New Generating and Counting Functions of Prime Numbers applied to Bertrand Theorem, Euler´s Product formula, Prime Numbers Products and Chebyschev 2 nd class function

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Abstract We present a set of novel prime numbers functions including a Generating Function and a Discriminating Function of Prime Numbers, neither requiring computer algorithms nor prime numbers tables and that are exact [1]. These two functions are instead defined in terms of ordinary elementary functions, therefore having the advantage of being readily computable, discrete differentiable and integrable, and hence applicable to any function that depend on integer numbers. Several relevant applications of these two new prime numbers functions are then presented, namely, (i) Obtaining a new and exact Prime Counting function, (ii) Applying Goldbach Conjecture to verify Bertrand Theorem on prime numbers, (iii) Readily evaluating the product of primes and composites numbers in any integer interval, (iv) Obtaining an accurate approximant to Euler´s Product formula, (v) obtaining an accurate approximant to the Chebyshev function of second kind, and finally introducing a new Primorial function defined for any integer.

Keywords: primes discriminating function, primes Generating Function, primes counting function, Goldbach conjecture, Bertrand theorem, prime numbers theorem, Euler´s product function, product of primes, Chebyshev function of second kind, primorial product, Primorial function

1 Introduction

Important efforts to obtain approximate mathematical formulae for functions of prime numbers persist to present days [1-7]. Thenceforth we here present a set of novel prime numbers functions [1] that are exact and computable as rightly demanded by Ribenboln [2], thus opening a door for new applications in the field of prime numbers based functions. Today some modern encryption systems – not the subject of the present work – exploit the Fundamental Theorem of Arithmetic" namely "a composite number can be expressed in one and only one way as a product of prime factors" [6], to codify information in terms of prime factors of large composite numbers. Prime numbers and their functions indeed belong to an advanced realm of intellectual conceptions, certainly to Analytical Number Theory [7-12], to Cryptography, and additionally they are related to some subfields of natural and formal sciences, where these numbers and their functions could play some role e.g. in discrete classical mechanics. A well-known statement of a prime numbers distribution is the *Bertrand Conjecture*: *There is at least one prime number between n and 2n, for all integer n*; a conjecture that became a theorem [5, 6] when later proved as such by Chebyschev. We have arranged the material of the present work as follows: In Section 2 we introduce our new prime function Ψ, whose purpose is to generate the exact prime numbers distributions in any integer interval. This primes generating function Ψ, being defined in terms of elementary functions, has the advantage of being exact and computable, as rightly demanded by Ribenboln [2] thus enabling us to apply both algebraic procedures, and all tools of discrete Differential and Integral Calculus in an altogether original way. Of course, this novel function enable us to operate within Discrete Classical Mechanics (see the section on Discussion and New Directions). Yet, the purpose of our prime generator Ψ is not to compete with algorithms on prime numbers, nor with tables of primes, objects that cannot be replaced into the analytic expression of any mathematical function, while our prime generator certainly can.

Our prime generating function Ψ is exact, and as such is validated here in finite integer intervals, and thoroughly tested in several ways, even by generating random prime numbers sets whose integer position in the infinite distribution of primes is randomly chosen using a sophisticated discrete differential algorithm, as presented in Section 2. In that section we also compare the performance of our function Ψ with of a commercial software algorithm. That comparison shows that our prime's generator is indeed exact. Our prime numbers generator Ψ is actually used in the construction of other emblematic functions of Number Theory, including an exact prime numbers Counting Gunction, also presented in Section 2. Section 3 is devoted to a verification of Bertrand Theorem, this time using our prime generator Ψ. To the effect we introduce there the so-called Goldbach Function – defined in terms of our function Ψ – a new function of primes that then allows us to verify Bertrand´s Theorem, and to find the probabilities of getting prime numbers in whatever integer intervals *(u,2u)*, as the initial integer *u* varies. In Section 4 we also present an accurate approximant to Euler's Product formula. This approximant is an auto-consistent relation that, being based on our prime functions, does not require the use of prime numbers tables or algorithms. Section 5 is devoted to obtain formulas for the products of prime and composite integers in any interval, using once again our prime generator function Ψ. In Section 6 we apply our prime discriminator Λ to obtain an accurate approximant to the Chebyshev function of second kind. Section 7 we devote to introduce a new primes *Primorial function* of integers that may advantageously replace the known *primorial product* of primes, the latter not being a function. In the final section we discuss the results obtained when applying our three prime functions to get approximants to relevant functions of prime numbers, and our applications of Bertrand Theorem to primes. We also announce two new lines of research that are now in progress in which our three new prime functions are applied to cases of discrete mechanics, particularly to the study of an object called the Goldbach particle.

2. Thrhe Novel and Exact Primes Functions

It is well-known that prime numbers have been studied since Euclid of Cyrene (350 BC), who showed us that there are infinite prime numbers, a case indeed of utmost relevance in Number Theory. Many of the known formulas derived thus far to obtain prime numbers are just versions of the famous *Sieve of Erathostenes* of Alexandría (ca. 280 BC) [6,-9]. In this section we introduce our new primes exact Discriminating Function Λ, our novel and exact primes Generating Function Ψ, and finally our primes Counting Function C. The primes generating Ψ might be considered to be just another version of the classical sieve, but one that shall be found instead to be analytic, not depending upon prime numbers tables or algorithms, thenceforth applicable in all mathematical instances of integer functions. For instance, our primes generating function allows you to explicitly calculate discreet derivatives and integrals (an example is shown in Fig. 3, Sub-section 2.3). To those effects we begin introducing below a set of required auxiliary functions: particularly, our prime numbers discriminating function.

2.1 A new Primes Discriminating function

We introduce a function that discern whether a given integer is or not prime with total exactitude. The definition of this prime discriminating function, denoted Λ below, is of course straightforward:

"$\Lambda(u) = 1$ if u is prime, while $\Lambda(u)=0$ if u is equal to 0, 1, or to any composite integer".

To obtain the mathematical expression of this new prime discriminating function we begin defining the following three simple functions of a positive real number u in terms of the well-known *floor*, or *integer part*, function $\lfloor u \rfloor$:

$$\Delta(u) = u - \lfloor u \rfloor, \tag{1}$$

$$h_1(u) = 1 + \left\lfloor \frac{u}{2} \right\rfloor. \tag{2}$$

$$h_2(u, m) = 1 + \left\lfloor \frac{(u + 2m - 1)}{2(2m - 1)} \right\rfloor \tag{3}$$

With these three functions we now construct the following three auxiliary functions Ω_i of u, and of the integer's m and n, in terms of the well-known function $\eta(u,u_0) = sign(u - u_0)$:

$$\Omega_0(u) = [\eta(|u|, 0)]^2 [\eta(|u|, 1)]^2 [1 - \eta(\Delta(u), 0)] \tag{4}$$

$$\Omega_1(u, m) = [\eta(|u|, 2m)]^2, \tag{5}$$

$$\Omega_2(u, m, n) = \{\eta[|u|, (2m - 1)(2n - 1)\ ]\}^2, \tag{6}$$

where $m > 2, n > 2$. With these three auxiliary functions we may now define our prime numbers discriminating function Λ as the double-product:

$$\Lambda(u) \equiv \Omega_0(u) \prod_{m=2}^{h_1(u)} \prod_{n=2}^{h_2(u,m)} [\Omega_1(u,m)\Omega_2(u,m,n)] \,. \tag{7}$$

This discriminating function Λ is exact and appears plotted below in Fig. 1, in the integer interval *[0, 101]*:

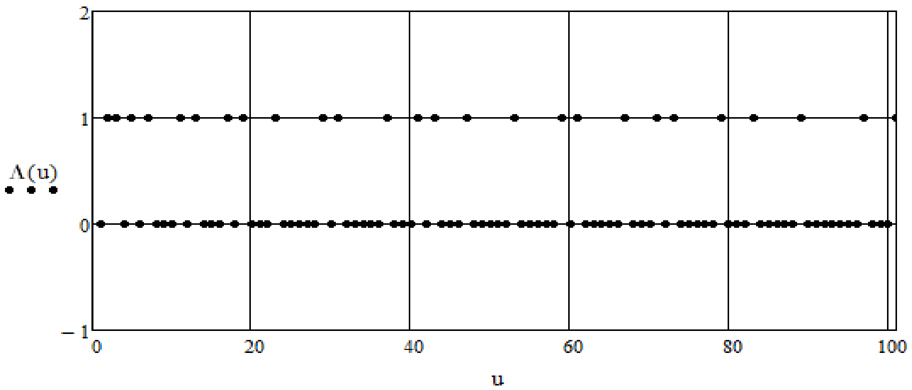


Fig 1 Plot of the prime discriminating function Λ in the integer interval *[1,101]*. The 26 primes in that domain appear plotted (dots) along the straight line $\Lambda(u)=1$, the remaining integers in that domain appear on the abscissa axis.

