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The Mask Game with Multiple Populations

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Abstract. Masks save lives. Therefore, whilst the culture of wearing masks is promoted, it is critical to understand the various aspects of how that culture is adopted. The main contribution of this paper is in the modeling of the mask game. Wearing a mask provides partial protection against epidemics at some cost of comfort. Players can be differentiated according to both their risk-state as well as their health-state (susceptible, infected and removed). We formulate the problem as a Bayesian game in which players know their own risk-state and ignore their own health state and the health and risk-states of their counterparts. Using ideas from evolutionary games, we reduce the problem to a one-shot equivalent game and describe the structure of the symmetric equilibria. We prove that the policies adopted by the players at such equilibria admit a threshold structure. More specifically, players wear masks only if their risk-state is equal to or bigger than a given threshold.

1 Introduction

Masks save lives [4, 7, 20, 26]. Before the COVID-19 pandemics which erupted in 2019, the vast majority of the world population had neither experienced social distancing nor the need of wearing masks. Since 2019, the situation has drastically changed: governments had to impose routinely large-scale mobility restrictions or mandate mask wearing in public spaces.

Nowadays, it is critical to understand the various aspects of how the culture of wearing masks is adopted. Such a study may in turn guide public policies and decision makers in crafting and adjusting measures that both incentivize adoption of and penalize resistance to masks.

In this paper, we study the multi-population mask game which is representative of competition and free riding situations that typically impact the results of interactions among individuals. The question of whether *to wear or not to wear masks* has been much discussed and has become central in the debate within the context of the global COVID-19 pandemics [24].

The mask game. The essence of the mask game is captured as follows. Wearing a mask provides protection (and reassurance) to the individual taking that action. At the same time, it protects others from contamination originated

at such particular individual [3]. While wearing a mask may cause discomfort (both physical and psychological), not wearing a mask results in taking risks of contamination. The probability that contamination results in a severe form of the disease varies between different sub-populations depending on their ages, health problems, sex, blood types etc. More generally, the evolution of pandemics is strongly impacted by the cooperative behavior of the sub-populations. In this context, cooperation may require social distancing and mobility limitation.

Why game theory? Game theoretic arguments are useful in various levels in order to understand the evolution of a pandemic and how to efficiently protect the population and its economical interests [5, 6, 14, 16, 22, 25, 29]. In this paper, we focus on modeling two behaviors within a population: cooperation and free riding. Free-riding is observed at various levels. First, among individuals that may not wear masks taking advantage of the fact that other individuals (in particular those for whom getting infected may be critical) actually wear masks. Second, among institutions from the same or different countries. For instance, non-cooperation and lack of coordination between public health institutions in different regions or countries has been reported in [11][5].

In previous works [27, 28], we have studied a vaccination game in which the context is that of malware epidemic diffusion in information networks. In such a context, herd immunity also occurs, and thus, games that describe competition and collaboration arise as natural tools for analysis. Unlike the scenario studied in this paper, in [27,28] players pay for getting a vaccination. Therein, we showed the existence of pure equilibrium in an SIS epidemic model that serves as a mean field approximation for the case of a population with a large number of individuals.

In [19], McNamara considered a version of the hawk-dove game where an animal knows its own fighting ability but not the ability of its opponent. Similarly, in this work we assume that players know their own risk-state, but not the risk-state of the rest of the population. In addition, we assume that the health-state is unknown (both for the individual and for the population), but that statistics about risk-states and health-states are common knowledge. Such statistics could be learned from experience, e.g., assuming that they change in a coarse time scale, noting that we leave learning aspects as subject for future work.

1.1 Our contribution

Our main contribution is twofold. First, we introduce and formulate the mask game in order to study cooperation and free-riding phenomena. Second, we analyze the dependence of these phenomena on risk groups of susceptible people. This work was motivated by various assumptions and recommendations from public health organizations for avoiding spreading the COVID-19 pandemics. We envision that these recommendations and the way they are applied, in turn, may benefit from the models introduced in this paper.

[5] https://epi.ufl.edu/articles/covid-19-epidemic-is-worsened-without-coordination.html

1.2 Safe distancing

COVID-19 has stimulated a huge amount of scientific research including experiments, simulations as well as mathematical models.[6] Consider, for instance, the problem of determining safe distancing [2, 12].

Safe distancing is defined differently in various countries.[7] The WHO defined it to be one meter and such distance was used in France and in China [10]; whereas in USA and India, such distance was set to two meters. Posing safety distance in terms of a single number could be far from accurate due to the fact that the duration of exposure is not taken into account [10, 13, 15]. Other approaches have been advocated to consider the time of exposure as an additional parameter that defines the safety of interactions. According to UK Health Security Agency, the chances of two individuals being infected when in contact with each other is low if 1) the distance between them is at least one meter and the contact duration does not exceed one minute; or 2) the distance is at least two meters and the duration of the contact does not exceed fifteen minutes.[8] In Appendix A, we present simple models for the probability of getting infected as a function of the interaction duration and the distance between the interacting players.

Our main results are based on a bi-linear payoff model, in which the effect of temporal and spatial aspects on safe distancing is indirectly captured through the model parameters, given as a function of infection rates with and without masks. The models presented in Appendix A, in contrast, motivate extensions of our main results, beyond the bi-linear payoff models, and their detailed analysis is left as subject for future work.

1.3 Other games that model evolution of cooperation

The study of cooperation using game theory has a long history and a rich literature. Most known approaches for this study have been the prisoner's dilemma [25], the Hawk-Dove game and the Chicken game [19]. These games have been adapted to various dynamic settings, see e.g. [1]. While these games provide the methodology for analyzing many cooperation issues, there are some fundamental aspects that they do not address. For example, by protecting oneself from the virus (through distancing, wearing masks, etc) one also protects society from the virus. Thus the players in the mask game have more incentive to act in a cooperative way.

