POTENTIAL METHOD AND PROJECTION THEOREMS FOR MACROSCOPIC HAUSDORFF DIMENSION

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Abstract. The macroscopic Hausdorff dimension $\mathrm{Dim}_H(E)$ of a set $E \subset \mathbb{R}^d$ was introduced by Barlow and Taylor to quantify a "fractal at large scales" behavior of unbounded, possibly discrete, sets E. We develop a method based on potential theory in order to estimate this dimension in $\mathbb{R}^d$. Then, we apply this method to obtain Marstrand-like projection theorems: given a set $E \subset \mathbb{R}^2$, for almost every $\theta \in [0, 2\pi]$, the projection of E on the straight line passing through 0 with angle θ has dimension equal to $\min(\mathrm{Dim}_H(E), 1)$.

AMS MSC: 28AXX, 31C40

Keywords: Macroscopic Hausdorff dimension; Fine properties of sets; Potential theory for dimensions; Marstrand-Mattila projection theorem.

1. Introduction

Fractal geometry provide a general framework for studying sets possessing either irregular or self-reproducing (deterministic or random, self-similar or self-affine) properties. Most definitions of fractal dimensions of sets included in $\mathbb{R}^d$ are based on the local properties (also known as microscopic) of the set. Taking into consideration that many statistical physics models are built on discrete spaces, Barlow and Taylor [1,2] introduced a new notion of dimension to study unbounded "fractal-like" sets on discrete space. This so-called macroscopic Hausdorff dimension (see Definition 2.2 below) has proved to be useful in quantifying the behavior at infinity of several objects, beyond the transient range of random walks in $\mathbb{Z}^d$ which was the original motivation of Barlow and Taylor in [2].

Macroscopic Hausdorff dimension is actually defined for every set (not only discrete) in $\mathbb{R}^d$ [2]. It is a discrete analog of Hausdorff dimension, and the word macroscopic comes from the fact that this dimension ignores the local structure of the sets. At the same time, the macroscopic Hausdorff dimension assesses the asymptotic behavior at infinity of the sets, so it is very relevant when one is interested in the description of infinite objects, how they fill the space "at large scale". The macroscopic Hausdorff dimension was a key tool used by Xiao et Zheng [17] in studying the range of a random walk in random environment. It is related to [12] where Khoshnevisan and Xiao are concerned with the macroscopic geometry of other random sets. In [11], Khoshnevisan, Kim and Xiao found out a multifractal behavior for the macroscopic dimension of tall peaks of solutions to stochastic PDEs. Georgiou *et* *el* [8] solved Barlow and Taylor question [2, Problem, p. 145] by qualifying the range of an arbitrary transient random walk. The macroscopic Hausdorff dimension was also useful for studying the large scale structure of sojourn sets associated to the Brownian motion [16], the fractional Brownian motion [3, 15], and the Rosenblatt process [4].

In this paper we are interested in building various methods for estimating the macroscopic Hausdorff dimension. Recalling the fact that macroscopic Hausdorff dimension is a discrete analog of the Hausdorff dimension, we start by stating the estimating methods used for the Hausdorff dimension. In most cases, when estimating the Hausdorff dimension of a set E, the difficult part consists in finding a suitable lower bound for $\dim_H(E)$. Various methods exist to find lower bounds for the standard Hausdorff dimension, and it is a natural question to ask whether these methods have their counterparts for the macroscopic Hausdorff dimension. The two usual techniques are the mass distribution principle and the potential theoretic method.

The mass distribution principle, see for instance [7, page 67], states that if a set $E \subset \mathbb{R}^d$ and a Borel finite measure μ are such that $\mu(E) = 1$ and $\mu(B(x, r)) \le Cr^s$ for every $x \in \mathbb{R}^d$ and $r > 0$, then the s-dimensional Hausdorff measure $\mathcal{H}^s(E)$ is larger than $\mu(E)/C$, and so E has at least Hausdorff dimension s.

The potential theoretic method is based on an integral analysis: if for some probability measure μ, $\mu(E) = 1$ and the integral $\displaystyle\iint_{(\mathbb{R}^d)^2} \frac{d\mu(x)d\mu(y)}{\|x-y\|_2^s}$ is finite, then again E has at least Hausdorff dimension s. In addition to bounding the Hausdorff dimension from below, the potential theoretic method plays a key role in proving the projection theorem.

The first aim of this paper is to establish similar results for the macroscopic Hausdorff dimension. This happens to be very easy for the mass distribution principle, and follows essentially from previous works. It is much more challenging for the potential theoretic method, and a careful analysis is needed.

As an application of the new potential theoretic method, we obtain a Marstrand-like projection theorem, describing the dimension of almost all projections on lines of sets $E \in \mathbb{R}^2$. Dealing with the dimensions of projections of Borel sets is a line of research that has a long history. It started with the investigation by Marstrand [13] of the projection theorem associated to the Hausdorff dimension. He dealt with orthogonal projections on linear subspaces and proved that

$$\text{for every Borel set } E \subset \mathbb{R}^2, \quad \dim_H(\mathrm{proj}_V E) = \min\{\dim_H E, 1\}$$

for almost every 1-dimensional subspaces V, where proj_V denotes the orthogonal projection onto V and $\dim_H E$ denotes the Hausdorff dimension of E. Afterwards Marstrand's results was proved by Kaufman but using potential theoretic methods [10]. Subsequently in 1975 Mattila extended these results to Borel sets $E \subset \mathbb{R}^n$ and almost all V in the Grassmannian $G(n, m)$ [14]. We prove analog results for the macroscopic Hausdorff dimension, using the potential theory method we developed above.

2. Definitions and statements of the results

Here and in the reset of the paper, let $(\mathbb{R}^d, \|.\|_2)$ be the d-dimensional Euclidean space equipped with the L^2- norm.

2.1. The macroscopic Hausdorff dimension. For $x \in \mathbb{R}^d$ and $r > 0$, $B(x,r)$ denotes the Euclidean ball with center x and radius r. For $E \subset \mathbb{R}^d$, the diameter of a set E is denoted by $|E|$.

Let us recall the definition of the Barlow-Taylor macroscopic Hausdorff dimension $\mathrm{Dim}_H(E)$ of a set $E \subseteq \mathbb{R}^d$, developed in [1,2].

Define, for all integer $n \in \mathbb{N}$, the n-th shell of $\mathbb{R}^d$ by

$$S_0 = B(0,1) \quad \text{and} \quad S_n := B(0,2^n) \setminus B(0,2^{n-1}) \text{ for all } n \geq 1. \tag{2.1}$$

Like the standard Hausdorff dimension, the macroscopic Hausdorff dimension $\mathrm{Dim}_H(E)$ aims at describing how a set E can be efficiently covered by balls. Since Dim_H is concerned only with large scale behaviors, Barlow and Taylor proposed to study the covers of the intersections $E \cap S_n$ by balls, for every $n \in \mathbb{N}$, and the balls used to cover the sets $E \cap S_n$ will all be of diameter at least 1. Again this is justified by the fact that this dimension is supposed to describe discrete sets (so small balls are not relevant).

To this end, let us introduce, for $E \subseteq \mathbb{R}^d$, the set of *covers* of E restricted to S_n defined by

$$\widetilde{\mathcal{C}}_n(E) = \left\{ \{B(x_i,r_i)\}_{i=1}^m : \ m \in \mathbb{N}, \, x_i \in S_n, \, r_i \geq 1, \, E \cap S_n \subset \bigcup_{i=1}^m B(x_i,r_i) \right\}.$$

Finally, for $s \geq 0$ and $n \in \mathbb{N}$, set

$$\widetilde{\nu}_n^s(E) = \inf \left\{ \sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^s : \{B_i = B(x_i,r_i)\}_{i=1}^m \in \widetilde{\mathcal{C}}_n(E) \right\}. \tag{2.2}$$

Observe that $\widetilde{\nu}_n^s$ is sub-additive, i.e. $\widetilde{\nu}_n^s(A \cup B) \leq \widetilde{\nu}_n^s(A) + \widetilde{\nu}_n^s(B)$ for every sets A and B, but is not a measure (because of the constraints on r_i).

Definition 2.1. When $\widetilde{\nu}_n^s(E) = \sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^s$ and $E \cap S_n \subset \bigcup_{i=1}^m B(x_i,r_i)$, the finite family of balls $\{B_i = B(x_i,r_i)\}_{i=1}^m$ is called an s-optimal cover of $E \cap S_n$.

The existence of optimal covers is not guaranteed. We will deal with this issue in Section 3.

We are now ready to define the Barlow-Taylor macroscopic Hausdorff dimension.

Definition 2.2. For every $s \geq 0$ and $E \subset \mathbb{R}^d$, define

$$\widetilde{\nu}^s(E) = \sum_{n \geq 1} \widetilde{\nu}_n^s(E).$$

The macroscopic Hausdorff dimension of $E \subset \mathbb{R}^d$ is defined by

$$\mathrm{Dim}_H(E) = \inf\{s \geq 0 : \widetilde{\nu}^s(E) < +\infty\}. \qquad (2.3)$$

One easily checks that $\mathrm{Dim}_H(E) \in [0, d]$ for all $E \subset \mathbb{R}^d$, that $\mathrm{Dim}_H(E) = 0$ when E is bounded, and that an alternative definition for $\mathrm{Dim}_H(E)$ is

$$\mathrm{Dim}_H(E) = \sup\{s \geq 0 : \widetilde{\nu}^s(E) = +\infty\},$$

where $\sup \emptyset = 0$ by convention. It is also standard that $\mathrm{Dim}_H(f(E)) \leq \mathrm{Dim}_H(E)$ for every Lipschitz mapping $f : \mathbb{R}^d \to \mathbb{R}^d$.

A key ingredient when working with the standard Hausdorff dimension is the existence of s-sets, i.e. sets $E \subset \mathbb{R}^d$ with Hausdorff dimension $\dim_H(E) = s$ and such that its s-Hausdorff measure $\mathcal{H}^s(E)$ is finite. We introduce a similar notion for the macroscopic Hausdorff dimension.

Definition 2.3. Let $s \geq 0$. A set $E \subset \mathbb{R}^d$ is called a macroscopic s-set when $\mathrm{Dim}_H(E) = s$ and $\widetilde{\nu}^s(E) < +\infty$.

We prove the existence of macroscopic s-sets.

Theorem 2.4. *Let $E \subset \mathbb{R}^d$ be such that $\widetilde{\nu}^s(E) = +\infty$. Then there exists a macroscopic s-set $\widetilde{E}$ such that $\widetilde{E} \subset E$.*

This extraction theorem is a key ingredient at various places in our proofs.

2.2. Methods to find lower bounds for $\mathbf{Dim}_H(E)$. For every set B and every measure μ, $\mu_{|B}$ stands for the restriction of μ on B, i.e. $\mu_{|B}(A) = \mu(A \cap B)$.

As recalled above, the mass distribution principle is a powerful, albeit simple, tool allowing to find a lower bound of the Hausdorff dimension by considering measures supported on the set, see [7, page 67]. We prove a similar result for the macroscopic Hausdorff dimension Dim_H.