Recently R. Shumacher [3] presented his so-called *Prime Number Double Product Formula*, actually a function denoted χ whose domain is the set of integers *x*. It is a true function defined as a double product [3] and whose expression includes the numbers *e* and π. Our discriminating function in Eq. (7), defined also as a double product, is in this particular sense, analogous to Schumacher function, in what respect to its purpose (according to Schumacher it is a *characteristic function of primes*). Thus, both functions must give a *0* if *x* is a composite and the number *1* if *x* is a prime, as correctly done by our discriminating function Λ, and shown in its plot in Fig.1. However, Schumacher double product function χ, when computer calculated, returns a zero when the argument *x* is a composite, but when the argument is a prime Schumacher function does not return the number *1* but instead a complex numbers of real part very close to one and of very small imaginary part *e.g.*

$$\chi(13) = 0.9999999999974 - 0.00000000000092i;$$
$$\chi(101)=1.00000000000081- 0.00000000035047i;$$

while our discriminating Λ function correctly gives: $\Lambda(13)=1$; $\Lambda(101)=1$. Even calculated with 20 decimals $\chi(1009)$ renders a complex number of real part 1 and with a imaginary part of order $10^{-21}i$; while our discriminating function gives exactly $\Lambda(1009)=1$. (All these calculations were done using a well-known commercial software, on a 16-bytes word computer)

2.3 The new Primes Generating function

The distribution of prime numbers on the set of integer numbers is frequently assumed to be random and has attracted mathematician's attention for centuries [4, 8, 12]. Several polynomial relations to generate primes are known, and the first that ougth to be mentioned is the well-known Leonard Euler polynomial formula (1772): $n^2 - n + 41$ giving a list of prime numbers for $n \leq 40$. Different prime numbers generating functions have been proposed since those Euler's days, some of them being recurrence formulae *e.g*. the 2008 Rowland´s formula [11] and the Shumacher one mentioned above [3].

We now proceed to formally define our new Prime Number Generating function Ψ in terms of our prime numbers discriminating function Λ in Eq. (7). As said above, our prime's generator is written in terms of elementary functions plus the *sign* function (yet, if required the *sign* function itself may be here replaced by one approximant, defined in terms of elementary functions too, as presented and shown in our previous work [14]) . This prime generator Ψ can be used either to generate all prime numbers in a given range of integers, or actually to tell us whether any integer *u*, is or not prime. Our prime`s generator function Ψ we define, in terms of our primes discriminating function Λ, as:

$$\Psi(u) \equiv u\Lambda(u) \tag{8}$$

whose first and second order *discrete* derivatives are, respectively:

$$\Psi_{d_1}(u) = \Psi(u+1) - \Psi(u) \tag{9}$$

$$\Psi_{d_2}(u) = \Psi(\mathrm{u}+2) - 2\Psi(\mathrm{u}+1) + \Psi(\mathrm{u}) \tag{10}$$

Three particular examples, illustrating, the application of our prime function Ψ, for three large integers are (actually using a prime number between two composites) are:

$$\Psi(1208777 - 2) = 0{\times}10^0;\ \Psi(1208777) = 1.208777{\times}10^6;$$

$$\Psi(120877 + 2) = 0{\times}10^0$$

Figure 3 shows a plot of this prime`s number generating function Ψ for integers *u* in the interval *[1,100]:* the dots along the inclined line in the plot corresponding of course to the prime numbers in that interval, while the zeroes along the abscissa axis correspond of course to composite integers in that interval.

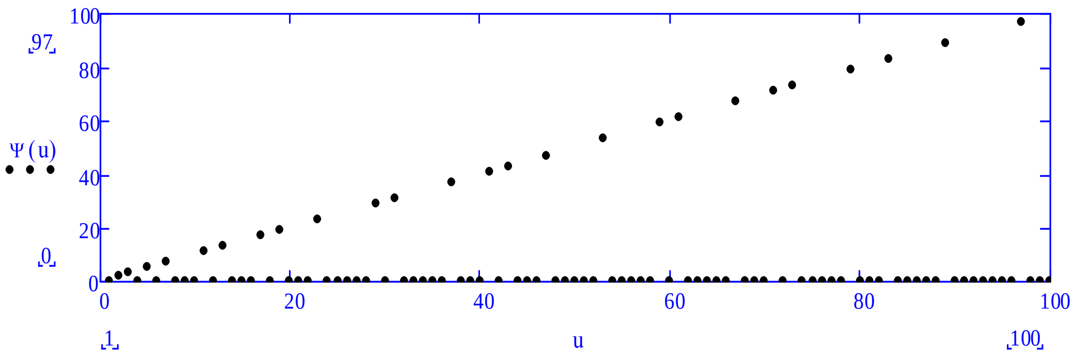


Fig. 2(a) Plot of the prime numbers in the integer interval (0, 100) given by our prime number generating function Ψ: as expected there are 25 dots lying on the inclined line.

As a second example we show in Fig. 2 (b) the plot of the prime numbers in the finite integer interval *(1000000, 1000100)*, such primes correctly determined using our prime generating function Ψ: this plot shows that there are only six prime numbers in that interval, as well as 94 dots along the abscissa axis, the.zeroes that correspond to the composites integers in the interval.

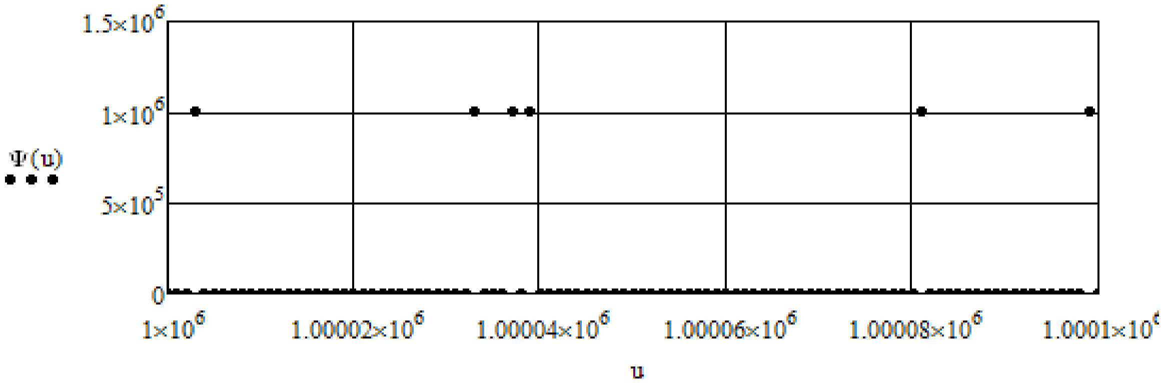


Fig. 2 (b) Plot of the six prime numbers that belong in the integer interval *(1000000, 1000100)* as given by the prime generating function Ψ. The dots along the abscissa axis correspond to the composite integers in that interval.

Related to the previous plot of prime numbers in Fig. 2(b), we now compare the fidelity of our computable prime generator with that of a prime algorithm of a commercial software package. For the comparison we choose again the same integer interval of in Fig. 2(b). Thus the first column in Table 1 (below) lists the ordinal integer *i–th* of the six primes in that interval, starting at $i=78499$. The 2nd Column of this table lists the corresponding six primes P_i given by a commercial software algorithm. For comparison the third column list the six primes already found with our prime numbers generator Ψ and shown in Fig. *2(b)*. It may be seen that our function does produce the same results given by the commercial software algorithm.

TABLE 1

$i\text{–}th$	P_i	$\Psi(P_i)$
78499	10000003	10000003
78500	10000033	10000033
78501	10000037	10000037
78502	10000039	10000039
78503	10000081	10000081
78503	10000089	10000089

Table 2 below is a more interesting test of the quality of our function Ψ as a prime number generator. In this second table we consider the same integer domain of Figure 2 (b) where our function Ψ found six primes. The test consists in checking whether our function would find: Firstly any prime numbers either *two integers below or two above each of the six primes*, and secondly *one order below or above*, respectively. The second column of the table shows that a single finding occurs when the ordinal is diminished by two**:** 1 000 037 is in effect the one prime within two ordinals below 1 000 039 (see Table 1). The third column shows again a single prime above 1000037. Thus we got a case of twin primes separated by an even integer (1000038). The 4rd. and 5th. column, each of six zeroes, shows that our function Ψ, as expected, *do not generate consecutive primes for each of the six primes* in the reference interval.

