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**Abstract**

A closure relation for moments equations in kinetic theory was recently introduced in [38], based on the study of the geometry of the set of moments. This relation was constructed from a projection of a moment vector toward the boundary of the set of moments and corresponds to approximating the underlying kinetic distribution as a sum of a chosen equilibrium distribution plus a sum of purely anisotropic Dirac distributions.

The present work generalizes this construction for kinetic equations involving unbounded velocities, i.e. to the Hamburger problem, and provides a deeper analysis of the resulting moment system. Especially, we provide representation results for moment vectors along the boundary of the moment set that implies the well-definition of the model. And the resulting moment model is shown to be weakly hyperbolic with peculiar properties of hyperbolicity and entropy of two subsystems, corresponding respectively to the equilibrium and to the purely anisotropic parts of the underlying kinetic distribution.

## 1 Introduction

This paper is a follow-up to [38] and aims at generalizing and analyzing the projective closures of moment equations.

Kinetic theory commonly describes the motion of a cloud of particles through its density $f$ in the phase space. This so-called distribution function depends on time $t \in \mathbb{R}^+$, position $x \in \mathbb{R}^d$ (the space dimension $d$ is set to one in all the paper) and on a state variable $s \in E$ which can model various quantities representing the particles such as the velocity $v \in \mathbb{R}^d$, the internal energy $I \in \mathbb{R}^+$ for polyatomic gases or the size $S \in \mathbb{R}^+$ for polydisperse sprays. In the spirit of Boltzmann's H-theorem, the kinetic equation commonly dissipates an entropy, which corresponds to saying that the dynamic tends to push the distribution function to have a certain form. This form, minimizing the entropy, corresponds to an equilibrium. In many applications in physics, the hypothesis of thermal equilibrium is made and the distribution with respect to this state variable is forgotten as only weaker information is required, i.e. the mean density, momentum and energy which are moments of the distribution function with respect to $s$. However, many applications in physics also involve out of equilibrium regimes which need to be accurately modeled and simulated.

The discretization of $f$ with respect to $s$ presents several difficulties. First, the computational cost is high because of the number of variables involved. Second, capturing appropriately the equilibria requires special treatment. The method of moment is a key technique for the reduction of kinetic equations into fluid models, which can be interpreted as such a discretization. The moments $\mathbf{f}$ are weighted integrals of the distribution function $f$ with respect to $s$. This technique consists in multiplying the kinetic equation by a set of basis functions of $s$ and integrating it. However, the resulting moment equations are commonly undetermined and require supplementary equations, so-called closure, to be well-posed. The most common idea to construct such closures consists in solving a truncated moment problem, i.e. finding a distribution function from its moments and computing the unknown terms based on this function. Such a construction also provides a kinetic interpretation to the moment model.

The problem of constructing a closure has been widely studied and many suggestions were provided depending on the field of applications and on the properties required for the approximation. We exhibit here a (non-exhaustive) list of them illustrating the main ideas. A first idea consisted in constructing the closure from the equilibrium distribution, it leads typically to Euler equations or Navier-Stocks through Chapman-Enskog expansion. A first alternative was proposed by Grad ([20]) and consisted in a polynomial perturbation of the Maxwellian. This leads to non-strictly hyperbolic equations. Some regularizations (see e.g. [44, 8] and references therein) dealt with this issue by a clever pertubation of the Maxwellian but it commonly leads to non-conservative equations. The idea of reconstructing the distribution minimizing the entropy ([30], see also e.g. [14, 22, 36] for other applications) was shown to satisfy most of the properties desired for a closure. Especially, the resulting moment equations are symmetric hyperbolic and possess an entropy directly related to the kinetic one. However, the numerical computation of this closure requires special consideration (see e.g. [23, 5, 4, 33, 39]). Furthermore, there exists moment vectors for which this closure cannot be constructed. This was circumvented recently ([1, 2]) by a clever modification of the entropy. Another approach, inspired by quadrature techniques to approximate integrals, consists in approximating the distribution function by a sum of Diracs. This quadrature-based method of moments (QMOM ; [34]) is very well-adapted to numerical applications, but it leads to weakly hyperbolic models, while strong hyperbolicity can be expected in some applications, and does not dissipate the entropy defined at the kinetic level (though other entropies can be found). Many extensions of this model were proposed to tackle e.g. the weak hyperbolicity, to capture certain Maxwellian regimes (see e.g. [31, 47, 9, 40, 17]).

In [38], a generic construction of closures was suggested by computing the projection of any vector of moments toward a certain direction headed to the boundary of the set of moments. This led to the interpretation of the closure as the sum of a chosen equilibrium distribution plus a sum of Dirac measures. However, two (strong) hypotheses were performed to obtain this distribution: 1-the equilibrium function was chosen constant and especially independent of the vector of moments from which the projection is performed; 2-the set of integration was chosen to be a 1D bounded interval. Furthermore, if numerical applications were presented illustrating the appropriate behavior of this model in various regimes, studying the hyperbolicity and the entropy of the resulting moment models was left as a perspective. The present paper extends the construction of this closure to moments on 1D unbounded sets, i.e. in Hamburger case (the construction in Stieltjes [43] case would follow similarly) and when the equilibrium function depends on the vector of moments, especially when the equilibrium is represented by a non-fixed Maxwellian. This study is completed with the study of hyperbolicity and entropy decay of the resulting moment model in the case of an equilibrium represented by an entropy-minimizing distribution. Especially, we show that the resulting model retains certain properties of the entropy-minimizing moment model and of the QMOM model, since the underlying distribution is a combination of those of these two models.

The paper is organized as follows. The next section recalls the context, i.e. the considered kinetic equation together with the construction of the projective closure and its application in Hausdorff case. The following section extends the framework of [38] in the case of Hamburger, i.e. when the set of integration is unbounded. Especially, it is devoted to the study of the set of moments and includes representations of the vectors along all the boundary of the realizability domain. Section 4 extends the construction of the projective closure when the equilibrium function depends on the vector of moments itself, including in the case of Hamburger with a Maxwellian as equilibrium function. Finally, Section 5 analyzes the hyperbolic and the entropic structure of the resulting model in a general framework. The last section is devoted to conclusive remarks.

# 2 Preliminary on the projective moment models

In this first part, the considered kinetic model is presented and the construction of the projective model from [38] is recalled with adapted notations.

## 2.1 The kinetic model

Consider a generic 1D kinetic equation of the form

$$\partial_t f + s\partial_x f = Q(f), \tag{1}$$

where the unknown $f$ is a distribution function depending on time $t \in \mathbb{R}^+$, position $x \in Z \subset \mathbb{R}$ and a state variable $s \in E \subset \mathbb{R}$.

The method below aims to propose an appropriate approximation of the variations of $f$ with respect to $s \in E \subset \mathbb{R}$. This state variable can be a velocity variable $v \in \mathbb{R}$ when modelling rarefied gases or plasmas, or direction of flight $\mu \in [-1, +1]$ when modelling radiation. The method below also applies to spray models, where the state variable commonly represents a size of droplets $S \in \mathbb{R}^+$ even though this variable appears differently in the kinetic model.

We will assume that (1) possesses the following features that we aim to preserve through the moment extraction presented in the next subsection

**(P1)** Together with appropriate initial and boundary condition, (1) possesses a non-negative solution $f \geq 0$, and it has finite moments at least up to a certain order $N+1$, i.e.

$$\forall i = 0, \ldots, N+1, \qquad \int_E s^i f(s) ds < \infty.$$

**(P2)** At fixed $s$, the left-hand side of (1) is a hyperbolic operator. Here, it is a transport operator at speed $s$.

**(P3)** Equation (1) is assumed to dissipate a strictly convex entropy function $\eta$, i.e.

$$\partial_t \mathcal{H}(f) + \partial_x \mathcal{J}(f) = \mathcal{D}(f) \leq 0,$$

where the entropy-entropy flux pair $(\mathcal{H}, \mathcal{J})$ and the entropy dissipation $\mathcal{D}$ read

$$\mathcal{H}(f) = \int_E \eta(f(s)) ds, \qquad \mathcal{J}(f) = \int_E s\eta(f(s)) ds,$$
$$\mathcal{D}(f) = \int_E \eta'(f(s)) Q(f(s)) ds.$$

Furthermore, the equality $\mathcal{D}(f) = 0$ holds if and only if $f = M$ equals a certain Maxwellian which will be specified later.

## 2.2 Projective moment model

We consider polynomial moment models up to order $N$. For this purpose, define

$$\mathbf{b}(s) = \mathbf{b}_N(s) := (1, \ldots, s^N)^T,$$

a basis of polynomials of $s$ up to degree $N$. Computing the moments up to order $N$ of (1) w.r.t. the $s$ variable provides the under-determined system

$$\partial_t \mathbf{f} + \partial_x \mathbf{F} = \mathbf{Q}, \tag{2}$$

where

$$\mathbf{f} = \int_E \mathbf{b}(s) f(s) ds, \qquad \mathbf{F} = \int_E s\mathbf{b}(s) f(s) ds, \qquad \mathbf{Q} = \int_E \mathbf{b}(s) Q(f)(s) ds. \tag{3}$$

The system (2-3) is under-determined and needs to be supplemented with a closure relation. The choice of this closure of course impacts on the existence of an underlying positivity solution, on the hyperbolicity of this system and on the existence of an entropy that correspond to the moment versions of the property (**P1-3**).

Following the property (**P1**) of the kinetic equation, the solution $f$ is assumed to be positive and integrable. In this direction, define the following set of functions which is assumed to contain the solution $f$ of (1) at all time

$$f \in L^1_{\mathbf{b}}(E)^+ = \left\{ f \in L^1(E) \quad \text{s.t.} \quad \mathbf{b}f \in L^1(E)^{N+1} \quad \text{and} \quad \operatorname*{ess\,inf}_{s \in E} f(s) > 0 \right\}$$

and the associated set of moments, also called realizability domain, in which we will aim to have the solution of (2) evolve

$$\mathcal{R}_{\mathbf{b}} = \left\{ \int_E \mathbf{b}(s) f(s) ds, \quad f \in L^1_{\mathbf{b}}(E)^+ \right\}.$$

In the next sections, we also widely exploit the following set obtained by replacing the integral in the definition of $\mathcal{R}_{\mathbf{b}}$ by a finite positive quadrature formula

$$\mathcal{R}_{\mathbf{b}}^{Q} = \left\{ \sum_{i=1}^{J} \alpha_i \mathbf{b}(s_i), \qquad J < \infty, \qquad (s_i)_{i=1,\dots,J} \in E^J, \quad (\alpha_i)_{i=1,\dots,J} \in (\mathbb{R}^+)^J \right\}.$$

For a vector $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$, the projective closure is constructed by minimizing the distance to some equilibrium function $f^{eq}$ (and to its moments denoted $\mathbf{f}^{eq}$). At this step, this function $f^{eq}$ can be a Maxwellian $M$ satisfying $\mathcal{D}(M) = 0$ or not. We make some additional hypotheses constraining the choice of $f^{eq}$:

**(H1)** It is positive and has finite moments up to order $N+1$, i.e. $f^{eq} \in L^1_{\mathbf{b}_{N+1}}(E)^+$.

**(H2)** It depends linearly on the moments $\mathbf{f}_0$ of order 0 and on the normalized moments $\mathbf{N} = \frac{\mathbf{f}}{\mathbf{f}_0}$ as

$$f^{eq} \equiv f^{eq}(\mathbf{f}) = \mathbf{f}_0 f^{eq}(\mathbf{N}).$$

**(H3)** Denote $H_{\mathbf{f},\mathbf{d}}$ the half line starting in $\mathbf{f}$ and directed by $\mathbf{d}$

$$H_{\mathbf{f},\mathbf{d}} := \left\{ \mathbf{f} + \alpha \mathbf{d}, \quad \alpha \in \mathbb{R}^+ \right\}.$$

Then we will suppose that for all $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$, the function

$$\begin{cases} \mathcal{R}_{\mathbf{b}} & \to L^1_{\mathbf{b}_{N+1}}(E)^+ \\ \mathbf{f} & \mapsto \frac{f^{eq}(\mathbf{f})}{\mathbf{f}^{eq}(\mathbf{f})_0}, \end{cases}$$

is constant along $H_{\mathbf{f},-\mathbf{f}^{eq}(\mathbf{f})}$.

Some further hypothesis on the equilibrium function will be made in the next section, related to the properties of hyperbolicity and entropy decay of the resulting closure.

Then, one decomposes all $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$ into

$$\mathbf{f} = \alpha_0 \mathbf{f}^{eq} + \mathbf{f}^p \quad \text{s.t.} \quad \mathbf{f}^p \in \partial \mathcal{R}_{\mathbf{b}}. \tag{4}$$

**Remark 1.** Hypothesis (**H3**) is *a priori* not necessary, but it simplifies the computations below, and it is natural and satisfied by all the functions $f^{eq}$ we have in mind. Especially, forall $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$ and $\alpha \in (-\infty, \alpha_0(\mathbf{f})]$, we can simply relate the value of the functions $\mathbf{f}^{eq}$, $\mathbf{f}^p$ and $\alpha_0$ (from the decomposition (4)) in $\mathbf{f}^\alpha := \mathbf{f} - \alpha \mathbf{f}^{eq}(\mathbf{f})$ to their value in $\mathbf{f}$ through

$$\mathbf{f}^{eq}(\mathbf{f}^\alpha) = \mathbf{f}^{eq}(\mathbf{f}), \qquad \mathbf{f}^p(\mathbf{f}^\alpha) = \mathbf{f}^p(\mathbf{f}) \quad \text{and} \quad \alpha_0(\mathbf{f}^\alpha) = \alpha_0(\mathbf{f}) - \alpha. \tag{5}$$

This is discussed with a particular choice of equilibrium function will be given in Section 4. See also Appendix A for another characterization of this hypothesis.

A representation of the projection $\mathbf{f}^p$ in the case of Hausdorff, i.e. with a bounded set of integration $E = [-1, +1]$, is first recalled from [38] before being extended to the unbounded set $E = \mathbb{R}$ in the next section.

### 2.3 Hausdorff case: $E = [-1, +1]$

In the following, we use extensively the notion of Riesz functional ([41]) to simplify notations.

**Definition 2.1.** The Riesz functional $R_{\mathbf{V}}$ associated to the vector $\mathbf{V}$ sends any polynomial $\boldsymbol{\lambda}^T\mathbf{b}$ onto $R_{\mathbf{V}}(\boldsymbol{\lambda}^T\mathbf{b}) = \boldsymbol{\lambda}^T\mathbf{V}$. We also apply it componentwise to vectors and matrices of polynomials.

When the set $E$ of integration is an interval, typically $E = [-1, 1]$, solving the Hausdorff problem (see e.g. [24, 3, 28, 11]) provides a characterization of realizable vectors $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$. This also provides the existence of a unique positive measure (or distribution) having such $\mathbf{f}^p \in \partial\mathcal{R}_{\mathbf{b}}$ for moments.

