I,\Omega^{\Gamma,pr}_{/\cdot})(-))}$$
$$\Gamma^{\vee,Hdg}_T\iota_T^{-1}(e'_*\mathcal{H}om^{\bullet}(\tilde{g}_I^*\hat{R}^{CH}(\rho^*_{\tilde{S}_I}L(i_{I*}j_I^*(F,W))),E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],\tilde{g}_J^*u^q_{IJ}(F,W))$$
$$\xrightarrow{\mathcal{H}om(T(\tilde{g}_I,R^{CH})(-)^{-1},-)}$$
$$\Gamma^{\vee,Hdg}_T\iota_T^{-1}(e'_*\mathcal{H}om^{\bullet}(\hat{R}^{CH}(\rho^*_{Y\times\tilde{S}_I}\tilde{g}_I^*L(i_{I*}j_I^*(F,W))),E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],\tilde{g}_J^*u^q_{IJ}(F,W))$$
$$\xrightarrow{T(\Gamma^{\vee,Hdg}_T,\Omega^{\Gamma,pr}_{/S})(F,W)}$$
$$\iota_T^{-1}(e'_*\mathcal{H}om^{\bullet}(\hat{R}^{CH}(\rho^*_{Y\times\tilde{S}_I}\Gamma^{\vee}_{T_I}\tilde{g}_I^*L(i_{I*}j_I^*(F,W))),E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],\tilde{g}_J^{*,\gamma}u^q_{IJ}(F,W))$$
$$\xrightarrow{(\mathcal{H}om(\hat{R}^{CH}_{Y\times\tilde{S}_I}(T^{q,\gamma}(D_{gI})(j_I^*(F,W))),E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}])}$$
$$\iota_T^{-1}(e'_*\mathcal{H}om^{\bullet}(\hat{R}^{CH}(\rho^*_{Y\times\tilde{S}_I}L(i'_{I*}j_I^{'*}g^*(F,W))),E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],u^q_{IJ}(g^*(F,W)))$$
$$\xrightarrow{\mathcal{H}om(\hat{R}^{CH}_{Y\times\tilde{S}_I}(Li'_{I*}j_I^{'*}(e)),-)}$$
$$\iota_T^{-1}(e'_*\mathcal{H}om^{\bullet}(\hat{R}^{CH}(\rho^*_{Y\times\tilde{S}_I}L(i'_{I*}j_I^{'*}(F',W))),E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/Y\times\tilde{S}_I},F_{DR}))[d_{YI}],u^q_{IJ}(F',W))$$
$$\xrightarrow{=:}\mathcal{F}^{FDR}_T(g^*M)$$

Definition 113.

- *Let $f : X \to S$ a morphism with $X,S \in \mathrm{Var}(k)$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. We have, for $M \in \mathrm{DA}_c(X)$, the following transformation map in $D(DRM(S))$*

$$T_*(f,\mathcal{F}^{FDR})(M) : \mathcal{F}^{FDR}_S(Rf_*M) \xrightarrow{\mathrm{ad}(f^{\hat{*}mod}_{Hdg},Rf^{Hdg}_*)(-)} Rf^{Hdg}_*f^{\hat{*}mod}_{Hdg}\mathcal{F}^{FDR}_S(Rf_*M)$$
$$\xrightarrow{T(f,\mathcal{F}^{FDR})(Rf_*M)} Rf^{Hdg}_*\mathcal{F}^{FDR}_X(f^*Rf_*M) \xrightarrow{\mathcal{F}^{FDR}_X(\mathrm{ad}(f^*,Rf_*)(M))} Rf^{Hdg}_*\mathcal{F}^{FDR}_X(M)$$

Clearly, for $p : Y \times S \to S$ a projection with $Y \in \mathrm{PSmVar}(\mathbb{C})$, we have, for $M \in \mathrm{DA}_c(Y \times S)$, $T_*(p,\mathcal{F}^{FDR})(M) = T_!(p,\mathcal{F}^{FDR})(M)[d_Y]$

- *Let $S \in \mathrm{Var}(k)$. Let $Y \in \mathrm{SmVar}(k)$ and $p : Y \times S \to S$ the projection. We have then, for $M \in \mathrm{DA}(Y \times S)$ the following transformation map in $D(DRM(S))$*

$$T_!(p,\mathcal{F}^{FDR})(M) : p^{Hdg}_!\mathcal{F}^{FDR}_{Y\times S}(M) \xrightarrow{\mathcal{F}^{FDR}_{Y\times S}(\mathrm{ad}(Lp_\sharp,p^*)(M))} Rp^{Hdg}_!\mathcal{F}^{FDR}_{Y\times S}(p^*Lp_\sharp M)$$
$$\xrightarrow{T(p,\mathcal{F}^{FDR})(Lp_\sharp(M,W))} Rp^{Hdg}_!p^{\hat{*}mod[-]}\mathcal{F}^{FDR}_S(Lp_\sharp M) \xrightarrow{T(p^{*mod},p^{\hat{*}mod})(-)} p^{Hdg}_!p^{*mod[-]}$$
$$\mathcal{F}^{FDR}_S(Lp_\sharp M) \xrightarrow{\mathrm{ad}(Rp^{Hdg}_!,p^{*mod[-]})(\mathcal{F}^{FDR}_S(Lp_\sharp M))} \mathcal{F}^{FDR}_S(Lp_\sharp M)$$

- *Let $f : X \to S$ a morphism with $X,S \in \mathrm{Var}(k)$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. We have then, using the second point, for $M \in \mathrm{DA}(X)$ the following transformation map in $D(DRM(S))$*

$$T_!(f,\mathcal{F}^{FDR})(M) : Rp^{Hdg}_!\mathcal{F}^{FDR}_X(M,W) := Rp^{Hdg}_!\mathcal{F}^{FDR}_{Y\times S}(l_*M)$$
$$\xrightarrow{T_!(p,\mathcal{F}^{FDR})(l_*M)} \mathcal{F}^{FDR}_S(Lp_\sharp l_*M) \xrightarrow{=} \mathcal{F}^{FDR}_S(Rf_!M)$$

- *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. We have, using the third point, for $M \in \mathrm{DA}(S)$, the following transformation map in in $D(DRM(X))$*

$$T^!(f, \mathcal{F}^{FDR})(M) : \mathcal{F}_X^{FDR}(f^!M) \xrightarrow{\mathrm{ad}(Rf_!^{Hdg}, f_{Hdg}^{*mod})(\mathcal{F}_X^{FDR}(f^!M))} f_{Hdg}^{*mod} Rf_!^{Hdg} \mathcal{F}_X^{FDR}(f^!M)$$
$$\xrightarrow{T_!(p_S, \mathcal{F}^{FDR})(\mathcal{F}^{FDR}(f^!M))} f_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(Rf_! f^! M) \xrightarrow{\mathcal{F}_S^{FDR}(\mathrm{ad}(Rf_!, f^!)(M))} f_{Hdg}^{*mod} \mathcal{F}_S^{FDR}(M)$$

- *Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We have, using definition 107 and the preceding point, denoting $\Delta_S : S \hookrightarrow S$ the diagonal closed embedding and $p_1 : S \times S \to S$, $p_2 : S \times S \to S$ the projections, for $M, N \in \mathrm{DA}(S)$ and $(F, W), (G, W) \in C_{fil}(\mathrm{Var}(k)^{sm}/S))$ such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$ and $(N, W) = D(\mathbb{A}^1, et)(G, W)$, the following transformation map in $D(DRM(S))$*

$$T(\mathcal{F}_S^{FDR}, \otimes)(M, N) : \mathcal{F}_S^{FDR}(M) \otimes_{O_S}^{Hdg} \mathcal{F}_S^{FDR}(N) := \Delta_S^{!Hdg}(p_1^{*mod} \mathcal{F}_S^{FDR}(M) \otimes_{O_{S\times S}} p_2^{*mod} \mathcal{F}_S^{FDR}(N)$$
$$\xrightarrow{T^!(p_1, \mathcal{F}_S^{FDR})(M) \otimes T^!(p_1, \mathcal{F}_S^{FDR})(M)} \Delta_S^{!Hdg}(\mathcal{F}_{S\times S}^{FDR}(p_1^! M) \otimes_{O_{S\times S}} \mathcal{F}_{S\times S}^{FDR}(p_2^! N)$$
$$\xrightarrow{(T(\otimes, \Omega)(\hat{R}^{CH}(\rho_{\tilde{S}_I \times \tilde{S}_J}^* L(i_I \times i_J)_*(j_I \times j_J)^* p_1^* F[2d_S]), \hat{R}^{CH}(\rho_{\tilde{S}_I \times \tilde{S}_J}^* L(i_I \times i_J)_*(j_I \times j_J)^* p_2^* F[2d_S])))}$$
$$\Delta_S^{!Hdg}(\mathcal{F}_{S\times S}^{FDR}(p_1^! M \otimes p_2^! N) \xrightarrow{T^!(\Delta_S, \mathcal{F}^{FDR})(p_1^! M \otimes p_2^! N)} \mathcal{F}_S^{FDR}(\Delta_S^!(p_1^! M \otimes p_2^! N)) = \mathcal{F}_S^{FDR}(M \otimes N)$$

where the last equality follows from the equality in $\mathrm{DA}(S)$

$$\Delta_S^!(p_1^! M \otimes p_2^! N) = \Delta_S^! p_1^! M \otimes \Delta_S^! p_2^! N = M \otimes N$$

Proposition 67. *Let $g : T \to S$ a morphism with $T, S \in \mathrm{Var}(k)$. Assume we have a factorization $g : T \xrightarrow{l} Y_2 \times S \xrightarrow{p_S} S$ with $Y_2 \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. Let $S = \cup_{i=1}^l S_i$ be an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$ Then, $T = \cup_{i=1}^l T_i$ with $T_i := g^{-1}(S_i)$ and we have closed embeddings $i'_i := i_i \circ l : T_i \hookrightarrow Y_2 \times \tilde{S}_i$, Moreover $\tilde{g}_I := p_{\tilde{S}_I} : Y \times \tilde{S}_I \to \tilde{S}_I$ is a lift of $g_I := g_{|T_I} : T_I \to S_I$. Let $f : X \to S$ a morphism with $X \in \mathrm{Var}(k)$ such that there exists a factorization $f : X \xrightarrow{l} Y_1 \times S \xrightarrow{p_S} S$, with $Y_1 \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. We have then the following commutative diagram whose squares are cartesians*

$$\begin{tikzcd}
f' : X_T \arrow[rr] \arrow[dr] \arrow[dd, "g'"'] & & Y_1 \times T \arrow[rr] \arrow[dr] & & T \arrow[dr] \arrow[dd] & \\
& Y_1 \times X \arrow[rr] \arrow[dl] & & Y_1 \times Y_2 \times S \arrow[rr] \arrow[dl] & & Y_2 \times S \arrow[dl] \\
f : X \arrow[rr] & & Y_1 \times S \arrow[rr] & & S &
\end{tikzcd}$$

Take a smooth compactification $\bar{Y}_1 \in \mathrm{PSmVar}(\mathbb{C})$ of Y_1, denote $\bar{X}_I \subset \bar{Y}_1 \times \tilde{S}_I$ the closure of X_I, and $Z_I := \bar{X}_I \backslash X_I$. Consider $F(X/S) := p_{S,\sharp} \Gamma_X^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S) \in C(\mathrm{Var}(k)^{sm}/S)$ and the isomorphism in $C(\mathrm{Var}(k)^{sm}/T)$

$$T(f, g, F(X/S)) : g^* F(X/S) := g^* p_{S,\sharp} \Gamma_X^\vee \mathbb{Z}(Y_1 \times S/Y_1 \times S) \xrightarrow{\sim}$$
$$p_{T,\sharp} \Gamma_{X_T}^\vee \mathbb{Z}(Y_1 \times T/Y_1 \times T) =: F(X_T/T).$$

which gives in $\mathrm{DA}(T)$ the isomorphism $T(f, g, F(X/S)) : g^* M(X/S) \xrightarrow{\sim} (X_T/T)$. Then the following diagram in $D(DRM(T)) \subset D_{\mathcal{D}(1,0)fil}(T/(Y_2 \times \tilde{S}_I))$, where the horizontal maps are given by proposition

64, commutes

$$\begin{array}{ccc}
g^{\hat{*}mod}_{Hdg}\iota_S^{-1}\mathcal{F}_S^{FDR}(M(X/S)) & \xrightarrow{g^{\hat{*}mod}_{Hdg}I(X/S)} & g^{\hat{*}mod}_{Hdg}Rf^{Hdg}_!(\Gamma^{\vee,Hdg}_{X_I}(O_{Y_1\times\tilde{S}_I},F_b)(d_{Y_1})[2d_{Y_1}],x_{IJ}(X/S)) \\
\downarrow {}'T(g,\mathcal{F}^{FDR})(M(X/S)) & & \downarrow T(p_{\tilde{S}_I},\gamma^{\vee,Hdg})(-) \\
 & & Rf'^{Hdg}_!g'^{\hat{*}mod}_{Hdg}(\Gamma^{\vee,Hdg}_{X_I}(O_{Y_1\times\tilde{S}_I},F_b)(d_{Y_1})[2d_{Y_1}],x_{IJ}(X/S)) \\
 & & \downarrow T(p^{\hat{*}mod}_{Y_1\times Y_2\times\tilde{S}_I,Hdg},p^{*mod}_{Y_1\times Y_2\times\tilde{S}_I,Hdg})(-) \\
\iota_T^{-1}\mathcal{F}_T^{FDR}(M(X_T/T)) & \xrightarrow{I(X_T/T)} & Rf'^{Hdg}_!(\Gamma^{\vee,Hdg}_{X_{T_I}}(O_{Y_2\times Y_1\times\tilde{S}_I},F_b)(d_{Y_{12}})[2d_{Y_{12}}],x_{IJ}(X_T/T)).
\end{array}$$

with $d_{Y_{12}}=d_{Y_1}+d_{Y_2}$.

Proof. Follows immediately from definition. □

Proposition 68. *Let $S\in\mathrm{Var}(k)$. Let $Y\in\mathrm{SmVar}(k)$ and $p:Y\times S\to S$ the projection. Let $S=\cup_{i=1}^l S_i$ an open cover such that there exist closed embeddings $i_i:S_i\hookrightarrow\tilde{S}_i$ with $\tilde{S}_i\in\mathrm{SmVar}(k)$. For $I\subset[1,\cdots l]$, we denote by $S_I=\cap_{i\in I}S_i$, $j_I^o:S_I\hookrightarrow S$ and $j_I:Y\times S_I\hookrightarrow Y\times S$ the open embeddings. We then have closed embeddings $i_I:Y\times S_I\hookrightarrow Y\times\tilde{S}_I$. and we denote by $p_{\tilde{S}_I}:Y\times\tilde{S}_I\to\tilde{S}_I$ the projections. Let $f':X'\to Y\times S$ a morphism, with $X'\in\mathrm{Var}(k)$ such that there exists a factorization $f':X'\xrightarrow{l'}Y'\times Y\times S\xrightarrow{p'}Y\times S$ with $Y'\in\mathrm{SmVar}(k)$, l' a closed embedding and p' the projection. Denoting $X'_I:=f'^{-1}(Y\times S_I)$, we have closed embeddings $i'_I:X'_I\hookrightarrow Y'\times Y\times\tilde{S}_I$ Consider*

$$F(X'/Y\times S):=p_{Y\times S,\sharp}\Gamma^{\vee}_{X'}\mathbb{Z}(Y'\times Y\times S/Y'\times Y\times S)\in C(\mathrm{Var}(k)^{sm}/Y\times S)$$

and $F(X'/S):=p_\sharp F(X'/Y\times S)\in C(\mathrm{Var}(k)^{sm}/S)$, so that $Lp_\sharp M(X'/Y\times S)[-2d_Y]=:M(X'/S)$. Then, the following diagram in $D(DRM(S))\subset D_{\mathcal{D}(1,0)fil}(S/(Y\times\tilde{S}_I))$, where the vertical maps are given by proposition 64, commutes

$$\begin{array}{ccc}
Rp^{Hdg}!\mathcal{F}^{FDR}_{Y\times S}(M(X'/Y\times S)) & \xrightarrow{T^!(p,\mathcal{F}^{FDR})(M(X'/Y\times S))} & \mathcal{F}^{FDR}_S(M(X'/S)) \\
\uparrow T(p^{\hat{*}mod}_{Hdg},p^{*mod}_{Hdg})(-)\circ Rp^{Hdg}!(I(X'/Y\times S)) & & \uparrow I(X'/S) \\
Rp^{Hdg}!Rf'^{Hdg}_!f'^{*mod}_{Hdg}\mathbb{Z}^{Hdg}_{Y\times S} & \xrightarrow{=} & Rf^{Hdg}_!f^{*mod}_{Hdg}\mathbb{Z}^{Hdg}_S
\end{array}.$$

Proof. Immediate from definition. □

Proposition 69. *Let $f_1:X_1\to S$, $f_2:X_2\to S$ two morphism with $X_1,X_2,S\in\mathrm{Var}(k)$. Assume that there exist factorizations $f_1:X_1\xrightarrow{l_1}Y_1\times S\xrightarrow{p_S}S$, $f_2:X_2\xrightarrow{l_2}Y_2\times S\xrightarrow{p_S}S$ with $Y_1,Y_2\in\mathrm{SmVar}(k)$, l_1,l_2 closed embeddings and p_S the projections. We have then the factorization*

$$f_{12}:=f_1\times f_2:X_{12}:=X_1\times_S X_2\xrightarrow{l_1\times l_2}Y_1\times Y_2\times S\xrightarrow{p_S}S$$

Let $S=\cup_{i=1}^l S_i$ an open affine covering and denote, for $I\subset[1,\cdots l]$, $S_I=\cap_{i\in I}S_i$ and $j_I:S_I\hookrightarrow S$ the open embedding. Let $i_i:S_i\hookrightarrow\tilde{S}_i$ closed embeddings, with $\tilde{S}_i\in\mathrm{SmVar}(k)$. We have then the following commutative diagram in $D(DRM(S))\subset D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$ where the vertical maps are given by proposition 64

$$\begin{array}{ccc}
\mathcal{F}^{FDR}_S(M(X_1/S))\otimes^{Hdg}_{O_S}\mathcal{F}^{FDR}_S(M(X_2/S)) & \xrightarrow{I(X_1/S)\otimes I(X_2/S)} & \begin{array}{c}Rf^{Hdg}_{1!}(\Gamma^{\vee,Hdg}_{X_{1I}}(O_{Y_1\times\tilde{S}_I},F_b)(d_2)[2d_1],x_{IJ}(X_1/S))\otimes_{O_S}\\ Rf^{Hdg}_{2!}(\Gamma^{\vee,Hdg}_{X_{2I}}(O_{Y_2\times\tilde{S}_I},F_b)(d_1)[2d_2],x_{IJ}(X_2/S))\end{array} \\
\downarrow T(\mathcal{F}^{FDR}_S,\otimes)(M(X_1/S),M(X_2/S)) & & \downarrow (Ew_{(Y_1\times\tilde{S}_I,Y_2\times\tilde{S}_I)/\tilde{S}_I}) \\
\mathcal{F}^{FDR}_S(M(X_1/S)\otimes M(X_2/S)=M(X_1\times_S X_2/S)) & \xrightarrow{I(X_{12}/S)} & Rf^{Hdg}_{12!}(\Gamma^{\vee,Hdg}_{X_{1I}\times_S X_{2I}}(O_{Y_1\times Y_2\times\tilde{S}_I},F_b)(d_{12})[2d_{12}],x_{IJ}(X_1/S))
\end{array}$$

with $d_1=d_{Y_1}$, $d_2=d_{Y_2}$ and $d_{12}=d_{Y_1}+d_{Y_2}$.

Proof. Immediate from definition. □

Theorem 53. *(i) Let $g : T \to S$ a morphism, with $S, T \in \mathrm{Var}(k)$. Assume we have a factorization $g : T \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. Let $M \in \mathrm{DA}_c(S)$. Then map in $D(DRM(T))$*

$$T(g, \mathcal{F}^{FDR})(M) : g^{\hat{*}mod}_{Hdg}\mathcal{F}^{FDR}_S(M) \xrightarrow{\sim} \mathcal{F}^{FDR}_T(g^*M)$$

given in definition 112 is an isomorphism.

(ii) Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. Then, for $M \in \mathrm{DA}_c(X)$, the map given in definition 113

$$T_!(f, \mathcal{F}^{FDR})(M) : Rf^{Hdg}_!\mathcal{F}^{FDR}_X(M) \xrightarrow{\sim} \mathcal{F}^{FDR}_S(Rf_!M)$$

is an isomorphism in $D(DRM(S))$.

(iii) Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. We have, for $M \in \mathrm{DA}_c(X)$, the map given in definition 113

$$T_*(f, \mathcal{F}^{FDR})(M) : \mathcal{F}^{FDR}_S(Rf_*M) \xrightarrow{\sim} Rf^{Hdg}_*\mathcal{F}^{FDR}_X(M)$$

is an isomorphism in $D(DRM(S))$.

(iv) Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$, S quasi-projective. Assume there exist a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p_S the projection. Then, for $M \in \mathrm{DA}_c(S)$, the map given in definition 113

$$T^!(f, \mathcal{F}^{FDR})(M) : \mathcal{F}^{FDR}_X(f^!M) \xrightarrow{\sim} f^{*mod}_{Hdg}\mathcal{F}^{FDR}_S(M)$$

is an isomorphism in $D(DRM(X))$.

