# Accurate HT-geometric automrophic congruences

Pascal Boyer

# ACCURATE KHT-GEOMETRIC AUTOMORPHIC CONGRUENCES

*by*

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***Abstract.*** — We first explain how to separate the various contributions coming either from torsion or from the distinct families of automorphic representations, to the modulo $l$ reduction of the cohomology of Harris-Taylor perverse sheaves. We then deduce the construction of accurate automorphic congruences for a similitude group $G/\mathbb{Q}$ with signature $(1, d-1)$.

## 1. Introduction

The first appearance of automorphic congruences can be trace back to Ramanujan's works on the $\tau$-functions. Now existence and construction of higher dimensional automorphic congruences play an essential role in particular to the Langlands program.

One possible geometric approach is to look at the cohomology groups of some $\overline{\mathbb{Z}}_l$-local system on a Shimura variety associated to some reductive group $G/\mathbb{Q}$, whose free quotients are expected to be of automorphic nature. The idea is then to take two (or two families of) such local systems whose modulo $l$ reductions are related, and be able to relate the modulo $l$ reductions of their cohomology groups. We then face two main problems.

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(a) The torsion may interfere and prevent us to relate the modulo $l$ reduction of the free quotients of the cohomology groups of our two $\overline{\mathbb{Z}}_l$-local systems.
(b) Even if we can manage about the torsion, the $\overline{\mathbb{Q}}_l$-cohomology of our local systems may involve different sorts of automorphic representations and we would only be able to construct non accurate automorphic congruences.

Usually to overcome (a), one localize at a non pseudo Eisenstein ideal but we then loose degenerate automorphic representations. In this article we are able, playing with all the Harris-Taylor local systems associated to one cuspidal representation, to deal with this two problems when $G/\mathbb{Q}$ is a similitude group with signature $(1, d-1)$ and the associated Shimura variety is of Kottwitz-Harris-Taylor type.

(a) More precisely we first prove the conjecture 5.10 of [**4**] which says that the modulo $l$ reduction of the torsion submodule of the cohomology groups of the Harris-Taylor perverse sheaves only depends on the modulo $l$ reduction of the perverse sheaf. This property is the main point of the article and rests on the fact that one can follow the torsion travelling into the cohomology groups of the family of Harris-Taylor perserve sheaves associated to a fixed irreducible cuspidal local representation, cf. lemma 5.8.
(b) Secondly, thanks to the computations of [**4**] recalled in §4, we are able to separate the contributions of the automorphic representations according to the shape of their local component at the place considered.

As an application, we are then able to prove the following statement about automorphic congruences. We start with a irreducible automorphic representation $\Pi$ of $G(\mathbb{A})$ with weight $\xi$, a irreducible algebraic representation of $G(\mathbb{Q})$, which locally at some prime number $p \neq l$ looks like

$$\Pi_p \simeq \Pi_p^v \otimes \mathrm{Speh}_s\big(\mathrm{St}_{t_1}(\pi_{v,1}) \times \cdots \mathrm{St}_{t_r}(\pi_{v,r})\big),$$

for some $s \geqslant 1$, where, cf. the convention of §3, $v|p$ is a fixed split place of the CM field $F$ defining $G$ and where $\pi_{v,1}, \cdots, \pi_{v,r}$ are irreducible cuspidal representations. We then consider both a weight $\xi'$ congruent to $\xi$ modulo $l$ and some local congruence $\pi'_{v,1}$ of $\pi_{v,1}$ which we suppose to be supercuspidal modulo $l$. We then prove, cf. the introduction of §6, the existence of a irreducible automorphic representation $\Pi'$ of $G(\mathbb{A})$ of weight $\xi'$ such that

– locally $\Pi'$ at the place $v$ is isomorphic to

$$\mathrm{Speh}_s\Big(\mathrm{St}_{t'_1}(\pi'_{v,1}) \times \cdots \times \mathrm{St}_{t'_{r'}}(\pi'_{v,r'})\Big)$$

with $t'_1 = t_1$ and where for $i = 2, \cdots, r'$, the $\pi'_{v,i}$ are irreducible cuspidal representations,

– globally $\Pi'$ share with $\Pi$ the same level outside $v$ and is weakly congruent to $\Pi$ in the sense that it shares the same Satake parameters at the unramified places.

More precisely theorem 6.2 gives a quantitative version in terms of an equality between multiplicities in the space of automorphic forms.

## Contents



## 2. Notations about representations of $GL_n(K)$

We fix a finite extension $K/\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and we denote by $|-|$ its absolute value.

***2.1. Definition***. — *Two representations $\pi$ and $\pi'$ of $GL_n(K)$ are said inertially equivalent and we denote by $\pi \sim^i \pi'$, if there exists a character $\chi : \mathbb{Z} \longrightarrow \bar{\mathbb{Q}}_l^\times$ such that*

$$\pi \simeq \pi' \otimes (\chi \circ \mathrm{val} \circ \det).$$

*We then denote by $e_\pi$ the order of the set of characters $\chi : \mathbb{Z} \longrightarrow \bar{\mathbb{Q}}_l^\times$, such that $\pi \otimes \chi \circ \mathrm{val}(\det) \simeq \pi$.*

For a representation $\pi$ of $GL_d(K)$ and $n \in \frac{1}{2}\mathbb{Z}$, set

$$\pi\{n\} := \pi \otimes q^{-n\,\mathrm{val} \circ \det}.$$

**2.2. Notations.** — *For $\pi_1$ and $\pi_2$ representations of respectively $GL_{n_1}(K)$ and $GL_{n_2}(K)$, we will denote by*

$$\pi_1 \times \pi_2 := \mathrm{ind}_{P_{n_1,n_1+n_2}(K)}^{GL_{n_1+n_2}(K)} \pi_1\{\frac{n_2}{2}\} \otimes \pi_2\{-\frac{n_1}{2}\},$$

*the normalized parabolic induced representation where for any sequence $\underline{r} = (0 < r_1 < r_2 < \cdots < r_k = d)$, we write $P_{\underline{r}}$ for the standard parabolic subgroup of $GL_d$ with Levi*

$$GL_{r_1} \times GL_{r_2-r_1} \times \cdots \times GL_{r_k-r_{k-1}}.$$

Recall that a representation $\varrho$ of $GL_d(K)$ is called *cuspidal* (resp. *supercuspidal*) if it is not a subspace (resp. subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero then these two notions coincides, but this is no more true for $\overline{\mathbb{F}}_l$.

**2.3. Definition.** — *(see* [**8**] *§9 and* [**2**] *§1.4) Let $g$ be a divisor of $d = sg$ and $\pi$ an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(K)$. The induced representation*

$$\pi\{\frac{1-s}{2}\} \times \pi\{\frac{3-s}{2}\} \times \cdots \times \pi\{\frac{s-1}{2}\}$$

*holds a unique irreducible quotient (resp. subspace) denoted* $\mathrm{St}_s(\pi)$ *(resp.* $\mathrm{Speh}_s(\pi)$*); it is a generalized Steinberg (resp. Speh) representation.*

Any generic irreducible representation $\Pi$ of $GL_d(K)$ is isomorphic to $\mathrm{St}_{t_1}(\pi_1) \times \cdots \times \mathrm{St}_{t_r}(\pi_r)$ where for $i = 1, \cdots, r$, the $\pi_i$ are irreducible cuspidal representations of $GL_{g_i}(K)$ and $t_i \geqslant 1$ are such that $\sum_{i=1}^r t_i g_i = d$. For $\Pi$ a irreducible generic representation and $s \geqslant 1$, we denote by

$$\mathrm{Speh}_s(\Pi) = \mathrm{Speh}_s\big(\mathrm{St}_{t_1}(\pi_1)\big) \times \cdots \times \mathrm{Speh}_s\big(\mathrm{St}_{t_r}(\pi_r)\big)$$

the Langlands quotient of the parabolic induced representation $\Pi\{\frac{1-s}{2}\} \times \Pi\{\frac{3-s}{2}\} \times \cdots \times \Pi\{\frac{s-1}{2}\}$. In terms of the local Langlands correspondence, if $\sigma$ is the representation of $\mathrm{Gal}(\bar{F}/F)$ associated to $\Pi$, then $\mathrm{Speh}_s(\Pi)$ corresponds to $\sigma \oplus \sigma(1) \oplus \cdots \oplus \sigma(s-1)$.

**2.4. Definition.** — *Let $D_{K,d}$ be the central division algebra over $K$ with invariant $1/d$ and with maximal order denoted by $\mathcal{D}_{K,d}$.*

The local Jacquet-Langlands correspondance is a bijection JL between the set of equivalence classes of irreducible admissible representations of

$D_{K,d}^{\times}$ and the one of irreducible admissible essentially square integrable representations of $GL_d(K)$.