**Proposition 1.** *Consider* $\mathbf{f} \in \mathbb{R}^{N+1}$. *Then* $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$ *if and only if*

$$\text{Even case } N = 2K: \qquad R_{\mathbf{f}}\left(\mathbf{b}_K\mathbf{b}_K^T\right) \quad \text{and} \quad R_{\mathbf{f}}\left((1-s^2)\mathbf{b}_{K-1}\mathbf{b}_{K-1}^T\right), \tag{6a}$$

$$\text{Odd case } N = 2K+1: \qquad R_{\mathbf{f}}\left((1+s)\mathbf{b}_K\mathbf{b}_K^T\right) \quad \text{and} \quad R_{\mathbf{f}}\left((1-s)\mathbf{b}_K\mathbf{b}_K^T\right) \tag{6b}$$

*are symmetric positive definite.*

**Proposition 2** ([11, 38])**.** *Consider* $\mathbf{f} \in \mathbb{R}^{N+1}$. *Then* $\mathbf{f} \in \partial\mathcal{R}_{\mathbf{b}}$ *if and only if the matrices* (6) *are symmetric positive semi-definite and at least one of them is singular.*

*In such a case, there exists a unique representing measure for* $\mathbf{f}$

$$\mathbf{f} = \sum_{i=1}^{J} \alpha_i \mathbf{b}(s_i) = \int_{-1}^{+1} \mathbf{b}(s) \sum_{i=1}^{J} \alpha_i \delta_{s_i}(s), \tag{7}$$

*where the masses* $\alpha_i > 0$ *are strictly positive, and the quadrature points* $s_i \in [-1, +1]$ *can be computed from the kernell of the singular matrix* (6) *and their number* $J \leq K$.

This also provides

$$\mathcal{R}_{\mathbf{b}} = int(\mathcal{R}_{\mathbf{b}}^Q) \subsetneq \mathcal{R}_{\mathbf{b}}^Q = \overline{\mathcal{R}_{\mathbf{b}}}.$$

by exploiting the convexity of these sets.

Using the decomposition (4), one simply obtains a representing measure for all $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$

$$\mathbf{f} = \alpha_0 \mathbf{f}^{eq} + \sum_{i=1}^{J} \alpha_i \mathbf{b}(s_i) = \int_{-1}^{+1} \mathbf{b}(s) \left( \alpha^0 f^{eq}(s) ds + \sum_{i=1}^{J} \alpha_i \delta_{s_i}(s) \right).$$

The projective closure is then obtained by replacing $f$ by this reconstruction in (3) which provides

$$\mathbf{F}(\mathbf{f}) = \int_E s\mathbf{b}(s)\alpha_0 f^{eq}(\mathbf{f})(s) ds + \sum_{i=1}^{J} \alpha_i s_i \mathbf{b}(s_i). \tag{8}$$

The presence of Dirac measures in the decomposition (4) does not offer a simple framework for the construction of the term $\mathbf{Q}(\mathbf{f})$ when the kinetic collision operator $Q$ is non-linear. Some intuitions on how to construct this term will be given in Section 5.2 by suggesting constraints this function needs to satisfy.

Concerning the previous results from [38]:

- The decomposition (4) was shown to exist for all $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$, to be unique and with $\alpha_0 > 0$. This was proved in the general case $E \subset \mathbb{R}$ but only for a fixed equilibrium function $f^{eq} \in L^1_{\mathbf{b}_{N+1}}(E)^+$ independent of the moment $\mathbf{f}$. This result extends directly when $f^{eq}$ depends on $\mathbf{f}$ but satisfies Hypothesis **(H3)**.

- The realizability domain was shown to be open, again in the general case $E \subset \mathbb{R}$. Especially, the vector $\mathbf{f}^p \in \partial\mathcal{R}_{\mathbf{b}}$ can not be realized by integrable functions. However, the representation for $\mathbf{f}^p \in \partial\mathcal{R}_{\mathbf{b}}$ as a sum of Dirac measures or distributions (7) was shown to be valid only in the case of Hausdorff $E = [-1, +1]$ compact.

- Analytic formulae were also provided for all of the parameters involved in decomposition (4,7), i.e. the Dirac masses $\alpha_i$, locations $s_i$ and $\alpha_0$, as functions of $\mathbf{f}$ and of the fixed $\mathbf{f}^{eq}$ independend of $\mathbf{f}$.

Concerning the present work:

- We extend the existence and uniqueness of a representation of the form (4) when the equilibrium function $f^{eq}$ depends on the unknown under reasonnable hypotheses. This is typically the case when the equilibrium is given by a Maxwellian corresponding to the fluid regime of a kinetic equation.

- So far, this closure is only restricted to moment models on bounded sets of integration $E = [-1, +1]$. The reason for this restriction is that vectors belonging to the boundary $\partial\mathcal{R}_{\mathbf{b}}$ of the realizability domain for moments on unbounded domains are not realized by such simple sum of Dirac deltas as in (7). This issue is investigated and we provide some representations for $\mathbf{f}^p \in \partial\mathcal{R}_{\mathbf{b}}$ in the case of Hamburger, i.e. when $E = \mathbb{R}$. These representations are only exploited in proofs in the rest of the paper, but they are not used for the construction of a closure because they can not be interpreted as positive distributions or measures.

- The computation of the parameters involved in the decomposition (4) is significantly more complicated when $f^{eq}$ depends on the moments $\mathbf{f}$. The development of an algorithm adapted to the computation of those parameters is not discussed in this paper and it is left as the main perspective after this work.

# 3 Extension of the formalism to Hamburger case: $E = \mathbb{R}$

In all this section, we denote similarly the number of moments in the even $N = 2K$ or odd $N = 2K + 1$ case. The difference between the two cases are discussed when appropriate.

## 3.1 Realizability and recursiveness

Again, the problem of moments in the sense of $L^1(\mathbb{R})^+$ has been solved in various manners (see e.g. [21, 11, 3]). We recall the one we will use below.

**Proposition 3.** *Consider* $\mathbf{f} \in \mathbb{R}^{N+1}$*. Then* $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$ *if and only if* $R_{\mathbf{f}}\left(\mathbf{b}_K \mathbf{b}_K^T\right)$ *is symmetric positive definite.*

In the odd case $N = 2K+1$, one remarks that this constraint only applies to the first moments $(\mathbf{f}_0, \ldots, \mathbf{f}_{2K})^T$ and the higher order moment $\mathbf{f}_{2K+1}$ is left free.

One deduces that the boundary is characterized by

$$\mathbf{f} \in \partial\mathcal{R}_{\mathbf{b}} \quad \Leftrightarrow \quad R_{\mathbf{f}}(\mathbf{b}_K \mathbf{b}_K^T) \quad \text{is symmetric positive and singular.}$$

However, contrarily to Proposition 1, not all the vectors on this boundary are realized by a sum of Diracs. Through the notion of recursiveness, R. Curto and L. Fialkow (see e.g. [11, 12, 15]) provided the description of the set $\mathcal{R}_{\mathbf{b}}^Q$ realized by Dirac measures which is closely related to $\mathcal{R}_{\mathbf{b}}$ also in the case of Hamburger. However, contrarily to Hausdorff case $\mathcal{R}_{\mathbf{b}}^Q \neq \overline{\mathcal{R}_{\mathbf{b}}}$. We first recall some results [11] and their reformulation from [38].

**Definition 3.1.** Denote $J = rank\ R_{\mathbf{f}}(\mathbf{b}_K \mathbf{b}_K^T)$. A vector $\mathbf{f} \in \mathbb{R}^{N+1}$ is positively recursively generated if

- $R_{\mathbf{f}}(\mathbf{b}_{J-1} \mathbf{b}_{J-1}^T)$ is symmetric positive definite.
- For all $j > J-1$, then

$$\begin{aligned} R_{\mathbf{f}}(s^{2j-1}) &= R_{\mathbf{f}}(s^{j-1}\mathbf{b}_{J-1}^T) R_{\mathbf{f}}(\mathbf{b}_{J-1}\mathbf{b}_{J-1}^T)^{-1} R_{\mathbf{f}}(s^j \mathbf{b}_{J-1}), \\ R_{\mathbf{f}}(s^{2j}) &= R_{\mathbf{f}}(s^{j}\mathbf{b}_{J-1}^T) R_{\mathbf{f}}(\mathbf{b}_{J-1}\mathbf{b}_{J-1}^T)^{-1} R_{\mathbf{f}}(s^j \mathbf{b}_{J-1}). \end{aligned}$$

One remarks again that this definition imposes no constraint on the moment of order $2J-1$ as the first equation for $j = J$ just rewrites $R_{\mathbf{f}}(s^{2J-1}) = R_{\mathbf{f}}(s^{2J-1})$.

Then the set of moments of Dirac measures is characterized by the following proposition.

**Proposition 4.** *Consider* $\mathbf{f} \in \mathbb{R}^{N+1}$*. The following assertions are equivalent:*

- $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}^Q$,
- $\mathbf{f}$ *is positively recursively generated,*
- $\mathbf{f} = \sum_{i=1}^{J} \alpha_i \mathbf{b}(s_i)$ *with* $J = rank\ R_{\mathbf{f}}(\mathbf{b}_K \mathbf{b}_K^T)$ *and* $\alpha_i > 0$ *for all* $i = 1, \ldots, J$.

Remarking that the case $J = K+1$ in the definition of positive recursiveness implies $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$ (through Proposition 3), then this result provides $\mathcal{R}_{\mathbf{b}} \subset \mathcal{R}_{\mathbf{b}}^Q$. Especially, it provides a representation for $\mathbf{f}^p \in \partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^Q$ as a sum of $J$ Dirac deltas where $J = rank\ R_{\mathbf{f}}(\mathbf{b}_K \mathbf{b}_K^T) < K+1$.

Decomposing the boundary

$$\partial\mathcal{R}_{\mathbf{b}} = \left(\partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^Q\right) \cup \left(\partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}_{\mathbf{b}}^Q\right),$$

one therefore obtains a representation for the first part of the boundary $\partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^Q$. However, contrarily to the case of Hausdorff, the other part $\partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}_{\mathbf{b}}^Q$ that is not represented through the last proposition is non-empty. Indeed, taking for instance $\mathbf{f} = (0,0,1)^T$ provides a positive semi-definite matrix $R_{\mathbf{f}}(\mathbf{b}_1 \mathbf{b}_1^T)$ while $R_{\mathbf{f}}(\mathbf{b}_0 \mathbf{b}_0^T) = 0$

and $\mathbf{f}$ is not positively recursively generated. Therefore, in the case of Hamburger, we only have

$$\mathcal{R}_{\mathbf{b}} \subsetneq \mathcal{R}_{\mathbf{b}}^{Q} \subsetneq \overline{\mathcal{R}_{\mathbf{b}}}.$$

Since there is no reason for the projection $\mathbf{f}^p \in \partial\mathcal{R}_{\mathbf{b}}$ defined in (4) to be restricted to $\partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^{Q}$, we need to find some other representation for the rest of the boundary $\partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}_{\mathbf{b}}^{Q}$. However, such a representation can neither be an integrable function (as it belongs not to the open set $\mathcal{R}_{\mathbf{b}}$) nor a discrete measure (as it does not belong to $\mathcal{R}_{\mathbf{b}}^{Q}$).

We first specify some more properties of vectors in $\partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}_{\mathbf{b}}^{Q}$ exploited to construct a representation.

## 3.2 Characterization of $\partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}_{\mathbf{b}}^{Q}$

First, we provide a lemma required in the proof of Proposition 5 below.

**Lemma 3.2.** *Consider $\mathbf{f} \in \partial\mathcal{R}_{\mathbf{b}}$ and $\mathbf{W} \in Ker\ R_{\mathbf{f}}(\mathbf{b}_K \mathbf{b}_K^T)$ such that $p := \mathbf{W}^T \mathbf{b}_K$ is of lowest (non-zero) possible degree. Then $p$ has distinct real roots.*

*Proof.* $p$ **has real roots:** By contradiction, suppose that $p$ has a pair of complex roots, then

$$p(s) = (s^2 + 2bs + c^2) q(s) \quad \text{with} \quad b^2 < c.$$

Then

$$\begin{aligned} p(s)^2 &= (s^2 + 2bs + c)^2 q(s)^2 \\ &= \left((s+b)^2 + (c - b^2)\right)^2 q(s)^2 \\ &= (s+b)^4\, q(s)^2 + 2\left(c - b^2\right)(s+b)^2\, q(s)^2 + \left(c - b^2\right)^2 q(s)^2. \end{aligned}$$

Each of these polynomials is squared. Then by the non-negativity hypothesis, $R_{\mathbf{f}}(p^2) = 0$ implies that $R_{\mathbf{f}}\left(q^2\right) = 0$. Especially, there exists $\tilde{\mathbf{W}} \in \mathbb{R}^{K+1}$ such that $q = \tilde{\mathbf{W}}^T \mathbf{b}_K$. This rewrites $\tilde{\mathbf{W}}^T R_{\mathbf{f}}(\mathbf{b}_K \mathbf{b}_K^T) \tilde{\mathbf{W}} = 0$, then by the non-negativity hypothesis $\tilde{\mathbf{W}} \in Ker\ R_{\mathbf{f}}(\mathbf{b}_K \mathbf{b}_K^T)$. But $deg\ q < deg\ p$ which violates the hypothesis.

$p$ **has distinct roots:** By contradiction, suppose that one of the roots of $p$ has a double multiplicity, then

$$p(s) = (s - s_1)^2 q(s).$$

Write $\tilde{\mathbf{W}}$ such that $q = \tilde{\mathbf{W}}^T \mathbf{b}_K$. Then, by hypothesis

$$\mathbf{W} R_{\mathbf{f}}(\mathbf{b}_K \mathbf{b}_K^T) \tilde{\mathbf{W}} = 0 = R_{\mathbf{f}}(pq), \quad \text{with} \quad p(s)q(s) = (s - s_1)^2 q(s)^2.$$

Finally, writing $\bar{\mathbf{W}}$ such that $\bar{\mathbf{W}}^T \mathbf{b}_K(s) = (s - s_1) q(s) =: \tilde{q}(s)$, then again $\bar{\mathbf{W}} \in Ker\ R_{\mathbf{f}}(\mathbf{b}_K \mathbf{b}_K^T)$ with $deg\ \tilde{q} < deg\ p$. $\square$

For $\mathbf{f}^p \in \partial\mathcal{R}_{\mathbf{b}}$, we observe the following properties on the dimension of the matrix $R_{\mathbf{f}^p}(\mathbf{b}_K \mathbf{b}_K^T)$ and its submatrices.