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[6] https://www.sciencemag.org/news/2020/03/mathematics-life-and-death-how-disease-models-shape-national-shutdowns-and-other

[7] See for instance https://theprint.in/theprint-essential/1m-1-5m-2m-the-different-levels-of-social-distancing-countries-are-following-amid-covid/449425/.

[8] https://www.gov.uk/government/publications/guidance-for-contacts-of-people-with-possible-or-confirmed-coronavirus-covid-19-infection-who-do-not-live-with-the-person

2 Model and problem formulation

2.1 Players, risk-states and health-states

Players We consider a countably infinite set of players. Players experiencing the COVID-19 might exhibit a variety of symptoms that can be broadly classified into mild, moderate, severe and critical. Severe and critical symptoms require emergency treatment and may cause death. Alternatively, mild and moderate symptoms can be treated with medication and self-isolation. In this context, there are different risk groups in the population given that severe and critical symptoms are often correlated with some age ranges, physical and health conditions, obesity and blood type, among others.

Risk-states. The risk-state of player i is denoted by $m(i)$. Risk-states are assumed to be ordered with lower states corresponding to lower risk of developing severe or critical symptoms. That is, the higher the risk-state, the smaller the probability to develop severe or critical symptoms conditioned on being infected. The probability of a given player to be in risk-state m is denoted by $q(m)$.

Health-states. The health state of player i is denoted by $h(i)$. The health states are S (susceptible); I (infected); or R (removed). The current state of a given player is ruled by the following dynamics:

- Susceptible (S): Players in this state are not immune to the SARS-CoV-2, and thus, might be infected.
- Infected (I): Players in this state have been infected and capable of infecting other players. Players remain in this state as long as they are asymptomatic or until they are quarantined.
- Recovered (R): Players in this state are not capable of infecting other players. Hence, either they have recovered from the virus (and assumed to be immunized), or they are deceased or they are quarantined.

Remark 1. We consider that the fraction of individuals in each of the health-states varies in a slow time scale. In particular, we denote by p the fraction of individuals at state I.

Information structure. A central assumption in the mask game is that player i does not know whether their own health-state is S or I. Moreover, players ignore both the health states (S or I) and risk-states m of the other players. Indeed, we assume that a newly infected player is contagious before symptoms develop. After some random time, an infected player enter the removed state and does not endanger other individuals anymore. E.g., the player may become symptomatic and/or observe a positive test result and therefore be quarantined. Let p be the probability that a player is initially in health-state S.

Formally, this is a Bayesian game [23] with a countably infinite set of players. There are $3M$ types of players, where 3 corresponds to the number of health-states and M to the number of risk-states. In this paper, we restrict to static Bayesian games, since we are interested in the Bayesian equilibrium for an arbitrary tagged player, rather than in dynamic properties of the population.

Table 1. Table of notation

Term	Description
i	player
M	number of risk-states (or classes)
$m(i)$	risk-state (or class) of player i, $m(i) \in \{1, \ldots, M\}$
$h(i)$	health-state of player i, $h(i) \in \{S, I, R\}$
NC	action 0, not cooperate (not wear mask)
CP	action 1, cooperate (wear mask)
$a(i)$	action of player i, $a(i) \in \{NC, CP\}$
$g(a)$	discomfort of action a; $g(CP) > 0$;
	$g(NC)$ can be positive to capture cost of fine (for not wearing masks)
$w(CP)$	$w(CP) \in [0, 1]$, where $w(CP) = 0$ (resp., $w(CP) = 1$) if
	CP provides maximum (resp., no) protection
$r(m)$	risk factor of type m, e.g., for getting severe form of disease,
	$r(1) < r(2) < \ldots < r(M)$
sets of policies	
U_p	set of pure policies; U_p has cardinality 2^M
U_r	set of randomized policies
U_m	set of mixed policies
actions and contact rates	
$u(m)$	$u(m) \in \{0, 1\}$, $u(m) = 1$ if tagged player of type m plays CP, 0 otherwise;
	under mixed policy, $u(m) \in [0, 1]$ is probability of playing CP
$v(l)$	$v(l) \in \{0, 1\}$, $v(l) = 1$ if a player of type l, different from the tagged player,
	plays CP; under mixed policy, $v(l) \in [0, 1]$ is probability of playing CP
β	baseline contact rate between susceptible or infected players
$\lambda_m(u, v)$	thinned contact rate towards tagged player u for a tagged player in risk-state m
	$\lambda_m(u,v) = \Big(u(m)w(u(m)) + (1-u(m))\Big)\beta \sum_{l=1}^{M} q(l)\Big(v(l)w(v(l)) + (1-v(l))\Big)$
$\mu(a, b)$	auxiliary function for thinned contact rate,
	$\mu(a, b) = \beta(wa + (1-a))(wb + (1-b))$
priors	
p	health-state prior, $p \in \mathbb{R}$, where $p = 1$ (resp., $p = 0$) if S (resp., I)
	occurs with probability 1
$q(m)$	risk-state prior; probability that encounter occurs with individual of type m

2.2 Actions and interactions

Player i is characterized by its risk-state $m(i)$ and by its action $a(i)$ (the choice of the action may depend on the state). We consider a set A containing two

possible actions: *cooperative*, i.e., $a(i) = CP$, which is to put the mask on, or *non-cooperative*, i.e., $a(i) = NC$, e.g., to go without mask. CP and NC are the possible pure actions of each player at each risk-state m and health-state S or I. A mixed action $u(m)$ adopted by a player in risk-state m is the probability that the player chooses action CP, i.e., to wear a mask, whereas $1 - u(m)$ is the probability to choose NC, i.e., not to wear a mask.

We consider in this paper interactions among randomly selected pairs of players. This is motivated by evolutionary game theory, and by a similar approach recently used for modeling the infection process in China [17].