***2.5. Notation***. — *For $\pi$ a cuspidal irreducible $\bar{\mathbb{Q}}_l$-representation of $GL_g(K)$ and for $t \geqslant 1$, we then denote by $\pi[t]_D$ the representation* $\mathrm{JL}^{-1}(\mathrm{St}_t(\pi))^{\vee}$ *of $D_{K,tg}^{\times}$.*

## 3. KHT-Shimura varieties and Harris-Taylor local systems

Let $F = F^+E$ be a CM field where $E/\mathbb{Q}$ is quadratic imaginary and $F^+/\mathbb{Q}$ totally real with a fixed real embedding $\tau : F^+ \hookrightarrow \mathbb{R}$. For a place $v$ of $F$, we will denote by

- $F_v$ the completion of $F$ at $v$,
- $\mathcal{O}_v$ the ring of integers of $F_v$,
- $\varpi_v$ a uniformizer,
- $q_v$ the order of the residual field $\kappa(v) = \mathcal{O}_v/(\varpi_v)$.

Let $B$ be a division algebra with center $F$, of dimension $d^2$ such that at every place $x$ of $F$, either $B_x$ is split or a local division algebra and suppose $B$ provided with an involution of second kind $*$ such that $*_{|F}$ is the complex conjugation. For any $\beta \in B^{*=-1}$, denote by $\sharp_\beta$ the involution $x \mapsto x^{\sharp_\beta} = \beta x^* \beta^{-1}$ and $G/\mathbb{Q}$ the group of similitudes, denoted $G_\tau$ in [**6**], defined for every $\mathbb{Q}$-algebra $R$ by

$$G(R) \simeq \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\}$$

with $B^{op} = B \otimes_{F,c} F$. If $x$ is a place of $\mathbb{Q}$ split $x = yy^c$ in $E$ then

$$G(\mathbb{Q}_x) \simeq (B_y^{op})^\times \times \mathbb{Q}_x^\times \simeq \mathbb{Q}_x^\times \times \prod_{z_i} (B_{z_i}^{op})^\times, \tag{3.1}$$

where, identifying places of $F^+$ over $x$ with places of $F$ over $y$, $x = \prod_i z_i$ in $F^+$.

**Convention**: for $x = yy^c$ a place of $\mathbb{Q}$ split in $E$ and $z$ a place of $F$ over $y$ as before, we shall make throughout the text, the following abuse of notation by denoting $G(F_z)$ in place of the factor $(B_z^{op})^\times$ in the formula (3.1).

In [**6**], the author justify the existence of $G$ like before such that moreover

- if $x$ is a place of $\mathbb{Q}$ non split in $E$ then $G(\mathbb{Q}_x)$ is quasi split;
- the invariants of $G(\mathbb{R})$ are $(1, d-1)$ for the embedding $\tau$ and $(0, d)$ for the others.

***3.2. Definition.*** — *Define* Spl *as the set of places $v$ of $F$ such that $p_v := v_{|\mathbb{Q}} \neq l$ is split in $E$ and $B_v^\times \simeq GL_d(F_v)$. For each $I \in \mathcal{I}$, write* Spl$(I)$ *the subset of* Spl *of places which do not divide the level $I$.*

As in [**6**] bottom of page 90, a compact open subgroup $U$ of $G(\mathbb{A}^\infty)$ is said *small enough* if there exists a place $x$ such that the projection from $U^v$ to $G(\mathbb{Q}_x)$ does not contain any element of finite order except identity.

***3.3. Notation.*** — *We denote by $\mathcal{I}$ the set of open compact subgroups small enough of $G(\mathbb{A}^\infty)$. For $I \in \mathcal{I}$, then $\mathrm{Sh}_{I,\eta} \longrightarrow \operatorname{Spec} F$ is the associated Shimura variety said of Kottwitz-Harris-Taylor type.*

In the sequel, $v$ will denote a fixed place of $F$ in Spl. For such a place $v$ the scheme $\mathrm{Sh}_{I,\eta}$ has a projective model $\mathrm{Sh}_{I,v}$ over $\operatorname{Spec} \mathcal{O}_v$ with special fiber $\mathrm{Sh}_{I,s_v}$. For $I$ going through $\mathcal{I}$, the projective system $(\mathrm{Sh}_{I,v})_{I\in\mathcal{I}}$ is naturally equipped with an action of $G(\mathbb{A}^\infty) \times \mathbb{Z}$ such that $w_v$ in the Weil group $W_v$ of $F_v$ acts by $-\deg(w_v) \in \mathbb{Z}$, where $\deg = \mathrm{val} \circ \mathrm{Art}^{-1}$ and $\mathrm{Art}^{-1} : W_v^{ab} \simeq F_v^\times$ is the Artin isomorphism which sends geometric Frobenius to uniformizers.

***3.4. Notations.*** — *(see [**1**] §1.3) For $I \in \mathcal{I}$, the Newton stratification of the geometric special fiber $\mathrm{Sh}_{I,\bar{s}_v}$ is denoted by*

$$\mathrm{Sh}_{I,\bar{s}_v} =: \mathrm{Sh}^{\geqslant 1}_{I,\bar{s}_v} \supset \mathrm{Sh}^{\geqslant 2}_{I,\bar{s}_v} \supset \cdots \supset \mathrm{Sh}^{\geqslant d}_{I,\bar{s}_v}$$

*where $\mathrm{Sh}^{=h}_{I,\bar{s}_v} := \mathrm{Sh}^{\geqslant h}_{I,\bar{s}_v} - \mathrm{Sh}^{\geqslant h+1}_{I,\bar{s}_v}$ is an affine scheme(1), smooth of pure dimension $d-h$ built up by the geometric points whose connected part of its Barsotti-Tate group is of rank $h$. For each $1 \leqslant h < d$, write*

$$i_h : \mathrm{Sh}^{\geqslant h}_{I,\bar{s}_v} \hookrightarrow \mathrm{Sh}^{\geqslant 1}_{I,\bar{s}_v}, \quad j^{\geqslant h} : \mathrm{Sh}^{=h}_{I,\bar{s}_v} \hookrightarrow \mathrm{Sh}^{\geqslant h}_{I,\bar{s}_v},$$

*and $j^{=h} = i_h \circ j^{\geqslant h}$.*

Consider now the ideals $I^v(n) := I^v K_v(n)$ where

$$K_v(n) := \mathrm{Ker}(GL_d(\mathcal{O}_v) \twoheadrightarrow GL_d(\mathcal{O}_v/\mathcal{M}_v^n)).$$

Recall then that $\mathrm{Sh}^{=h}_{I^v(n),\bar{s}_v}$ is geometrically induced under the action of the parabolic subgroup $P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$. Concretely this means that there exists a closed subscheme $\mathrm{Sh}^{=h}_{I^v(n),\bar{s}_v,\overline{1_h}}$ stabilized by the Hecke action of $P_{h,d}(F_v)$ and such that

$$\mathrm{Sh}^{=h}_{I^v(n),\bar{s}_v} = \mathrm{Sh}^{=h}_{I^v(n),\bar{s}_v,\overline{1_h}} \times_{P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)} GL_d(\mathcal{O}_v/\mathcal{M}_v^n),$$

(1) see for example [**7**]

meaning that $\mathrm{Sh}^{=h}_{I^v(n),\bar{s}_v}$ is the disjoint union of copies of $\mathrm{Sh}^{=h}_{I^v(n),\bar{s}_v,\overline{1_h}}$ indexed by $GL_d(\mathcal{O}_v/\mathcal{M}_v^n)/P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$ and exchanged by the action of $GL_d(\mathcal{O}_v/\mathcal{M}_v^n)$.

***3.5. Notations***. — *For $1 \leqslant t \leqslant s_g := \lfloor d/g \rfloor$, let $\Pi_t$ be any representation of $GL_{d-tg}(F_v)$. We then denote by*

$$\widetilde{HT}_1(\pi_v, \Pi_t) := \mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}$$

*the Harris-Taylor local system on the Newton stratum $\mathrm{Sh}^{=tg}_{I,\bar{s}_v,\overline{1_{tg}}}$ where*

- *$\mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}}}$ is defined thanks to Igusa varieties attached to the representation $\pi_v[t]_D$,*
- *$\Xi : \frac{1}{2}\mathbb{Z} \longrightarrow \overline{\mathbb{Z}}_l^\times$ is defined by $\Xi(\frac{1}{2}) = q^{1/2}$.*

*We also introduce the induced version*

$$\widetilde{HT}(\pi_v, \Pi_t) := \Big( \mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}} \Big) \times_{P_{tg,d}(F_v)} GL_d(F_v),$$

*where the unipotent radical of $P_{tg,d}(F_v)$ acts trivially and the action of*

$$(g^{\infty,v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times P_{tg,d}(F_v) \times W_v$$

*is given*

- *by the action of $g_v^c$ on $\Pi_t$ and $\deg(\sigma_v) \in \mathbb{Z}$ on $\Xi^{\frac{tg-d}{2}}$, and*
- *the action of $(g^{\infty,v}, g_v^{et}, \mathrm{val}(\det g_v^c) - \deg \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times GL_{d-tg}(F_v) \times \mathbb{Z}$ on $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Xi^{\frac{tg-d}{2}}$.*

