Eliciting Multiple Prior Beliefs

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Abstract

Despite the increasing importance of multiple priors in various domains of economics and the significant theoretical advances concerning them, choice-based incentive-compatible multiple-prior elicitation largely remains an open problem. This paper develops a solution, comprising a preference-based identification of a subject's probability interval for an event, and two procedures for eliciting it. The method does not rely on specific assumptions about subjects' ambiguity attitudes or probabilistic sophistication. To demonstrate its feasibility, we implement it in two incentivized experiments to elicit the multiple-prior equivalent of subjects' cumulative distribution functions over continuous-valued sources of uncertainty. We find a predominance of non-degenerate probability intervals among subjects for all explored sources, with intervals being wider for less familiar sources. Finally, we use our method to undertake the first elicitation of the mixture coefficient in the Hurwicz α-maxmin EU model that fully controls for beliefs.

Key words: Multiple Priors, Belief Measurement, α-maxmin EU, Imprecise Probability.

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Contents

1. Introduction

The standard Bayesian model of decision under uncertainty in economics stipulates that a decision maker's beliefs are fully captured by a single probability measure over the states of the world (Savage, 1954; Anscombe and Aumann, 1963). However, in the face of contrary empirical evidence, starting with Ellsberg (1961)'s famous examples, more general theories have weakened the standard assumption of probabilistic beliefs. One of the most popular models involves as a 'belief component' a set of priors over the states of the world (Gilboa and Schmeidler, 1989). Multiple prior models have found a growing number of applications in macroeconomics (Ilut and Schneider, 2014), finance (Garlappi et al., 2007; Epstein and Schneider, 2010), mechanism design (Bose and Renou, 2014), econometrics (Manski, 2003, 2013), health economics (Giustinelli et al., 2021), but also beyond economics, in climate science (Kriegler et al., 2009), risk analysis (Cox, 2012) and uncertainty communication (Dieckmann et al., 2010), including by central banks (Carney et al., 2019). Despite obvious theoretical appeal, empirical applications of multiple prior models still have to operate in the absence of appropriate choice-based procedures for eliciting their 'belief component'. To date, almost all attempts to operationalize multiple prior elicitation have focused on subjects' *stated* probability intervals for individual events (Giustinelli et al., 2021; Kriegler et al., 2009), and hence involve procedures that are neither choice based nor incentive compatible. This paper proposes a choice-based incentive-compatible elicitation method for probability intervals, and implements it in two laboratory experiments.

Our proposal is inspired by the matching probability (MP) method for determining the subjective probability of an uncertain event (Borel, 1939; Anscombe and Aumann, 1963). Under Subjective Expected Utility (SEU), the subjective probability of a target event E coincides with its MP, which can be inferred from preferences between a bet on E and bets on events generated by extraneous random devices with known probability, e.g. the colour of a randomly drawn ball from an urn of known composition. Specifically, for urns containing only red or blue balls, the MP is given by the proportion r of red balls such that the subject is indifferent between the gamble that pays out a monetary prize z if E occurs, and nothing otherwise, and the gamble on the urn with proportion r of red balls that pays z if the next ball drawn from the urn is red (Abdellaoui et al., 2005; Dimmock et al., 2015).

Our insight for eliciting probability intervals is to use extraneous random devices with *interval-valued* rather than precise probabilities. To illustrate, consider an urn containing only red and blue balls, where all that is known is that at least proportion r of the balls in the urn are red, and at least proportion b are blue (with $r + b \leq 1$). Here, the probabilities of getting red or blue on the next draw from the urn are summarized by the intervals $[r, 1-b]$ and $[b, 1-r]$, respectively. Under the popular Hurwicz α-maxmin expected utility model, a single indifference between the gamble on the target event E and bets on such urns does not suffice to identify the subject's probability interval for E. However, we show that the latter can be identified from a *pair* of correctly-chosen indifferences involving bets concerning E and such 'interval-valued' urns. As explained in detail in Section 2.4, we thus propose a preference-based association of an 'interval-valued' urn to each event, which coincides with the subject's probability interval for the event under Hurwicz α-maxmin expected utility. This *matching probability interval* notion resolves the problem of choice-based incentive-compatible probability-interval elicitation in theory.

Our elicitation approach is theoretically robust, as can be seen in the weakness of the assumptions underlying it. On the one hand, it operates under Hurwicz α-maxmin expected utility, and hence under a range of ambiguity attitudes. In particular, this model is consistent with recent experimental evidence suggesting that people may exhibit ambiguity aversion in some choices and be ambiguity seeking in others (Kocher et al., 2018). On the other hand, our approach does not require that subjects' preferences—or sets of priors—are generated by precise probabilistic beliefs, i.e. that they are probabilistically sophisticated (Machina and Schmeidler, 1992; Chew and Sagi, 2006). Multiple priors clearly come to the fore in situations where assumptions of this sort are unwarranted. Elicitation under Hurwicz α-maxmin expected utility in the absence of probabilistic sophistication faces well-known theoretical difficulties with the identification of this model (e.g. Ghirardato et al., 2004; Eichberger et al., 2011), and one contribution of our approach is to provide a new resolution of them (Section 5). Hill (2021) develops the axiomatic underpinnings of an extension of the approach taken here, in a more general setup.

We then operationalise elicitation of matching probability intervals in a laboratory setting. Since probability intervals are two-dimensional, standard techniques for eliciting MPs, which are one-dimensional, cannot be used. We develop two methods for eliciting matching probability intervals: a two-dimensional extension of well-known (one-dimensional) choice lists, and an adaptive binary-choice procedure, which can be thought of as an interval analogue of the bisection or staircase method for eliciting matching probabilities. We implement these methods in tandem, following the hybrid elicitation approach adopted by Abdellaoui et al. (2021). Many elicitation applications in economics and beyond require subjects' probability distributions or cumulative distribution functions (CDFs) over a continuous variable of interest (e.g. US inflation in 2023, Eurozone GDP in 2022, average global temperature in 2030). Motivated by this observation, we implement our methods on two pairs of sources of uncertainty of the latter sort, to elicit the interval-valued CDFs generated by subjects' multiple priors. Interval-valued CDFs are commonly used in applications to go beyond the assumption of precise subjective probabilities (Karanki et al., 2009); our elicitation of CDFs provides a test of our approach, showing that it can operate in such contexts.

Our central findings attest to the feasibility of the approach. Our method yields generally consistent results, eliciting, for the vast majority of subjects, non-degenerate interval-valued CDFs. Our elicitation suggests that imprecise beliefs—i.e. non-singleton intervals for some events—are widespread, with only a handful of subjects having fully precise probabilities for all elicited events. This finding, which is consistent with elicitations using stated probability intervals (Giustinelli et al., 2021), attests to the relevance of multiple-prior belief elicitation.

Moreover, by eliciting subjects' beliefs for two similar sources which intuitively differ in familiarity or predictability (e.g. the temperature in Paris and in Sydney for subjects in Paris), our elicitations provide insight into the relationship between intuitive familiarity or predictability and probability intervals. Specifically, we observe that the width of subjective probability intervals is typically larger for intuitively less familiar or less predictable sources. Again, the reasonableness of this correlation corroborates the solidity of our method.

Finally, we connect our elicited beliefs with the Hurwicz α-maxmin EU model, and perform what to our knowledge is the first elicitation of the mixture coefficient α in that model that fully controls for beliefs without making particular assumptions about their form, such as probabilistic sophistication.

The paper is structured as follows. Section 2 sets out the theoretical background and presents the central planks of our methods (the 'matching probability interval' notion, the two-dimensional choice lists and the binary-choice procedure), with the relevant theoretical results. Section 3 sets out our experimental implementations, in the form of two studies. Section 4 contains our results and supporting analyses, whereas in Section 5 we discuss connected issues, related literature and future directions. Proofs, data analyses and experimental details are contained in the Appendices.

2. Theoretical Background

In this section, we first set out the general setup, the objects of elicitation and the underlying decision model (Sections 2.1–2.3). Then we present in turn the elements of our method. First, we propose an analogue of MPs for probability intervals, and show that they are sufficient to yield the subject's probability interval for an event, in theory (Section 2.4). Then we turn to implementation, presenting an extended notion of choice list for probability intervals (Section 2.5) and a binary-choice procedure, reminiscent of the bisection procedure for matching probabilities (Section 2.6).