(v) Let $S \in \mathrm{Var}(k)$. Then, for $M, N \in \mathrm{DA}_c(S)$, the map in $D(DRM(S))$

$$T(\mathcal{F}^{FDR}_S, \otimes)(M, N) : \mathcal{F}^{FDR}_S(M) \otimes^{Hdg}_{O_S} \mathcal{F}^{FDR}_S(N) \xrightarrow{\sim} \mathcal{F}^{FDR}_S(M \otimes N)$$

given in definition 113 is an isomorphism.

Proof. The proof is similar to the complex case : follows from [4] by proposition 67 and proposition 68, more precisely :
(i):follows from proposition 67 and proposition 64.
(ii):follows from proposition 68
(iii),(iv): see [10].
(v):follows from proposition 69. □

We have the following easy proposition

Proposition 70. *Let $S \in \mathrm{Var}(k)$ and $S = \cup_{i=1}^l S_i$ an open affine covering and denote, for $I \subset [1, \cdots l]$, $S_I = \cap_{i \in I} S_i$ and $j_I : S_I \hookrightarrow S$ the open embedding. Let $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings, with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We have, for $M, N \in \mathrm{DA}(S)$ and $F, G \in C(\mathrm{Var}(k)^{sm}/S)$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following commutative diagram in $D_{O_S fil,\mathcal{D},\infty}(S/(\tilde{S}_I))$*

$$\begin{tikzcd}
\mathcal{F}^{GM}_S(L\mathbb{D}_S M) \otimes^L_{O_S} \mathcal{F}^{GM}_S(L\mathbb{D}_S N) \arrow[r, "T(\mathcal{F}^{GM}_S, \mathcal{F}^{FDR}_S)(M) \otimes T(\mathcal{F}^{GM}_S, \mathcal{F}^{FDR}_S)(N)"] \arrow[d, "T(\mathcal{F}^{GM}_S, \otimes)(L\mathbb{D}_S M, L\mathbb{D}_S N)"] & \mathcal{F}^{FDR}_S(M) \otimes^{Hdg}_{O_S} \mathcal{F}^{FDR}_S(N) \arrow[d, "T(\mathcal{F}^{FDR}_S, \otimes)(M, N)"] \\
\mathcal{F}^{GM}_S(L\mathbb{D}_S(M \otimes N)) \arrow[r, "T(\mathcal{F}^{GM}_S, \mathcal{F}^{FDR}_S)(M \otimes N)"] & \mathcal{F}^{FDR}_S(M \otimes N)
\end{tikzcd}$$

Proof. Immediate from definition. □

8 The Hodge realization functors for relative motives over a field k of characteristic 0

8.1 The Hodge realization functor for relative motives over a subfield $k \subset \mathbb{C}$

Let $k \subset \mathbb{C}$ a subfield. We have for $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(k)$ we have the commutative diagram of site

$$\begin{tikzcd}
\mathrm{AnSp}(\mathbb{C})/T^{an}_{\mathbb{C}} \arrow[rr, "\mathrm{An}_T:=\mathrm{An}_T\circ(\pi_{k/\mathbb{C}}(-))"] \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & & \mathrm{Var}(k)/T \arrow[dr, "\rho_T"] \arrow[dd, "P(f)"] & \\
& \mathrm{AnSp}(\mathbb{C})^{sm}/T^{an}_{\mathbb{C}} \arrow[rr, "\mathrm{An}_T"] \arrow[dd, "P(f)"] & & \mathrm{Var}(k)^{sm}/T \arrow[dd, "P(f)"] \\
\mathrm{AnSp}(\mathbb{C})/S^{an}_{\mathbb{C}} \arrow[rr, "\mathrm{An}_S:=\mathrm{An}_S\circ(\pi_{k/\mathbb{C}}(-))"] \arrow[dr, "\rho_S"] & & \mathrm{Var}(k)/S \arrow[dr, "\rho_S"] & \\
& \mathrm{AnSp}(\mathbb{C})^{sm}/S^{an}_{\mathbb{C}} \arrow[rr, "\mathrm{An}_S"] & & \mathrm{Var}(k)^{sm}/S
\end{tikzcd} .$$

This gives for $s : \mathcal{I} \to \mathcal{J}$ a functor with $\mathcal{I}, \mathcal{J} \in \mathrm{Cat}$ and $f : T_\bullet \to S_{s(\bullet)}$ a morphism of diagram of algebraic varieties with $T_\bullet \in \mathrm{Fun}(\mathcal{I}, \mathrm{Var}(k))$, $S_\bullet \in \mathrm{Fun}(\mathcal{J}, \mathrm{Var}(k))$ the commutative diagram of sites

$$Dia^{12}(S) := \begin{tikzcd}
\mathrm{AnSp}(\mathbb{C})/T^{an}_{\bullet,\mathbb{C}} \arrow[rr, "\mathrm{An}_{T_\bullet}:=\mathrm{An}_{T_\bullet}\circ(\pi_{k/\mathbb{C}}(-))"] \arrow[dr, "\rho_{T_\bullet}"] \arrow[dd, "P(f_\bullet)"] & & \mathrm{Var}(k)/T_\bullet \arrow[dr, "\rho_{T_\bullet}"] \arrow[dd, "P(f_\bullet)"] & \\
& \mathrm{AnSp}(\mathbb{C})^{sm}/T^{an}_{\bullet,\mathbb{C}} \arrow[rr, "\mathrm{An}_{T_\bullet}"] \arrow[dd, "P(f_\bullet)"] & & \mathrm{Var}(k)^{sm}/T_\bullet \arrow[dd, "P(f_\bullet)"] \\
\mathrm{AnSp}(\mathbb{C})/S^{an}_{\bullet,\mathbb{C}} \arrow[rr, "\mathrm{An}_{S_\bullet}:=\mathrm{An}_{S_\bullet}\circ(\pi_{k/\mathbb{C}}(-))"] \arrow[dr, "\rho_{S_\bullet}"] & & \mathrm{Var}(k)/S_\bullet \arrow[dr, "\rho_{S_\bullet}"] & \\
& \mathrm{AnSp}(\mathbb{C})^{sm}/S^{an}_{\bullet,\mathbb{C}} \arrow[rr, "\mathrm{An}_{S_\bullet}"] & & \mathrm{Var}(k)^{sm}/S_\bullet
\end{tikzcd} .$$

8.1.1 The Betti realization functor

Let $k \subset \mathbb{C}$ a subfield.

Definition 114. *Let $S \in \mathrm{Var}(k)$.*

(i) The Ayoub's Betti realization functor is

$$\mathrm{Bti}^*_S : \mathrm{DA}(S) \to D(S^{an}_{\mathbb{C}}) \ , \ M \in \mathrm{DA}(S) \mapsto \mathrm{Bti}^*_S M = Re(S^{an}_{\mathbb{C}})_* \mathrm{An}^*_S M = e(S^{an}_{\mathbb{C}})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}^*_S F$$

where $F \in C(\mathrm{Var}(k)^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(ii) In [9], we define the Betti realization functor as

$$\widetilde{\mathrm{Bti}}^*_S : \mathrm{DA}(S) \to D(S^{an}) = D(S^{cw}) \ , \ M \mapsto \widetilde{\mathrm{Bti}}^*_S M = Re(S^{cw})_* \widetilde{\mathrm{Cw}}^*_S M = e(S^{cw})_* \underline{\mathrm{sing}}_{\mathbb{I}^*} \widetilde{\mathrm{Cw}}^*_S F$$

where $F \in C(\mathrm{Var}(k)^{sm}/S)$ is such that $M = D(\mathbb{A}^1, et)(F)$.

(iii) For the Corti-Hanamura weight structure on $\mathrm{DA}^-(S)$*, we have by functoriality of (i) the functor*

$$\mathrm{Bti}_S^* : \mathrm{DA}^-(S) \to D_{fil}(S_{\mathbb{C}}^{an}) \ , \ M \mapsto (\mathrm{Bti}_S^* M, W) := \mathrm{Bti}_S^*(M, W) := e(S^{an})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_S^*(F, W)$$

where $(F, W) \in C_{fil}(\mathrm{Var}(k)^{sm}/S)$ *is such that* $(M, W) = D(\mathbb{A}^1, et)(F, W)$*.*

Note that by [9], An_S^* *and* $\widetilde{\mathrm{Cw}}_S^*$ *derive trivially.*

Note that, by considering the explicit $\mathbb{D}_S^1$ local model for presheaves on $\mathrm{AnSp}(\mathbb{C})^{sm}/S_{\mathbb{C}}^{an}$, $\mathrm{Bti}_S^*(\mathrm{DA}^-(S)) \subset D^-(S^{an})$; by considering the explicit $\mathbb{I}_S^1$ local model for presheaves on CW^{sm}/S^{cw}, $\widetilde{\mathrm{Bti}}_S^*(\mathrm{DA}^-(S)) \subset D^-(S_{\mathbb{C}}^{an})$.

Let $f : T \to S$ a morphism, with $T, S \in \mathrm{Var}(k)$. We have, for $M \in \mathrm{DA}(S)$, $(F, W) \in C_{fil}(\mathrm{Var}(k)^{sm}/S)$ such that $(M, W) = D(\mathbb{A}^1, et)(F, W)$, and an equivalence $(\mathbb{A}^1, et)$ local $e : f^*(F, W) \to (F', W)$ with $(F', W) \in C_{fil}(\mathrm{Var}(k)^{sm}/S)$ such that $(f^*M, W) = D(\mathbb{A}^1, et)(F', W)$ the following canonical transformation map in $D_{fil}(T_{\mathbb{C}}^{an})$:

$$\begin{aligned} T^0(f, \mathrm{Bti})(M, W) : f^* \mathrm{Bti}_S^*(M, W) &:= f^* e(S_{\mathbb{C}}^{an})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_S^*(F, W) \\ &\xrightarrow{T(f,e)(-)} e(T_{\mathbb{C}}^{an})_* f^* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_S^*(F, W) \\ \xrightarrow{e(T_{\mathbb{C}}^{an})_* T(f,c)(F,W)} e(T^{an})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} f^* \mathrm{An}_T^*(F, W) &\xrightarrow{=} e(T_{\mathbb{C}}^{an})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_T^* f^*(F, W) \\ &\xrightarrow{e(T_{\mathbb{C}}^{an})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_T^* e} e(T_{\mathbb{C}}^{an})_* \underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}_T^*(F', W) =: \mathrm{Bti}_T^* f^*(M, W). \end{aligned}$$

Definition 115. *Let* $f : T \to S$ *a morphism, with* $T, S \in \mathrm{Var}(k)$*. Consider the graph factorization* $f : T \xrightarrow{l} T \times S \xrightarrow{p} S$ *of* f *with* l *the graph closed embedding and* p *the projection. We have, for* $M \in \mathrm{DA}_c(S)$*, the following canonical transformation map in* $D_{fil,c}(T_{\mathbb{C}}^{an})$*:*

$$\begin{aligned} T(f, \mathrm{Bti})(M, W) : f^{*w} \mathrm{Bti}_S^*(M, W) &:= l^* \Gamma_T^{\vee, w} p^* \mathrm{Bti}_S^*(F, W) \\ \xrightarrow{T^0(p,\mathrm{Bti})(-)} l^* \Gamma_T^{\vee, w} \mathrm{Bti}_{T \times S}^* p^*(F, W) &\xrightarrow{\gamma_T^\vee(p^*(F,W))} l^* \Gamma_T^{\vee, w} \mathrm{Bti}_{T \times S}^* \Gamma_T^\vee p^*(F, W) \\ \xrightarrow{=} l^* \mathrm{Bti}_{T \times S}^* \Gamma_T^\vee p^*(F, W) &\xrightarrow{T^0(l,\mathrm{Bti})(-)} \mathrm{Bti}_T^* l^* \Gamma_T^\vee p^*(F, W) = \mathrm{Bti}_T^* f^*(M, W). \end{aligned}$$

where we use definition 6.

Definition 116.

- *Let* $f : X \to S$ *a morphism, with* $X, S \in \mathrm{Var}(k)$*. We have, for* $M \in \mathrm{DA}_c(X)$*, the following transformation map in* $D_{fil,c}(S_{\mathbb{C}}^{an})$

$$\begin{aligned} T_*(f, \mathrm{Bti})(M, W) : \mathrm{Bti}_S^*(Rf_*(M, W)) &\xrightarrow{\mathrm{ad}(f^*, Rf_{*w})(\mathrm{Bti}_S^*(Rf_*(M,W)))} Rf_{*w} f^{*w} \mathrm{Bti}_S^*(Rf_*(M, W)) \\ \xrightarrow{T(f,\mathrm{Bti})(Rf_*(M,W))} Rf_{*w} \mathrm{Bti}_X^*(f^* Rf_*(M, W)) &\xrightarrow{\mathrm{Bti}_X^*(\mathrm{ad}(f^*, Rf_*)(M,W))} Rf_{*w} \mathrm{Bti}_X^*(M, W) \end{aligned}$$

  Clearly if $l : Z \hookrightarrow S$ *is a closed embedding, then* $T_*(l, \mathrm{Bti})(M, W)$ *is an isomorphism since* $\mathrm{ad}(l^*, l_*)(-) : l^* l_*(M, W) \to (M, W)$ *is an isomorphism (see section 3).*

- *Let* $f : X \to S$ *a morphism with* $X, S \in \mathrm{Var}(k)$*. Assume there exist a factorization* $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ *with* $Y \in \mathrm{SmVar}(k)$*,* l *a closed embedding and* p_S *the projection. We have then, for* $M \in \mathrm{DA}_c(X)$*, using theorem 54 for closed embeddings, the following transformation map in* $D_{fil}((Y \times S)_{\mathbb{C}}^{an})$

$$\begin{aligned} T_!(f, \mathrm{Bti})(M) : Rf_{!w} \mathrm{Bti}_X^*(M, W) &= Rp_{S!w} l_* \mathrm{Bti}_X^*(M, W) \\ &\xrightarrow{T_*(l,\mathrm{Bti})(M,W)^{-1}} Rp_{S!w} \mathrm{Bti}(Y \times S)^*(l_*(M, W)) \\ \xrightarrow{\mathrm{Bti}(Y \times S)^* \mathrm{ad}(Lp_{S\sharp}, p_S^*)(l_*(M,W))} Rp_{S!w} \mathrm{Bti}(Y \times S)^*(p_S^* Lp_{S\sharp} l_*(M, W)) &\xrightarrow{T(p_S,\mathrm{Bti})(p_{S\sharp} l_*(M,W))} \\ Rp_{S!w} p_S^* \mathrm{Bti}(Y \times S)^*(Lp_{S\sharp} l_*(M, W)) &= Rp_{S!w} p_S^{!w} \mathrm{Bti}(Y \times S)^*(Rf_!(M, W)) \\ &\xrightarrow{\mathrm{ad}(Rp_{S!w}, p_S^{!w})(-)} \mathrm{Bti}(Y \times S)^*(Rf_!(M, W)) \end{aligned}$$

Clearly, for $f : X \to S$ a proper morphism, with $X, S \in \mathrm{Var}(k)$ we have, for $M \in \mathrm{DA}_c(Y \times S)$, $T_!(f, \mathrm{Bti})(M, W) = T_*(f, \mathrm{Bti})(M, W)$.

- *Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$. We have, using the second point, for $M \in \mathrm{DA}(S)$, the following transformation map in $D_{fil}(X_{\mathbb{C}}^{an})$*

$$T^!(f,\mathrm{Bti})(M,W) : \mathrm{Bti}_X^*(f^!(M,W)) \xrightarrow{\mathrm{ad}(f_!,Rf^!)(\mathrm{Bti}_X^*(f^!(M,W)))} f^{!w}Rf_{!w}\,\mathrm{Bti}_X^*(f^!(M,W))$$
$$\xrightarrow{T_!(f,\mathrm{Bti})((f^!(M,W)))} f^{!w}\,\mathrm{Bti}_S^*(f_!f^!(M,W)) \xrightarrow{\mathrm{Bti}_S^*(\mathrm{ad}(f_!,f^!)(M,W))} f^{!w}\,\mathrm{Bti}_S^*(M,W)$$

- *Let $S \in \mathrm{Var}(k)$. We have, for $M, N \in \mathrm{DA}(S)$ and $F, G \in C_(\mathrm{Var}(k)^{sm}/S))$ such that $M = D(\mathbb{A}^1, et)(F)$ and $N = D(\mathbb{A}^1, et)(G)$, the following transformation map in $D_{fil}(S_{\mathbb{C}}^{an})$*

$$\mathrm{Bti}_S^*(M,W)\otimes \mathrm{Bti}_S^*(N,W) := (e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*}\,\mathrm{An}_S^*(F,W))\otimes(e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*}\,\mathrm{An}_S^*(G,F))$$
$$\xrightarrow{T(\mathrm{sing}_{D^*},\otimes)(\mathrm{An}_S^*(F,W),\mathrm{An}_S^*(G,F))} e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*}\,\mathrm{An}_S^*((F,W)\otimes(G,W)) =: \mathrm{Bti}_S^*((M,W)\otimes(N,W))$$

Theorem 54. *(i) Let $f : X \to S$ a morphism, with $X, S \in \mathrm{Var}(k)$. For $M \in \mathrm{DA}_c(S)$,*

$$T(f,\mathrm{Bti})(M,W) : f^{*w}\,\mathrm{Bti}_S^*(M,W) \xrightarrow{\sim} \mathrm{Bti}_X^*\, f^*(M,W)$$

is an isomorphism in $D_{fil}(X_{\mathbb{C}}^{an})$.

(ii) Let $f : X \to S$ a morphism, with $X, S \in \mathrm{Var}(k)$. For $M \in \mathrm{DA}_c(X)$,

$$T_!(f,\mathrm{Bti})(M,W) : Rf_{!w}\,\mathrm{Bti}_X^*(M,W) \xrightarrow{\sim} \mathrm{Bti}_S^*\, Rf_!(M,W)$$

is an isomorphism.

(iii) Let $f : X \to S$ a morphism, with $X, S \in \mathrm{Var}(k)$. For $M \in \mathrm{DA}_c(X)$,

$$T_*(f,\mathrm{Bti})(M,W) : Rf_{*w}\,\mathrm{Bti}_X^*(M,W) \xrightarrow{\sim} \mathrm{Bti}_S^*\, Rf_*(M,W)$$

is an isomorphism.

(iv) Let $f : X \to S$ a morphism, with $X, S \in \mathrm{Var}(k)$. For $M \in \mathrm{DA}_c(S)$,

$$T^!(f,\mathrm{Bti})(M,W) : f^{!w}\,\mathrm{Bti}_S^*(M,W) \xrightarrow{\sim} \mathrm{Bti}_X^*\, f^!(M,W)$$

is an isomorphism.

(v) Let $S \in \mathrm{Var}(k)$. For $M, N \in \mathrm{DA}_c(S)$,

$$T(\otimes,\mathrm{Bti})(M,W) : \mathrm{Bti}_S^*(M,W)\otimes \mathrm{Bti}_S^*\, N \xrightarrow{\sim} \mathrm{Bti}_X^*((M,W)\otimes(N,W))$$

is an isomorphism.

Proof. By functoriality it reduced to the case of Corti-Hanamura motives which is then obvious. □

The main result on the Betti realization functor is the following

Theorem 55. *(i) We have $\mathrm{Bti}_S^* = \widetilde{\mathrm{Bti}}_S^*$ on $\mathrm{DA}^-(S)$*

(ii) The canonical transformations $T(f, \mathrm{Bti})$, for $f : T \to S$ a morphism in $\mathrm{Var}(k)$, define a morphism of 2 functor

$$\mathrm{Bti}_\cdot^* : \mathrm{DA}(\cdot) \to D((\cdot)_{\mathbb{C}}^{an}),\ S \in \mathrm{Var}(k) \mapsto \mathrm{Bti}_S^* : \mathrm{DA}(S) \to D(S_{\mathbb{C}}^{an})$$

which is a morphism of homotopic 2 functor.

(ii)' The canonical transformations $T(f, \mathrm{Bti})$, for $f : T \to S$ a morphism in $\mathrm{Var}(k)$, define a morphism of 2 functor

$$\mathrm{Bti}^*_{\cdot} : \mathrm{DA}_c(\cdot) \to D_{fil}((\cdot)^{an}_{\mathbb{C}}),\ S \in \mathrm{Var}(k) \mapsto \mathrm{Bti}^*_S : \mathrm{DA}(S) \to D_{fil}(S^{an}_{\mathbb{C}})$$

which is a morphism of homotopic 2 functor.