*We also introduce*

$$HT(\pi_v, \Pi_t)_{\overline{1_{tg}}} := \widetilde{HT}(\pi_v, \Pi_t)_{\overline{1_{tg}}}[d-tg],$$

*and the perverse sheaf*

$$P(t, \pi_v)_{\overline{1_{tg}}} := j^{=tg}_{1,!*} HT(\pi_v, \mathrm{St}_t(\pi_v))_{\overline{1_{tg}}} \otimes \mathbb{L}(\pi_v),$$

*and their induced version, $HT(\pi_v, \Pi_t)$ and $P(t, \pi_v)$, where*

$$j^{=h} = i^h \circ j^{\geqslant h} : \mathrm{Sh}^{=h}_{I,\bar{s}_v} \hookrightarrow \mathrm{Sh}^{\geqslant h}_{I,\bar{s}_v} \hookrightarrow \mathrm{Sh}_{I,\bar{s}_v}$$

*and $\mathbb{L}^\vee$, the dual of $\mathbb{L}$, is the local Langlands correspondence.*

For $\overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$ coefficients, we will mention it in the index of the local system, as for example $HT_{\overline{\mathbb{Q}}_l}(\pi_v, \Pi_t)$ or $HT_{\overline{\mathbb{F}}_l}(\pi_v, \Pi_t)$.

## 4. $\overline{\mathbb{Q}}_l$-cohomology groups

From now on, we fix a prime number $l$ unramified in $E$. Let us first recall some known facts about irreducible algebraic representations of $G$, see for example [**6**] p.97. Let $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}_l$ be a fixed embedding and et write $\Phi$ the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ whose restriction to $E$ equals $\sigma_0$. There exists then an explicit bijection between irreducible algebraic representations $\xi$ of $G$ over $\overline{\mathbb{Q}}_l$ and $(d+1)$-uple $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$ where $a_0 \in \mathbb{Z}$ and for all $\sigma \in \Phi$, we have $\overrightarrow{a_\sigma} = (a_{\sigma,1} \leqslant \cdots \leqslant a_{\sigma,d})$.

For $K \subset \overline{\mathbb{Q}}_l$ a finite extension of $\mathbb{Q}_l$ such that the representation $\iota^{-1} \circ \xi$ of highest weight $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$, is defined over $K$, write $W_{\xi,K}$ the space of this representation and $W_{\xi,\mathcal{O}}$ a stable lattice under the action of the maximal open compact subgroup $G(\mathbb{Z}_l)$, where $\mathcal{O}$ is the ring of integers of $K$ with uniformizer $\lambda$.

*Remark.* if $\xi$ is supposed to be $l$-small, in the sense that for all $\sigma \in \Phi$ and all $1 \leqslant i < j \leqslant n$ we have $0 \leqslant a_{\tau,j} - a_{\tau,i} < l$, then such a stable lattice is unique up to a homothety.

***4.1. Notation***. — *We will denote by $V_{\xi,\mathcal{O}/\lambda^n}$ the local system on $\mathrm{Sh}_{I,v}$ as well as*

$$V_{\xi,\mathcal{O}} = \lim_{\overleftarrow{n}} V_{\xi,\mathcal{O}/\lambda^n} \quad \textit{and} \quad V_{\xi,K} = V_{\xi,\mathcal{O}} \otimes_{\mathcal{O}} K.$$

*For the $\overline{\mathbb{Z}}_l$ and $\overline{\mathbb{Q}}_l$ version, we will write respectively $V_{\xi,\overline{\mathbb{Z}}_l}$ and $V_{\xi,\overline{\mathbb{Q}}_l}$.*

*Remark.* The representation $\xi$ is said *regular* if its parameter $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$ verifies for all $\sigma \in \Phi$ that $a_{\sigma,1} < \cdots < a_{\sigma,d}$.

***4.2. Definition***. — *For a $\overline{\mathbb{Z}}_l$-sheaf $\mathcal{F}$ on $\mathrm{Sh}_{I,v}$, we will denote by $\mathcal{F}_\xi$ the sheaf $\mathcal{F} \otimes V_{\xi,\overline{\mathbb{Z}}_l}$.*

***4.3. Definition***. — *Let $\xi$ be a irreducible algebraic $\mathbb{C}$-representation with finite dimension of $G$. Then a irreducible $\mathbb{C}$-representation $\Pi_\infty$ of $G(\mathbb{A}_\infty)$ is said $\xi$-cohomological if there exists an integer $i$ such that*

$$H^i((\mathrm{Lie}\ G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U_\tau, \Pi_\infty \otimes \xi^\vee) \neq (0)$$

*where $U_\tau$ is a maximal open compact modulo center subgroup of $G(\mathbb{R})$, cf. [**6**] p.92. We then denote by $d_\xi^i(\Pi_\infty)$ the dimension of this cohomology group.*

*Remark.* If $\xi$ has $\bar{\mathbb{Q}}_l$-coefficients, then a irreducible $\bar{\mathbb{Q}}_l$-representation $\Pi^\infty$ of $G(\mathbb{A}^\infty)$ is said $\xi$-cohomolocal if there exists a $\mathbb{C}$-representation $\Pi_\infty$ of

$G(\mathbb{A}_\infty)$ such that $\iota_l\Big(\Pi^\infty\Big) \otimes \Pi_\infty$ is an automorphic $\mathbb{C}$-representation of $G(\mathbb{A})$, where $\iota_l : \overline{\mathbb{Q}}_l \simeq \mathbb{C}$.

Recall that $G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times GL_d(F_v) \times \prod_{i=2}^r (B_{v_i}^{op})^\times$, for a fixed place $v$ of $F$ above $p$. For $\Pi$ a irreducible representation of $G(\mathbb{A})$, its component for the similitude factor le facteur $\mathbb{Q}_p^\times$ is denoted as in [**6**], $\Pi_{p,0}$: as all the open compact subgroup of $\mathcal{I}$ contain $\mathbb{Z}_p^\times$, the representation $\Pi$ which appear in the cohomology groups will all verify that $(\Pi_{p,0})_{|\mathbb{Z}_p^\times} = 1$. We now consider

- an admissible irreducible representation $\Pi$ of $G(\mathbb{A})$ with multiplicity $m(\Pi)$ in the space of automorphic forms,
- and let $\pi_v$ be a cuspidal irreducible representation of $GL_g(F_v)$.
- We also fix a finite level $I$ and we denote by $S$ the set of places $x$ such that $I_x$ is maximal and we moreover impose $v \in S$. We then denote by $\mathbb{T}^S$ the $\overline{\mathbb{Z}}_l$-unramified Hecke algebra outside $S$.

From [**2**] and more precisely from [**4**], we now recall the main results about the $\overline{\mathbb{Q}}_l$-cohomology groups of Harris-Taylor perverse sheaves. We first introduce some notations.

- For any $\overline{\mathbb{Q}}_l$-perverse sheaf $P$ on the projective system of schemes $\mathrm{Sh}^{=h}_{I^v(n),\bar{s}_v}$ with $I^v(n) := I^v K_v(n)$ where $K_v(n) := \mathrm{Ker}(GL_d(\mathcal{O}_v) \twoheadrightarrow GL_d(\mathcal{O}_v/\mathcal{M}_v^n))$, we consider
$$H^i(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, P) := \varinjlim_n H^i(\mathrm{Sh}_{I^v(n),\bar{s}_v}, P).$$
It is then equipped with an action of $\mathbb{T}^S_{\overline{\mathbb{Q}}_l} \times GL_d(F_v) \times W_v$ and its image in the Grothendieck group of admissible representations of $\mathbb{T}^S \times GL_d(F_v) \times W_v$ will be denoted $[H^i(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, P)]$.
- For a fixed maximal ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}^S_{\overline{\mathbb{Q}}_l}$, and a $\overline{\mathbb{Q}}_l$-perverse sheaf $P$ as above, we then denote by $H^i(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, P)_{\widetilde{\mathfrak{m}}}$ the localization at $\widetilde{\mathfrak{m}}$ and $[H^i((\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, P))]_{\widetilde{\mathfrak{m}}}$ its image in the Grothendieck group of admissible representations of $GL_d(F_v) \times W_v$. If $\widetilde{\mathfrak{m}}$ is associated to a irrreductible representation $\Pi^{\infty,v}$ of $G(\mathbb{A}^{\infty,v})$, we might also denote it by $[H^i((\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, P)]\{\Pi^{\infty,v}\}$.
- In particular for $HT(\pi_v, \Pi_t)$ a Harris-Taylor local system, we denote by $[H^i_c(HT(\pi_v, \Pi_t))]_{\widetilde{\mathfrak{m}}}$ (resp. $[H^i_{!*}(HT(\pi_v, \Pi_t))]_{\widetilde{\mathfrak{m}}}$), the image of $H^i(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, j^{=h}_! HT(\pi_v, \Pi_t))_{\widetilde{\mathfrak{m}}}$ (resp. $H^i(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j^{=h}_{!*} HT(\pi_v, \Pi_t))_{\widetilde{\mathfrak{m}}}$)

in the Grothendieck group of $GL_d(F_v) \times W_v$ admissible representations. We also use a similar notation for $HT_{1_{tg}}(\pi_v, \Pi_t)$ by replacing $GL_d(F_v)$ by $P_{tg,d}(F_v)$.