2.1. Preliminaries

We consider decision making situations where the objects of choice are two-outcome prospects that pay a fixed monetary outcome z if an event occurs, and nothing otherwise. Prospects with general winning event E and winning amount z are denoted $(z, E, 0)$ and called *bets*. The *complementary* bet, which pays out when the event E does not occur, is denoted $(0, E, z)$.

As mentioned previously, we use extraneous interval-valued random devices realised by urns containing red and blue balls with partial information about the composition. For instance, consider the urn where subjects are only told that at least a proportion r of its balls are red, at least a proportion b are blue (with $r + b \leq 1$) but receive no information about the colour composition of the remaining balls (except that each is either red or blue). For such an urn, the information only allows assignment of the interval $[r, 1 - b]$ for the probability of the next ball drawn from the urn being red; similarly, there is the interval $[b, 1-r]$ for the next ball being blue. Using these probability intervals for parametrization, we denote the urn with at least proportion r of red balls and at least proportion b of blue balls by $[r, 1-b]$. We denote the set of such *interval-valued urns* by $\mathcal{I}$.[1]

Each urn $[r, 1-b]$ in $\mathcal{I}$ can be related to two (sorts of) prospects. One is the prospect which pays z if the next ball drawn from the urn is red, and nothing otherwise. For such a prospect, the probability of winning is characterized by the interval $[r, 1-b]$; we denote this prospect by $(z, [r, 1-b], 0)$. The other prospect involves the complementary bet on this urn—that is, the bet on the next ball drawn from it being blue. We denote this prospect by $(0, [r, 1-b], z)$. Note that the probability of *losing* here is characterised by the interval $[r, 1-b]$, so the probability of winning is given by $[b, 1-r]$; this prospect is thus essentially equivalent to $(z, [b, 1-r], 0)$. Since these prospects all involve objectively given information about the probability of winning, albeit in interval rather than precise probability form, we call them *interval lotteries* (IL).[2] Standard lotteries correspond to the special case where the composition of the urn is fully known—i.e. $r = 1-b$. So, for instance, the *matching probability* (MP) of an event E can be defined in this setup as the r such that $(z, [r, r], 0) \sim (z, E, 0)$.

The set $\mathcal{I}$ of interval-valued urns can be visually represented by the black-edged triangle in Figure 1. Each point (x, y) represents the urn $[x, y]$—i.e. with at least proportion x of red balls and at least proportion $1-y$ of blue ones. As such, it represents two interval lotteries: the bet on red, $(z, [x, y], 0)$, where all that is known is that the winning probability is in the interval $[x, y]$, and the bet on blue $(0, [x, y], z)$, where all that is known is that the losing probability is in this interval. Standard lotteries and urns with fully known composition correspond to the points on the diagonal $(x = y)$.

2.2. Upper and lower probabilities and CDFs

The sources of uncertainty considered here are real-valued variables, e.g. the daily minimum temperature in Paris between November and March. In the precise probability case, elicitation aims at revealing the subjective probability over the variable, which can be represented as a subjective cumulative distribution function (CDF). One common way of doing so is by eliciting subjective probabilities of events corresponding to the variable lying below certain fixed values. For a variable taking values in a real interval T, the events considered are of the form $E_t = \{t' \in T : t' \leq t\}$. For future reference, we call these *cumulative events*. We now set out the aim of the corresponding exercise for multiple priors.

Multiple prior belief representations involve a convex, closed set $\mathcal{C}$ of probability mea-

[1] Formally: $\mathcal{I} = \{[x, y] : (x, y) \in \mathbb{R}^2,\ 0 \leq x \leq y \leq 1\}$.

[2] Our notion of interval lottery is distinct from that used by Gul and Pesendorfer (2014). They use 'interval lottery' to denote (precise) probability measures over the set of intervals of (monetary) prizes; here, 'interval lottery' denotes assignments of probability intervals to (fully determined, precise) outcomes. In particular, the interval lotteries $(z, [r, 1-b], 0)$ used here clearly do not belong to the concept used by Gul and Pesendorfer (zero probability is assigned to each outcome in the interior of the interval $[0, z]$).

sures: measures over the values of variable of interest, in our case.[3] For each event E_t, the set of priors generates a *probability interval* $\{p(E_t) : p \in \mathcal{C}\} = [\underline{p}(E_t), \overline{p}(E_t)]$, where $\underline{p}(E_t) = \min\{p(E_t) : p \in \mathcal{C}\}$ and $\overline{p}(E_t) = \max\{p(E_t) : p \in \mathcal{C}\}$ are the *lower* and *upper probabilities for* E_t respectively. As is well-known, a set of priors contains more information than the collection of upper and lower probabilities for all events generated from it, but the latter (or sometimes less) is often sufficient for applications, and sometimes preferable insofar as it is easier to communicate.

For continuous-valued variables, CDFs are often used. Recall that for a probability measure $p \in \Delta(T)$, the CDF is defined as $F_p(t) = p(\{t' \in T : t' \leq t\}) = p(E_t)$; when the probability measure is a subjective probability, this is the corresponding subjective CDF. In this context, a set of priors $\mathcal{C}$ generates the *interval-valued CDF* $F_{\mathcal{C}}(t) = \{p(E_t) : p \in \mathcal{C}\}$, which takes the probability interval corresponding to E_t as value, for each t. This can be visually represented in terms of two (real-valued) functions: the *lower CDF*, $\underline{F_{\mathcal{C}}}(t) = \min\{p(E_t) : p \in \mathcal{C}\} = \underline{p}(E_t)$, and the *upper CDF*, $\overline{F_{\mathcal{C}}}(t) = \max\{p(E_t) : p \in \mathcal{C}\} = \overline{p}(E_t)$. Although, like upper and lower probabilities, they involve an information loss as compared to sets of priors, these are widely used for representing, communicating and studying sets of priors over continuous variables, where they often go under the name of distribution bands or p-boxes (Berger et al., 2000; Karanki et al., 2009). In the implementation of our elicitation procedure conducted here, our aim is to elicit subjective upper and lower CDFs for the variables considered.

2.3. Decision model

For the purposes of presentation, we will focus on one of the most popular and general models of decision under uncertainty involving sets of priors, the Hurwicz α-maxmin EU model. (In Section 5 and Appendix A.3, we discuss how our proposals extend to generalisations.) Under the α-maxmin model, a bet $(z, E, 0)$ is evaluated according to:

$$\alpha\underline{p}(E).u(z) + (1-\alpha)\overline{p}(E).u(z) \tag{1}$$

where $\overline{p}(E)$ and $\underline{p}(E)$ are the upper and lower probabilities of E generated by the subjects' set of priors, as defined above, and u is a utility function normalized so that $u(0) = 0$. The coefficient α is often taken to reflect ambiguity attitude in this model, with $\alpha > \frac{1}{2}$ associated with typical ambiguity aversion and $\alpha < \frac{1}{2}$ with typical ambiguity seeking. For illustration, the standard behavior in the Ellsberg two-urn example can be accommodated by $\alpha > \frac{1}{2}$ but not by $\alpha < \frac{1}{2}$. This model coincides with the Gilboa-Schmeidler maxmin-EU model when $\alpha = 1$; whenever $\alpha \neq 1$, the model does not satisfy the Gilboa-Schmeidler uncertainty aversion axiom—which can be thought of as characterizing universal ambiguity aversion. Hence, it can accommodate ambiguity seeking behavior in certain choices (even for $1 > \alpha > \frac{1}{2}$). Since typical findings suggest some ambiguity seeking behavior, but not in situations that give reason to believe that $\alpha < \frac{1}{2}$, we take $\alpha > \frac{1}{2}$ to be typical and assume that preferences are represented according to (1) with $\alpha > \frac{1}{2}$ in the sequel (except where specified). As discussed in Section A.3, the full strength

[3]Technically, $\mathcal{C} \subseteq \Delta(T)$, the set of probability measures over T.