Proof. (i): See [9]
(ii) and (ii)':Follows from theorem 54. □

Remark 11. *For $X \in \mathrm{Var}(k)$, the quasi-isomorphisms*

$$\mathbb{Z}\mathrm{Hom}(\bar{\mathbb{D}}^{\bullet}_{k\,et}, X) \xrightarrow{\mathrm{An}^*} \mathbb{Z}\mathrm{Hom}(\bar{\mathbb{D}}_{\mathbb{C}}^n(0,1), X^{an}_{\mathbb{C}}) \xrightarrow{\mathrm{Hom}(i, X^{cw}_{\mathbb{C}})} \mathbb{Z}\mathrm{Hom}([0,1]^n, X^{cw}_{\mathbb{C}}),$$

where,

$$\bar{\mathbb{D}}^n_{et} := (e : U \to \mathbb{A}^n_k, \bar{\mathbb{D}}^n(0,1) \subset e(U)) \in \mathrm{Fun}(\mathcal{V}^{et}_{\mathbb{A}^n_k}(\bar{\mathbb{D}}^n(0,1)), \mathrm{Var}(k))$$

is the system of etale neighborhood of the closed ball $\bar{\mathbb{D}}^n_k(0,1) \subset \mathbb{A}^n_k$, and $i : [0,1]^n \hookrightarrow \bar{\mathbb{D}}^n_{\mathbb{C}}(0,1)$ is the closed embedding, shows that a closed singular chain $\alpha \in \mathbb{Z}\mathrm{Hom}^n([0,1]^n, X^{cw}_{\mathbb{C}})$, is homologue to a closed singular chain

$$\beta = \alpha + \partial\gamma = \tilde{\beta}_{|[0,1]^n} \in \mathbb{Z}\mathrm{Hom}^n(\Delta^n, X^{cw}_{\mathbb{C}})$$

which is the restriction by the closed embedding $[0,1]^n \hookrightarrow U^{cw}_{\mathbb{C}} \xrightarrow{e} \mathbb{A}^n_{\mathbb{C}}$, where $e : U \to \mathbb{A}^n_k$ an etale morphism with $U \in \mathrm{Var}(k)$, of a a complex algebraic morphism $\tilde{\beta} : U_{\mathbb{C}} \to X_{\mathbb{C}}$ defined over k. Hence $\beta([0,1]^n) = \tilde{\beta}([0,1]^n) \subset X$ is the restriction of a real algebraic subset of dimension n in $Res_{\mathbb{R}}(X)$ (after restriction a scalar that is under the identification $\mathbb{C} \simeq \mathbb{R}^2$) defined over k.

8.1.2 The complex Hodge realization functor for relative motives over a subfield $k \subset \mathbb{C}$

Let $k \subset \mathbb{C}$ a subfield.

Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embedding $i_i : S \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Recall (see section 5.2) that $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$ is the category

- whose set of objects is the set of triples $\{(((M_I, F, W), u_{IJ}), (K, W), \alpha)\}$ with

$$((M_I, F, W), u_{IJ}) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)),\ (K, W) \in D_{fil,c,k}(S^{an}_{\mathbb{C}}),$$
$$\alpha : T(S/(\tilde{S}_I))((K, W) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}}) \to DR(S)^{[-]}(((M_I, W), u_{IJ})^{an})$$

  where α is an morphism in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$,

- and whose set of morphisms consists of

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

  where $\phi_D : ((M_1, F, W), u_{IJ}) \to ((M_2, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \to (K_2, W)$ are morphisms and

$$\theta = (\theta^{\bullet}, I(DR(S)(\phi_D^{an})) \circ I(\alpha_1), I(\alpha_2) \circ I(\phi_C \otimes I)) :$$
$$I(T(S/(\tilde{S}_I))((K_1, W) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}}))[1] \to I(DR(S)(((M_{2I}, W), u_{IJ})^{an}))$$

  is an homotopy, $I : D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}})) \to K_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$ being the injective resolution functor, and for

  - $\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$
  - $\phi' = (\phi'_D, \phi'_C, [\theta']) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \to (((M_{3I}, F, W), u_{IJ}), (K_3, W), \alpha_3)$

the composition law is given by

$$\phi' \circ \phi := (\phi'_D \circ \phi_D, \phi'_C \circ \phi_C, I(DR(S)(\phi_D^{'an})) \circ [\theta] + [\theta'] \circ I(\phi_C \otimes I)[1]) : \\ (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{3I}, F, W), u_{IJ}), (K_3, W), \alpha_3),$$

in particular for $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S_{\mathbb{C}}^{an})$,

$$I_{(((M_I,F,W),u_{IJ}),(K,W),\alpha)} = ((I_{M_I}), I_K, 0),$$

together with the localization functor

$$(D(zar), I) : C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S_{\mathbb{C}}^{an}) \to D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S_{\mathbb{C}}^{an}) \\ \to D_{\mathcal{D}(1,0)fil,rh,\infty}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S_{\mathbb{C}}^{an}).$$

Note that if $\phi = (\phi_D, \phi_C, [\theta]) : (((M_1, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_2, F, W), u_{IJ}), (K_2, W), \alpha_2)$ is a morphism in $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S_{\mathbb{C}}^{an})$ such that ϕ_D and ϕ_C are isomorphisms then ϕ is an isomorphism (see remark 5). Moreover,

- For $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S_{\mathbb{C}}^{an})$, we set

$$(((M_I, F, W), u_{IJ}), (K, W), \alpha)[1] := (((M_I, F, W), u_{IJ})[1], (K, W)[1], \alpha[1]).$$

- For

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

  a morphism in $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S_{\mathbb{C}}^{an})$, we set (see [11] definition 3.12)

$$\mathrm{Cone}(\phi) := (\mathrm{Cone}(\phi_D), \mathrm{Cone}(\phi_C), ((\alpha_1, \theta), (\alpha_2, 0))) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S_{\mathbb{C}}^{an}),$$

  $((\alpha_1, \theta), (\alpha_2, 0))$ being the matrix given by the composition law, together with the canonical maps

  - $c_1(-) = (c_1(\phi_D), c_1(\phi_C), 0) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \to \mathrm{Cone}(\phi)$
  - $c_2(-) = (c_2(\phi_D), c_2(\phi_C), 0) : \mathrm{Cone}(\phi) \to (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1)[1]$.

Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embedding $i_i : S \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Consider the category

$$(D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil,c,k}(\tilde{S}_{I,\mathbb{C}}^{an})) \in \mathrm{Fun}(\Gamma(\tilde{S}_I), \mathrm{TriCat})$$

such that

$$(D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil,c,k}(\tilde{S}_{I,\mathbb{C}}^{an}))(\tilde{S}_I) = D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil,c,k}(\tilde{S}_{I,\mathbb{C}}^{an})$$

- whose objects are $(((M_I, F, W), (K_I, W), \alpha_I), u_{IJ}) \in (D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil,c,k}(\tilde{S}_{I,\mathbb{C}}^{an}))$ such that

$$((M_I, F, W), (K_I, W), \alpha_I) \in D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil,c,k}(\tilde{S}_{I,\mathbb{C}}^{an}) =: \mathcal{D}(\tilde{S}_I)$$

  and for $I \subset J$,

$$u_{IJ} : ((M_I, F, W), (K_I, W), \alpha_I) \to \\ p_{IJ*}((M_J, F, W), (K_J, W), \alpha_J) := (p_{IJ*}(M_J, F, W), p_{IJ*}(K_J, W), p_{IJ*}\alpha_J)$$

  are morphisms in $\mathcal{D}(\tilde{S}_I)$,

- whose morphisms $m = (m_I) : (((M_I, F, W), (K_I, W), \alpha_I), u_{IJ}) \to (((M'_I, F, W), (K'_I, W), \alpha'_I), v_{IJ})$ is a family of morphism such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $\mathcal{D}(\tilde{S}_I)$

We have then the identity functor

$$\begin{aligned}I_S : D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}}) \to (D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil,c,k}(\tilde{S}^{an}_{I,\mathbb{C}})), \\ (((M_I,F,W),u_{IJ}),(K,W),\alpha) \mapsto (((M_I,F,W),i_{I*}j_I^*(K,W),j_I^*\alpha),(u_{IJ},I,0)), \\ m=(m_I,n) \mapsto m=(m_I,i_*j_I^*n)\end{aligned}$$

which is a full embedding since by definition for $((M_I,F,W),u_{IJ}) \in D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$,

$$u_{IJ} : (M_I,F,W) \to p_{IJ*}(M_J,F,W)$$

are filtered Zariski local equivalences, i.e. isomorphisms in $D_{\mathcal{D}(1,0)fil}(\tilde{S}_I)$, and hence for $((M_I,F,W),u_{IJ}),(K,W),\alpha) \in D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$,

$$\begin{aligned}(u_{IJ},I,0) : ((M_I,F,W),i_{I*}j_I^*(K,W),j_I^*\alpha) \to \\ p_{IJ*}((M_J,F,W),i_{J*}j_J^*(K,W),j_J^*\alpha) = (p_{IJ*}(M_J,F,W),i_{I*}j_I^*(K,W),j_I^*\alpha)\end{aligned}$$

are isomorphisms in $\mathcal{D}(\tilde{S}_I)$.

Definition 117. *For $h : U \to S$ a smooth morphism with $S,U \in \mathrm{SmVar}(k)$ and $h : U \xrightarrow{n} X \xrightarrow{f} S$ a compactification of h with n an open embedding, $X \in \mathrm{SmVar}(k)$ such that $D := X \backslash U = \cup_{i=1}^s D_i \subset X$ is a normal crossing divisor, we denote by, using definition 76 and definition 105*

$$\begin{aligned}I(U/S) : h_{!Hdg}h^{!Hdg}\mathbb{Z}^{Hdg}_S \xrightarrow{:=} \\ (p_{S*}E_{zar}(\Omega^\bullet_{X\times S/S} \otimes_{O_{X\times S}} (n\times I)_{!Hdg}\Gamma^{\vee,Hdg}_U(O_{U\times S},F_b)),\mathbb{D}_S h_*E_{usu}\mathbb{Q}_{U^{an}_{\mathbb{C}}},h_!\alpha(U,\delta)) \\ \xrightarrow{((DR(X\times S/S)(\mathrm{ad}((n\times I)_{!Hdg},(n\times I)^*)(-)),0),I,0)} \\ (\mathrm{Cone}((\Omega^{\Gamma,pr}_{/S}(i_{D_i}\times I))_{i\in[1,\ldots,s]} : p_{S*}E_{zar}(\Omega^\bullet_{X\times S/S} \otimes_{O_{X\times S}} \Gamma^{\vee,Hdg}_X(O_{X\times S},F_b)) \to \\ (\cdots \to (p_{S*}E_{zar}(\Omega^\bullet_{D_I\times S/S} \otimes_{O_{D_I\times S}} \Gamma^{\vee,Hdg}_{D_I}(O_{D_I\times S},F_b))) \to \cdots)),\mathbb{D}_S h_*E_{usu}\mathbb{Z}_{U^{an}_{\mathbb{C}}},h_!\alpha(U,\delta)) \\ \xrightarrow{=:} (\mathcal{F}^{FDR}_S(\mathbb{Z}(U/S)),\mathrm{Bti}^*_S\,\mathbb{Z}(U/S),\alpha(\mathbb{Z}(U/S)))\end{aligned}$$

the canonical isomorphism in $D_{\mathcal{D}fil}(S) \times_I D_{c,k}(S^{an}_{\mathbb{C}})$, where

- *we recall that (see section 6.1)*

$$h^{!Hdg}\mathbb{Z}^{Hdg}_S = (\Gamma^{\vee,Hdg}_U(O_{U\times S},F_b),\mathbb{Z}_{U^{an}_{\mathbb{C}}},\alpha(U)) \in HM_{gm,k,\mathbb{C}}(U),$$

- *$i_{D_i} : D_i \hookrightarrow X$ are the closed embeddings,*
- *$\alpha(\mathbb{Z}(U/S)) := h_!\alpha(U,\delta) := T^w(h,\otimes)(-) \circ h_!\alpha(U,\delta)$ (see definition 68), with*

$$\begin{aligned}\alpha(U,\delta) := (DR(U)(\Omega_{(U\times U/U)/(U/pt)}(\Gamma^{\vee,Hdg}_U(O_{U\times U}))))^{-1} \circ \alpha(U) : \\ \mathbb{C}_{U^{an}_{\mathbb{C}}} \to DR(U)((p_{U*}E_{zar}(\Omega^\bullet_{U\times U/U} \otimes_{O_{U\times U}} \Gamma^{\vee,Hdg}_U(O_{U\times U})))^{an}),\end{aligned}$$

by the way we note that the following diagram in $C(U^{an}_{\mathbb{C}})$ commutes

$$\begin{tikzcd}
\mathbb{C}_{U^{an}_{\mathbb{C}}} \arrow[rr, "\alpha(U)"] & & \Omega^\bullet_{U^{an}_{\mathbb{C}}} =: DR(U)(O^{an}_U) \\
p_{U*}E_{usu}\Gamma^\vee_U\mathbb{C}_{U\times U^{an}_{\mathbb{C}}} \arrow[u, "\mathrm{ad}(\delta^*_U,\delta_{U*})(-)"] \arrow[r, "\alpha(U\times U)"] & DR(U)((p_{U*}E_{zar}(\Omega^\bullet_{U\times U/U} \otimes_{O_{U\times U}} \Gamma^{\vee,Hdg}_U(O_{U\times U})))^{an}) \arrow[ur, "DR(U)(\Omega_{(U\times U/U)/(U/pt)}(\Gamma^{\vee,Hdg}_U(O_{U\times U})))"'] &
\end{tikzcd}$$

Lemma 9. *Let $S \in \mathrm{SmVar}(k)$. Let $g : U'/S \to U/S$ a morphism with $U/S := (U,h), U'/S := (U',h) \in \mathrm{Var}(k)^{sm}/S$. Let $h : U \xrightarrow{n} X \xrightarrow{f} S$ a compactification of h with n an open embedding, $X \in \mathrm{SmVar}(k)$ such that $D := X\backslash U = \cup_{i=1}^{s} D_i \subset X$ is a normal crossing divisor, Let $h' : U \xrightarrow{n'} X' \xrightarrow{f'} S$ a compactification of h' with n' an open embedding, $X' \in \mathrm{SmVar}(k)$ such that $D' := X\backslash U = \cup_{i=1}^{s} D_i \subset X$ is a normal crossing divisor and such that $g : U' \to U$ extend to $\bar{g} : X' \to X$, see definition-proposition 3. Then, using definition 117, the following diagram in $D_{\mathcal{D}fil}(S) \times_I D_{c,k}(S^{an}_{\mathbb{C}})$ commutes*

$$\begin{array}{ccc} h'_{!Hdg} h'^{!Hdg} \mathbb{Z}^{Hdg}_S & \xrightarrow{I(U'/S)} & (\mathcal{F}^{FDR}_S(\mathbb{Z}(U'/S)), \mathrm{Bti}^*_S \mathbb{Z}(U'/S), \alpha(\mathbb{Z}(U'/S))) \\ \downarrow{\scriptstyle \mathrm{ad}(g_{!Hdg}, g^{!Hdg})(h^{!Hdg}\mathbb{Z}^{Hdg}_S)} & & \downarrow{\scriptstyle (\Omega^{\Gamma,pr}_{/S}(R^{CH}_S(g)), \mathrm{Bti}^*_S(g), \theta(g))} \\ h_{!Hdg} h^{!Hdg} \mathbb{Z}^{Hdg}_S & \xrightarrow{I(U/S)} & (\mathcal{F}^{FDR}_S(\mathbb{Z}(U/S)), \mathrm{Bti}^*_S \mathbb{Z}(U/S), \alpha(\mathbb{Z}(U/S))) \end{array}$$

where

$$\theta(g) := R_{\mathcal{D}}([\Gamma_g]) : I(\mathrm{Bti}^*_S \mathbb{Z}(U'/S) \otimes \mathbb{C})[1] \to I(DR(S)(o_F \mathcal{F}^{FDR}_S(\mathbb{Z}(U/S))^{an}))$$

is the homotopy given by the third term of the Deligne homology class of the graph $\Gamma_g \subset U' \times_S U$ (see definition 77) and $o_F : C_{\mathcal{D}fil}(S) \to C_{\mathcal{D}}(S)$ is the forgetful functor and we recall (see section 6.1) that $I : C(S^{an}_{\mathcal{C}}/\tilde{S}^{an}_{I,\mathbb{C}}) \to K(S^{an}_{\mathcal{C}}/\tilde{S}^{an}_{I,\mathbb{C}})$ is the injective resolution functor.

Proof. Immediate from definition. □

We now define the Hodge realization functor.

Definition 118. *Let $k \subset \mathbb{C}$ a subfield. Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^{s} S_i$ an open cover such that there exists closed embedding $i_i : S \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We define the Hodge realization functor, using definition 108, definition 114, and lemma 9*

$$\mathcal{F}^{Hdg}_S := (\mathcal{F}^{FDR}_S, \mathrm{Bti}^*_S \otimes \mathbb{Q}) : C(\mathrm{Var}(k)^{sm}/S) \to D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))^0 \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$$

first on objects and then on morphisms :

- *for $F \in C(\mathrm{Var}(k)^{sm}/S)$, taking $(F,W) \in C_{fil}(\mathrm{Var}(k)^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F,W)$ gives the weight structure on $D(\mathbb{A}^1, et)(F)$,*

$$\begin{aligned} \mathcal{F}^{Hdg}_S(F) &:= (\mathcal{F}^{FDR}_S(F,W), \mathrm{Bti}^*_S(F,W) \otimes \mathbb{Q}, \alpha(F)) := \\ (e(S)_*\mathcal{H}om((\hat{R}^{CH}_{\tilde{S}_I}(\rho^*_{\tilde{S}_I} Li_{I*}j^*_I(F,W)), &\hat{R}^{CH}(T^q(D_{IJ})(-))), (E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}), T_{IJ})), \\ e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}^*_S L(F,W), \alpha(F)) &\in D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}}) \end{aligned}$$

where $\alpha(F)$ is the map in $D_{fil}(S^{an}_{\mathbb{C}}/(\tilde{S}^{an}_{I,\mathbb{C}}))$, writing for short $DR(S) := DR(S)^{[-]} := (DR(\tilde{S}_I)[-d_{\tilde{S}_I}])$

$$\begin{aligned} \alpha(F) : T(S/(\tilde{S}_I))((\mathrm{Bti}^*_S(M,W)) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}}) &:= (i_{I*}j^*_I((e(S)_*\underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}^*_S L(F,W)) \otimes \mathbb{C}_S), I) \\ &\xrightarrow{=} (e(\tilde{S}_I)_*\underline{\mathrm{sing}}_{\mathbb{D}^*} \mathrm{An}^*_{\tilde{S}_I} Li_{I*}j^*_I(F,W) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}}, T(p_{IJ}, \mathrm{An})(Li_{I*}j^*_I(F,W))) \\ \xrightarrow{=} ((((\cdot \to \oplus_{(U_{I\alpha},h_{I\alpha})\in V_I} h_{I\alpha!}h^!_{I\alpha}\mathbb{C}_{\tilde{S}^{an}_{I,\mathbb{C}}} &\xrightarrow{\mathrm{ad}(g^{\bullet!}_{I,\alpha,\beta}, g^{\bullet}_{I,\alpha,\beta!})(-)} \oplus_{(U_{I\alpha},h_{I\alpha})\in V_I} h_{I\alpha!}h^!_{I\alpha}\mathbb{C}_{\tilde{S}^{an}_{I,\mathbb{C}}} \to \cdot), u_{IJ})), W) \\ &\xrightarrow{(\alpha(\mathbb{Z}(U_{I\alpha}/\tilde{S}_I)), \theta(g^{\bullet}_{I,\alpha,\beta}))} \\ DR(S)(o_F(e(S)_*\mathcal{H}om((\hat{R}^{CH}_{\tilde{S}_I}(\rho^*_{\tilde{S}_I} Li_{I*}j^*_I(F,W)), \hat{R}^{CH}(T^q(D_{IJ})(-))), &(E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}), T_{IJ})))^{an}) \\ &\xrightarrow{=:} DR(S)((o_F\mathcal{F}^{FDR}_S(M,W))^{an}) \end{aligned}$$

with

$$\mathcal{F}_S^{DR}(M) := o_F\mathcal{F}_S^{FDR}(M) \in D_{\mathcal{D}0fil}(S/(\tilde{S}_I)),$$

where $o_F : D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \to D_{\mathcal{D}0fil}(S/(\tilde{S}_I))$ *is the forgetful functor, using lemma 9,*

$$(\alpha(\mathbb{Z}(U_{I\alpha}/S)), \theta(g^{\bullet}_{I,\alpha,\beta}))$$

being the matrix given inductively by the composition law in $D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil,c,k}(\tilde{S}^{an}_{I,\mathbb{C}})$*, that is we have the following isomorphism in* $(D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil,c,k}(\tilde{S}^{an}_{I,\mathbb{C}}))$*, denoting for short* $V_I := \mathrm{Var}(k)^{sm}/\tilde{S}_I$

$$\begin{aligned}(I^{\bullet}(U_{I\alpha}/\tilde{S}_I)) : (((\cdot \to \oplus_{(U_{I\alpha},h_{I\alpha})\in V_I} h_{I\alpha !Hdg}h^{!Hdg}_{I\alpha}\mathbb{Q}^{Hdg}_{\tilde{S}_I} \xrightarrow{\mathrm{ad}(g^{\bullet,!Hdg}_{I,\alpha,\beta}, g^{\bullet}_{I,\alpha,\beta !Hdg})(-)} \\ \oplus_{(U_{I\alpha},h_{I\alpha})\in V_I} h_{I\alpha !Hdg}h^{!Hdg}_{I\alpha}\mathbb{Q}^{Hdg}_{\tilde{S}_I} \to \cdot), u_{IJ}), W) \\ \xrightarrow{\sim} I_S(\mathcal{F}^{Hdg}_S(F) := (\mathcal{F}^{FDR}_S(F,W), \mathrm{Bti}^*_{\tilde{S}_I} Li_{I*}j^*_I(F,W)\otimes\mathbb{Q}, \alpha(F)))\end{aligned}$$

where we denote by $g^n_{I,\alpha,\beta} : U_{I\alpha} \to U_{I\beta}$ *which satisfy* $h_{I\beta}\circ g^n_{I,\alpha,\beta} = h_{I\alpha}$ *the morphisms in the canonical projective resolution*

$$\begin{aligned}q : Li_{I*}j^*_I(F,W) := ((\cdots \to \oplus_{(U_{I\alpha},h_{I\alpha})\in \mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \xrightarrow{(\mathbb{Z}(g^{\bullet}_{I,\alpha,\beta}))} \\ \oplus_{(U_{I\alpha},h_{I\alpha})\in \mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \to \cdots), W) \to i_{I*}j^*_I(F,W),\end{aligned}$$