Pairwise encounters between susceptible or infected players occur according to a Poisson process with rate β. Let $q(m)$ be the fraction of individuals of risk-state m, where $\sum_{m=1}^{M} q(m) = 1$. Then, each tagged player encounters players of risk-state m according to a Poisson process with parameter $\beta \cdot q(m)$. *As we assume that the fraction of infected individuals changes on a slow time scale, the rate $\beta \cdot q(m)$ of interactions between a tagged player and players of risk-state m is assumed to be constant and does not depend on players' actions.*

Thinning. Adopting a cooperative behavior does not eliminate the risk of being infected. In our model, we assume that the result of choosing pure actions a and b by players i and j respectively, is to reduce the contact rate according to the value $w(a)w(b)\beta$ where $0 < w(CP) < w(NC) = 1$. That is, based on the pure actions a and b, players i and j can block on average a fraction $1 - w(a)w(b)$ of potential infections in repeated interactions between them. This provides the relevant measure of infection rate between them which we use in the rest of the paper. The resulting effect is that of *thinning* the interaction process: in general, thinning a point process of interactions means reducing the interaction rate by deleting independently the events of interaction according to independent Bernoulli trials. As a result, the actual interaction process is still an independent Poisson process but with rate $w(a)w(b)\beta$.

2.3 Partial information and policies

A susceptible player involved in encounters with an infected player receives a payoff that depends on the pair (a, m) of its action a and its own risk-state m, as well as on the action b (but not the risk-state) of the other player. As a result, players have only partial information, and the payoff is calculated as a mean taken over the actual health-states of the two players, which are assumed to be unknown to the two interacting players. They have however knowledge on the probability distribution over the unobserved parameters.

Assumption 1 *Players know statistics of population risk level (i.e., they know the distribution of risk) but they do not know the specific risk level of their neighbors.*

This is a Bayesian game and it can be solved by an equivalent full information game, once we replace the original actions with some rules to choose the actions

based on each players risk-state; the pure actions of this full information game are called policies. The full information game is also called the extended game.

Definition 1 (Pure policy). *A pure policy of a player is a map from the set of M possible risk-states to the set A of pure actions (CP and NC).*

For instance for $M = 2$, a pure policy $u(1) = 0$ and $u(2) = 1$ prescribes playing NC in risk-state 1 and playing CP in risk-state 2. Note that the policy depends on the player own risk-state. It does not depend on the health-state since a player does not know its own health-state (nor that of the player he is interacting with).

Assumption 2 *Players know their own risk-state, but they do not know their health-state.*

Definition 2 (Randomized stationary policy). *A randomized stationary policy u is a map that assigns to each m a probability $u(m)$ of playing CP. The action NC is then played with probability $1 - u(m)$.*

Definition 3 (Mixed policy). *A mixed policy is a randomized stationary policy, given as a probability distribution over the set of pure policies.*

Continuing on the $M = 2$ example, the randomized policy u such that $u(1) = 0$ and $u(2) = 1/2$ is obtained by randomizing with probability $1/2$ between pure policies u_1 and u_2, where $u_1(1) = u_2(1) = 1$ and $u_1(2) = 1$ and $u_2(2) = 0$, i.e., $u = (u_1 + u_2)/2$. In Section 5, we shall prove that a randomized symmetric equilibria of the mask game can be always obtained by mixing two consecutive pure threshold policies.[9]

We denote by U_p, U_r and U_m the classes of pure, randomized and of mixed policies, respectively. Note that U_p has cardinality 2^M.

2.4 The extended game: health-state and risk-state priors

The extended game has the same set of players. Instead of choosing an action directly among $\{CP, NC\}$, a player chooses a policy. The game is played as follows. First, a type (i.e., state) is assigned to each player, independently, according to priors $q(m)$ (for the risk-state) and p (for the health-state). Then, players are paired uniformly at random and interact.

The priors as well as the policies used by players in different states are common knowledge (those are the beliefs about the other player's states). The utility of a player is not known to other players since it depends on their unknown state, but we can compute the expected utility using Bayes rule because the prior distribution is known. Thus the extended game has perfect information and the Nash equilibrium of the extended game provides the Bayesian equilibrium in the original problem. We illustrate below how to express the thinned contact rate in the extended game in terms of policies.

[9] Two threshold policies are consecutive if their thresholds differ by one.

2.5 Contact rates in terms of policies

To compute the thinned contact rate as a function of policies, we focus on a tagged player k. Assume that all players except player k decide to wear or not to wear a mask according to a pure policy v whereas player k adopts pure policy u.

The thinned contact rate towards player k on the event that player k is of risk type m and that it encounters a player with risk class l is given by

$$\tilde{\lambda}_m(u, v, l) := \Big(u(m)w(u(m)) + 1 - u(m)\Big) \cdot \Big(v(l)w(v(l)) + 1 - v(l)\Big)\beta \cdot q(l) \quad (1)$$

where $v(l)$ is a pure policy played by all players except a tagged player k,

$$v(l) = \begin{cases} 1, \text{ if a player of type } l \text{ plays } CP \\ 0, \text{ otherwise} \end{cases} \quad (2)$$

and $u(m)$ is a pure policy played by player k,

$$u(m) = \begin{cases} 1, \text{ if player } k \text{ of type } m \text{ plays } CP \\ 0, \text{ otherwise} \end{cases} \quad (3)$$

where $l \in \{1, ..., M\}$ and $m \in \{1, ..., M\}$.

Here the first term in the above product is the contribution of the mask of player k to his own thinned contact rate and the second term is the contribution of the mask of the player that interacts with k to the thinned contact rate towards player k. Note that $w(CP) > 0$ accounts for imperfect protection in the cooperative case since the masks do not guarantee zero risk of contamination.