Table 2

i	$\Psi[P_i\text{-}2]$	$\Psi[P_i+2\ ]$	$\Psi[\ P_i\text{-}1]$	$\Psi[P_i+1]$
78499	*0*	*0*	*0*	*0*
78500	*0*	*0*	*0*	*0*
78501	*0*	*1000039*	*0*	*0*
78502	*1000037*	*0*	*0*	*0*
78503	*0*	*0*	*0*	*0*
78504	*0*	*0*	*0*	*0*

To further demonstrate, in a strking way, the quality and reliability of our prime number generating function ψ we present Table 3 below. It is a table that shows the results of applying our prime generator function to **64** *randomly* generated integers u_n, $n\in[1,64]$, that randomly happened to fall in the integer interval $[13, 2591491]$. These 64 random integers u_n were generated using a sophisticated *computer-programmed discrete differential random formalism* (printed below Table 3).This program uses as input a pair of integers parameter $(K_i\ \mathbf{,}\ u_i)$ to generate four sets of 16 random integers. Table 3 is thus divided in four **sets of 16** integers, each showing the expected correct result $\psi(u_n)=u_n$ – given by our generator function Ψ– whenever u_n is prime, along the 3rd. column of each set, otherwise giving the expected zeroes in that column. For instance: for the *16th* run of the program our random formalism gave us the integer $u_{n=}1270197$ which is not prime (in effect, it factorizes as $1319 \times 107 \times 9$) and you find the expected $\psi(1270197)=0$ in the 3rd. column of the first set, of the left column of Table 3. This gives strong evidence of a finite but completely random distribution of primes, in a finite integer interval, which is successfully predicted by our prime numbers generator function Ψ.

TABLE 3

n =	u_n =	$\Psi(u_n)$ =	n =	u_n =	$\Psi(u_n)$ =
1	35	0	17	13	13
2	71	71	18	27	0
3	143	0	19	57	0
4	289	0	20	115	0
5	583	0	21	241	241
6	1169	0	22	491	491
7	2351	2351	23	1023	0
8	4735	0	24	2139	0
9	9565	0	25	4311	0
10	19173	0	26	9071	0
11	38597	0	27	18729	0
12	77915	0	28	38455	0
13	156137	0	29	81473	0
14	312863	312863	30	168543	0
15	631507	631507	31	347025	0
16	1270197	0	32	730789	730789

n =	u_n =	$\Psi(u_n)$ =	n =	u_n =	$\Psi(u_n)$ =
33	73	73	49	9	0
34	147	0	50	19	19
35	297	0	51	39	0
36	597	0	52	81	0
37	1195	0	53	173	173
38	2403	0	54	361	0
39	4831	4831	55	765	0
40	9745	0	56	1661	0
41	19619	0	57	3529	3529
42	39571	0	58	7457	7457
43	79229	79229	59	15189	0
44	158961	0	60	32065	0
45	318841	318841	61	65687	65687
46	638585	0	62	139319	0
47	1287937	0	63	294927	0
48	2591491	0	64	619019	619019

2.3.1 Integer Numbers Generator Program based on a Random Differential Discrete formalism (used to construct Table 3).

Table 3 was generated by applying the program listed below this paragraph in the integer intervals *[1, 16]*, *[17,32]*, *[33,48]* and *[49,64]*. For each of these four intervals the program first generates a pair of random integers K and u (shown in the table as u_1, u_{17}, u_{33} and u_{49}).

$$\boldsymbol{n = 1, 2, 3, \ldots 16}$$

$$\boldsymbol{K = 1 + trunc[rnd(100)]}$$

$$u_1 = 2\, trunc\left[rnd\left(\frac{K}{2}\right)\right] + 1$$

$$u_{n+1} = \left\{2\left[u_n + trunc\left(rnd\left(\frac{u_n}{K}\right)\right)\right] + 1\right\}$$

Another relevant and key advantage of our prime number functions presented above – both the Discriminator Λ and the Generator Ψ – is the possibility of applying them to study discrete dynamical systems whose motion law is determined by a distribution of prime numbers. In Fig. 3 we show a plot of the first discrete derivative of our analytic prime number generating function Ψ in the integer interval *[1,100]*

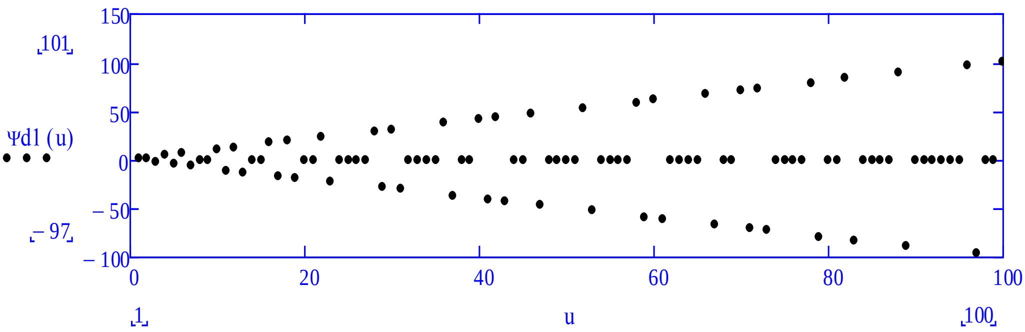


Fig. 3 Plot of the first discrete derivative of our prime numbers generating function Ψ in the integer domain *[1,100]*.

This discrete plot confirms that our generating function Ψ will allow obtaining the exact discrete derivatives associated to any distribution of prime numbers. It thus becomes possible to study any discrete dynamic mechanics model, even after defining either the Lagrangian or the Action of such discrete models, and evaluating these two classical functions by simply inserting in their expressions our prime generator when required. It should also become particularly useful if you need to locally analyze sets of primes in finite intervals, as in the case of the Bertrand Theorem [15,16], already mentioned in Section 3, and its applications [17] (see Section 3)

We must emphasize again to our would-be readers that the main aim of our computable prime numbers generating function *is neither to compete with algorithms* that calculate prime numbers *nor with tables of primes*. Such algorithms and tables are not even analytic and cannot be placed, for instance, into the analytic expression of a physics formula *e.g* into Wien displacement. Moreover, we must also emphasize that our prime number generating function Ψ can be used to generate all the prime numbers in a given integer interval in *strictly ordered succession*, giving a zero each time the integer *u* in the domain is not a prime. This is a true novelty that we have verified up to $u_{max} = 10^7$ using a modest computer. Readers at large may easily test our function Ψ on larger integer domains using large frame computers.

2.4 A new Prime Numbers Counting Function and the Prime Number Theorem

For the last three centuries one of the desideratum of Number Theory [5, 6, 8] has been to obtain a relation that would give us the number of primes equal or less than any integer *x*, *i.e.* the ideal counting function *Π(x)* that will return the exact number of prime numbers not exceeding a given *x*. In 1763 K. F. Gauss, after analyzing cases of distribution of the primes up to *large integers x*, conjectured his simple Primes Count estimate *x/ln x* in such distributions, so that we may write:

$$\Pi(x) \approx \frac{x}{\ln x} \tag{11}$$

Years later Gauss once again conjectured [4, 5,10] that: for large integers *x* the number of primes *Π(x)* is also approximately given by the following function defined as a *logarithm-integral*, and known as the *Li(x)* function*:*

$$Li(x) \equiv \int_2^x \frac{dt}{\ln t} \approx \Pi(x) \tag{12}$$

From Eq. (12), one may therefore write:

$$lim_{x\to\infty}\left[\frac{\Pi(x)}{\left(\frac{x}{\ln x}\right)}\right] = 1 \tag{13}$$

a relation known as the *Prime Number Theorem* later proved independently by C-J de la Vallée Poussain and J. Hadamard (*ca.*1900) [5,6,10]. The *first-order* asymptotic Li_a approximation to the *Li* function thenceforth being the first Gauss conjecture mentioned above *i.e.*:

$$Li_a(x) = \frac{x}{\ln x} \tag{14}$$

2.4.1. The new Prime Numbers Counting function

Our discriminating function *Λ*, presented in Sub-section 2.1, allows us to define a new prime counting function $C(u,u_{in})$ which should give us the exact number of primes in any integer interval $[u_{in}, u]$ starting at any initial u_{in}:

$$C(u, u_{in}) \equiv \sum_{j=u_{in}}^{u} \Lambda(j) \ . \tag{15}$$

With this new function $C(u,u_{in})$ we may readily find the number of primes in any interval, *e.g.* for the interval *(1000, 2)* we get the correct result *C(1000,2)=168*. In Fig. 4 we have plotted this *exact* prime counting function *C* (the stair-case line) in the integer interval $u \in [1,102]$. For comparison, the figure also shows the number of primes given by the first order asymptotic Gauss prime numbers counting function $Li_a(x) = x/\ln(x)$ (the dotted line). As a simple comparison, the relative errors of Gauss approximate counting function and of our primes counting function for *x =10000* are, respectively:

$$\frac{\frac{10000}{\ln 10000}-168}{168} = -0.1383, \qquad \text{and} \qquad \frac{C(10000,2)-168}{168} = 0 \tag{16}$$

Note that in Fig. 4 we have also plotted Gauss function Li_G. This figure illustrates that both Gauss Li_G function (dotted curve) and its first-order asymptotic expansion Li_a are just approximations to the exact number of primes not exceeding a known prime value, as given by our exact prime number counting function (the staircase line): ours being therefore an optimum prime counting function.