- *for* $m : F_1 \to F_2$ *a morphism in* $C(\mathrm{Var}(k)^{sm}/S)$*, taking* $(F_1,W),(F_2,W) \in C_{fil}(\mathrm{Var}(k)^{sm}/S)$ *such that* $D(\mathbb{A}^1,et)(F_2,W)$ *gives the weight structure on* $D(\mathbb{A}^1,et)(F_2)$ $D(\mathbb{A}^1,et)(F_1,W)$ *gives the weight structure on* $D(\mathbb{A}^1,et)(F_1)$ *and such that* $m : (F_1,W)\to(F_2,W)$ *is a filtered morphism, the morphism* $\mathcal{F}^{Hdg}_S(m)$ *in* $D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$ *is given by*

$$\begin{aligned}\mathcal{F}^{Hdg}_S(m) : &= I_S^{-,-1}((I^{\bullet}(U_{I\alpha}/(\tilde{S}_I))) \circ (\mathrm{ad}(l^{\bullet !Hdg}_{I\alpha,\beta}, l^{\bullet}_{I\alpha,\beta !Hdg})(\mathbb{Q}^{Hdg}_{U_{I\alpha}})) \circ (I^{\bullet}(U_{I\alpha}/(\tilde{S}_I)))^{-1}) \\ &= (\mathcal{F}^{FDR}_S(m), \mathrm{Bti}^*_S(m)\otimes\mathbb{Q}, \theta(m) := (\theta(l_{I\alpha,\beta}))) : \mathcal{F}^{Hdg}_S(F_1) \to \mathcal{F}^{Hdg}_S(F_2)\end{aligned}$$

using lemma 9, that is we have the following commutative diagram in $(D_{\mathcal{D}(1,0)fil}(\tilde{S}_I)\times_I D_{fil,c,k}(\tilde{S}^{an}_{I,\mathbb{C}}))$*, denoting for short* $V_I := \mathrm{Var}(k)^{sm}/\tilde{S}_I$*,*

$$\begin{tikzcd}(((\cdot \to \oplus_{(U_{I\alpha},h_{I\alpha})\in V_I} h_{I\alpha !Hdg}h^{!Hdg}_{I\alpha}\mathbb{Q}^{Hdg}_{\tilde{S}_I} \arrow[r, "A^{Hdg}_{g^{\bullet}_{1I,\alpha,\beta}}"] \arrow[d, "\mathrm{ad}(l^{\bullet !Hdg}_{I,\alpha,\beta}, l^{I,\bullet}_{\alpha,\beta !Hdg})(-)"] & \oplus_{(U_{I\alpha},h_{I\alpha})\in V_I} h_{I\alpha !Hdg}h^{!Hdg}_{I\alpha}\mathbb{Q}^{Hdg}_{\tilde{S}_I} \to \cdot), u_{IJ}), W) \arrow[r, "(I^{\bullet}(U_{I\alpha}/\tilde{S}_I))"] & \mathcal{F}^{Hdg}_S(F_1) \arrow[d, "\mathcal{F}^{Hdg}_S(m)=(\mathcal{F}^{FDR}_S(m), \mathrm{Bti}^*_S(m), (\theta(l_{I\alpha,\beta})))"] \\ (((\cdot \to \oplus_{(U_{I\alpha},h_{I\alpha})\in V_I} h_{I\alpha !Hdg}h^{!Hdg}_{I\alpha}\mathbb{Q}^{Hdg}_{\tilde{S}_I} \arrow[r, "A^{Hdg}_{g^{\bullet}_{2I,\alpha,\beta}}"] & \oplus_{(U_{I\alpha},h_{I\alpha})\in V_I} h_{I\alpha !Hdg}h^{!Hdg}_{I\alpha}\mathbb{Q}^{Hdg}_{\tilde{S}_I} \to \cdot), u_{IJ}), W) \arrow[r, "(I^{\bullet}(U_{\alpha}/\tilde{S}_I))"] & \mathcal{F}^{Hdg}_S(F_2)\end{tikzcd}$$

where

- *we denoted for short* $A^{Hdg}_{g^{\bullet}_{1I,\alpha,\beta}} := \mathrm{ad}(g^{\bullet,!Hdg}_{1I,\alpha,\beta}, g^{\bullet}_{1I,\alpha,\beta !Hdg})(h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{\tilde{S}_I})$
- *we denoted for short* $A^{Hdg}_{g^{\bullet}_{2I,\alpha,\beta}} := \mathrm{ad}(g^{\bullet,!Hdg}_{2I,\alpha,\beta}, g^{\bullet}_{2I,\alpha,\beta !Hdg})(h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{\tilde{S}_I})$
- *we denote by* $g^n_{1I,\alpha,\beta} : U_{I\alpha} \to U_{I\beta}$*, which satisfy* $h_{I\beta}\circ g^n_{1I,\alpha,\beta} = h_{I\alpha}$*, the morphisms in the canonical projective resolution*

$$\begin{aligned}q : Li_{I*}j^*_I(F_1,W) := ((\cdots \to \oplus_{(U_{I\alpha},h_{I\alpha})\in \mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \xrightarrow{(\mathbb{Z}(g^{\bullet}_{1I,\alpha,\beta}))} \\ \oplus_{(U_{I\alpha},h_{I\alpha})\in \mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \to \cdots), W) \to i_{I*}j^*_I(F_1,W)\end{aligned}$$

– *we denote by* $g^n_{2I,\alpha,\beta} : U_{I\alpha} \to U_{I\beta}$, *which satisfy* $h_{I\beta} \circ g^n_{2I,\alpha,\beta} = h_\alpha$, *the morphisms in the canonical projective resolution*

$$q : Li_{I*}j_I^*(F_2, W) := ((\cdots \to \oplus_{(U_{I\alpha},h_{I\alpha})\in\mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \xrightarrow{(\mathbb{Z}(g^\bullet_{2I,\alpha,\beta}))} \oplus_{(U_{I\alpha},h_{I\alpha})\in\mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \to \cdots), W) \to i_{I*}j_I^*(F_2, W)$$

– *we denote by* $l^n_{I\alpha,\beta} : U_{I\alpha} \to U_{I\beta}$ *which satisfy* $h_{I\beta} \circ l^n_{I\alpha,\beta} = h_{I\alpha}$ *and* $l^{n+1}_{I\alpha,\beta} \circ g^n_{1I\alpha,\beta} = g^n_{2I\alpha,\beta} \circ l^n_{I\alpha,\beta}$ *the morphisms in the morphism of canonical projective resolutions*

$$Li_{I*}j_I^*(m) : Li_{I*}j_I^*(F_1, W) := ((\cdots \to \oplus_{(U_{I\alpha},h_{I\alpha})\in\mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \to \cdots), W) \xrightarrow{(\mathbb{Z}(l^\bullet_{I\alpha,\beta}))} ((\cdots \to \oplus_{(U_{I\alpha},h_{I\alpha})\in\mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_\alpha/\tilde{S}_I) \to \cdots), W) =: Li_{I*}j_I^*(F_2, W),$$

– *the maps* $I^\bullet(U_{I\alpha})$ *are given by definition 117 and lemma 9.*

Obviously $\mathcal{F}_S^{Hdg}(F[1]) = \mathcal{F}_S^{Hdg}(F)[1]$ *and* $\mathcal{F}_S^{Hdg}(\mathrm{Cone}(m)) = \mathrm{Cone}(\mathcal{F}_S^{Hdg}(m))$. *This functor induces by proposition 62 and remark 5 the functor*

$$\mathcal{F}_S^{Hdg} := (\mathcal{F}_S^{FDR}, \mathrm{Bti}_S^* \otimes \mathbb{Q}) : \mathrm{DA}(S) \to D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S_{\mathbb{C}}^{an}),$$
$$M = D(\mathbb{A}^1, et)(F) \mapsto \mathcal{F}_S^{Hdg}(M) := \mathcal{F}_S^{Hdg}(F) = (\mathcal{F}_S^{FDR}(M), \mathrm{Bti}_S^* M \otimes \mathbb{Q}, \alpha(M)),$$

with $\alpha(M) = \alpha(F)$.

We now give the functoriality with respect to the five operation using the De Rahm realization case and the Betti realization case :

Proposition 71. *(i) Let* $g : T \to S$ *a morphism with* $T, S \in \mathrm{Var}(k)$. *Assume there exists a factorization* $g : T \xrightarrow{l} Y \times S \xrightarrow{p} S$, *with* $Y \in \mathrm{SmVar}(k)$, l *a closed embedding and* p *the projection. Let* $S = \cup_{i\in I}S_i$ *an open cover and* $i_i : S_i \hookrightarrow \tilde{S}_i$ *closed embeddings with* $\tilde{S}_i \in \mathrm{SmVar}(k)$. *Then,* $\tilde{g}_I : Y \times \tilde{S}_I \to \tilde{S}_I$ *is a lift of* $g_I = g_{|T_I} : T_I \to S_I$ *and we have closed embeddings* $i'_I := i_I \circ l \circ j'_I : T_I \hookrightarrow Y \times \tilde{S}_I$. *Then, for* $M \in DA_c(S)$, *the following diagram commutes :*

$$\begin{tikzcd} g^{*w}\,\mathrm{Bti}_S^* M \otimes \mathbb{C} \arrow[r, "g^*(\alpha(M))"] \arrow[d, "T(g,bti)(M)"'] & DR(T)^{[-]}((g^{\hat{*}mod}_{Hdg}\mathcal{F}_S^{DR}(M))^{an}) \arrow[d, "DR(T)^{[-]}((T(g,\mathcal{F}^{FDR})(M))^{an})"] \\ \mathrm{Bti}_T^* g^*M \otimes \mathbb{C} \arrow[r, "\alpha(g^*M)"] & DR(T)^{[-]}((\mathcal{F}_T^{DR}(g^*M))^{an}) \end{tikzcd},$$

see section 5, definition 112 and definition 115

(ii) Let $f : T \to S$ *a morphism with* $T, S \in \mathrm{QPVar}(k)$. *Then, for* $M \in DA_c(T)$,*the following diagram commutes :*

$$\begin{tikzcd} Rf_{*w}\,\mathrm{Bti}_T^* M \otimes \mathbb{C} \arrow[r, "f_*(\alpha(M)"] & DR(S)^{[-]}((Rf_*^{Hdg}\mathcal{F}_T^{DR}(M))^{an}) \\ \mathrm{Bti}_S^* Rf_*M \otimes \mathbb{C} \arrow[r, "\alpha(Rf_*M)"] \arrow[u, "T_*(f,bti)(M)"] & DR(S)^{[-]}((\mathcal{F}_S^{DR}(Rf_*M))^{an}) \arrow[u, "DR(S)^{[-]}((T_*(f,\mathcal{F}^{FDR})(M))^{an})"'] \end{tikzcd}$$

see section 5, definition 113 and definition 116 *(iii) Let $f : T \to S$ a morphism with $T, S \in \mathrm{QPVar}(k)$. Then, for $M \in DA_c(T)$,the following diagram commutes :*

$$\begin{tikzcd}
Rf_{!w}\,\mathrm{Bti}^*_T M \otimes \mathbb{C} \arrow[r, "f_!(\alpha(M))"] \arrow[d, "T_!(f,bti)(M)"'] & DR(S)^{[-]}((Rf^{Hdg}_! \mathcal{F}^T_{DR}(M))^{an}) \arrow[d, "DR(S)^{[-]}((T_!(f,\mathcal{F}_{DR})(M))^{an})"] \\
\mathrm{Bti}^*_S Rf_! M \otimes \mathbb{C} \arrow[r, "\alpha(Rf_!M)"] & DR(S)^{[-]}((\mathcal{F}^S_{DR}(Rf_!M))^{an})
\end{tikzcd}$$

see section 5, definition 113 and definition 116.

(iv) Let $f : T \to S$ a morphism with $T, S \in \mathrm{QPVar}(k)$. Then, for $M \in DA_c(S)$,the following diagram commutes :

$$\begin{tikzcd}
f^{!w}\,\mathrm{Bti}^*_S M \otimes \mathbb{C} \arrow[r, "f^!(\alpha(M))"] & DR(T)^{[-]}((f^{*mod}_{Hdg} \mathcal{F}^{DR}_S(M))^{an}) \\
\mathrm{Bti}^*_T f^! M \otimes \mathbb{C} \arrow[r, "\alpha(f^!M)"] \arrow[u, "T^!(f,bti)(M)"] & DR(T)^{[-]}((\mathcal{F}^{DR}_T(f^!M))^{an}) \arrow[u, "DR^{[-]}(T)((T^!(g,\mathcal{F}^{FDR})(M))^{an})"']
\end{tikzcd}$$

see section 5, definition 113 and definition 116.

(v) Let $S \in \mathrm{Var}(k)$. Then, for $M, N \in DA_c(S)$,the following diagram commutes :

$$\begin{tikzcd}
\mathrm{Bti}^*_S M \otimes \mathrm{Bti}^*_S N \otimes \mathbb{C} \arrow[r, "\alpha(M)\otimes\alpha(N)"] \arrow[d, "T(\otimes,bti)(M,N)"'] & DR(S)((\mathcal{F}^{DR}_S(M) \otimes_{O_S} \mathcal{F}^{DR}_S(N))^{an}) \arrow[d, "DR(S)((T(\otimes,\mathcal{F}^{DR})(M,N))^{an})"] \\
\mathrm{Bti}^*_S(M \otimes N) \otimes \mathbb{C} \arrow[r, "(\alpha(M\otimes N))"] & DR(S)((\mathcal{F}^S_{DR}(M \otimes N))^{an})
\end{tikzcd}$$

see definition 113 and definition 116.

Proof. (i): Follows from the following commutative diagram in $(D_{\mathcal{D}(1,0)fil}(Y \times \tilde{S}_I) \times_I D_{fil,c,k}(Y \times \tilde{S}^{an}_{I,\mathbb{C}}))$,

$$\begin{tikzcd}
\begin{array}{c}(((\to \oplus_{(U_{I\alpha},h_{I\alpha})\in V_I} \tilde{g}^{*Hdg}_I h_{I\alpha!Hdg} h^{!Hdg}_{I\alpha} \mathbb{Z}^{Hdg}_{\tilde{S}_I} \xrightarrow{A^{Hdg}_{g^\bullet_{I,\alpha,\beta}}} \\ \oplus_{(U_{I\alpha},h_{I\alpha})\in V_I} h_{I\alpha!Hdg} h^{!Hdg}_{I\alpha} \mathbb{Z}^{Hdg}_{\tilde{S}_I} \to), u_{IJ}), W)\end{array} \arrow[r, "(\tilde{g}^{*Hdg}_I I^\bullet(U_{I\alpha}/\tilde{S}_I))"] \arrow[d, "T^{Hdg}(\tilde{g}_I,h_I)(-)"'] & \begin{array}{c}(g^{\hat{*}mod}_{Hdg} \mathcal{F}^{FDR}_T(F), \\ g^{*w}\,\mathrm{Bti}^*_S(F,W), g^*(\alpha(F)))\end{array} \arrow[d, "(T(g,\mathcal{F}^{FDR})(M){,}T(g{,}\,\mathrm{Bti})(M){,}0)"'] \\
\begin{array}{c}(((\to \oplus_{(U'_{I\alpha},h_{I\alpha})\in W_I} h'_{I\alpha!Hdg} h'^{!Hdg}_{I\alpha} \mathbb{Z}^{Hdg}_{Y\times\tilde{S}_I} \xrightarrow{A^{Hdg}_{g'^\bullet_{I,\alpha,\beta}}} \\ \oplus_{(U'_{I\alpha},h'_{I\alpha})\in W_I} h'_{I\alpha!Hdg} h'^{!Hdg}_{I\alpha} \mathbb{Z}^{Hdg}_{Y\times\tilde{S}_I} \to), u_{IJ}), W)\end{array} \arrow[r, "(I^\bullet(U'_\alpha/Y\times\tilde{S}_I))"] & \begin{array}{c}(\mathcal{F}^{FDR}_T(g^*F), \\ \mathrm{Bti}^*_T(g^*F,W), \alpha(g^*F))\end{array}
\end{tikzcd}$$

where, we have denoted for short $V_I := \mathrm{Var}(k)^{sm}/\tilde{S}_I$ and $W_I := \mathrm{Var}(k)^{sm}/Y \times \tilde{S}_I$,

- we denoted for short $A^{Hdg}_{g^\bullet_{I,\alpha,\beta}} := \mathrm{ad}(g^{\bullet,!Hdg}_{I,\alpha,\beta}, g^\bullet_{I,\alpha,\beta!Hdg})(h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{\tilde{S}_I})$
- we denoted for short $A^{Hdg}_{g'^\bullet_{I,\alpha,\beta}} := \mathrm{ad}(g'^{\bullet,!Hdg}_{I,\alpha,\beta}, g'^\bullet_{I,\alpha,\beta!Hdg})(h'^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{Y\times\tilde{S}_I})$
- we denote by $g^n_{I,\alpha,\beta} : U_{I\alpha} \to U_{I\beta}$, which satisfy $h_{I\beta} \circ g^n_{I,\alpha,\beta} = h_{I\alpha}$, the morphisms in the canonical projective resolution

$$q : Li_{I*}j^*_I(F,W) := (\cdots \to \oplus_{(U_{I\alpha},h_{I\alpha})\in \mathrm{Var}(k)^{sm}/\tilde{S}_I} \mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \xrightarrow{(\mathbb{Z}(g^\bullet_{I,\alpha,\beta}))} \oplus_{(U_{I\alpha},h_{I\alpha})\in \mathrm{Var}(k)^{sm}/\tilde{S}_I} \mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \to \cdots) \to i_{I*}j^*_I(F,W)$$

- we denote by $g'^{n}_{I,\alpha,\beta} : U'_{I\alpha} \to U'_{I\beta}$, which satisfy $h'_{I\beta} \circ g'^{n}_{I,\alpha,\beta} = h'_{\alpha}$, the morphisms in the canonical projective resolution

$$q : Li'_{I*}j'^{*}_{I}(g^*F, W) := (\cdots \to \oplus_{(U'_{I\alpha},h'_{I\alpha})\in\mathrm{Var}(k)^{sm}/Y\times\tilde{S}_I}\mathbb{Z}(U'_{I\alpha}/Y\times\tilde{S}_I) \xrightarrow{(\mathbb{Z}(g'^{\bullet}_{I,\alpha,\beta}))} \oplus_{(U'_{I\alpha},h'_{I\alpha})\in\mathrm{Var}(k)^{sm}/Y\times\tilde{S}_I}\mathbb{Z}(U'_{I\alpha}/Y\times\tilde{S}_I) \to \cdots) \to i'_{I*}j'^{*}_{I}(g^*F, W)$$

(ii): Follows from (i) by adjonction.
(iii): The closed embedding case is given by (ii) and the smooth projection case follows from (i) by adjonction.
(iv): Follows from (iii) by adjonction.
(v):Obvious

□

We can now state the following key proposition and the main theorem:

Proposition 72. *Let $k \subset \mathbb{C}$ a subfield.*

(i) Let $S \in \mathrm{Var}(k)$. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then we have the isomorphism in $D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$

$$\mathcal{F}^{Hdg}_S(\mathbb{Z}_S) \xrightarrow{:=} (\mathcal{F}^{FDR}_S(\mathbb{Z}_S, W), \mathrm{Bti}^*_S(\mathbb{Z}_S, W)\otimes\mathbb{Q}, \alpha(\mathbb{Z}_S)) \xrightarrow{((\Omega^{\Gamma,pr}_{/\tilde{S}_I}(\hat{R}^{CH}(\mathrm{ad}(i^*_I, i_{I*})(\Gamma^{\vee,w}_{S_I}\mathbb{Z}_{\tilde{S}_I})))), I, 0)}$$
$$I^{-1}_S((e(S)_*\mathcal{H}om((\hat{R}^{CH}(\Gamma^{\vee,w}_{S_I}\mathbb{Z}_{\tilde{S}_I}), \hat{R}^{CH}(x_{IJ})), (E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}), T_{IJ})), T(S/(\tilde{S}_I))(\mathbb{Q}^w_{S^{an}_{\mathbb{C}}}), \Gamma^{\vee,w}_{S_I}\alpha(\tilde{S}_I, \delta)))$$
$$\xrightarrow{=} \iota_S((\Gamma^{\vee,Hdg}_{S_I}(O_{\tilde{S}_I}, F_b), x_{IJ}), \mathbb{Q}^w_{S^{an}_{\mathbb{C}}}, \alpha(S)) =: \iota_S(\mathbb{Q}^{Hdg}_S)$$

with (see section 6.1) $j^*_I\mathbb{Q}^w_{S^{an}_{\mathbb{C}}} = i^*_I\Gamma^{\vee,w}_{S_I}\mathbb{Q}_{\tilde{S}_I}$ and

$$\alpha(S) : T(S/(\tilde{S}_I))((\mathbb{Q}^w_{S^{an}_{\mathbb{C}}}) \otimes \mathbb{C}_{S^{an}_{\mathbb{C}}}) \xrightarrow{=} (\Gamma^{\vee,w}_{S_I}\mathbb{C}_{\tilde{S}^{an}_{I,\mathbb{C}}}, t_{IJ}) \xrightarrow{(\Gamma^{\vee}_{S_I}\alpha(\tilde{S}_I))} DR(S)(o_F(\Gamma^{\vee,Hdg}_{S_I}(O_{\tilde{S}_I}, F_b), x_{IJ}))$$

