On the pushable chromatic number of various types of grids

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Abstract

Pushing a vertex in an oriented graph means reversing the direction of all the arcs incident to that vertex, resulting in another oriented graph. The pushable chromatic number of an oriented graph $\overrightarrow{G}$ is the order of a smallest (in terms of vertices) oriented graph $\overrightarrow{H}$ such that, by pushing vertices in $\overrightarrow{G}$, we can obtain an oriented graph $\overrightarrow{G}'$ that admits an oriented $\overrightarrow{H}$-colouring, *i.e.*, a vertex-mapping $\phi : V(\overrightarrow{G}') \to V(\overrightarrow{H})$ preserving the arcs (for every arc $\overrightarrow{uv}$ of $\overrightarrow{G}'$, the arc $\overrightarrow{\phi(u)\phi(v)}$ exists in $\overrightarrow{H}$). This notion extends to (undirected) graphs and families of graphs: the pushable chromatic number of a graph is the maximum pushable chromatic number over all its orientations, while the pushable chromatic number of a family of graphs is the maximum pushable chromatic number over all its members.

We here initiate the study of the pushable chromatic number of several types of grids. For hexagonal grids, we determine that the pushable chromatic number is exactly 4. For square grids, we show that the pushable chromatic number is 5 or 6. For triangular grids, we prove that the pushable chromatic number lies in between 7 and 12.

The pushable chromatic number of graphs is, together with the oriented chromatic number, the 2-edge-coloured chromatic number, and the signed chromatic number, part of a group of four chromatic parameters that tend to behave in a very comparable way in general. Following the current work, all of these four parameters have now been investigated in the context of several grids. We take this occasion to summarise the current knowledge on the behaviour of these four chromatic parameters in these graphs.

Keywords: oriented graph; oriented colouring; pushable chromatic number; grid.

1. Introduction

In this work, we initiate the study of the **pushable chromatic number** of various types of grid-like graphs, including **hexagonal grids**, **square grids**, and **triangular grids**. Before we can elaborate more on our results, we start by recalling a few concepts; in case some of the upcoming definitions and notions are unclear, we refer the reader to Section 2, in which some of these are reminded formally.

Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be two oriented graphs. An $\overrightarrow{H}$*-colouring* ϕ of $\overrightarrow{G}$ is a vertex-mapping $\phi : V(\overrightarrow{G}) \to V(\overrightarrow{H})$ (that essentially "colours" the vertices of $\overrightarrow{G}$ with those of $\overrightarrow{H}$). We say that ϕ is an *oriented colouring* if ϕ forms, essentially, an homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$, thus preserving the arcs, *i.e.*, we have $\overrightarrow{\phi(u)\phi(v)} \in A(\overrightarrow{H})$ whenever $\overrightarrow{uv} \in A(\overrightarrow{G})$. The

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oriented chromatic number of $\overrightarrow{G}$, denoted by $\chi_o(\overrightarrow{G})$, is the order of the smallest (in terms of vertices) $\overrightarrow{H}$ such that $\overrightarrow{G}$ admits oriented $\overrightarrow{H}$-colourings. This last notion extends to (undirected) graphs, the *oriented chromatic number* $\chi_o(G)$ of a graph G referring to the largest value of $\chi_o(\overrightarrow{G})$ over all orientations $\overrightarrow{G}$ of G. In turns, for a family $\mathcal{F}$ of graphs, the *oriented chromatic number* $\chi_o(\mathcal{F})$ of $\mathcal{F}$ is the largest value of $\chi_o(G)$ for some $G \in \mathcal{F}$.

Pushable graphs are oriented graphs coming together with a *pushing operation. Pushing* a vertex v in an oriented graph $\overrightarrow{G}$ means reversing the direction of all arcs incident to v, resulting in an *equivalent* oriented graph $\overrightarrow{G}'$ (being an orientation of the same graph). The *pushable chromatic number* $\chi_p(\overrightarrow{G})$ of $\overrightarrow{G}$ is now the smallest value of $\chi_o(\overrightarrow{G}')$ over all oriented graphs $\overrightarrow{G}'$ that are equivalent to $\overrightarrow{G}$ through pushing vertices. Again, the *pushable chromatic number* $\chi_p(G)$ of a graph G is the maximum pushable chromatic number over all its orientations, while the *pushable chromatic number* $\chi_p(\mathcal{F})$ of a graph family $\mathcal{F}$ is the maximum pushable chromatic number over all its members.

The notion of pushable chromatic number was first introduced by Klostermeyer and MacGillivray in 2004 [16]. Since then, several of its aspects have been investigated in the literature. General properties of the pushable chromatic number were studied in several references [4, 13, 24]. The pushing operation itself was featured in a number of works, such as [15, 17, 18, 23] to name a few. Regarding the pushable chromatic number of particular graph classes, partial results were obtained in the context of e.g. outerplanar graphs, 2-trees, planar graphs, planar graphs with girth properties, graphs with bounded acyclic chromatic number, and graphs with bounded maximum degree [2, 13, 16, 24]. For more details on the oriented chromatic number of oriented graphs, we refer the interested reader to the recent survey [25] by Sopena.

In this work, we initiate the study of the pushable chromatic number of several types of grids (hexagonal grids, square grids, triangular grids), which, to the best of our knowledge, has not been considered as such in this context. It is worthwhile mentioning, however, that some of the properties of grids allow to derive first bounds from previous works. For instance, grids being generally of maximum degree, or being sometimes planar (which is the case for all three types of grids considered herein), existing upper bounds for graphs with such properties apply directly to our context. As will be recalled later, there are also strong connections between the pushable chromatic number and the oriented chromatic number, from which we can also deduce upper bounds. Still, we give dedicated arguments providing better bounds. As will also be reminded in a later section, the oriented chromatic number and the pushable chromatic number are actually part (together with the 2-edge-coloured chromatic number and the signed chromatic number) of a set of four chromatic numbers that are known to behave similarly, the pushable chromatic number being actually the last one of these four parameters to be investigated specifically in the context of grids.

This work is organised as follows. We start with preliminary Section 2, in which notions and tools to be used throughout, in particular related to oriented colourings and the pushable chromatic number, are given or recalled. We then start by considering hexagonal grids in Section 3, for which we determine that the pushable chromatic number is exactly 4. Next, we consider square grids, in Section 4, for which we show the pushable chromatic number is 5 or 6. Lastly, we focus on triangular grids in Section 5, their pushable chromatic being showed to lie in between 7 and 12. As mentioned earlier, this work closes some gap in the study of grids in the context of four related chromatic parameters. We thus take this opportunity to summarise, in Section 6, how these four parameters compare in this specific context. In that section, we also raise perspectives for further work on this topic.

2. Preliminaries

2.1. General terminology and notation

Throughout this work, the term "graph" refers to a simple undirected graph. For a graph G, we denote by $V(G)$ and $E(G)$ its *vertex set* and *edge set*, respectively. Two vertices u and v of G are *neighbours* (or said to be *adjacent*) if $uv \in E(G)$. We say that the edge uv is *incident* to u and v. The *degree* of a vertex v, denoted by $d(v)$, is the number of its neighbours in G. The *minimum degree* $\delta(G)$ and *maximum degree* $\Delta(G)$ of G refer to the minimum and maximum degree value, respectively, over all vertices of G.

An *orientation* $\overrightarrow{G}$ of G is obtained by assigning one of the two possible directions to every edge uv of G, resulting in an *arc* either $\overrightarrow{uv}$ from u to v, or $\overrightarrow{vu}$ from v to u. We denote by $A(\overrightarrow{G})$ the *arc set* of $\overrightarrow{G}$. An orientation is more generally called an *oriented graph*. To make the distinction between graphs and oriented graphs clear, we will voluntarily employ overhead arrows in the latter case, particularly when dealing with symbols associated to orientations (e.g. $\overrightarrow{G}$) and arcs (e.g. $\overrightarrow{uv}$). Note that, in such instances, this allows to retrieve information on the *underlying graph* directly (e.g. G from $\overrightarrow{G}$ and uv from $\overrightarrow{uv}$). An orientation of a cycle $(v_1, \dots, v_n, v_1)$ such that all arcs follow the same direction (*i.e.*, either $\overrightarrow{v_1v_2}, \dots, \overrightarrow{v_nv_1}$ or $\overrightarrow{v_2v_1}, \dots, \overrightarrow{v_1v_n}$ are arcs) is called a *directed cycle*.

Whenever employing a term or notation for graphs in the context of an oriented graph $\overrightarrow{G}$, it should be understood that we mean that term or notation for G, the graph underlying $\overrightarrow{G}$. Regarding more specific terms and notations, for an oriented graph $\overrightarrow{G}$ and an arc $\overrightarrow{uv}$ of $\overrightarrow{G}$, we say that $\overrightarrow{uv}$ is *out-going from* u and *in-coming to* v. In that case, we call v an *out-neighbour* of u, and u an *in-neighbour* of v. For a vertex v of $\overrightarrow{G}$, we denote by $d^-(v)$ and $d^+(v)$ the *in-degree* and *out-degree* of v, respectively, being its number of in-neighbours and out-neighbours, respectively. We say that v is a *source* if $d^-(v) = 0$, while we say that v is a *sink* if $d^+(v) = 0$. The *minimum in-degree* $\delta^-(\overrightarrow{G})$ and *maximum in-degree* $\Delta^-(\overrightarrow{G})$ of $\overrightarrow{G}$ refer to the minimum and maximum in-degree value, respectively, over all vertices of $\overrightarrow{G}$. The *minimum out-degree* $\delta^+(\overrightarrow{G})$ and *maximum out-degree* $\Delta^+(\overrightarrow{G})$ of $\overrightarrow{G}$ are defined analogously, with respect to the out-degrees of the vertices of $\overrightarrow{G}$.