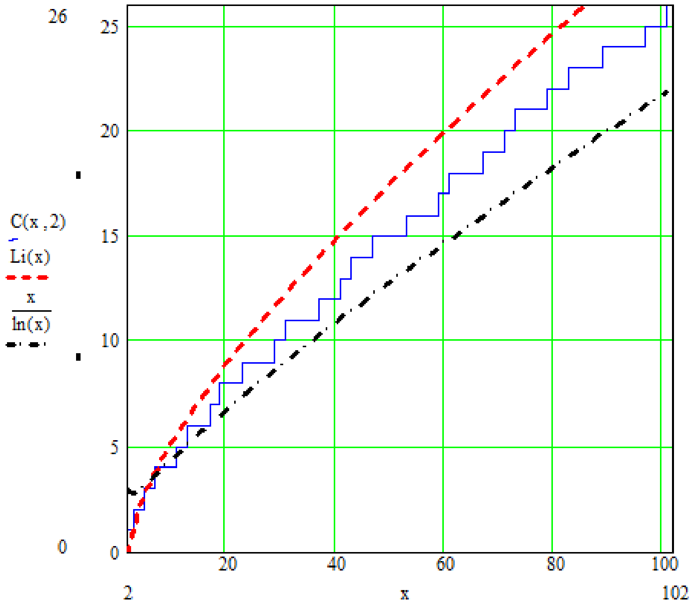


Fig. 4 Plot of the prime counting function *C(x,2)* in the interval *[1,100]* (*stepped line*). It may be seen that for *u=100* our counting function gives the exact result *C(100,2)=25*. For comparison, also plotted appears the Gauss function *Li* (*dash curve*) and the first order asymptotic approximation *x/ln(x)* (*dash--dot curve*). One finds that *Li(100)=21.27* while its asymptotic approximation gives $Li_a(100) \equiv x/\ln(x) = 29.08$.

Using a modest personal computer (64 bits) and our prime counting function one may easily find that the number of primes in say the interval $(2, 10^5)$ is $C(10^5,2)=9592$, which is the correct result. Another comparison between the three functions is observed in Fig. 4: Gauss integral approximant gives $Li(10^5)= 9629.62$ (with *relative error* $\sim 10^{-3}$) for that interval, showing it to be of acceptable accuracy. The first-order asymptotic approximation *Eq.(15)*, gives instead $Li_a\ (10^5)= 8685.89$, being thus a poorer approximant to our prime counting function *C(x,2)*, and of course to the ideal counter $\pi(x,2)$.

3. Primes Generating Function Ψ applied to verify Bertrand Theorem

As already said in the Introduction the Bertrand theorem was first proved by Chebyshev (1855) [5]. It refers to the *Bertrand Conjecture* on the existence of primes [15-17] in some particular integer intervals namely: *There is at least one prime number between u and 2u for all* $u \in \mathbb{N}$. Using our primes discriminating function Λ it is straightforward to find the finite number of primes C_p in any of such *Bertrand integer* *intervals (u,2u)*. In effect, our function simply allows us to write the novel prime function:

$$C_p(u) = \sum_u^{2u} \Lambda(k). \tag{17}$$

Two examples, for u=*9* and for the much larger integer u=*10000* being:

(i) u= *9*: $\Psi \Rightarrow C_p(9) = 3$, the counted prime numbers being $\{11, 13, 17\}$,
(ii) and the two-orders larger count $C_p(10000) = 1033$.

Analogously the number of composite integers in the same interval is given by:

$$C_c(u) = \sum_u^{2u}[1 - \Lambda(k)], \tag{18}$$

a relation which gives us:

(i) $C_c(9) = 7$, the composites being {9,10,12,14,15,16,18},
(ii) $C_c(10000) = 8968$

That is there are 1033 primes and 8968 composites between 10000 and 20000.

It becomes interesting to plot in Fig. 5 these two Bertrand counting functions C_p and C_c of integers (and in addition the sum C_p+ C_c). Note in the plot that both counts increase with the final integer u, yet not monotonically.

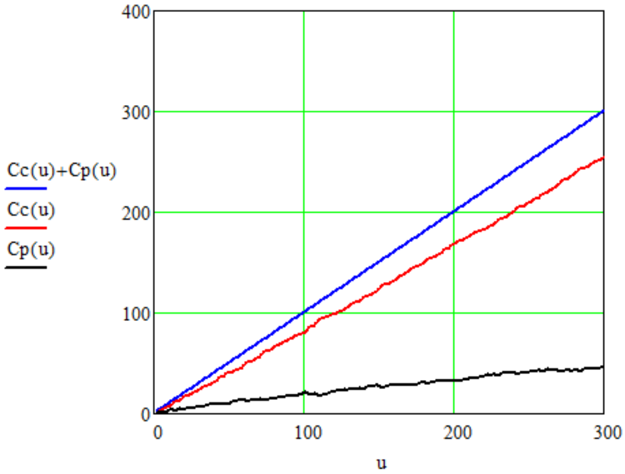


Fig. 5 Plot of the Bertrand count functions C_p (brown line) and C_c (red line) and their sum C_p +C_c (blue line) in the integer interval *(2,300).*

3.1 Bertrand Probabilities for Primes and Composites

It would be interesting to calculate what we call the *Bertrand probabilities* P_p, P_c of finding primes and composites in any *Bertrand integer* interval (*u,2u)*, respectively. By definition we have that the probability P_p for finding prime numbers in that interval (*u,2u)* is:

$$P_p(u) = \frac{C_p(u)}{C_p(u)+C_c(u)}\,, \tag{19}$$

while the probability for composites is analogously defined:

$$P_c(u) = \frac{C_c(u)}{C_p(u)+C_c(u)}. \tag{20}$$

Thus for instance the probability of finding a prime between *100* and *200* is:

$$P_p(100) = 0.207920792079208,$$

while the probability for composites is:

$$P_c(100) = 0.792079207920792,$$

and of course we get: $P_p(100) + P_c(100) = 1.000000000$.

In the following Figure 6 we plot the two probabilities P_p and P_c , of finding either prime or composites in the interval *(u,2u)* for *u* in the interval *(2,300).*

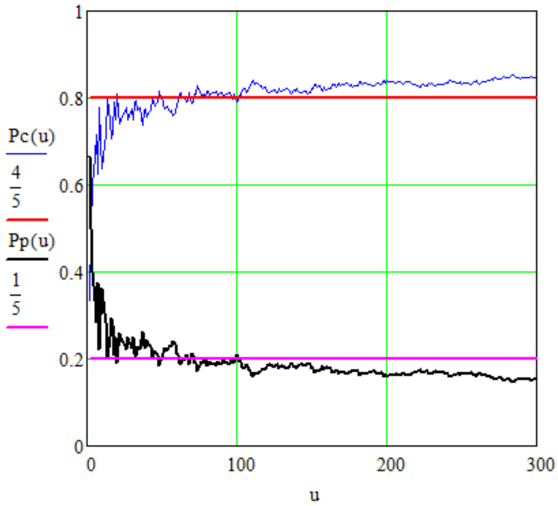


Fig. 6 Plot of the Bertrand probabilities $P_{p,}\, P_c$ for *u* in the interval *(2,300)*. It shows that the probability P_p for finding primes in the interval *(u,2u)* decreases below *1/5* as *u* grows large.

We may finally plot the quotient of the two probabilities $P_{p,}\, P_c$. As shown in the following figure this quotient appears to decrease below ¼ as u → ∞:

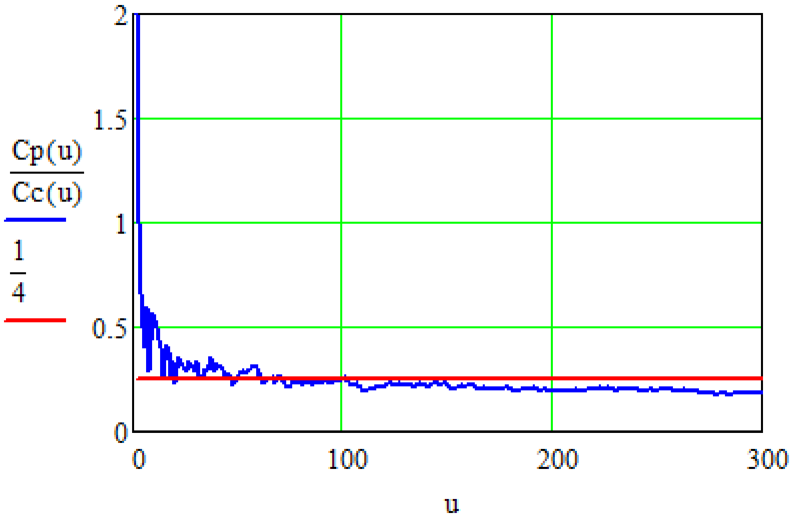


Fig. 7 Plot of the quotient $C_{p,}/C_c$ of the two Bertrand probabilities in the interval (2,300) showing its decreasing value as *u* gets large.