For convenience, let the auxiliary function $\mu(a, b)$ be defined as follows,

$$\mu(a, b) = (wa + (1 - a))(wb + (1 - b))\beta. \quad (4)$$

In the remainder of this paper, except otherwise noted, w is assumed to be a constant that is functionally independent of u, v and l. Indeed, under a pure policy, w multiplies $a \in \{0, 1\}$ and $b \in \{0, 1\}$, and has effect only when $a = 1$ or $b = 1$. In that case, $1 - w \in [0, 1]$ denotes the level of protection of masks, $w := w(CP)$. Then,

$$\tilde{\lambda}_m(u, v, l) = \mu(u(m), v(l)) \cdot q(l).$$

Note that $\mu(a, b) = \beta$ in the baseline case, where no player wears masks,

$$\mu(NC, NC) = \beta. \quad (5)$$

Remark 2. We focus on mixed policies as every matrix game in strategic normal form has an equilibrium in mixed policies. Existence of equilibrium is guaranteed for mixed policies, which can be interpreted as expected policies across a population of players playing pure strategies.

Remark 3. We consider pairwise interaction across players. The above rates follow from a thinning argument, e.g., where the coefficient $u(l)w(u(l)) + 1 - u(l)$ equals $w(1) := w \geq 0$ if $u(l) = 1$ and 1 if $u(l) = 0$.

Aggregate rate. The aggregate thinned contact rate towards a tagged player with risk-state m is

$$\lambda_m(u,v) := \sum_{l=1}^{M} \tilde{\lambda}_m(u,v,l) = \sum_{l=1}^{M} \mu(u(m),v(l)) \cdot q(l) \tag{6}$$

$$= \Big(u(m)w(u(m)) + 1 - u(m)\Big)\beta \sum_{l=1}^{M} q(l)\Big(v(l)w(v(l)) + 1 - v(l)\Big) \tag{7}$$

for $u, v \in U_p$.

2.6 Risk and discomfort

Next, we introduce formally the notion of risk and discomfort and we define the cost of wearing or not a mask accordingly.

Risk factor. Define $r(m)$ to be the risk factor, shortly risk, of being in state m. For example, it may stand for the probability for a player in risk-state m that becomes infected, to get the severe form of COVID-19. We enumerate m such that r is monotone increasing.

Discomfort. $g(a)$ is the discomfort of a given action a of a player. $g(CP)$ is positive due to the fact that cooperation requires to put masks and requires distancing. $g(NC)$ can include also non-physical type of discomfort, e.g., the risk of receiving a fine for those who do not wear masks.

Cost. Each player faces a cost given by the product of the rate of getting infected times the risk factor plus the discomfort for wearing a mask. Given that some player k is in risk-state m, that all other players follow policy v except player k that adopts policy u, the cost for player k is given by

$$\phi_m(u,v) = r(m)\lambda_m(u,v) + g(u(m)) \tag{8}$$

where the definition of λ has been given in (6).

We next uncondition the cost with respect to class m over the distribution q for the risk-state:

$$\Phi(u,v) := \sum_{m=1}^{M} q(m)\phi_m(u,v) = \sum_{m=1}^{M} q(m)(r(m)\lambda_m(u,v) + g(u(m))). \tag{9}$$

The above cost functional is further analyzed next.

2.7 No a priori information on health-states

The cost functional Φ can be expressed as a square matrix characterizing a game where player k chooses a row of the matrix and a second player chooses a column. The entries of the matrix are the utility obtained by the tagged player k when it uses policy u and it interacts with a player that uses policy v.

Assumption 3 *We consider a population game with infinitely many players but finitely many classes of players.*

Games with infinitely many players and finitely many classes are typical in evolutionary games.

Nash equilibrium. The considered matrix game is symmetric and hence we do not need to include in the matrix the utility of the second player. Since this is a standard symmetric matrix game, it has a (symmetric) equilibrium $\hat{u}$ in mixed strategies [21].

Bayesian equilibrium. The partial information problem has a solution concept known as Bayesian Nash equilibrium. We adapt this concept to the setting of infinite set of players with a finite number of categories.

A mixed policy $\hat{u} \in U_m$ is said to be a Bayesian equilibrium if and only if for any player i and for each risk-state m it holds

$$\hat{u} \in \text{argmax}_u \phi_m(u, \hat{u}). \tag{10}$$

Note that the argmax is taken over the set of policies. Thus $\hat{u} \in U_m$ is a policy such that no player can increase their utility by deviating unilaterally. In the rest of the paper we focus on a symmetric equilibrium, wherein all players adopt the same policy. In this case sufficient and necessary condition for $\hat{u}$ to be an equilibrium is given by

$$\hat{u} \in \text{argmax}_u \Phi(u, \hat{u}). \tag{11}$$

Indeed, the above condition holds if and only if, for each risk-state m, (10) holds.

The role of a priori. The Nash equilibrium and Bayesian equilibrium of the above extended game *do not assume any a priori on the health-state of interacting players* and r has the meaning of an expected risk over the health-state distributions of the two players[10]. Actually, the population game we shall describe in the next sections relies on a standard payoff maximization for which the same formalism is used *when the a priori health-state distribution is known to players.*

[10] Such statistics could be learned from experience, e.g., assuming that they change in a coarse time scale; we leave such learning aspects as subject for future work.

2.8 Accounting for a priori information on health-states

Next, we capture the interactions across players of different health-states, i.e., susceptible S and infected I. In particular, we adapt the payoff expression (9) to account for health-states. Then, assuming that the a priori health-state distribution is known to players, we introduce the mask game which is considered in the remainder of this paper.

The payoff f_{kj} to a player at health-state k when paired with a player in health-state j is given by:

$$f_{kj}(m,a,b) = \begin{cases} -r(m)\cdot\mu(a,b) - g(a), & \text{if } kj = SI \\ -g(a), & \text{if } kj \in \{II, SS, IS\}. \end{cases} \tag{12}$$

As before, $r(m)$ is the risk factor for being of type m. In the context of COVID-19, $r(m)$ is the risk for a player of becoming infected by a severe form of COVID-19 given that it gets infected.

Note that an interaction between two players of the same health-state or of a player in health-state I with a player with health-state S does not include the component $r(m)\cdot\mu(a,b)$ in the payoff since there is no risk of infection: either the player is already infected or it interacts with a susceptible player; in both cases it cannot change its state from S to I.