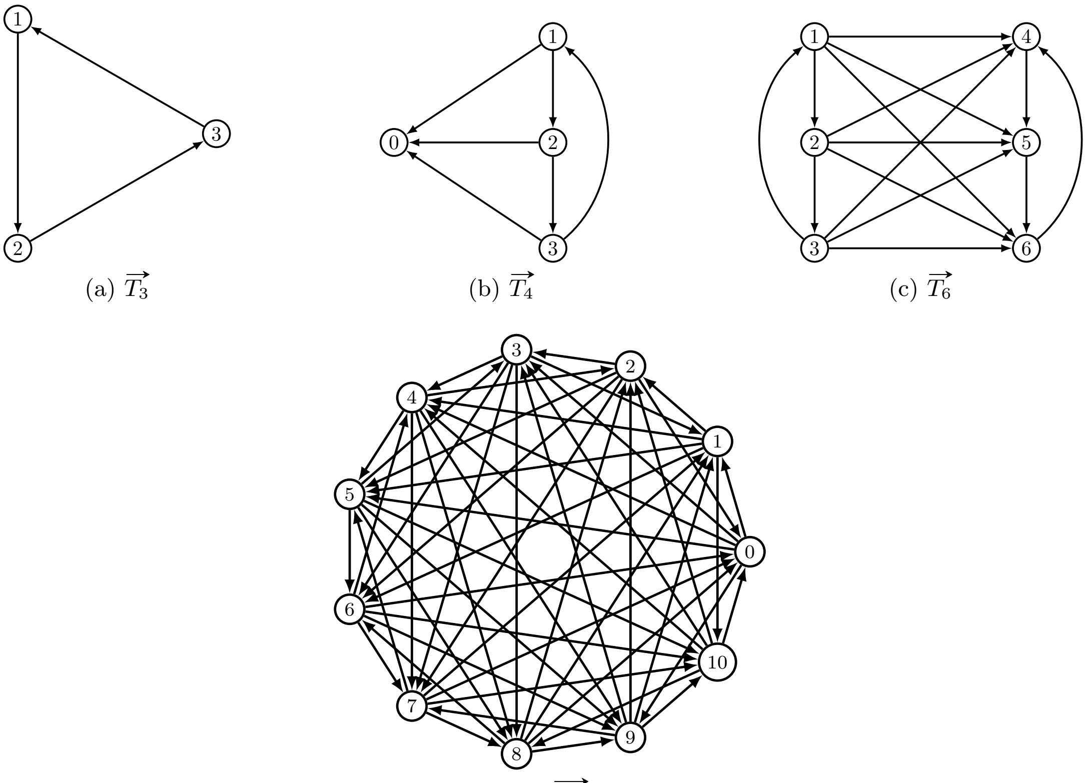
2.2. Types of grids

Let $n \geq 1$ and $m \geq 1$ be two fixed positive integers. The *square grid with n rows and m columns*, denoted by $S(n,m)$, is the Cartesian product $P_n \,\square\, P_m$ of the paths of order n and m, respectively. That is, for every $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, the grid $S(n,m)$ contains a vertex (i,j) being part of the ith row and jth column. Every such vertex (i,j) is adjacent to the at most four vertices $(i-1,j)$, $(i+1,j)$, $(i,j-1)$ and $(i,j+1)$ that exist. In particular, $\Delta(S(n,m)) \leq 4$.

In this work, we also deal with two other types of grids, namely hexagonal grids and triangular grids, which are, essentially, subgraphs and supergraphs, respectively, of square grids. To make our exposition more legible, we deal with these two types of grids using the same terminology as that we introduced for square grids, although we are aware it can be less natural at times (especially when talking about rows and columns).

- The *hexagonal grid with n rows and m columns*, denoted by $H(n,m)$, is obtained from $S(n,m)$ by removing every edge $(i,j)(i+1,j)$ such that i and j have the same parity. In particular, $\Delta(H(n,m)) \leq 3$.

- The *triangular grid with n rows and m columns*, denoted by $T(n,m)$, is obtained from $S(n,m)$ by adding every possible diagonal edge $(i,j)(i-1,j-1)$. In particular, $\Delta(T(n,m)) \leq 6$.



(a) $\overrightarrow{T_3}$ (b) $\overrightarrow{T_4}$ (c) $\overrightarrow{T_6}$

(d) $\overrightarrow{P_{11}}$

Figure 1: Some of the oriented graphs $\overrightarrow{H}$ that will be used to design $\overrightarrow{H}$-colourings.

Whenever considering one of the types of grids above, we implicitly assume it is embedded in the plane in the natural planar way (*i.e.*, vertices in a same row lie on a same horizontal line and vertices in a same column lie on a same vertical line – note, indeed, that our terminology for the vertices yield notions of top, bottom, left and right for any grid, in the natural way).

2.3. Colouring oriented and pushable graphs

Some of our upper bounds on the pushable chromatic number will be established through $\overrightarrow{H}$-colourings for particular oriented graphs $\overrightarrow{H}$, depicted in Figure 1 (their vertices being dealt with, throughout this work, through the terminology introduced in that figure). Particularly, the oriented graph $\overrightarrow{T_3}$, depicted in (a), is already known to be of interest, for instance due to the following result (which actually generalises to all oriented trees, as observed in several earlier works, see e.g. [25]):

Theorem 2.1. *Every oriented path admits an oriented $\overrightarrow{T_3}$-colouring.*

Proof. Assuming the consecutive vertices of the underlying path are $v_1, \dots, v_n$ from one end v_1 to the other v_n, an oriented $\overrightarrow{T_3}$-colouring can be obtained by simply assigning colour 1 to v_1, and then assigning a valid colour to $v_2, \dots, v_n$ consecutively, following that order. One such valid colour always exists, as, for every vertex v_i considered in that course, only the colour assigned to v_{i-1} brings a constraint for the choice of v_i's colour, and every vertex of $\overrightarrow{T_3}$ has in-degree and out-degree 1. $\square$

When considering the pushable chromatic number, a tricky aspect to take into account is the pushing operation. One way to get rid of this subtlety, is by considering oriented

$\overrightarrow{H}$-colourings for some particular oriented graphs $\overrightarrow{H}$. Namely, given $\overrightarrow{H}$, we denote by $R(\overrightarrow{H})$ the *anti-twin* of $\overrightarrow{H}$, being the oriented graph obtained starting from two vertex-disjoint copies of $\overrightarrow{H}$, one with vertices $u_1, \dots, u_n$ and the other with vertices $v_1, \dots, v_n$ (where the vertex-mapping $f : \{u_1, \dots, u_n\} \to \{v_1, \dots, v_n\}$ where $f(u_i) = v_i$ for every $i \in \{1, \dots, n\}$ is an isomorphism between the two copies), and then, for every arc $\overrightarrow{u_i u_j}$, adding both arcs $\overrightarrow{u_j v_i}$ and $\overrightarrow{v_j u_i}$ between the two copies. The property of interest is the following:

Theorem 2.2 (Klostermeyer, MacGillivray [16]). *Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be two oriented graphs. Pushing vertices in $\overrightarrow{G}$ to reach an orientation of G that is orientedly $\overrightarrow{H}$-colourable is possible, if and only if, $\overrightarrow{G}$ is orientedly $R(\overrightarrow{H})$-colourable.*

In particular, a consequence of Theorem 2.2 is the following:

Corollary 2.3 (Klostermeyer, MacGillivray [16]). *If $\overrightarrow{G}$ is an oriented graph, then*

$$\chi_{\mathrm{p}}(\overrightarrow{G}) \leq \chi_{\mathrm{o}}(\overrightarrow{G}) \leq 2\chi_{\mathrm{p}}(\overrightarrow{G}).$$

The pushing operation actually yields equivalence classes over the orientations of a given graph, that can be reached from each other through pushing vertices. In particular, we say that two orientations $\overrightarrow{G}$ and $\overrightarrow{G}'$ of a graph G are *push-equivalent*, if one can obtain $\overrightarrow{G}'$ from $\overrightarrow{G}$ by pushing vertices of $\overrightarrow{G}$. Note that this notion of equivalence through the pushing operation is the same as the notion of equivalence we gave in Section 1; however, we introduce the former term as it emphasises that we are actually referring to the pushing operation. An oriented graph from an equivalence class is called a *representative* of that class. One point of interest for these notions is the following:

Observation 2.4 (Klostermeyer, MacGillivray [16]). *If an oriented graph $\overrightarrow{G}$ can have its vertices pushed so that the resulting oriented graph is orientedly $\overrightarrow{H}$-colourable for some oriented graph $\overrightarrow{H}$, then $\overrightarrow{G}$ can also have its vertices pushed so that the resulting oriented graph is orientedly $\overrightarrow{H}'$-colourable for any oriented graph $\overrightarrow{H}'$ that is push-equivalent to $\overrightarrow{H}$.*