Proposition 2.5 (Macroscopic mass distribution principle)**.** *Let E be a Borel subset of $\mathbb{R}^d$ and $s > 0$. Suppose that there exists a Radon measure μ on $\mathbb{R}^d$ such that $\mu(E) = +\infty$ and a constant $c > 0$ such that for all $n \in \mathbb{N}$, $x \in S_n$ and $1 \leq r \leq 2^n$,*

$$\mu_{|S_n}(B(x,r)) \leq c \left(\frac{r}{2^n}\right)^s.$$

Then, for all $n \in \mathbb{N}$, $\widetilde{\nu}^s_n(E) \geq \dfrac{\mu_{|S_n}(E)}{c}$ and $Dim_H(E) \geq s$.

The proof of the macroscopic mass distribution principle is not complicated. Although it was not exactly stated before as we write it, it essentially follows directly from previous results, and so it is not so innovative.

This is not the case for the potential method below. Let us first introduce the macroscopic s-energy of a measure.

Definition 2.6. Let $s \geq 0$, and let μ be a finite mass distribution on $\mathbb{R}^d$. The macroscopic (μ, s)-potential at a point x is defined as

$$\phi_\mu^s(x) := \int_{\mathbb{R}^d} \frac{d\mu(y)}{\|x-y\|_2^s \vee 1}. \tag{2.4}$$

The macroscopic s-energy of μ is

$$I_s(\mu) := \int_{\mathbb{R}^d} \phi_\mu^s(x) d\mu(x) = \iint_{(\mathbb{R}^d)^2} \frac{d\mu(x)d\mu(y)}{\|x-y\|_2^s \vee 1}. \tag{2.5}$$

In the case of standard Hausdorff dimension, in the integrals (2.4) and (2.5), the quantity $\|x-y\|_2^s \vee 1$ is simply $\|x-y\|_2^s$. This modification is justified by the fact that Dim_H is not concerned with local behavior, so we are not interested in small interactions $\|x-y\|_2 < 1$.

Theorem 2.7. *Let E be a subset of $\mathbb{R}^d$.*

(1) If there exists a Radon measure μ on $\mathbb{R}^d$ such that $\mu(E) = +\infty$ and if

$$\sum_{n\geq 0} 2^{ns} I_s(\mu_{|S_n}) < +\infty,$$

then $\widetilde{\nu}^s(E) = +\infty$ and $\mathrm{Dim}_H(E) \geq s$.

(2) If $\widetilde{\nu}^s(E) = +\infty$, then for all $0 < \varepsilon < s$ there exists a Radon measure μ^ε on $\mathbb{R}^d$ such that $\mu^\varepsilon(E) = +\infty$ and $\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu^\varepsilon_{|S_n}) < +\infty$.

The potential theoretic methods we demonstrated in Theorem 2.7 are very comparable to the ones established for the standard Hausdorff dimension [6, Theorem 4.13]. Unlike the standard Hausdorff dimension case, for the macroscopic Hausdorff dimension, we consider the measure μ is define on $\mathbb{R}^d$, and we focus on the restriction of μ on every annulus S_n. For this reason, we deal with sums over n.

2.3. Application to projections. Projection theorems for Hausdorff dimensions have recently regained a lot of attention after some breakthroughs by M. Hochman and P. Shmerkin [9] and others, who used these theorems to tackle many long-standing questions in geometric measure theory and dynamical systems. It is quite satisfactory that they have natural counterparts in terms of macroscopic Hausdorff dimensions, as stated in the following theorem.

Theorem 2.8. *Let $E \subset \mathbb{R}^2$ be a Borel set. Define L_θ as the straight line passing through 0 with angle θ, and $\mathrm{proj}_\theta E$ as the orthogonal projection of E onto L_θ.*

(a) If $\mathrm{Dim}_H(E) < 1$, then $\mathrm{Dim}_H(\mathrm{proj}_\theta E) = \mathrm{Dim}_H(E)$ for Lebesgue almost every $\theta \in [0, \pi]$.

(b) If $\mathrm{Dim}_H(E) \geq 1$, then $\mathrm{Dim}_H(\mathrm{proj}_\theta E) = 1$ for Lebesgue almost every $\theta \in [0, \pi]$.

As in the standard Hausdorff dimension case, the proof is based on a subtle use of the potential method and Theorem 2.7.

It can be expected that Theorem 2.8 can be extended in higher dimensional spaces, and that both Theorem 2.7 and Theorem 2.8 are useful in other situations that the one we describe here.

The structure of the paper is as follows. The main three results, Theorems 2.4, 2.7 and 2.8 are established in Sections 4, 5, and 6 respectively. Some necessary technical properties of the macroscopic Hausdorff dimension are proved in Section 3.

3. First properties of Macroscopic Hausdorff Dimension

3.1. An alternative definition for the macroscopic Hausdorff dimension. We will use an alternative, easier to handle with, definition for the macroscopic Hausdorff dimension, based on a simple modification of the $\widetilde{\nu}_n^s$ quantities. We restrict ourselves to covers centered on integer points, with integer radii. We show that, up to a constants, this does not modify the values of the quantities involved in the computations, and the value of the macroscopic Hausdorff dimension is left unchanged.

We introduce for $E \subseteq \mathbb{R}^d$ and $n \geq 0$, the set of *proper covers* of E restricted to S_n by

$$\mathcal{C}_n(E) = \left\{ \{B(x_i, r_i)\}_{i=1}^m : \ m \in \mathbb{N}, \, x_i \in \mathbb{Z}^d \cap S_n, \, r_i \in \mathbb{N}^*, \, E \cap S_n \subset \bigcup_{i=1}^m B(x_i, r_i) \right\}.$$

Definition 3.1. For every $s \geq 0$, $n \geq 0$ and $E \subset \mathbb{R}^d$, define

$$\nu_n^s(E) = \inf \left\{ \sum_{i=1}^m \left(\frac{r_i}{2^n} \right)^s : \{B_i = B(x_i, r_i)\}_{i=1}^m \in \mathcal{C}_n(E) \right\} \tag{3.1}$$

and

$$\nu^s(E) = \sum_{n \geq 1} \nu_n^s(E). \tag{3.2}$$

Due to the fact that the x_i are (multi)-integers, as well as the r_i, the above infimum (3.1) in $\nu_n^s(E)$ is reached for some cover $\{B_i = B(x_i, r_i)\}_{i=1}^m \in \mathcal{C}_n(E)$.

Observe that ν_n^s is still sub-additive, i.e. $\nu_n^s(A \cup B) \leq \nu_n^s(A) + \nu_n^s(B)$ for every sets A and B.

Lemma 3.2. *For every $n \geq 0$, every set $E \subset \mathbb{R}^d$, one has*

$$\widetilde{\nu}_n^s(E) \leq \nu_n^s(E) \leq (2 + \sqrt{d})^s \widetilde{\nu}_n^s(E). \tag{3.3}$$

In particular, one still has

$$Dim_H(E) = \inf \{ s \geq 0 : \nu^s(E) < +\infty \} = \sup \{ s \geq 0 : \nu^s(E) = +\infty \}. \tag{3.4}$$

Proof. The fact that $\mathcal{C}_n(E) \subset \widetilde{\mathcal{C}}_n(E)$ implies directly that $\widetilde{\nu}_n^s(E) \leq \nu_n^s(E)$.

Now, let $\{B(\tilde{x}_i, \tilde{r}_i)\}_{i=1}^m \in \widetilde{\mathcal{C}}_n(E)$. Each ball $B(\tilde{x}_i, \tilde{r}_i)$ is included in a ball $B(x_i, \tilde{r}_i + \sqrt{d})$, where $x_i \in \mathbb{Z}^d \cap E_n$. So $\left\{B\left(x_i, \left\lceil \tilde{r}_i + \sqrt{d} \right\rceil\right)\right\}_{i=1}^m \in \mathcal{C}_n(E)$, and using that $\left\lceil \tilde{r}_i + \sqrt{d} \right\rceil \leq \tilde{r}_i + \sqrt{d} + 1 \leq (2 + \sqrt{d})\tilde{r}_i$ (since $\tilde{r}_i \geq 1$), one has

$$\sum_{i=1}^m \left(\frac{\tilde{r}_i + \sqrt{d}}{2^n}\right)^s \leq (2 + \sqrt{d})^s \sum_{i=1}^m \left(\frac{\tilde{r}_i}{2^n}\right)^s.$$

This holds for any cover $\{B(\tilde{x}_i, \tilde{r}_i)\}_{i=1}^m \in \widetilde{\mathcal{C}}_n(E)$, so $\nu_n^s(E) \leq (2 + \sqrt{d})^s \widetilde{\nu}_n^s(E)$. $\square$

Lemma 3.2 shows in particular that the convergence/divergence properties of $\widetilde{\nu}^s(E)$ and $\nu^s(E)$ are identical.

The main advantage of dealing with $\nu^s(E)$ is the existence of optimal proper s-covers, i.e. covers $\{B_i = B(x_i, r_i)\}_{i=1}^m \in \mathcal{C}_n(E)$ such that $\nu_n^s(E) = \sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^s$. These optimal covers exists because x_i and r_i are positive integers.

In our further analysis, the size of the balls of optimal covers will matter, justifying the following definition.

Definition 3.3. For $E \subset \mathbb{Z}^d$, $n \in \mathbb{N}$ and $0 < s < d$, define

$$\beta_n^s(E) := \max \left\{ \max_{1 \leq i \leq p} \frac{r_i}{2^n} \,:\, (B(x_i, r_i))_{i=1}^p \text{ is an } s\text{-optimal proper cover of } E \cap S_n \right\}.$$

The quantity $\beta_n^s(E)$ will be important, in particular for Theorem 2.7 about potential methods and for the projection Theorem 2.8.

3.2. Some preliminary results. We first prove two propositions that will be needed later.

Proposition 3.4. *Let μ_n be a Borel measure on S_n, $E \subset \mathbb{R}^d$ be a Borel set and $0 < c < +\infty$ be a constant.*

a) If $\max_{r \in \mathbb{N}^*} \dfrac{\mu_n(B(x,r))}{(r/2^n)^s} \leq c$ for all $x \in E \cap S_n$, then $\nu_n^s(E) \geq \dfrac{\mu_n(E)}{c2^s}$.

b) If $\max_{r \in \mathbb{N}^*} \dfrac{\mu_n(B(x,r))}{(r/2^n)^s} > c$ for all $x \in E \cap S_n$, then $\nu_n^s(E) \leq \dfrac{(5(1+\sqrt{d}/2))^s}{c} \mu_n(S_n)$.