Definition 4 (Mask game). *The mask game is a game in which players, knowing their own risk-state, but without knowing whether they are in health-state S or I, make decisions so as to maximize their expected payoff. The payoff to a tagged player after each interaction, in turn, is a function of that player risk-state, m, his action, a, and the opponent action, b,*

$$f(m,a,b) = -p(1-p)r(m)\mu(a,b) - g(a). \tag{13}$$

The payoff of the mask game can be alternatively expressed as

$$f(m,a,b) = -\gamma r(m)(wa+1-a)(wb+1-b) - g(a) \tag{14}$$

where

$$\gamma := p(1-p)\beta. \tag{15}$$

Note that γ is a baseline infection rate, obtained from the baseline contact rate multiplied by a factor $p(1-p)$ to capture contacts wherein one of the players is infected and the other is not. Then, the first term of the payoff function (14) is obtained after thinning such baseline infection rate, to account for the actions of the players, noting that a fraction $r(m)$ of the infections results in a severe form of the disease.

Common knowledge and priors. Note that parameters q, p, w, g, r, β, utility functions and information structure are assumed to be common knowledge. In particular, two of such parameters represent the a priori partial information that we assume to be known to every player about the system state. The first is the

baseline contact rate modulated by the probability that a contact involves an infected and a non-infected player, $\gamma = p(1-p)\beta$. The second is the distribution of risk-states among individuals, $q(m)$. The interpretation in the mask game is that individuals know a priori the probability of meeting different risk types.

Time scale. We have assumed the mask game is played at a timescale that is much faster than the virus diffusion dynamics. That is, the health-state distribution, captured by parameter p in our model, does not change significantly over time in order for individuals to take a decision on their actions.

3 Equilibrium for the single-population case (single risk-state, $m = 1$)

Let us restrict to the case when only one risk category is present, i.e., $m = 1$. Then the mask game is represented by the 2×2 matrix game:

	CP	NC
CP	A/A	B/C
NC	C/B	D/D

(16)

Consider an arbitrary player i that is either cooperative (CP, plays 1st row) or not (NC, plays 2nd row), and it encounters another player j who may be cooperative (CP, plays 1st column) or non-cooperative (NC plays 2nd column). Let a and b be the actions of the row and column players. The utility of the row player is A if $a = b = CP$, D if $a = b = NC$, B if $a = CP$ and $b = NC$ and finally C if $a = NC$ and $b = CP$.

Recall that

$$\mu(a, b) = \beta(wa + (1 - a))(wb + (1 - b))$$

so that:

$$\mu(a,b) = \begin{cases} \beta w^2, & \text{if } a = b = CP = 1, \\ \beta, & \text{if } a = b = NC = 0, \\ \beta w, & \text{otherwise.} \end{cases} \tag{17}$$

Let $r = r(1) = 1$ with no loss of generality.

It follows from (13) and (17) that the entries of the matrix for the mask game are given by

$$\begin{aligned} A &= -p(1-p)\beta w^2 - g(CP) = -\gamma w^2 - g(CP) \\ B &= -p(1-p)\beta w - g(CP) = -\gamma w - g(CP) \\ C &= -p(1-p)\beta w - g(NC) = -\gamma w - g(NC) \\ D &= -p(1-p)\beta - g(NC) = -\gamma - g(NC) \end{aligned}$$

We can immediately observe that $B < A$ since $w < 1$. In the rest of the discussion let us denote

$$\Delta g = g(CP) - g(NC).$$

3.1 Lack of cooperation in the context of Prisoner's Dilemma

In the Prisoner's dilemma game, (CP, CP) provides smaller utility than (NC, NC). In the mask game, this condition translates into $A < D$, i.e.,

$$\gamma(1 - w^2) < \Delta g.$$

Theorem 1. *If (CP, CP) provides smaller utility than (NC, NC), in the mask game, players will not cooperate under the Nash equilibrium.*

Proof. If $C > A$ then playing cooperatively $a = b = CP$ is not an equilibrium since any player profits from unilaterally deviating from cooperative behavior. Conversely, if $C \leq A$ then deviating unilaterally from cooperation is not profitable and thus $a = b = CP$ is an equilibrium. Similarly, $B < D$ if and only if the non-cooperative behavior is an equilibrium.

However, $C \leq A$ requires $\Delta g \leq \gamma w(1-w)$ which is not possible since $A < D$. Finally, since $B < A$ is always verified, if $A < D$, then $B < D$ holds so that (NC, NC) is the symmetric equilibrium for $\gamma(1 - w^2) < \Delta g$. □

In the Prisoner's dilemma game the cooperative solution is defined as the one that is best for everyone (and the point is to show that it is unstable). In the mask game, in contrast, cooperation is predefined as CP irrespectively of whether a higher or lower utility is obtained by playing cooperatively. Nonetheless, as shown above, as far as (CP, CP) provides smaller utility than (NC, NC) the same sort of dilemma that occurs under the Prisoner's dilemma will also hold under the mask game.

3.2 Cooperation in the context of Hawk-and-Dove game

In the Hawk-and-Dove (HD) game, an encounter between two hawks provides smaller utility than an encounter between two doves. Assume that $A > D$ in the mask game (noting that CP and NC correspond to dove and hawk, respectively). Then the utility for a player in an encounter of two CP players is larger than the one of two non-cooperative ones.[11]

Assuming that $A > D$ in the mask game means

$$\gamma(1 - w^2) > \Delta g.$$

We are interested in symmetric equilibria of the matrix game in (16).

- *Case $D \geq B$ and $A \leq C$:* playing NC is the unique pure equilibrium
- *Case $D \leq B$:* both (NC, CP) and (CP, NC) are pure equilibria but they are not symmetric; playing CP is not an equilibrium.

[11] Strictly speaking, in the HD game, we should require also $A \leq C$, since playing hawk against dove brings larger utility.