Observation 2.4 implies that the number of tournaments $\overrightarrow{H}$ of order k that one has to consider for proving that a graph has pushable chromatic number at most k, can be reduced to such $\overrightarrow{H}$'s that are pairwise not push-equivalent (*i.e.*, unique representatives of all equivalence classes). In the context of the current paper, we consider small values of k only, for which the number of equivalence classes to consider can be checked to be very small (the notion of motion for oriented cycles that we introduce in later Observation 3.2, is an example of a tool that can be used to facilitate the formal checking of this fact):

Observation 2.5. *The following statements are true:*

- *There is only one equivalence class of tournaments on* 3 *vertices, i.e., all tournaments on* 3 *vertices are push-equivalent.*
- *There are two equivalence classes of tournaments on* 4 *vertices (two representatives are depicted in Figure 7, (a) and (b)).*
- *There are two equivalence classes of tournaments on* 5 *vertices (two representatives are depicted in Figure 9, (a) and (b)).*
- *There are six equivalence classes of tournaments on* 6 *vertices (six representatives are depicted in Figure 8, (a) to (f)).*

2.4. Paley and Tromp constructions

For proving that families of oriented graphs are, in general, orientedly $\overrightarrow{H}$-colourable for some $\overrightarrow{H}$, it is generally preferable to consider such $\overrightarrow{H}$'s having a structure as regular and symmetric as possible. Regarding these thoughts, a few constructions of oriented graphs, which we recall now, are known to be worth considering.

For a prime power $p \equiv 3 \bmod 4$, the *Paley tournament* on p vertices, denoted by $\overrightarrow{P_p}$, is the oriented graph with vertex set $\mathbb{Z}/p\mathbb{Z} = \{0, \dots, p-1\}$ in which $\overrightarrow{uv}$ is an arc if and only if $v - u$ is a non-zero quadratic residue in $\mathbb{Z}/p\mathbb{Z}$. Figure 1(d) shows $\overrightarrow{P_{11}}$ as an example. The structure of Paley tournaments is so regular and symmetric, that they are known to have several properties of interest. In our case, we are interested in the following notions. For some $n \geq 1$, an *orientation n-vector* is a sequence $S = (\alpha_1, \dots, \alpha_n) \in \{-, +\}^n$. Let $\overrightarrow{G}$ be an oriented graph. For a *vertex n-vector* $X = (v_1, \dots, v_n)$, being a sequence of n pairwise distinct vertices of $\overrightarrow{G}$, we say that a vertex $u \in V(G)$ *complies* with X with respect to S, if, for every $i \in \{1, \dots, n\}$, we have the arc $\overrightarrow{uv_i}$ if $\alpha_i = -$, and the arc $\overrightarrow{v_iu}$ otherwise. Now, for some $p, q \geq 1$, we say that $\overrightarrow{G}$ has *Property $P_{p,q}$*, if for every orientation p-vector S and every vertex p-vector X, there are at least q vertices of $\overrightarrow{G}$ that comply with X with respect to S. Back to Paley tournaments, we are interested in the result below. Be aware that [11] is not the first place in which this result ever appeared (see e.g. [6]); however, we advise the reader to refer to [11] because the terminology used there by the authors is closer to ours, and also the proof they provide is fully self-contained.

Theorem 2.6 (Dybizbański, Ochem, Pinlou, Szepietowski [11])**.** *Every Paley tournament $\overrightarrow{P_p}$ has Properties $P_{1,\frac{p-1}{2}}$ and $P_{2,\frac{p-3}{4}}$.*

Another interesting construction has been commonly used in this context, that of Tromp. For an oriented graph $\overrightarrow{G}$, the *Tromp graph* $T(\overrightarrow{G})$ of $\overrightarrow{G}$ is obtained starting from $\overrightarrow{G}$, by then adding a new vertex ∞ dominating all vertices of $\overrightarrow{G}$ (*i.e.*, $\overrightarrow{\infty v}$ is an arc for every $v \in V(\overrightarrow{G})$), and considering the anti-twin of the resulting oriented graph. Besides several other properties, the Tromp construction is known to have interesting properties when combined with Paley tournaments:

Theorem 2.7 (Dybizbański, Ochem, Pinlou, Szepietowski [11])**.** *If the Paley tournament $\overrightarrow{P_p}$ has Property $P_{n,q}$, then the Tromp graph $T(\overrightarrow{P_p})$ has Property $P_{n+1,q}$.*

3. Hexagonal grids

In this section, we establish that the pushable chromatic number of hexagonal grids is precisely 4. We start off by proving the upper bound.

Theorem 3.1. *For every hexagonal grid H, we have $\chi_{\mathrm{p}}(H) \leq 4$.*

Proof. Let $\overrightarrow{H}$ be an orientation of any hexagonal grid $H = H(n, m)$ with n rows and m columns. We prove the result by showing that we can push vertices in $\overrightarrow{H}$, so that what results is an orientation $\overrightarrow{H}'$ of H that is orientedly $\overrightarrow{T_4}$-colourable, where $\overrightarrow{T_4}$ refers to the orientation of K_4 depicted in Figure 1(b).

To describe these vertices we need to push, consider first S, the set containing the following vertices of $\overrightarrow{H}$:

- for every odd row i of $\overrightarrow{H}$, S contains the vertices $(i, 1), (i, 5), (i, 9), \dots$.
- for every even row i, S contains the vertices $(i, 3), (i, 7), (i, 11), \dots$.

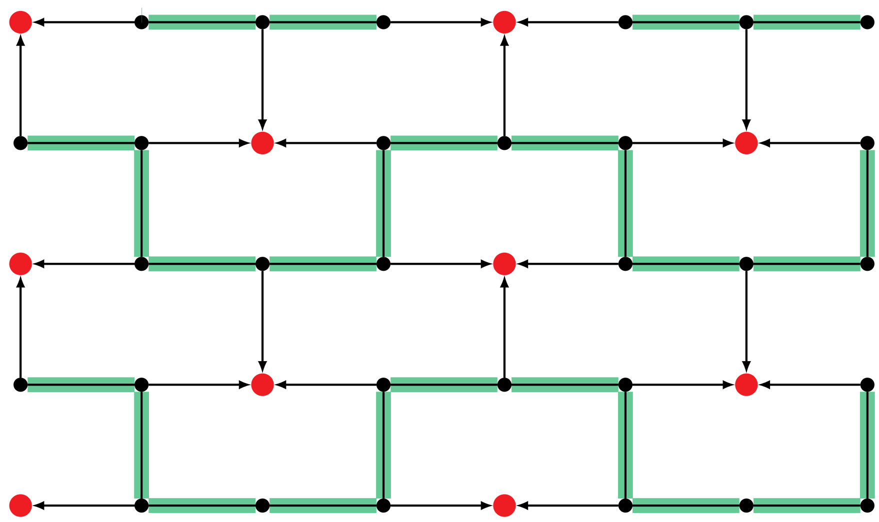


Figure 2: Illustration of the terminology used in the proof of Theorem 3.1. Red vertices are vertices of the set S. Green edges are edges of the linear forest $F = H - S$. By pushing black vertices, we can make sure to have the depicted arcs. Edges represent arcs that can be oriented in any direction.

Note that S is an independent set. Also, every vertex of $V(\overrightarrow{H}) \setminus S$ is adjacent to precisely one vertex of S, and $\overrightarrow{F} = \overrightarrow{H} - S$ is an oriented linear forest (*i.e.*, an oriented forest of paths), see Figure 2. Due to all these properties, note that we can, from $\overrightarrow{H}$, push vertices in $V(\overrightarrow{H}) - S$ to reach an orientation $\overrightarrow{H}'$ of H in which all vertices of S are sinks. Now, to obtain an oriented $\overrightarrow{T_4}$-colouring ϕ of $\overrightarrow{H}'$, it suffices to assign colour 0 to the vertices of S, and to assign colours in $\{1, 2, 3\}$ to the vertices of $V(\overrightarrow{F})$ so that these colours form an oriented $\overrightarrow{T_3}$-colouring of $\overrightarrow{F}$ (which is possible, by Theorem 2.1, as the vertices $1, 2, 3$ in $\overrightarrow{T_4}$ induce $\overrightarrow{T_3}$). The vertices of S being sinks in $\overrightarrow{H}'$ (and so is vertex 0 of $\overrightarrow{T_4}$), ϕ is indeed oriented. The stated bound thus follows. □

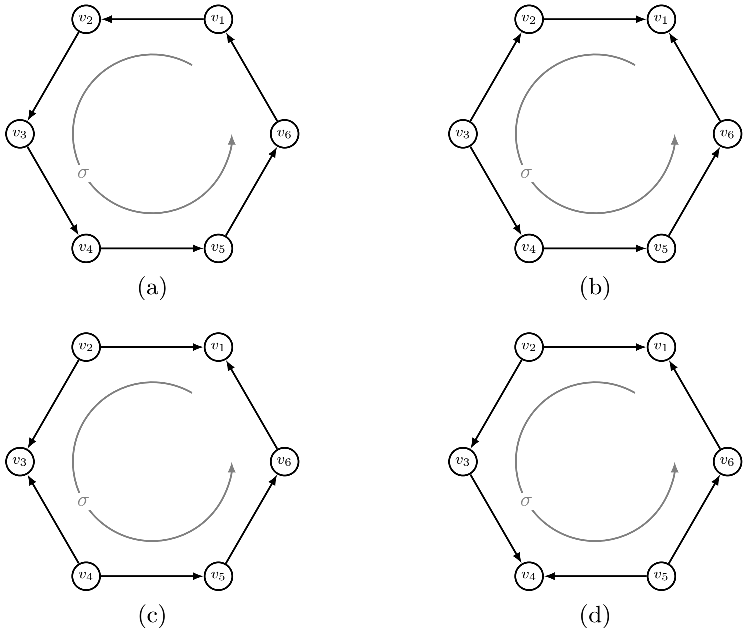
We now focus on proving that the bound in Theorem 3.1 is tight in general, in the sense that some hexagonal grids have pushable chromatic number strictly more than 3. For that, we need to introduce a few concepts and ideas beforehand.