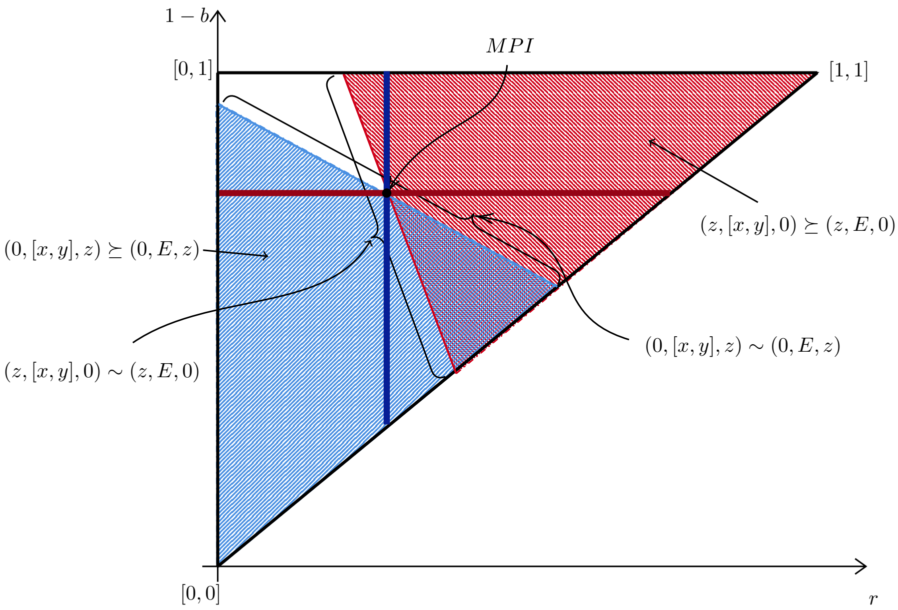


Figure 1: Matching Probability Interval in space $\mathcal{I}$ of interval-valued urns, for an event E.

of this assumption is not required for the central elements of the elicitation method. Our aim is to elicit $\underline{p}(E)$ and $\overline{p}(E)$ for selected events.

We also assume the same representation for interval lotteries: i.e. preferences concerning them are represented by an evaluation of $(z, [r, 1-b], 0)$ by:

$$\alpha r u(z) + (1-\alpha)(1-b)u(z) \tag{2}$$

2.4. Theory: Matching Probability Intervals

Our approach is based on the following notion. The *matching probability interval (MPI)* of an event E is an $[r, 1-b] \in \mathcal{I}$ such that:

$$(z, [r, 1-b], 0) \sim (z, E, 0) \tag{3}$$

$$(0, [r, 1-b], z) \sim (0, E, z) \tag{4}$$

Plugging these indifferences into (1) and (2) yields the following equations:

$$\begin{aligned} \alpha r + (1-\alpha)(1-b) &= \alpha \underline{p}(E) + (1-\alpha)\overline{p}(E), \\ \alpha(1-(1-b)) + (1-\alpha)(1-r) &= \alpha(1-\overline{p}(E)) + (1-\alpha)(1-\underline{p}(E)). \end{aligned} \tag{5}$$

Clearly, these equations are satisfied by $r = \underline{p}(E)$, $1-b = \overline{p}(E)$. Moreover, whenever there is a unique pair $r, 1-b$ satisfying them, then there is a unique matching probability interval, which indicates precisely the subjective probability interval for E: i.e. $[\underline{p}(E), \overline{p}(E)] = [r, 1-b]$. Under the α-maxmin EU model with $\alpha \neq \frac{1}{2}$, the MPI is unique (Proposition 3, Appendix A.1). So to elicit the subjects' probability interval for the event E, it suffices to find the MPI of E.

The MPI can be illustrated in Figure 1. The red hatched area represents the upper contour set of the bet $(z, E, 0)$ in the space of interval lotteries corresponding to bets on red: that is, the set of (x, y) such that $(z, [x, y], 0) \succeq (z, E, 0)$. The blue hatched area is the upper contour set of the complementary bet $(0, E, z)$ in the space of complementary ILs (corresponding to bets on blue): that, it is the set of (x, y) such that $(0, [x, y], z) \succeq (0, E, z)$. The boundaries of these sets (the diagonal red and blue lines respectively) represent the indifference curves of $(z, E, 0)$ (resp. $(0, E, z)$), in the space of 'red' (resp. 'blue') ILs. The matching probability interval corresponds to the black point at the intersection of these two lines.

This Figure also brings out the contribution of ILs as compared to standard lotteries and the long-standing identification problem for the α-maxmin EU model (Section 5). The MP of the bet $(z, E, 0)$ is given by the point where the red indifference curve meets the diagonal; clearly eliciting it is insufficient to pin down the subject's probability interval for E. Similarly, the MP of the complementary bet $(0, E, z)$ is given by the point where the blue indifference curve meets the diagonal. Eliciting both of these MPs is sufficient to pin down the subject's probability interval (as the intersection of the indifference curves) *only if the slope of the indifference curves is known: but this is determined by the mixture coefficient α in (1), which also needs to be elicited.* The use of ILs, and the notion of MPI built upon it, allows elicitation of the subjective probability interval without requiring elicitation of the mixture coefficient α. Indeed, we shall use our probability interval elicitation in tandem with MPs to estimate subjects' α (Section 4.5).

The notion of MPI resolves the challenge of incentive-compatible probability-interval elicitation *in theory.* Under SEU, eliciting preferences between the bet on E and each of a range of lotteries suffices to find the MP, which corresponds to the subject's subjective probability. Similarly here, obtaining the subject's preferences between each pair consisting of a bet (for or against E) and an IL provides the MPI—and hence the subject's probability interval for E—as the point satisfying (3) and (4). It is well known that there are fully incentive-compatible mechanisms for eliciting such preferences. For instance: the subject states her preference between each pair consisting of a bet (for or against E) and an IL; a random bet (for or against E) and IL are then chosen and she is remunerated according to the prospect which she stated as more preferred between the two. By the standard argument, it is in the subject's best interest to report her preferences truthfully, for if not there is a chance of receiving her less preferred prospect in the choice that is 'played for real'. These elicited preferences provide, *inter alia*, the MPI and hence the subject's probability interval for E.

Of course, implementation typically requires a method involving fewer preference questions. This is especially challenging for probability intervals, since the target is a point in a two-dimensional space, whereas elicitation of the precise probability of an event only needs to search a one-dimensional space. To operationalise our approach, we develop two more parsimonious methods for eliciting MPIs, whilst making no claim to have exhausted all possibilities.

2.5. Implementation 1: 2D Choice lists

Consider any MPI $[r, 1-b]$ of an event E, so that the indifference (3) is satisfied. Since $\alpha > 0$ in the representation (1), it follows that:

$$\begin{aligned} (z,[q,1-b],0) &\succ (z,E,0) && \text{for all } q > r \\ (z,[q,1-b],0) &\prec (z,E,0) && \text{for all } q < r \end{aligned} \tag{6}$$

On Figure 1, this determines the preferences on the 'red' ILs corresponding to the bold red (horizontal) line. To the left of the MPI, the bet on E is preferred to the IL corresponding to the bet on red from the urn $[q, 1-b]$ (i.e. with probability $[q, 1-b]$ of winning); to the right of the MPI, the IL is preferred to the bet; and at the MPI, the two are indifferent.

Similar reasoning applies to complementary bets. By the indifference (4) and the fact that $\alpha > 0$, for an MPI $[r, 1-b]$ of E, it follows that:

$$\begin{aligned} (0,[r,q],z) &\prec (0,E,z) && \text{for all } q > 1-b \\ (0,[r,q],z) &\succ (0,E,z) && \text{for all } q < 1-b \end{aligned} \tag{7}$$

On Figure 1, this determines the preferences on the (complementary) 'blue' ILs corresponding to the bold blue (vertical) line. Above the MPI, the bet against E is preferred to the IL corresponding to the bet on blue from urn $[r, q]$ (i.e. with probability $[r, q]$ of losing); below the MPI, the IL is preferred to the bet; and at the MPI, the two are indifferent.

It follows that the only points supporting the specified preference patterns on the corresponding horizontal and vertical lines are MPIs.

Proposition 1. *For any event E, let $[r, 1-b] \in \mathcal{I}$ be such that:*

$$\begin{aligned} (z,[q,1-b],0) &\succ (z,E,0) && \textit{for all } q > r \\ (z,[q,1-b],0) &\prec (z,E,0) && \textit{for all } q < r \\ (0,[r,q],z) &\prec (0,E,z) && \textit{for all } q > 1-b \\ (0,[r,q],z) &\succ (0,E,z) && \textit{for all } q < 1-b \end{aligned}$$

Then $[r, 1-b]$ is a matching probability interval of E.