(ii) Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$, X quasi-projective. Consider a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y = \mathbb{P}^{N,o} \subset \mathbb{P}^N$ an open subset, l a closed embedding and p_S the projection. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(\mathbb{C})$. Recall that $S_I := \cap_{i\in I}S_i$, $X_I = f^{-1}(S_I)$, and $\tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. Then, using proposition 71(iii), the maps of definition 113 and definition 116 gives an isomorphism in $D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{an}_{\mathbb{C}})$

$$(T_!(f, \mathcal{F}^{FDR})(\mathbb{Z}_X, W), T_!(f, \mathrm{Bti})(\mathbb{Z}_X, W), 0) :$$
$$\mathcal{F}^{Hdg}_S(M^{BM}(X/S)) := (\mathcal{F}^{FDR}_S(Rf_!(\mathbb{Z}_X, W)), \mathrm{Bti}^*_S\, Rf_!(\mathbb{Z}_X, W)\otimes\mathbb{Q}, \alpha(Rf_!\mathbb{Z}_X))$$
$$\xrightarrow{\sim} (Rf_{Hdg!}(\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I}, F_b), x_{IJ}(X/S)), Rf_{!w}\mathbb{Q}^w_{X^{an}}, f_!(\alpha(X))) =: \iota_S(Rf_{!Hdg}\mathbb{Q}^{Hdg}_X).$$

with

$$\mathbb{Q}^{Hdg}_X := ((\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I}, F_b), x_{IJ}(X/Y\times S)), \mathbb{Q}^w_{X^{an}_{\mathbb{C}}}, \alpha(X)) \in C(MHM_{gm,k,\mathbb{C}}(X))$$

Proof. (i):Follows from proposition 62.
(ii): Follows from (i) by proposition 71(iii),theorem 53(i) and theorem 54(i). □

The main theorem of this section is the following :

Theorem 56. *Let $k \subset \mathbb{C}$ a subfield.*

(i) For $S \in \mathrm{Var}(k)$, we have $\mathcal{F}_S^{Hdg}(\mathrm{DA}_c(S)) \subset D(MHM_{gm,k,\mathbb{C}}(S))$,

$$\iota_S : D(MHM_{gm,k,\mathbb{C}}(S)) \hookrightarrow D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S_{\mathbb{C}}^{an})$$

being a full embedding by theorem 44.

(ii) The Hodge realization functor $\mathcal{F}_{Hdg}(-)$ define a morphism of 2-functor on $\mathrm{Var}(k)$

$$\mathcal{F}_{-}^{Hdg} : \mathrm{Var}(k) \to (\mathrm{DA}_c(-) \to D(MHM_{gm,k,\mathbb{C}}(-)))$$

whose restriction to $\mathrm{QPVar}(k)$ is an homotopic 2-functor in sense of Ayoub. More precisely,

(ii0) for $g : T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(k)$, and $M \in \mathrm{DA}_c(S)$, the the maps of definition 112 and of definition 115 induce an isomorphism in $D(MHM_{gm,k,\mathbb{C}}(T))$

$$\begin{aligned} T(g, \mathcal{F}^{Hdg})(M) &:= (T(g, \mathcal{F}^{FDR})(M), T(g, bti)(M), 0) : \\ g^{\hat{*}Hdg}\mathcal{F}_S^{Hdg}(M) &:= \iota_T^{-1}(g_{Hdg}^{\hat{*}mod}\mathcal{F}_S^{FDR}(M), g^* \mathrm{Bti}_S(M) \otimes \mathbb{Q}, g^*(\alpha(M))) \\ &\xrightarrow{\sim} \iota_T^{-1}(\mathcal{F}_T^{FDR}(g^*M), \mathrm{Bti}_T^*(g^*M) \otimes \mathbb{Q}, \alpha(g^*M)) =: \mathcal{F}_T^{Hdg}(g^*M), \end{aligned}$$

(ii1) for $f : T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(k)$, and $M \in \mathrm{DA}_c(T)$, the maps of definition 113 and of definition 116 induce an isomorphism in $D(MHM_{gm,k,\mathbb{C}}(S))$

$$\begin{aligned} T_*(f, \mathcal{F}^{Hdg})(M) &:= (T_*(f, \mathcal{F}^{FDR})(M), T_*(f, bti)(M), 0) : \\ Rf_{Hdg*}\mathcal{F}_T^{Hdg}(M) &:= \iota_S^{-1}(Rf_*^{Hdg}\mathcal{F}_T^{FDR}(M), Rf_* \mathrm{Bti}_T(M) \otimes \mathbb{Q}, f_*(\alpha(M))) \\ &\xrightarrow{\sim} \iota_S^{-1}(\mathcal{F}_S^{FDR}(Rf_*M), \mathrm{Bti}_S^*(Rf_*M) \otimes \mathbb{Q}, \alpha(Rf_*M)) =: \mathcal{F}_S^{Hdg}(Rf_*M), \end{aligned}$$

(ii2) for $f : T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(k)$, and $M \in \mathrm{DA}_c(T)$, the maps of definition 113 and of definition 116 induce an isomorphism in $D(MHM_{gm,k,\mathbb{C}}(S))$

$$\begin{aligned} T_!(f, \mathcal{F}^{Hdg})(M) &:= (T_!(f, \mathcal{F}^{FDR})(M), T_!(f, bti)(M), 0) : \\ Rf_{!Hdg}\mathcal{F}_T^{Hdg}(M) &:= \iota_S^{-1}(Rf_!^{Hdg}\mathcal{F}_T^{FDR}(M), Rf_! \mathrm{Bti}_T^*(M) \otimes \mathbb{Q}, f_!(\alpha(M))) \\ &\xrightarrow{\sim} \iota_S^{-1}(\mathcal{F}_S^{FDR}(Rf_!M), \mathrm{Bti}_S^*(Rf_!M) \otimes \mathbb{Q}, \alpha(f_!M)) =: \mathcal{F}_S^{Hdg}(Rf_!M), \end{aligned}$$

(ii3) for $f : T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(k)$, and $M \in \mathrm{DA}_c(S)$, the maps of definition 113 and of definition 116 induce an isomorphism in $D(MHM_{gm,k,\mathbb{C}}(T))$

$$\begin{aligned} T^!(f, \mathcal{F}^{Hdg})(M) &:= (T^!(f, \mathcal{F}^{FDR})(M), T^!(f, bti)(M), 0) : \\ f^{*Hdg}\mathcal{F}_S^{Hdg}(M) &:= \iota_T^{-1}(f_{Hdg}^{*mod}\mathcal{F}_S^{FDR}(M), f^! \mathrm{Bti}_S(M) \otimes \mathbb{Q}, f^!(\alpha(M))) \\ &\xrightarrow{\sim} \iota_T^{-1}(\mathcal{F}_T^{FDR}(f^!M), \mathrm{Bti}_T^*(f^!M) \otimes \mathbb{Q}, \alpha(f^!M)) =: \mathcal{F}_T^{Hdg}(f^!M), \end{aligned}$$

(ii4) for $S \in \mathrm{Var}(k)$, and $M, N \in \mathrm{DA}_c(S)$, the maps of definition 113 and of definition 116 induce an isomorphism in $D(MHM_{gm,k,\mathbb{C}}(S))$

$$\begin{aligned} T(\otimes, \mathcal{F}^{Hdg})(M, N) &:= (T(\otimes, \mathcal{F}_S^{FDR})(M, N), T(\otimes, bti)(M, N), 0) : \\ \iota_S^{-1}(\mathcal{F}_S^{FDR}(M) &\otimes_{O_S}^{Hdg} \mathcal{F}_S^{FDR}(N), \mathrm{Bti}_S(M) \otimes \mathrm{Bti}_S(N) \otimes \mathbb{Q}, \alpha(M) \otimes \alpha(N)) \\ &\xrightarrow{\sim} \mathcal{F}_S^{Hdg}(M \otimes N) := \iota_S^{-1}(\mathcal{F}_S^{FDR}(M \otimes N), \mathrm{Bti}_S(M \otimes N) \otimes \mathbb{Q}, \alpha(M \otimes N)). \end{aligned}$$

(iii) For $S \in \mathrm{Var}(k)$, the following diagram commutes :

$$\begin{tikzcd} \mathrm{Var}(k)/S \arrow[r, "MH(/S)"] \arrow[d, "M(/S)"'] & D(MHM_{gm,k,\mathbb{C}}(S)) \arrow[d, "\iota^S"] \\ \mathrm{DA}(S) \arrow[r, "\mathcal{F}_S^{Hdg}"] & D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S_{\mathbb{C}}^{an}) \end{tikzcd}$$

Proof. (i): Let $M \in \mathrm{DA}_c(S)$. There exist by definition of constructible motives an isomorphism in $\mathrm{DA}(S)$

$$w(M) : M \xrightarrow{\sim} \mathrm{Cone}(M(X_0/S)[d_0] \xrightarrow{m_1} \cdots \xrightarrow{m_m} M(X_m/S)[d_m]),$$

with $f_n : X_n \to S$ morphisms and $X_n \in \mathrm{QPVar}(k)$. This gives the isomorphism in $D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S_{\mathbb{C}}^{an})$

$$\mathcal{F}_S^{Hdg}(w(M)) : \mathcal{F}_S^{Hdg}(M) \xrightarrow{\sim} \mathrm{Cone}(\mathcal{F}_S^{Hdg}(M(X_0/S))[d_0] \xrightarrow{\mathcal{F}_S^{Hdg}(m_1)} \cdots \xrightarrow{\mathcal{F}_S^{Hdg}(m_1)} \mathcal{F}_S^{Hdg}(M(X_m/S))[d_m]),$$

On the other hand, by proposition 72(i), we have

$$\mathcal{F}_S^{Hdg}(M(X_n/S)) \xrightarrow{\sim} Rf_{!Hdg}\mathbb{Q}_X^{Hdg} \in D(MHM_{gm,k,\mathbb{C}}(S)).$$

This prove (i).
(ii0): Follows from theorem 53(i), proposition 71(i) and theorem 54.
(ii1): Follows from theorem 53(iii), proposition 71(ii), and theorem 54(iii).
(ii2):Follows from theorem 53(ii), proposition 71(iii), and theorem 54(ii).
(ii3): Follows from theorem 53(iv), proposition 71(iv), and theorem 54(iv).
(ii4):Follows from theorem 53(v), proposition 71(v) and theorem 54(v).
(iii): By (ii), for $g : X'/S \to X/S$ a morphism, with $X', X, S \in \mathrm{Var}(k)$ and $X/S = (X, f)$, $X'/S = (X', f')$, we have by adjonction the following commutative diagram

$$\begin{tikzcd}
\mathcal{F}_S^{Hdg}(M(X'/S) = f'_! f'^! \mathbb{Z}_S = f_! g_! g^! f^! \mathbb{Z}_S) \arrow[r, "\mathcal{F}_S^{Hdg}(M(/S)(g) = f_! \mathrm{ad}(g_!, g^!)(f^! \mathbb{Z}_S))"] \arrow[d, "T_!(f', \mathcal{F}^{Hdg})(f'^! M(X'/S)) \circ T^!(f', \mathcal{F}^{Hdg})(M(X'/S))"'] & \mathcal{F}_S^{Hdg}(M(X/S) = f_! f^! \mathbb{Z}_S) \arrow[d, "T_!(f, \mathcal{F}^{Hdg})(f^! M(X/S)) \circ T^!(f, \mathcal{F}^{Hdg})(M(X/S))"'] \\
MH(X'/S) := Rf'_{!Hdg} f'^{!Hdg} \mathbb{Z}_S^{Hdg} = f_{!Hdg} g_{!Hdg} g^{!Hdg} f^{!Hdg} \mathbb{Z}_S^{Hdg} \arrow[r, "f_{!Hdg} \mathrm{ad}(g_{!Hdg}, g^{!Hdg})(f^{!Hdg} \mathbb{Z}_S^{Hdg})"] & MH(X/S) := f_{!Hdg} f^{!Hdg} \mathbb{Z}_S^{Hdg}
\end{tikzcd}.$$

where the left and right columns are isomorphisms by (ii). This proves (iii). $\square$

The theorem 56 gives immediately the following :

Corollary 5. *Let $k \subset \mathbb{C}$ a subfield. Let $f : U \to S$, $f' : U' \to S$ morphisms, with $U, U', S \in \mathrm{Var}(k)$ irreducible, U' smooth. Let $\bar{S} \in \mathrm{PVar}(k)$ a compactification of S. Let $\bar{X}, \bar{X}' \in \mathrm{PVar}(k)$ compactification of U and U' respectively, such that f (resp. f') extend to a morphism $\bar{f} : \bar{X} \to \bar{S}$, resp. $\bar{f}' : \bar{X}' \to \bar{S}$. Denote $\bar{D} = \bar{X} \backslash U$ and $\bar{D}' = \bar{X}' \backslash U'$ and $\bar{E} = (\bar{D} \times_{\bar{S}} \bar{X}') \cup (\bar{X} \times_{\bar{S}} \bar{D}')$. Denote $i : \bar{D} \hookrightarrow \bar{X}$, $i' : \bar{D} \hookrightarrow \bar{X}$ denote the closed embeddings and $j : U \hookrightarrow \bar{X}$, $j' : U' \hookrightarrow \bar{X}'$ the open embeddings. Denote $d = \dim(U)$ and $d' = \dim(U')$. We have the following commutative diagram in $D(\mathbb{Z})$*

$$\begin{tikzcd}
RHom^\bullet_{\mathrm{DA}(\bar{S})}(M(U'/\bar{S}), M((\bar{X}, \bar{D})/\bar{S})) \arrow[r, "\mathcal{F}_S^{Hdg}(-,-)"] \arrow[d, "RI(-,-)"] & RHom^\bullet_{DMHM(\bar{S})}(f'_{!Hdg}\mathbb{Z}_{U'}^{Hdg}, f_{*Hdg}\mathbb{Z}_U^{Hdg}) \arrow[d, "RI(-,-)"] \\
RHom^\bullet(M(\mathrm{pt}), M(\bar{X}' \times_{\bar{S}} \bar{X}, \bar{E})(d')[2d']) \arrow[r, "\mathcal{F}_{Hdg}^{\mathrm{pt}}(-,-)"] \arrow[d, "l"] & RHom^\bullet(\mathbb{Z}_{\mathrm{pt}}^{Hdg}, a_{U' \times_S U!}\mathbb{Z}_{U \times_S U'}^{Hdg}(d')[2d']) \arrow[d, "l"] \\
\mathcal{Z}_d(\bar{X}' \times_{\bar{S}} \bar{X}, E, \bullet) \arrow[r, "\mathcal{R}^d_{\bar{X}' \times_{\bar{S}} \bar{X}}"] & C^{\mathcal{D}}_{2d+\bullet}(\bar{X}' \times_{\bar{S}} \bar{X}, E, Z(d))
\end{tikzcd}$$

where

$$M((\bar{X}, \bar{D})/\bar{S}) := \mathrm{Cone}(\mathrm{ad}(i_*, i^!) : M(\bar{D}/\bar{S}) \to M(\bar{X}/\bar{S})) = \bar{f}_* j_* E_{et}(\mathbb{Z}(U/U)) \in \mathrm{DA}(\bar{S})$$

and l the isomorphisms given by canonical embedding of complexes.

Proof. The upper square of this diagram follows from theorem 56(ii). On the other side, the lower square follows from the absolute case. $\square$

8.2 The p adic Hodge realization functor for relative motives over a subfield $k \subset \mathbb{C}_p$

Let p a prime number. Let $k \subset \mathbb{C}_p$ a subfield.

For $S \in \mathrm{Var}(k)$, we have the analytical functor

$$\mathrm{an}_S^{*mod} : C_{\mathcal{D}}(S) \to C_{D(O_{S^{an}_{\mathbb{C}_p}})}(S^{an,pet}_{\mathbb{C}_p}),\ M \mapsto M^{an} := \mathrm{an}_S^{*mod} M$$

given by the morphism of ringed topological spaces $\mathrm{an}_S : S^{an}_{\mathbb{C}_p} \xrightarrow{\mathrm{an}_S} S_{\mathbb{C}_p} \xrightarrow{\pi_{k/\mathbb{C}_p}(S)} S$. For $S \in \mathrm{Var}(k)$, we denote by for short $O\mathbb{B}_{dr,S} := O\mathbb{B}_{dr,S^{an}_{\mathbb{C}_p}} := O\mathbb{B}_{dr,R_{\mathbb{C}_p}(S^{an}_{\mathbb{C}_p})}$. where $R_{\mathbb{C}_p} : \mathrm{AnSp}(\mathbb{C}_p) \to \mathrm{AdSp}/(\mathbb{C}_p, O_{\mathbb{C}_p})$ is the canonical functor (see section 2). For $\mathcal{S} \in \mathrm{Cat}$ a site, p a prime number, we recall (see section 2) the functor

$$(-) \otimes \mathbb{Z}_p : C(\mathcal{S}) \to C_{\mathbb{Z}_p}(\mathcal{S}) \subset C(\mathbb{N} \times \mathcal{S}) = \mathrm{PSh}(\mathcal{S}, \mathrm{Fun}(\mathbb{N}, C(\mathbb{Z}))),\ K \mapsto K \otimes \mathbb{Z}_p := (K \otimes \mathbb{Z}/p^n\mathbb{Z})_{n \in \mathbb{N}}.$$

Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i \in I} S_i$ an open cover such that there exists closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_I \in \mathrm{SmVar}(k)$. We have the category $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$

- whose set of objects is the set of triples $\{(((M_I, F, W), u_{IJ}), (K, W), \alpha)\}$ with

$$((M_I, F, W), u_{IJ}) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)),\ (K, W) \in D_{\mathbb{Z}_p fil,c,k}(S^{et}),$$
$$\alpha : \mathbb{B}_{dr,(\tilde{S}_I)}(K, W) \to F^0 DR(S)^{[-]}(((M_I, F, W), u_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I}, F), t_{IJ}))$$

  where α is a morphism in $D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$,

- and whose set of morphisms consists of

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

  where $\phi_D : ((M_1, F, W), u_{IJ}) \to ((M_2, F, W), u_{IJ})$ and $\phi_C : (K_1, W) \to (K_2, W)$ are morphisms and

$$\theta = (\theta^\bullet, I(F^0 DR(S)(\phi_D^{an}) \times I) \circ I(\alpha_1), I(\alpha_2) \circ I(\mathbb{B}_{dr,(\tilde{S}_I)}(\phi_C))) :$$
$$I(\mathbb{B}_{dr,(\tilde{S}_I)}(K_1, W))[1] \to I(F^0 DR(S)(((M_{2I}, F, W), u_{IJ})^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I}, F), t_{IJ}))$$

  is an homotopy, $I : D_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p})) \to K_{\mathbb{B}_{dr},G,fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$ being the injective resolution functor, and for

  - $\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$
  - $\phi' = (\phi'_D, \phi'_C, [\theta']) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \to (((M_{3I}, F, W), u_{IJ}), (K_3, W), \alpha_3)$

  the composition law is given by

$$\phi' \circ \phi := (\phi'_D \circ \phi_D, \phi'_C \circ \phi_C, I(DR(S)(\phi_D^{'an} \otimes I)) \circ [\theta] + [\theta'] \circ I(\mathbb{B}_{dr,(\tilde{S}_I)}(\phi_C))[1]) :$$
$$(((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{3I}, F, W), u_{IJ}), (K_3, W), \alpha_3),$$

  in particular for $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$,

$$I_{(((M_I,F,W),u_{IJ}),(K,W),\alpha)} = ((I_{M_I}), I_K, 0),$$

and also the category $D_{\mathcal{D}(1,0)fil,rh,\infty}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$ defined in the same way, together with the localization functor

$$(D(zar), I) : C_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et}) \to D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$$
$$\to D_{\mathcal{D}(1,0)fil,rh,\infty}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et}).$$

Note that if $\phi = (\phi_D, \phi_C, [\theta]) : (((M_1, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_2, F, W), u_{IJ}), (K_2, W), \alpha_2)$ is a morphism in $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$ such that ϕ_D and ϕ_C are isomorphism then ϕ is an isomorphism (see remark 8). Moreover,

- For $(((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$, we set

$$(((M_I, F, W), u_{IJ}), (K, W), \alpha)[1] := (((M_I, F, W), u_{IJ})[1], (K, W)[1], \alpha[1]).$$

- For

$$\phi = (\phi_D, \phi_C, [\theta]) : (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1) \to (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2)$$

  a morphism in $D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$, we set (see [11] definition 3.12)

$$\mathrm{Cone}(\phi) := (\mathrm{Cone}(\phi_D), \mathrm{Cone}(\phi_C), ((\alpha_1, \theta), (\alpha_2, 0))) \in D_{\mathcal{D}(1,0)fil,rh}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et}),$$

  $((\alpha_1, \theta), (\alpha_2, 0))$ being the matrix given by the composition law, together with the canonical maps

  - $c_1(-) = (c_1(\phi_D), c_1(\phi_C), 0) : (((M_{2I}, F, W), u_{IJ}), (K_2, W), \alpha_2) \to \mathrm{Cone}(\phi)$
  - $c_2(-) = (c_2(\phi_D), c_2(\phi_C), 0) : \mathrm{Cone}(\phi) \to (((M_{1I}, F, W), u_{IJ}), (K_1, W), \alpha_1)[1]$.

Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embedding $i_i : S \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Consider the category

$$(D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{\mathbb{Z}_p fil,c,k}(\tilde{S}_I^{et})) \in \mathrm{Fun}(\Gamma(\tilde{S}_I), \mathrm{TriCat})$$

such that

$$(D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{\mathbb{Z}_p fil,c,k}(\tilde{S}_I^{et}))(\tilde{S}_I) = D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{\mathbb{Z}_p fil,c,k}(\tilde{S}_I^{et})$$

- whose objects are $(((M_I, F, W), (K_I, W), \alpha_I), u_{IJ}) \in (D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{\mathbb{Z}_p fil,c,k}(\tilde{S}_I^{et}))$ such that

$$((M_I, F, W), (K_I, W), \alpha_I) \in D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{\mathbb{Z}_p fil,c,k}(\tilde{S}_I^{et}) =: \mathcal{D}_p(\tilde{S}_I)$$

  and for $I \subset J$,

$$u_{IJ} : ((M_I, F, W), (K_I, W), \alpha_I) \to$$
$$p_{IJ*}((M_J, F, W), (K_J, W), \alpha_J) := (p_{IJ*}(M_J, F, W), p_{IJ*}(K_J, W), p_{IJ*}\alpha_J)$$

  are morphisms in $\mathcal{D}_p(\tilde{S}_I)$,

- whose morphisms $m = (m_I) : (((M_I, F, W), (K_I, W), \alpha_I), u_{IJ}) \to (((M'_I, F, W), (K'_I, W), \alpha'_I), v_{IJ})$ is a family of morphism such that $v_{IJ} \circ m_I = p_{IJ*} m_J \circ u_{IJ}$ in $\mathcal{D}_p(\tilde{S}_I)$

We have then the identity functor

$$I_S : D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et}) \to (D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{\mathbb{Z}_p fil,c,k}(\tilde{S}_I^{et})),$$
$$(((M_I, F, W), u_{IJ}), (K, W), \alpha) \mapsto (((M_I, F, W), i_{I*} j_I^* (K, W), j_I^* \alpha), (u_{IJ}, I, 0)),$$
$$m = (m_I, n) \mapsto m = (m_I, i_* j_I^* n)$$

which is a full embedding since by definition, for $((M_I, F, W), u_{IJ}) \in D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))$,

$$u_{IJ} : (M_I, F, W) \to p_{IJ*}(M_J, F, W)$$

are filtered Zariski local equivalence, i.e. isomorphisms in $D_{\mathcal{D}(1,0)fil}(\tilde{S}_I)$, and hence for $((M_I, F, W), u_{IJ}), (K, W), \alpha) \in D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$,

$$(u_{IJ}, I, 0) : ((M_I, F, W), i_{I*} j_I^* (K, W), j_I^* \alpha) \to$$
$$p_{IJ*}((M_J, F, W), i_{J*} j_J^* (K, W), j_J^* \alpha) = (p_{IJ*}(M_J, F, W), i_{I*} j_I^* (K, W), j_I^* \alpha)$$

are isomorphisms in $\mathcal{D}_p(\tilde{S}_I)$.

Definition 119. *For $h : U \to S$ a smooth morphism with $S, U \in \mathrm{SmVar}(k)$ and $h : U \xrightarrow{n} X \xrightarrow{f} S$ a compactification of h with n an open embedding, $X \in \mathrm{SmVar}(k)$ such that $D := X\backslash U = \cup_{i=1}^s D_i \subset X$ is a normal crossing divisor, we denote by, using definition 76 and definition 105*

$$\begin{aligned}
&I_p(U/S) : h_{!Hdg} h^{!Hdg} \mathbb{Z}_{p,S}^{Hdg} \xrightarrow{:=} \\
&(p_{S*} E_{zar}(\Omega^\bullet_{X\times S/S} \otimes_{O_{X\times S}} (n\times I)_{!Hdg} \Gamma_U^{\vee,Hdg}(O_{U\times S}, F_b)), \mathbb{D}_S h_* E_{et} \mathbb{Z}_{p,U^{et}}, h_! \alpha(U,\delta)) \\
&\xrightarrow{((DR(X\times S/S)(\mathrm{ad}((n\times I)_{!Hdg},(n\times I)^*)(-)),0),I,0)} \\
&(\mathrm{Cone}((\Omega^{\Gamma,pr}_{/S}(i_{D_i}\times I))_{i\in[1,\ldots,s]} : p_{S*} E_{zar}(\Omega^\bullet_{X\times S/S} \otimes_{O_{X\times S}} \Gamma_X^{\vee,Hdg}(O_{X\times S}, F_b)) \to \\
&(\cdots \to (p_{S*} E_{zar}(\Omega^\bullet_{D_I\times S/S} \otimes_{O_{D_I\times S}} \Gamma_{D_I}^{\vee,Hdg}(O_{D_I\times S}, F_b))) \to \cdots)), \mathbb{D}_S h_* E_{et} \mathbb{Z}_{p,U^{et}}, h_! \alpha(U,\delta)) \\
&\xrightarrow{=:} (\mathcal{F}_S^{FDR}(\mathbb{Z}(U/S)), Rh_! \mathbb{Z}_{p,U^{et}}, \alpha(\mathbb{Z}(U/S)))
\end{aligned}$$

the canonical isomorphism in $D_{\mathcal{D}fil}(S) \times_I D_{\mathbb{Z}_p,c,k}(S^{et})$, where

- *we recall that (see section 6.2)*
$$h^{!Hdg} \mathbb{Z}_{p,S}^{Hdg} = (\Gamma_U^{\vee,Hdg}(O_{U\times S}, F_b), \mathbb{Z}_{p,U^{et}}, \alpha(U)) \in HM_{gm,k,\mathbb{C}_p}(U),$$
- *$i_{D_i} : D_i \hookrightarrow X$ are the closed embeddings,*
- *$\alpha(\mathbb{Z}(U/S)) := h_! \alpha(U,\delta) := T^w(h,\otimes)(-) \circ h_! \alpha(U,\delta)$ (see definition 86), with*
$$\begin{aligned}
&\alpha(U,\delta) := (DR(U)(\Omega_{(U\times U/U)/(U/pt)}(\Gamma_U^{\vee,Hdg}(O_{U\times U}, F_b))) \otimes I)^{-1} \circ \alpha(U) : \\
&\mathbb{B}_{dr,U} \to DR(U)((p_{U*} E_{zar}(\Omega^\bullet_{U\times U/U} \otimes_{O_{U\times U}} \Gamma_U^{\vee,Hdg}(O_{U\times U})))^{an} \otimes_{O_U} (O\mathbb{B}_{dr,U}, F)),
\end{aligned}$$

  by the way we note that the following diagram in $C(U^{an,pet}_{\mathbb{C}_p})$ commutes

$$\begin{tikzcd}
\mathbb{B}_{dr,U} \arrow[r, "\alpha(U)"] & F^0((\Omega^\bullet_{U^{an}_{\mathbb{C}_p}}, F_b) \otimes_{O_U} (O\mathbb{B}_{dr,U}, F)) =: F^0 DR(U)(O\mathbb{B}_{dr,U}, F) \\
 & E_{et} F^0 DR(U)((p_{U*} E_{zar}(\Omega^\bullet_{U\times U/U} \otimes_{O_{U\times U}} \Gamma_U^{\vee,Hdg}(O_{U\times U}, F_b)))^{an} \otimes_{O_U} (O\mathbb{B}_{dr,U}, F)) \arrow[u, "DR(U)(\Omega_{(U\times U/U)/(U/pt)}(\Gamma_U^{\vee,Hdg}(O_{U\times U}, F_b))\otimes I)"'] \\
p_{U*} E_{et} \Gamma_U^{\vee} \mathbb{B}_{dr,U\times U} \arrow[uu, "\mathrm{ad}(\delta_U^{*mod}, \delta_{U*})(-)"] \arrow[r, "\alpha(U\times U)"] & p_{U*} E_{et} F^0 DR(U\times U)((\Omega^\bullet_{U\times U/U} \otimes_{O_{U\times U}} \Gamma_U^{\vee,Hdg}(O_{U\times U}, F_b))^{an} \otimes_{O_{U\times U}} (O\mathbb{B}_{dr,U\times U}, F)) \arrow[u, "T^w(p_U,\otimes)(-)^{-1}"']
\end{tikzcd}$$

Lemma 10. *Let $S \in \mathrm{SmVar}(k)$. Let $g : U'/S \to U/S$ o morphism with $U/S := (U,h), U'/S := (U',h) \in \mathrm{Var}(k)^{sm}/S$. Let $h : U \xrightarrow{n} X \xrightarrow{f} S$ a compactification of h with n an open embedding, $X \in \mathrm{SmVar}(k)$ such that $D := X\backslash U = \cup_{i=1}^s D_i \subset X$ is a normal crossing divisor, Let $h' : U \xrightarrow{n'} X' \xrightarrow{f'} S$ a compactification of h' with n' an open embedding, $X' \in \mathrm{SmVar}(k)$ such that $D' := X\backslash U = \cup_{i=1}^s D_i \subset X$ is a normal crossing divisor and such that $g : U' \to U$ extend to $\bar{g} : X' \to X$, see definition-proposition 3. Then, using definition 119, the following diagram in $D_{\mathcal{D}fil}(S) \times_I D_{c,k}(S^{et})$ commutes*

$$\begin{tikzcd}
h'_{!Hdg} h'^{!Hdg} \mathbb{Z}_{p,S}^{Hdg} \arrow[r, "I_p(U'/S)"] \arrow[d, "\mathrm{ad}(g_{!Hdg}, g^{!Hdg})(h^{!Hdg}\mathbb{Z}_{p,S}^{Hdg})"'] & (\mathcal{F}_S^{FDR}(\mathbb{Z}(U'/S)), e(S^{et})_* C_*(\mathbb{Z}(U'/S) \otimes \mathbb{Z}_p), \alpha(\mathbb{Z}(U'/S))) \arrow[d, "(\Omega^{\Gamma,pr}_{/S}(R_S^{CH}(g)), Re(S^{et})_*\mathbb{Z}(g), \theta(g))"] \\
h_{!Hdg} h^{!Hdg} \mathbb{Z}_{p,S}^{Hdg} \arrow[r, "I_p(U/S)"] & (\mathcal{F}_S^{FDR}(\mathbb{Z}(U/S)), e(S^{et})_* C_*(\mathbb{Z}(U/S) \otimes \mathbb{Z}_p), \alpha(\mathbb{Z}(U/S)))
\end{tikzcd}$$

where

$$\theta(g) := R_{\mathcal{D}}([\Gamma_g]) : I(\mathbb{B}_{dr,S}(h'_!\mathbb{Z}_{p,U'^{et}}))[1] \to I(F^0DR(S)(\mathcal{F}_S^{FDR}(\mathbb{Z}(U/S))^{an} \otimes_{O_S} (O\mathbb{B}_{dr,S}, F)))$$

is the homotopy given by the third term of the syntomic homology class of the graph $\Gamma_g \subset U' \times_S U$, (see definition 95 and we recall (see section 6.2) that $I : C_{B_{dr},G}(S^{an}_{\mathbb{C}_p}/(\tilde{S}^{an}_{I,\mathbb{C}_p})) \to K_{B_{dr},G}(S^{an}_{\mathbb{C}_p}/(\tilde{S}^{an}_{I,\mathbb{C}_p}))$ is the injective resolution functor.

Proof. Immediate from definition. □

We can now define the p adic Hodge realization functor for motives :

Definition 120. *Let $k \subset \mathbb{C}_p$ a subfield. Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i=1}^s S_i$ an open cover such that there exists closed embedding $i_i : S \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. We define the Hodge realization functor as, using definition 108,*

$$\mathcal{F}_S^{Hdg} := (\mathcal{F}_S^{FDR}, e(S^{et})_*C_*L \otimes \mathbb{Z}_p) : C(\mathrm{Var}(k)^{sm}/S) \to D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_pfil,c,k}(S^{et}),$$
$$F \mapsto \mathcal{F}_S^{Hdg}(F) := (\mathcal{F}_S^{FDR}(F,W), e(S^{et})_*C_*(L(F,W) \otimes \mathbb{Z}_p), \alpha(F)),$$

first on objects and then on morphisms :

- *for $F \in C(\mathrm{Var}(k)^{sm}/S)$, taking $(F,W) \in C_{fil}(\mathrm{Var}(k)^{sm}/S)$ such that $D(\mathbb{A}^1, et)(F,W)$ gives the weight structure on $D(\mathbb{A}^1, et)(F)$,*

$$\mathcal{F}_S^{Hdg}(F) := (\mathcal{F}_S^{FDR}(F,W), e(S^{et})_*C_*(L(F,W) \otimes \mathbb{Z}_p), \alpha(F)) :=$$
$$(e(S)_*\mathcal{H}om((\hat{R}^{CH}_{\tilde{S}_I}(\rho^*_{\tilde{S}_I}Li_{I*}j_I^*(F,W)), \hat{R}^{CH}(T^q(D_{IJ})(-))), (E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}), T_{IJ})),$$
$$e(S^{et})_*C_*(L(F,W) \otimes \mathbb{Z}_p), \alpha(F)) \in D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_pfil,c,k}(S^{et})$$

where $\alpha(F)$ is the map in $D_{\mathbb{B}_{dr}fil}(S^{an,pet}_{\mathbb{C}_p}/(\tilde{S}^{an,pet}_{I,\mathbb{C}_p}))$ writing for short $DR(S) := DR(S)^{[-]} := (DR(\tilde{S}_I)[-d_{\tilde{S}_I}])$

$$\alpha(F) : \mathbb{B}_{dr,(\tilde{S}_I)}(Re(S^{et})_*((M,W) \otimes^L \mathbb{Z}_p)) := \mathbb{B}_{dr,(\tilde{S}_I)}((i_{I*}j_I^*e(S^{et})_*C_*(L(F,W) \otimes \mathbb{Z}_p)), I)$$
$$\xrightarrow{=} \mathbb{B}_{dr,(\tilde{S}_I)}((e(\tilde{S}^{et}_{I,\bar{k}})_*C_*(Li_{I*}j_I^*(F,W) \otimes \mathbb{Z}_p)), I)$$
$$\xrightarrow{=} ((((\cdot \to \oplus_{(U_{I\alpha},h_{I\alpha}) \in V_I}\mathbb{B}_{dr,\tilde{S}_I}(h_{I\alpha!}h^!_{I\alpha}\mathbb{Z}_{p,\tilde{S}^{et}_{I,\bar{k}}}) \xrightarrow{\mathbb{B}_{dr,\tilde{S}_I}(\mathrm{ad}(g^{\bullet !}_{I,\alpha,\beta}, g^{\bullet}_{I,\alpha,\beta !})(-))}$$
$$\oplus_{(U_{I\alpha},h_{I\alpha}) \in V_I}\mathbb{B}_{dr,\tilde{S}_I}(h_{I\alpha!}h^!_{I\alpha}\mathbb{Z}_{p,\tilde{S}^{et}_{I,\bar{k}}}) \to \cdot), u_{IJ})), W)$$
$$\xrightarrow{(\alpha(\mathbb{Z}(U_{I\alpha}/\tilde{S}_I)), \theta(g^{\bullet}_{I,\alpha,\beta}))}$$
$$F^0DR(S)((e(S)_*\mathcal{H}om((\hat{R}^{CH}_{\tilde{S}_I}(\rho^*_{\tilde{S}_I}Li_{I*}j_I^*(F,W)), \hat{R}^{CH}(T^q(D_{IJ})(-))), (E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}), T_{IJ})))^{an}$$
$$\otimes_{O_S}((O\mathbb{B}_{dr,\tilde{S}_I}, F), t_{IJ}))$$
$$\xrightarrow{=:} F^0DR(S)((\mathcal{F}_S^{FDR}(M,W))^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I}, F), t_{IJ}))$$

using lemma 10,

$$(\alpha(\mathbb{Z}(U_{I\alpha}/S)), \theta(g^{\bullet}_{I,\alpha,\beta}))$$

being the matrix given inductively by the composition law in $D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil,c,k}(\tilde{S}^{et}_I)$, that is we have the following isomorphism in $(D_{\mathcal{D}(1,0)fil}(\tilde{S}_I) \times_I D_{fil,c,k}(\tilde{S}^{et}_I))$, denoting for short $V_I :=$ $\mathrm{Var}(k)^{sm}/\tilde{S}_I$

$$(I^{\bullet}(U_{I\alpha}/\tilde{S}_I)) : (((\cdot \to \oplus_{(U_{I\alpha},h_{I\alpha}) \in V_I}h_{I\alpha!Hdg}h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{p,\tilde{S}_I} \xrightarrow{\mathrm{ad}(g^{\bullet,!Hdg}_{I,\alpha,\beta}, g^{\bullet}_{I,\alpha,\beta!Hdg})(-)}$$
$$\oplus_{(U_{I\alpha},h_{I\alpha}) \in V_I}h_{I\alpha!Hdg}h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{p,\tilde{S}_I} \to \cdot), u_{IJ}), W)$$
$$\xrightarrow{\sim} I_S(\mathcal{F}_S^{Hdg}(F) := (\mathcal{F}_S^{FDR}(F,W), e(\tilde{S}^{et}_I)_*C_*(Li_{I*}j_I^*(F,W) \otimes \mathbb{Z}_p), \alpha(F)))$$

where we denote by $g^n_{I,\alpha,\beta}:U_{I\alpha}\to U_{I\beta}$ *which satisfy* $h_{I\beta}\circ g^n_{I,\alpha,\beta}=h_{I\alpha}$ *the morphisms in the canonical projective resolution*

$$q:Li_{I*}j_I^*(F,W):=((\cdots\to\oplus_{(U_{I\alpha},h_{I\alpha})\in\mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I)\xrightarrow{(\mathbb{Z}(g^{\bullet}_{I,\alpha,\beta}))}\oplus_{(U_{I\alpha},h_{I\alpha})\in\mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I)\to\cdots),W)\to i_{I*}j_I^*(F,W),$$