It is also well known, cf. lemma 3.2 of [**4**] that if $\Pi$ is a irreducible automorphic representation of $G(\mathbb{A})$ then its local component $\Pi_v$ is isomorphic to

$$\mathrm{Speh}_s(\mathrm{St}_{t_1}(\pi_{1,z})) \times \cdots \times \mathrm{Speh}_s(\mathrm{St}_{t_u}(\pi_{u,z})) \simeq \mathrm{Speh}_s\Big(\mathrm{St}_{t_1}(\pi_{1,z}) \times \cdots \times \mathrm{St}_{t_u}(\pi_{u,z})\Big)$$

where the $\pi_{i,z}$ are irreducible cuspidal representations.

***4.4. Notation***. — *For $\pi_v$ a irreducible cuspidal representation of $GL_g(F_v)$, we denote by $\mathcal{A}_{\xi,\pi_v}(r,s)$ the set of equivalence classes of automorphic irreducible representations of $G(\mathbb{A})$ which are $\xi$-cohomological and such that*

- $(\Pi_{p,0})_{|\mathbb{Z}_p^\times} = 1$,
- *its local component at $v$ looks like $\mathrm{Speh}_s(\mathrm{St}_t(\pi'_v)) \times ?$ with $r = s+t-1$, $\pi'_v \sim^i \pi_v$ and ? is a irreducible representation of $GL_{d-stg}(F_v)$ which we do not want to precise.*

*Remark.* An automorphic irreducible representation $\Pi$ of $G(\mathbb{A})$ which is $\xi$-cohomological and such that

$$\Pi_v \simeq \mathrm{Speh}_s\Big(\mathrm{St}_{t_1}(\pi_{1,v}) \times \cdots \times \mathrm{St}_{t_u}(\pi_{u,v})\Big)$$

belongs to $\mathcal{A}_{\xi,\pi_{v,i}}(s+t_i-1, s)$ for $i = 1, \cdots, u$.

***4.5. Proposition***. — *(proposition 3.6 of [**4**])*
*Let $\pi_v$ be a irreducible cuspidal representation of $GL_g(F_v)$ and $1 \leqslant r \leqslant d/g$. Then we have*

$$[H^i_{!*}(HT_{\overline{1_{rg}},\xi}(\pi_v, \Pi_r))]\{\Pi^{\infty,v}\} = \frac{e_{\pi_v} \sharp \operatorname{Ker}^1(\mathbb{Q}, G)}{d}$$
$$\sum_{\substack{(s,t) \\ \Pi \in \mathcal{A}_{\xi,\pi_v}(s+t-1,s)}} \sum_{\Pi' \in \mathcal{U}_{G,\xi}(\Pi^{\infty,v})} \sum_{m_{s,t}(r,i)=1} m(\Pi') d_\xi(\Pi'_\infty) \Big(\Pi_r \otimes R_{\pi_v}(s,t)(r,i)(\Pi_v)\Big)$$

*where*

- *$\mathcal{U}_{G,\xi}(\Pi^{\infty,v})$ is the set of equivalence classes of automorphic, $\xi$-cohomological, irreducible representations $\Pi'$ of $G(\mathbb{A})$ such that $(\Pi')^{\infty,v} \simeq \Pi^{\infty,v}$;*

– $\Pi_v$ *is the local component of all the* $\Pi' \in \mathcal{U}_{G,\xi}(\Pi^{\infty,v})$ *such that* $d_\xi(\Pi'_\infty) \neq 0$, *is the common value of the* $d^i_\xi(\Pi'_\infty)$ *for* $i \equiv s \mod 2$, *cf. corollary VI.2.2 of* [**6**] *and corollary 3.3 of* [**4**]

*Concerning* $R_{\pi_v}(s,t)(r,i)(\Pi_v)$ *as a sum of representations of* $GL_{d-rg}(F_v)\times\mathbb{Z}$, *for* $\Pi_v \simeq \mathrm{Speh}_s(\mathrm{St}_{t_1}(\pi_{1,v})) \times \cdots \times \mathrm{Speh}_s(\mathrm{St}_{t_u}(\pi_{u,v}))$, *it is given by the formula*

$$R_{\pi_v}(r,i)(\Pi_v) = \sum_{k:\ \pi_{k,v}\sim_i\pi_v} m_{s,t_k}(r,i) R_{\pi_v}(s,t_k)(r,i)(\Pi_v,k) \otimes \Big(\xi_k \otimes \Xi^{i/2}\Big)$$

*where*

- *the characters* $\xi_k$ *are such that* $\pi_{k,v} \simeq \pi_v \otimes \xi_k \circ \mathrm{val} \circ \det$;
- $R_{\pi_v}(s,t_k)(r,i)(\Pi_v,k)$ *can be written as*

$$\begin{aligned} R_{\pi_v}(s,t_k)(r,i)(\Pi_v,k) := \mathrm{Speh}_s(\mathrm{St}_{t_1}(\pi_{1,v})) \times \cdots \times \mathrm{Speh}_s(\mathrm{St}_{t_{k-1}}(\pi_{k-1,v})) \\ \times R_{\pi_{k,v}}(s,t_k)(r,i) \times \\ \mathrm{Speh}_s(\mathrm{St}_{t_{k+1}}(\pi_{k+1,v})) \times \cdots \times \mathrm{Speh}_s(\mathrm{St}_{t_u}(\pi_{u,v})). \end{aligned}$$

- $R_{\pi_{k,v}}(s,t_k)(r,i)$ *is a representation of* $GL_{d-rg}(F_v)$ *which can be computed combinatorially as explained below*
- *and* $m_{s,t}(r,i) \in \{0,1\}$ *is given in the next definition.*

The representation $R_{\pi_{k,v}}(s,t_k)(r,i)$ is computed as follows: we first apply the Jacquet functor $J_{P^{op}_{rg,d}}$ to $\mathrm{Speh}_s(\mathrm{St}_{t_k}(\pi_{k,v}))$ which can be written as a sum

$$\sum \langle a_1 \rangle \otimes \langle a_2 \rangle$$

where $a_1, a_2$ are multisegments in the Zelevinsky line of $\pi_v$; the precise computation is given in corollary 1.5.6 of [**2**]. For $\tau_v$ an irreducible representation of $D^\times_{v,h}$, we then consider

$$\begin{array}{rcl} R_{\tau_v} : \mathrm{Groth}\Big(GL_h(F_v) \times GL_{d-h}(F_v)\Big) & \longrightarrow & \mathrm{Groth}\Big(D^\times_{v,h}/\mathcal{D}^\times_{v,h} \times GL_{d-h}(F_v)\Big) \\ \alpha \otimes \beta & \mapsto & \mathrm{vol}(D^\times_{v,tg}/F_v^\times)^{-1} \sum_\psi \mathrm{Tr}\ \alpha(\varphi_{\tau\otimes\psi^\vee})\psi \otimes \beta, \end{array}$$

where $\psi$ describe the set of character of $F_v^\times$ and $\varphi_{\pi_v[t]_D}$ is, as defined by Deligne, Kazhdan and Vignéras, a pseudo-coefficient for $\mathrm{St}_t(\pi_v)$. We then write

$$R_{\pi_{k,v}[r]_D}\Big(\langle a_1 \rangle\Big) a_2 = \sum_\psi \epsilon_\psi \psi \otimes \Pi_\psi \in \mathrm{Groth}\Big(F_v^\times \times GL_{(st-r)g}(F_v)\Big)$$

with $\epsilon_\psi \in \{-1,1\}$ and where the $\psi$ such that $\Pi_\psi$ are non zero, looks like $|-|^{j/2}$ with $j \in \mathbb{Z}$. Then we have

$$R_{\pi_{k,v}}(s,t_k)(r,i) = \Pi_{|-|^{-i/2}}.$$

*Remark.* We do not need the precise computation of $R_{\pi_v}(s,t)(r,i)$, we just want to use the fact that if $\pi'_v$ is any irreducible cuspidal representation, then $R_{\pi'_v}(s,t)(r,i)$ is obtained from $R_{\pi_v}(s,t)(r,i)$ by replacing $\pi_v$ by $\pi'_v$ in its combinatorial description. In particular if the modulo $l$ reduction of $\pi'_v$ is isomorphic to the modulo $l$ reduction of $\pi_v$, then the modulo $l$ reduction of $R_{\pi_v}(s,t)(r,i)$ and those of $R_{\pi'_v}(s,t)(r,i)$, are isomorphic.