**Proposition 5.** *Consider $\mathbf{f} \in \partial\mathcal{R}_{\mathbf{b}}$ and write $J = rank\ R_{\mathbf{f}}(\mathbf{b}_K \mathbf{b}_K^T)$. The following assertions are equivalent:*

- $\mathbf{f} \notin \mathcal{R}_{\mathbf{b}}^{Q}$,
- $R_{\mathbf{f}}(\mathbf{b}_{2K-1})$ *is positively recursively generated, but* $\mathbf{f}$ *is not,*

- *rank* $R_{\mathbf{f}}(\mathbf{b}_i\mathbf{b}_i^T) = J$ *for all* $i = J-1, \dots, K$,

- $\mathbf{f} = \mathbf{f}^{rec} + \mathbf{f}^{nrec}$ *with* $\mathbf{f}^{rec} = \sum_{i=1}^{J} \alpha_i \mathbf{b}(s_i)$ *positively recursively generated and*

$$\mathbf{f}^{nrec} = \begin{cases} \alpha_K \mathbf{e}^{2K} & \textit{if} \quad N = 2K \quad \textit{even case}, \\ \alpha_K \mathbf{e}^{2K} + \beta_K \mathbf{e}^{2K+1} & \textit{if} \quad N = 2K+1 \quad \textit{odd case}, \end{cases} \tag{9}$$

  *non-positively recursively generated with* $\alpha_i > 0$ *and* $\mathbf{e}^i := (0, \dots, 0, 1, 0, \dots, 0)^T$ *is the* $i$*-th vector of the canonical basis.*

*Proof.* Denote $\tilde{J} \leq K$ the first integer such that $R_{\mathbf{f}}(\mathbf{b}_{\tilde{J}-1}\mathbf{b}_{\tilde{J}-1}^T)$ is positive definite and $R_{\mathbf{f}}(\mathbf{b}_{\tilde{J}}\mathbf{b}_{\tilde{J}}^T)$ singular, i.e. such that $rank\ R_{\mathbf{f}}(\mathbf{b}_{\tilde{J}-1}\mathbf{b}_{\tilde{J}-1}^T) = rank\ R_{\mathbf{f}}(\mathbf{b}_{\tilde{J}}\mathbf{b}_{\tilde{J}}^T) = \tilde{J}$.

Either $\tilde{J} = J$ which corresponds to having $\mathbf{f}$ recursively generated, or $\tilde{J} < J$ and there exists $\tilde{K} > \tilde{J}$ such that $rank\ R_{\mathbf{f}}(\mathbf{b}_{\tilde{J}-1}\mathbf{b}_{\tilde{J}-1}^T) < rank\ R_{\mathbf{f}}(\mathbf{b}_{\tilde{K}}\mathbf{b}_{\tilde{K}}^T) = \tilde{J} + 1$.

We first prove that $\tilde{K} = K$ which implies $\tilde{J} = J - 1$ through rank conditions:

By contradiction, suppose that $\tilde{K} < K$. By construction, $rank\ R_{\mathbf{f}}(\mathbf{b}_i\mathbf{b}_i^T) = \tilde{J}$ for all $i = \tilde{J}-1, \dots, \tilde{K}-1$. Especially $rank\ R_{\mathbf{f}}(\mathbf{b}_{\tilde{K}-1}\mathbf{b}_{\tilde{K}-1}^T) = rank\ R_{\mathbf{f}}(\mathbf{b}_{\tilde{K}-2}\mathbf{b}_{\tilde{K}-2}^T)$. Then there exists $\mathbf{W} \in Ker\ R_{\mathbf{f}}(\mathbf{b}_{\tilde{K}-1}\mathbf{b}_{\tilde{K}-1})$ such that $deg(\mathbf{W}^T\mathbf{b}_{\tilde{K}-1}) = \tilde{K} - 1$.

Write $\tilde{\mathbf{W}} = (\mathbf{W}^T, 0, \dots, 0)^T$ such that

$$\tilde{\mathbf{W}}^T R_{\mathbf{f}}(\mathbf{b}_K\mathbf{b}_K^T)\tilde{\mathbf{W}} = 0 = \mathbf{W}^T R_{\mathbf{f}}(\mathbf{b}_{\tilde{K}-1}\mathbf{b}_{\tilde{K}-1}^T)\mathbf{W}.$$

As $R_{\mathbf{f}}(\mathbf{b}_K\mathbf{b}_K^T)$ is non-negative, then $\tilde{\mathbf{W}} \in Ker\ R_{\mathbf{f}}(\mathbf{b}_K\mathbf{b}_K)$. Write the polynomial $p := \mathbf{W}^T\mathbf{b}_{\tilde{K}-1} = \tilde{\mathbf{W}}^T\mathbf{b}_K$ and define $\bar{\mathbf{W}}$ such that $\bar{\mathbf{W}}^T\mathbf{b}_K(s) = sp(s)$. Then

$$\bar{\mathbf{W}}^T R_{\mathbf{f}}(\mathbf{b}_K\mathbf{b}_K^T)\bar{\mathbf{W}} = R_{\mathbf{f}}(pq) \quad \text{with} \quad q(s) = p(s)s^2.$$

Since $deg\ (q) = \tilde{K}+1 \leq K$, then $q \in Span(\mathbf{b}_K)$ and $R_{\mathbf{f}}(pq) = 0 = \bar{\mathbf{W}}^T R_{\mathbf{f}}(\mathbf{b}_K\mathbf{b}_K^T)\bar{\mathbf{W}}$. Again, the non-negativity provides $\bar{\mathbf{W}} \in Ker\ R_{\mathbf{f}}(\mathbf{b}_K\mathbf{b}_K^T)$.

Then, defining $\tilde{p}(s) := sp(s)$, we have for all $\tilde{q} \in \mathbb{R}_{\tilde{K}+1}[X]$ that $R_{\mathbf{f}}(\tilde{p}\tilde{q}) = 0$ and $rank\ R_{\mathbf{f}}(\mathbf{b}_{\tilde{K}-1}\mathbf{b}_{\tilde{K}-1}^T) = rank\ R_{\mathbf{f}}(\mathbf{b}_{\tilde{K}}\mathbf{b}_{\tilde{K}}^T)$ which conflicts with the definition of $\tilde{K}$.

Therefore, $\tilde{K} = K$, and $rank\ R_{\mathbf{f}}(\mathbf{b}_i\mathbf{b}_i^T) = J$ for all $i = J-1, \dots, K-1$, and there exists $P$ orthogonal such that

$$R_{\mathbf{f}}\left(\mathbf{b}_{K-1}\mathbf{b}_{K-1}^T\right) = P Diag\left(\lambda_1, \dots, \lambda_J,\ 0, \dots, 0\right) P^T.$$

Then, writing $\tilde{\mathbf{b}}_K(s) = \left(\mathbf{b}_{K-1}(s)^T P,\ s^K\right)^T$, one has

$$R_{\mathbf{f}}(\tilde{\mathbf{b}}_K\tilde{\mathbf{b}}_K^T) = \left(\begin{array}{c|c|c|c} Diag(\lambda_1, \dots, \lambda_J) & 0_{\mathbb{R}^{J\times(K-J-1)}} & 0_{\mathbb{R}^J} & 0_{\mathbb{R}^J} \\ \hline 0^T_{\mathbb{R}^{(K-J-1)\times J}} & 0_{\mathbb{R}^{(K-J-1)\times(K-J-1)}} & 0_{\mathbb{R}^{(K-J-1)}} & 0_{\mathbb{R}^{(K-J-1)}} \\ \hline 0^T_{\mathbb{R}^J} & 0^T_{\mathbb{R}^{(K-J-1)}} & 0 & a \\ \hline 0^T_{\mathbb{R}^J} & 0^T_{\mathbb{R}^{(K-J-1)}} & a & b \end{array}\right),$$

where only the components $R_{\mathbf{f}}\left((\mathbf{b}_K)_K(\tilde{\mathbf{b}}_K)_{K-1}\right) = a$ and $R_{\mathbf{f}}\left((\mathbf{b}_K)_K^2\right) = b$ that appear not in the matrix $R_{\mathbf{f}}(\mathbf{b}_{K-1}\mathbf{b}_{K-1})$ and are *a priori* non-zero, i.e. the last two components of the last row or column of $R_{\mathbf{f}}(\tilde{\mathbf{b}}_K\tilde{\mathbf{b}}_K^T)$.

By contradiction, suppose that $a \neq 0$, and define $V = (0, \dots, 0, -\frac{b}{a}, 1)^T$, then we have $V^T R_{\mathbf{f}}(\tilde{\mathbf{b}}_K\tilde{\mathbf{b}}_K^T)V = -b$, which would prevent $R_{\mathbf{f}}(\tilde{\mathbf{b}}_K\tilde{\mathbf{b}}_K^T)$ from being positive. Therefore, $a = 0$ and $R_{\mathbf{f}}((\mathbf{b}_K)_K\mathbf{b}_{K-1}) \in Span\left(R_{\mathbf{f}}(\mathbf{b}_{K-1}\mathbf{b}_{K-1}^T)\right)$. This implies that $R_{\mathbf{f}}(\mathbf{b}_{2K-1})$ is positively recursively generated. $\square$

As a consequence, any realizable vector $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$ can be decomposed under the form

$$\mathbf{f} = \alpha_0 \mathbf{f}^{eq} + \mathbf{f}^{rec} + \alpha_K \mathbf{f}^{nrec}, \tag{10}$$

where $\mathbf{f}^{rec} \in \partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^{Q}$ is positively recursively generated (a sum of $\mathbf{b}(s_i)$), $\alpha_K \geq 0$ is non-negative and $\mathbf{f}^{nrec}$ is of the form (9).

If $\mathbf{f}^{eq}$ and $\mathbf{f}^{rec}$ have well-understood representation, writing $\mathbf{f}^{nrec}$ in terms of moments is more problematic. Several methods to include this term in a representation are provided in the next subsection.

## 3.3 Some representation results including $\partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}_{\mathbf{b}}^{Q}$

In the next section, we focus on a particular choice of equilibrium function $f^{eq}$ that provides naturally a projection $\mathbf{f}^p \in \partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}_{\mathbf{b}}^{Q}$. However, we provide some partial results of representation used (in proofs) in the next sections. These results provide some mathematical understanding and could also be used to construct other closures, even if they do not provide a straightforward kinetic interpretation.

### 3.3.1 Representation using the limit of a sequence of Dirac distributions

The first representation of any vector $\mathbf{f}^p \in \partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}_{\mathbf{b}}^{Q}$ follows simply from the definition of this boundary as $\partial\mathcal{R}_{\mathbf{b}} = \overline{\mathcal{R}_{\mathbf{b}}} \backslash \mathcal{R}_{\mathbf{b}}$ and therefore $\mathbf{f}^p \in \overline{\mathcal{R}_{\mathbf{b}}} \backslash \mathcal{R}_{\mathbf{b}}^{Q}$. Since it belongs to the closure set of the realizability domain, any vector on the boundary is the limit of a sequence of realizable vectors. A first proposition is recalled.

**Proposition 6.** *Considering moments up to an even $N = 2J$ or odd $N = 2J - 1$ order, the realizability domain is represented by exactly $J$ different Dirac measures*

$$\mathcal{R}_{\mathbf{b}_N} = \left\{ \sum_{i=1}^{J} \alpha_i \mathbf{b}(s_i), \qquad \alpha_i > 0, \qquad s_i \in E \subset \mathbb{R} \quad s.t. \quad s_i \neq s_j \quad \forall i \neq j \right\}.$$

This result has been formulated in various manners in the literature. The proof can be found for instance in [13, 11].

Based on such a representation of realizable vectors, we may interpret any vector on the boundary $\partial\mathcal{R}_{\mathbf{b}}$ as the limit of a sequence in $\mathcal{R}_{\mathbf{b}}$. In this spirit, one verifies for instance that the last canonical vector reads

$$\mathbf{e}^N = \lim_{\epsilon \to 0} \epsilon^N \mathbf{b}_N(\epsilon^{-1}) = \lim_{\epsilon \to 0} \int_{\mathbb{R}} \epsilon^N \mathbf{b}_N(s) \delta_{\epsilon^{-1}}(s)$$

which provides a representation of $\mathbf{f}^{nrec} = \mathbf{e}^N$. This can be extended with the following result.

**Corollary 1.** *Consider a vector $\mathbf{f} \in \partial\mathcal{R}_{\mathbf{b}}$ with even $N = 2K$ or odd $N = 2K + 1$ order moments. Then,*

- *Either $\mathbf{f} \in \partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^{Q}$ then $\mathbf{f}$ can be represented by less Dirac measures*

$$\mathbf{f} = \sum_{i=1}^{J} \alpha_i \mathbf{b}(s_i), \tag{11a}$$

  *with $J < K$ such that $\alpha_i > 0$ and $s_i \neq s_j$ for all $i \neq j$.*

- *Or* $\mathbf{f} \in \partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}_{\mathbf{b}}^{Q}$ *then* $\mathbf{f}$ *can be represented by*

$$
\begin{aligned}
\mathbf{f} = &\sum_{i=1}^{J} \alpha_i \mathbf{b}(s_i) \\
&+ \lim_{\epsilon \to 0} \left\{ \begin{array}{ll} \alpha_K \epsilon^{2K} \mathbf{b}(\pm\epsilon^{-1}) & \textit{even case } N = 2K, \\ \alpha_K \left( \frac{\epsilon^{2K} + \beta_K \epsilon^{2K+1}}{2} \mathbf{b}(\epsilon^{-1}) + \frac{\epsilon^{2K} - \beta_K \epsilon^{2K+1}}{2} \mathbf{b}(-\epsilon^{-1}) \right) & \\ & \textit{odd case } N = 2K+1, \end{array} \right.
\end{aligned}
\tag{11b}
$$

*with* $J < K - 1$ *such that* $\alpha_i > 0$ *and* $s_i \neq s_j$ *for all* $i \neq j$ *and* $\alpha_K > 0$.

*Proof.* The case $\mathbf{f} \in \partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^{Q}$ was treated in Proposition 4.

If $\mathbf{f} \in \partial\mathcal{R}_{\mathbf{b}}$, Corollary 1 states that

$$
\mathbf{f} = \sum_{i=1}^{J} \alpha_i \mathbf{b}(s_i) + \alpha_K \left\{ \begin{array}{lll} \mathbf{e}^{2K} & \text{if} \quad N = 2K & \text{even case}, \\ \mathbf{e}^{2K} + \beta_K \mathbf{e}^{2K+1} & \text{if} \quad N = 2K+1 & \text{odd case}, \end{array} \right.
$$

with $\alpha_K \geq 0$. If $\alpha_K = 0$, one easily verifies that $\mathbf{f}$ is recursively generated and is therefore in $\mathbb{R}_{\mathbf{b}}^{Q}$. Then $\mathbf{f} \in \partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}_{\mathbf{b}}^{Q}$ imposes $\alpha_K > 0$. In the even case, one can represent

$$
\mathbf{e}^{2K} = \lim_{\epsilon \to 0} \epsilon^{2K} \mathbf{b}_{2K}(\epsilon^{-1}),
$$

and in the odd case

$$
\begin{aligned}
\left(\mathbf{e}^{2K} + \beta_K \mathbf{e}^{2K+1}\right) = \lim_{\epsilon \to 0} \frac{1}{2} \Big[ &\left(\epsilon^{2K} + \beta_K \epsilon^{2K+1}\right) \mathbf{b}_{2K+1}(\epsilon^{-1}) \\
&+ \left(\epsilon^{2K} - \beta_K \epsilon^{2K+1}\right) \mathbf{b}_{2K+1}(-\epsilon^{-1}) \Big],
\end{aligned}
$$

which correspond to the limit of the sum of two Dirac measures with positive masses (if $\epsilon$ is small enough). □

In the odd case of (11), the moment of order $2K$ of the two Dirac measures tend to $\pm\infty$ in the limit $\epsilon \to 0$, but their sum vanishes to the finite value $\alpha_K$.