Proof. a) Let $\{B(x_i, r_i)\}_{i=1}^m \in \mathcal{C}_n(E)$. For each $1 \leq i \leq m$, there exists $y_i \in B(x_i, r_i) \cap E \cap S_n$ such that $B(x_i, r_i) \subset B(y_i, 2r_i)$, so

$$\mu_n(B(x_i, r_i)) \leq \mu_n(B(y_i, 2r_i)) \leq c \left(\frac{2r_i}{2^n}\right)^s = c2^s \left(\frac{r_i}{2^n}\right)^s.$$

Then,

$$\mu_n(E \cap S_n) \leq \sum_{i=1}^m \mu_n(B(x_i, r_i)) \leq c2^s \sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^s,$$

which is true for all covers $\{B(x_i,r_i)\}_{i=1}^m \in \mathcal{C}_n(E)$. Finally, taking the infimum over all elements of $\mathcal{C}_n(E)$, one gets

$$\mu_n(E) = \mu_n(E \cap S_n) \leq c2^s \nu_n^s(E).$$

b) Consider the family of balls

$$\mathcal{B}_n = \left\{ B(x,r) : x \in E \cap S_n, \, r \in \{1,2,...,2^n\} \text{ and } \mu_n(B(x,r)) > c\left(\frac{r}{2^n}\right)^s \right\}.$$

Then

$$E \cap S_n \subset \bigcup_{B(x,r) \in \mathcal{B}_n} B(x,r).$$

Now, we invoke the following $5r$-covering Lemma [5, Lemma 4.8].

Lemma 3.5. *Let $\mathcal{B}$ be a family of balls in $\mathbb{R}^N$ and suppose that $\sup_{B\in\mathcal{B}} d(B) < \infty$. Then there exists a countable sub-family of disjoint balls $\mathcal{B}_0$ of $\mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{i\in\mathcal{B}_0} 5B_i.$$

Using the previous lemma, there exists a finite family $(B_i = B(x_i,r_i))_{i=1,\dots,m}$ of disjoint balls, all elements of $\mathcal{B}_n$, such that $\bigcup_{B\in\mathcal{B}_n} B \subset \bigcup_{i=1}^m 5B_i$. The finiteness of the family comes from the boundedness of S_n and the fact that the balls all have a diameter greater than 1. Up to a small translation of each x_i by a vector of length at most $\sqrt{d}/2$, one can assume that $x_i \in \mathbb{Z}^d$ and that

$$\bigcup_{B\in\mathcal{B}_n} B \subset \bigcup_{i=1}^m 5B\left(x_i, \left\lceil r_i + \sqrt{d}/2 \right\rceil\right).$$

With the translations that we added, some balls $B \in \mathcal{B}_n$ may intersect, but this does not affect our argument.

Using the definition of $\nu_n^s(E)$, one finally gets

$$\begin{aligned}
\nu_n^s(E) &\leq \sum_{i=1}^m \left(\frac{5\left\lceil r_i + \sqrt{d}/2 \right\rceil}{2^n} \right)^s \leq (5(2+\sqrt{d}/2))^s \sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^s \\
&\leq \frac{(5(2+\sqrt{d}/2))^s}{c} \sum_{i=1}^m \mu_n(B_i) \leq \frac{(5(2+\sqrt{d}/2))^s}{c} \mu_n(S_n),
\end{aligned}$$

where the last equality comes from the disjointness of the B_is. □

The following proposition guarantees that given a measure μ on a set E, there exists a smaller set $F \subset E$ such that the measure μ has a controlled local scaling behavior on F.

Proposition 3.6. *Let $E \subset \mathbb{R}^d$ be a Borel set. Then, for every $0 < s \leq d$ there exists a constant $c_s > 0$ (depending only on s) and a set $\emptyset \neq F \subset E$ such that for every $n \geq 1$,* *(a)* $\frac{4}{5}\nu_n^s(E) \leq \nu_n^s(F) \leq \nu_n^s(E)$

(b) $\nu_n^s\left(F \cap B\left(x,r\right)\right) \leq c_s\left(\frac{r}{2^n}\right)^s$ *for all* $x \in \mathbb{Z}^d \cap E_n$ *and* $r \geq 1$.

Proof. Let $E \subset \mathbb{R}^d$ and set for every $n \geq 1$

$$F_n := \left\{ x \in E \cap S_n : \max_{r \geq 1} \frac{\nu_n^s\left(E \cap B(x,r)\right)}{\left(r/2^n\right)^s} > 5(5(2+\sqrt{d}/2))^s \right\}.$$

Using Proposition 3.4 (b) applied to the set F_n and the measure $\mu_n(A) = \nu_n^s(E \cap A)$, one gets

$$\mu_n(F_n) \leq (5(2+\sqrt{d}/2))^s 5^{-1}(2+\sqrt{d}/2))^{-s}\mu_n(S_n) = \frac{1}{5}\mu_n(E).$$

Then $\mu_n(E\backslash F_n) \geq \frac{4}{5}\mu_n(E)$, i.e. as soon as $E \cap S_n$ is not empty, $(E\backslash F_n) \cap S_n \neq \emptyset$. Finally, the set $F = \bigcup_{n \geq 0} E\backslash F_n$ satisfies the two conditions mentioned above, with the constant $c_s = 5(5(2+\sqrt{d}/2))^s$. □

3.3. Proof of the mass distribution principle : Proposition 2.5. For $n \in \mathbb{N}$, let $\{B(x_i,r_i)\}_{i=1}^m \in \widetilde{\mathcal{C}}_n(E)$, then

$$\mu_{|S_n}(E \cap S_n) \leq \mu_{|S_n}\left(\bigcup_{i=1}^m B(x_i,r_i)\right) \leq \sum_{i=1}^m \mu_{|S_n}(B(x_i,r_i)) \leq c\sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^s.$$

Taking infimum over all proper covers $\{B(x_i,r_i)\}_{i=1}^m \in \widetilde{\mathcal{C}}_n(E)$, one gets

$$\frac{\mu_{|S_n}(E \cap S_n)}{c} \leq \nu_n^s(E).$$

Then $\widetilde{\nu}^s(E) \geq \dfrac{\sum_{n\geq 0}\mu_{|S_n}(E)}{c} = \dfrac{\mu(E)}{c} = +\infty$ and so $\mathrm{Dim}_H\left(E\right) \geq s$.

Observe that the same proof works if $\widetilde{\mathcal{C}}_n(E)$ and $\widetilde{\nu}_n^s(E)$ are replaced respectively by $\mathcal{C}_n(E)$ and $\nu_n^s(E)$.

4. Subsets of finite macroscopic measure

In this section, we prove a stronger version than Theorem 2.4, more precisely:

Theorem 4.1. *Let* $E \subset \mathbb{R}^d$ *such that* $\nu^s(E) = +\infty$. *Then there exists a macroscopic* s*-set* $\widetilde{E}$ *such that* $\widetilde{E} \subset E$ *and* $\lim_{n\to+\infty}\sup_{t\in[0,d]}\beta_n^t(\widetilde{E}) = 0$.

Observe that we can either work with $\widetilde{\nu}^s$ or ν^s, since $(\widetilde{\nu}^s(E) < +\infty) \Leftrightarrow (\nu^s(E) < +\infty)$. We choose to work with ν^s, and in this case $\beta_n^s(\widetilde{E})$ is defined without ambiguity.

We start with three technical lemmas, that will later help us extract a macroscopic s-set and prove the projection theorem.

Lemma 4.2. *Let $(a_n)_{n\geq 1}$ be a bounded sequence of positive real numbers, such that $\lim_{n\to+\infty} A_n := \sum_{k=1}^{+\infty} a_k = +\infty$. For every $\varepsilon > 0$, $\sum_{n=1}^{+\infty} \frac{a_n}{A_n^{1+\varepsilon}} < +\infty$ and $\sum_{n=1}^{+\infty} \frac{u_n}{A_n} = +\infty$.*

This is a standard exercise, we prove it for completness.

Proof. Let $\varepsilon > 0$. For $n \geq 2$ and $\varepsilon > 0$, one has $]\int_{A_{n-1}}^{A_n} \frac{dx}{x^{1+\varepsilon}} \geq \int_{A_{n-1}}^{A_n} \frac{dx}{A_n^{1+\varepsilon}} = \frac{u_n}{A_n^{1+\varepsilon}}$. Then, $\frac{1}{\varepsilon}\frac{1}{A_1^\varepsilon} \geq \frac{1}{\varepsilon}\left(\frac{1}{A_1^\varepsilon} - \frac{1}{A_n^\varepsilon}\right) = \int_{A_1}^{A_n} \frac{dx}{x^{1+\varepsilon}} \geq \sum_{k=2}^{n} \frac{a_k}{A_k^{1+\varepsilon}}$. So the sums $\sum_{k=1}^{n} \frac{a_n}{A_n^{1+\varepsilon}}$ are uniformly bounded and the series converges. Similarly, $\ln(A_n) - \ln(A_1) = \int_{A_n}^{A_1} \frac{dx}{x} \leq \sum_{k=2}^{n} \frac{a_k}{A_{k-1}}$. Since $A_n \to +\infty$ as $n \to +\infty$, the series $\sum_{k=2}^{n} \frac{a_k}{A_{k-1}}$ diverges. Also, since (a_n) is bounded, $A_n \sim A_{n-1}$ and the series $\sum_{k=2}^{n} \frac{a_k}{A_k}$ diverges. □

Lemma 4.3. *Let $(a_n)_{n\geq 1}$ be a positive sequence converging to zero, $(b_n)_{n\geq 1}$ be a bounded sequence of positive real numbers, such that $\sum_{n\geq 1} a_n b_n = +\infty$. Then, there exists a sequence $(c_n)_{n\geq 1}$ such that:*

(1) either $c_n = b_n$, or $c_n = 0$,
(2) $\sum_{n\geq 1} a_n c_n = +\infty$,
(3) $\sum_{n\geq 1} a_n^2 c_n < +\infty$.

Proof. We assume without loss of generality that $0 \leq a_n, b_n < 1$ for every n, and that $(a_n)_{n\in\mathbb{N}}$ is a non-increasing sequence.

For $j \geq 0$, let us call $D_j = \{n \geq 0 : 2^{-j-1} \leq a_n < 2^{-j}\}$, and $B_j = \sum_{n\in D_j} b_n$. We call $d_j = \max(D_j)$, which is finite since $a_n \to 0$. Observe that the integer sets D_j are arranged in increasing order: $d_j + 1 = \min(D_{j+1})$. Also, one has

$$\frac{1}{2}\sum_{j=0}^{+\infty} 2^{-j} B_j \leq \sum_{n\geq 0} a_n b_n = \sum_{j=0}^{+\infty} \sum_{n\in D_j} a_n b_n \leq \sum_{j=0}^{+\infty} 2^{-j} B_j,$$

so that $\sum_{j=0}^{+\infty} 2^{-j} B_j = +\infty$.

We put $n_1 = 0$, $j_1 = 1$, and $c_n = 0$ for every $n \in D_0 \cup D_1$.

Remark that $\sum_{n\geq d_1+1} a_n b_n \geq 1/2 \sum_{j\geq 2} 2^{-j} B_j = +\infty$.