A symmetric mixed equilibrium is obtained by solving for u

$$Au + B(1-u) = uC + (1-u)D,$$

where u is the probability of playing CP. This gives $u(A-B-C+D)+(B-D) = 0$ and thus

$$u = \frac{D-B}{(A-C)+(D-B)}$$

When it exists, this corresponds also to the unique evolutionary stable strategy (ESS) of the game. However, because

$$(A-C)+(D-B) = -\gamma(1-w)^2,$$

it requires that

$$D - B = -\gamma(1-w) + \Delta g \leq 0.$$

To complete the analysis of all possible parameter ranges for the game, we consider the following two additional cases:

- *Case $D \leq B$ and $A \leq C$:* the mixed equilibrium introduced above is a Nash equilibrium.
- *Case $A \geq C$:* the condition $A \geq C$ implies

  $$D - B = -\gamma(1-w) + \Delta g \leq -\gamma w(1-w) + \Delta g = -(A-C) \leq 0$$

  but no interior equilibrium is possible since $p < 1$ requires $A < C$. Note that in this case, strictly speaking, the game becomes a coordination game where CP is the only pure equilibrium.

3.3 Summary of equilibria for matrix game

The following theorem follows from the results derived in Sections 3.1 and 3.2.

Theorem 2. *The possible symmetric equilibria for (the matrix form of) the mask game are enlisted below:*

$$u = \begin{cases} 0 & \text{if} \quad \gamma\,(1-w) \leq \Delta g \\ \frac{\gamma(1-w)-\Delta g}{\gamma(1-w)^2} & \text{if} \quad \gamma\, w(1-w) \leq \Delta g \leq \gamma\,(1-w) \\ 1 & \text{if} \quad \Delta g \leq \gamma\, w(1-w) \end{cases} \tag{18}$$

Remark 4. In the case $\Delta g \leq \gamma\, w(1-w)$ the only pure equilibrium is also the social optimum, and the quantity $\gamma\, w(1-w)$ can be interpreted as a threshold on the minimum penalty to impose for not wearing a mask in order to attain full cooperation.

3.4 Special case where masks provide maximum protection ($w = 0$)

Next, let us consider the special case where $w = 0$.[12] In that case, we have $A = -g(CP)$ and $C = -g(NC)$. Let us further assume $\Delta g > 0$, i.e., $A < C$. The mask game in matrix form is given by (19). Even though masks provide perfect protection, players may end up not wearing them.The rationale goes as follows: wearing a mask provides both individual protection and protection to neighbors. When mask provides maximum protection, players have no incentive to incur the discomfort of wearing a mask when their neighbors are already wearing it, so in equilibrium a positive fraction of the players will not wear masks. There are two possible cases to consider:

- case 1) $\Delta g > \gamma$: we have a single equilibrium (NC, NC) corresponding to $u = 0$.
- case 2) $0 < \Delta g < \gamma$: we have two asymmetric equilibria, namely (NC, CP) and (CP, NC). In addition, we also have a mixed symmetric equilibrium

$$u = 1 - \frac{\Delta g}{\gamma}.$$

Note that in all cases a positive fraction of the players does not wear masks, as they either benefit from their neighbors wearing masks or the discomfort of wearing masks it too high and no one wears masks.

	CP	NC
CP	$-g(CP)/-g(CP)$	$-g(CP)/-g(NC)$
NC	$-g(NC)/-g(CP)$	$-g(NC)-\gamma/-g(NC)-\gamma$

(19)

4 Equilibrium for the multi-population case (multiple risk-states, $m \geq 2$)

In this section, we provide the general structure of symmetric equilibria for the mask game. The results that follow hold for any $m \geq 2$.

Let us consider $m \geq 2$ and the distribution $q(m)$ over the risk categories. In order to simplify the discussion, it is convenient to consider the equivalent fitness function

$$f(m, a, b) = -p(1-p)\mu(a, b) - \frac{g(a)}{r(m)} \tag{20}$$

Note that maximizing the above equation is equivalent to maximizing (13).

As in the case $m = 1$, it is possible to provide a matrix representation of the mask game. For instance, in the case of two risk categories, $m = 2$, the game is

[12] According to [4], face masks significantly reduce the risk of SARS-CoV-2 infection compared to social distancing. There is a very low risk of infection when everyone wears a face mask, even if it doesn't fit perfectly on the face.

represented by a 4×4 matrix which writes

	(CP,1)	(NC,1)	(CP,2)	(NC,2)
(CP,1)	A_1/A_1	B_1/C_1	A_1/A_2	B_1/C_2
(NC,1)	C_1/B_1	D_1/D_1	C_1/B_2	D_1/D_2
(CP,2)	A_2/A_1	B_2/C_1	A_2/A_2	B_2/C_2
(NC,2)	C_2/B_1	D_2/D_1	C_2/B_2	D_2/D_2

(21)

Note that the invariance along the fitness of player 1 between columns $(CP, 1)$ and $(CP, 2)$, and between columns $(NC, 1)$ and $(NC, 2)$, follows from the fact that players do not have information on the type of the opponents they encounter. The same holds for the fitness of player 2 between rows $(CP, 1)$ and $(CP, 2)$, and between rows $(NC, 1)$ and $(NC, 2)$.

From the monotonicity of r it follows that for every pair of risk categories $m \leq m'$ it holds

$$A_{m'} - A_m = B_{m'} - B_m = g(CP)\left(\frac{1}{r(m)} - \frac{1}{r(m')}\right) > 0 \tag{22}$$
$$C_{m'} - C_m = D_{m'} - D_m = g(NC)\left(\frac{1}{r(m)} - \frac{1}{r(m')}\right) > 0$$

Let denote $u_m := u(m)$ the probability of playing CP at the equilibrium. Let $\mathcal{M}_u$ the set of risk-states for which $0 < u_m < 1$ for each $m \in \mathcal{M}$. We say that a policy u is *mixing* if $\mathcal{M}_u \neq \emptyset$.