The assumptions on $i$ in proposition 3.6.1 of [**2**] are contained in the following definition of $m_{s,t}(r,i)$.

***4.6. Definition***. — *The point with coordinates $(r,i)$ such that $m_{s,t}(r,i) = 1$, are contained in the convex hull of the polygon with edges $(s+t-1,0)$, $(t,\pm(s-1))$ and $(1,\pm(s-t))$ if $s \geqslant t$ (resp. $(t-s+1,0)$ if $t \geqslant s$); inside it for a fixed $r$, the indexes $i$ start from the boundary en grows by 2, i.e. $m_{s,t}(r,i)=1$ if and only if*

- $\max\{1, s+t-1-2(s-1)\} \leqslant r \leqslant s+t-1$;
- *if* $t \leqslant r \leqslant s+t-1$ *then* $0 \leqslant |i| \leqslant s+t-1-r$ *and* $i \equiv s+t-1-r \mod 2$;
- *if* $\max\{1, s+t-1-2(s-1)\} \leqslant r \leqslant t$ *then* $0 \leqslant |i| \leqslant s-1-(t-r)$ *and* $i \equiv s-t-1+r \mod 2$,

*as represented in figures 1 and 2.*

*Remark.* For

$$\Pi_v \simeq \mathrm{Speh}_s(\mathrm{St}_{t_1}(\pi_{1,v})) \times \cdots \times \mathrm{Speh}_s(\mathrm{St}_{t_u}(\pi_{u,v}))$$

the set of $(r,i)$ such that $[H^i_{!*}(HT_{\overline{1_{rg}},\xi}(\pi_v,\Pi_r))]\{\Pi^{\infty,v}\} \neq 0$ is obtained by superposition of the $u$ previous diagrams as in the figure 3. More precisely for a fixed $(r,i)$, the contribution of diagram of $\mathrm{Speh}_s(\mathrm{St}_{t_k}(\pi_{k,v}))$ is the same as in the starting point $(s+t_k-1,0)$ after replacing $\mathrm{Speh}_s(\mathrm{St}_{t_k}(\pi_{k,v}))$ by $R_{\pi_k}(s,t_k)(r,i)$. We can then trace back any

$$R_{\pi_k}(s,t_k)(r,i)(\Pi'_v) \otimes \left(\xi_k \otimes \Xi^{i/2}\right)$$

of $[H^i_{!*}(HT_{\overline{1_{rg}},\xi}(\pi_v,\Pi_r))]\{\Pi^{\infty,v}\}$, to

$$R_{\pi_k}(s+t_k,0)(r',0)(\Pi'_v) \otimes \left(\xi_k \otimes \Xi^0\right)$$

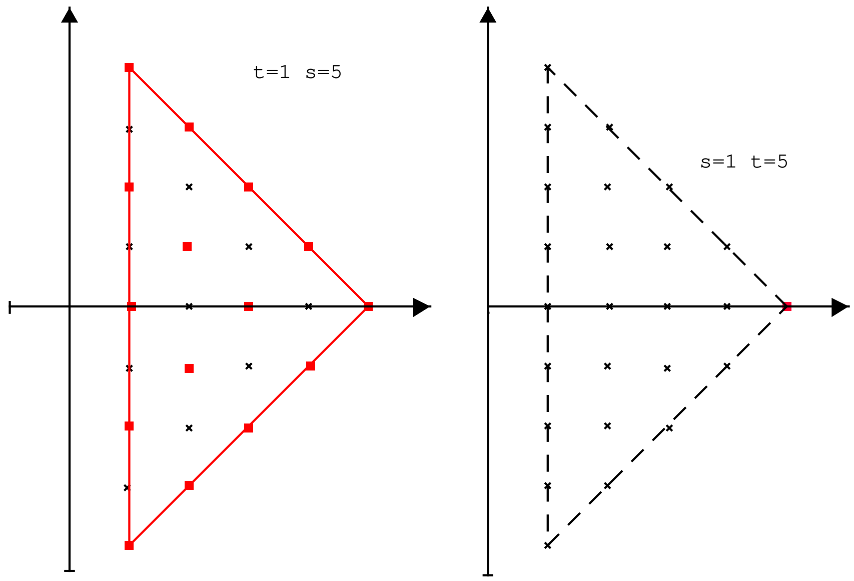


Figure 1. The squares indicate the $(r,i)$ such that $m_{s,t}(r,i)=1$ for a Speh ($t=1$) at left and a Steinberg ($s=1$) on the right

of $[H^0({}^pj_{!*}^{\geqslant(s+t_k)g}\mathcal{F}_{\overline{\mathbb{Q}}_l,\xi}(\pi_v,s+t_k)_1[d-(s+t_k)g])]\{\Pi^{\infty,v}\}$. Note although that for $i=0$, some of the constituants of $[H^0_{!*}(HT_{\overline{1_{rg}},\xi}(\pi_v,\Pi_r))]\{\Pi^{\infty,v}\}$ may or may not come from $r'>r$.

*Comments about the exemple of figure 3 for $r=4$:*

- $\mathrm{Speh}_4(\pi_v)\times\mathrm{Speh}_4(\mathrm{St}_3(\pi_v))\times R_{\pi_v}(4,5)(4,0)$ comes from $(8,0)$;
- $\mathrm{Speh}_4(\pi_v)\times R_{\pi_v}(4,3)(4,0)\times\mathrm{Speh}_4(\mathrm{St}_5(\pi_v))$ comes from $(6,0)$;
- $R_{\pi_v}(4,1)(4,0)\times\mathrm{Speh}_4(\mathrm{St}_3(\pi_v))\times\mathrm{Speh}_4(\mathrm{St}_5(\pi_v))$ does not come from any $(r',0)$ for $r'>4$.

## 5. Integral Harris-Taylor perverse sheaves

We now consider a fixed irreducible cuspidal representation $\pi_v$ such that its modulo $l$ reduction is still supercuspidal. Then for any $t\geqslant 1$, the representation $\pi_v[t]_D$ remains irreducible modulo $l$ so that $\mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}}}$ has an unique, up to homothety, stable $\overline{\mathbb{Z}}_l$-lattice. Then for a $\overline{\mathbb{Z}}_l$-representation $\Pi_t$ of $GL_{tg}(F_v)$, we have a well defined $\overline{\mathbb{Z}}_l$-local system $HT(\pi_v,\Pi_t)$.

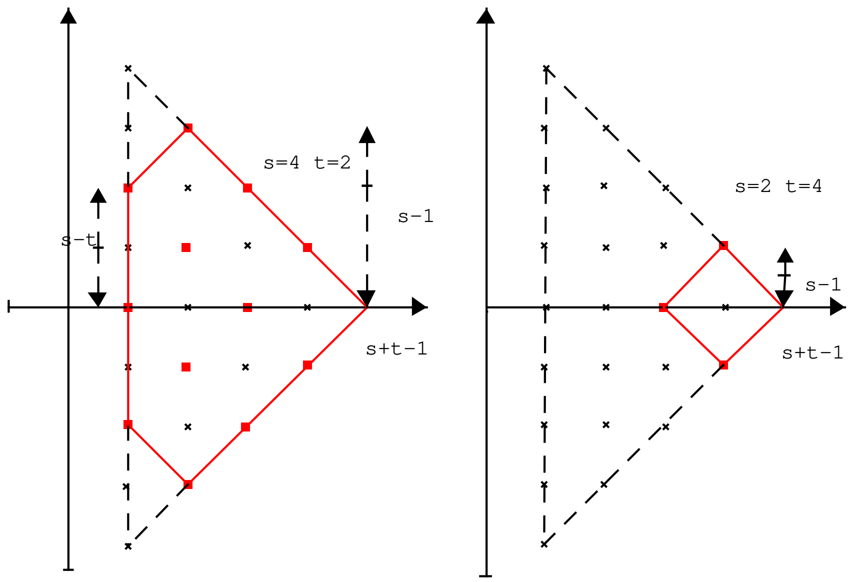


FIGURE 2. $m_{s,t}(r,i)=1$ when $s \geqslant t$ at left and $t \geqslant s$ on the right

*Remark.* $\Pi_t$ is called the infinitesimal part of the Harris-Taylor local system and it does not play any role here, we only mention it as it appears everywhere in [**4**] or [**1**].

Over $\overline{\mathbb{Z}}_l$, we have two natural $t$-structures denoted $p$ and $p+$ which are exchanged by Grothendieck-Verdier duality. We then have two notions of intermediate extension, ${}^p j_{!*}$ and ${}^{p+} j_{!*}$. In [**3**] we explain, using the Newton stratification, how to construct $\overline{\mathbb{Z}}_l$-filtrations of perverse sheaves, with torsion free graded parts. In [**5**], read the introduction of loc. cit. for the statement of the results, we then prove the following results.

- When the modulo $l$ reduction of $\pi_v$ remains supercuspidal, the two intermediate extensions ${}^p j^{=tg}_{!*} HT(\pi_v, \Pi_t)$ and ${}^{p+} j^{=tg}_{!*} HT(\pi_v, \Pi_t)$ are equals, as it is, essentially formally, the case when $\pi_v$ is a character.