Let us call n_2 the first integer n such that $\sum_{n=d_1+1}^{n_2} a_n b_n > 1/2$. Observing that for $n \geq d_1 + 1$, $a_n b_n \leq 2^{-1}$, one necessarily has $1/2 < \sum_{n=d_1+1}^{n_2} a_n b_n < 1$.

We call j_2 the unique integer such that $n_2 \in D_{j_2}$, and we put $c_n = b_n$ for every $n \in \{d_1+1, ..., n_2\}$, and $c_n = 0$ for every $n \in \{n_2+1, ..., d_{j_2}\}$. By construction,

$$1/2 < \sum_{j=j_1+1}^{j_2} \sum_{n \in D_j} a_n c_n < 1.$$

We iterate the construction. Assume that we have built two finite sequences of integers $(n_k)_{k=1,...,p}$ and $(j_k)_{k=1,...,p}$ such that:

(1) for $k = 1, ..., p-1$, $j_{k+1} > j_k$, and for $k = 1, ..., p$, $n_k \in D_{j_k}$
(2) for $k = 1, ..., p$, $c_n = b_n$ if $n \in \{d_{j_{k-1}}+1, ..., n_k\}$, and $c_n = 0$ if $n \in \{n_k + 1, ..., d_{j_k}\}$,
(3) for $k = 1, ..., p$, one has

$$1/(k+1) < \sum_{j=j_{k-1}+1}^{j_k} \sum_{n \in D_j} a_n c_n < 2/k. \tag{4.1}$$

Let us call n_{p+1} the first integer such that $\sum_{n=d_p+1}^{n_{p+1}} a_n b_n > 1/(p+2)$. Observing that for $n \geq d_p + 1$, $a_n b_n \leq 2^{-j_p} \leq 1/(p+1)$ (since $j_p \geq p$), one necessarily has $1/(p+2) < \sum_{n=d_p+1}^{n_{p+1}} a_n b_n < 1/(p+2) + 1/(p+1) \leq 2/(p+1)$.

We call j_{p+1} the unique integer such that $n_{p+1} \in D_{j_{p+1}}$, and we put $c_n = b_n$ for every $n \in \{d_p+1, ..., n_{p+1}\}$, and $c_n = 0$ for every $n \in \{n_{p+1}+1, ..., d_{j_{p+1}}\}$. Clearly, these n_{p+1} and j_{p+1} satisfy the recurrence properties.

Now, gathering the information, we deduce by (4.1) that

$$\sum_{n \geq 0} a_n c_n = \sum_{k=1}^{+\infty} \sum_{j=j_{k-1}+1}^{j_k} \sum_{n \in D_j} a_n c_n \geq \sum_{k=1}^{+\infty} 1/(k+1) = +\infty$$

and, using that $a_n \leq 2^{-j}$when $n \geq D_j$, and that $j_{k-1} \geq k-1$,

$$\begin{aligned} \sum_{n \geq 0} a_n^2 c_n &= \sum_{k=1}^{+\infty} \sum_{j=j_{k-1}+1}^{j_k} \sum_{n \in D_j} a_n^2 c_n \leq \sum_{k=1}^{+\infty} \sum_{j=j_{k-1}+1}^{j_k} 2^{-j} \sum_{n \in D_j} a_n c_n \\ &\leq \sum_{k=1}^{+\infty} 2^{-k+1}/(k+1) < +\infty. \end{aligned}$$

This concludes the proof. □

The same lines of computations can certainly be adapted to impose $\sum_{n \geq 0} a_n c_n = +\infty$ and $\sum_{n \geq 0} h(a_n) c_n < +\infty$ for any map $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(x) = o(x)$ when $x \to 0^+$.

As a first step toward Theorem 4.1, we reduce the problem to sets that can be covered by small sets only.

Proposition 4.4. *Let $E \subset \mathbb{R}^d$ such that $\nu^s(E) = +\infty$. Then, given $\alpha > 0$, there exists a set $\bar{E}$ such that $\nu^s(\bar{E}) = +\infty$ and $\lim_{n\to+\infty} \sup_{t\in[0,d]} \beta_n^t(\widetilde{E}) = 0$.*

Proof. It is an application of Lemma 4.2.

Call $A_n = \sum_{k=1}^n \nu_k^s(E)$ and $\alpha_n = A_n^{-1}$. By assumption, $\alpha_n \to 0$ when $n \to +\infty$.

For every $n \geq 1$, S_n can be covered by at most $2\alpha_n^{-1}$ balls of diameter $2^n \alpha_n^{1/d}$. Call $\mathcal{A}_n$ such a family of sets. One obviously has

$$\nu_n^s(E) \leq \sum_{A\in\mathcal{A}_n} \nu_n^s(E\cap A)$$

Thus there must exist $A_n \in \mathcal{A}_n$ such that $\nu_n^s(E\cap A_n) \geq \alpha_n \nu_n^s(E)$. Then one defines the set $\widetilde{E}$ as

$$\widetilde{E} = \bigcup_{n\geq 1} E \cap A_n.$$

By Lemma 4.2,

$$\sum_{n\geq 0} \nu_n^s(\widetilde{E}) \geq \sum_{n\geq 0} \nu_n^s(E\cap A_n) \geq \sum_{n\geq 0} \alpha_n \nu_n^s(E) = +\infty.$$

Now, it is clear that for every n, $|\widetilde{E} \cap S_n| \leq 2^n \alpha_n^{1/d}$, so by Definition 3.3, for every $t > 0$

$$\beta_n^t(\widetilde{E}) \leq \alpha_n^{1/d}.$$

Actually, this implies more: necessarily $\nu_n^s(\widetilde{E}) \leq \alpha_n^{s/d}$. In particular, $\beta_n^t(\widetilde{E}) \to 0$ as $n \to +\infty$ uniformly in t.

□

Finally, we prove Theorem 4.1.

Proof. Let E be such that $\nu^s(E) = +\infty$. By Proposition 4.4, one also assumes that $\lim_{n\to+\infty} \sup_{s\in[0,d]} \beta_n^s(\widetilde{E}) = 0$, and that item (3) holds for some $\alpha > 0$. This two facts will not be used in this proof, but will be key in the next section.

Observe that since for every n $\nu_n^s(E) \leq 1$, then $A_n := \sum_{k=0}^{n} \nu_k^s(E) \leq n$.

The idea consists in replacing E by a set $\widetilde{E}$ such that $\nu_n^s(\widetilde{E}) \sim b_n \nu_n^s(E)$, such that $\sum_{n\geq 1} \nu_n^s(\widetilde{E}) < +\infty$ but b_n is "as large as possible". Lemma 4.2 helps to build such a sequence.

First, for every $\varepsilon > 0$, denote by

$$B_n^\varepsilon = \sum_{k\geq n} \frac{\nu_k^s(E)}{A_k^{1+\varepsilon}}$$

By Lemma 4.2, one knows that $B_n^\varepsilon \to 0$ as $n \to \infty$, for every $\varepsilon > 0$.

We build iteratively a non-increasing sequence $(\varepsilon_n)_{n\geq 0} \subset \mathbb{R}^+$, and a sequence of integers $(n_k)_{k\geq 1}$.

Consider n_1 as the smallest positive integer such that $B_{n_1}^{\frac{1}{4}} \leq 1$ and set $\varepsilon_n = \frac{1}{2}$ for all $0 \leq n \leq n_1$.

Next we proceed by induction to build $(\varepsilon_n)_{n\geq 0}$ and $(n_k)_{k\geq 1}$.

Assume that $n_1 < n_2 < ... < n_p$ are defined.

Define n_{p+1} as the smallest integer such that

$$n_p < n_{p+1} \text{ and } B_{n_{p+1}}^{\frac{1}{2^p}} \leq \frac{1}{2^p}. \tag{4.2}$$

Put $\varepsilon_n = \dfrac{1}{2^{p+1}}$ for all $n_p < n \leq n_{p+1}$. Finally, let

$$b_n = \min\left\{1/2, (A_n)^{-(1+\varepsilon_n)}\right\}. \tag{4.3}$$

Then by construction of ε_n, one has:

(i) $\varepsilon_n \to 0$ as $n \to +\infty$,

(ii) By (4.2), and the fact that $B_{n_k}^{\frac{1}{2^{k+1}}} \leq B_{n_k}^{\frac{1}{2^k}} \leq 2^{-k-1}$,

$$\sum_{n\geq 0} b_n \nu_n^s(E) \leq \sum_{n\geq 0} \frac{\nu_n^s(E)}{A_n^{1+\varepsilon_n}} \leq \sum_{n=0}^{n_1} \frac{\nu_n^s(E)}{A_n^{1+\frac{1}{2}}} + \sum_{k\geq 1} \sum_{n=n_k+1}^{n_{k+1}} \frac{\nu_n^s(E)}{A_n^{1+\frac{1}{2^{k+1}}}} \tag{4.4}$$

$$\leq \sum_{n=0}^{n_1} \frac{\nu_n^s(E)}{A_n^{\frac{3}{2}}} + \sum_{k\geq 1} B_{n_k}^{\frac{1}{2^{k+1}}} \leq \sum_{n=0}^{n_1} \frac{\nu_n^s(E)}{(A_n)^{\frac{3}{2}}} + \sum_{k\geq 1} \frac{1}{2^{k-1}} < +\infty. \tag{4.5}$$

Next, we construct a set $\widetilde{E} \subset E$ such that for all $n \in \mathbb{N}$, one has

$$|\nu_n^s(\widetilde{E}) - b_n \nu_n^s(E)| \leq 2^{-ns}.$$

To achieve this, observe that by Definition 2.1, S_n contains a finite number of lattice points, and denote by $M_{n,d}$ their cardinality. These points are denote by x_i for $i \in \{1, \dots, M_{n,d}\}$.

Consider the following function:

$$\begin{aligned} g_n : \{0, 1, \dots, M_{n,d}\} &\longrightarrow \mathbb{R}^+ \\ m &\longmapsto \nu_n^s\left(\bigcup_{i=1}^m E \cap B(x_i, 1)\right). \end{aligned}$$

where $g_n(0) = 0$ by convention. It is clear that g_n is non-decreasing, and ranges from 0 to $\nu_n^s(E)$. Moreover, for all $m \in \{1, \dots, M_{n,d} - 1\}$, if $\{B(y_j, r_j)\}_{j=1}^p$ is an s-optimal cover of $\bigcup_{i=1}^m E \cap B(x_i, 1)$, then $\left\{(B(y_j, r_j))_{j=1}^p, B(x_{m+1}, 1)\right\}$ is a proper cover of $\bigcup_{i=1}^{m+1} E \cap B(x_i, 1)$ (not necessarily optimal). Using these two covers, one gets

$$g_n(m+1) - g_n(m) \leq \left(\sum_{j=1}^{p} \left(\frac{r_j}{2^n} \right)^s + \frac{1}{2^{ns}} \right) - \sum_{j=1}^{p} \left(\frac{r_j}{2^n} \right)^s \leq 2^{-ns}.$$

Hence, g_n has only small increments.