This representation can be useful for proofs, as in Section 4 below. However, it is useless for the construction of a closure since all moments of order superior to $N$ are unbounded in this limit. It is therefore a bad candidate for a closure representation, as it does not provide finite higher order moments.

One remarks one more important representation result of the boundary.

**Corollary 2.** *The part of the boundary* $\partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^{Q}$ *is dense in* $\partial\mathcal{R}_{\mathbf{b}}$.

*Proof.* One simply observes that the number of deltas, i.e. at most $J - 1$ in the even case or $J$ in the odd case, before taking the limit in (11) is strictly lower than the number in the interior $\mathcal{R}_{\mathbf{b}}^{Q}$, i.e. $J$ in the even case or $J + 1$ in the odd case. Therefore, before the limit, the representation is in $\partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^{Q}$. □

### 3.3.2 Representation with derivatives of a Dirac distribution

As functions and discrete measures are rejected to represent $\mathbf{f}^{nrec} \in \partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}_{\mathbf{b}}^{Q}$, we interpret moments in the sense of (differentiable) distributions in this paragraph.

**Proposition 7.** *Consider $\mathbf{f} \in \partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}_{\mathbf{b}}^{Q}$. Then*

$$\mathbf{f} = \sum_{i=1}^{J} \alpha_i \mathbf{b}(s_i) + \frac{\alpha_K}{2K!} \mathbf{b}^{(2K)}(s_K),$$

*where $\alpha_i > 0$, the velocities $s_i$ are the roots of the polynomial $p$ of lowest (non-zero) degree such that $R_{\mathbf{f}}(p^2) = 0$ and*

- *Even case $N = 2K$: $s_K \in \mathbb{R}$ can have any real value,*
- *Odd case $N = 2K+1$:*

$$s_K = (2K+1) \frac{R_{\mathbf{f}}(s^{2K+1}) - \sum\limits_{i=1}^{J} \alpha_i s_i^{2K+1}}{R_{\mathbf{f}}(s^{2K}) - \sum\limits_{i=1}^{J} \alpha_i s_i^{2K}}.$$

*Proof.* Corollary 1 provides that $(\mathbf{f}_0, \dots, \mathbf{f}_{2K-1})^T$ is positively recursively generated and is therefore represented by a sum of Dirac distributions. Finally, only the last component $R_{\mathbf{f}}(s^{2K}) \neq \sum\limits_{i=1}^{J} \alpha_i s_i^{2K}$. This can be corrected by adding a vector $\mathbf{b}^{(2K)}(s_K)$ which has only zero components except those in $2K$ (and potentially $2K+1$ in the odd case). Especially, adding this term to the sum does not alter $R_{\mathbf{f}}(\mathbf{b}_{2K-1})$. Removing all the first eigenvectors provides

$$R_{\mathbf{f}}(\mathbf{b}_K \mathbf{b}_K^T) - \sum_{i=1}^{J} \alpha_i \mathbf{b}_K(s_i) \mathbf{b}_K(s_i)^T = Diag(0, \dots, 0, \alpha_K).$$

This provides $\alpha_K = R_{\mathbf{f}}(s^{2K}) - \sum\limits_{i=1}^{J} \alpha_i s_i^{2K} > 0$. Rewrite

$$Diag(0, \dots, 0, \alpha_K) = \frac{\alpha_K}{2K!} (\mathbf{b}_K \mathbf{b}_K^T)^{(2K)}(s_K)$$

and the matrix can be corrected by $R_{\mathbf{f}}(\mathbf{b}_K \mathbf{b}_K^T)$ where $\mathbf{f} = \frac{\alpha_K}{2K!} \mathbf{b}^{(2K)}(s_K)$ for some $s_K$. In the even case, no other moment is known, then $\mathbf{b}^{(2K)}(s_K) = (0, \dots, 0, 2K!)^T$ for all $s_K$. In the odd case, we still have $R_{\mathbf{f}}(s^{K+1}\mathbf{b}_K) \in Span(R_{\mathbf{f}}(\mathbf{b}_K \mathbf{b}_K^T))$ following Corollary 1, then

$$R_{\mathbf{f}}(s^{2K}) - \sum_{i=1}^{J} \alpha_i s_i^{2K} = \alpha_K, \qquad R_{\mathbf{f}}(s^{2K+1}) - \sum_{i=1}^{J} \alpha_i s_i^{2K+1} = (2K+1)\alpha_K s_K,$$

which provides a unique $s_K$. □

This provides a representation for all $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$ which is valid when the projection $\mathbf{f}^p \in \partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^{Q}$ and which continuously degenerates onto the known one when $\mathbf{f}^p \in \partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}_{\mathbf{b}}^{Q}$. One may easily construct a closure out of this representation in a general framework. However, it is not entirely satisfactory for several reasons:

- The positivity of a distribution in $\mathcal{D}(\mathbb{R})'$ is understood here as the positivity of the integral against all the positive test functions in $\mathcal{D}(\mathbb{R})$. This representation is not positive since the last term $\mathbf{b}^{(2K)}(s)$ is not represented by a positive distribution but only by some derivative of a positive distribution. Such a derivative of a distribution is not a positive distribution, and higher order moments of such a representation do not satisfy the realizability property of Proposition 3.

- In terms of application, if the sum of Diracs can have various interpretations in the physics community (think e.g. of Klimontovitch equation or PIC simulations in plasma, or more generally a BBGKY derivation in kinetic theory), the present reconstruction can only be understood as some mathematical tool.

- Finally, the non-uniqueness of the velocity of this last term in the even case is unpleasant the moment system is not closed at this step and there is yet no reason to fix a certain value to $s_K$.

### 3.3.3 Representation with a modified equilibrium

Another idea consists in merging this last order term $\mathbf{f}^{nrec}$ in the equilibrium $\mathbf{f}^{eq}$ and defining a modified equilibrium

$$\tilde{\mathbf{f}}^{eq} := \mathbf{f}^{eq} + \alpha_K \mathbf{f}^{nrec}.$$

Decompose $\tilde{\mathbf{f}}^{eq}$ under the form

$$\tilde{\mathbf{f}}^{eq} = \frac{1}{1+\alpha_K}\mathbf{f}^{eq} + \frac{\alpha_K}{1+\alpha_K}\mathbf{f}^{p,eq},$$

where $\mathbf{f}^{p,eq}$ satisfies

$$\forall i = 0, \ldots, N-1, \qquad \mathbf{f}^{p,eq}_i = \mathbf{f}^{eq}_i \quad \text{and} \quad \mathbf{f}^{p,eq}_N = \mathbf{f}^{eq}_N + (1+\alpha_K).$$

There remains to find an appropriate representation for $\mathbf{f}^{p,eq}_i \in \mathcal{R}_{\mathbf{b}}$. This construction does not simplify the representation problem as we still need a representation of a realizable vector, but it can be used to obtain the properties of the present projective closure with another representation such as Grad's ([20]) or quadrature methods (QMOM ; [34, 47]). Especially, recent works ([40, 17]) aimed at turning this last method strongly hyperbolic. Instead of enforcing strong hyperbolicity, the present idea would provide an exact representation of the equilibria.

This idea is not pushed forward in the rest of the paper and is only left as a perspective. In the next section, we focus on a representation of the form (4) that always enforces $\mathbf{f}^p \in \partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}^Q_{\mathbf{b}}$.

# 4 A parametrization of $\mathcal{R}_{\mathbf{b}}$ adapted to the projective closure

In order to avoid the difficulty of the representation over $\partial\mathcal{R}_{\mathbf{b}} \backslash \mathcal{R}^Q_{\mathbf{b}}$, we now construct some appropriate equilibrium function such that $\mathbf{f}^{nrec} = 0$ in (9), or equivalently such that the projection $\mathbf{f}^p \in \partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}^Q_{\mathbf{b}}$.

## 4.1 Preliminary

In the first place, the equilibrium functions we have in mind are the Maxwellians. But the present results are more general. We make the following additional hypothesis on $f^{eq}$ which is satisfied by all the Maxwellians:

**(H4)** The equilibrium $f^{eq}$ is defined from $M$ parameters $\boldsymbol{\lambda} \in \mathbb{R}^M$ such that

$$(N+1) - M =: 2J \quad \text{is even}$$

and is of the form

$$f^{eq} = (\eta^*)'(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}), \tag{12}$$

where $\eta^*$ is the Legendre dual of the strictly convex entropy $\eta$. It satisfies $(\eta^*)' = (\eta')^{-1}$ and such a distribution corresponds to one given by an entropy minimization process ([6, 35, 30, 26, 42]).

Following the decomposition (4), our goal now is to reconstruct a measure of the form

$$d\mu(s) = (\eta^*)' \left( \boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s) \right) ds + \sum_{i=1}^{J} \alpha_i \delta_{s_i}(s) \tag{13}$$

from a vector $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$. Remark that we have chosen to seek a decomposition with $J$ Dirac measures such that the number of parameters on this decomposition equals the number of moments. Let us denote

$$\mathbf{v} := (\boldsymbol{\lambda}, \alpha_1, s_1, \dots, \alpha_J, s_J) \in E_{\mathbf{v}} \subset \mathbb{R}^{N+1}$$

the vector of parameters required to define the reconstruction. The set $E_{\mathbf{v}}$ of parameters $\mathbf{v}$ will be defined in the next subsection.

**Remark 2.** Considering a vector $\mathbf{f} \in \mathcal{R}_{\mathbf{b}_{M-1}}$ of size $M$, the distribution (12) is commonly obtained as the unique minimizer (see e.g. [6, 35, 30, 26, 42]) of

$$f^{eq} = \underset{\substack{f \in L^1_{\boldsymbol{b}_M}(E)^+ \\ \int_E \mathbf{b}_{M-1} f = \mathbf{f} \in \mathcal{R}_{\mathbf{b}_{M-1}}}}{\operatorname{arginf}} \int_E \eta(f),$$

and the coefficients $\boldsymbol{\lambda} \in \Lambda \subset \mathbb{R}^M$ correspond to the Lagrange multipliers associated with the moment constraints. In the present context with $\mathbf{f} \in \mathcal{R}_{\mathbf{b}_N}$ with $N > M-1$, one remarks:

- Adapting this optimization result, the equilibrium function $f^{eq}$ solves

$$f^{eq} = \underset{\substack{f \in L^1_{\boldsymbol{b}}(E)^+ \\ \int_E p^2 \mathbf{b}_{M-1} f = R_{\boldsymbol{f}}(p^2 \mathbf{b}_{M-1})}}{\operatorname{arginf}} \int_E p^2 \eta(f) \qquad \text{where} \quad p(s) = \prod_{i=1}^{J} (s - s_i).$$

  However, this provides little information since the roots $s_i$ of $p$ are also unknowns in our decomposition and further attempts relating the present decomposition to an optimization problem are left as perspectives.

- Furthermore, the coefficients $\boldsymbol{\lambda}$ in (13) do not correspond to the Lagrange multipliers associated with the reduced moment constraints

$$\int_E \mathbf{b}_{M-1} f^{eq} = R_{\mathbf{f}}(\mathbf{b}_{M-1})$$

because $R_{\mathbf{f}}(\mathbf{b}_{M-1})$ are the moments of $f^{eq}$ and the sum $\sum_i \alpha_i \delta_{s_i}$ together.

After choosing such an equilibrium function, **(H2)** rewrites:

**(H2')** The strictly convex entropy $\eta$ satisfies

$$\forall (p, K) \in \mathbb{R}_M[X] \times \mathbb{R}^+, \quad \exists! \tilde{K} \in \mathbb{R}^+ \quad \text{s.t.} \quad K(\eta^*)'(p) = (\eta^*)'(\tilde{K}p).$$

Indeed, with such a constraint, choosing

$$K = \left( \int_E (\eta^*)' \left( \boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s) \right) ds \right)^{-1}$$

provides **(H2)**.

**Remark 3.** Condition (**H2'**) holds for instance for the entropies

- Boltzmann/Shanons' $\eta(f) = f \log f - f$, that gives $(\eta^*)'(p) = \exp(p)$,
- quadratic $\eta(f) = f^2$, that gives $(\eta^*)'(p) = p$
- or Burgs' $\eta(f) = -\log f$, that gives $(\eta^*)'(p) = p^{-1}$.

However, other physical entropies such as those of Fermi-Dirac or Bose-Einstein $\eta(p) = (1 \pm f)\log(1 \pm f) \mp f \log f$ provide $(\eta^*)'(p) = (\exp(\mp p) \pm 1)^{-1}$ which do not enforce this property.

## 4.2 Bijection between moments $f$ and parameters v

Let us first define the appropriate set $E_{\mathbf{v}}$ of parameters $\mathbf{v}$ for the reconstruction (14).

### 4.2.1 Definition of the set $E_{\mathbf{v}}$ of parameters

First, the set of appropriate Lagrange multipliers is denoted $\Lambda$ and following classical optimization result ([26, 42]), we set

$$\Lambda := \left\{ \boldsymbol{\lambda} \in \mathbb{R}^M \quad \text{s.t.} \quad (\eta^*)' \left( \boldsymbol{\lambda}^T \mathbf{b}_{M-1} \right) \in L^1_{\mathbf{b}_{N+1}}(E)^+ \right\}$$

for its moments $\mathbf{f}^{eq}$ to be well-defined.

Second, in order to distinguish a Dirac from another in (14), we require that they all have non-zero (and positive) masses and all different locations. Without loss of generalities, let us order these locations and define

$$E_{\alpha,s} := \{(\alpha_i, s_i)_{i=1,\dots,J} \quad \text{s.t.} \quad \forall i, \quad \alpha_i > 0, \quad s_i \in E \quad \text{s.t.} \quad s_1 < s_2 < \dots < s_J\}.$$

Finally, the considered set of parameters $E_{\mathbf{v}}$ yields

$$E_{\mathbf{v}} = \Lambda \times E_{\alpha,s}.$$

By abuse, let us denote $\mathbf{f}$ the function that sends $\mathbf{v} \in E_{\mathbf{v}}$ onto $\mathcal{R}_{\mathbf{b}}$ as

$$\mathbf{f} : \left\{ \begin{array}{rcl} E_{\mathbf{v}} & \to & \mathcal{R}_{\mathbf{b}} \\ \mathbf{v} = (\boldsymbol{\lambda}, \alpha_1, s_1, \dots, \alpha_J, s_J) & \mapsto & \mathbf{f}(\mathbf{v}) = \displaystyle\int_E \mathbf{b}(\eta^*)' \left(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}\right) + \sum_{i=1}^{J} \alpha_i \mathbf{b}(s_i). \end{array} \right. \tag{14}$$

Its Jacobian reads (in the form of a concatenation of a $N \times M$ matrix and vectors)

$$J_{\mathbf{v}}\mathbf{f}(\mathbf{v}) = \left( \int_E \mathbf{b}\mathbf{b}_{M-1}^T (\eta^*)'' \left(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}\right),\ \mathbf{b}(s_1),\ \alpha_1 \mathbf{b}'(s_1), \dots,\ \mathbf{b}(s_J),\ \alpha_J \mathbf{b}'(s_J) \right). \tag{15}$$

For this Jacobian to be well-defined, we therefore need to add another hypothesis.