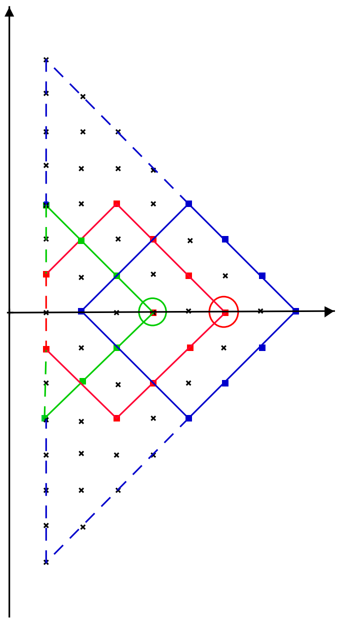

FIGURE 3. Superposition to compute $m(r,i)$ for $\Pi_v \simeq \mathrm{Speh}_4(\pi_v) \times \mathrm{Speh}_4(\mathrm{St}_3(\pi_v)) \times \mathrm{Speh}_4(\mathrm{St}_5(\pi_v))$

– The following resolution of ${}^p j_{!*}^{=t} HT(\pi_v, \Pi_t)$, proved over $\overline{\mathbb{Q}}_l$ in [**1**], is still valid over $\overline{\mathbb{Z}}_l$:

$$0 \to j_!^{=s} HT(\pi_v, \Pi_t\{\frac{t-s}{2}\}) \times \mathrm{Speh}_{s-t}(\pi_v\{t/2\})) \otimes \Xi^{\frac{s-t}{2}} \longrightarrow \cdots \\ \longrightarrow j_!^{=t+1} HT(\pi_v, \Pi_t\{-1/2\} \times \pi_v\{t/2\}) \otimes \Xi^{\frac{1}{2}} \longrightarrow \\ j_!^{=t} HT(\pi_v, \Pi_t) \longrightarrow {}^p j_{!*}^{=t} HT(\pi_v, \Pi_t) \to 0. \quad (5.1)$$

This resolution is equivalent to the fact that the cohomology sheaves of ${}^p j_{!*}^{=t} HT(\pi_v, \Pi_t)$ are torsion free.

– We have a filtration

$$\mathrm{Fil}_!^{(t-s)}(\pi_v, \Pi_t) \hookrightarrow \mathrm{Fil}_!^{t-s+1}(\pi_v, \Pi_t) \hookrightarrow \cdots \\ \hookrightarrow \mathrm{Fil}_!^{0}(\pi_v, \Pi_t) = j_!^{=tg} HT(\pi_v, \Pi_t), \quad (5.2)$$

with graded parts $\mathrm{gr}_!^{-\delta}(\pi_v,\Pi_t) \simeq {}^p j_{!*}^{\geqslant(t+\delta)g} HT(\pi_v,\Pi_t \overrightarrow{\times} \mathrm{St}_\delta(\pi_v)) \otimes \Xi^{\delta/2}$.

– By the adjunction property, the map

$$j_!^{=(t+\delta)g} HT(\pi_v,\Pi_t\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\pi_v\{t/2\})) \otimes \Xi^{\delta/2}$$
$$\longrightarrow j_!^{=(t+\delta-1)g} HT(\pi_v,\Pi_t\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\pi_v\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}} \quad (5.3)$$

is given by

$$HT(\pi_v,\Pi_t\{\frac{-\delta}{2}\} \times \mathrm{Speh}_\delta(\pi_v\{t/2\})) \otimes \Xi^{\delta/2} \longrightarrow$$
$${}^p i^{(t+\delta)g,!} j_!^{=(t+\delta-1)g} HT(\pi_v,\Pi_t\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\pi_v\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}}. \quad (5.4)$$

***5.5. Proposition***. — *By the main result of* [**5**], *we have*

$${}^p i^{(t+\delta)g,!} j_!^{=(t+\delta-1)g} HT(\pi_v,\Pi_t\{\frac{1-\delta}{2}\} \times \mathrm{Speh}_{\delta-1}(\pi_v\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}}$$
$$\simeq HT\Big(\pi_v,\Pi_t\{\frac{-\delta}{2}\}) \times \big(\mathrm{Speh}_{\delta-1}(\pi_v\{-1/2\}) \times \pi_v\{\frac{\delta-1}{2}\}\big)\{t/2\}\Big) \otimes \Xi^{\delta/2}. \quad (5.6)$$

*Proof*. — As it is not explicitly stated in [**5**], we then give a short proof of this fact. By duality we are led to understand ${}^p i^{(t'+1)g,*} j_*^{=t'g} HT(\pi_v,\Pi_{t'})$. From [**5**], we have a filtration of $j_*^{=t'g} HT(\pi_v,\Pi_{t'})$ with successive graded parts

– ${}^{p+} j_{!*}^{=t'g} HT(\pi_v,\Pi_{t'})$ which is equals to ${}^p j_{!*}^{=t'g} HT(\pi_v,\Pi_{t'})$ as the modulo $l$ reduction of $\pi_v$ is supposed to be supercuspidal;
– ${}^{p+} j_{!*}^{=(t'+1)g} HT(\pi_v,\Pi_{t'}\{1/2\} \times \pi_v\{-t'/2\}) \otimes \Xi^{-1/2}$ such that when taking ${}^p i^{(t'+1)g,*}$ we obtain $HT(\pi_v,\Pi_{t'}\{1/2\} \times \pi_v\{-t'/2\}) \otimes \Xi^{-1/2}$,
– the following graded parts give 0 after taking ${}^p i^{(t'+1)g,*}$.

We then have a long exact sequence

$$0 \to {}^p i^{(t'+1)g,*}\, {}^p j_{!*}^{=t'g} HT(\pi_v,\Pi_{t'}) \longrightarrow {}^p i^{(t'+1)g,*} j_*^{=t'g} HT(\pi_v,\Pi_{t'}) \longrightarrow$$
$$HT(\pi_v,\Pi_{t'}\{1/2\} \times \pi_v\{-t'/2\}) \otimes \Xi^{-1/2} \longrightarrow \cdots$$

where ${}^p i^{(t'+1)g,*}\, {}^p j_{!*}^{=t'g} HT(\pi_v,\Pi_{t'})$ is zero as it is free by [**5**] and a $p$ intermediate extension. By taking $t' = t+\delta-1$ et $\Pi_{t'} = \Pi_t\{\frac{1-\delta}{2}\} \times$

$\mathrm{Speh}_{\delta-1}(\pi_v\{t/2\}$ and after dualization, we then obtained the statement of the proposition. $\square$

In particular, up to homothety, the map (5.6), and so those of (5.4), is unique. Finally as the map of (5.1) are strict, the given maps (5.3) are uniquely determined, that is if we forget the infinitesimal parts, these maps are independent of the chosen $t$ in (5.1).

We fix a level $I^v$ outside $v$ and a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S$: recall that fixing $\mathfrak{m}$ means that we focus on liftings $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, i.e. on $\xi$-cohomological automorphic representations $\Pi$ of $G(\mathbb{A})$ such that their modulo $l$ Satake parameters are prescribed by $\mathfrak{m}$. As usual we then denote with an index $\mathfrak{m}$ for the localization $\otimes_{\mathbb{T}^S} \mathbb{T}^S_{\mathfrak{m}}$.

***5.7. Notation***. — *For $1 \leqslant h \leqslant d$, we denote by $i(h)$ the smaller index $i$ such that $H^i(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j^{=h}_{!*} HT_\xi(\pi_v,\Pi_h))_{\mathfrak{m}}$ has non trivial torsion: if it does not exists then set $i(h)=+\infty$.*

*Remark.* By duality, as ${}^p j^{=h}_{!*} = {}^{p+} j^{=h}_{!*}$ for Harris-Taylor local systems associated to a irreducible cuspidal representation whose modulo $l$ reduction remains supercuspidal, note that when $i_I(h)$ is finite then $i(h) \leqslant 0$.

We now suppose for the rest of this section, that $I^v$ and $\mathfrak{m}$ are chosen so that there exists $1 \leqslant t \leqslant s_g$ with $i(t)$ finite and we denote by $t_0$ the bigger such $t$.