Recalling (4.3), $0 = g_n(0) \leq b_n \nu_n^s(E) \leq \nu_n^s(E) = g_n(M_{n,d})$, so there must exist an integer $m_n \in \{1, \dots, M_{n,d}\}$ such that

$$b_n \nu_n^s(E) \leq g_n(m_n) \leq b_n \nu_n^s(E) + 2^{-ns}.$$

Put

$$\widetilde{E}_n = \bigcup_{i=1}^{m_n} E \cap B(x_i, 1) \text{ and } \widetilde{E} = \bigcup_{n \geq 0} \widetilde{E}_n. \tag{4.6}$$

Then by construction, $\widetilde{E} \subset E$, and for all $n \in \mathbb{N}$one has

$$b_n \nu_n^s(E) \leq \nu_n^s(\widetilde{E}) \leq b_n \nu_n^s(E) + 2^{-ns}.$$

And so, by (4.5),

$$\nu^s(\widetilde{E}) = \sum_{n \geq 0} \nu_n^s(\widetilde{E}) \leq \sum_{n \geq 0} \left(b_n \nu_n^s(E) + 2^{-ns} \right) < +\infty.$$

To complete the proof, it is enough to show that for all $\varepsilon > 0$, $\nu^{s-\varepsilon}(\widetilde{E}) = +\infty$. To this end, fix $\varepsilon > 0$, and let $(B(x_i, r_i))_{i=1}^m$ be an optimal $(s-\varepsilon)$-cover of $\widetilde{E} \cap S_n$, and assume that for this specific cover, $\beta_n^{s-\varepsilon}(\widetilde{E})$ is reached, i.e. there exists $i \in \{1, ..., m\}$ such that $r_i = 2^n \beta_n^{s-\varepsilon}(\widetilde{E})$. In particular, $\nu_n^{s-\varepsilon}(\widetilde{E}) \geq (\beta_n^{s-\varepsilon}(\widetilde{E}))^{s-\varepsilon}$.

One sees that

$$\nu_n^{s-\varepsilon}(\widetilde{E}) = \sum_{i=1}^{m} \left(\frac{r_i}{2^n} \right)^{s-\varepsilon} \geq \sum_{i=1}^{m} \left(\frac{r_i}{2^n} \right)^{s} \cdot (\beta_n^{s-\varepsilon}(\widetilde{E}))^{-\varepsilon} \geq (\beta_n^{s-\varepsilon}(\widetilde{E}))^{-\varepsilon} \cdot \nu_n^s(\widetilde{E}). \tag{4.7}$$

Two cases are separated.

On the one hand, If $\beta_n^{s-\varepsilon}(\widetilde{E}) \leq \sqrt[s]{\dfrac{\nu_n^s(E)}{A_n}}$, then (4.7) yields

$$\begin{aligned} \nu_n^{s-\varepsilon}(\widetilde{E}) &\geq \left(\frac{A_n}{\nu_n^s(E)} \right)^{\varepsilon/s} \cdot \nu_n^s(\widetilde{E}) \geq \left(\frac{A_n}{\nu_n^s(E)} \right)^{\varepsilon/s} \cdot b_n \cdot \nu_n^s(E) \\ &\geq \frac{(\nu_n^s(E))^{1-\varepsilon/s}}{A_n^{1+\varepsilon_n-\varepsilon/s}} \geq \frac{\nu_n^s(E)}{A_n^{1+\varepsilon_n-\varepsilon/s}}. \end{aligned} \tag{4.8}$$

where the fact that $\nu_n^s(E) \leq 1$ has been used in the last step.

On the other hand, if $\beta_n^{s-\varepsilon}(\widetilde{E}) \geq \sqrt[s]{\dfrac{\nu_n^s(E)}{A_n}}$, one has

$$\nu_n^{s-\varepsilon}(\widetilde{E}) \geq (\beta_n^{s-\varepsilon}(\widetilde{E}))^{s-\varepsilon} \geq \frac{(\nu_n^s(E))^{1-\varepsilon/s}}{A_n^{1-\varepsilon/s}} \geq \frac{\nu_n^s(E)}{A_n^{1-\varepsilon/s}} \tag{4.9}$$

Finally, using the fact that $\varepsilon_n \to 0$ together with the lower bounds (4.8) and (4.9), one gets that for every large n, $\nu_n^{s-\varepsilon}(\widetilde{E}) \geq \dfrac{\nu_n^s(E)}{A_n}$. By Lemma 4.2, $\displaystyle\sum_{n\geq 0} \frac{\nu_n^s(E)}{A_n} = +\infty$, hence $\nu^{s-\varepsilon}(\widetilde{E}) = \sum_{n\geq 0} \nu_n^{s-\varepsilon}(\widetilde{E}) = +\infty$.

This holds for every $\varepsilon > 0$, so $\mathrm{Dim}_H\left(\widetilde{E}\right) = s$. □

5. Potential Methods

5.1. First part of Theorem 2.7. Consider $E \subset \mathbb{R}^d$, and assume that there exists a Radon measure μ on $\mathbb{R}^d$ such that $\mu(E) = +\infty$ and $\displaystyle\sum_{n\geq 0} 2^{ns} I_s(\mu_{|S_n}) < +\infty$. We prove that $\nu^s(E) = +\infty$, which implies that $\widetilde{\nu}^s(E) = +\infty$ and $\mathrm{Dim}_H(E) \geq s$.

For $n \in \mathbb{N}$, we write $\mu_n = \mu_{|S_n}$, and define

$$\phi_{\mu_n}^s := \int_{\mathbb{R}^d} \frac{d\mu_n(y)}{\|x-y\|_2^s \vee 1} \qquad \text{and } E_n = \left\{ x \in E \cap S_n : \max_{r\geq 1} \frac{\mu_n(B(x,r))}{\left(\frac{r}{2^n}\right)^s} \leq 1 \right\}$$

For every $x \in E_n^c$, there exists an integer r_x such that $\dfrac{\mu_n(B(x,r_x))}{\left(\frac{r_x}{2^n}\right)^s} \geq 1$. One has

$$\phi_{\mu_n}^s(x) = \int_{\mathbb{R}^d} \frac{d\mu_n(y)}{\|x-y\|_2^s \vee 1} \geq \int_{B(x,r_x)} \frac{d\mu_n(y)}{\|x-y\|_2^s \vee 1} \geq \frac{\mu_n(B(x,r_x))}{r_x^s} \geq \frac{1}{2^{ns}}.$$

Then $I_s(\mu_n) \geq \displaystyle\int_{E_n^c} \phi_{\mu_n}^s(x) d\mu_n(x) \geq \frac{1}{2^{ns}} \mu_n(E_n^c)$, which implies that

$$\sum_{n\geq 0} \mu_n(E_n^c) \leq \sum_{n\geq 0} 2^{ns} I_s(\mu_n) < +\infty.$$

But as $E \cap S_n = E_n \cup E_n^c$ and $\displaystyle\sum_{n\geq 0} \mu_n(E\cap S_n) = +\infty$, then $\displaystyle\sum_{n\geq 0} \mu_n(E_n) = +\infty$. Moreover, by Proposition 3.4 a), one has $\nu_n^s(E_n) \geq \frac{\mu_n(E_n)}{2^s}$. Finally, $\nu^s(E) = \sum_{n\geq 0} \nu_n^s(E_n) = +\infty$ which gives that $\mathrm{Dim}_H(E) \geq s$.

5.2. Second part of Theorem 2.7. This is the most delicate part. Assume now that $\widetilde{\nu}^s(E) = +\infty$, and fix $0 < \varepsilon < s$.

Our goal is to build a Radon measure μ^ε on $\mathbb{R}^d$ such that $\mu^\varepsilon(E) = +\infty$ and $\displaystyle\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu^\varepsilon_{|S_n}) < +\infty$. We are going to build each measure $\mu_n^\varepsilon = \mu^\varepsilon_{|S_n}$.

For this, we use the results we previously proved.

By Theorem 4.1 there exists a set $E_1 \subset E$ such that $\lim_{n\to+\infty} \sup_{t\in[0,d]} \beta_n^t(E_1) = 0$ and $\nu^s(E_1) = +\infty$.

Then by Theorem 2.4, there exists a macroscopic s-set $E_2 \subset E_1$ such that $\mathrm{Dim}_H\,(E_2) = s$ and $\nu^s(E_2) < +\infty$.

Consider an optimal $(s-\frac{\varepsilon}{2})$-cover $\{B(x_i,r_i)\}_{i=1}^m$ of $E_2 \cap S_n$. One sees that

$$\begin{aligned}\left(\beta_n^{s-\varepsilon/2}(E_2)\right)^{\frac{\varepsilon}{4}} \nu_n^{s-\frac{\varepsilon}{2}}(E_2) &= \left(\beta_n^{s-\varepsilon/2}(E_2)\right)^{\frac{\varepsilon}{4}} \sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^{s-\frac{\varepsilon}{2}} \\ &= \left(\beta_n^{s-\varepsilon/2}(E_2)\right)^{\frac{\varepsilon}{4}} \sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^{s-\frac{\varepsilon}{4}} \left(\frac{r_i}{2^n}\right)^{-\frac{\varepsilon}{4}} \\ &\geq \sum_{i=1}^m \left(\frac{r_i}{2^n}\right)^{s-\frac{\varepsilon}{4}} \geq \nu_n^{s-\frac{\varepsilon}{4}}(E_2),\end{aligned}$$

where we used that $\beta_n^{s-\varepsilon/2}(E_2) \geq \dfrac{r_i}{2^n}$. Recalling that $\mathrm{Dim}_H\,(E_2) = s$, it follows that $\sum_{n\geq} \left(\beta_n^{s-\varepsilon/2}(E_2)\right)^{\frac{\varepsilon}{4}} \nu_n^{s-\frac{\varepsilon}{2}}(E_2) = +\infty$. Moreover as $E_2 \subset E_1$, then $\beta_n^{s-\varepsilon/2}(E_2) \to 0$ as $n \to +\infty$.

Setting $a_n = \left(\beta_n^{s-\varepsilon/2}(E_2)\right)^{\frac{\varepsilon}{4}}$ and $b_n = \nu^n_{s-\frac{\varepsilon}{2}}(E_2)$, one then sees that the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ satisfies the assumptions of Lemma 4.3. Consider the sequence $(c_n)_{n\geq 1}$ given by this Lemma, and define the set $E_3 \subset E_2$ as follows: for every $n \geq 1$,

- if $c_n = 0$, then $E_3 \cap S_n = \emptyset$,
- if $c_n = b_n$, then $E_3 \cap S_n = E_2 \cap S_n$.