Let $\overrightarrow{G}$ be an oriented graph, and $\overrightarrow{C}$ be an oriented cycle on $x \geq 3$ vertices in $\overrightarrow{G}$. We define a *motion* σ for $\overrightarrow{C}$ as one of the two possible "virtual" orientations of the cycle $(v_0, \dots, v_{x-1}, v_0)$ underlying $\overrightarrow{C}$ as a directed cycle. So, by σ, either $\overrightarrow{v_0v_1}, \dots, \overrightarrow{v_{x-1}v_0}$ are all arcs (in which case we can write e.g. $\sigma = (v_0, \dots, v_{x-1}, v_0)$), or $\overrightarrow{v_1v_0}, \dots, \overrightarrow{v_0v_{x-1}}$ are all arcs (in which case we can write e.g. $\sigma = (v_{x-1}, \dots, v_0, v_{x-1})$). We can now classify the arcs of $\overrightarrow{C}$ into two groups, $F(\overrightarrow{C})$ and $B(\overrightarrow{C})$, depending on whether they meet the motion σ: an arc $\overrightarrow{uv}$ of $\overrightarrow{C}$ is said to be a *forward arc* (in $F(\overrightarrow{C})$) if that arc meets σ, while $\overrightarrow{uv}$ is said to be a *backward arc* (in $B(\overrightarrow{C})$) otherwise. The *parity* of $\overrightarrow{C}$ (with respect to σ) is *even* if $|F(\overrightarrow{C})|$ is even, and *odd* otherwise.

The main point for considering the parity of oriented cycles by motions, is that this parameter is invariant under pushing vertices. That is, for fixed motions, pushing vertices in an oriented graph has no effect on the parity of its oriented cycles. This is significant; notably it yields a necessary condition for two oriented graphs to be push-equivalent.

Observation 3.2. *Let $\overrightarrow{G}$ be an oriented graph, and $\overrightarrow{C}$ be an oriented cycle of $\overrightarrow{G}$ with motion σ. The parity of $\overrightarrow{C}$ by σ is preserved upon pushing vertices of $\overrightarrow{G}$.*

Proof. Note that changing the direction of arcs of $\overrightarrow{C}$ can only be done through pushing vertices of $\overrightarrow{C}$. Assume now a vertex v of $\overrightarrow{C}$ is pushed, resulting in another orientation $\overrightarrow{C}'$



(a) (b) (c) (d)

Figure 3: The four canonical orientations of $C_6 = (v_1, \dots, v_6, v_1)$ with even parity w.r.t. the motion $\sigma = (v_1, \dots, v_6, v_1)$.

of C. Note that pushing v changed only the direction of the two arcs e and f of $\overrightarrow{C}$ incident to v. Then, regarding σ, if $e \in F(\overrightarrow{C})$ then $e \in B(\overrightarrow{C}')$, and *vice versa*, and similarly for f. This implies that $\overrightarrow{C}$ and $\overrightarrow{C}'$ have the same parity (with respect to σ). $\square$

We now consider the oriented $\overrightarrow{T_3}$-colourability of orientations of C_6, the cycle of length 6, with even parity. Be aware that, although the terminology and the setting are different, the proof arguments we use are rather common when it comes to dealing with oriented cycles (see e.g. Proposition 10 in [25]).

Observation 3.3. *Let $\overrightarrow{C}$ be any orientation of C_6 with motion σ. If $\overrightarrow{C}$ is of even parity with respect to σ and $\overrightarrow{C}$ is orientedly $\overrightarrow{T_3}$-colourable, then $\overrightarrow{C}$ must be a directed cycle.*

Proof. Assume the vertices of $\overrightarrow{C}$ are $v_1, \dots, v_6$, forming a cycle $(v_1, \dots, v_6, v_1)$ in C (being thus C_6). We can also assume the motion σ is $(v_1, \dots, v_6, v_1)$. For $\overrightarrow{C}$ to be of even parity with respect to σ, the set $F(\overrightarrow{C})$ must have cardinality 0, 2, 4 or 6. In what follows, we deal only with the four orientations of C_6 depicted in Figure 3 (for which $|F(\overrightarrow{C})|$ is 6 or 4); indeed, it can be observed that all the other cases are either equivalent to one of these cases (in the isomorphic sense), or can/cannot be orientedly $\overrightarrow{T_3}$-coloured by similar arguments as in one of these canonical cases we focus on.

We consider each case for $\overrightarrow{C}$ in Figure 3 separately:

(a) In this case, setting $\phi(v_1) = 1$, $\phi(v_2) = 2$, $\phi(v_3) = 3$, $\phi(v_4) = 1$, $\phi(v_5) = 2$, $\phi(v_6) = 3$ results in an oriented $\overrightarrow{T_3}$-colouring ϕ of $\overrightarrow{C}$.

(b) Assume $\overrightarrow{C}$ admits an oriented $\overrightarrow{T_3}$-colouring ϕ. Since $\overrightarrow{T_3}$ is vertex-transitive[1], we can assume $\phi(v_3) = 1$, which implies that $\phi(v_2) = \phi(v_4) = 2$, $\phi(v_5) = 3$, $\phi(v_6) = 1$ and

[1] Recall that a graph or oriented graph is *vertex-transitive* if any two of its vertices are equivalent under some element of its automorphism group.

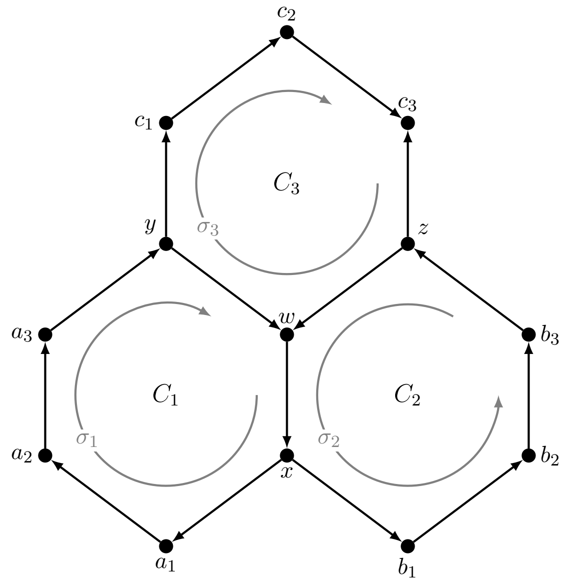


Figure 4: An oriented hexagonal grid with pushable chromatic number strictly more than 3.

$\phi(v_1) = 2$. But then $\phi(v_2) = \phi(v_1) = 2$ while v_1 and v_2 are adjacent, which is a contradiction.

(c) Assume $\overrightarrow{C}$ admits an oriented $\overrightarrow{T_3}$-colouring ϕ. We can assume $\phi(v_4) = 1$, which implies that $\phi(v_3) = \phi(v_5) = 2$, $\phi(v_6) = 3$ and $\phi(v_1) = 1$. But then vertex $\phi(v_2)$ must have, in $\overrightarrow{T_3}$, both an incident out-going arc to vertex 2 and one to 1, while $\overrightarrow{T_3}$ has maximum out-degree 1, a contradiction.

(d) Assume $\overrightarrow{C}$ admits an oriented $\overrightarrow{T_3}$-colouring ϕ. We can assume $\phi(v_2) = 1$, which implies that $\phi(v_1) = \phi(v_3) = 2$, $\phi(v_6) = 1$, $\phi(v_5) = 3$ and $\phi(v_4) = 3$. But then $\phi(v_4) = \phi(v_5)$ while v_4 and v_5 are adjacent, which is yet another contradiction.

Thus, for $\overrightarrow{C}$ to be orientedly $\overrightarrow{T_3}$-colourable, $\overrightarrow{C}$ must be a directed cycle. □

We are now ready to show, in conjunction with previous Theorem 3.1, that, in general, hexagonal grids have pushable chromatic number 4.