The red (horizontal) and blue (vertical) bold lines can thus be thought of as a pair of choice lists, and the MPI is the switching point on each of them. We henceforth refer to the combination of the two as a *2D choice list*. Inspired by this observation, consider the following mechanism for eliciting a subject's MPI for an event E. A subject reports an interval-valued urn $[r, 1-b]$ for E. She is then remunerated as follows. First, an urn $[x, y]$ is chosen at random from the 2D choice list.[4] Then she 'receives' or 'plays' a bet or IL according to the following scheme:

- if $y = 1-b$, $x < r$, then she gets $(z, E, 0)$ (i.e. she 'plays' the bet on E)

[4] I.e. the interval is chosen at random from $\{[x,y] : (x,y) \in \mathcal{I}, y = 1-b\} \cup \{[x,y] : (x,y) \in \mathcal{I}, x = r\}$, the union of the horizontal and vertical lines going through $(r, 1-b)$ in Figure 1.

- if $y = 1 - b$, $x \geq r$, then she gets $(z, [x, y], 0)$ (i.e. she 'plays' this IL)
- if $x = r$, $y < 1 - b$, then she gets $(0, [x, y], z)$ (i.e. she 'plays' this IL)
- if $x = r$, $y \geq 1 - b$, then she gets $(0, E, z)$ (i.e. she 'plays' the bet against E)

It follows from the previous Proposition that this mechanism is incentive compatible in the following sense: on each choice list, reporting the urn reflecting one's true upper or lower probability is in one's best interest—it weakly dominates any other report in the respective choice list. Hence asking a subject for an urn $[r, 1 - b]$ such that, in each of the branches on the 2D choice list, she prefers the option she would receive under the mechanism incentivises reporting a MPI. Since precise probabilities (and SEU) are a special case of multiple priors (respectively, α-maxmin EU), this mechanism functions equally for Bayesian decision makers, who are incentivised to report their precise probabilities.

Note that despite the higher complexity involved in eliciting probability intervals as opposed to precise probability values, this incentive mechanism is as parsimonious as standard choice lists for MPs. In the latter, MPs are determined by the switching point, i.e. the maximum probability for which the subject prefers the bet over the lottery with that probability. Similarly, the proposed probability-interval incentive mechanism only asks for a single point, which is the switching point on each branch of the 2D choice list. In standard MP choice lists, the switching point determines the preferences in the rest of the choice list by stochastic dominance. Similarly here, the elicited point determines the other preferences in the 2D choice list according to following property, which can be thought of as a probability-interval analogue of stochastic dominance.

Definition 1 (Lower Stochastic Dominance)**.** For all urns $[r, 1-b]$ and $[r', 1-b]$, $(z, [r, 1-b], 0) \prec (z, [r', 1 - b], 0)$ whenever $r < r'$.

This says that, between ILs $(z, [r, 1-b], 0)$ and $(z, [r', 1-b], 0)$ corresponding to bets on red from urns with the same minimum proportion of blue balls, the decision maker prefers the prospect where the minimum proportion of red balls is higher. It is straightforward to check that this is the property behind preference patterns (6) and (7).

2.6. Implementation 2: Binary-choice procedure

Our second elicitation technique, which we implement in tandem with 2D choice lists (Section 3.4), is a 'bisection-style' binary-choice procedure for identifying the MPI. Here we set out its general principles; full details are provided in Appendix A.2. The logic can again be illustrated on Figure 1. The space of interval-valued urns is divided into four preference-defined areas, summarised in Table 1. The procedure is based on the following observation.

Proposition 2. *Suppose preferences are represented according to (1) with $\alpha > \frac{1}{2}$, and let E be an event.*

Name	Preferences	Colour (in Figure 1)
R-B	$(z,[x,y],0) \succeq (z,E,0)$ $(0,[x,y],z) \succeq (0,E,z)$	Red & Blue
W	$(z,[x,y],0) \preceq (z,E,0)$ $(0,[x,y],z) \preceq (0,E,z)$	White (neither Red nor Blue)
R	$(z,[x,y],0) \succeq (z,E,0)$ $(0,[x,y],z) \preceq (0,E,z)$	Red
B	$(z,[x,y],0) \preceq (z,E,0)$ $(0,[x,y],z) \succeq (0,E,z)$	Blue

Table 1: Preference-based division of $\mathcal{I}$

a. *For any point $[x,y]$ in the R-B region (i.e. such that the corresponding preferences in Table 1 hold, for E), $\underline{p}(E) \leq x$ and $\overline{p}(E) \geq y$. Moreover, for any point $[x,y]$ in the W region, $\underline{p}(E) \geq x$ and $\overline{p}(E) \leq y$.*

b. *For any point $[x,y]$ in the R region (i.e. such that the corresponding preferences in Table 1 hold, for E), every $[x',y']$ with $x' \geq x$ and $y' \geq y$ is also in R. Moreover, for any point $[x,y]$ in the B region, every $[x',y']$ with $x' \leq x$ and $y' \leq y$ is also in B.*

It follows from part a. that if the experimenter has found a point (interval-valued urn) $[x_{RB}, y_{RB}]$ in the R-B region (i.e. such that there is the preference pattern in Table 1, row 1), and a point $[x_W, y_W]$ in the W region, then the MPI is contained in the 'box generated' by these points, i.e. it is in the set $\{[x,y] : x_W \leq x \leq x_{RB}, y_{RB} \leq y \leq y_W\}$. The procedure works by searching the smallest such generated box for further points in R-B or W, in order to 'reduce' the size of the boxes and hence 'home into' the MPI. In this sense, it is analogous to the bisection procedure for MPs, where preferences indicate that the MP is in a particular interval, and the procedure searches to reduce the width of that interval.

Note that a similar result to Proposition 2 a. for the R and B regions does not hold.[5] However, by part b. it can be concluded, for any point (interval-valued urn) $[x,y]$ in R, that every point North-East of $[x,y]$ is also in R, and similarly for a point in B. So, if the experimenter has just discovered a point in R (i.e. the elicited preferences for the relevant urn are as set out in Table 1), then, to seek a point in R-B or W, she needs to look South-West of this point; and analogously for points in B. The procedure works, when at a point in R and B, by performing a bisection along one-dimensional cuts of the space $\mathcal{I}$ guided by this observation, until a point in R-B or W is found. Then the procedure between the closest R-B and W points re-applies, in an attempt to generate a 'smaller' box. Details are available in Appendix A.2.

One final important property of the procedure used is an in-built 'precision bias'. Whenever

[5]This can be seen by considering the horizontal and vertical bold lines in Figure 1 to define four quadrants, and by noting that there are both red and blue areas in the upper left-hand and lower right-hand quadrants.

	Type of Source of uncertainty	Treatments	# Events	Elicited	# subjects
EXP 1	Minimum winter temperature (°C)	Where: Paris; Sydney	4	MPIs, MPs (Paris)	80
EXP 2	Entrance exam grade (/20)	Exam: Maths, Contraction	5	MPIs	52

Table 2: Summary of studies

no point in the areas R-B or W has been found, and hence at a stage when it is searching for points in these areas, the procedure deliberately moves closer to the space of precise urns (the 45° line in Figure 1); again, see Appendix A.2 for details. In this way, if there is any misclassification of subjects, the tendency would be for the procedure to represent them as more precise than they actually are.

3. Experimental Methods

We carried out two experiments in which we used our method to elicit upper and lower CDFs for various sources of uncertainty. Both experiments involved two comparable yet different sources of uncertainty (Table 2). EXP 1 implemented a faster elicitation, eliciting probability intervals for fewer points per source. This left time for standard matching probability elicitation for the same events, which yields insights into the α-maxmin EU model (see Section 4.5). By contrast, EXP 2 implemented a slower upper and lower CDF elicitation, eliciting more points per source. It also involved an omnibus confirmation screen, allowing subjects to confirm or revise all choices concerning events in a source, after elicitation. In EXP 2, no MPs were elicited.