***5.8. Lemma***. — *For $1 \leqslant t \leqslant t_0$, we have $i(t) = t - t_0$ and*

$$H^{i(t)}_{tor}(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j^{=tg}_{!*} HT(\pi_v,\Pi_t)) \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l \simeq$$
$$H^0_{tor}(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j^{=t_0 g}_{!*} HT(\pi_v, \Pi_t\{\frac{t_0-t}{2}\} \times \mathrm{Speh}_{t-t_0}(\pi_v)\{\frac{t}{2}\}\Xi^{\frac{t-t_0}{2}}) \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l. \tag{5.9}$$

*Proof.* — Note first that for every $t_0 < t \leqslant s_g$, then the cohomology groups of $j^{=hg}_! HT_\xi(\pi_v,\Pi_h)$ are torsion free. Indeed consider the spectral sequence associated to the filtration (5.2): by definition of $t_0$, note that the $E_1^{p,q}$ are torsion free. Moreover as explained in [**2**] over $\overline{\mathbb{Q}}_l$, we remark that all the $d_r^{p,q}$ for $p+q \geqslant 0$ are zero. So torsion can also appear in degree $\leqslant 0$ but as the open Newton stratum $\mathrm{Sh}^{=hg}_{I,\bar{s}_v}$ are affine, then the cohomology of $j^{=hg}_! HT_\xi(\pi_v,\Pi_h)$ is zero in degree $<0$ and torsion free for $i=0$.

Consider then the spectral sequence associated to the resolution (5.1) for $t > t_0$: its $E_1$ terms are torsion free and it degenerates at $E_2$. As by hypothesis the aims of this spectral sequence is free and equals to only one $E_2$ terms, we deduce that all the maps

$$H^0\big(\mathrm{Sh}_{I^v(\infty),\bar s_v}, j_!^{=(t+\delta)g} HT_\xi(\pi_v, \Pi_t\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\pi_v\{t/2\})) \otimes \Xi^{\delta/2}\big)_{\mathfrak{m}} \longrightarrow H^0\big(\mathrm{Sh}_{I^v(\infty),\bar s_v}, j_!^{=(t+\delta-1)g} HT_\xi(\pi_v, \Pi_h\{\frac{1-\delta}{2}\} \times \mathrm{Speh}_{\delta-1}(\pi_v\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}}\big)_{\mathfrak{m}} \quad (5.10)$$

are strict. Then from the previous fact stressed after (5.6), this property remains true when we consider the associated spectral sequence for $1 \leqslant t' \leqslant t_0$.

Consider now $t = t_0$ where we know the torsion to be non trivial. From what was observed above we then deduce that the map

$$H^0\big(\mathrm{Sh}_{I^v(\infty),\bar s_v}, j_!^{=(t_0+1)g} HT_\xi(\pi_v, \Pi_{t_0}\{\frac{-1}{2}\}) \times \pi_v\{t_0/2\})) \otimes \Xi^{1/2}\big)_{\mathfrak{m}} \longrightarrow H^0\big(\mathrm{Sh}_{I^v(\infty),\bar s_v}, j_!^{=t_0 g} HT_\xi(\pi_v, \Pi_{t_0})\big)_{\mathfrak{m}} \quad (5.11)$$

has a non trivial torsion cokernel so that $i(t_0) = 0$.

Finally for any $1 \leqslant t \leqslant t_0$, the map like (5.11) for $t+\delta-1 < t_0$ are strict so that the $H^i(\mathrm{Sh}_{I^v(\infty),\bar s_v}, {}^p j_{!*}^{=tg} HT_\xi(\pi_v, \Pi_t))_{\mathfrak{m}}$ are zero for $i < t - t_0$ while when $t + \delta - 1 = t_0$ its cokernel has non trivial torsion which gives then a non trivial torsion class in $H^{t-t_0}(\mathrm{Sh}_{I^v(\infty),\bar s_v}, {}^p j_{!*}^{=tg} HT_\xi(\pi_v, \Pi_t))_{\mathfrak{m}}$. □

## 6. Automorphic congruences

From now on we consider

- two irreducible algebraic representations $\xi$ and $\xi'$ defined over some finite extension $K/\mathbb{Q}_l$ with stable lattice $W_{\xi,\mathcal{O}}$ and $W_{\xi',\mathcal{O}}$ such that modulo they become isomorphic modulo $\lambda$ any uniformizer of $\mathcal{O}$;
- two irreducible cuspidal representations $\pi_v$ and $\pi'_v$ such that their modulo $l$ reduction are isomorphic and supercuspidal,
- and as before, a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S$.

For $V$ a $\overline{\mathbb{Z}}_l$-free module, we denote by $r_l(V) = V \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ its modulo $l$ reduction.

***6.1. Notation***. — *For $\psi_v$ a $\overline{\mathbb{F}}_l$-representation of $GL_h(F_v)$, we denote by $\dim_{\overline{\mathbb{F}}_l,n}\psi$ the set*

$$\Big\{\dim_{\overline{\mathbb{F}}_l} \psi_v^{K_v(n)},\ n \in \mathbb{N}\Big\}.$$

***6.2. Theorem.*** — *(cf. conjecture 5.2.1 of* [**4**]*)*
*Let* $r$ *be maximal such that there exists* $s$ *and* $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ *with* $\Pi_{\widetilde{\mathfrak{m}}} \in \mathcal{A}_{\xi,\pi_v}(r,s)$ *and* $\Pi_{\widetilde{\mathfrak{m}},S}^{I^v} \neq (0)$*. Then we have*

$$\sum_{\Pi \in \mathcal{A}_{\xi,\pi_v}(r,s)} m(\Pi) d_\xi(\Pi_\infty) \dim_{\overline{\mathbb{Q}}_l} (\Pi^{\infty,v})^{I^v} \dim_{\overline{\mathbb{F}}_l,n} r_l\big(R_{\pi_v}(r,r)(\Pi_v)\big)$$
$$=$$
$$\sum_{\Pi' \in \mathcal{A}_{\xi',\pi'_v}(r,s)} m(\Pi') d_\xi(\Pi'_\infty) \dim_{\overline{\mathbb{Q}}_l} (\Pi'^{,\infty,v})^{I^v} \dim_{\overline{\mathbb{F}}_l,n} r_l\big(R_{\pi_v}(r,r)(\Pi'_v)\big). \quad (6.3)$$

*Remark.* The result is still valid without the maximality hypothesis on $r$ but it is more tricky to expose, cf. the remark after the proof.

A qualitative version of the previous theorem could be formulated as follows.

- Given a irreducible $\xi$-cohomological automorphic representation $\Pi$ of $G(\mathbb{A})$ such that $\Pi_v \simeq \mathrm{Speh}_s(\mathrm{St}_t(\pi_v)) \times ?$, with $\pi_v$ cuspidal with modulo $l$ reduction still supercuspidal,
- and $\pi'_v$ such that its modulo $l$ reduction is isomorphic to those of $\pi_v$,

then there exists a irreducible $\xi'$-cohomological automorphic representation $\Pi'$ such that

- outside $v$, $\Pi$ and $\Pi'$ share the same level $I^v$,
- their also share the same modulo $l$ Satake parameters at the places outside $S$, i.e. they are weakly automorphic congruent,
- at $v$, we have $\Pi'_v \simeq \mathrm{Speh}_s(\mathrm{St}_t(\pi_v)) \times ?'$,

where by convention, the symbols ? and ?′ mean any representation we do not want to precise.

The strategy of the proof is the same as in [**4**]. The main point to notice is that the left hand side of (6.3) is a part of the image of $H^0(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j_{!*}^{=rg} HT_\xi(\pi_v,\Pi_r))_{\mathfrak{m}}$. In [**4**] §5.3, we look at the particular case where the localized cohomology groups at $\mathfrak{m}$ are concentrated in degree 0: this is for example the case when $\xi$ is very regular, or if $\overline{\rho}_{\mathfrak{m}}$ is irreducible. Then

$$[H^*(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j_{!*}^{=rg} HT_\xi(\pi_v,\Pi_r))_{\mathfrak{m}}] = [H^0(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j_{!*}^{=rg} HT_\xi(\pi_v,\Pi_r))_{\mathfrak{m}}],$$

and its modulo $l$ reduction is then equal to

$$H^*(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, \mathbb{F}^p j_{!*}^{=rg} HT_{\xi'}(\pi'_v,\Pi_r))_{\mathfrak{m}} = H^*(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, j_{!*}^{=rg} HT_{\xi',\overline{\mathbb{F}}_l}(\pi'_v,\Pi_r))_{\mathfrak{m}}$$
$$= r_l\Big(H^0(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j_{!*}^{=rg} HT_{\xi'}(\pi'_v,\Pi_r))_{\mathfrak{m}}\Big),$$

where $\mathbb{F}(\bullet) = \bullet \otimes^{\mathbb{L}}_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ and by the main result of [**5**], as ${}^p j_{!*}^{=rg} HT_{\xi'}(\pi'_v,\Pi_r) \simeq {}^{p+} j_{!*}^{=rg} HT_{\xi'}(\pi'_v,\Pi_r)$ for $\pi'_v$ such that $r_l(\pi'_v)$ remains supercuspidal, then

$$\mathbb{F}^p j_{!*}^{=rg} HT_{\xi'}(\pi'_v,\Pi_r) \simeq j_{!*}^{=rg} HT_{\xi',\overline{\mathbb{F}}_l}(\pi'_v,\Pi_r).$$

To isolate the various contribution in these cohomology groups, we start from $r=d$ towards $r=1$ so that for a fixed $r$, as for $r'>r$, the modulo $l$ reduction of elements of $\mathcal{A}_{\xi,\pi_v}(r',1)$ was, by an inductive argument, already identified with those of $\mathcal{A}_{\xi',\pi'_v}(r',1)$, we can then identify the contribution of $\mathcal{A}_{\xi,\pi_v}(r,1)$ in the modulo $l$ reduction of $[H^*(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j_{!*}^{=rg} HT_\xi(\pi_v,\Pi_r))_{\mathfrak{m}}]$ to that of $\mathcal{A}_{\xi',\pi'_v}(r,1)$.