It is immediate from the construction and Lemma 4.3 that $c_n = \nu_n^{s-\varepsilon/2}(E_3)$ and

$$\begin{aligned}&\sum_{n\geq} \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} \nu_n^{s-\frac{\varepsilon}{2}}(E_3) = +\infty \\ \text{and } &\sum_{n\geq} \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{2}} \nu_n^{s-\frac{\varepsilon}{2}}(E_3) < +\infty \qquad (5.1)\end{aligned}$$

Finally, by Proposition 3.6, there exists $\emptyset \neq E_4 \subset E_3 \subset E$ such that for all $n \in \mathbb{N}$,

$$\frac{4}{5}\nu_n^{s-\frac{\varepsilon}{2}}(E_3) \leq \nu_n^{s-\frac{\varepsilon}{2}}(E_4) \leq \nu_n^{s-\frac{\varepsilon}{2}}(E_3) \qquad (5.2)$$

$$\text{and} \qquad \nu_n^{s-\frac{\varepsilon}{2}}(E_4 \cap B(x,r)) \leq c_s \left(\frac{r}{2^n}\right)^{s-\frac{\varepsilon}{2}} \qquad (5.3)$$

for all $x \in \mathbb{Z}^d$ and $r \geq 1$.

Define the measures $\mu_n^\varepsilon(A) := \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} \nu_n^{s-\frac{\varepsilon}{2}}(E_4 \cap A)$. Then by our construction and (5.2), one has

$$\sum_{n\geq 0} \mu_n^\varepsilon(E \cap S_n) = \sum_{n\geq 0} \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} \nu_n^{s-\frac{\varepsilon}{2}}(E_4)$$
$$\geq \frac{4}{5} \sum_{n\geq 0} \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} \nu_n^{s-\frac{\varepsilon}{2}}(E_3) = +\infty.$$

We are left to prove that

$$\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^\varepsilon) = \sum_{n\geq 0} 2^{n(s-\varepsilon)} \int_{\mathbb{R}^d} \phi_{\mu_n^\varepsilon}^{s-\varepsilon}(x) d\mu_n^\varepsilon(x) < +\infty$$

For $x \in S_n$, one can write

$$\phi_{s-\varepsilon}^{\mu_n^\varepsilon}(x) = \int_{S_n} \frac{d\mu_n^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon} \vee 1}$$

Every $y \in S_n$ belongs to the ball $B(x, 2^{n+1})$. For $1 \leq r \leq 2^{n+1}$, denote by $m_n^\varepsilon(r) = \mu_n^\varepsilon(B(x,r))$. By (5.3), one has

$$m_n^\varepsilon(r) = \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} \nu_n^{s-\frac{\varepsilon}{2}}\left(E_4 \cap B(x,r)\right) \leq c_s \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} \left(\frac{r}{2^n}\right)^{s-\frac{\varepsilon}{2}}. \qquad (5.4)$$

Using the fact that $B(x, 2^{n+1}) = \bigcup_{r=1}^{2^{n+1}} B(x,r) \setminus B(x, r-1)$, one has

$$\phi_{s-\varepsilon}^{\mu_n^\varepsilon}(x) \leq \sum_{r=1}^{2^{n+1}} \int_{B(x,r)\setminus B(x,r-1)} \frac{d\mu_n^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon} \vee 1}$$
$$= \mu_n^\varepsilon(B(x,1)) + \sum_{r=2}^{2^{n+1}} \int_{B(x,r)\setminus B(x,r-1)} \frac{d\mu_n^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon}}.$$

One the one hand, by (5.2), $\mu_n^\varepsilon(B(x,1)) \leq c_s \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\frac{\varepsilon}{2})}$. On the other hand,

$$\sum_{r=2}^{2^{n+1}} \int_{B(x,r)\setminus B(x,r-1)} \frac{d\mu_n^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon}}$$
$$= \sum_{r=2}^{2^{n+1}} \int_{r-1}^{r} t^{\varepsilon-s} dm_n^\varepsilon(t)$$
$$= \sum_{r=2}^{2^{n+1}} \left(\left[t^{\varepsilon-s} m_n^\varepsilon(t)\right]_{r-1}^{r} + (s-\varepsilon) \int_{r-1}^{r} t^{\varepsilon-s-1} m_n^\varepsilon(t) dt \right)$$

$$\leq c_s \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\frac{\varepsilon}{2})} \sum_{r=1}^{2^n} \left(\left[t^{\frac{\varepsilon}{2}}\right]_{r-1}^{r} + (s-\varepsilon)\int_{r-1}^{r} t^{\frac{\varepsilon}{2}-1}dt \right)$$

$$\leq c_s \left(1+2\frac{s-\varepsilon}{\varepsilon}\right) \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\frac{\varepsilon}{2})} \sum_{r=1}^{2^{n+1}} \left(r^{\frac{\varepsilon}{2}} - (r-1)^{\frac{\varepsilon}{2}}\right)$$

$$\leq C \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\varepsilon)}.$$

for some constant C. So

$$\phi_{s-\varepsilon}^{\mu_n^\varepsilon}(x) \leq c_s, \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\frac{\varepsilon}{2})} + C \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\varepsilon)} \leq \widetilde{C} \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\varepsilon)}.$$

Moving to the integral, one gets

$$I_{s-\varepsilon}(\mu_n^\varepsilon) = \int_{\mathbb{R}^d} \phi_{s-\varepsilon}^{\mu_n^\varepsilon}(x) d\mu_n^\varepsilon(x) \leq C \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} 2^{-n(s-\varepsilon)} \mu_n^\varepsilon(E_4).$$

Finally, recalling (5.1), (5.2), (5.3) and the definition of μ_n^ε, one has

$$\begin{aligned} \sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^\varepsilon) &\leq C \sum_{n\geq 0} \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{4}} \mu_n^\varepsilon(E_4) \\ &\leq C \sum_{n\geq 0} \left(\beta_n^{s-\frac{\varepsilon}{2}}(E_2)\right)^{\frac{\varepsilon}{2}} \nu_{s-\frac{\varepsilon}{2}}^n(E_4) < +\infty \end{aligned}$$

as desired.

6. Projection of a Set

In this section we are considering the orthogonal projection of sets in $\mathbb{R}^2$ and we aim at proving the projection Theorem 2.8 for the macroscopic Hausdorff dimension.

Let us introduce some notations.

For every $\theta \in [0, 2\pi]$, call $e_\theta = (\cos\theta, \sin\theta)$ the vector with angle θ, and L_θ the straight line in $\mathbb{R}^2$ with angle θ passing through the origin.

Then, recall that $proj_\theta : \mathbb{R}^2 \to L_\theta$ is the orthogonal projection onto L_θ.

6.1. Case where $\mathrm{Dim}_H(E) \geq 1$. Let us start by proving item b) of Theorem 2.8, assuming that item a) is proved.

Consider $E \subset \mathbb{R}^2$ with $\mathrm{Dim}_H(E) \geq 1$.

By Theorem 4.1, for every $p \geq 2$, there exists $E_p \subset E$ such that $\mathrm{Dim}_H(E_p) = 1 - 1/p$. For each set E_p, by item a), there exists a set $\Theta_p \subset [0, \pi]$ of full Lebesgue measure such that for every $\theta \in \Theta_p$, $\mathrm{Dim}_H(proj_\theta(E_p)) = 1 - 1/p$. In particular, this implies that $\mathrm{Dim}_H(proj_\theta(E)) \geq 1 - 1/p$.

Consider now the set $\Theta = \bigcap_{p\geq 2} \Theta_p$. The above arguments show that Θ is still of full Lebesgue measure in $[0, \pi]$, and that for every $\theta \in \Theta$, $\mathrm{Dim}_H(proj_\theta(E)) \geq 1$. Since obviously $\mathrm{Dim}_H(proj_\theta(E))$ is always less than 1 (since it is included in L_θ), the result follows.

6.2. First extractions when $\mathrm{Dim}_H(E) < 1$. Fix a set $E \subset \mathbb{R}^2$ with $0 < \mathrm{Dim}_H(E) = s < 1$. The rest of the section is devoted to prove that $\mathrm{Dim}_H(\mathrm{proj}_\theta E) = \mathrm{Dim}_H(E)$ for almost every $\theta \in [0,\pi]$.

Writing $L_\theta = \{\lambda e_\theta : \lambda \in \mathbb{R}\}$, we can define the n-th shells inside L_θ as $S_n^\theta = \{v = (x,y) \in L_\theta : \|v\|_2 \in [2^{n-1}, 2^n]\}$. Identifying L_θ with $\mathbb{R}$, the results we obtained before in dimension 1 apply to L_θ and S_n^θ.

We are going to project 2-dimensional measures onto the lines L_θ. For this, let us define for every $n \geq 0$ the cylinders

$$C_n^\theta := \mathrm{proj}_\theta^{-1} S_n^\theta. \tag{6.1}$$

We are going to prove that for every $0 < \varepsilon < s$, the set

$$\Theta_{s-\varepsilon} = \{\theta \in [0,\pi] : \mathrm{Dim}_H(proj_\theta(E)) \geq s - \varepsilon\} \tag{6.2}$$

has full Lebesgue measure. The conclusion then follows using the same argument as the one used to prove item b). More precisely, from the properties above, $\Theta := \bigcap_{p\geq 1} \Theta_{s-1/p}$ has full Lebesgue measure, and for every $\theta \in \Theta$, $\mathrm{Dim}_H(proj_\theta(E)) \geq s$. But since $proj_\theta$ is a Lipschitz mapping, $\mathrm{Dim}_H(proj_\theta(E)) \leq s = \mathrm{Dim}_H(E)$. Finally one gets $\mathrm{Dim}_H(\mathrm{proj}_\theta E) = \mathrm{Dim}_H(E)$ for almost all $\theta \in [0,\pi]$.

Fix $0 < \varepsilon < s$.

Applying Theorem 2.7(2), there exists a Borel measure μ^ε supported by E such that

$$\sum_{n\geq 0} \mu_n^\varepsilon(E \cap S_n) = +\infty, \tag{6.3}$$

$$\text{and} \quad \sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^\varepsilon) < +\infty, \tag{6.4}$$

where μ_n^ε is a simplified notation for $\mu^\varepsilon_{|S_n}$. Observe that in fact, via the finer Theorem 4.1 and Proposition 4.4, we can impose that $\lim_{n\to+\infty} \mu_n^\varepsilon(E \cap S_n) = 0$.

We need to impose an additional condition on μ^ε, namely that

$$\sum_{n\geq 0} 2^{-n} \mu_n^\varepsilon(E \cap S_n) \left(\sum_{k=0}^{n} 2^k \mu_k^\varepsilon(E \cap S_k) \right) < +\infty. \tag{6.5}$$

This is achieved thanks to the following lemma.

Lemma 6.1. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two positive sequences converging to zero, such that $\sum_{n\geq 1} a_n = +\infty$ and $\sum_{n\geq 1} a_n b_n = +\infty$. There exists a sequence $(c_n)_{n\geq 1}$ such that:*

(1) either $c_n = a_n$, or $c_n = 0$,
(2) $\sum_{n\geq 1} c_n = +\infty$,
(3) $\sum_{n\geq 1} c_n b_n < +\infty$.

Proof. Again, without loss of generality, we assume that $0 < a_n, b_n < 1$. Let us call $D_j = \{n \geq 0 : 2^{-j-1} \leq b_n < 2^{-j}\}$, for $j \geq 0$.