3.1. Subjects

132 subjects (undergraduate students) were recruited from two French academic institutions: 80 from university of Paris 1 for EXP 1 and 52 from HEC Paris Business School for EXP 2 (Table 2). Subjects' choices were collected through computer-based individual interviews that lasted about one hour in each of the two studies. Each individual interview started with a video presentation of the experimental instructions, followed by comprehension questions and one training MPI elicitation task (on an event not involved in the ensuing experiment).[6] Appendix C reports the typical screenshots faced by subjects for the tasks. In both experiments, subjects were told that there were no right or wrong answers, and that they could ask any question regarding the experiment. Differences in experimental instructions between the experiments are explained in the sequel.

[6]The video presentations are available upon request.

3.2. Sources of uncertainty

Each experiment involved two comparable sources of uncertainty, with one treatment for each source (Table 2). The type of source in EXP 1 was the minimum daily temperature over the previous November–March period; the sources differed in the city whose temperature was of interest—Paris, where the experiment was carried out, and Sydney. EXP 2 involved marks in two of the previous year's entrance exams for admission at undergraduate level through the 'ECS' and 'ECE' entrance streams to a prominent French business school, HEC Paris. The subjects in the experiment—students admitted at this level at the school—had sat the exams either the same year or the previous year. The sources differed in the exam considered: the probability and statistics exam (officially called 'Mathématiques II'), which is generally considered to be 'objectively marked', and the 'Contraction' exam—a summary of a philosophical or literary text—whose marking is considered more 'subjective', 'random' and 'unpredictable' among candidates and students. Indeed, the marks in the latter exam have higher variance.[7]

Each source of uncertainty involves a variable (temperature in °C, mark out of 20), the aim was to elicit subjects' multiple prior beliefs—in the form of the generated upper and lower CDFs (Section 2.2)—over the variable. In each experiment, the subject chose a number at the beginning of the experiment[8] which identified, according to a spreadsheet to which the subject would only have access at the end of the experiment, a random day D between 1 November and 31 March of the previous Winter (in EXP 1), and a random candidate C for entry to HEC Paris in the previous Spring (in EXP 2). We estimated upper and lower CDFs by eliciting subjects' upper and lower probabilities for cumulative events, i.e. events E_{t_i} of the form: "the minimum temperature on day D in Paris (resp. Sydney) was less than or equal to t_i", or "candidate C obtained a mark less than or equal to t_i in the Maths (resp. Contraction) exam" (Section 2.2), for various fixed values of t_i given in Table 3.[9]

Note that the events used pertained to time periods several months before subjects participated in the experiment (temperature the previous Winter, for subjects taking part in the experiment in Spring; exams sat the previous Spring, for subjects taking part in the experiment in Autumn). Moreover, there is a natural difference in the familiarity with or predictability of the sources involved in each experiment—with Paris's weather being more familiar to Paris subjects than Sydney's, and Maths considered a more predictable exam than Contraction. Finally, we had access to the real data for all the sources, which were used for incentivisation (Section 3.5).[10]

[7]The variance of marks for Maths is 3.77, where it is 9.92 for Contraction.

[8]They chose a number between 1 and 150 in EXP1 (the number of days in the period considered), 1 and 456 (the number of candidates) in EXP 2.

[9]For each source in EXP 1, we chose temperature values close to the 10%, 33%, 66% and 90% percentiles of the true distribution. For EXP 2, we used the same values for both sources (Maths and Contraction), picked so they would seem to reasonably scan the range and correspond to comparable points in the true distribution over Contraction scores, where they were at the 3% 15% 33% 68% and 86% percentiles. They were at the 0%, 0%, 2%, 21% and 60% of the true distribution of Maths scores.

[10]For the weather, the data source was Météo France (Paris Orly meteofrance.fr) and the Australian Bureau

	Treatment	Events $E_{t_i} = \{t' \in T : t' \leq t'\}$ for t_i:
EXP 1	Paris	$-2, 2, 5, 8$
	Sydney	$15, 18, 20, 22$
EXP 2	Maths	$7, 10, 12, 15, 17$
	Contraction	$7, 10, 12, 15, 17$

Table 3: Sources of uncertainty and events

3.3. Choice Tasks

EXP 1

EXP 1 consisted of three blocks of tasks. Each of the first two blocks concerned a single source (Paris or Sydney), and involved the elicitation of the upper and lower probabilities for each of the events in the source (Table 3). The order of these two blocks was randomized. In each block, the subject first declared, in an non-incentivized manner and using a scrollbar, her estimated maximum and minimum values for the minimum temperature on the unidentified day selected. This is standard procedure in expert elicitation for unbounded sources, aimed at combatting anchoring bias (Morgan, 2014), and played no role in our elicitation. Then the elicitation procedure set out in Section 2 and implemented as described in Section 3.4 was applied for each event in the source. Within each block, the two extreme events (i.e. lowest and highest temperature points) were asked first, in a random order, followed by the other two events, in a random order.

The final block involved the elicitation of MPs for the events in Paris treatment. MPs were elicited for each event E_{t_i} in this source and its complement $E^c_{t_i}$ (Table 3). The order of elicitations was randomized in this block.

EXP 2

EXP 2 consisted of two blocks of tasks, corresponding to the first two blocks of EXP 1. Each of the blocks concerned a single source (Maths or Contraction), and involved the elicitation of the upper and lower probabilities for each of the events in the source (Table 3). The order of the blocks was randomized. In each block, the elicitation procedure set out in Section 2 and implemented as described in Section 3.4 was applied for each event in the source. The order of events in each block was randomized. Each block ended with an omnibus confirmation screen, in which the interval-valued urns elicited for each of the events in the source were displayed and graphed, and the subject was given the opportunity to go back and modify any of her responses for the events in the source (Section 3.4). This screen, the sources and the larger number of

of Metereology (Sydney Observatory Hill bom.gov.au); for the marks, they were provided by HEC admission services.

events elicited per source were the central differences with respect to EXP 1.

3.4. Elicitation procedures

Upper and lower probabilities in EXP 1 and EXP 2

Our elicitation procedure follows the general hybrid structure adopted by Abdellaoui et al. (2021) for MP elicitation, under which a bisection procedure is used to aid subjects to fill in responses on a choice list, which they then confirm or modify. For each event E_{t_i} (Table 3), we first applied the binary-choice procedure set out in Section 2.6 and Appendix A.2. Each step of the procedure involved an event E_{t_i} and a 100-ball urn with a specified minimum number of blue and red balls, where nothing was known about the colour of the remaining balls. At each step, two choices were elicited from subjects: their choice in the decision between the bet on the event E_{t_i} and the bet on the next ball drawn from the urn being red, and their choice in the decision between the bet on $E^c_{t_i}$ (or against E_{t_i}) and the bet on the next ball drawn from the same urn being blue. (Details on the display are provided in Appendix C.) The urn proposed in the next step depended on the preferences elicited in the previous step according to the procedure (Section 2.6 and Appendix A.2). The subjective probability interval for E_{t_i} elicited at the end of the procedure is deduced from the preferences over such bets, as specified in the cited sections. The procedure continued until the interval was estimated to a precision of 0.15 if it was not degenerate, 0.05 if it was degenerate (i.e. corresponding to a precise probability), or up to 12 steps, whichever came first.

At the end of the binary-choice procedure, the 'confirmation' 2D-choice list described in Section 2.5 was displayed for verification. Although the 'space of choices' to be confirmed is the two-dimensional 'cross' in Figure 1, we implemented it via a one-dimensional scrollbar-based display with two cursors (see Appendix C). The cursors specified the minimum number of red and blue balls respectively, and hence together determined an interval-valued urn. They were initially set at the values determined by the binary-choice procedure. To confirm the whole 2D-choice list, the subject had to scan all the associated choices. When moving the red cursor, the blue cursor remained fixed at the pre-specified value. This accentuates the separate nature of the cursors, which cannot be moved in tandem. By moving the red cursor, the subject scanned all the urns with the same minimum number of blue balls but differing minimum numbers of red balls, i.e. the choices represented by the bold red horizontal line in Figure 1. During this scan the corresponding choices between the bets on the event E_{t_i} and the bets on red from the urn were displayed, with the 'chosen bet', specified as in Section 2.5, being indicated (i.e. the bet on the urn when there are more red balls than the provisionally elicited point; the bet on the event otherwise). The subject also had to scan the choices associated with moving the blue cursor—there, the red cursor (and hence minimum number of red balls) was held fixed. When moving the blue cursor, the choices between the bets on $E^c_{t_i}$ and the bets on blue from the urn were displayed (with the corresponding choice, following the logic in Section 2.5). This corresponds to scanning the choices represented by the bold blue vertical line in Figure 1. By clicking on the appropriate bet (on the event or the urn) in any of the displayed choices, subjects could revise their reported preferences, hence modifying the specified position of the fixed cursors (and the associated provisionally elicited point). After such modifications, subjects had to reconfirm all of the associated choices, by moving one and then the other cursor, before moving on to the next stage of the experiment. The precision of the scrollbar, and hence subject responses, was to the nearest 0.01 (to the precise minimum number of red and blue balls out of 100 respectively).