- *for* $m:F_1\to F_2$ *a morphism in* $C(\mathrm{Var}(k)^{sm}/S)$, *taking* $(F_1,W),(F_2,W)\in C_{fil}(\mathrm{Var}(k)^{sm}/S)$ *such that* $D(\mathbb{A}^1,et)(F_2,W)$ *gives the weight structure on* $D(\mathbb{A}^1,et)(F_2)$ $D(\mathbb{A}^1,et)(F_1,W)$ *gives the weight structure on* $D(\mathbb{A}^1,et)(F_1)$ *and such that* $m:(F_1,W)\to(F_2,W)$ *is a filtered morphism, the morphism* $\mathcal{F}_S^{Hdg}(m)$ *in* $D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I))\times_I D_{fil,c,k}(S^{et})$ *is given by*

$$\begin{aligned}\mathcal{F}_S^{Hdg}(m): &= I_S^{-,-^{-1}}((I^{\bullet}(U_{I\alpha}/(\tilde{S}_I)))\circ(\mathrm{ad}(l^{\bullet!Hdg}_{I\alpha,\beta},l^{\bullet}_{I\alpha,\beta!Hdg})(\mathbb{Z}^{Hdg}_{p,U_{I\alpha}}))\circ(I^{\bullet}(U_{I\alpha}/(\tilde{S}_I)))^{-1})\\ &= (\mathcal{F}_S^{FDR}(m),Re(S^{et})_*\mathbb{Z}(m),\theta(m):=(\theta(l_{I\alpha,\beta}))):\mathcal{F}_S^{Hdg}(F_1)\to\mathcal{F}_S^{Hdg}(F_2)\end{aligned}$$

using lemma 9, that is we have the following commutative diagram in $(D_{\mathcal{D}(1,0)fil}(\tilde{S}_I)\times_I D_{fil,c,k}(\tilde{S}_I^{et}))$, *denoting for short* $V_I:=\mathrm{Var}(k)^{sm}/\tilde{S}_I$,

$$\begin{array}{ccc}(((\cdot\to\oplus_{(U_{I\alpha},h_{I\alpha})\in V_I}h_{I\alpha!Hdg}h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{p,\tilde{S}_I}\xrightarrow{A^{Hdg}_{g^{\bullet}_{1I,\alpha,\beta}}}\oplus_{(U_{I\alpha},h_{I\alpha})\in V_I}h_{I\alpha!Hdg}h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{p,\tilde{S}_I}\to\cdot),u_{IJ}),W) & \xrightarrow{(I^{\bullet}(U_{I\alpha}/\tilde{S}_I))} & \mathcal{F}_S^{Hdg}(F_1)\\ \downarrow{\scriptstyle \mathrm{ad}(l^{\bullet!Hdg}_{I,\alpha,\beta},l^{I,\bullet}_{\alpha,\beta!Hdg})(-)} & & \downarrow{\scriptstyle \mathcal{F}_S^{Hdg}(m)=(\mathcal{F}_S^{FDR}(m),Re(S^{et})_*\mathbb{Z}(m),(\theta(l_{I\alpha,\beta})))}\\ (((\cdot\to\oplus_{(U_{I\alpha},h_{I\alpha})\in V_I}h_{I\alpha!Hdg}h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{p,\tilde{S}_I}\xrightarrow{A^{Hdg}_{g^{\bullet}_{2I,\alpha,\beta}}}\oplus_{(U_{I\alpha},h_{I\alpha})\in V_I}h_{I\alpha!Hdg}h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{p,\tilde{S}_I}\to\cdot),u_{IJ}),W) & \xrightarrow{(I^{\bullet}(U_{\alpha}/\tilde{S}_I))} & \mathcal{F}_S^{Hdg}(F_2)\end{array}$$

where

- *we denoted for short* $A^{Hdg}_{g^{\bullet}_{1I,\alpha,\beta}}:=\mathrm{ad}(g^{\bullet,!Hdg}_{1I,\alpha,\beta},g^{\bullet}_{1I,\alpha,\beta!Hdg})(h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{\tilde{S}_I})$
- *we denoted for short* $A^{Hdg}_{g^{\bullet}_{2I,\alpha,\beta}}:=\mathrm{ad}(g^{\bullet,!Hdg}_{2I,\alpha,\beta},g^{\bullet}_{2I,\alpha,\beta!Hdg})(h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{\tilde{S}_I})$
- *we denote by* $g^n_{1I,\alpha,\beta}:U_{I\alpha}\to U_{I\beta}$, *which satisfy* $h_{I\beta}\circ g^n_{1I,\alpha,\beta}=h_{I\alpha}$, *the morphisms in the canonical projective resolution*

$$q:Li_{I*}j_I^*(F_1,W):=((\cdots\to\oplus_{(U_{I\alpha},h_{I\alpha})\in\mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I)\xrightarrow{(\mathbb{Z}(g^{\bullet}_{1I,\alpha,\beta}))}\oplus_{(U_{I\alpha},h_{I\alpha})\in\mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I)\to\cdots),W)\to i_{I*}j_I^*(F_1,W)$$

- *we denote by* $g^n_{2I,\alpha,\beta}:U_{I\alpha}\to U_{I\beta}$, *which satisfy* $h_{I\beta}\circ g^n_{2I,\alpha,\beta}=h_{\alpha}$, *the morphisms in the canonical projective resolution*

$$q:Li_{I*}j_I^*(F_2,W):=((\cdots\to\oplus_{(U_{I\alpha},h_{I\alpha})\in\mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I)\xrightarrow{(\mathbb{Z}(g^{\bullet}_{2I,\alpha,\beta}))}\oplus_{(U_{I\alpha},h_{I\alpha})\in\mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I)\to\cdots),W)\to i_{I*}j_I^*(F_2,W)$$

- *we denote by* $l^n_{I\alpha,\beta}:U_{I\alpha}\to U_{I\beta}$ *which satisfy* $h_{I\beta}\circ l^n_{I\alpha,\beta}=h_{I\alpha}$ *and* $l^{n+1}_{I\alpha,\beta}\circ g^n_{1I\alpha,\beta}=g^n_{2I\alpha,\beta}\circ l^n_{I\alpha,\beta}$ *the morphisms in the morphism of canonical projective resolutions*

$$Li_{I*}j_I^*(m):Li_{I*}j_I^*(F_1,W):=((\cdots\to\oplus_{(U_{I\alpha},h_{I\alpha})\in\mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I)\to\cdots),W)\xrightarrow{(\mathbb{Z}(l^{\bullet}_{I\alpha,\beta}))}((\cdots\to\oplus_{(U_{I\alpha},h_{I\alpha})\in\mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{\alpha}/\tilde{S}_I)\to\cdots),W)=:Li_{I*}j_I^*(F_2,W),$$

- *the maps* $I^{\bullet}(U_{I\alpha})$ *are given by definition 117 and lemma 9.*

Obviously $\mathcal{F}_S^{Hdg}(F[1]) = \mathcal{F}_S^{Hdg}(F)[1]$ *and* $\mathcal{F}_S^{Hdg}(\mathrm{Cone}(m)) = \mathrm{Cone}(\mathcal{F}_S^{Hdg}(m))$. *This functor induces by proposition 62 and remark 8 the functor*

$$\mathcal{F}_S^{Hdg} := (\mathcal{F}_S^{FDR}, e(S^{et})_* C_*(L \otimes \mathbb{Z}_p) : \mathrm{DA}(S) \to D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et}),$$
$$M = D(\mathbb{A}^1, et)(F) \mapsto \mathcal{F}_S^{Hdg}(M) := \mathcal{F}_S^{Hdg}(F) = (\mathcal{F}_S^{FDR}(M), Re(S^{et})_*(M \otimes^L \mathbb{Z}_p), \alpha(M)),$$

with $\alpha(M) = \alpha(F)$.

We now give the functoriality with respect to the five operation using the De Rahm realization case and the etale realization case :

Proposition 73. *Let* p *a prime number. Consider an embedding* $k \subset \mathbb{C}_p$.

(i) Let $g : T \to S$ *a morphism with* $T, S \in \mathrm{Var}(k)$. *Assume there exists a factorization* $g : T \xrightarrow{l} Y \times S \xrightarrow{p} S$, *with* $Y \in \mathrm{SmVar}(k)$, l *a closed embedding and* p *the projection. Let* $S = \cup_{i \in I} S_i$ *an open cover and* $i_i : S_i \hookrightarrow \tilde{S}_i$ *closed embeddings with* $\tilde{S}_i \in \mathrm{SmVar}(k)$. *Then,* $\tilde{g}_I : Y \times \tilde{S}_I \to \tilde{S}_I$ *is a lift of* $g_I = g_{|T_I} : T_I \to S_I$ *and we have closed embeddings* $i'_I := i_I \circ l \circ j'_I : T_I \hookrightarrow Y \times \tilde{S}_I$. *Then, for* $M = D(\mathbb{A}^1, et)(F) \in DA_c(S)$, *the following diagram commutes :*

$$\begin{array}{ccc}
\mathbb{B}_{dr,(Y\times\tilde{S}_I)}(g^{*w} Re(S^{et})_*(M \otimes^L \mathbb{Z}_p)) & \xrightarrow{g^*(\alpha(M))} & F^0 DR(T)^{[-]}((g_{Hdg}^{\hat{*}mod} \mathcal{F}_S^{FDR}(M))^{an} \otimes_{O_T} ((O\mathbb{B}_{dr,Y\times\tilde{S}_I}, F), t_{IJ})) \\
\downarrow{\scriptstyle \mathbb{B}_{dr,(Y\times\tilde{S}_I)}(T^*(g,e)(M\otimes\mathbb{Z}_p))} & & \downarrow{\scriptstyle DR(T)^{[-]}((T(g,\mathcal{F}^{FDR})(M))^{an}\otimes I)} \\
\mathbb{B}_{dr,(Y\times\tilde{S}_I)}(Re(T^{et})_* g^*(M \otimes^L \mathbb{Z}_p)) & \xrightarrow{\alpha(g^*M)} & F^0 DR(T)^{[-]}((\mathcal{F}_T^{FDR}(g^*M))^{an} \otimes_{O_T} ((O\mathbb{B}_{dr,Y\times\tilde{S}_I}, F), t_{IJ})),
\end{array}$$

see definition 112 and definition 86.

(ii) Let $f : T \to S$ *a morphism with* $T, S \in \mathrm{Var}(k)$. *Assume there exists a factorization* $f : T \xrightarrow{l} Y \times S \xrightarrow{p} S$, *with* $Y \in \mathrm{SmVar}(k)$, l *a closed embedding and* p *the projection. Let* $S = \cup_{i \in I} S_i$ *an open cover and* $i_i : S_i \hookrightarrow \tilde{S}_i$ *closed embeddings with* $\tilde{S}_i \in \mathrm{SmVar}(k)$. *Then, for* $M = D(\mathbb{A}^1, et)(F) \in DA_c(T)$,*the following diagram commutes :*

$$\begin{array}{ccc}
\mathbb{B}_{dr,(\tilde{S}_I)}(Rf_{*w} Re(T^{et})_*(M \otimes^L \mathbb{Z}_p)) & \xrightarrow{f_*(\alpha(M))} & F^0 DR(S)^{[-]}((Rf_*^{Hdg} \mathcal{F}_T^{FDR}(M))^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I}, F), t_{IJ})) \\
\uparrow{\scriptstyle \mathbb{B}_{dr,(\tilde{S}_I)}(T_*(f,e)(M\otimes\mathbb{Z}_p))} & & \uparrow{\scriptstyle DR(S)^{[-]}((T_*(f,\mathcal{F}^{FDR})(M))^{an}\otimes I)} \\
\mathbb{B}_{dr,(\tilde{S}_I)}(Re(S^{et})_* Rf_*(M \otimes^L \mathbb{Z}_p)) & \xrightarrow{\alpha(Rf_*M)} & F^0 DR(S)^{[-]}((\mathcal{F}_S^{FDR}(Rf_*M))^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I}, F), t_{IJ}))
\end{array}$$

see definition 113 and definition 86.

(iii) Let $f : T \to S$ *a morphism with* $T, S \in \mathrm{Var}(k)$. *Assume there exists a factorization* $f : T \xrightarrow{l} Y \times S \xrightarrow{p} S$, *with* $Y \in \mathrm{SmVar}(k)$, l *a closed embedding and* p *the projection. Let* $S = \cup_{i \in I} S_i$ *an open cover and* $i_i : S_i \hookrightarrow \tilde{S}_i$ *closed embeddings with* $\tilde{S}_i \in \mathrm{SmVar}(k)$. *Then, for* $M = D(\mathbb{A}^1, et)(F) \in DA_c(T)$,*the following diagram commutes :*

$$\begin{array}{ccc}
\mathbb{B}_{dr,(\tilde{S}_I)}(Rf_{!w} Re(T^{et})_*(M \otimes^L \mathbb{Z}_p)) & \xrightarrow{f_!(\alpha(M))} & F^0 DR(S)^{[-]}((Rf_!^{Hdg} \mathcal{F}_{FDR}^T(M))^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I}, F), t_{IJ})) \\
\downarrow{\scriptstyle \mathbb{B}_{dr,(\tilde{S}_I)}(T_!(f,e)(M\otimes\mathbb{Z}_p))} & & \downarrow{\scriptstyle DR(S)^{[-]}((T_!(f,\mathcal{F}_{FDR})(M))^{an}\otimes I)} \\
\mathbb{B}_{dr,(\tilde{S}_I)}(Re(S^{et})_* Rf_!(M \otimes^L \mathbb{Z}_p)) & \xrightarrow{\alpha(Rf_!M)} & F^0 DR(S)^{[-]}((\mathcal{F}_{FDR}^S(Rf_!M))^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I}, F), t_{IJ}))
\end{array}$$

see definition 113 and definition 86.

(iv) Let $f : T \to S$ a morphism with $T, S \in \mathrm{Var}(k)$. Assume there exists a factorization $f : T \xrightarrow{l} Y \times S \xrightarrow{p} S$, with $Y \in \mathrm{SmVar}(k)$, l a closed embedding and p the projection. Let $S = \cup_{i\in I} S_i$ an open cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then, for $M = D(\mathbb{A}^1, et)(F) \in DA_c(S)$,the following diagram commutes :

$$\begin{array}{ccc}
\mathbb{B}_{dr,(Y\times\tilde{S}_I)}(f^{!w}Re(S^{et})_*(M \otimes^L \mathbb{Z}_p)) & \xrightarrow{f^!(\alpha(M))} & F^0DR(T)^{[-]}((f^{*mod}_{Hdg}\mathcal{F}^{FDR}_S(M))^{an} \otimes_{O_T} ((O\mathbb{B}_{dr,Y\times\tilde{S}_I}, F), t_{IJ})) \\
\uparrow{\scriptstyle \mathbb{B}_{dr,(Y\times\tilde{S}_I)}(T^!(f,e)(M\otimes\mathbb{Z}_p))} & & \uparrow{\scriptstyle DR^{[-]}(T)((T^!(g,\mathcal{F}^{FDR})(M))^{an}\otimes I)} \\
\mathbb{B}_{dr,(Y\times\tilde{S}_I)}(Re(T^{et})_* f^!(M \otimes^L \mathbb{Z}_p)) & \xrightarrow{\alpha(f^!M)} & F^0DR(T)^{[-]}((\mathcal{F}^{FDR}_T(f^!M))^{an} \otimes_{O_T} ((O\mathbb{B}_{dr,Y\times\tilde{S}_I}, F), t_{IJ}))
\end{array}$$

see definition 113 and definition 86.

(v) Let $S \in \mathrm{Var}(k)$. Let $S = \cup_{i\in I} S_i$ an open cover and $i_i : S_i \hookrightarrow \tilde{S}_i$ closed embeddings with $\tilde{S}_i \in$ $\mathrm{SmVar}(k)$. Then, for $M, N \in DA_c(S)$,the following diagram commutes :

$$\begin{array}{ccc}
\mathbb{B}_{dr,(\tilde{S}_I)}(Re(S^{et})_*(M \otimes^L \mathbb{Z}_p)) \otimes_{\mathbb{B}_{dr,S}} \mathbb{B}_{dr,(\tilde{S}_I)}(Re(S^{et})_*(N \otimes^L \mathbb{Z}_p)) & \xrightarrow{\alpha(M)\otimes\alpha(N)} & F^0DR(S)((\mathcal{F}^{FDR}_S(M) \otimes^{Hdg}_{O_S} \mathcal{F}^{FDR}_S(N))^{an} \otimes_{O_S} (((O\mathbb{B}_{dr,\tilde{S}_I}, F), t_{IJ}) \\
\downarrow{\scriptstyle T(\otimes,\mathbb{B}_{dr})(Re(S^{et})_*M\otimes\mathbb{Z}_p, Re(S^{et})_*N\otimes\mathbb{Z}_p)} & & \downarrow{\scriptstyle DR(S)((T(\otimes,\mathcal{F}^{FDR})(M,N))^{an}\otimes I)} \\
\mathbb{B}_{dr,(\tilde{S}_I)}(Re(S^{et})_*((M \otimes N) \otimes^L \mathbb{Z}_p)) & \xrightarrow{(\alpha(M\otimes N))} & F^0DR(S)((\mathcal{F}^S_{FDR}(M \otimes N))^{an} \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I}, F), t_{IJ}))
\end{array}$$

see definition 113 and definition 86.

Proof. (i): Follows from the following commutative diagram in $(D_{\mathcal{D}(1,0)fil}(Y \times \tilde{S}_I) \times_I D_{fil,c,k}(Y \times \tilde{S}^{et}_I))$,

$$\begin{array}{ccc}
(((\to \oplus_{(U_{I\alpha},h_{I\alpha})\in V_I} \tilde{g}_I^{*Hdg} h_{I\alpha!Hdg} h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{p,\tilde{S}_I} \xrightarrow{A^{Hdg}_{g^\bullet_{I,\alpha,\beta}}} \oplus_{(U_{I\alpha},h_{I\alpha})\in V_I} h_{I\alpha!Hdg} h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{p,\tilde{S}_I} \to), u_{IJ}), W) & \xrightarrow{(\tilde{g}_I^{*Hdg} I^\bullet(U_{I\alpha}/\tilde{S}_I))} & (g^{\hat{*}mod}_{Hdg}\mathcal{F}^{FDR}_T(F), g^{*w}e(S^{et})_*C_*L(F,W), g^*(\alpha(F))) \\
\downarrow{\scriptstyle T^{Hdg}(\tilde{g}_I,h_I)(-)} & & \downarrow{\scriptstyle (T(g,\mathcal{F}^{FDR})(M), T(g,e)(M), 0)} \\
(((\to \oplus_{(U'_{I\alpha},h_{I\alpha})\in W_I} h'_{I\alpha!Hdg} h'^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{p,Y\times\tilde{S}_I} \xrightarrow{A^{Hdg}_{g'^\bullet_{I,\alpha,\beta}}} \oplus_{(U'_{I\alpha},h'_{I\alpha})\in W_I} h'_{I\alpha!Hdg} h'^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{p,Y\times\tilde{S}_I} \to), u_{IJ}), W) & \xrightarrow{(I^\bullet(U'_\alpha/Y\times\tilde{S}_I))} & (\mathcal{F}^{FDR}_T(g^*F), e(T^{et})_*C_*L(g^*F,W), \alpha(g^*F))
\end{array}$$

where, we have denoted for short $V_I := \mathrm{Var}(k)^{sm}/\tilde{S}_I$ and $W_I := \mathrm{Var}(k)^{sm}/Y \times \tilde{S}_I$,

- we denoted for short $A^{Hdg}_{g^\bullet_{I,\alpha,\beta}} := \mathrm{ad}(g^{\bullet,!Hdg}_{I,\alpha,\beta}, g^\bullet_{I,\alpha,\beta!Hdg})(h^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{p,\tilde{S}_I})$
- we denoted for short $A^{Hdg}_{g'^\bullet_{I,\alpha,\beta}} := \mathrm{ad}(g'^{\bullet,!Hdg}_{I,\alpha,\beta}, g'^\bullet_{I,\alpha,\beta!Hdg})(h'^{!Hdg}_{I\alpha}\mathbb{Z}^{Hdg}_{p,Y\times\tilde{S}_I})$
- we denote by $g^n_{I,\alpha,\beta} : U_{I\alpha} \to U_{I\beta}$, which satisfy $h_{I\beta} \circ g^n_{I,\alpha,\beta} = h_{I\alpha}$, the morphisms in the canonical projective resolution

$$q : Li_{I*}j_I^*(F,W) := (\cdots \to \oplus_{(U_{I\alpha},h_{I\alpha})\in \mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \xrightarrow{(\mathbb{Z}(g^\bullet_{I,\alpha,\beta}))} \oplus_{(U_{I\alpha},h_{I\alpha})\in \mathrm{Var}(k)^{sm}/\tilde{S}_I}\mathbb{Z}(U_{I\alpha}/\tilde{S}_I) \to \cdots) \to i_{I*}j_I^*(F,W)$$

- we denote by $g'^n_{I,\alpha,\beta} : U'_{I\alpha} \to U'_{I\beta}$, which satisfy $h'_{I\beta} \circ g'^n_{I,\alpha,\beta} = h'_\alpha$, the morphisms in the canonical projective resolution

$$q : Li'_{I*}j'^*_I(g^*F,W) := (\cdots \to \oplus_{(U'_{I\alpha},h'_{I\alpha})\in \mathrm{Var}(k)^{sm}/Y\times\tilde{S}_I}\mathbb{Z}(U'_{I\alpha}/Y\times\tilde{S}_I) \xrightarrow{(\mathbb{Z}(g'^\bullet_{I,\alpha,\beta}))} \oplus_{(U'_{I\alpha},h'_{I\alpha})\in \mathrm{Var}(k)^{sm}/Y\times\tilde{S}_I}\mathbb{Z}(U'_{I\alpha}/Y\times\tilde{S}_I) \to \cdots) \to i'_{I*}j'^*_I(g^*F,W)$$

(ii): Follows from (i) by adjonction.
(iii): The closed embedding case is given by (ii) and the smooth projection case follows from (i) by adjonction.
(iv): Follows from (iii) by adjonction.
(v):Obvious

□

Proposition 74. *Let p a prime number. Consider an embedding $k \subset \mathbb{C}_p$.*