The indifference condition for $m \in \mathcal{M}$ writes

$$\sum_{j=1}^{M} u_j q(j) A_m + (1 - u_j) q(j) B_m = \sum_{j=1}^{M} u_j q(j) C_m + (1 - u_j) q(j) D_m, \qquad m \in \mathcal{M} \tag{23}$$

which leads to the following linear system

$$\sum_{j=1}^{M} u_j q(j) \left(A_m - C_m - B_m + D_m\right) + \left(B_m - D_m\right) = 0, \qquad m \in \mathcal{M} \tag{24}$$

The corresponding homogeneous system is given by

$$\sum_{j=1}^{M} u_j q(j) \left(A_m - C_m - B_m + D_m\right) = 0, \qquad m \in \mathcal{M} \tag{25}$$

We can hence observe that it must hold $|\mathcal{M}| \leq 1$, i.e., any interior equilibrium where $0 < u_m, u_{m'} < 1$ for some pair of risk categories $m \neq m'$ is not possible.

Lemma 1. *There is at most one risk-state for which a symmetric equilibrium u is mixing, i.e., $|M_u| \leq 1$.*

Proof. Without loss of generality, let us assume that $\mathcal{M} = \{m, m'\}$, for some $m \neq m'$. In order for the system (24) to be compatible, since the homogeneous

system (25) has rank 1, we need to require that the complete system (24) has same rank. However, this requires $B_m - D_m = B_{m'} - D_{m'}$, which in turns requires $g(NC) = g(CP)$, so that $B_m - D_m = B_{m'} - D_{m'} = \gamma(1-w)$. Now, since $A_m - C_m - B_m + D_m = A_{m'} - C_{m'} - B_{m'} + D_{m'} = -\gamma(1-w)^2$, we obtain in turn the relation $\sum_{j=1}^{M} u_j q(j)\gamma(1-w)^2 - \gamma(1-w) = 0$ which requires

$$\sum_{j=1}^{M} u_j q(j) = \frac{1}{1-w}$$

which leads to a contradiction. Indeed, if $w \in [0,1]$ then the right hand side of the inequality is greater than 1 while the left hand side is smaller than or equal to 1. Otherwise, if $w = 0$ then the only solution would be $u_j = 1$ for all $j \in \{1, \ldots, M\}$ so that mixing is impossible as well. Thus there can be at most one risk category for which the optimal policy is a mixing policy. □

Furthermore, we can observe that

Lemma 2. *1.* $u(m') = 0 \Rightarrow u(m) = 0$ *for all* $m \leq m'$
2. $u(m') = 1 \Rightarrow u(m) = 1$ *for all* $m \geq m'$

Proof. Let us assume that for risk category m it holds $u(m) = 0$. Then, for all $m \leq m'$ it holds $u(m) = 0$. Otherwise, it would be possible to deviate from NC: this is not possible since we would in turn obtain a contradiction from (22). Similarly, if for risk category m' it holds $u(m') = 1$, then for all $m \geq m'$ it holds $u(m) = 1$ as well. □

We hence summarize the previous results in the following

Theorem 3 (Threshold structure). *A symmetric equilibrium* $\hat{u}$ *of the mask game has a threshold structure in* m*: there exists risk category index* $1 \leq m^* \leq M$ *such that*

$$\hat{u}(m) = \begin{cases} 0, & \text{if } m < m^* \\ 1, & \text{if } m > m^*. \end{cases} \tag{26}$$

where m^* *is non-increasing with* $\gamma = p(1-p)\beta$.
Furthermore, mixing is permitted at m^* *iff* $\gamma\, w(1-w) \leq \frac{\Delta g}{r(m^*)} \leq \gamma\,(1-w)$ *and the following holds*

$$u_{m^*} = \frac{1}{q(m^*)} \left(\frac{\gamma(1-w) - \Delta g}{\gamma(1-w)^2} - \sum_{m > m^*} q(m) \right) \in (0,1) \tag{27}$$

We note that all symmetric equilibria of the game can be obtained by randomizing two consecutive pure threshold policies.

Finally we can provide the following sufficient condition for the existence of a uniform type of symmetric equilibrium, i.e., where every player plays either CP or NC.

Corollary 1. *The following conditions are necessary and sufficient for the existence of a pure uniform symmetric equilibrium*

– *always wear mask:* $\frac{\Delta g}{r(1)} \leq w\gamma\,(1-w) \Rightarrow \hat{u}(m) = 1$ *for all* $m = 1, \ldots, M$
– *never wear mask:* $\frac{\Delta g}{r(M)} \geq \gamma\,(1-w) \Rightarrow \hat{u}(m) = 0$ *for all* $m = 1, \ldots, M$

Remark 5. We observe that inequality $\Delta g / r(1) \leq \gamma\, w(1-w)$ can be satisfied either for sufficiently large γ or for sufficiently small Δg, e.g., because of fines for not wearing a mask. However, the two parameters are not equivalent: due to physical reasons $\gamma < +\infty$, so that when $r(m') = 0$ for some class m' (and all classes $m \leq m'$), e.g., when the risk of contracting a severe form of COVID-19 is negligible for such class, such condition cannot hold.

5 Concluding remarks and competition at other levels

We describe briefly other competing phenomena in COVID-19 that are worthwhile investigating with game theoretical methods.

Vaccination game: in previous work [27, 28] we studied the vaccination game in the context of e-viruses. Herd immunity occurs also in that context and games describing competition and collaboration are natural to define and study. We envision that the extension of ideas in [27, 28], where we showed the existence of pure equilibrium in an SIS epidemic model, to the realm of the mask game is an interesting avenue for future investigation.

The strain game: new variants of viruses compete with each other. We shall collect data so as to develop an evolutionary model of competition between virus strains. The main modeling difficulty here is to decide who a player i is. Is it a single virus or is it all the viruses of a given strain? or perhaps all the strains that have branched out of a common virus? Some of those questions have been addressed in [8], and we envision a refinement of those answers using the theory of group evolutionary games [1].