Omnibus confirmation screen in EXP 2

In EXP 2, after the procedure described above was completed for all the events in the source, the subject was asked to confirm all the elicited values, and given the opportunity to modify responses. The confirmation screen displayed the interval-valued urns elicited for the five events in the source. Moreover, this information was summarized in a graph displaying the minimum number of red and blue balls for each event (see Appendix C). Hovering the mouse over the points on the graph caused the associated interval-valued urn to be highlighted. By clicking on the point on the graph or the urn, the subject could access the corresponding two-cursor scrollbar confirmation screen at the end of the binary-choice procedure for that event, where she could change her choices in exactly the same way as set out above.

Matching probabilities in EXP 1

The MP of the bet on a given event was elicited through a two-step procedure, from which our multiple prior elicitation procedure was inspired. First, a candidate MP was determined through a bisection process (Abdellaoui et al., 2008) that consisted in a chained sequence of binary choices between the bet on the event and an urn whose composition was fully known. Starting with a binary choice between $(z, [\frac{1}{2}, \frac{1}{2}], 0)$ and $(z, E, 0)$, it then asks a binary choice with the midpoint of the lower (respectively upper) interval $[0, \frac{1}{2}]$ (resp. $[\frac{1}{2}, 1]$) whenever the subject chooses the former (resp. latter) option, and so on. The displays used were similar to those described above. Then the complete confirmation (one-dimensional, single cursor) scrollbar-based choice list, filled in according to the prior bisection choices, was displayed for verification. The precision of the elicited MP was to the nearest 0.05.

3.5. Incentivizing subjects

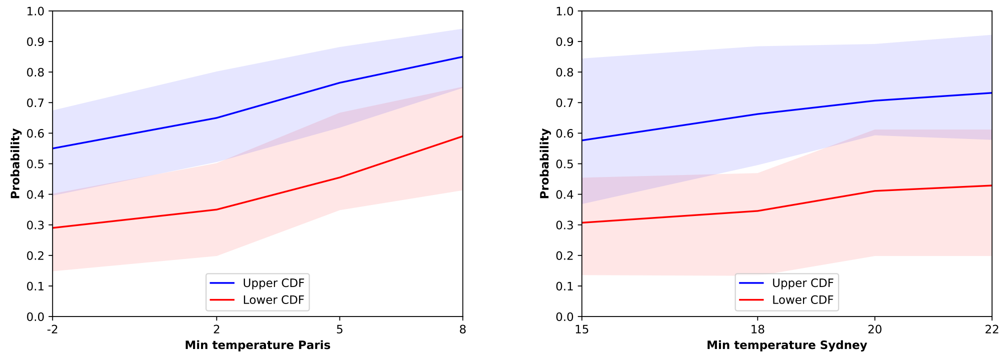
Participants in all studies received a flat payment of €10. Additionally, a random incentive system was implemented, which was entirely analogous to those standardly used to implement elicitation of matching probabilities. As noted above, after the presentation of the instructions and before the beginning of the experiment, the subject chose a number from a given range, which identified an individual case of the variable of interest (the day, if the source was minimum temperature; the candidate, if the source was the mark). The exact case identified was specified according to a spreadsheet that would only be revealed at the end of the experiment. This is in concordance with the approach set out by Johnson et al. (2021), who argue that it reduces hedging motivations, given the well-known fact that ambiguity models are indifferent to ex ante hedging. At the end of the experiment, a choice list (a 2D-choice list or MP-choice list in EXP 1; a 2D-choice list in EXP 2) and choice on it were chosen at random by the computer. The subject was then paid according to the decision she had made on that choice. If she had chosen, say, the bet on the event that the minimum temperature in Paris is less than or equal to 2°C, then the day which she chose was revealed, as well as daily temperature data for the November–March period, and she won if the minimum temperature on that day was indeed 2°C or less; if not, she lost. If she had chosen the urn, then she composed the appropriate urn—she counted the specified minimum numbers of red and blue balls, with the remaining balls coming from pre-constructed Ellsberg urns (of unknown composition). Then a ball was drawn from the constructed urn, and she was paid according to whether she bet on the color of that ball or not. All bets yielded 20€ if won, and nothing otherwise.

4. Results

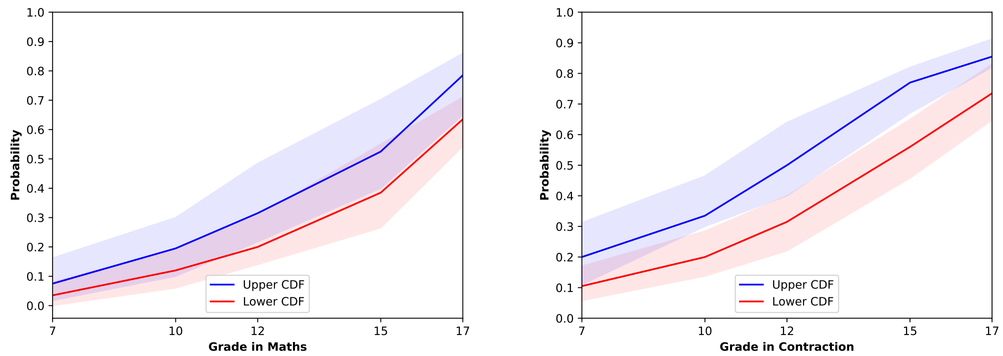
4.1. Descriptive Statistics and Performance

Figure 2 plots the median, 25% and 75% quantile upper and lower CDFs for all elicited events and both experiments (see Tables 5–6 in Appendix B.1 for basic descriptive statistics). This Figure already gives some early indications about our results, and the performance of our elicitation method.

First of all, it reports 'well-behaved' upper and lower CDFs with probabilities differing across subjects and events—thus suggesting the consistency of the method. The elicited points for both upper and lower CDFs were also consistent across the successive steps of the elicitation procedure: the binary-choice procedure and the confirmation 2D choice list (Tables 5–8, Appendix B.1). Moreover, it suggests that, in the aggregate, upper and lower CDFs are increasing, as they should be. Figure 3 plots descriptive statistics of the empirical distribution of individual Kendall τ_b rank correlation coefficients between the size of events and the upper (resp. lower) probabilities or MPs elicited for each source (see also Table 10, Appendix B.1). As is clear from the Figure, the median Kendall τ_b is far greater than 0 for all sources, pointing to increasing upper and lower CDFs. There is however a notable difference between EXP 1 and EXP 2. In EXP 2, where subjects were given the opportunity to confirm all their replies on all the 2D choice lists for a source (Section 3.4), CDFs were strictly increasing for the vast majority of subjects. In EXP 1, where 2D choice lists were confirmed after consideration of the event and there was no opportunity to reconfirm later, there were more violations of monotonicity. Comparison of the (cognitively less demanding) MPs with upper and lower CDFs in the Paris treatment, whose median Kendall ranks are similar (Figure 3a), suggests that such violations were not unique to the elicitation method proposed here. As could have been expected, the frequency of monotonicity violations appears to increase with the difficulty of the choice task, with the MP task being arguably easier than that for probability-interval elicitation, and the task for Paris,



(a) EXP 1



(b) EXP 2

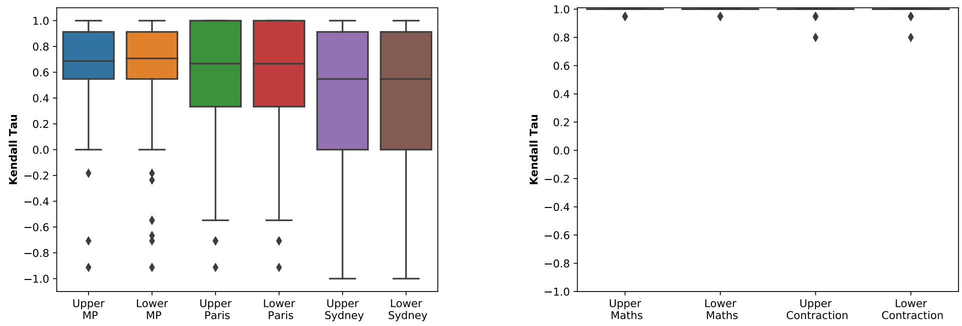
Figure 2: Median, 25% and 75% quantile ranges of upper and lower CDFs

the more familiar source for our subjects, being easier than that for Sydney.