3.2. Goldbach Conjecture and the Bertrand Theorem

We now define the following discrete function *G(λ;b)* where *b* is a free integer parameter, *λ* an integer variable in the interval *λ∈(0, b-2)*, and *Ψ* our prime generator:

$$G(\lambda;b) \equiv \psi(b-\lambda)+\psi(b+\lambda)-2b, \tag{21}$$

a novel prime function that we ave baptized the *Goldbach function* because: *If G(λ;b)=0 the even number 2b is equal to the sum of the two prime numbers* $\psi(b-\lambda)$ *and* $\psi(b+\lambda)$, a relation that simply expresses the well-known *Goldbach Conjecture* [5, 6,12]. This is a novel form of writing this conjecture, one that shall be shown below to be useful. Let us first show the discrete function *G(λ;b)* for the parameter value *b=100* and *λ ∈(0,98)*. Thus, our function *G* appears as a discrete plot in Fig. 8 for *b=100*: For instance the third point (blue dot) corresponds to *λ=39*. It therefore represents the even number *200* written as the sum of the prime numbers *139 and 61*.

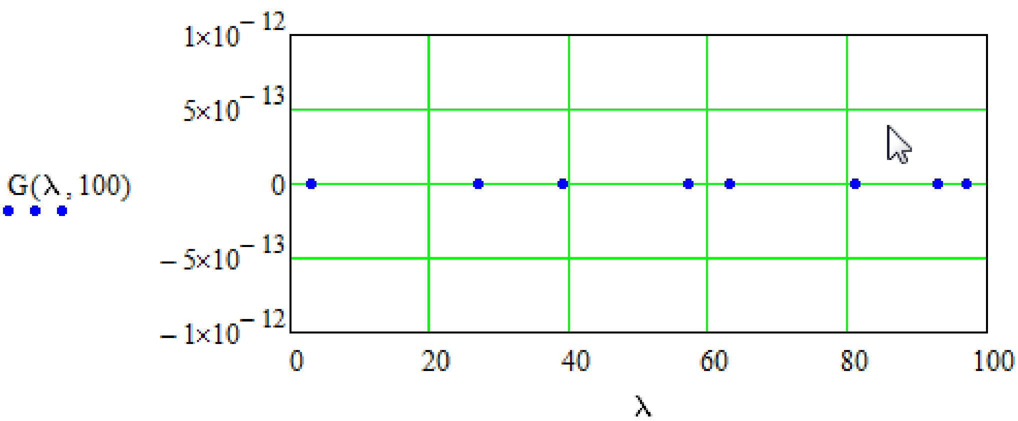


Fig. 8 Plot of the discrete Goldbach function $G(\lambda;b)$ for the parameter value $b= 100$. The function gives eight points (blue dots) that represent eight ways of obtaining the even number *200* as a sum of two odd integers (see the text)

Bertrand Theorem can now be verified as a direct consequence of the Goldbach Conjecture. In effect in the following Figure 9 (a) we have plotted the prime numbers $\psi(b-\lambda)$ and $\psi(b+\lambda)$, for the integer b*=100*, and λ in the interval λ*=0, 1,…98.* The blue dots represent the 22 prime numbers in the Bertrand interval *[100, 200]*. The red dots in the plot of Figure 9(a) represent the 25 primes in the interval *[2,100]*.The sum of the ordinates of those blue and red points *of equal abscissae* λ *(e.g. for* λ*=39)*, is equal to *2b=200*, as expected from the Goldbach conjecture [18].

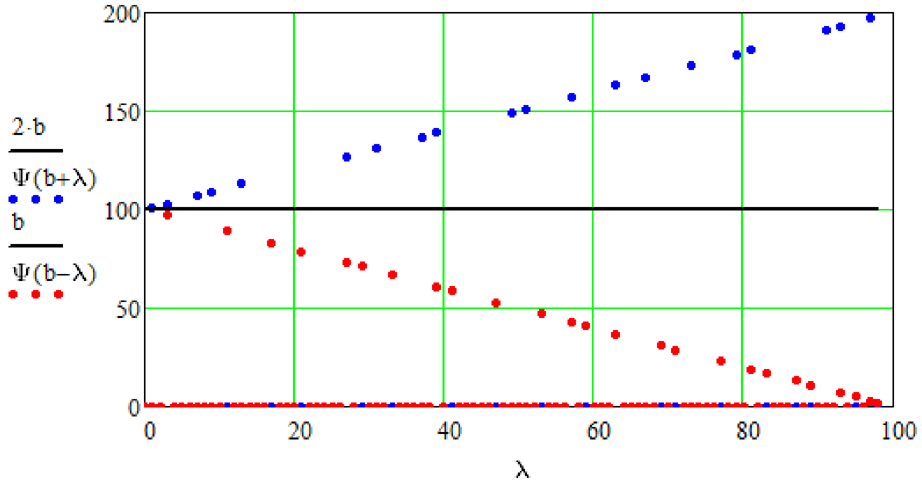


Fig. 9(a) Functions $\psi(b-\lambda)$ and $\psi(b+\lambda)$ *vs.* the integer variable λ for b*=100*. For the abscissa λ*= 67*: the coordinates of the corresponding blue and red dots are: *(67,37)* and *(67,163)* respectively, as expected form Goldbach Conjecture. The blue dots represent 22 prime numbers in the Bertrand interval *(100, 200)*.

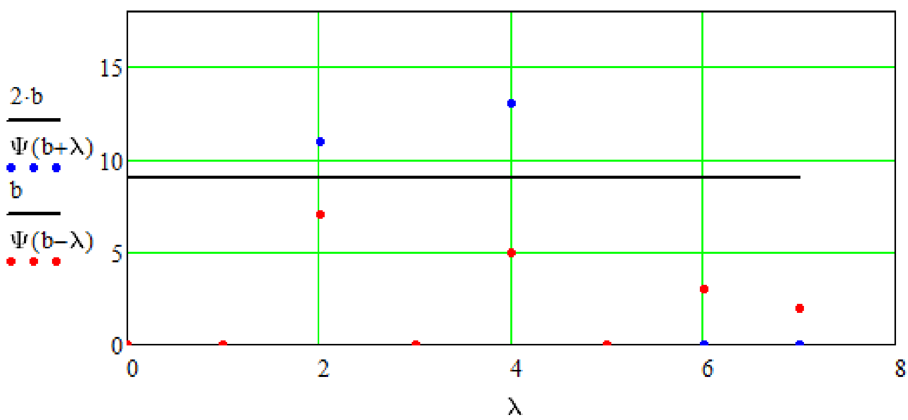


Fig. 9(b) Functions $\psi(b-\lambda)$ and $\psi(b+\lambda)$ *vs.* the integer variable λ for b*=9.* The plot shows the two primes 11 and 13 in the interval *(9, 18).* Red dots represent the four primes in the interval *(2,9).*The prime 17 does not appear plotted since the sum 17+3 ≠ 2(9) i.e. these two primes do not satisfy Golbach´s Conjecture (their sum is not *2b*) .

4. Euler`s Prime Numbers formula: an approximation

Let us begin this section recalling Euler's remarkable real function (*ca.*1734) $\xi(\sigma)$, defined [5,6,10] for arguments $\sigma \in \mathbb{N}$,:

$$\xi(\sigma) = \sum_{n=1}^{\infty}\left(\frac{1}{n^{\sigma}}\right), \tag{22}$$

which for the integers $\sigma = 2, 4$ and 6 converges to the three notable values [19]:

$$\xi(2) = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}; \quad \xi(4) = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots = \frac{\pi^4}{90};$$

$$\xi(6) = \frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\cdots = \frac{\pi^6}{945}$$

By invoking the Fundamental Theorem of Arithmetic [6] Euler later derived another formula for its real function ξ, but now remarkably *derived in terms of prime numbers p, and as an infinite product* [5,10], instead of the infinite sum in Eq. (22):

$$\xi(\sigma) = \prod_{p=2}^{\infty}\left(\frac{p^{\sigma}}{p^{\sigma}-1}\right), \quad \sigma > 1 \tag{23}$$

Thus after incorporating this Euler's product formula into Eq. (18) we may write:

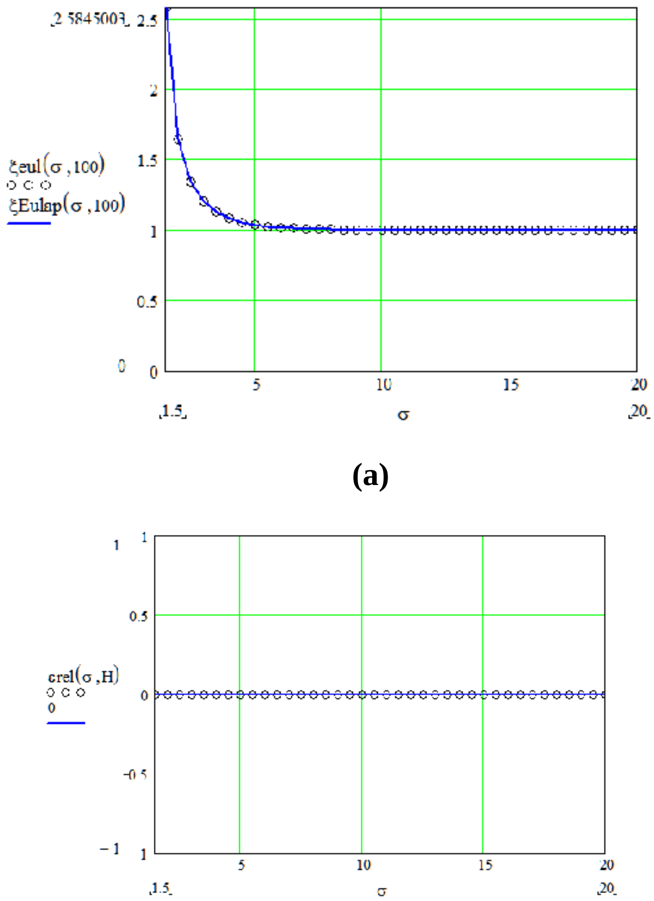
$$\xi(\sigma) \equiv \sum_{n=1}^{\infty}\left(\frac{1}{n^{\sigma}}\right) = \prod_{p}\left(\frac{p^{\sigma}}{p^{\sigma}-1}\right), \sigma > 1 \tag{24a}$$

which we may evaluate only up to a finite upper prime number limit $q_{max}(H) = P(H)$, the latter being the *H-th* prime number, to get approximated values ξ_{eul} using the following Eq. 24 (b)

$$\xi_{eul}(\sigma) \equiv \sum_{n=1}^{q_{max}(H)}\left(\frac{1}{n^{\sigma}}\right) = \prod_{j=1}^{H}\left(\frac{P(j)^{\sigma}}{P(j)^{\sigma}-1}\right), \quad \sigma > 1 \tag{24b}$$

Below we introduce our novel and original approximant to Euler´s product function Eq (23), in terms of our analytic Prime Number Generating function Ψ and using a finite upper limit in the product, instead of the infinite limit in Eq (23). We name this new approximant as the *self-consistent approximated Euler product* formula ξ_{Eulap}, and define it as the product from *2* to the $q_{max}(H)$*'th* prime number:

$$\xi_{Eulap}(\sigma, H) = \prod_{q=2}^{q_{max}(H)}\left\{\left[\frac{\Psi(q)^{\sigma}+\left[\frac{\Psi(q)}{q}-1\right]}{\Psi(q)^{\sigma}-1}\right]\right\}. \tag{25}$$



(a)

(b)

Fig. 10 (a) Coincident plots of Euler's product exact function ξ_{ex} and our approximant ξ_{euapp} as functions of the real number σ; **(b)** Plot of the relative error of our approximant to Euler's product formula (two plots for the upper bound *H=100* of the approximant)

In Eq. (25) the integer index *q* in the product runs from 2 up to *P(H)*. Our Eq. (25) is thus a self-consistent expression that does not depend upon tables or any external algorithm, and as shown below it is a very accurate approximant. For instance the $H = 100^{th}$ prime is $q_{max}(100)=541$ and for this value our approximant *Eq.(21)* gives:

$$\xi_{Eulap}(2{,}100) = 1.644515221724293$$

which is exactly the value obtained above using Eqs.(22). Other approximate values given by our *Eq.(21)* approximant are:

$$\xi_{Eulap}(4{,}0{,}100) = 1.0823232233369194,\ \xi_{Eulap}(3{,}0{,}100) = 1.202056602179509, \quad (26)$$

which compare very well with the values given by the exact Euler's product formula:

$$\xi_{eul}(4{,}0{,}100) = \frac{\pi^4}{90} = \ 1.082323233711138,\ \xi_{eul}(3{,}0{,}100)1.20205690315959, \qquad (27)$$

respectively. A typical relative error of our approximant to Euler's product is rather small:

$$\frac{\pi^2/_6 - \xi_{Eulap}(2,0,100)}{\pi^2/_6} = 0.0002546\ .$$

5. Product of integer numbers

5.1 A prime product

Prime product formulae, or their approximants, are important in number theory [6, 10, 20]. Such products allow us to obtain a novel application of our prime generating Ψ and our discriminating Λ functions. That is to obtain the product of a succession of prime numbers $P(x,x_i)$ starting at *any integer* x_i and on the integer interval (x_i, x). This novel product function R_p is given by the relation:

$$R_p(x, x_i) = \prod_{k=x_i}^{x} [\Psi(k)]^{\Lambda(k)}\ . \tag{28}$$

Two examples being: (i) $R_p(20,2) = 9\ 699\ 690 = 2.3.5.7.11.13.17.19$

(ii) $R_p(31,11) = 955\ 049\ 953 = 11.13.17.19.23.29.31$

The product R_p for the interval that starts at $x_i = 2$, and up to $x=20$, appears plotted in Fig. 11 below

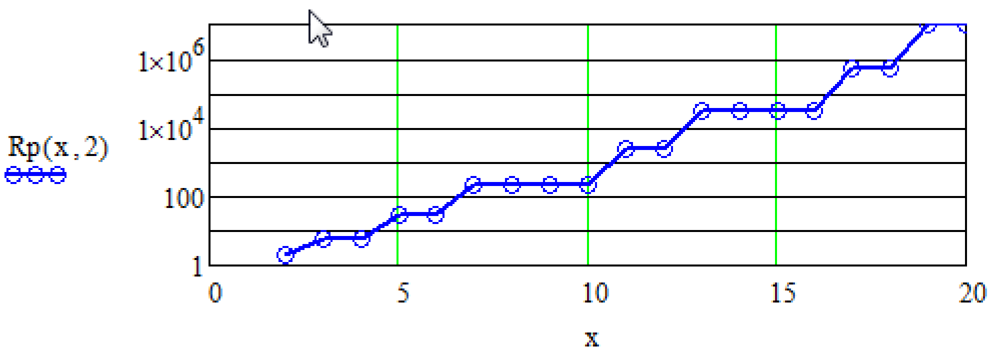


Fig. 11 Product of the prime numbers in the integer interval *(2, x)* up to *x=20*

5.2 Product of Composites

Analogously we may apply our prime numbers generating function Ψ and prime discriminating function Λ to get the product of a succession of composite numbers $R_c(x,x_i)$ starting at *any integer* x_i and for the integer interval (x_i,x). This product is defined as:

$$R_c(x, x_i) = \prod_{k=x_i}^{x} [k - \Psi(k)]^{[1-\Lambda(k)]}. \tag{29}$$

Two examples being:

$R_c(20,2) = 250822656000 = 4.6.8.9.12.14.16.18.20$

Rc(31,11)= 23726443732992000000

This composites product R_c appears plotted for the interval *(2, 20)* in Fig. 12 below:

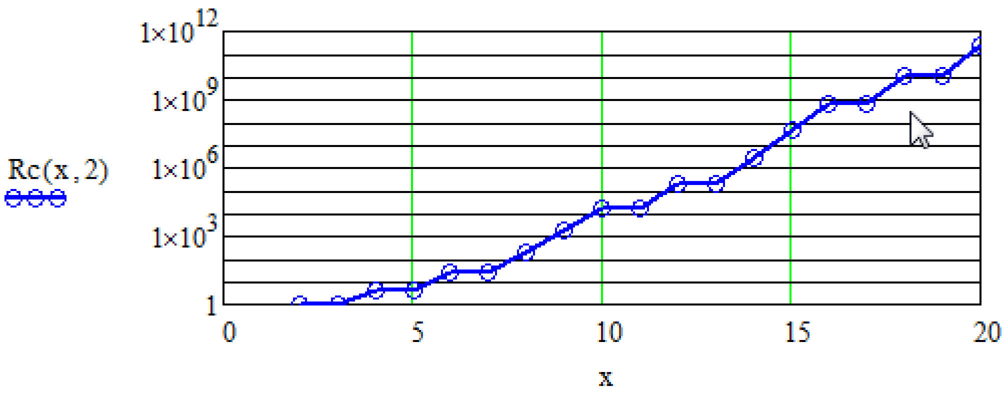


Fig. 12 Plot of the product R_c of the composite numbers in the integer domain *(2, 20)*

6. Approximant to the Chebyschev Function of Second Kind

Equation (30) below is an *exact* formula for evaluating the well-known Chebyshev Function of second kind [21-23] in terms of the prime function *P(j)*, and the *integer part* or *floor* function. The *P(j)* function of course simply gives the *j-th* prime, e.g. *P(1)= 2* and *P(100)= 541*. In this exact Chebyshev Function the upper limit *Π(x,2)* of the sum is non other than the prime function *Π*, i.e. this limit is the exact number of primes not exceeding a given integer threshold *x*, a number that has to be read from a table of prime numbers. This simply means that the exact Chebyshev Function of second kind *is not self-consistent*.