Theorem 3.4. *For every hexagonal grid H having three faces sharing a common vertex, we have $\chi_{\mathrm{p}}(H) > 3$.*

Proof. Assume the claim is wrong, and assume H contains three faces (being C_6), which we denote by C_1, C_2, C_3, having a common vertex w. Assume $C_1 = (w, x, a_1, a_2, a_3, y, w)$, $C_2 = (w, x, b_1, b_2, b_3, z, w)$ and $C_3 = (w, y, c_1, c_2, c_3, z, w)$. We get to a contradiction by showing that, even when reducing H to the vertices of C_1, C_2, C_3, there are orientations of H that cannot be orientedly $\overrightarrow{T_3}$-coloured, even if we are allowed to push vertices. Recall that, due to Observations 2.4 and 2.5, focusing on $\overrightarrow{T_3}$ only is indeed sufficient.

Let $\sigma_1 = (w, x, a_1, a_2, a_3, y, w)$, $\sigma_2 = (w, x, b_1, b_2, b_3, z, w)$ and $\sigma_3 = (w, y, c_1, c_2, c_3, z, w)$ be motions for C_1, C_2, C_3, respectively, for any orientation of H. We consider the orientation $\overrightarrow{H}$ of H that is depicted in Figure 4. Note that, in $\overrightarrow{H}$, the orientations of C_1, C_2, C_3 are of even parity with respect to $\sigma_1, \sigma_2, \sigma_3$. Besides, recall that, by Observation 3.2, the parity (with respect to $\sigma_1, \sigma_2, \sigma_3$) of any orientation of C_1, C_2, C_3 obtained through pushing vertices in $\overrightarrow{H}$ must remain even. Now, if we had $\chi_{\mathrm{p}}(\overrightarrow{H}) \leq 3$, then it means that we

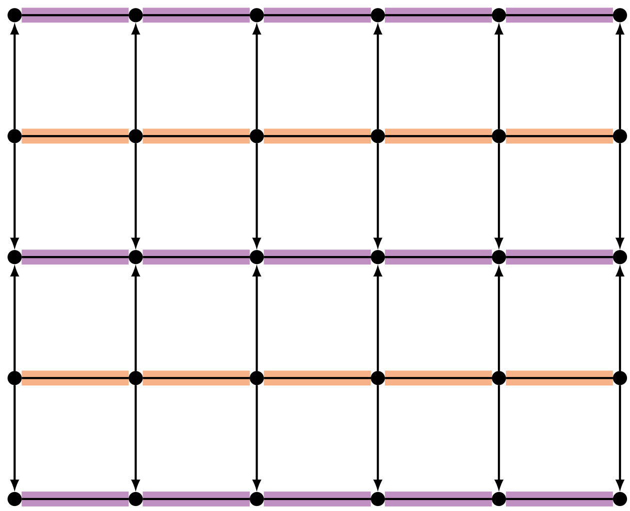


Figure 5: Illustration of the kind of orientation that must be reached, though pushing vertices, in the proof of Theorem 4.1. Vertices in the purple area are dominated by their neighbours in the orange area. Edges represent arcs that can be oriented in any direction.

would be able to push vertices in $\overrightarrow{H}$ to reach an orientation $\overrightarrow{H}'$ of H that is orientedly $\overrightarrow{T_3}$-colourable. By Observation 3.3, this orientation $\overrightarrow{H}'$ must verify that the resulting orientations of C_1, C_2, C_3 must all form directed cycles. One can check that $\overrightarrow{H}'$ cannot verify this for all three oriented cycles. □

4. Square grids

We here prove that the pushable chromatic number of square grids is either 5 or 6. We establish the following upper bound first.

Theorem 4.1. *For every square grid S, we have $\chi_{\mathrm{p}}(S) \leq 6$.*

Proof. Let $\overrightarrow{S}$ be an orientation of a square grid $S = S(n, m)$ with n rows and m columns. To establish the upper bound, we prove that $\overrightarrow{S}$ can have some vertices pushed, so that the resulting orientation $\overrightarrow{S}'$ of S is orientedly $\overrightarrow{T_6}$-colourable (where, recall, $\overrightarrow{T_6}$ is the oriented graph from Figure 1(c)).

By first pushing vertices in the second row (if needed), then pushing vertices in the third row (if needed), and so on row after row, it can be checked that, from $\overrightarrow{S}$, we can obtain an orientation $\overrightarrow{S}'$ of S such that every arc between a vertex (i, j) and a vertex $(i + 1, j)$ is oriented towards (i, j) if i is odd, and towards $(i + 1, j)$ otherwise. In other words, in such an orientation, the vertices in even rows dominate their neighbours in odd rows. Now, to obtain an oriented $\overrightarrow{T_6}$-colouring of $\overrightarrow{S}'$, it suffices to colour separately, in an oriented way, the oriented graph induced by the vertices from the even rows with colours $1, 2, 3$, and the oriented graph induced by the vertices from the odd rows with colours $4, 5, 6$. Note that this is possible by Theorem 2.1, since vertices $1, 2, 3$ in $\overrightarrow{T_6}$ induce $\overrightarrow{T_3}$, and similarly for vertices $4, 5, 6$, while the oriented graph induced by the vertices from the even rows (and similarly for those from the odd rows) of $\overrightarrow{S}'$ is an oriented linear forest. The resulting $\overrightarrow{T_6}$-colouring of $\overrightarrow{S}'$ is indeed an oriented $\overrightarrow{T_6}$-colouring, as, in $\overrightarrow{T_6}$, every arc between a vertex in $\{1, 2, 3\}$ and a vertex in $\{4, 5, 6\}$ is oriented towards the latter. □

We now turn to establishing our lower bound, 5, on the pushable chromatic number of square grids. First off, because hexagonal grids are subgraphs of square grids, Theorem 3.4

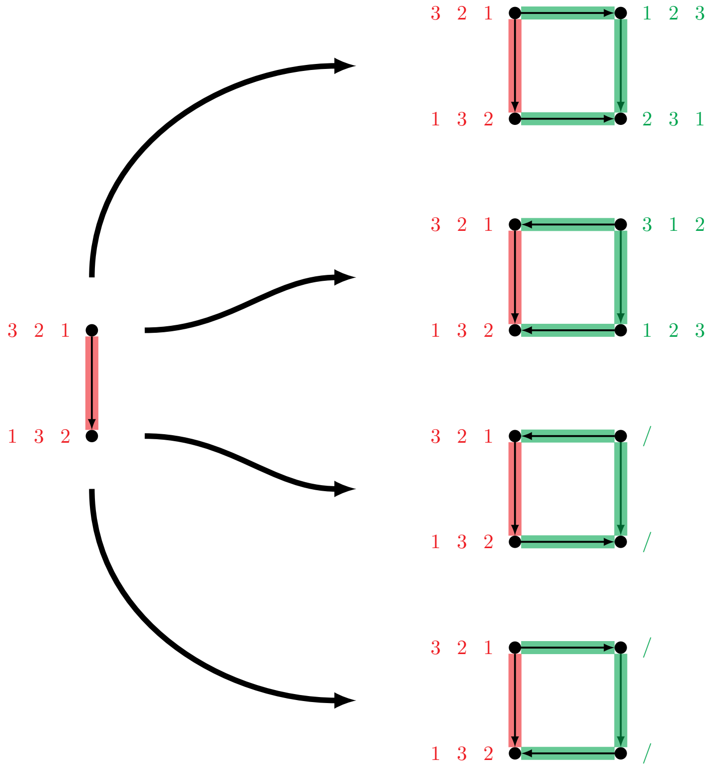


Figure 6: Illustration of the heuristic process we used to design oriented square grids that are not orientedly $\overrightarrow{H}$-colourable for some oriented graph $\overrightarrow{H}$. In the depicted situation, we construct such an oriented square grid with two rows, for $\overrightarrow{H} = \overrightarrow{T_3}$. We start (on the left) from one column and two rows oriented from top to bottom (in red), and we compute all possible ways to orientedly $\overrightarrow{H}$-colour it (in the figure, two colours on top of each other form such a valid pair of colours). We then extend (on the right) that oriented square grid to one with two rows and one more column, by adding the three arcs in green. The arc of this second column being oriented from top to bottom, the directions of two other arcs remain to be chosen, *i.e.*, those joining vertices in different columns. For each of the four ways to orient these two arcs, we look at the possibilities for all oriented $\overrightarrow{H}$-colourings of the first column to be extended to the second column. In the two first possibilities (the top ones), the three ways to orientedly $\overrightarrow{H}$-colour the first column lead to three compatible ways to colour the second column. In the two last possibilities (the bottom ones), the three ways to orientedly $\overrightarrow{H}$-colour the first column cannot be extended to an oriented $\overrightarrow{H}$-colouring of the second column. These resulting two oriented square grids are thus not orientedly $\overrightarrow{H}$-colourable.

implies that the pushable chromatic number of square grids is strictly more than 3. To verify that the pushable chromatic number of square grids is actually strictly more than 4, we proceeded following a computer-based approach. The general ideas are the following (see Figure 6 for an illustration).

To prove that square grids have pushable chromatic number strictly more than some k, one way to proceed is by considering every possible tournament $\overrightarrow{H}$ on k vertices, and by constructing an oriented square grid $S(\overrightarrow{H})$ that is not orientedly $R(\overrightarrow{H})$-colourable (where, recall, R refers to the anti-twin construction introduced right before Theorem 2.2). With such an $S(\overrightarrow{H})$ in hand for every $\overrightarrow{H}$, any oriented square grid containing a copy of all $S(\overrightarrow{H})$'s then has pushable chromatic number strictly more than k, by Theorem 2.2.