4.2. Bayesian analysis

We also adopt a standard Bayesian approach, estimating hyperparameters for upper and lower CDFs using a MCMC procedure. We run estimations for each source under the assumption that upper and lower CDFs follow a (truncated) normal distribution, and under a Beta distribution (Table 14, Appendix B.2). As shown in Table 4, the Beta distribution has the best goodness of fit under both the AIC and BIC criteria for the sources in EXP 1, whereas the truncated Normal distribution performs better according to both criteria for the sources in EXP 2. Henceforth, we present the results under these distributions (the analyses under the other distributions are given in Appendix B.2).

Figure 4 plots 1000 MCMC samples for each of the upper and lower distributions, for each source. (Statistics on the distributions of parameters are given in Tables 15–22, Appendix B.2.) They suggest that the proposed elicitation technique supports parametric estimation of subjective probability intervals in the population, insofar as they chime with expectations given the nature of the events. For instance, they suggest that the dispersion of subjective upper and lower probabilities is larger for the temperature source (EXP 1) than the grade source (EXP 2),



(a) EXP 1

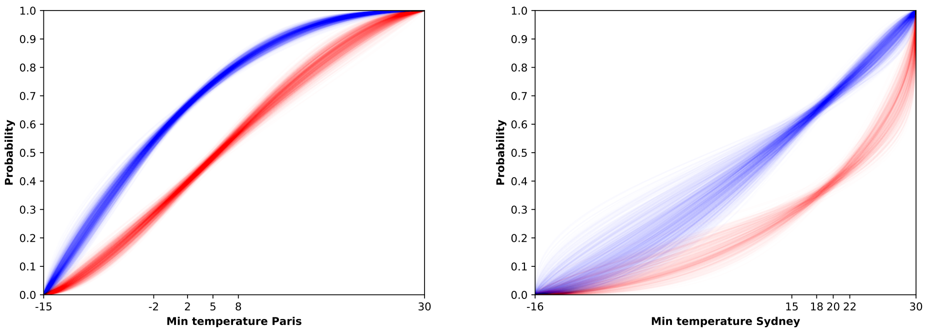
(b) EXP 2

Figure 3: Individual-level Kendall τ_b for upper and lower CDFs for the sources for each experiment, and for the MPs in EXP 1, Paris treatment. In the case of MPs, the Lower MP is calculated using the MPs of the events E_t (and should be increasing in t), whereas the Kendall τ_b for upper MPs are calculated using one minus the MPs of complementary events E_t^c (which should be increasing with t).

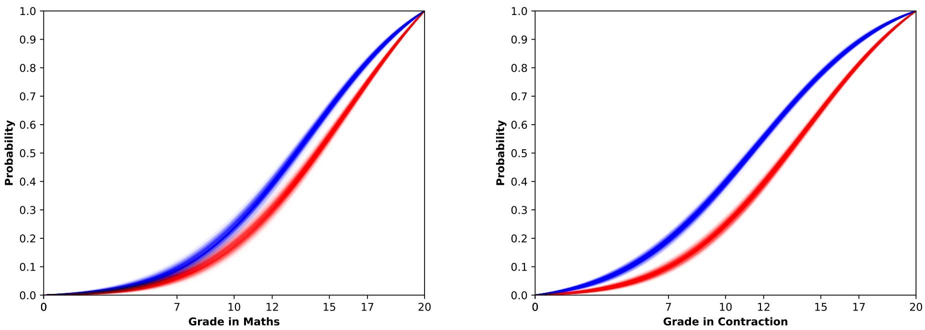
Note: The Kendall τ_b is an indicator of ordinal association: the value 1 indicates that the CDFs or MPs are strictly increasing; 0 suggests that there is no association between the elicited probability and the size of the event; -1 indicates a strictly decreasing relationship between the two.

		Paris	Sydney	Mathematics	Contraction
AIC	normal	706.65	700.79	411.22	385.52
	beta	648.26	684.36	416.18	390.64
BIC	normal	711.42	705.56	415.12	389.42
	beta	653.02	689.12	420.08	394.54

Table 4: AIC and BIC under (truncated) normal and Beta specifications for CDFs (Table 14).



(a) EXP 1



(b) EXP 2

Figure 4: Bayesian estimation of lower and upper CDFs: plots of 1000 samples from MCMC. (Beta distribution for EXP 1; Truncated Normal distribution for EXP 2)

which could be related to the fact that all subjects in EXP 2 had sat both exams, and were very interested in the marking, several months before. Also, within EXP 1, there is more dispersion in the estimated distributions for Sydney than for Paris, as would be expected given the less familiar nature of the former source for Paris subjects.[11]

4.3. Imprecision

Both the graphs of raw data (Figure 2) and those emerging from the Bayesian analysis (Figure 4) suggest that subjects' beliefs are often *imprecise*: i.e. there is a gap between their upper and lower probabilities. Indeed, two-sided Kolmogorov-Smirnoff tests of the hypothesis that the median upper and lower CDFs are drawn from the same distribution reject the hypothesis for each source ($p < 0.0001$ in all cases), suggesting a gap between upper and lower CDFs. For further analysis, we define the following index. For an event E from a given source (e.g. minimum temperature in Paris), we say that a subject's *imprecision concerning* E is $\overline{p}(E) - \underline{p}(E)$, i.e. the width of her (elicited) probability interval for E. A subject's *Imprecision Index* for a source is defined to be her average imprecision across all elicited events in the source:

[11]More precisely, it is clear from Tables 16 and 18 that the standard deviations of the parameters for the Paris source are lower than for Sydney.