$$\Psi_{Chex}(x) = \sum_{j=1}^{\Pi(x,2)} \left\{ floor \left[\frac{\ln(x)}{ln[P(j)]} \right] ln[P(j)] \right\}, \tag{30}$$

Below, in Eq. (31) we present our approximant to Chebyshev Function of second kind. In it we have replaced the function *P(j)* with a simple algebraic expression written in terms of our exact Prime Numbers Generating function ψ and our Prime Discriminating Function Λ, both defined in Section 2, i.e. in Eq (31) we inserted $[\psi(q) + e(1 - \Lambda(q))]$. instead of *P(j)*. This gives us an accurate analytical approximant function Ψ_{Cheap} to Chebyshev's Function, namely:

$$\Psi_{Cheap}(x) = \sum_{q=2}^{x} \left\{ floor \left[\frac{\ln(x)}{ln[\Psi(q)+e(1-\Lambda(q))]} \right] ln[\psi(q) + e(1 - \Lambda(q))]\, \Lambda(q) \right\}, \tag{31}$$

where *e* is the base of the natural logarithms. Also note, in our approximant Ψ_{cheap} to that Chebyshev's function, that the finite upper limit of the sum is the variable *x*, instead of the function value *Π(x)*. Thus, our approximant Ψ_{Cheapp} is a *self-consistent* expression that *does not require the use of tables of prime numbers*. It instead uses our primes discriminating function Λ and our prime generator Ψ. This does represents a notable computational advantage when compared with the exact expression of Chebyshev Function of second kind given in Eq (30).

The good accuracy of our approximant to Chebyschev 2^{nd}. kind function can be demonstrated considering the values, given by the two functions (30) and (31) for say the j=100^{th} prime. Up to an accuracy of $\mathbf{10^{-14}}$ we get the same results:

$$\Psi_{Cheex}(j) = 94.0453112293574 \quad \text{and} \quad \Psi_{Cheap}(j) = 94.0453112293574.$$

Another important result of our approximant to Chebyschev 2nd. kind function is the following limit that it indeed satisfies:

$$\lim_{x\to\infty}\left[\frac{\Psi_{Cheap}(x)}{x}\right] = 1, \tag{32}$$

which amounts, according to Hardy and Wright [10] to the *verification of the prime number theorem*. This truly relevant attribute of our approximant Ψ_{Cheapp} is clearly shown in Fig. 13 below.

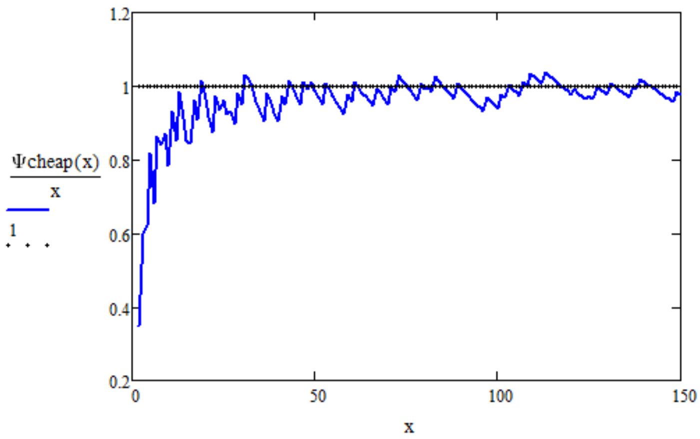
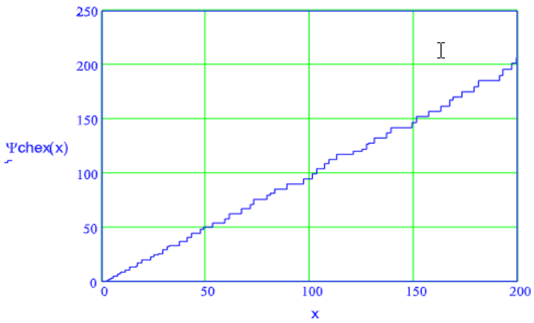
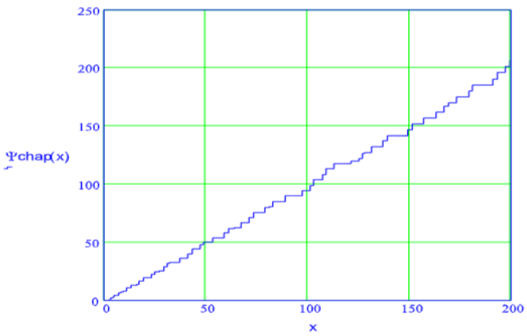


Fig. 13 The function $\Psi_{Cheapp(x)/x}$ plotted in the integer interval *(2,150)* showing its asymptotic behaviour towards its limit *1* (equivalent to verifying the prime number therorem)

In Figs. 14 (a) and 14(b), and for the sake of comparison, we have plotted the exact Chebyshev function of 2^{nd}. kind and our approximant to it, respectively; both plotted in the integer interval *[0,200]*. It may be seen that our approximant does reproduce the plot of the exact function. This assertion is validated by Fig. 14 below.



(a)



(b)

Fig. 14 Comparison plots of the exact Chebyschev 2nd kind function **(a)** and of our approximant to that function **(b),** Eq. (45), in the real domain *[0,200]*.

In figure 15 we have plotted the relative error of our approximant Ψ_{Cheapp} when compared with the exact Chebyshev function in the interval *[0,200]*. It may be seen that the accuracy of our approximant (Eq. (45)) which uses our Prime Number function Λ is very high, the relative error being less than 1×10^{-13} for the whole interval.

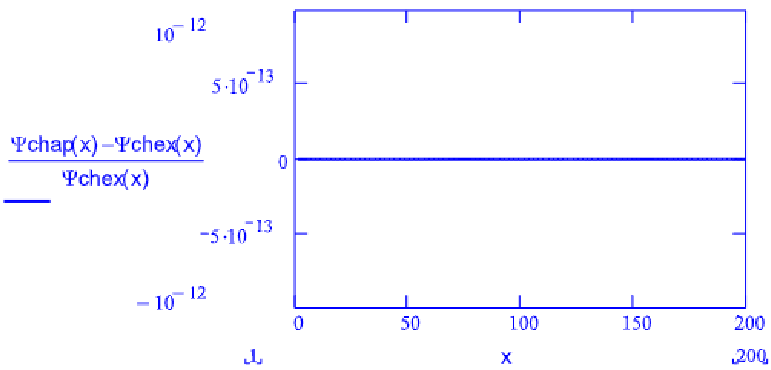


Fig. 15 Relative error (order of 10^{-13}) of our approximant to the Chebychev function, w.r.t. the exact function, in the interval *[0,200]* showing the high accuracy, of our approximant, Eq. (31) that does not use tables of primes..

In Figure A-1 of the Appendix 1 we again compare both functions, Eqs (30) and (31), but for an interval of higher values of the independent variable, namely $x \in [5000, 5050]$, those plots show the same order of accuracy of our approximant in Eq (31).

Finally note that Hardy and Wright [10] have stated that the Chebyschev function of 2nd. kind is in some ways better to apply than the prime counting π function since te first deals with a product of primes instead of counting them.

7 The primorial product $p_n\#$.

The well-known *primorial* product [24], denoted $p_n\#$, is defined as the product of the first n consecutive prime numbers p_n, thus returning the product of the n primes:

$$p_n\# = \prod_{k=1}^{n} p_k. \tag{33}$$

Such product, of course, *is not a function*, and demands to use a table of the primes p_k in order to evaluate the value $p_n\#$ of the *primordial*: that is it is not a *self-consistent* operation. The results of this *primordial* coincides with that of our previous *function* R_p, given in Eq. (28), for obtaining the product of primes, provided one sets $x_i=1$ in that equation. However, note that our function R_p is directly computable, and self-consistent, because it is defined in terms of our discriminating function Λ and our prime generating function Ψ: thus our prime product function R_p does not require the use of any table of primes.

Having said that we may now *redefine* a new *Primorial*, yet this time as a true function, whose domain is actually the set of integer numbers N. We define this new prime function as follows:

$$Primorial(x) = \prod_{x=2}^{x} [\Psi(k)]^{\Lambda(k)}, \;\; x \in N. \tag{34}$$

In Fig. 16 we have plotted, in logarithmic scale, this integer *Primordial(x)* function. It may be seen that it is an interesting stepped function: the plotted plateau heights being *2, 6. 30, 180, 1260, 2310, 30030.* These constants plateau values of our primes *Primorial* function are explained by the fact that there are only composite integers in between consecutive primes, *e.g*. the plateau of height 180 corresponds to the prime product $2 \times 3 \times 5 \times 6$.