The question now, is how to design oriented square grids that are not orientedly $\overrightarrow{H}$-colourable for some fixed $\overrightarrow{H}$. To ease the search for such oriented square grids, we have designed the following algorithm. We start from an oriented square grid $G(n, 1)$ with n rows (for some fixed n) and only one column (*i.e.*, an oriented path on n consecutive vertices $(1, 1), \ldots, (n, 1)$), which we orient from "top" ($(1, 1)$) to "bottom" ($(n, 1)$). For this oriented square grid $G(n,1)$, we compute all possible ways $(\phi((1,1)),\dots,\phi((n,1)))$ to orientedly $\overrightarrow{H}$-colour its vertices (via a colouring ϕ), resulting in a set $\mathcal{P}_1$ of n-tuples.

We now extend this $G(n,1)$ to an oriented square grid $G(n,2)$ with n rows and two columns, by essentially starting from $G(n,1)$, adding the arcs of the second column oriented from top $((1,2))$ to bottom $((n,2))$, and choosing the worst possible orientation for the n arcs joining vertices of the first column and vertices of the second column. By "worst possible orientation", we mean in terms of possible ways to orientedly $\overrightarrow{H}$-colour the vertices of the second column, taking into account that vertices of the first column must be assigned colours forming an n-tuple from $\mathcal{P}_1$ by any oriented $\overrightarrow{H}$-colouring of $G(n,2)$. In other words, we look at all the ways to colour the vertices of the second column (these ways lie in $\mathcal{P}_1$, due to the orientation of the column from top to bottom), and any of them is valid is there exists one possible way to colour the vertices of the first column that is "compatible" with it. This yields ways $(\phi((1,2)),\dots,\phi((n,2)))$ to orientedly $\overrightarrow{H}$-colour the second column (via a colouring ϕ), which are plausible with respect to the first column, resulting in a set $\mathcal{P}_2 \subseteq \mathcal{P}_1$ of n-tuples. The oriented square grid $G(n,2)$ we keep to go on, is one minimising the corresponding $|\mathcal{P}_2|$.

We go on like this repeatedly, adding, to $G(n,m-1)$, a new mth column oriented from top to bottom, and considering the worst possible orientation of the arcs joining vertices from that new mth column and vertices from the previous $(m-1)$th one. That is, we look at all the ways to colour this mth column with n-tuples from $\mathcal{P}_1$, and keep those compatible with at least one n-tuple from $\mathcal{P}_{m-1}$, resulting in $\mathcal{P}_m$ (being a subset of $\mathcal{P}_1$). The orientation of the arcs joining vertices in the mth and $(m-1)$th columns we keep, is one for which $|\mathcal{P}_m|$ is minimum; this results in $G(n,m)$.

If at some point we reach a situation where $|\mathcal{P}_m| = 0$, then we essentially constructed an oriented square grid $G(n,m)$ on n rows and m columns that is not orientedly $\overrightarrow{H}$-colourable. This is because the sets $\mathcal{P}_1,\dots,\mathcal{P}_m$ essentially represent all possible ways to orientedly $\overrightarrow{H}$-colour the vertices, and the fact that $|\mathcal{P}_m| = 0$ means that there is no oriented $\overrightarrow{H}$-colouring of the first $m-1$ columns that complies with an oriented $\overrightarrow{H}$-colouring of the mth column.

Now, to prove, in conjunction with the approach above, that some oriented square grids have pushable chromatic number strictly more than 4, it suffices to provide, for every tournament $\overrightarrow{H}$ on 4 vertices, an oriented square grid that cannot have its vertices pushed so that the resulting oriented square grid admits an oriented $\overrightarrow{H}$-colouring. By Observation 2.2, we can actually drop the pushing consideration by considering the existence of an oriented $R(\overrightarrow{H})$-colouring instead. By Observation 2.4, we do not have to consider all tournaments $\overrightarrow{H}$ on 4 vertices, and can only consider one representative of each of the equivalence classes (for the pushing operation). There are only two such equivalence classes, as mentioned in Observation 2.5, having as representatives the tournaments depicted in Figure 7, (a) and (b). In (c) and (d), we also depict two oriented square grids obtained through our approach (through computer programs), which cannot be orientedly coloured by the anti-twin of (a) and (b), respectively. Thus we deduce that any oriented square grid containing these two, has pushable chromatic number strictly more than 4.

Theorem 4.2. *There exist oriented square grids with pushable chromatic number at least* 5.

5. Triangular grids

We now consider triangular grids, for which we prove the pushable chromatic number lies in between 7 and 12. We start by proving the upper bound.

Theorem 5.1. *For every triangular grid T, we have $\chi_{\mathrm{p}}(T) \leq 12$.*

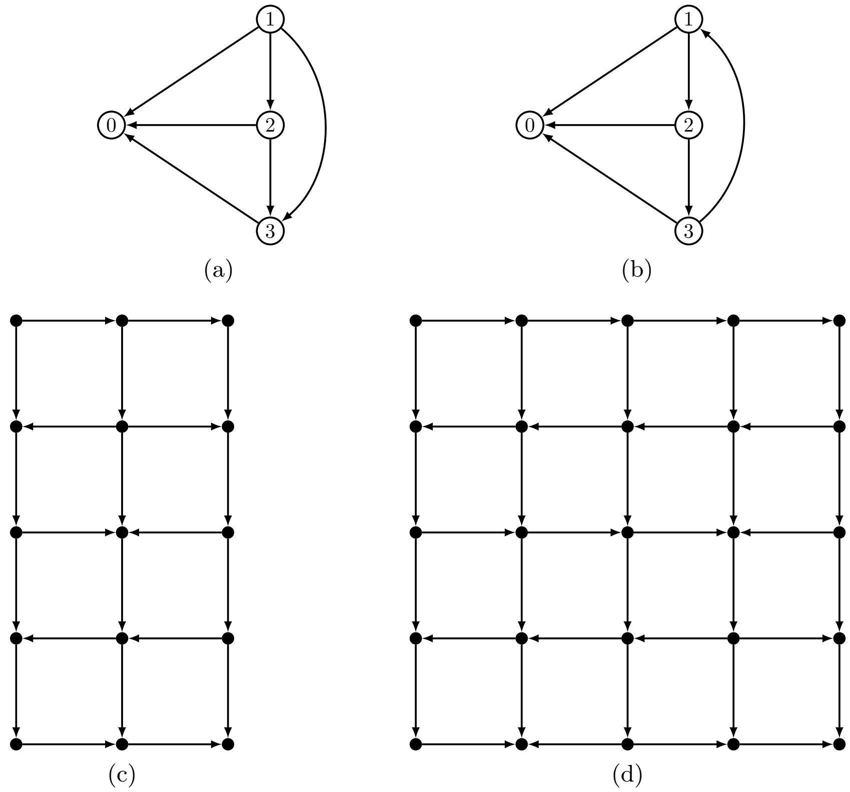


Figure 7: Representatives of the two equivalence classes (with respect to the pushing operation) for the tournaments on 4 vertices ((a) and (b)), and two oriented square grids ((c) and (d)) that cannot be orientedly coloured by them, even if some vertices can be pushed. (c) is the example for (a), while (d) is the example for (b).

Proof. Let $\overrightarrow{T}$ be an orientation of a triangular grid $T = T(n, m)$ with n rows and m columns. We get the result by proving that $\overrightarrow{T}$ admits an oriented $\overrightarrow{H}$-colouring ϕ, where $\overrightarrow{H}$ refers to $T(\overrightarrow{P_{11}})$, the Tromp graph of the Paley tournament on 11 vertices. Note that this indeed implies the claimed bound by Theorem 2.2, as $\overrightarrow{H}$ is the anti-twin of the oriented graph on 12 vertices obtained from $\overrightarrow{P_{11}}$ by adding a universal vertex being an out-neighbour of all other vertices. The main arguments we use rely on the fact that, by Theorem 2.6, $\overrightarrow{P_{11}}$ has Property $P_{2,2}$, and thus that, by Theorem 2.7, $T(\overrightarrow{P_{11}})$ (and thus $\overrightarrow{H}$) has Property $P_{3,2}$.

We build ϕ by $\overrightarrow{H}$-colouring the vertices of the rows of $\overrightarrow{T}$ one after another in order starting from the first one, and, whenever considering a new row, starting with the vertex of the mth column, then considering that of the $(m-1)$th column, and so on until the vertex of the first column is considered (resulting in all vertices of the row to be coloured). Note that, proceeding that way, whenever considering a new vertex (i, j), at most three of its neighbours, $(i, j+1)$, $(i-1, j)$ and $(i-1, j-1)$ (if they exist), have been considered earlier in the process, and thus assigned a colour by ϕ. An additional condition we will require (besides the partial colouring being oriented at any point), when colouring the vertices, is that every two vertices $(i-1, j-1)$ and $(i, j+1)$ at distance 2 are assigned distinct colours by ϕ. This is to guarantee that at least one colour will be assignable to (i, j), as, if we had $\phi((i-1, j-1)) = \phi((i, j+1))$ and the arcs $\overrightarrow{(i-1, j-1)(i, j)}$ and $\overrightarrow{(i, j)(i, j+1)}$, then it would be impossible to colour (i, j) in an oriented way (without pushing vertices).