**(H5)** The strictly convex entropy $\eta$ satisfies

$$\forall \boldsymbol{\lambda} \in \Lambda, \quad (\eta^*)''(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}) \in L^1_{\mathbf{b}_{N+M}}(E).$$

**Remark 4.**

- This requirement could be refined to a condition on $\eta$ using convex analysis. We only remark that this holds when $\eta$ is Boltzmann entropy on $E = \mathbb{R}$ or with all the entropies mentioned in Remark 3 when $E = [-1, 1]$, i.e. for all the applications we have in mind.
- The integrability against $\mathbf{b}_{N+M-1}$ is sufficient for the Jacobian to be well-defined, but we will also use this hypothesis in the next section for the construction of the closure. This closure will require one order of integrability higher.
- By convexity of $\eta$ and the properties of the Legendre transformation, $(\eta^*)''$ is positive.

**Proposition 8.** *Under hypothesis (**H5**), the Jacobian $J_{\mathbf{v}}\mathbf{f}(\mathbf{v})$ is invertible for all $\mathbf{v} \in E_{\mathbf{v}}$.*

*Proof.* Define for instance a vector of polynomials $\mathbf{l}$ such that for all $i \leq J$

$$\mathbf{l}_{2i-1}(s) = \prod_{j \neq i} \frac{(s - s_j)^2}{(s_i - s_j)^2}, \qquad \mathbf{l}_{2i}(s) = (s - s_i) \prod_{j \neq i} \frac{(s - s_j)^2}{(s_i - s_j)^2},$$

and the remaining ones, for all $i > 2J$

$$\mathbf{l}_i = \prod_{j=1}^{J} (s - s_j)^2 \mathbf{b}_{i-2J}.$$

The Hermite interpolation polynomials form a basis as long as the interpolation points are all different, i.e. $s_i \neq s_j$ for all $i \neq j$. Then one easily verify that the components of the $\mathbf{l}$ are a basis of $\mathbb{R}_N[X]$. This provides a change of basis matrix $P$ such that

$$P J_{\mathbf{v}}\mathbf{f}(\mathbf{v}) P^T = \left( \begin{array}{c|c} * & Diag(J_1, \dots, J_J) \\ \hline \displaystyle\int_E \prod_{j=1}^{J} (s - s_j)^2 \mathbf{b}_{M-1} \mathbf{b}_{M-1}^T (\eta^*)'' \left(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}\right) & 0_{J \times 2M} \end{array} \right),$$

$$J_i = \begin{pmatrix} 1 & \alpha_i \mathbf{l}_i'(s_i) \\ 0 & \alpha_i \end{pmatrix}.$$

The right part of this matrix (composed of Jordan blocks) is composed of linearly independent vectors as long as all $\alpha_i \neq 0$ and $s_i \neq s_j$ for $i \neq j$.

The block on the bottom left $\int_E \prod_{j=1}^{J}(s - s_j)^2 \mathbf{b}_{M-1}\mathbf{b}_{M-1}^T(\eta^*)'' \left(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}\right)$ is symmetric positive definite, then is also composed of linearly independent vectors, which are also independent of those on the right side of the matrix.

Together, this provides the invertibility of $J_{\mathbf{v}}\mathbf{f}(\mathbf{v})$. □

**Corollary 3.** *The function (14) is a $C^1$-diffeomorphism from $E_{\mathbf{v}}$ into $\mathbf{f}(E_{\mathbf{v}}) \subset \mathcal{R}_{\mathbf{b}}$.*

*Proof.* The bijection follows directly from the previous computation, and one observes that $\mathbf{f}(\mathbf{v}) \in \mathcal{R}_{\mathbf{b}}$ is realizable as long as the masses $\alpha_i > 0$. □

**Remark 5.** Hypothesis **(H3)** always holds for an equilibrium function of the form (12) satisfying Hypothesis **(H2')** (see also Remark 3). This simply follows from the fact that the function (14) is a bijection.

#### 4.2.2 Density of $\mathbf{f}(E_{\mathbf{v}})$ into $\mathcal{R}_{\mathbf{b}}$

In order to study the part of $\mathcal{R}_{\mathbf{b}}$ that can be represented by a representation of the form (14), we focus on the boundary of the set $\mathbf{f}(E_{\mathbf{v}})$. For this purpose, let us decompose the boundary $E_{\mathbf{v}}$ into

$$\partial E_{\mathbf{v}} = (\partial\Lambda \times E_{\alpha,s}) \cup (\Lambda \times \partial E_{\alpha,s}). \tag{16}$$

By abuse of notation (since $\mathbf{f}$ is not defined on the boundary), we denote $\mathbf{f}(\partial E_{\mathbf{v}}) \equiv \partial\mathbf{f}(E_{\mathbf{v}})$ the boundary of the set of moments represented by (14). And we denote $\mathbf{f}(\partial\Lambda \times E_{\alpha,s})$ and $\mathbf{f}(\Lambda \times \partial E_{\alpha,s})$ the parts of this boundary corresponding to the decomposition (16).

At this step, the inclusion $\mathbf{f}(E_{\mathbf{v}}) \subset \mathcal{R}_{\mathbf{b}}$ ensures not that any realizable vector $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$ possesses a decomposition of the form (14) with positive masses $\alpha_i$ and an equilibrium function of the form (12). In fact, this never holds and we only have

$$\mathbf{f}(E_{\mathbf{v}}) \subsetneq \mathcal{R}_{\mathbf{b}}.$$

Two illustrative counterexamples are provided to explain this issue:

- Consider a distribution of the form (14) but with less than $J = \frac{N+1-M}{2}$ Dirac measures. This also corresponds to a boundary distribution $\mathbf{f}(\mathbf{v})$ with $\mathbf{v} \in \partial E_{\mathbf{v}}$, i.e. $\mathbf{v} \notin E_{\mathbf{v}}$. This can be represented in two manners in (14): a Dirac with a zero mass $\alpha_J = 0$ at any location $s_J \in \mathbb{R}$, or several Dirac measures at the same location $s_i = s_{i+1}$. Especially, this implies the non-uniqueness of the parameters in such a limit and therefore the loss of the bijective property along this boundary.

  Such vectors belong to $\mathbf{f}(\Lambda \times \partial E_{\alpha,s})$.

- M. Junk ([25, 26]) exhibited realizable vectors that can not be represented by an entropy minimization reconstruction in the case of Boltzmann entropy for the problem of Hamburger $E = \mathbb{R}$. This led to the fact that $\Lambda$ is open and possesses a boundary $\partial\Lambda\backslash\Lambda \neq \emptyset$ and there exists some $\boldsymbol{\lambda}^s \in \partial\Lambda\backslash\Lambda$ such that

$$\mathbf{f}^{\lim} = \lim_{\boldsymbol{\lambda}\to\boldsymbol{\lambda}^s} \left(\int_{\mathbb{R}} \mathbf{b}_{M-1}(\eta^*)' \left(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}\right)\right) \in \mathcal{R}_{\mathbf{b}_{M-1}} \tag{17a}$$

is realizable, but that have no representation of the form

$$\nexists \boldsymbol{\lambda} \in \Lambda \quad \text{s.t.} \quad \mathbf{f}^{\lim} = \int_{\mathbb{R}} \mathbf{b}_{M-1}(\eta^*)' \left( \boldsymbol{\lambda}^T \mathbf{b}_{M-1} \right). \tag{17b}$$

Since (14) is a bijection, adding some $\alpha_i \mathbf{b}(s_i)$ in (17a) and taking the same limit on $\boldsymbol{\lambda}$ also leads to a realizable vector that can not be represented under the form (17b).

Such vectors belong to $\mathbf{f}(\partial\Lambda \times E_{\alpha,s})$.

Concerning the second point, we focus on a case (see hypothesis in Theorem 4.1 below) avoiding the appearance of such a Junk line ([25]). This hypothesis could be softened, as one only needs such a Junk line not to open onto a part of the realizability domain $\mathcal{R}_{\mathbf{b}} \backslash \mathbf{f}(E_{\mathbf{v}})$ that can not be represented by our representation (14) and that has a non-empty interior in $\mathbb{R}^{N+1}$. For simplicity, this discussion is avoided here.

Concerning the first point, we may still quantify the part of the realizability domain not covered by the present representation (14).

**Theorem 4.1.** *Suppose that $\Lambda \neq \emptyset$ and $\partial\Lambda \cap \Lambda = \emptyset$. Then $\mathbf{f}(E_{\mathbf{v}})$ is dense in $\mathcal{R}_{\mathbf{b}}$.*

In order to keep the main results highlighted in the core of the text, the (essentially technical) proof of this theorem is given in Appendix B.

**Remark 6.** The hypothesis $\Lambda \cap \partial\Lambda = \emptyset$ was also exploited by M. Junk in [26]. It is satisfied when

- the equilibrium $f^{eq}$ corresponds to a Maxwellians, i.e. when using Boltzmann entropy $(\eta^*)' = \exp$ for the Hamburger problem $E = \mathbb{R}$ and where the equilibrium is modeled with $M = 3$ parameters,
- or with all the entropies discussed in Remark 3 for the Hausdorff problem $E = [-1, +1]$.

The other cases involving a Junk line, e.g. when the equilibrium is modelled with higher entropy minimizer for Hamburger problem, would require further considerations in order to prove that such a part of the boundary $\mathbf{f}\left(\partial\Lambda \times E_{\alpha,s}\right)$ does not open onto a domain $\mathcal{R}_{\mathbf{b}} \backslash \mathbf{f}(E_{\mathbf{v}})$ with non-empty interior. This can be expected. However, it would require further analysis to prove this result, and this is not discussed here.

### 4.3 Discussion on the set $\mathcal{R}_{\mathbf{b}} \backslash \mathbf{f}(E_{\mathbf{v}})$

In the spirit of [25], the existence of such realizable vectors $\mathbf{f} \in \mathcal{R}_{\mathbf{b}} \backslash \mathbf{f}(E_{\mathbf{v}})$ may seem problematic. The existence and uniqueness of parameters $\mathbf{v} \in E_{\mathbf{v}}$ such that $\mathbf{f} = \mathbf{f}(\mathbf{v})$ are not ensured.

In fact, the previous result can be refined, and we prove the existence and uniqueness of a representation of the form (13) for such vectors just by removing the Dirac measures having zero mass.

**Proposition 9.** *Under the hypothesis of Theorem 4.1, for all $\mathbf{f} \in \mathcal{R}_{\mathbf{b}}$, there exists a unique representation of the form*

$$\mathbf{f} = \int_E \mathbf{b}(s)(\eta^*)' \left( \boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s) \right) ds + \sum_{i=1}^{\tilde{J}} \alpha_i \mathbf{b}(s_i), \tag{18}$$

*with $\boldsymbol{\lambda} \in \Lambda$, $\tilde{J} \leq J$ and such that $(\alpha_i)_{i=1,\dots,\tilde{J}} \in (\mathbb{R}^{*+})^{\tilde{J}}$ are all strictly positive and $(s_i)_{i=1,\dots,\tilde{J}} \in E^{\tilde{J}}$.*

*Proof.* The case $\mathbf{f} \in \mathcal{R}_{\mathbf{b}} \cap \mathbf{f}(E_{\mathbf{v}})$ is covered in Theorem 4.1 with $\tilde{J} = J$.

In the case $\mathbf{f} \in \mathcal{R}_{\mathbf{b}} \backslash \mathbf{f}(E_{\mathbf{v}})$, then $\mathbf{f} \in \mathcal{R}_{\mathbf{b}} \cap \mathbf{f}(\partial E_{\mathbf{v}})$. Using the decomposition (16), either $\mathbf{f} \in \mathcal{R}_{\mathbf{b}} \cap \mathbf{f}(\Lambda \times \partial E_{\alpha,s})$ or $\mathbf{f} \in \mathcal{R}_{\mathbf{b}} \cap \mathbf{f}(\partial\Lambda \times E_{\alpha,s})$.

Lemma B.1 in Appendix B provides that $\mathbf{f}(\partial\Lambda \times E_{\alpha,s}) = \partial\mathcal{R}_{\mathbf{b}}$. But, since this set is open $\mathcal{R}_{\mathbf{b}} \cap \mathbf{f}(\partial\Lambda \times E_{\alpha,s}) = \mathcal{R}_{\mathbf{b}} \cap \partial\mathcal{R}_{\mathbf{b}} = \emptyset$.

Finally, the case $\mathbf{f} \in \mathcal{R}_{\mathbf{b}} \cap \mathbf{f}(\partial\Lambda \times E_{\alpha,s})$ corresponds to the case where $\mathbf{f}$ is of the form (18) with less Dirac measures $\tilde{J} < J$ (so we have existence of the parameters). In that case, simply consider the reduced vector $R_{\mathbf{f}}(\mathbf{b}_{M-1+2\tilde{J}})$. We can use Theorem 4.1 again on this vector to obtain the uniqueness of $\tilde{J}$ parameters $(\alpha_i)_{i=1,\dots,\tilde{J}} \in (\mathbb{R}^{*+})^{\tilde{J}}$ and $(s_i)_{i=1,\dots,\tilde{J}} \in E^J$ such that (18) holds. □

This provides that the realizability domain is in bijection with the set of measures of the form (13), as long as the equilibrium function creates no Junk line.

# 5 Hyperbolicity and entropic structure

Now, we discuss the hyperbolicity of the projective models and the notions of entropy that can be expected through this construction.

## 5.1 Weak hyperbolicity and wave speeds of the full system

We first study the hyperbolic structure of the full moment model through the following proposition.