Put $c_n = 0$ for every $n \in D_0 \cup D_1$, and $n_0 = 0$, $j_0 = 1$.

We know that $\sum_{j\geq 2} \sum_{n\in D_j} a_n b_n = +\infty$. We go through each D_j in increasing order. Consider the first couple (n_1, j_1) such that $n_1 \in D_{j_1}$ and $\sum_{j=2}^{j_1-1} \sum_{n\in D_j} a_n b_n + \sum_{n\in D_{j_1}, n\leq n_1} a_n b_n \geq 1/2$. Put $c_n = a_n$ for all $n \in \bigcup_{j=2}^{j_1-1} D_j \cup \{n \in D_{j_1} : n \leq n_1\}$, and $c_n = 0$ for all $n \in \{n \in D_{j_1} : n > n_1\}$. By our choice,

$$1/2 \leq \sum_{j=0}^{j_1} \sum_{n\in D_j} c_n b_n = \sum_{j=2}^{j_1-1} \sum_{n\in D_j} a_n b_n + \sum_{n\in D_{j_1}, n\leq n_1} a_n b_n < 1.$$

We then iterate the process: assume that we have built two finite sequences of integers $(n_k)_{k=1,...,p}$ and $(j_k)_{k=1,...,p}$ such that

(1) for $k = 1, ..., p-1$, $j_{k+1} > j_k$, and for $k = 1, ..., p$, $n_k \in D_{j_k}$
(2) for $k = 1, ..., p$, $c_n = a_n$ if $n \in \bigcup_{j=j_{k-1}}^{j_k-1} D_j \cup \{n \in D_{j_k} : n \leq n_k\}$, and $c_n = 0$ for all $n \in \{n \in D_{j_k} : n > n_k\}$.
(3) for $k = 1, ..., p$, one has

$$2^{-k} \leq \sum_{j=j_{k-1}}^{j_k} \sum_{n\in D_j} c_n b_n < 2^{-k+1}. \tag{6.6}$$

We know that $\sum_{j\geq j_p+1} \sum_{n\in D_j} a_n b_n = +\infty$. Consider the first couple (n_{p+1}, j_{p+1}) such that $n_{p+1} \in D_{j_{p+1}}$ and $\sum_{j=j_p}^{j_{p+1}-1} \sum_{n\in D_j} a_n b_n + \sum_{n\in D_{j_{p+1}}, n\leq n_{p+1}} a_n b_n \geq 2^{-(p+1)}$. Put $c_n = a_n$ for all $n \in \bigcup_{j=j_p}^{j_{p+1}-1} D_j \cup \{n \in D_{j_{p+1}} : n \leq n_{p+1}\}$, and $c_n = 0$ for all $n \in \{n \in D_{j_{p+1}} : n > n_{p+1}\}$. Then, since for all the selected integers n, $a_n b_n \leq 2^{-j_{p+1}} \leq 2^{-(p+1)}$, (6.6) holds true.

Collecting the information, on one hand one has by (6.6)

$$\sum_{n\geq 0} c_n b_n = \sum_{k\geq 1} \sum_{j=j_{k-1}}^{j_k} \sum_{n\in D_j} c_n b_n \leq \sum_{k\geq 1} 2^{-k+1} < +\infty.$$

On the other hand, since $j_k \geq k+1$, one sees that for each $n \in D_j$ for $j \in \{j_{k-1}, ...j_k\}$, $b_n \leq 2^{-k}$, so again by (6.6),

$$\sum_{n\geq 0} c_n = \sum_{k\geq 1} \sum_{j=j_{k-1}}^{j_k} \sum_{n\in D_j} c_n \geq \sum_{k\geq 1} 2^k \sum_{j=j_{k-1}}^{j_k} \sum_{n\in D_j} c_n b_n \geq \sum_{k\geq 1} 1 = +\infty,$$

hence the result. $\square$

Setting $a_n = \mu_n^\varepsilon(E)$, then $(a_n)_{n\geq 0}$ tends to zero when n tends to infinity. Define then

$$b_n = 2^{-n} \sum_{k=0}^{n} 2^k a_k.$$

Since $\sum_{k=0}^{n} 2^k \sim 2^n$, $(b_n)_{n\geq 0}$ is a generalized Caesaro mean associated with the sequence $(a_n)_{n\geq 0}$, and converges to zero when n tends to infinity.

So either $\sum_{n\geq 1} a_n b_n < +\infty$, and (6.5) is true, or $\sum_{n\geq 1} a_n b_n = +\infty$ and we are exactly in the situation of Lemma 6.1: there exists a sequence $(c_n)_{n\geq 1}$ such that:

(1) either $c_n = a_n$, or $c_n = 0$,
(2) $\sum_{n\geq 1} c_n = +\infty$,
(3) $\sum_{n\geq 1} c_n b_n < +\infty$.

Setting $\widetilde{E} = \bigcup_{n\geq 0: a_n = c_n} E \cap S_n$, by construction one has $\mu^\varepsilon(\widetilde{E}) = \sum_{n\geq 1} c_n = +\infty$, and since $\mu_k^\varepsilon(\widetilde{E} \cap S_k) = c_k \leq a_k = \mu_k^\varepsilon(E \cap S_k)$, one has

$$\sum_{n\geq 0} 2^{-n} \mu_n^\varepsilon(\widetilde{E} \cap S_n) \left(\sum_{k=0}^{n} 2^k \mu_k^\varepsilon(\widetilde{E} \cap S_k) \right) \leq \sum_{n\geq 1} c_n b_n < +\infty,$$

hence (6.5) is obtained for $\widetilde{E}$. This property will be used at the very end of the proof of Proposition 6.4 only. It is obvious that if Theorem 2.8 is proved for this smaller set E, it is also true for the original set.

Finally, observe that, replacing $\widetilde{E}$ by $\bigcup_{n\geq 0} \widetilde{E} \cap S_{2n}$ or $\bigcup_{n\geq 0} \widetilde{E} \cap S_{2n+1}$, one can assume in addition to (6.3), (6.4) and (6.5) that

$$\text{if } S_n \neq \emptyset, \quad \text{then } S_{n-1} = S_{n+1} = \emptyset. \tag{6.7}$$

To resume this section, we have proved that the original set E contains a subset, still denoted by E for simplification, and a measure μ^ε supported by E such that (6.3), (6.4), (6.5) and (6.7) simultaneously hold.

6.3. Final proof of item a) of Theorem 2.8. Consider the set E obtained after extraction above. For all $\theta \in [0, \pi]$, $k \geq n$ and $A \subset L_\theta$, we focus on the restriction of μ_k^ε on $C_n(\theta)$

$$(\mu_k^\varepsilon)_{|C_n^\theta}(A) := \mu_k^\varepsilon(\{x \in E \cap S_k \,:\, \mathrm{proj}_\theta x \in A \cap S_n^\theta\}),$$

Equivalently for each non-negative function f, one has

$$\int_{-\infty}^{+\infty} f(t) d(\mu_k^\varepsilon)_{|C_n^\theta}(t) \;=\; \int_{C_n^\theta \cap S_k} f(x.e_\theta) d\mu_k^\varepsilon(x).$$

where $x.e_\theta$ denotes the scalar product. Since e_θ is unitary, we identify $x.e_\theta$ with $\mathrm{proj}_\theta x$, the orthogonal projection of x onto L_θ.

Definition 6.2. The projected measure $\mu^{\varepsilon,\theta}$ is defined as $\mu^{\varepsilon,\theta} = \sum_{n\geq 1} \mu_n^{\varepsilon,\theta}$, where

$$\mu_n^{\varepsilon,\theta} = \sum_{k\geq n} (\mu_k^{\varepsilon})_{|C_n^\theta}. \tag{6.8}$$

Note that each $\mu_n^{\varepsilon,\theta}$ is a measure supported on $\mathrm{proj}_\theta E \cap S_n^\theta$.

We are going to prove that for almost all $\theta \in [0,\pi]$,

$$\sum_{n\geq 0} \mu_n^{\varepsilon,\theta}(\mathrm{proj}_\theta E) = +\infty \text{ and } \sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^{\varepsilon,\theta}) < \infty. \tag{6.9}$$

for almost all $\theta \in [0,\pi]$. Then item a) of Theorem 2.7 will allow us to conclude that the set $\Theta_{s-\varepsilon}$ defined by (6.2) has full Lebesgue measure, as announced.

This is the purpose of the next two propositions.

Proposition 6.3. *For every $\theta \in [0,\pi]$,*

$$\mu^{\varepsilon,\theta}(proj_\theta E) = +\infty. \tag{6.10}$$

Proof. This simply follows from the observation that

$$\mu^{\varepsilon,\theta}(\mathrm{proj}_\theta E) = \sum_{n\geq 0} \mu_n^{\varepsilon,\theta}(\mathrm{proj}_\theta E) = \sum_{n\geq 0}\sum_{k\geq n} (\mu_k^{\varepsilon})_{|C_n^\theta}(E) \geq \sum_{n\geq 0} \mu_n^{\varepsilon}(E) = +\infty,$$

since the union of the $(C_n^\theta)_{n\geq 1}$ cover $\mathbb{R}^2$ (there are small overlaps (their borders) between the C_n^θ). Hence the result. □

So the first part of (6.9) is proved.

Let us move to the second part. Observe that even if $\mu^{\varepsilon,\theta}(\mathrm{proj}_\theta E) = +\infty$, it is likely that $\mathrm{proj}_\theta E$ has dimension less than $\mathrm{Dim}_H(E)$. A trivial example is when the s-dimensional set E is included in a straight line of angle ϕ passing through 0, and $\theta = \phi + \pi/2$.

Proposition 6.4. *One has*

$$\mathbb{E}_\theta \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^{\varepsilon,\theta}) \right] < +\infty. \tag{6.11}$$

Proof. Remark that if (6.11) is proved, then $\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^{\varepsilon,\theta}) < +\infty$ for Lebesgue almost every $\theta \in [0,\pi]$, so (6.9) and item a) of Theorem 2.8 are proved.

We start with the following lemma.

Lemma 6.5. *There exists a constant $C_0 > 0$ such that the following holds. Let $x \in S_k$ for some $k \geq 0$. For all $0 \leq n \leq k$, the set $J_{x,n} = \{\theta \in [0,\pi] : x \in C_k^\theta\}$ is an interval modulo π, and $|J_{x,n}| \leq C_0 2^{n-k}$.*

Proof. The fact that $J_{x,k}$ is an interval is obvious.