In the general case where $s$ is not necessary equal to 1 but its congruence modulo 2 is fixed, the first problem is that $\mathcal{A}_{\xi,\pi_v}(r,s)$ and $\mathcal{A}_{\xi,\pi_v}(r,s')$ with $s>s'$ first contribute to $[H^0(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j_{!*}^{=rg} HT_\xi(\pi_v,\Pi_r))_{\mathfrak{m}}]$ at the same time. To separate them, the only solution seems to look at $[H^{1-s}(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j_{!*}^{=(r+1-s)g} HT_\xi(\pi_v,\Pi_r))_{\mathfrak{m}}]$ where only $\mathcal{A}_{\xi,\pi_v}(r,s)$ contributes. We are then led to argue on individual cohomology groups where we recall the following short exact sequence for a torsion free $\overline{\mathbb{Z}}_l$-perverse sheaf $P$ over a $\overline{\mathbb{F}}_p$-scheme $X$:

$$0 \to \mathbb{F}H^i(X,P) \longrightarrow H^i(X,\mathbb{F}P) \longrightarrow H^{i+1}(X,P)[l] \to 0,$$

where in our situation $P = {}^p j_{!*}^{=tg} HT(\pi_v,\Pi_t)$ and, as recall before, as the $p$ and $p+$ intermediate extension of $HT(\pi_v,\Pi_t)$ are isomorphic, then

$$\mathbb{F}^p j_{!*}^{=tg} HT(\pi_v,\Pi_t) \simeq j_{!*}^{=tg} HT_{\overline{\mathbb{F}}_l}(\pi_v,\Pi_t).$$

We then must face the problem of understanding the $l$-torsion and its contribution in the modulo $l$ reduction of the cohomology. In [**4**] §5.4, we formulate the conjecture that, for a fixed representation $\Pi_t$ of $GL_{tg}(F_v)$, the modulo $l$ reduction of the torsion in each cohomology group of ${}^p j_{!*}^{=tg} HT(\pi_v,\Pi_t)$ should depend only on the modulo $l$ reduction of $\pi_v$ and we explain how this conjecture implies the previous theorem.

*Proof.* — We first look at $H^0(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, {}^p j_{!*}^{=rg} HT_\xi(\pi_v,\Pi_t))_{\mathfrak{m}}$ which, by maximality of $r$ and the spectral sequence associated to (5.1), is isomorphic to $H^0(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, j_!^{=rg} HT_\xi(\pi_v,\Pi_t))_{\mathfrak{m}}$ and so is torsion free. To this cohomology group contribute the irreducible automorphic representations of $\mathcal{A}_{\xi,\pi_v}(r,s)$ with $r=s+t-1$ for some set

$$\mathcal{B}_\xi(\pi_v)=\Big\{(s,t) \text{ s. t. } \exists \widetilde{\mathfrak{m}}\subset\mathfrak{m}, \Pi_{\widetilde{\mathfrak{m}}}\in\mathcal{A}_{\xi,\pi_v}(r,s) \text{ and } (\Pi_{\widetilde{\mathfrak{m}}}^{\infty,v})^{I_S}\neq(0)\Big\}.$$

The qualitative version of the theorem asks to prove that $\mathcal{B}_\xi(\pi_v)=\mathcal{B}_{\xi'}(\pi'_v)$ and the quantitative one then follows from the formula of the multiplicities in 4.5.

(a) About the quantitative version, consider first the easiest case when the qualitative version tells us $\mathcal{B}_\xi(\pi_v)=\mathcal{B}_{\xi'}(\pi'_v)=\{(s,t)\}$. Then the modulo $l$ reduction of

$$[H^0(\mathrm{Sh}_{I^v(\infty),\bar{s}_v},{}^pj_{!*}^{=rg}HT_\xi(\pi_v,\Pi_t))_{\mathfrak{m}}]=[H^*(\mathrm{Sh}_{I^v(\infty),\bar{s}_v},{}^pj_{!*}^{=rg}HT_\xi(\pi_v,\Pi_t))_{\mathfrak{m}}]$$

is equal to that of $H^*(\mathrm{Sh}_{I^v(\infty),\bar{s}_v},{}^pj_{!*}^{=rg}HT_{\xi'}(\pi'_v,\Pi_t))_{\mathfrak{m}}$ which is also equal to that of $H^0(\mathrm{Sh}_{I^v(\infty),\bar{s}_v},{}^pj_{!*}^{=rg}HT_{\xi'}(\pi'_v,\Pi_t))_{\mathfrak{m}}$. We can then conclude, as these two cohomology groups are torsion free, from the explicit computation of the multiplicities in 4.5.

(b) We now focus on the sequence of the dimensions

$$d_{k,n}=\dim_{\overline{\mathbb{F}}_l}H^k(\mathrm{Sh}_{I^v(n),\bar{s}_v},{}^pj_{!*}^{=(r-k)g}HT_\xi(\pi_v,\mathbb{1}_{r-k}))_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l,$$

where $\mathbb{1}_{r-k}$ is the trivial representation of $GL_{(r-k)g}(F_v)$, for $k=0,\cdots,r-1$ and $n\in\mathbb{N}$. More specifically we will focus on the $d_{k,n}-d_{k+1,n}$. Note the following facts:

- By maximality of $r$, we have $H^{k+1}(\mathrm{Sh}_{I^v(n),\bar{s}_v},{}^pj_{!*}^{=(r-k)g}HT_\xi(\pi_v,\mathbb{1}))_{\mathfrak{m}}=(0)$ so that

$$\begin{aligned}H^k(\mathrm{Sh}_{I^v(n),\bar{s}_v},{}^pj_{!*}^{=(r-k)g}HT_\xi(\pi_v,\mathbb{1}_{r-k}))_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l\\ \simeq H^k(\mathrm{Sh}_{I^v(n),\bar{s}_v},j_{!*}^{=(r-k)g}HT_{\xi,\overline{\mathbb{F}}_l}(\pi_v,\mathbb{1}_{r-k}))_{\mathfrak{m}}.\end{aligned}$$

- By lemma 5.8, the dimension of the modulo $l$ reduction of the torsion of $H^k(\mathrm{Sh}_{I^v(n),\bar{s}_v},{}^pj_{!*}^{=(r-k)g}HT_\xi(\pi_v,\mathbb{1}))_{\mathfrak{m}}$ is prescribed by those of $H^1(\mathrm{Sh}_{I^v(n),\bar{s}_v},{}^pj_{!*}^{=(r-1)g}HT_\xi(\pi_v,\mathbb{1}))_{\mathfrak{m}}$.
- From 4.5 and the description in 4.6 of the $m_{s,t}(r,i)$, the behavior of the contribution to the sequence $d_{k,n}-d_{k+1,n}$ of the modulo $l$ reduction of the free quotients of each cohomology groups is completely determined by $\mathcal{B}_\xi(\pi_v)$. Indeed note that the contribution of some automorphic representation $\Pi$ such that $\Pi_v\simeq\mathrm{Speh}_s(\mathrm{St}_t(\pi_v))\times?$ with $s+t-1=r$, only contributes to $d_{k,n}$ for $k=0,\cdots,s-1$ so

that $d_{s-1,n} - d_{s,n}$, after eliminating the contribution of the torsion part, will detect such $\Pi$.

So as the contributions of the torsion to the sequence $(d_{k,n})_{k\geqslant 1, n\in\mathbb{N}}$ depends only on $(d_{1,n})_{n\in\mathbb{N}}$, we can then infer the elements $(s,t) \in \mathcal{B}_\xi(\pi_v)$ (resp. $\mathcal{B}_{\xi'}(\pi'_v)$) for $s \geqslant 2$ and prove the quantitative previous theorem for $s \geqslant 2$. To deduce the result for $s = 1$, we then look at $(d_{0,n})_{n\in\mathbb{N}}$ where the contribution of the torsion is zero and where, from what we have already obtained before, the contribution of the free quotient of each of the $s \geqslant 2$ are the same for $\pi_v$ and $\pi'_v$. The remaining part for $\pi_v$ and $\pi'_v$ should then match, which gives us the case of $s = 1$.

$\square$

*Remark.* We could keep on tracing the contribution of the torsion submodules, to the modulo $l$ reduction of the others cohomology groups. Once the contribution coming from $r$ maximal is understood, we could consider the next $r' < r$ and repeat the previous argument taking into account what is already known about the contribution of what is related to $r$.

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