To start with, we note that, $\overrightarrow{H}$ containing $\overrightarrow{T_3}$ as a subgraph, all vertices from the first row of $\overrightarrow{T}$ can be coloured in an oriented way, by Theorem 2.1. Now assume that, in the general case, the vertices from the first $i-1$ rows have been properly assigned a colour by

ϕ, and consider extending ϕ to the vertices of the ith row.

- Observe first that we can extend ϕ to (i, m), as, $\overrightarrow{H}$ having Property $P_{3,2}$, there are at least two colours that comply with those assigned to $(i-1, m)$ and $(i-1, m-1)$ (the only two possible coloured neighbours of (i, m) – which must be assigned distinct colours, since they are adjacent), and, among these two colours, we can choose one that is different from that assigned to $(i-1, m-2)$, as required.

- More generally speaking, assume we are considering vertex (i, j), where all vertices of the $(i-1)$th row have been properly coloured, and similarly for all vertices $(i, j+1), \dots, (i, m)$. As mentioned earlier, (i, j) has at most three coloured vertices in its neighbourhood. Also, among these at most three vertices, the colours of $(i, j+1)$ and $(i-1, j-1)$, if they exist, were chosen different by ϕ. So, the at most three colours by ϕ around (i, j) must be different. Due to $\overrightarrow{H}$ having Property $P_{3,2}$, there are at least two colours that comply with those around (i, j), and among these two colours, there is one that is different from that assigned to $(i-1, j-2)$ (if that vertex exists). We assign that colour by ϕ to (i, j), so that the process can go on.

Once all vertices are coloured, ϕ is an oriented $\overrightarrow{H}$-colouring. Thus, $\chi_{\mathrm{p}}(T) \leq 12$. □

Regarding lower bounds, recall that, triangular grids being supergraphs of square grids, Theorem 4.2 implies that some triangular grids have pushable chromatic number at least 5. Through essentially the same approach as that we used for square grids (described right before the statement of Theorem 4.2 – the only difference here being that one has to consider diagonal edges when adding columns in the generating algorithm), we were able to verify, through computer programs, that some triangular grids have pushable chromatic number at least 7. As reported in Observation 2.5, there are indeed, through the pushing operation, six equivalence classes of tournaments on 6 vertices. See Figure 8, (a) to (f), for representatives of these classes. For each of these representatives $\overrightarrow{H}$, our approach led us to an oriented triangular grid that is not orientedly $R(\overrightarrow{H})$-colourable. See Figure 8, (g) to (l), for these. Any oriented triangular grid containing these smaller six oriented triangular grids thus has pushable chromatic number at least 7.

Theorem 5.2. *There exist oriented triangular grids with pushable chromatic number at least* 7.

6. Discussion

A 2-*edge-coloured graph* (G, σ) is a graph G together with a *signature* $\sigma : E(G) \to \{-, +\}$ making every edge either *negative* (assigned sign $-$) or *positive* (assigned sign $+$). In some contexts, 2-edge-coloured graphs also come with a *switching operation*, where switching a vertex means changing the polarity of each of its incident edges (positive edges become negative, and *vice versa*). In the context of a 2-edge-coloured graph in which vertices can be switched, we use the term *signed graph* instead.

2-edge-coloured graphs and signed graphs are very similar to oriented graphs and pushable graphs, in that the former can be perceived as a static version of the latter (the pushing and switching operations being exclusive to the dynamic versions). Most of the aspects we have discussed in this work regarding oriented graphs and pushable graphs, actually have a counterpart in 2-edge-coloured graphs and signed graphs. Notably, the notion of equivalence classes for 2-edge-coloured graphs through the switching operation is reminiscent of that for oriented graphs through the pushing operation. There are also notions of colourings

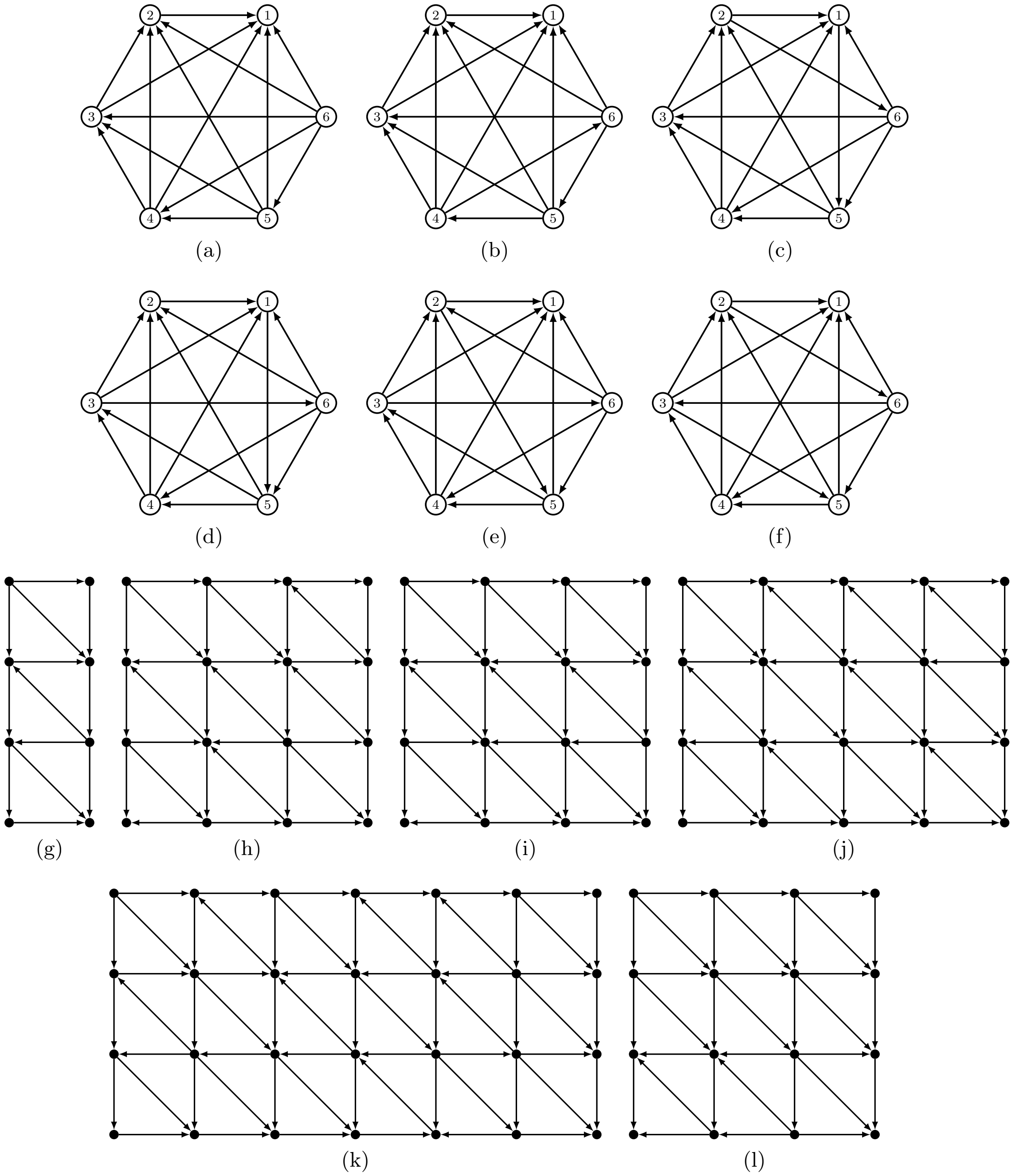


Figure 8: Representatives of the six equivalence classes (with respect to the pushing operation) for the tournaments on 6 vertices ((a) to (f)), and six oriented square grids ((g) to (l)) that cannot be orientedly coloured by them, even if some vertices can be pushed. (g) is the example for (a), (h) is the example for (b), etc.

and chromatic number, that arise in a quite similar fashion. Namely, given two 2-edge-coloured graphs (G, σ) and (H, π), a *signed* (H, π)*-colouring* $\phi : V((G, \sigma)) \to V((H, \pi))$ of (G, σ) assigns vertices of (H, π) as colours to the vertices of (G, σ), and it is additionally required that edges and their signs are preserved by ϕ, *i.e.*, $uv \in E((G, \sigma))$ if and only if $\phi(u)\phi(v) \in E((H, \pi))$, and $\sigma(uv) = \pi(\phi(u)\phi(v))$. The 2*-edge-coloured chromatic number* $\chi_{2\mathrm{ec}}((G, \sigma))$ of (G, σ) is then the order of the smallest (in terms of order)

	Hexagonal grids	**Square grids**	**Triangular grids**
Oriented	$\chi_o = 6$ [5, 19]	$8 \leq \chi_o \leq 11$ [9, 19]	$8 \leq \chi_o \leq 24$
Pushable	$\chi_p = 4$ **Thms 3.1** and **3.4**	$5 \leq \chi_p \leq 6$ **Thms 4.1** and **4.2**	$7 \leq \chi_p \leq 12$ **Thms 5.1** and **5.2**
2-edge-coloured	$4 \leq \chi_{2ec} \leq 8$	$8 \leq \chi_{2ec} \leq 9$ [1, 8]	$8 \leq \chi_{2ec} \leq 20$
Signed	$\chi_s = 4$ [14]	$5 \leq \chi_s \leq 6$ [10]	$6 \leq \chi_s \leq 10$ [14]

Table 1: Summary of all bounds known so far on the oriented, pushable, 2-edge-coloured and signed chromatic numbers of hexagonal, square and triangular grids. Cells in green are those for which the corresponding chromatic number was completely determined. Cells in yellow are those for which we only know bounds on the corresponding chromatic number, obtained through dedicated studies. Cells in red are those for which the corresponding chromatic number did not receive any dedicated attention to date, and for which the best bounds we know are thus deduced from other ones.