Let $x = (u, v) \in S_k$. We study the case where $x_1 \geq 0$, the case $x_1 < 0$ being symmetric. Using polar coordinates, one has $x = (r\cos\theta_0, r\sin\theta_0)$ for some $2^{k-1} \leq r \leq 2^k$ and $\theta_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then the projection of x on L_θ is given by:

$$\mathrm{proj}_\theta x = (r\cos(\theta-\theta_0)\cos\theta, r\cos(\theta-\theta_0)\sin\theta).$$

Recall (6.1), one sees that for $0 \leq n \leq k$,

$$\begin{aligned} x \in C_n^\theta &\iff 2^{n-1} \leq r\cos(\theta-\theta_0) \leq 2^n \\ &\iff \frac{2^{n-1}}{r} \leq \cos(\theta-\theta_0) \leq \frac{2^n}{r} \\ &\iff 2^{n-k} \leq \cos(\theta-\theta_0) \leq \min\{1, 2^{n-k+1}\} \\ &\iff \theta \in \left[\theta_0 + \arccos\left(2^{n-k}\right), \theta_0 + \arccos\left(\min\{1, 2^{n-k+1}\}\right)\right] \quad \mod \pi. \end{aligned}$$

Denote by $J_{n,x} := \left[\theta_0 + \arccos\left(\frac{1}{2}2^{n-k}\right), \theta_0 + \arccos\left(\min\{1, 2^{n-k+1}\}\right)\right]$. The Taylor development $\arccos(y) = \frac{\pi}{2} - y + o(y)$ yields that $|J_{n,x}| = 2^{n-k}(1 + o(1))$. $\square$

From the proof, it also follows that $|J_{x,n}| \sim C2^{n-k}$ when n/k is quite small.

Let us study (6.11). One has

$$\begin{aligned} &\mathbb{E}_\theta\left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^{\varepsilon,\theta})\right] \\ &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^{\varepsilon,\theta})\right] d\theta \\ &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \int_{S_n^\theta}\int_{S_n^\theta} \frac{d\mu_n^{\varepsilon,\theta}(u)\, d\mu_n^{\varepsilon,\theta}(v)}{|u-v|^{s-\varepsilon} \vee 1}\right] d\theta \\ &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{j,k\geq n} \int_{E\cap S_j\cap C_n^\theta}\int_{E\cap S_k\cap C_n^\theta} \frac{d\mu_k^\varepsilon(x)\, d\mu_j^\varepsilon(y)}{|x\cdot e_\theta - y\cdot e_\theta|^{s-\varepsilon} \vee 1}\right] d\theta \\ &:= I_1 + 2I_2 \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k\geq n} \iint_{(E\cap S_j\cap C_n^\theta)^2} \frac{d\mu_k^\varepsilon(x)\, d\mu_k^\varepsilon(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon} \vee 1}\right] d\theta \\ I_2 &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k>j\geq n} \int_{E\cap S_j\cap C_n^\theta}\int_{E\cap S_k\cap C_n^\theta} \frac{d\mu_k^\varepsilon(x)\, d\mu_j^\varepsilon(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon} \vee 1}\right] d\theta. \end{aligned}$$

Starting with I_1, one has

$$\begin{aligned}
I_1 &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k\geq n} \iint_{(E\cap S_k\cap C_n^\theta)^2} \frac{d\mu_k^\varepsilon(x)\, d\mu_k^\varepsilon(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon}\vee 1}\right] d\theta \\
&= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k\geq n} \iint_{(E\cap S_k)^2} \frac{\mathbb{1}_{C_n^\theta}(x)\mathbb{1}_{C_n^\theta}(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon}\vee 1} d\mu_k^\varepsilon(x)\, d\mu_k^\varepsilon(y)\right] d\theta \\
&= \sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k\geq n} \iint_{(E\cap S_k)^2} \int_0^\pi \frac{\mathbb{1}_{C_n^\theta}(x)\mathbb{1}_{C_n^\theta}(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon}\vee 1} d\theta d\mu_k^\varepsilon(x)\, d\mu_k(y) \\
&\leq \sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k\geq n} \iint_{(E\cap S_k)^2} \left[\int_0^\pi \frac{\mathbb{1}_{x\in C_n^\theta}(\theta)\mathbb{1}_{y\in C_n^\theta}(\theta)}{|\tau_{x-y}\cdot e_\theta|^{s-\varepsilon}} d\theta\right] \frac{d\mu_k^\varepsilon(x)\, d\mu_k^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon}\vee 1},
\end{aligned}$$

where τ_{x-y} is the unit vector in the direction of $x-y$. By Lemma 6.5, when $x\in S_k$ one has $\mathbb{1}_{x\in C_n^\theta}(\theta) = \mathbb{1}_{J_{n,x}}(\theta)$. Then

$$\int_0^\pi \frac{\mathbb{1}_{x\in C_n^\theta}(\theta)\mathbb{1}_{y\in C_n^\theta}(\theta)}{|\tau_{x-y}\cdot e_\theta|^{s-\varepsilon}} d\theta = \int_{J_{n,x}\cap J_{n,y}} \frac{d\theta}{|\cos(\widehat{\tau_{x-y}, e_\theta})|^{s-\varepsilon}}.$$

By Lemma 6.5, the interval $J_{n,x}\cap J_{n,y}$ has length smaller than $C_0 2^{n-k}$. So the integral above is taken over an interval of length at most $C_0 2^{n-k}$. Moreover, as $s<1$, the integral reaches its largest value when θ close to $\frac{\pi}{2}$. Thus

$$\int_0^\pi \frac{\mathbb{1}_{x\in C_n^\theta}(\theta)\mathbb{1}_{y\in C_n^\theta}(\theta)}{|\tau_{x-y}\cdot e_\theta|^{s-\varepsilon}} d\theta \leq \int_{\frac{\pi}{2}-C_0 2^{n-k}}^{\frac{\pi}{2}+C_0 2^{n-k}} \frac{d\theta}{|\cos(\theta)|^{s-\varepsilon}} \leq \int_{-C_0 2^{n-k}}^{C_0 2^{n-k}} \frac{d\theta}{|\theta|^{s-\varepsilon}} = C 2^{(n-k)(1-s+\varepsilon)}. \tag{6.12}$$

where $C>0$ is some positive constant. Then going back to I_1 and using 6.12, one gets

$$\begin{aligned}
I_1 &\leq C \sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k\geq n} 2^{(n-k)(1-s+\varepsilon)} \iint_{(E\cap S_k)^2} \frac{d\mu_k^\varepsilon(x)\, d\mu_k^\varepsilon(y)}{\|x-y\|_2^\varepsilon \vee 1} \\
&= C \sum_{n\geq 0}\sum_{k\geq n} 2^{n+k(s+\varepsilon-1)} \iint_{(E\cap S_k)^2} \frac{d\mu_k^\varepsilon(x)\, d\mu_k^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon}\vee 1} \\
&= C \sum_{n\geq 0} 2^{n(s+\varepsilon-1)} \sum_{k=0}^{n} 2^k \iint_{(E\cap S_n)^2} \frac{d\mu_n^\varepsilon(x)\, d\mu_n^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon}\vee 1} \\
&\leq 2C \sum_{n\geq 0} 2^{n(s-\varepsilon)} I_{s-\varepsilon}(\mu_n^\varepsilon) < +\infty,
\end{aligned}$$

which is finite by (6.4).

Moving to I_2, the same manipulations as above for I_1 yield

$$
\begin{aligned}
I_2 &= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k>j\geq n} \int_{E\cap S_j\cap C_n^\theta} \int_{E\cap S_k\cap C_n^\theta} \frac{d\mu_k^\varepsilon(x)\, d\mu_j^\varepsilon(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon}\vee 1}\right] d\theta. \\
&= \int_0^\pi \left[\sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k>j\geq n} \int_{E\cap S_j} \int_{E\cap S_k} \frac{\mathbb{1}_{C_n^\theta}(x)\mathbb{1}_{C_n^\theta}(y)}{|(x-y)\cdot e_\theta|^{s-\varepsilon}} d\mu_k^\varepsilon(x)\, d\mu_j^\varepsilon(y)\right] d\theta \\
&= \sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k>j\geq n} \int_{E\cap S_j} \int_{E\cap S_k} \left[\int_0^\pi \frac{\mathbb{1}_{J_{n,x}}(\theta)\mathbb{1}_{J_{n,y}}(\theta)}{|\tau_{x-y}\cdot e_\theta|^{s-\varepsilon}} d\theta\right] \frac{d\mu_k^\varepsilon(x)\, d\mu_j^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon}}.
\end{aligned}
$$

As before, by Lemma 6.5, $|J_{k,x}| \leq 2^{n-k}$ and $|J_{j,y}| \leq 2^{n-j}$ for all $x \in S_k \cap C_n^\theta$ and $y \in S_j \cap C_n^\theta$). Then, as $k \geq j+1$, the same argument as in (6.12) yields

$$
\int_0^\pi \frac{\mathbb{1}_{x\in C_n^\theta}(\theta)\mathbb{1}_{y\in C_n^\theta}(\theta)}{|\tau_{x-y}\cdot e_\theta|^{s-\varepsilon}} d\theta \leq C2^{(n-k)(1-s+\varepsilon)}. \tag{6.13}
$$

for some $C > 0$.

Next, we make use of equation (6.7) : indeed, it is not possible that μ_j^ε and μ_{j+1}^ε are simultaneously non-zero. Hence, for $x \in S_k$ and $y \in S_j$ such that $j < k$ and μ_j^ε and μ_k^ε not both equal to zero, then necessarily $|k-j| \geq 2$ and $2^{k-2} \leq \|x-y\|_2 \leq 2^{k+1}$. This implies in particular that

$$
\int_{E\cap S_j} \int_{E\cap S_k} \frac{d\mu_k^\varepsilon(x)\, d\mu_j^\varepsilon(y)}{\|x-y\|_2^{s-\varepsilon}} \leq C2^{-k(s-\varepsilon)} \mu_k^\varepsilon(E\cap S_k)\, \mu_j^\varepsilon(E\cap S_j), \tag{6.14}
$$

the inequality being in fact close to be sharp.

Finally, combining (6.14) and (6.13)), one gets that for some $C' > 0$,

$$
\begin{aligned}
I_2 &\leq C' \sum_{n\geq 0} 2^{n(s-\varepsilon)} \sum_{k>j\geq n} 2^{(n-k)(1-s+\varepsilon)} 2^{-k(s-\varepsilon)} \mu_k^\varepsilon(E\cap S_k)\, \mu_j^\varepsilon(E\cap S_j) \\
&= C' \sum_{n\geq 0} 2^n \sum_{k>j\geq n} 2^{-k} \mu_j^\varepsilon(E\cap S_j) \mu_k^\varepsilon(E\cap S_k) \\
&= C' \sum_{j\geq 0} \left(\sum_{n=0}^{j} 2^n\right) \mu_n^\varepsilon(E\cap S_n) \sum_{k\geq n+1} 2^{-k} \mu_k^\varepsilon(E\cap S_k) \\
&\leq C' \sum_{n\geq 0} 2^n \mu_n^\varepsilon(E\cap S_n) \sum_{k\geq n+1} 2^{-k} \mu_k^\varepsilon(E\cap S_k) \\
&\leq C' \sum_{n\geq 0} 2^{-n} \mu_n^\varepsilon(E\cap S_n) \left(\sum_{k=0}^{n} 2^k \mu_k^\varepsilon(E\cap S_k)\right).
\end{aligned}
$$

This last double sum is finite, because the set E was chosen so that (6.5) holds true. This concludes the proof. $\square$

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