**Theorem 5.1.** *Under the hypotheses of Theorem 4.1, the left-hand-side of (2,8) is weakly hyperbolic as long as $\mathbf{f} \in \mathbf{f}(E_{\mathbf{v}})$. Especially, the Jacobian of the flux is similar to a matrix of the form*

$$J_{\mathbf{f}}\mathbf{F} = P \left( \begin{array}{c|c} A^{eq} & 0_{\mathbb{R}^{2K\times 2K}} \\ \hline B & A^s \end{array} \right) P^{-1},$$

*where*

$$A^{eq} = \left( \int_E l(s)^2 (\mathbf{b}_{M-1}\mathbf{b}_{M-1}^T)(s)(\eta^*)'' \left( \boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s) ds \right) \right)^{-1} \times \left( \int_E s l(s)^2 (\mathbf{b}_{M-1}\mathbf{b}_{M-1}^T)(s)(\eta^*)'' \left( \boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s) \right) ds \right),$$

$$A^s = Diag(A_{\alpha_1,s_1}, \dots, A_{\alpha_K,s_K}), \quad A_{\alpha_k,s_k} = \begin{pmatrix} s_k & \alpha_k \\ 0 & s_k \end{pmatrix}, \quad l(s) = \prod_{k=1}^{K} (s - s^k),$$

*and some matrix $B \in \mathbb{R}^{2J\times(M-1)}$. Especially, the spectrum of $J_{\mathbf{f}}\mathbf{F}$ is composed of the eigenvalues of $A^{eq}$ (multiplicity one) and each $s_k$ of multiplicity two.*

*Proof.* Under the hypotheses of Theorem 4.1, we can decompose

$$\mathbf{f} = \int_E \mathbf{b}(s)(\eta^*)' \left( \boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s) \right) ds + \sum_{i=1}^{K} \alpha_i \mathbf{b}(s_i),$$

where $J_{\mathbf{v}}\mathbf{f}$ is computed in (15). Performing the same decomposition and computation for the flux reads

$$
\begin{aligned}
\mathbf{F} &= \int_E s\mathbf{b}(s)(\eta^*)' \left(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s)\right) ds + \sum_{i=1}^{K} \alpha_i s_i \mathbf{b}(s_i),\\
J_{\mathbf{v}}\mathbf{F} &= \Bigg( \int_E s(\mathbf{b}\mathbf{b}_{M-1}^T)(s)(\eta^*)'' \left(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s)\right) ds, \qquad (19)\\
&\qquad s_1\mathbf{b}(s_1),\ \alpha_1\left(s_1\mathbf{b}'(s_1) + \mathbf{b}(s_1)\right), \dots,\ \mathbf{b}(s_J),\ \alpha_J\left(s_J\mathbf{b}'(s_J) + \mathbf{b}(s_J)\right) \Bigg).
\end{aligned}
$$

Define the matrix $P$ such that

$$
P\mathbf{b} = \left( l^2 \mathbf{b}_{M-1}^T,\ \tilde{\mathbf{b}}^T \right)^T,
$$

$$
\tilde{\mathbf{b}}(s) = \left( l_1^2(s),\ (s-s_1)l_1^2(s),\ \dots,\ l_J^2(s),\ (s-s_J)l_J^2(s) \right)^T, \qquad l_i(s) = \prod_{\substack{j=1\\ j\neq i}}^{J} \frac{s-s_j}{s-s_i}.
$$

This new vector of polynomials forms a basis, thus $P$ is invertible and we can decompose

$$
J_{\mathbf{f}}\mathbf{F} = (J_{\mathbf{v}}\mathbf{f}) \left[ (PJ_{\mathbf{v}}\mathbf{f})^{-1} (PJ_{\mathbf{v}}\mathbf{F}) \right] (J_{\mathbf{v}}\mathbf{f})^{-1}.
$$

Computing the blocks provides

$$
\begin{aligned}
(PJ_{\mathbf{v}}\mathbf{f}) &= \left( \begin{array}{c|c} \int_E l^2(s)(\mathbf{b}_{M-1}\mathbf{b}_{M-1}^T)(s)(\eta^*)'' \left(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s)\right) ds & 0_{\mathbb{R}^{N-M+1\times 2J}} \\ \hline \int_E (\tilde{\mathbf{b}}\mathbf{b}_{M-1}^T)(s)(\eta^*)'' \left(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s)\right) ds & \mathcal{B} \end{array} \right),\\
(PJ_{\mathbf{v}}\mathbf{F}) &= \left( \begin{array}{c|c} \int_E s l^2(s)(\mathbf{b}_{M-1}\mathbf{b}_{M-1}^T)(s)(\eta^*)'' \left(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s)\right) ds & 0_{\mathbb{R}^{N-M+1\times 2J}} \\ \hline \int_E s(\tilde{\mathbf{b}}\mathbf{b}_{M-1}^T)(s)(\eta^*)'' \left(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s)\right) ds & \mathcal{C} \end{array} \right),\\
\mathcal{B} &= Diag(B_{\alpha_1,s_1}, \dots, B_{\alpha_J,s_J}), \qquad \mathcal{C} = Diag(C_{\alpha_1,s_1}, \dots, C_{\alpha_J,s_J}),\\
B_{\alpha_i,s_i} &= \begin{pmatrix} 1 & 2\alpha_i l_i'(s_i) \\ 0 & \alpha_i \end{pmatrix}, \qquad C_{\alpha_i,s_i} = \begin{pmatrix} s_1 & \alpha_i(1 + 2s_i l_i'(s_i)) \\ 0 & \alpha_i \end{pmatrix}.
\end{aligned}
$$

Computing the inverse $(PJ_{\mathbf{v}}\mathbf{f})^{-1}$ of the triangular block matrix $(PJ_{\mathbf{v}}\mathbf{f})$ and multiplying by $(PJ_{\mathbf{v}}\mathbf{F})$ provides the result. Remark that the matrix $B$ can be computed explicitly, but its value does not impact on the eigenstructure of $J_{\mathbf{f}}\mathbf{F}$.

□

Based on the considered representation involving the propagation of Dirac distributions, weak hyperbolicity was highly expected. Indeed, such weakly hyperbolic models are known to create and propagate $\delta$-shocks (see e.g. [29, 7, 16]), i.e. the solution involves Dirac measures propagated with the flow. In the present case, those $\delta$-shocks are exactly those appearing in (4). The same phenomenon appears with the QMOM method ([34]), which also provides a weakly hyperbolic model as this reconstruction is only composed of Dirac distributions. However, the equilibrium part in (4) provides other benefits that are studied in the next subsection.

From [30], the wave speeds of the equilibrium system, i.e. without the delta in (4), are simply the eigenvalues of $A^{eq}$ where the polynomial $l^2$ is set to 1.

We provide two corollaries for the applications we have in mind.

**Corollary 4.** *Suppose that $N = 2J+1$ is odd and $f^{eq} \in L_{\mathbf{b}_{N+1}}(E)^+$ is independent of $\mathbf{f}$ and such that $\mathbf{f}^s \in \partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^Q$. Then the wave speeds are the $s_i$ (of multiplicity two) and*

$$s^{eq} := \frac{\int_E s \prod_{i=1}^{J} (s - s_i)^2 f^{eq}(s) ds}{\int_E \prod_{i=1}^{J} (s - s_i)^2 f^{eq}(s) ds}.$$

*Proof.* Under the hypothesis, $\mathbf{f}^s$ can be represented by $J$ Dirac measures and there is a bijection between $E_{\alpha,s}$ and $\partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^Q$ such that we can use the decomposition (14) with the appropriate number of parameters. The rest of the proofs is identical. ☐

The hypothesis $\mathbf{f}^s \in \partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^Q$ always hold when $E$ is compact, e.g. in the case of Hausdorff problem $E = [-1, +1]$ for radiative transfer which was studied in [38].

In the case of Hamburger $E = \mathbb{R}$, e.g. for rarefied gases, one may try to use a precomputation of a Maxwellian, for instance by solving Euler equations, and use the present projective method to study the perturbation from the thermodynamic equilibrium. However, at this step, there is no reason for this Maxwellian to satisfy $\mathbf{f}^s \in \partial\mathcal{R}_{\mathbf{b}} \cap \mathcal{R}_{\mathbf{b}}^Q$. For this purpose, one may use a Maxwellian that depends on the moments $\mathbf{f}$. The difference between those two approaches relies on the parameters of this Maxwellian, either a precomputation, therefore independent of $\mathbf{f}$ and easier to compute numerically, or a non-linear function of $\mathbf{f}$ fully coupled with the location and the masses (and *a priori* $\rho \neq \mathbf{f}_0$, $u \neq \frac{\mathbf{f}_1}{\mathbf{f}_0}$ and $T \neq \frac{\mathbf{f}_2}{\mathbf{f}_0} - \left(\frac{\mathbf{f}_1}{\mathbf{f}_0}\right)^2$).

**Corollary 5.** *Suppose that $N = 2J+1$ is odd and $f^{eq}$ is a Maxwellian, i.e. of the form*

$$f^{eq}(s) = \frac{\rho}{\sqrt{2\pi T}} \exp\left(-\frac{(s-u)^2}{2T}\right)$$

*where the density $\rho$, the velocity $u$ and the temperature $T$ depend on $\mathbf{f}$. Then, the wave speeds are the $s_i$ (of multiplicity 2) and the eigenvalues of*

$$A^{eq} = \left(\int_E \prod_{k=1}^{K} (s - s^k)^2 (\mathbf{b}_2 \mathbf{b}_2^T)(s) \exp\left(-\frac{(s-u)^2}{2T}\right) ds\right)^{-1} \times \left(\int_E s \prod_{k=1}^{K} (s - s^k)^2 (\mathbf{b}_2 \mathbf{b}_2^T)(s) \exp\left(-\frac{(s-u)^2}{2T}\right) ds\right).$$

This is a direct application of Theorem 5.1. One may find a simpler form of the wave speeds of this system by using the change of variables $\sigma = \frac{s-u}{\sqrt{2T}}$ in those integrals.

## 5.2 Entropy dissipation

We provide here partial results around the entropy dissipation of the present closure. At this point, we have not provided any hints on how to construct the moments of the collision operator. Contrarily to many closures in the literature, one can not define it just by plugging the chosen reconstruction in the definition of the moments of the collision operator (3), because this reconstruction involves Dirac measures which are commonly inappropriate for most collision operators (typically for Boltzmann or BGK).

We provide here two constructions providing some intuitions on constraints to impose on this $\mathbf{Q}$ for the closed moment equations to have appropriate entropy dissipation.

#### 5.2.1 Symmetric hyperbolicity of the equilibrium subsystem

The equilibrium part of the reconstruction is closely related to the entropy minimizing closure. In [30], such a closure was shown to lead to a Godunov-Mock ([19, 37, 18]) symmetric form

$$A(\boldsymbol{\lambda})\partial_t\boldsymbol{\lambda} + B(\boldsymbol{\lambda})\partial_x\boldsymbol{\lambda} = \mathbf{D}_\lambda,$$

with $A$ symmetric definite positive and $B$ symmetric. The definition of those relate the hyperbolic moment equations with its underlying kinetic interpretation. Following [27, 45, 10], one commonly constructs a closure $\mathbf{Q}$ with such a closure, such that it leads to a definition of the entropy dissipation operator $\mathbf{D}_\lambda$ (depending on $\boldsymbol{\lambda}$) that has non-positive components.

This property also inspired the construction of the $\varphi$-divergence closure in [1, 2].

We exhibit here a property of the equilibrium function that leads to a similar Godunov-Mock form.

**Proposition 10.** *For a given function $\mathbf{Q}(\mathbf{f})$, suppose that the function $\mathbf{f}$ satisfies (2,8) in a neighborhood of $(x,t)$ and such that it belongs to $\mathbf{f}(x,t) \in \mathbf{f}(E_{\mathbf{v}})$. Then $\boldsymbol{\lambda}$ satisfy the symmetric hyperbolic equation*

$$A(\mathbf{v})\partial_t\boldsymbol{\lambda} + B(\mathbf{v})\partial_x\boldsymbol{\lambda} = \mathbf{D}_\lambda(\mathbf{v}), \tag{20}$$

*with $A$ symmetric positive definite and $B$ symmetric. Those matrices and the operator $\mathbf{D}_\lambda$ depend on both $\boldsymbol{\lambda}$ and on $(\alpha_i, s_i)_{i=1,\dots,J}$ and are defined by*

$$\begin{aligned} A(\mathbf{v}) &= \int_E l^2(s)(\mathbf{b}_{M-1}\mathbf{b}_{M-1}^T)(s)(\eta^*)''(\boldsymbol{\lambda}^T\mathbf{b}_{M-1}(s))ds,\\ B(\mathbf{v}) &= \int_E s l^2(s)(\mathbf{b}_{M-1}\mathbf{b}_{M-1}^T)(s)(\eta^*)''(\boldsymbol{\lambda}^T\mathbf{b}_{M-1}(s))ds,\\ \mathbf{D}_\lambda(\mathbf{v}) &= R_{\mathbf{Q}}(l^2\mathbf{b}_{M-1}). \end{aligned}$$

*Proof.* We have

$$J_{\mathbf{v}}\mathbf{f}\partial_t\mathbf{v} + J_{\mathbf{v}}\mathbf{F}\partial_x\mathbf{v} = \mathbf{Q}, \tag{21}$$

where $J_{\mathbf{v}}\mathbf{f}$ is defined in (15) and $J_{\mathbf{v}}\mathbf{F}$ in (19). Define the matrix $P$ such that $P\mathbf{b} = l^2\mathbf{b}_{M-1}$, then

$$PJ_{\mathbf{v}}\mathbf{f} = \left(A(\mathbf{v}), 0_{\mathbb{R}^{M-1}}, \dots, 0_{\mathbb{R}^{M-1}}\right), \qquad PJ_{\mathbf{v}}\mathbf{F} = \left(B(\mathbf{v}), 0_{\mathbb{R}^{M-1}}, \dots, 0_{\mathbb{R}^{M-1}}\right).$$

Therefore multiplying (21) on the left by $P$ reads (20). □

#### 5.2.2 Entropy equation of the Dirac subsystems

Similarly, the singular part of the reconstruction is closely related to the QMOM closure ([34]).

In [7], it was shown that a system of moments represented by one Dirac measure satisfied an additional equation of the form

$$\partial_t \left(\alpha_1 H(s_1)\right) + \partial_x \left(\alpha_1 s_1 H(s_1)\right) = D_{H,\alpha,s},$$

for any convex function $H$. Commonly one choose to impose a non-positive entropy dissipation term $D_{H,\alpha,s} \leq 0$. This idea extends for multiple Dirac measures, e.g. in [9] under the form

$$\partial_t \left( \sum_{i=1}^{J} \alpha_i H_i(s_i) \right) + \partial_x \left( \sum_{i=1}^{J} \alpha_i s_i H_i(s_i) \right) = D_{H,\alpha,s}(\mathbf{v}),$$

for any convex functions $H_i$.

We exhibit here a property of the singular part of the reconstruction that leads to a similar entropy equation.

**Proposition 11.** *For a given function $\mathbf{Q}(\mathbf{f})$ and given convex functions $(H_i)_{i=1,\dots,J}$, suppose that the function $\mathbf{f}$ satisfies (2,8) in a neighborhood of $(x,t)$ and such that it belongs to $\mathbf{f}(x,t) \in \mathbf{f}(E_{\mathbf{v}})$.*

*Then the coefficients $(\alpha_i, s_i)_{i=1,\dots,J}$ satisfy an equation of the form*

$$\partial_t \left( \sum_{i=1}^{J} \alpha_i H_i(s_i) \right) + \partial_x \left( \sum_{i=1}^{J} \alpha_i s_i H_i(s_i) \right) = D_{H,\alpha,s}(\mathbf{v}). \tag{22}$$

*The operator $D_{H,\alpha,s}$ depends on $\boldsymbol{\lambda}$, $(\alpha_i, s_i)_{i=1,\dots,J}$ and on the choice of $(H_i)_{i=1,\dots,J}$, it is of the form*

$$D_{H,\alpha,s}(\mathbf{v}) = R_{\mathbf{Q}}(p) + K \partial_x \boldsymbol{\lambda}_{M-1},$$

*for some $p \in \mathbb{R}_N[X]$ and $K \in \mathbb{R}$.*

We have found little interpretation for the values of $p$ and $K$ and their construction is only detailed through the proof.