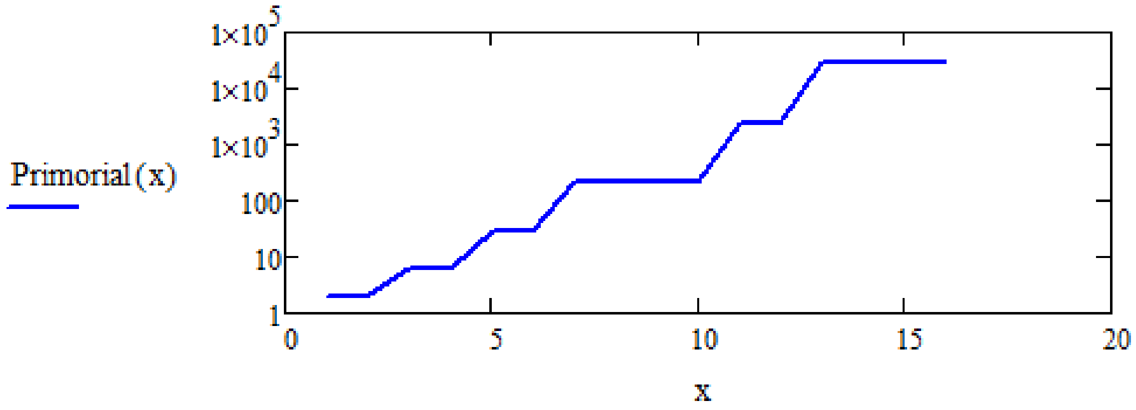


Fig. 16 Plot of the *Primorial(x)* function in the interval of integers *(2-20)*, the plateau heights are explained in the text.

We now introduce a novel prime numbers function *b(x)*, defined in terms of our *Primorial(x)* function on the integers *N*:

$$b(x) \equiv Primorial(x)^{\left\{\frac{1}{\Psi(k)^{\Lambda(k)}}\right\}}. \tag{35}$$

This function is analogous to one already presented (just as a plot) of the simpler primorial product $p_n\#$, in [23, 24]. This new function *b(x)*, defined now in terms of our prime discriminating and generating functions Λ, Ψ, is therefore a *true* function that has interesting properties. In effect this function *b(x)*, after some initial oscillations in the open interval *(0,~2)*, has the following appealing asymptotic behaviour:

$$\lim_{x\to\infty} b(x) = e, \tag{36}$$

as illustrated in Fig.17, where the blue dots are the values of the function *b(x)* while the dashed line represents of course the constant *e=2.71828184*, base of the natural logarithms.

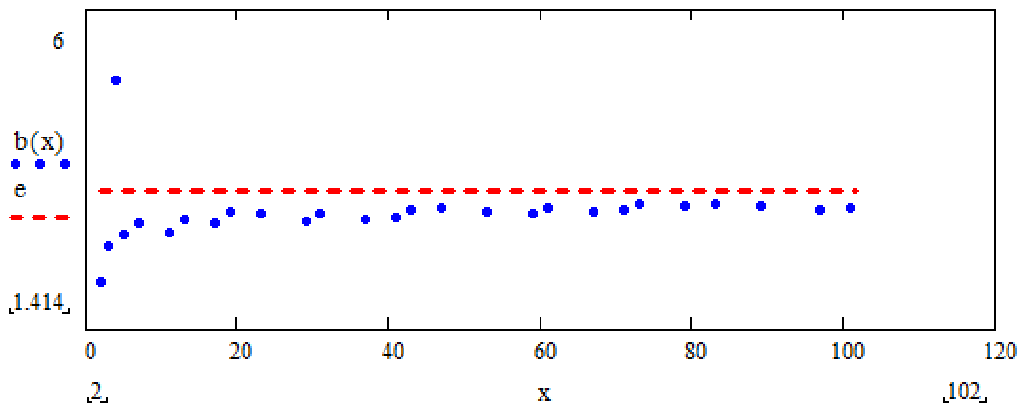


Fig. 17 Plot of the function *b(x)* (blue dots) showing its asymptotic limit at the constant *e* (red dashes)

8. Discussion and New Directions

In Section 2 of this work we have presented a novel prime numbers generating function, denoted *Ψ*, that is computable and gives the exact list of primes in any given integer interval, thus being better than most published prime generators. This generating function of primes is defined using another novel prime number discriminating function *Λ*, the latter written in terms of elementary functions, also presented in Section 2. Not dependent upon prime numbers tables or algorithms, these two functions are easier-to-use tools to calculate, or even to define, new prime numbers mathematical functions and relations, as well as for applications in basic sciences e.g. in physics, as in the case of discrete mechanics. We presented solid evidence (see Table 1) of the exactitude of our prime discriminating function by generating 64 random integers lying in the arbitrarily chosen integer domain *[13,2591491]*, and then successfully applying that discriminating function to determine whether these random integers belong or not to the primes distribution in that interval (Section 2, Table 1). We also introduced a new Prime Numbers Counting Function $C(u,2)$, that renders the exact number of primes from *2* and up any integer u, or in any integer interval (u_i ,u_f) for any initial integer u_i, and that can be easily implemented using a personal computer. It does generate the well-known staircase-like plot of primes when plotted for any integer interval. In Section 3 we have applied this prime counting function $C(2,u)$ to obtain the number of primes in Bertrand Theorem type of integer intervals $(u_i,2u_i)$ for any initial integer u_i. We also applied it to count the composite integers in such intervals. Also, in Section 3 we introduced a new prime number function, that we called the *Goldbach function*, since its definition is based on the well-known Goldbach Conjecture on prime numbers. We then used this Goldbach function to verify Bertrand Theorem on prime numbers.

Both, our new prime numbers counting and generating functions have been used in Section 4 to derive finite approximants to Euler's Product formula [Eqs. (24b), (25)].These approximants are shown to be of good accuracy, achieving relative errors of order 10^{-7}, even when calculated with relatively small upper bound values of the finite product and sum in their definitions, and using modest personal computers. Much better results should be obtained using mainframe computers nowadays available at large, to run the computer programs implementing our three prime numbers functions and the derived Euler´s product approximant with higher upper bounds.

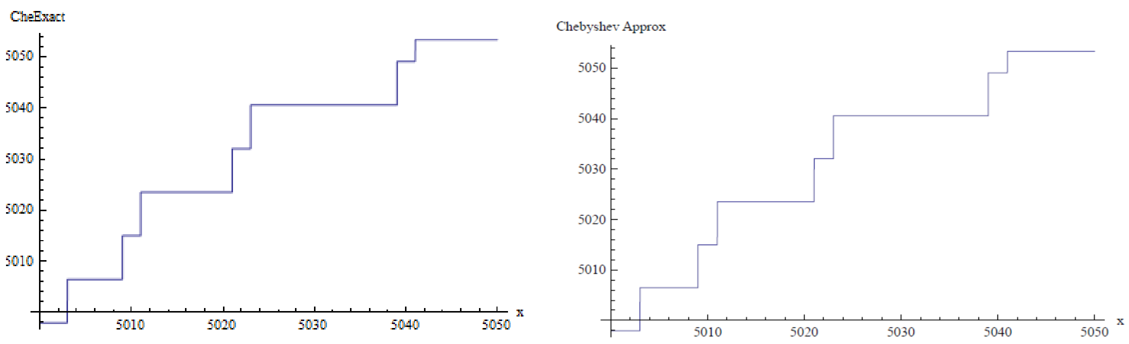
In Section 5 we applied our Primes Discriminating and Generating Function to construct a new function that calculates the product of any number of primes, or even composites, in integer intervals. Once again, this new prime's function can be used without resorting to tables or algorithms for prime numbers.

In Section 6, we defined a finite self-consistent approximant, to the important Chebyschev function of 2^{nd}. kind, that does not require prime number tables or numerical algorithms, but that instead is written in terms of our prime number counting function $C(2,x)$. The latter successfully having replaced the well-known primes $\pi(x)$ in the exact expression of the Chebyshev function of 2^{nd}. kind, thus obtaining an approximant that eliminates the need to use tables of primes. Moreover, this approximant function is very accurate giving zero relative error when compared with the exact function. Finally, in Section 7, after recalling the *primorial product* $p_n\#$ of primes, we introduced a *Primorial function* of integers *x*, to evaluate such products, based on our primes generator Ψ and discriminating function Λ, *Primorial(x)* being therefore a function that does not require the use of table of prime numbers. We then defined a new prime numbers function *b(x)* in terms of this *Primorial function P(x)*, and showed that it does tend to the limit *e* when the variable *x* tends to infinity (plot analogous to the one in [24] for the product $p_n\#$).

In work now in progress we are applying our prime numbers functions to the study of a discrete two-dimensional mechanical model of an object defined to have two prime numbers coordinates, an object that we have named the *Goldbach Particle*. We are also extending our previous work [1], to apply the classical mechanics Hamilton Principle to that Goldbach particle.

Appendix 1

Below, and for comparison, we plot the Chesbyshev exact function, Eq (30) and our approximant Eq (31) to that function in the real domain [5000, 5050]



(a) **(b)**

Fig. A1 Comparison plots of the exact Chebyschev function **(a)** in Eq. (30), and of our approximant Chebyschev Aprox **(b),** in Eq. (31), in the interval *[5000, 5050]* with relative accuracy of order of 10^{-13}.

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