Politics of vaccination. Competition between countries, regions and cities consists another avenue for future work [18]. Both vertical as well as horizontal competition shall be addressed. In particular, it is key to identify incentive mechanisms for collaboration and synchronization between countries.

Finally, in our analysis we have made use of Poisson assumptions on the contact process. These standard models have been used in the context of COVID-19 (e.g., equation (3) in [9]) and it would be interesting to test how sensitive are the results to the Poisson assumption.

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A On the probability of getting infected

In the main body of this paper, we have considered the players' payoff is provided as a bi-linear model which can be interpreted as the rate at which a player gets infected as a function of their strategies. In what follows, we present more detailed models to capture the dependence on space and/or time of the probability at which players get infected.

In particular, note the key role played by the thinning argument presented in Section 2.2. As the models presented in the main body of the paper deal essentially with rates, such thinning argument yields a bi-linear payoff model, e.g., (4). Alternatively, the infection probability models considered in this appendix describe the expression of infection probabilities as a function of given point processes. The players' payoff is represented by the infection probability. We provide a more detailed discussion on the probability of infection as a function of the spatial an temporal interaction of individuals. The thinning argument however in this case yields non-linear relationships between actions and infection probabilities. The study of the following alternative models, and the resulting games, is left as subject for future work.

A.1 Infection probabilities and payoffs

Consider any player i. Let j be the nth player that is in contact with player i. It is assumed that player j is infected. If the contact duration is too short or if the distance between i and j is too large then the chances of i getting infected is small.

We introduce first a model for the probability of getting infected as a function of the duration of the exposure (temporal considerations) and of the distance between the infected and infecting node (spatial considerations). We then combine the two into a single spatio-temporal model.

A.2 A model that restricts to temporal considerations

Assumption 4 *The probability of getting infected depends on the duration for which players interact.*

Assume that the duration of the contact between i and j is an independent random variable denoted by τ_n with expectation $\bar{\tau}$. Assume that the contact duration has to exceed some minimum value y in order for player i to be infected. Such a temporal model provides the following expression

$$Q_1 = P(\tau_n > y) = \exp\left(-\frac{y}{\bar{\tau}}\right) \tag{28}$$

which describes the probability of a player in health-state S getting infected due to long exposure periods.

A.3 Models that restrict to spatial considerations

Assumption 5 *The probability of getting infected depends on the distance at which players interact.*

Assume that an interaction between two players in health-states S and I respectively, results in the player in state S getting infected if she is closer than a given threshold ρ to a player in state I.

Assume that the distance between interacting players is obtained from a Poisson process with parameter δ (the Poisson assumption is discussed, for instance, in [9]). We model both the case in which the Poisson process is defined on the line and the case where it is defined on the plane. In the case of the line, the intensity δ of the process has units of $meter^{-1}$ and in the case of the plane, units are $meter^{-2}$.

Linear model: in the case of a Poisson location process on the line, the following expression for the probability of getting infected

$$Q_2^{(l)} = \exp\left(-\frac{\rho}{\delta}\right) \tag{29}$$

if the location distribution lives on the line.

Planar model. In the case of a Poisson location process on the plane, an interaction between a player at health-state S with a player at health-state I leads to infecting the player in the susceptible state S if the distance to the closest player during the interaction is very small, i.e., smaller than some threshold denoted by ρ. Infection does not occur if and only if no point of the Poisson process (with spatial rate δ) falls into the circle of radius ρ. The probability of this event is given by

$$Q_2^{(p)} = \exp\left(-\frac{\pi\rho^2}{2\delta}\right) \tag{30}$$

where, as before, δ is the intensity of the process.

A.4 Spatio-temporal model

Assumption 6 *The probability of getting infected depends both on the duration and distance at which players interact.*

Combining the spatial and temporal aspects and assuming that the location and duration of interactions are independent, the following expression describes the probability for susceptible player i to be infected during a given contact with player j:

$$Q_3 = Q_1 Q_2. \tag{31}$$

where Q_3 is the infection probability accounting for temporal and spatial aspects, factor Q_1 is as in (28) and the spatial infection factor Q_2 can be given, e.g., as in (29) or (30).

A.5 Beyond bi-linear payoff models

Next, we revisit the thinning argument introduced in equation (4).

Replace δ by $\delta/(w(a)w(b))$ and $\bar{\tau}$ by $\bar{\tau}/(w(a)w(b))$, where a and b are the actions of players i and j. Then, we obtain μ for all $w(a)$ and $w(b)$ (see end of Section 2.2). For example, when infections occur in a line, according to the spatial model (29), we have

$$Q_2^{(l)} = \exp(-\rho w(a)w(b)/\delta) \tag{32}$$

Similarly, the model based on temporal considerations gives

$$Q_1 = \exp(-y w(a)w(b)/\bar{\tau}) \tag{33}$$

Thus, the model that combines spatial and temporal considerations is given as follows:

$$Q_3 = \mu(a,b) = \exp(-y w(a)w(b)/\bar{\tau}))\exp(-\rho w(a)w(b)/\delta). \tag{34}$$

The above model extends the thinning method presented before for the bi-linear payoff model; the analysis of the resulting game is left as subject for future work.

A.6 Additional remarks

Daily infection probability: if we assume that player i has k contacts with infected players per day then the probability of not getting infected in a day is $(1-\mu)^k$, where k could be itself random. For example it can have the distribution of the number interactions in which a player i is involved per day, i.e. a Poisson random variable with parameter s interactions per day.

Additional elements impacting infection probability: the probability of being infected at a given interaction depends on a number of factors, including whether the interaction occurs indoors or outdoors, weather and location. The estimate of the infection probability can be assessed conditional on those aspects. Then, the ultimate infection probability during a given day can be assessed using the arguments presented in the above paragraph together with a mobility model, which is out of the scope of this paper.