*Proof.* The polynomials (of degree $N = 2J + M - 1$) orthogonal to $Span(\mathbf{b}_{M-1})$ with respect to the measure $(\eta^*)''(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s))ds$ form a space of dimension $2J$. For our purpose, an appropriate basis of those can be obtained by projecting onto this orthogonal set the Hermite interpolation polynomials, i.e. the polynomials $(\mathbf{p}_i, \mathbf{q}_i)_{i=1,\dots,J}$ such that $\mathbf{p}_i(s_j) = 0 = \mathbf{q}_i(s_j)$ for $j \neq i$ and $\mathbf{q}_i(s_i) = 0 \neq \mathbf{q}_i'(s_i)$. For completeness, we provide a technique to construct them and eventually obtain $K$ and $p$.

Define the Lagrange polynomials (non-normalized yet)

$$\mathbf{l}_i(s) = \prod_{j \neq i} (s - s_j)$$

annihilating at all $s_j$ for $j \neq i$. Then, define $\tilde{\mathbf{p}}_{i,0}(s) = 1$ and for $j = 1, \dots, M$

$$\tilde{\mathbf{p}}_{i,j}(s) = \left( s^j - \sum_{k=0}^{j-1} \tilde{\mathbf{p}}_{i,k}(s) \frac{\int_E \tilde{\mathbf{p}}_{i,k}(s) \mathbf{l}_i(s)^2 s^j (\eta^*)'' \left(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s)\right) ds}{\int_E \tilde{\mathbf{p}}_{i,k}(s)^2 \mathbf{l}_i(s)^2 (\eta^*)'' \left(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s)\right) ds} \right).$$

Then, we denote $\tilde{\mathbf{p}}_i := \mathbf{l}_i^2 \tilde{\mathbf{p}}_{i,M}$ and $\tilde{\mathbf{q}}_i := \mathbf{l}_i^2 \tilde{\mathbf{p}}_{i,M+1}$ the polynomials of degree $M + 2(J-1) = N-1$ and $N$ annihilating at all $s_j$ with $j \neq i$ orthogonal to all polynomials of degree respectively $M-1$ and $M$ with respect to the measure $(\eta^*)''(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s))ds$.

Define $\tilde{P} \in \mathbb{R}^{N\times N}$ such that

$$\tilde{P}\mathbf{b} = \left(\mathbf{b}_{M-1}^T, \tilde{\mathbf{p}}_1, \tilde{\mathbf{q}}_1, \dots, \tilde{\mathbf{p}}_J, \tilde{\mathbf{q}}_J\right)^T.$$

One computes

$$\begin{aligned}
\tilde{P} J_{\mathbf{v}} \mathbf{f} &= \left(\begin{array}{c|c} B_0 & U \\ \hline 0_{\mathbb{R}^{2J\times(M-1)}} & Diag(B_1, \dots, B_J) \end{array}\right), \\
B_0 &= \int_E \mathbf{b}_{M-1}(s)\mathbf{b}_{M-1}(s)^T (\eta^*)''(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s))ds, \\
U &= \left(\mathbf{b}_{M-1}(s_1), \alpha_1 \mathbf{b}'_{M-1}(s_1), \dots, \mathbf{b}_{M-1}(s_J), \alpha_J \mathbf{b}'_{M-1}(s_J)\right), \\
B_i &= \left(\begin{array}{c|c} \tilde{\mathbf{p}}_i(s_i) & \alpha_i \tilde{\mathbf{p}}'_i(s_i) \\ \hline \tilde{\mathbf{q}}_i(s_i) & \alpha_i \tilde{\mathbf{q}}'_i(s_i) \end{array}\right) \qquad \forall i = 1, \dots, J,
\end{aligned}$$

and similarly

$$\begin{aligned}
\tilde{P} J_{\mathbf{v}} \mathbf{F} &= \left(\begin{array}{c|c} C_0 & V \\ \hline L & Diag(C_1, \dots, C_J) \end{array}\right), \\
C_0 &= \int_E s\mathbf{b}_{M-1}(s)\mathbf{b}_{M-1}(s)^T (\eta^*)''(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s))ds, \\
V &= \big(s_1\mathbf{b}_{M-1}(s_1), \alpha_1(\mathbf{b}'_{M-1}(s_1) + s_1\mathbf{b}_{M-1}(s_1)), \\
&\qquad\qquad \dots, \alpha_J(\mathbf{b}'_{M-1}(s_J) + s_J\mathbf{b}_{M-1}(s_J))\big), \\
L &= \int_E \left(\begin{array}{cccc} 0 & \dots & 0 & s^M \tilde{\mathbf{p}}_1(s) \\ 0 & \dots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline 0 & \dots & 0 & s^M \tilde{\mathbf{p}}_J(s) \\ 0 & \dots & 0 & 0 \end{array}\right) (\eta^*)''(\boldsymbol{\lambda}^T \mathbf{b}_{M-1}(s))ds, \\
C_i &= \left(\begin{array}{c|c} s_i\tilde{\mathbf{p}}_i(s_i) & \alpha_i(s_i\tilde{\mathbf{p}}'_i(s_i) + \tilde{\mathbf{p}}_i(s_i)) \\ \hline s_i\tilde{\mathbf{q}}_i(s_i) & \alpha_i(s_i\tilde{\mathbf{q}}'_i(s_i) + \tilde{\mathbf{q}}_i(s_i)) \end{array}\right) \qquad \forall i = 1, \dots, J.
\end{aligned}$$

Remark that, by construction of $\tilde{\mathbf{p}}_i$, only the integrals involving $\tilde{\mathbf{p}}_i$ and $s^M$ do not cancel out in the matrix $L$.

For simplicity, define

$$\tilde{B}_i = \begin{pmatrix} \tilde{\mathbf{p}}_i(s_i) & \tilde{\mathbf{p}}'_i(s_i) \\ \tilde{\mathbf{q}}_i(s_i) & \tilde{\mathbf{q}}'_i(s_i) \end{pmatrix}.$$

We prove that this matrix is invertible by contradiction. Assuming $\tilde{B}_i$ is singular and denoting $V_0$ a zero left-eigenvector, then the polynomial

$$V^0 \begin{pmatrix} \tilde{\mathbf{p}}_i \\ \tilde{\mathbf{q}}_i \end{pmatrix} =: W^0\mathbf{b},$$

satisfies

$$W^0 J_{\mathbf{v}} \in Span\left(\int_E \mathbf{b}_N \mathbf{b}_{M-1}^T (\eta^*)''(\boldsymbol{\lambda}^T \mathbf{b}_{M-1})\right).$$

And especially there exists a zero left-eigenvector to $J_{\mathbf{v}}\mathbf{f}$ which contradicts that $\mathbf{f} \in \mathbf{f}(E_{\mathbf{v}})$. Therefore $\tilde{B}_i$ is invertible and

$$\tilde{B}_i^{-1} = \frac{1}{\tilde{\mathbf{p}}_i(s_i)\tilde{\mathbf{q}}_i'(s_i) - \tilde{\mathbf{p}}_i'(s_i)\tilde{\mathbf{q}}_i(s_i)} \begin{pmatrix} \tilde{\mathbf{q}}_i'(s_i) & -\tilde{\mathbf{p}}_i'(s_i) \\ -\tilde{\mathbf{q}}_i(s_i) & \tilde{\mathbf{p}}_i(s_i) \end{pmatrix}.$$

Lastly, define $(\mathbf{p}_i, \mathbf{q}_i)^T = \tilde{B}_i^{-1}(\tilde{\mathbf{p}}_i, \tilde{\mathbf{q}}_i) =: P^i\mathbf{b}$. Then multiplying (21) by $P^i$ provides the system

$$\partial_t\alpha_i + (s_i\partial_x\alpha_i + \alpha_i\partial_x s_i) + \kappa(\mathbf{p}_i)\partial_x\boldsymbol{\lambda}_{M-1} = R_{\mathbf{Q}}(\mathbf{p}_i), \tag{23a}$$

$$\alpha_i(\partial_t s_i + \alpha_i s_i\partial_x s_i) \qquad + \kappa(\mathbf{q}_i)\partial_x\boldsymbol{\lambda}_{M-1} = R_{\mathbf{Q}}(\mathbf{q}_i), \tag{23b}$$

where

$$\kappa(\mathbf{p}_i) = \int_E s^M\mathbf{p}_i(s)(\eta^*)''(\boldsymbol{\lambda}^T\mathbf{b}_{M-1}(s))ds = \frac{\kappa(\tilde{\mathbf{p}}_i)\tilde{\mathbf{q}}_i'(s_i)}{\tilde{\mathbf{q}}_i'(s_i)\tilde{\mathbf{p}}_i(s_i) - \tilde{\mathbf{p}}_i'(s_i)\tilde{\mathbf{q}}_i(s_i)}, \tag{23c}$$

$$\kappa(\mathbf{q}_i) = \int_E s^M\mathbf{q}_i(s)(\eta^*)''(\boldsymbol{\lambda}^T\mathbf{b}_{M-1}(s))ds = -\frac{\kappa(\tilde{\mathbf{p}}_i)\tilde{\mathbf{q}}_i(s_i)}{\tilde{\mathbf{q}}_i'(s_i)\tilde{\mathbf{p}}_i(s_i) - \tilde{\mathbf{p}}_i'(s_i)\tilde{\mathbf{q}}_i(s_i)}. \tag{23d}$$

Summing over $i$ the equations $H_i'(s_i)\times$(23b)$+H_i(s_i)\times$(23a) provides the result with

$$p = \sum_{i=1}^{J} H_i'(s_i)\mathbf{q}_i + H(s_i)\mathbf{p}_i, \tag{24a}$$

$$K = -\kappa(p) = \int_E \sum_{i=1}^{J}\left(\frac{H_i'(s_i)\tilde{\mathbf{q}}_i(s_i) - H(s_i)\tilde{\mathbf{q}}_i'(s_i)}{\tilde{\mathbf{q}}_i'(s_i)\tilde{\mathbf{p}}_i(s_i) - \tilde{\mathbf{p}}_i'(s_i)\tilde{\mathbf{q}}_i(s_i)}\right)\tilde{\mathbf{p}}_i(s)s^M(\eta^*)''\left(\boldsymbol{\lambda}^T\mathbf{b}_{M-1}(s)\right)ds. \tag{24b}$$

□

One can interpret the polynomials $\mathbf{p}_i$ and $\mathbf{q}_i$ as the projections of the Hermite interpolation polynomials anihilating at all $s_j$ for $j \neq i$ and such that the value of its derivative for $\mathbf{p}_i$ or the value of the polynomial itself for $\mathbf{q}_i$ is zero in $s_i$, onto the orthogonal complement of $\mathbb{R}_{M-1}[X]$ in $\mathbb{R}_N[X]$ with respect to the $L^2$ scalar product weighted with $(\eta^*)''(\boldsymbol{\lambda}^T\mathbf{b}_{M-1})$ and normalized after projection.

Remark that the spatial derivative of the highest degree term $\boldsymbol{\lambda}_{M-1}$ in the equilibrium functions remains in the entropy equation (22) after computations. Of course, there is little chance that this derivative term remains signed along the simulation. Therefore, imposing a non-positive dissipation term $D_{H,\alpha,s}$ requires further assumptions, especially on the convex functions $H_i$.

**Corollary 6.** *For a given function $\mathbf{Q}(\mathbf{f})$, suppose that the function $\mathbf{f}$ satisfies (2,8) in a neighborhood of $(x,t)$ and such that it belongs to $\mathbf{f}(x,t) \in \mathbf{f}(E_{\mathbf{v}})$ and consider a convex function $H_i$ such that*

$$H_i(s_i) = K_i\tilde{\mathbf{q}}_i(s_i), \qquad H_i'(s_i) = K_i\tilde{\mathbf{q}}_i'(s_i),$$

*where $\tilde{\mathbf{q}}_i \in \mathbb{R}_N[X]$ is constructed in the proof of Proposition 11 as a non-trivial polynomial such that*

$$\tilde{\mathbf{q}}_i(s_j) = 0 = \tilde{\mathbf{q}}_i'(s_j) \quad \forall i \neq j, \qquad \text{and} \qquad \int_E \tilde{\mathbf{q}}_i(s)\mathbf{b}_M(s)(\eta^*)''(\boldsymbol{\lambda}\mathbf{b}_{M-1}(s))\,ds = 0_{\mathbb{R}^{M+1}}.$$

*Then*

$$\partial_t(\alpha_i H_i(s_i)) + \partial_x(\alpha_i s_i H_i(s_i)) = K_i R_{\mathbf{Q}}(\tilde{\mathbf{q}}_i).$$

*Proof.* Computing

$$
\begin{aligned}
H_i'(s_i)\mathbf{q}_i + H_i(s_i)\mathbf{p}_i = \frac{K_i}{\tilde{\mathbf{p}}_i(s_i)\tilde{\mathbf{q}}_i'(s_i) - \tilde{\mathbf{p}}_i'(s_i)\tilde{\mathbf{q}}_i(s_i)} \big[ & \tilde{\mathbf{q}}_i'(s_i)\left(-\tilde{\mathbf{q}}_i(s_i)\tilde{\mathbf{p}}_i + \tilde{\mathbf{p}}_i(s_i)\tilde{\mathbf{q}}_i\right) \\
& + \tilde{\mathbf{q}}_i(s_i)\left(\tilde{\mathbf{q}}_i'(s_i)\tilde{\mathbf{p}}_i - \tilde{\mathbf{p}}_i'(s_i)\tilde{\mathbf{q}}_i\right)\big] \\
= K_i\tilde{\mathbf{q}}_i. &
\end{aligned}
$$

Since the term in $\tilde{\mathbf{p}}_i$ in $p$ is zero, then $\kappa(p) = 0$, and only the term $R_{\mathbf{Q}}(p)$ remains in $D_{H,\alpha,s}$ which simplifies into $R_{\mathbf{Q}}(\tilde{\mathbf{q}}_i)$ □

**Remark 7.**

- Only the value of $H_i(s_i)$ and its derivative $H_i'(s_i)$ at $s_i$ impacts on this result.
- Therefore, one easily finds a function satisfying this property, e.g. by constructing a quadratic polynomial from these two data together with a constant $K_i$ such that $\text{sign}(K_i) = -\text{sign}(R_{\mathbf{Q}}(\tilde{\mathbf{q}}_i))$, to obtain a non-positive dissipation term $D_{H,\alpha,s} \leq 0$.
- This result was written for the $i$-th Dirac term, but this can of course be adapted to any combination of deltas.