(i) Let $S \in \mathrm{Var}(k)$. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Then we have the isomorphism in $D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{fil,c,k}(S^{et})$

$$\mathcal{F}_S^{Hdg}(\mathbb{Z}_S) \xrightarrow{:=} (\mathcal{F}_S^{FDR}(\mathbb{Z}_S), e(S^{et})_*(\mathbb{Z}_S \otimes \mathbb{Z}_p), \alpha(\mathbb{Z}_S)) \xrightarrow{((\Omega^{\Gamma,pr}_{/\tilde{S}_I}(\hat{R}^{CH}(\mathrm{ad}(i_I^*,i_{I*})(\Gamma^{\vee,w}_{S_I}\mathbb{Z}_{\tilde{S}_I})))),I,0)}$$
$$I_S^{-1}((e(S)_*\mathcal{H}om((\hat{R}^{CH}(\Gamma^{\vee,w}_{S_I}\mathbb{Z}_{\tilde{S}_I}), \hat{R}^{CH}(x_{IJ})), (E_{zar}(\Omega^{\bullet,\Gamma,pr}_{/\tilde{S}_I}, F_{DR}), T_{IJ})), T(S/(\tilde{S}_I))(\mathbb{Z}^w_{p,S^{et}}), \alpha(\tilde{S}_I, \delta)))$$
$$\xrightarrow{=} \iota_S((\Gamma^{\vee,Hdg}_{S_I}(O_{\tilde{S}_I}, F_b), x_{IJ}), \mathbb{Z}^w_{p,S^{et}}, \alpha(S)) =: \iota_S(\mathbb{Z}_S^{Hdg})$$

with

$$\alpha(S) : \mathbb{B}_{dr,(\tilde{S}_I)}(\mathbb{Z}_{S^{et}} \otimes \mathbb{Z}_p) := \mathbb{B}_{dr,(\tilde{S}_I)}(\Gamma^{\vee,w}_{S_I}\mathbb{Z}_{p,\tilde{S}_I^{et}}, x_{IJ})$$
$$\xrightarrow{(\Gamma^{\vee}_{S_I}\alpha(\tilde{S}_I))} F^0DR(S)((\Gamma^{\vee,Hdg}_{S_I}(O_{\tilde{S}_I}, F_b), x_{IJ}) \otimes_{O_S} ((O\mathbb{B}_{dr,\tilde{S}_I}, F), t_{IJ}))$$

(ii) Let $f : X \to S$ a morphism with $X, S \in \mathrm{Var}(k)$, X quasi-projective. Consider a factorization $f : X \xrightarrow{l} Y \times S \xrightarrow{p_S} S$ with $Y = \mathbb{P}^{N,o} \subset \mathbb{P}^N$ an open subset, l a closed embedding and p_S the projection. Let $S = \cup_i S_i$ an open cover such that there exist closed embeddings $i_i : S_i \hookrightarrow \tilde{S}_i$ with $\tilde{S}_i \in \mathrm{SmVar}(k)$. Recall that $S_I := \cap_{i\in I} S_i$, $X_I = f^{-1}(S_I)$, and $\tilde{S}_I := \Pi_{i\in I}\tilde{S}_i$. Then, using proposition 73(iii), the map of definition 113 gives an isomorphism in $D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$

$$(T_!(f, \mathcal{F}^{FDR})(\mathbb{Z}_X), T_!(e,f)(\mathbb{Z}_{p,X^{et}}))) :$$
$$\mathcal{F}_S^{Hdg}(M^{BM}(X/S)) := (\mathcal{F}_S^{FDR}(Rf_!\mathbb{Z}_X), e(S^{et})_*Rf_!(\mathbb{Z}_X \otimes \mathbb{Z}_p), \alpha(Rf_!\mathbb{Z}_X))$$
$$\xrightarrow{\sim} (Rf_{Hdg!}(\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I}, F_b), x_{IJ}(X/S)), Rf_{!w}\mathbb{Z}^w_{p,X^{et}}, f_!(\alpha(X))) =: \iota_S(Rf_{!Hdg}(\mathbb{Z}^{Hdg}_{p,X})).$$

with

$$\mathbb{Z}^{Hdg}_{p,X} := ((\Gamma^{\vee,Hdg}_{X_I}(O_{Y\times\tilde{S}_I}, F_b), x_{IJ}(X/S)), \mathbb{Z}^w_{p,X^{et}}, \alpha(X)) \in C(MHM_{gm,k,\mathbb{C}_p}(X))$$

Proof. Follows from proposition 73(iii) and theorem 53. □

The main theorem of this section is the following :

Theorem 57. *Let p a prime number. Let $k \subset \mathbb{C}_p$ a subfield.*

(i) For $S \in \mathrm{Var}(k)$, we have $\mathcal{F}_S^{Hdg}(\mathrm{DA}_c(S)) \subset D(MHM_{gm,k,\mathbb{C}_p}(S))$,

$$\iota_S : D(MHM_{gm,k,\mathbb{C}_p}(S)) \hookrightarrow D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$$

being a full embedding by theorem 49.

(ii) The Hodge realization functor $\mathcal{F}_{Hdg}(-)$ define a morphism of 2-functor on $\mathrm{Var}(k)$

$$\mathcal{F}_-^{Hdg} : \mathrm{Var}(k) \to (\mathrm{DA}_c(-) \to D(MHM_{gm,k,\mathbb{C}_p}(-)))$$

whose restriction to $\mathrm{QPVar}(\mathbb{C})$ is an homotopic 2-functor in sense of Ayoub. More precisely, *(ii0) for $g : T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(k)$, and $M \in \mathrm{DA}_c(S)$, the the map of definition 112 induces an isomorphism in $D(MHM_{gm,k,\mathbb{C}_p}(T))$*

$$\begin{aligned}T(g, \mathcal{F}^{Hdg})(M) := (T(g, \mathcal{F}^{FDR})(M), T(g, e)(M \otimes^L \mathbb{Z}_p), 0) :\\ g^{\hat{*}Hdg}\mathcal{F}_S^{Hdg}(M) := \iota_T^{-1}(g_{Hdg}^{\hat{*}mod}\mathcal{F}_S^{FDR}(M), g^* Re(S^{et})_*(M \otimes^L \mathbb{Z}_p), g^*(\alpha(M)))\\ \xrightarrow{\sim} \iota_T^{-1}(\mathcal{F}_T^{FDR}(g^*M), Re(T^{et})_* g^*(M \otimes^L \mathbb{Z}_p), \alpha(g^*M)) =: \mathcal{F}_T^{Hdg}(g^*M),\end{aligned}$$

(ii1) for $f : T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(k)$, and $M \in \mathrm{DA}_c(T)$, the map of definition 113 induces an isomorphism in $D(MHM_{gm,k,\mathbb{C}_p}(S))$

$$\begin{aligned}T_*(f, \mathcal{F}^{Hdg})(M) := (T_*(f, \mathcal{F}^{FDR})(M), I, 0) :\\ Rf_{Hdg*}\mathcal{F}_T^{Hdg}(M) := \iota_S^{-1}(Rf_*^{Hdg}\mathcal{F}_T^{FDR}(M), Rf_* Re(T^{et})_*(M \otimes^L \mathbb{Z}_p), f_*(\alpha(M)))\\ \xrightarrow{\sim} \iota_S^{-1}(\mathcal{F}_S^{FDR}(Rf_*M), Re(S^{et})_* Rf_*(M \otimes^L \mathbb{Z}_p), \alpha(Rf_*M)) =: \mathcal{F}_S^{Hdg}(Rf_*M),\end{aligned}$$

(ii2) for $f : T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(k)$, and $M \in \mathrm{DA}_c(T)$, the map of definition 113 induces an isomorphism in $D(MHM_{gm,k,\mathbb{C}_p}(S))$

$$\begin{aligned}T_!(f, \mathcal{F}^{Hdg})(M) := (T_!(f, \mathcal{F}^{FDR})(M), T_!(f, e)(M \otimes^L \mathbb{Z}_p), 0) :\\ Rf_{!Hdg}\mathcal{F}_T^{Hdg}(M) := \iota_S^{-1}(Rf_!^{Hdg}\mathcal{F}_T^{FDR}(M), Rf_! Re(T^{et})_*(M \otimes^L \mathbb{Z}_p), f_!(\alpha(M)))\\ \xrightarrow{\sim} \iota_S^{-1}(\mathcal{F}_S^{FDR}(Rf_!M), Re(S^{et})_* Rf_! M \otimes \mathbb{Z}_p, \alpha(f_!M)) =: \mathcal{F}_T^{Hdg}(f_!M),\end{aligned}$$

(ii3) for $f : T \to S$ a morphism, with $T, S \in \mathrm{QPVar}(k)$, and $M \in \mathrm{DA}_c(S)$, the map of definition 113 induces an isomorphism in $D(MHM_{gm,k,\mathbb{C}_p}(T))$

$$\begin{aligned}T^!(f, \mathcal{F}^{Hdg})(M) := (T^!(f, \mathcal{F}^{FDR})(M), T^!(f, e)(M \otimes^L \mathbb{Z}_p), 0) :\\ f^{*Hdg}\mathcal{F}_S^{Hdg}(M) := \iota_T^{-1}(f_{Hdg}^{*mod}\mathcal{F}_S^{FDR}(M), f^! Re(S^{et})_*(M \otimes^L \mathbb{Z}_p), f^!(\alpha(M)))\\ \xrightarrow{\sim} \iota_T^{-1}(\mathcal{F}_T^{FDR}(f^!M), Re(T^{et})_* f^!(M \otimes^L \mathbb{Z}_p), \alpha(f^!M)) =: \mathcal{F}_T^{Hdg}(f^!M),\end{aligned}$$

(ii4) for $S \in \mathrm{Var}(k)$, and $M, N \in \mathrm{DA}_c(S)$, the map of definition 113 induces an isomorphism in $D(MHM_{gm,k,\mathbb{C}_p}(S))$

$$\begin{aligned}T(\otimes, \mathcal{F}^{Hdg})(M, N) := (T(\otimes, \mathcal{F}_S^{Hdg})(M, N), I, 0) : \mathcal{F}_S^{Hdg}(M) \otimes_{O_S}^{Hdg} \mathcal{F}_S^{Hdg}(N) :=\\ \iota_S^{-1}(\mathcal{F}_S^{FDR}(M) \otimes_{O_S}^{Hdg} \mathcal{F}_S^{FDR}(N), Re(S^{et})_*(M \otimes^L \mathbb{Z}_p) \otimes Re(S^{et})_*(N \otimes^L \mathbb{Z}_p), \alpha(M) \otimes \alpha(N))\\ \xrightarrow{\sim} \iota_S^{-1}\mathcal{F}_S^{Hdg}(M \otimes N) := \iota_S^{-1}(\mathcal{F}_S^{FDR}(M \otimes N), Re(S^{et})_*((M \otimes N) \otimes^L \mathbb{Z}_p), \alpha(M \otimes N)).\end{aligned}$$

(iii) For $S \in \mathrm{Var}(k)$, the following diagram commutes :

$$\begin{tikzcd} \mathrm{Var}(k)/S \arrow[r, "MH(/S)"] \arrow[d, "M(/S)"'] & D(MHM_{gm,k,\mathbb{C}_p}(S)) \arrow[d, "\iota^S"] \\ \mathrm{DA}(S) \arrow[r, "\mathcal{F}_S^{Hdg}"] & D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et}) \end{tikzcd}$$

Proof. (i):Let $M \in \mathrm{DA}_c(S)$. There exist by definition of constructible motives an isomorphism in $\mathrm{DA}(S)$

$$w(M) : M \xrightarrow{\sim} \mathrm{Cone}(M(X_0/S)[d_0] \xrightarrow{m_1} \cdots \xrightarrow{m_m} M(X_m/S)[d_m]),$$

with $f_n : X_n \to S$ morphisms and $X_n \in \mathrm{QPVar}(k)$. This gives the isomorphism in $D_{\mathcal{D}(1,0)fil}(S/(\tilde{S}_I)) \times_I D_{\mathbb{Z}_p fil,c,k}(S^{et})$

$$\mathcal{F}_S^{Hdg}(w(M)) : \mathcal{F}_S^{Hdg}(M) \xrightarrow{\sim} \mathrm{Cone}(\mathcal{F}_S^{Hdg}(M(X_0/S))[d_0] \xrightarrow{\mathcal{F}_S^{Hdg}(m_1)} \cdots \xrightarrow{\mathcal{F}_S^{Hdg}(m_1)} \mathcal{F}_S^{Hdg}(M(X_m/S))[d_m]),$$

On the other hand, by proposition 74(i), we have

$$\mathcal{F}_S^{Hdg}(M(X_n/S)) \xrightarrow{\sim} Rf_{!Hdg}\mathbb{Z}_{p,X}^{Hdg} \in D(MHM_{gm,k,\mathbb{C}_p}(S)).$$

This prove (i).
(ii0): Follows from theorem 53(i) and proposition 73(i).
(ii1): Follows from theorem 53(iii) and proposition 73(ii).
(ii2):Follows from theorem 53(ii) and proposition 73(iii).
(ii3): Follows from theorem 53(iv) and proposition 73(iv).
(ii4):Follows from theorem 53(v) and proposition 73(v).
(iii): By (ii), for $g : X'/S \to X/S$ a morphism, with $X', X, S \in \mathrm{Var}(k)$ and $X/S = (X, f)$, $X'/S = (X', f')$, we have by adjonction the following commutative diagram

$$\begin{tikzcd}
\mathcal{F}_S^{Hdg}(M(X'/S) = f'_!f'^!\mathbb{Z}_S = f_!g_!g^!f^!\mathbb{Z}_S) \arrow[r, "\mathcal{F}_S^{Hdg}(M(/S)(g)=f_!\,\mathrm{ad}(g_!,g^!)(f^!\mathbb{Z}_S))"] \arrow[d, "T_!(f',\mathcal{F}^{Hdg})(f'^!M(X'/S))\circ T^!(f',\mathcal{F}^{Hdg})(M(X'/S))"'] & \mathcal{F}_S^{Hdg}(M(X/S) = f_!f^!\mathbb{Z}_S) \arrow[d, "T_!(f,\mathcal{F}^{Hdg})(f^!M(X/S))\circ T^!(f,\mathcal{F}^{Hdg})(M(X/S))"'] \\
MH(X'/S) := Rf'_{!Hdg}f'^{!Hdg}\mathbb{Z}_{p,S}^{Hdg} = f_{!Hdg}g_{!Hdg}g^{!Hdg}f^{!Hdg}\mathbb{Z}_{p,S}^{Hdg} \arrow[r, "f_{!Hdg}\,\mathrm{ad}(g_{!Hdg},g^{!Hdg})(f^{!Hdg}\mathbb{Z}_{p,S}^{Hdg})"] & MH(X/S) := f_{!Hdg}f^{!Hdg}\mathbb{Z}_{p,S}^{Hdg}
\end{tikzcd}.$$

where the left and right columns are isomorphisms by (ii). This proves (iii). ☐

The theorem 57 gives immediately the following :

Corollary 6. *Let p a prime number. Let $k \subset \mathbb{C}_p$ a subfield. Let $f : U \to S$, $f' : U' \to S$ morphisms, with $U, U', S \in \mathrm{Var}(k)$ irreducible, U' smooth. Let $\bar{S} \in \mathrm{PVar}(k)$ a compactification of S. Let $\bar{X}, \bar{X}' \in \mathrm{PVar}(k)$ compactification of U and U' respectively, such that f (resp. f') extend to a morphism $\bar{f} : \bar{X} \to \bar{S}$, resp. $\bar{f}' : \bar{X}' \to \bar{S}$. Denote $\bar{D} = \bar{X}\backslash U$ and $\bar{D}' = \bar{X}'\backslash U'$ and $\bar{E} = (\bar{D} \times_{\bar{S}} \bar{X}') \cup (\bar{X} \times_{\bar{S}} \bar{D}')$. Denote $i : \bar{D} \hookrightarrow \bar{X}$, $i' : \bar{D} \hookrightarrow \bar{X}$ denote the closed embeddings and $j : U \hookrightarrow \bar{X}$, $j' : U' \hookrightarrow \bar{X}'$ the open embeddings. Denote $d = \dim(U)$ and $d' = \dim(U')$. We have the following commutative diagram in $D(\mathbb{Z})$*

$$\begin{tikzcd}
RHom^\bullet_{\mathrm{DA}(\bar{S})}(M(U'/\bar{S}), M((\bar{X}, \bar{D})/\bar{S})) \arrow[r, "\mathcal{F}_S^{Hdg(-,-)}"] \arrow[d, "RI(-,-)"] & RHom^\bullet_{D(MHM_{gm,k,\mathbb{C}_p}(\bar{S}))}(f'_{!Hdg}\mathbb{Z}_{p,U'}^{Hdg}, f_{*Hdg}\mathbb{Z}_{p,U}^{Hdg}) \arrow[d, "RI(-,-)"] \\
RHom^\bullet(M(\mathrm{pt}), M(\bar{X}' \times_{\bar{S}} \bar{X}, \bar{E})(d')[2d']) \arrow[r, "\mathcal{F}_{Hdg}^{\mathrm{pt}\,(-,-)}"] \arrow[d, "l"] & RHom^\bullet(\mathbb{Z}_{\mathrm{pt}}^{Hdg}, a_{U'\times_S U!}\mathbb{Z}_{p,U\times_S U'}^{Hdg}(d')[2d']) \arrow[d, "l"] \\
\mathcal{Z}_d(\bar{X}' \times_{\bar{S}} \bar{X}, E, \bullet) \arrow[r, "\mathcal{R}^d_{\bar{X}'\times_{\bar{S}}\bar{X}}"] & C^{syn}_{2d+\bullet}(\bar{X}' \times_{\bar{S}} \bar{X}, E, Z(d))
\end{tikzcd}$$

where

$$M((\bar{X}, \bar{D})/\bar{S}) := \mathrm{Cone}(\mathrm{ad}(i_*, i^!) : M(\bar{D}/\bar{S}) \to M(\bar{X}/\bar{S})) = \bar{f}_*j_*E_{et}(\mathbb{Z}(U/U)) \in \mathrm{DA}(\bar{S})$$

and l the isomorphisms given by canonical embedding of complexes.

Proof. The upper square of this diagram follows from theorem 57(ii). On the other side, the lower square follows from the absolute case. ☐

References

[1] J.Ayoub, *Note sur les opérations de Grothendieck et la realisation de Betti*, Journal of the Institute of Mathematics of Jussieu, Volume 9, Issue 02, April 2010, pp.225-263.

[2] J.Ayoub, *L'algebre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle I*, Journal die reine (Crelles Journal), Volume 2014, Issue 693, pp.1-149.

[3] J.Ayoub, *L'algebre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle II*, Journal die reine (Crelles Journal), Volume 2014, Issue 693, pp.151-126.

[4] J.Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique I et II*, Société de Mathématiques de France, Astérisque, Volume 314 and 315, 2006.

[5] A.Beilinson, *How to glue perverse sheaves, in K-theory,arithmetic and geometry*, Lecture Notes in Mathematics, Vol.1289, Springer-Verlag, Berlin, 1987.

[6] A.Beilinson, *On the derived category of perverse sheaves, in K-theory, arithmetic and geometry*, Lecture Notes in Mathematics, Vol.1289, Springer-Verlag, Berlin, 1987, 27-41.

[7] Bhatwadekar, Rao, *On a question of Quillen* AMS 1983

[8] M.V.Bondarko,*Weight structure vs. t-structure ; weight filtrations, spectral sequences, and complexes (for motives and in general)*, Journal of K-theory, Volume 6, Issue 3,pp.387-504 , 2010

[9] J.Bouali, *On the realization functor of the derived category of mixed motives*, arxiv preprint 1706.04545, 2017-arxiv.org.

[10] J.Bouali, *The Hodge realization functor for relative motives of complex algebraic varieties*, arxiv preprint

[11] J.Cirici, F.Guillen, *Homotopy theory of mixed Hodge Complexes*, Preprint (2013) arxiv:1304.6236.

[12] D.C.Cisinski, F.Deglise, *Triangulated categories of mixed motived*, arxiv preprint 0912.2110, 2009-arxiv.org.

[13] A.Corti, M.Hanamura, *Motivic decomposition and intersection Chow groups I*, Duke math, J.103 (2000) 459-522.

[14] Coutinho, *A primer of algebraic D-modules*, London Mathematical Society, Cambrige university press, 1995

[15] H.Esnault, E.Viehweg, *Deligne-Beilinson cohomology* Perspect.Math.vol.4,Academic Press, 43-91, 1988

[16] F.Jin, *Borel-Moore Motivic homology and weight structure on mixed motives*, Math.Z.283 (2016), no.3, 1149-1183.

[17] H.Hironaka, *Resolution of singularities of an algebraic variety over a field of caracteristic zero*, Ann. of Math. 79(1964), 205-326

[18] R.Hotta, K.Takeuchi, T.Tanisaki, *D-Modules, Perverse Sheaves, and Representation Theory*, Birkhauser Verlag, 2008.

[19] R.Huber, *Etale cohomology of rigid analytic varieties and adic spaces*, Springer, 1996.

[20] M.Kashiwara, *Vanishing cycle sheaves and holonomic systems of differential equations*, Springer Lecture Note, 1016,(1983),134-142.

[21] S.Li, X.Pan, *Logaritmic De Rham comparaison theorem for open rigid spaces*, Forum of Mathematics, Sigma 7 E32, 2019.

[22] J.Milne, *Etale cohomology*, PMS, Volume 33, 1980.

[23] C.Peters, J.Steenbrink, *Mixed Hodge Structures*, Volume 52, Springer, 2008.

[24] M.Popa, *The V-filtration on D-modules*

[25] M.Saito, *Mixed Hodge Modules*, Proc. Japan Acad. Ser, A.Math.Sci.,Volume 62,Number 9,360-363, 1986.

[26] M.Saito, *Module de Hodge polarizable*, Publ.Res.Inst.Math.Sci 24(1988), no.6,849-995(1989).

[27] P.Scholze, *p-adic Hodge theory for rigid analytic varieties*, Forum of Mathematics, Pi,1,el,(2013).

LAGA UMR CNRS 7539
Université Paris 13, Sorbonne Paris Cité, 99 av Jean-Baptiste Clement,
93430 Villetaneuse, France,
bouali@math.univ-paris13.fr