2-edge-coloured graph (H, π) such that signed (H, π)-colourings of (G, σ) exist. Taking the switching operation into account, the *signed chromatic number* $\chi_s((G, \sigma))$ of (G, σ) is the order of the smallest 2-edge-coloured graph (H, π) such that (G, σ) can have some of its vertices switched to reach a 2-edge-coloured graph that is signedly (H, π)-colourable. Similarly as for the oriented chromatic number and the pushable chromatic number, the 2-edge-coloured chromatic number and the signed chromatic number extend naturally to graphs and families of graphs, through considering the worst possible signatures. We refer the reader to [22], a survey on the topic, for more details.

Oriented graphs and 2-edge-coloured graphs (and similarly their dynamic counterparts, pushable graphs and signed graphs) might look quite similar, in that they share the same nature, being obtained from a usual graph by choosing one of two possible configurations (a direction or the other, or a sign or the other) for every edge. This is illustrated notably by the fact that very similar results can be found in both contexts; a perfect illustration, which is worth mentioning as it will be useful below, is the fact that the straight signed counterpart of Corollary 2.3 holds (see [21]): for every 2-edge-coloured graph (G, σ), we have $\chi_s((G, \sigma)) \leq \chi_{2ec}((G, \sigma)) \leq 2\chi_s((G, \sigma))$. An interesting line of research has actually been to investigate whether these types of objects differ significantly in some contexts. For instance, it is known that, for a given graph, its oriented chromatic number can be arbitrarily larger than its 2-edge-coloured chromatic number, and *vice versa* [3], though the two parameters can differ by a multiplicative factor being at most the chromatic number [20]. Actually, even in the context of square grids, it was already observed in [1] that these two parameters can actually differ for a given grid.

We have gathered, in Table 1, what is currently known on the four chromatic parameters of the types of grids we considered in this work. The cells in green contain chromatic parameters that have been fully determined. The cells in yellow contain chromatic parameters for which only bounds are known, which were obtained through dedicated studies. The cells in red contain chromatic parameters for which, to the best of our knowledge, no specific studies were lead to date. The bounds given in such cases follow from several relationships, between the parameters (through Corollary 2.3 and its signed counterpart), or between the types of grids (through the fact, here, that square grids are subgraphs of triangular grids). Recall that the oriented chromatic number and the 2-edge-coloured chromatic should be considered in parallel, as they hold as static versions of the two other, more dynamic chromatic parameters (the pushable chromatic number and the signed chromatic number). A few things are worth highlighting:

- Generally speaking, hexagonal grids, particularly due to their lower maximum degree, form, to date, the most understood type of grids, as their oriented, pushable and signed chromatic numbers were determined. Square grids are certainly the type of grids that received the most attention throughout the years, which resulted in the gaps between the known lower bounds and upper bounds being, most of the time, almost closed. Due to their larger maximum degree, triangular grids are definitely more complicated to deal with (with the current methods we have), and, consequently, less is known about the four chromatic parameters here.

- To the best of our knowledge, there are only three combinations of a chromatic parameter and a grid type, to which no specific study was dedicated. Two of these combinations involve triangular grids, while the last one involves hexagonal grids. In each of these cases, note that the chromatic parameter involved is a static one, while the dynamic version of that parameter was investigated for that type of grids (allowing to deduce bounds, notably from Corollary 2.3 and its signed counterpart). It is worth mentioning also that, although the grids we consider here have very low maximum degree, we do not seem to get better bounds from existing results on graphs with bounded maximum degree (such as [7]) or with particular properties (such as planarity).

- Regarding similarities and discrepancies between the "oriented world" and the "signed world", being categorical would be a bit daring at this point, as most of the four chromatic parameters are not fully understood for most types of grids. As mentioned earlier, we know best about hexagonal grids, and it seems that there could be no general differences between the oriented context and the signed context. To be certain about that fact, it would be necessary to investigate whether the 2-edge-coloured chromatic number of hexagonal grids is 6 or not. Regarding square grids, we note first that the bounds known for the pushable and signed chromatic numbers coincide, which could be an indication that, maybe, here as well these two contexts are pretty similar. Regarding the other two more static parameters, the situation seems to be clearer for the 2-edge-coloured chromatic number than it is for the oriented chromatic number. This last fact is actually surprising, as [12] was the very first work on this whole topic (back in 2003), and the upper bound of 11 from that work has never been improved upon since then. Lastly, regarding triangular grids, it is even harder to take things for sure, as our knowledge here is very partial. It would be necessary to investigate their oriented and 2-edge-coloured chromatic numbers to have more insight on this question. Regarding their pushable and signed chromatic numbers, we note that the best bounds we have at the moment do not bring any reason for expecting any significant difference between the two.

Regarding the possible directions that one could consider next to improve our knowledge from Table 1, apart from considering the cases for which only derived bounds are known, we believe that determining the pushable chromatic number of square grids would be the most interesting one. Towards this objective, we actually tried to push further the approach described right before Theorem 4.2, to show that grids have pushable chromatic 6. It turns out that, over all tournaments on 5 vertices, there are only two equivalence classes with respect to the pushing operation, two representatives of which are depicted in Figures 9, (a) and (b). Thus, to show that grids have pushable chromatic number 6, it would be sufficient, for each $\overrightarrow{H}$ of these two representatives, to come up with an example of an oriented square grid that admits no oriented $R(\overrightarrow{H})$-colouring. Our approach led us to construct one such

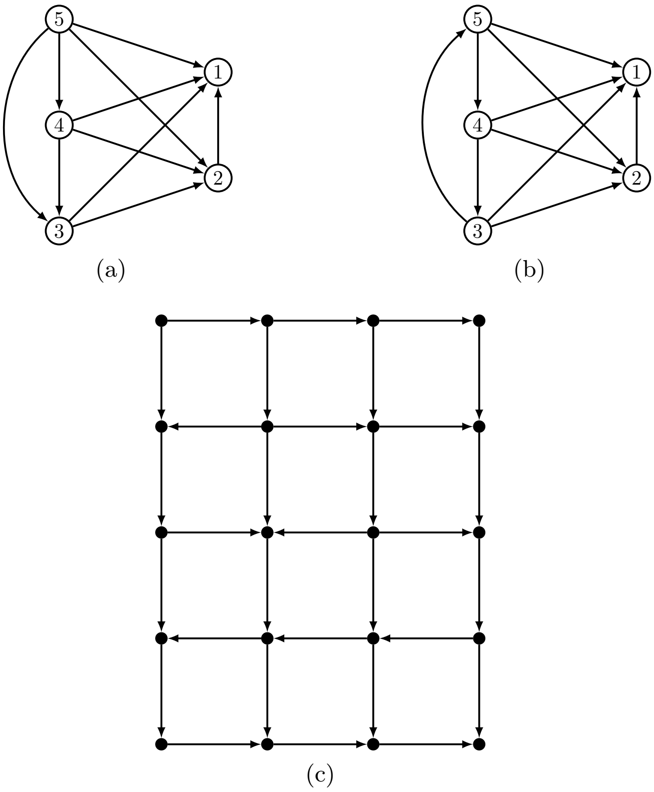


Figure 9: Representatives of the two equivalence classes (with respect to the pushing operation) for the tournaments on 5 vertices ((a) and (b)), and one oriented square grid ((c)) that cannot be orientedly coloured by (a), even if some vertices can be pushed.

oriented square grid for the representative in Figure 9(a), which we depict in (c). Thus, the question of whether grids have pushable chromatic number 5, reduces to the following:

Question 6.1. *Can every oriented square grid have some of its vertices pushed, so that the resulting oriented square grid admits an oriented $\overrightarrow{H}$-colouring, where $\overrightarrow{H}$ is the tournament in Figure 9(b)?*

Let us mention that our approach for constructing bad oriented square grids did not allow us to come up with a counterexample to Question 6.1, and, worse than that, it actually failed by far (in the sense that the tournament in Figure 9(b) is very resilient to our approach). From our experimentations, we would actually not be too surprised either, if Question 6.1 turned out to be answered positively. Let us mention, however, that a positive answer to Question 6.1 would, through Corollary 2.3, imply that all oriented square grids admit oriented $R(\overrightarrow{H})$-colourings (where $\overrightarrow{H}$ is the tournament in Figure 9(b)), and thus that the oriented chromatic number of square grids is at most 10, improving on the nearly 20-year-old upper bound from [12]. This would be quite surprising, as $R(\overrightarrow{H})$ is not a tournament.

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