# 6 Conclusion and perspectives

## 6.1 Conclusive remarks

We have extended the framework for the construction of the projective closure initiated in [38] in several directions:

- First, it was restricted to the Hausdorff problem (i.e. with $E = [-1, +1]$ for the set of integration) to the Hamburger case (i.e. with an unbounded set of integration $E = \mathbb{R}$). Extension to other bounded or unbounded 1D sets (Stieltjes $E = \mathbb{R}^+$ or Toeplitz $E = S^1$) present no more difficulties.
- Several representation results were provided along the boundary of the realizability domain. Since vectors along this boundary are known not to be vectors of moments of integrable functions, these representations consist in giving other interpretations of those vectors. In practice, these representations are a limit of Dirac measures, a sum of Dirac measures plus a derivative of one, and a modification of the equilibrium function such that the projection naturally avoids the difficulties of representation along the boundary.

In this generalized framework, the well-definition of the projective closure was exhibited by providing a parametrization of the realizability domain well-adapted to our closure. Especially, we showed that the projective closure is based on a one-to-one function from the realizability domain and the set of measures of the desired form.

Finally, the hyperbolic and the entropic structure of the resulting moment model was analyzed. It was shown to be weakly hyperbolic, and its wave speeds were computed. We exhibited interesting entropic properties of two subsystems related to the equilibrium part and to the purely anisotropic part of the underlying distribution.

## 6.2 Numerical perspectives

Contrarily to [38] where analytical formulae for the projective closure were given in the case of a fixed equilibrium and simulation were performed, the present extension to Hamburger case is still restricted to a theoretical study. And the first perspective we have in mind are numerical. Here, we have shown that the closure exists when the equilibrium distribution is a Maxwellian (or any distribution of the form given by an entropy minimization process), but computing numerically the parameters of such a distribution remains an open problem.

Numerical investigations in this direction can be related to the construction of QMOM method (with its extensions ; see e.g. [34, 32, 31, 46, 47, 9, 17] and references therein) which also requires to compute the masses and locations of Dirac distribution. As a comparison:

- The QMOM closure for moments up to order 4 involves a reconstruction of the form
$$f_{QMOM} = \alpha_1 \delta_{s_1} + \alpha_2 \delta_{s_2}, \tag{25a}$$
for moments up to order 5 where $\alpha_i$ and $s_i$ can be computed from $\mathbf{f}$.

- The EQMOM ([47]) closure for moments up to order 5 involves a reconstruction of the form
$$f_{EQMOM}(s) = \rho_1 \exp\left(-\sigma(s_1 - s)^2\right) + \rho_2 \exp\left(-\sigma(s_2 - s)^2\right), \tag{25b}$$
where $\rho_i$, $s_i$ and $\sigma$ can be computed from $\mathbf{f}$.

- The first non-trivial reconstruction of the present closure for moments up to order 5 would be of the form
$$f_{\Pi_N}(s) = \rho \exp\left(-\sigma(s - s_1)^2\right) + \alpha \delta_{s_2}(s) \tag{25c}$$
where we aim to compute the (unique set of) parameters $\rho$, $s_1$, $\sigma$, $\alpha$ and $s_2$ as functions of $\mathbf{f}$.

In terms of properties, most closures are known to become ill-conditioned in the limit $\mathbf{f}$ approaching $\partial \mathcal{R}_{\mathbf{b}}$. This is typically the case for the entropy-minimizing closure which has good properties at the continuous level (symmetric hyperbolicity and kinetic entropy decay). The QMOM closures can take several values in this limit (see [9] and computations in Appendix B), but they can be computed and this closure has better numerical properties in this limit. The present closure is in between those two, and the idea would be to keep the number of parameters $\boldsymbol{\lambda}$ low enough in order to preserve the good continuous properties (symmetric hyperbolicity and kinetic entropy dissipation) of the entropy-based closure while not deteriorating the good numerical properties of the Dirac part. The first model we have in mind is the one obtained from the reconstruction (25c).

# A Discussion on Hypothesis (H3)

This hypothesis states that the equilibrium moment vector is constant along a ray starting in $\mathbf{f}$ and pointed toward $-\mathbf{f}^{eq}(\mathbf{f})$. This can be characterized by

**Proposition 12.** *Suppose that Hypothesis* ***(H2)*** *holds and that $f^{eq}$ is a $C^1$ function of $\mathbf{f}$. Then, Hypothesis* ***(H3)*** *holds if and only if*

$$J_{\mathbf{f}}\left(\frac{f^{eq}}{\mathbf{f}_0^{eq}}\right)(\mathbf{f})\mathbf{f}^{eq}(\mathbf{f}) = 0_{\mathbb{R}^{N+1}}. \tag{26}$$

*Proof.* One reformulates

$$\frac{f^{eq}(\mathbf{f})}{\mathbf{f}^{eq}(\mathbf{f})_0} = \frac{f^{eq}(\mathbf{f}-\alpha\mathbf{f}^{eq}(\mathbf{f}))}{\mathbf{f}^{eq}(\mathbf{f}-\alpha\mathbf{f}^{eq}(\mathbf{f}))_0}.$$

Then

$$\frac{1}{\alpha}\left(\frac{f^{eq}}{\mathbf{f}_0^{eq}}(\mathbf{f}-\alpha\mathbf{f}^{eq}(\mathbf{f})) - \frac{f^{eq}}{\mathbf{f}_0^{eq}}(\mathbf{f})\right) = 0.$$

Having $\alpha$ tend to zero read (26). □

This characterization is not simpler nor more general than the original formulation of **(H3)**. However, it provides a numerical characterization of this hypothesis. And following Remark 5, this hypothesis is always satisfied with the equilibrium (12). Therefore, it provides an additional numerical information on $f^{eq}$, and therefore on the parameters $\boldsymbol{\lambda}$ that could be used for numerical applications.

# B Proof of Theorem 4.1

We prove here that the set of moments of the form (14) is dense in $\mathcal{R}_{\mathbf{b}}$

## B.1 Correspondance of $\partial\mathcal{R}_{\mathbf{b}}$ and $\mathbf{f}\left((\partial\Lambda\backslash\Lambda)\times E_{\alpha,s}\right)$

Before proving this result, we provide a technical lemma.

**Lemma B.1.** *Suppose that $\Lambda \neq \emptyset$ and $\partial\Lambda\cap\Lambda = \emptyset$. Then*

$$\mathbf{f}\left((\partial\Lambda\backslash\Lambda)\times E_{\alpha,s}\right)\cap\mathbb{R}^{N+1} = \partial\mathcal{R}_{\mathbf{b}}\cap\mathbb{R}^{N+1}.$$

*Proof.* Define the application

$$\mathbf{g}_K : \left\{\begin{array}{ll} \Lambda & \to \mathcal{R}_{\mathbf{b}_K} \\ \boldsymbol{\lambda} & \mapsto \displaystyle\int_E \mathbf{b}_K(\eta^*)'(\boldsymbol{\lambda}^T\mathbf{b}_{M-1}) \end{array}\right.$$

for some integer $K$.

By hypothesis, $\Lambda$ is a non-empty open set. Following optimization results (see [26, 6]), this provides that $\mathbf{g}_{M-1}$ is a bijection. Especially, any point on the boundary $\mathbf{f}^s \in \partial\mathcal{R}_{\mathbf{b}_{M-1}}$ can be represented as a limit for some $\boldsymbol{\lambda}^s \in \partial\Lambda$ of

$$\mathbf{f}^s = \lim_{\boldsymbol{\lambda}\to\boldsymbol{\lambda}^s}\int_E \mathbf{b}_{M-1}(\eta^*)'(\boldsymbol{\lambda}^T\mathbf{b}_{M-1}).$$

According to Corollary 1, the vector $\mathbf{f}^s$ correspond either to the moments of the sum of $K$ Dirac measures, where $K < \frac{M-1}{2}$ in the case $M-1$ even or $K < \frac{M-2}{2}$ in the case $M-1$ odd, or to those moments plus the limit of one or two Dirac measures of the form (11).

Under the integrability assumption (**H5**) on $(\eta^*)'(\boldsymbol{\lambda}^T\mathbf{b}_{M-1})$, then its moments of higher order also exists. Therefore, the only $\boldsymbol{\lambda}^s \in \partial\Lambda$ such that $\lim_{\boldsymbol{\lambda}\to\boldsymbol{\lambda}^s} \mathbf{g}_N(\boldsymbol{\lambda})$ is finite are those that match the moments at least up to order $N$ of a sum of $K$ Dirac measures.

Adding the remaining $J$ Dirac measures from the reconstruction (14) and observing that this holds for all location $s_i$ and positive mass $\alpha_i$ provides that all boundary vector $\mathbf{f} \in \partial\mathcal{R}_{\mathbf{b}}$ can be represented as a limit of $\lim_{\mathbf{v}\to\mathbf{v}^s} \mathbf{f}(\mathbf{v})$ for some $\mathbf{v}^s \in ((\partial\Lambda\backslash\Lambda) \times E_{\alpha,s})$. □

## B.2 Decomposing the boundary

With Corollary 3, the function (14) is a $C^1$-diffeomorphism from $E_{\mathbf{v}}$, i.e. an open subset of $\mathbb{R}^{N+1}$ into $\mathbf{f}(E_{\mathbf{v}})$, i.e. another open subset of $\mathcal{R}_{\mathbf{b}} \subset \mathbb{R}^{N+1}$. Then the boundary of the arriving set $\partial\mathbf{f}(E_{\mathbf{v}})$ match with $\mathbf{f}(\mathbf{v})$ in the limit $\mathbf{v} \to \partial E_{\mathbf{v}}$.

Using the decomposition (16) of this boundary, we have

$$\partial\mathbf{f}(E_{\mathbf{v}}) = \mathbf{f}(\partial\Lambda \times E_{\alpha,s}) \cup \mathbf{f}(\Lambda \times \partial E_{\alpha,s}).$$

Using Lemma B.1, the first part $\mathbf{f}(\partial\Lambda \times E_{\alpha,s}) = \partial\mathcal{R}_{\mathbf{b}}$. There remains to show that the remaining part $\mathbf{f}(\Lambda \times \partial E_{\alpha,s})$ does not open onto a domain $\mathcal{R}_{\mathbf{b}}\backslash\mathbf{f}(E_{\mathbf{v}})$ with non-empty interior in $\mathbb{R}^{N+1}$. In particular, we show that for all $\mathbf{f} \in \partial\mathbf{f}(E_{\mathbf{v}})\backslash\partial\mathcal{R}_{\mathbf{b}}$ and all direction $\mathbf{d} \in \mathbb{R}^{N+1}$, there exists $\epsilon > 0$ such that for all $0 < \delta < \epsilon$, then $\mathbf{f} + \delta\mathbf{d} \in \overline{\mathbf{f}(E_{\mathbf{v}})}$.

## B.3 Tangent and normal vectors to the boundary $\mathbf{f}(\Lambda \times \partial E_{\alpha,s})$

Any vector $\mathbf{f}^s \in \mathbf{f}(\Lambda \times \partial E_{\alpha,s})$ can be written under the form

$$\mathbf{f}^s = \int_E \mathbf{b}(\eta^*)'(\boldsymbol{\lambda}\mathbf{b}_{M-1}) + \sum_{i=1}^{J-1} \alpha_i \mathbf{b}(s_i) \tag{27}$$

with at least one less Dirac measure than (14). It can therefore be written as the limit

$$\mathbf{f}^s = \int_E \mathbf{b}(\eta^*)'(\boldsymbol{\lambda}\mathbf{b}_{M-1}) + \sum_{i=1}^{J-1} \alpha_i \mathbf{b}(s_i) + \lim_{\epsilon\to0^+} \epsilon\mathbf{b}(s_J)$$

for any $s_J \in \mathbb{R}$ different from the other $(s_i)_{i=1,\dots,J-1}$. From this representation, computing the Jacobian $J\mathbf{f}(\boldsymbol{\lambda}, \alpha_1, s_1, \dots, \alpha_{J-1}, s_{J-1}, \epsilon, s_J)$ before the limit provides a basis of $\mathbb{R}^{N+1}$ of the form (15). By definition of the Jacobian, adding any of the column vectors composing this Jacobian multiplied by an infinitesimal $\epsilon$ to $\mathbf{f}^s$ corresponds to having the corresponding parameter vary in $\mathbf{f}^s$. One easily verifies that having $\mathbf{f}^s$ vary along the direction associated to $\boldsymbol{\lambda}$ or any of the $\alpha_i$ or $s_i$ for $i = 1, \dots, J-1$ does not alter the representation (14). Neither does the variation along $s_J$. However, one obtains that $-\mathbf{b}(s_J)$ is a normal direction to the boundary $\mathbf{f}(\Lambda \times \partial E_{\alpha,s})$ that points outward.

## B.4 No hole in $\mathbf{f}(\Lambda \times E_{\alpha,s})$

Let us study the half-line $H_{\mathbf{f}^s,-\mathbf{b}(s_J)}$ starting in $\mathbf{f}^s$ and pointed toward $-\mathbf{b}(s_J)$. Since $\mathbf{f}^s \in \mathcal{R}_{\mathbf{b}}$, then this ray intersects the boundary, i.e. $\mathcal{R}_{\mathbf{b}} \cap H_{\mathbf{f}^s,-\mathbf{b}(s_j)}$ is a singleton. Since $\mathbf{f}(E_{\mathbf{v}})$ is open and satisfies $\partial\mathcal{R}_{\mathbf{b}} \subset \partial\mathbf{f}(E_{\mathbf{v}})$, then this half-line also intersects $\partial\mathbf{f}(E_{\mathbf{v}})$, and since $\mathbf{f}(\partial\Lambda \times E_{\alpha,s}) = \partial\mathcal{R}_{\mathbf{b}}$, then it intersect the other part of the boundary $\mathbf{f}(\Lambda \times \partial E_{\alpha,s})$. This implies that there exists $\delta \geq 0$ such that

$$\mathbf{f}^s - \delta\mathbf{b}(s_J) =: \mathbf{g}^s \in \mathbf{f}(\Lambda \times \partial E_{\alpha,s}).$$

Especially, this $\mathbf{g}^s$ is also of the form of (27). This provides that

$$\int_E \mathbf{b}(\eta^*)'(\boldsymbol{\lambda}^T\mathbf{b}_{M-1}) + \sum_{i=1}^{J-1}\alpha_i\mathbf{b}(s_i) - \delta\mathbf{b}(s_J) = \int_E \mathbf{b}(\eta^*)'(\boldsymbol{\lambda}'^T\mathbf{b}_{M-1}) + \sum_{i=1}^{J-1}\alpha_i'\mathbf{b}(s_i') \tag{28}$$

for some $\boldsymbol{\lambda}' \in \Lambda$ and $\alpha_i' \geq 0$. However, one observes that $\mathbf{f}$ is also a $C^1$-diffeomorphism over the set of the form $\Lambda \times E_{\alpha,s}$ where the constraint on $\alpha_J$ is replaced by a negativity constraint. Especially, this provides the uniqueness of a representation of the form (28) and therefore the equalities $\boldsymbol{\lambda}' = \boldsymbol{\lambda}$, $\alpha_i' = \alpha_i$ and $s_i' = s_i$. This means that $\mathbf{f}^s \pm \delta\mathbf{b}(s_J) \in \mathbf{f}(E_{\mathbf{v}})$ remains represented by a distribution of the form (14) on both side of $\mathbf{f}^s$ for $\delta$ sufficiently small.

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