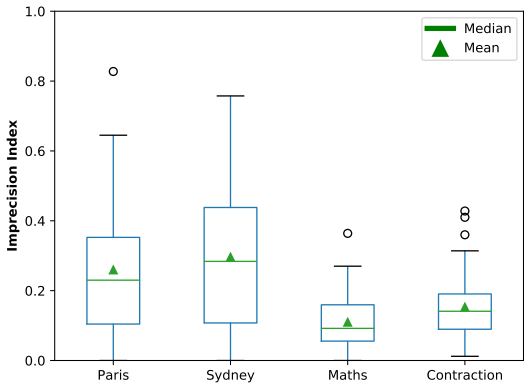


Figure 5: Imprecision Index (Eq. (8)) across sources in EXP 1 and EXP 2

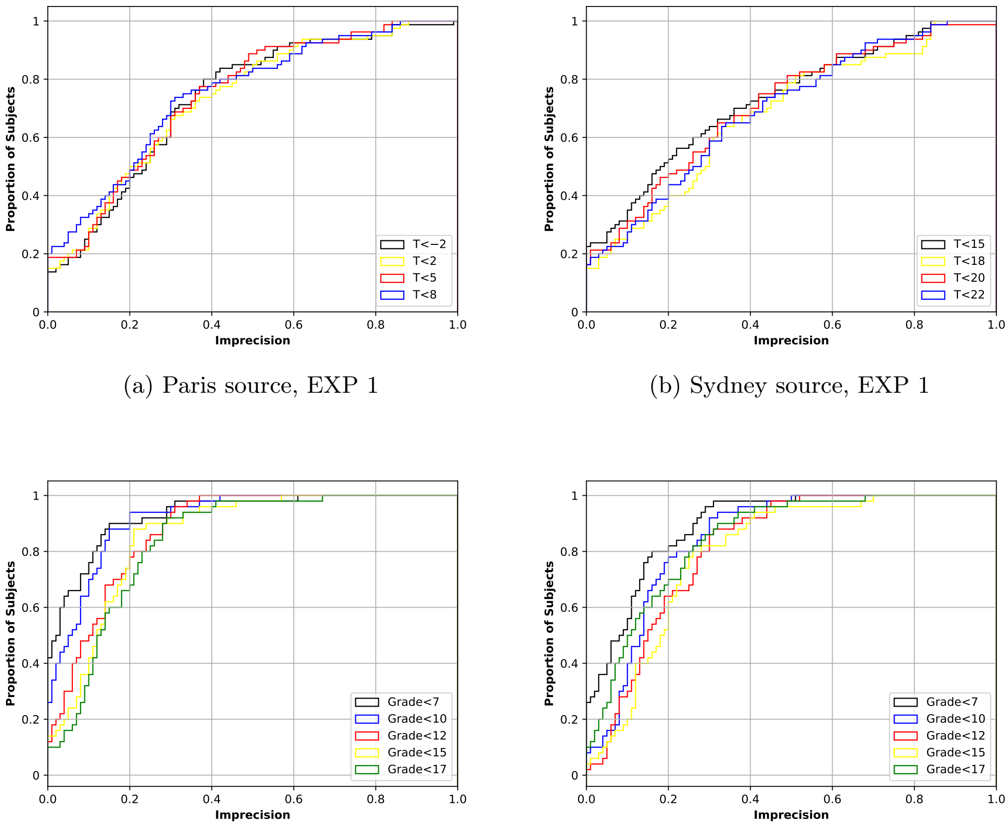
$$II = \frac{1}{n}\sum_{i=1}^{n}\left(\overline{p}(E_n) - \underline{p}(E_n)\right). \tag{8}$$

This clearly gives an indication of how imprecise the subject's beliefs are, on average, for events in the source. Naturally, an SEU decision maker will assign precise probabilities to all events, and hence have an imprecision index of 0 (for all sources).

Figure 5 displays the mean, median, 25% and 75% quantile, and max and min Imprecision Indices across all sources in both experiments (see also Table 11, Appendix B.1). It clearly suggests a tendency towards imprecision, with mean and median Imprecision Indices greater than 0.1 for all sources. Two-sided binomial tests reject the hypothesis of equal probability for the Imprecision Index to be equal to vs. greater than 0 for each source ($p < 0.0001$ in all cases), with a clear majority of subjects—74 out of 80 in EXP 1, and 49 out of 52 in EXP 2—having strictly positive Imprecision Indices.

The general message of widespread imprecision is confirmed by data on the number of precise events—events for which the subject's elicited upper and lower probabilities coincide (Table 12, Appendix B.1). Not more than around 5% of subjects gave precise probabilities for all events in a single source. Only 2 subjects (out of the 132 participating in both experiments) gave precise probabilities for all events elicited. The data in Table 12 also allows a check on the extent to which this imprecision could be driven by the binary-choice procedure, insofar as it gives the number of precise events after the binary-choice procedure and before the confirmation 2D choice list, as well as after confirmation. The general finding of few fully precise subjects holds both before and after the confirmation stage. Moreover, relatively few subjects change to fully precise probabilities for all events of the source (at most 3 out of 80, for Sydney in EXP 1), with several fully precise subjects introducing imprecision during the confirmation stage, especially in EXP 2.

Delving further, we also investigate imprecision at the event level within sources. Figure 6 plots CDFs of the imprecision for each elicited event in each of the experiments and sources,



(a) Paris source, EXP 1

(b) Sydney source, EXP 1

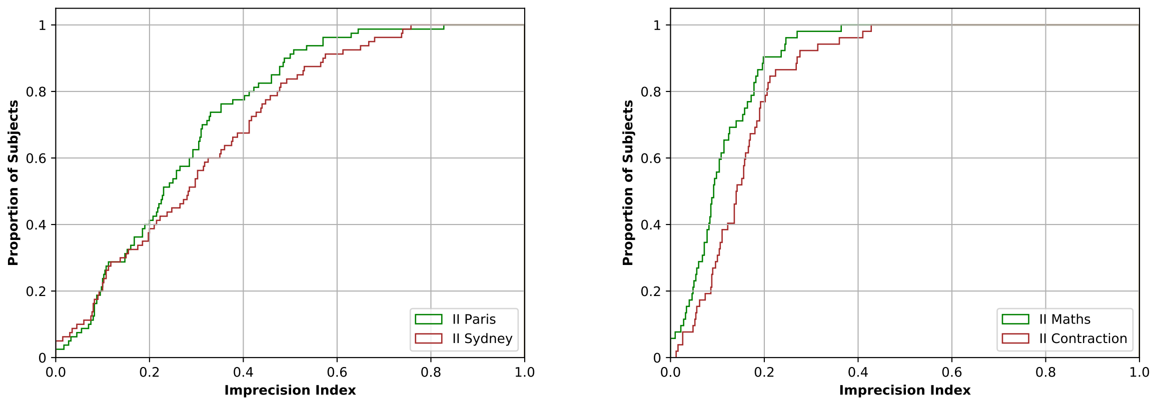
(c) Maths source, EXP 2

(d) Contraction source, EXP 2

Figure 6: CDFs of Imprecision across subjects, for each elicited event

across subjects. One-way ANOVAs of the imprecision (dependent variable) against the event (factor) reject the null hypothesis of identical imprecision across all events for the sources in EXP 2 ($p < 0.001$ for Maths; $p = 0.003$ for Contraction), whilst failing to reject it for the sources in EXP 1 (Table 13, Appendix B.1.1). This suggests not only that imprecision is widespread, but that imprecision may be event dependent within sources, as one would expect if some events are intuitively more uncertain than others. For instance, the least imprecise event in EXP 2 involves, for both sources, the lowest grade, where many subjects are presumably more sure of their judgements.

In summary, the development of a method for eliciting multiple priors does not emerge from this analysis as devoid of relevance: rather, it reveals that, when given the possibility to 'express' the imprecision implied by non-degenerate probability intervals, many subjects do, at least for the events considered here. Moreover, at least within some sources, the extent of imprecision may depend on the event.



(a) EXP 1 (b) EXP 2

Figure 7: CDFs of Imprecision Index (Eq. (8))

4.4. Imprecision and familiarity

One reasonable hypothesis is that *ceteris paribus* subjects' beliefs are more imprecise concerning events with which they are less familiar, or about which they feel as if they have less knowledge. In terms of multiple priors models, this corresponds to the probability intervals for the events being wider. Since, as explained in Section 3, each of our experiments features two sources with which our subjects will typically have different levels of familiarity, or which they naturally consider as having different levels of predictability, a natural conjecture would be that imprecision would be larger for Sydney than Paris, and for the Contraction grade than the Maths one. After all, Paris subjects are less familiar with the weather in Sydney than that in Paris; and the Contraction exam is generally considered to be 'less predictable' than the Maths one (Section 3.2).

Figure 7 plots the CDFs of the Imprecision Index defined above (Eq. (8)) across subjects, for the pair of sources in each experiment. A two-sided paired t-test barely fails to reject the null hypothesis of identical Imprecision Indices across the sources in EXP 1 ($p = 0.0895$), whilst it rejects it for EXP 2 ($p = 0.0016$). A two-sided Binomial test with null hypothesis that an equal number of subjects have larger Imprecision Index under one source than the other fails to reject the null hypothesis for EXP 1 ($p = 0.576$), but rejects it for EXP 2 ($p = 0.017$). Rerunning the latter test using the results of the Bayesian analysis—i.e. for each experiment and source, using the posterior distribution over parameters obtained from the Bayesian estimation to sample 1000 tuples of parameters determining the upper and lower distributions, and computing, for each tuple, the Imprecision Index as defined in Eq. (8)—yields a rejection of the null hypothesis across the sources for both EXP 1 and EXP 2 ($p < 0.001$ in both cases). These findings confirm the expected relationship between imprecision and predictability in EXP 2: indeed, in Figure 7, the CDF for Contraction—known as the less predictable exam—is entirely to the right of that for Math, indicating a larger Imprecision Index. They also point to a similar relationship between imprecision and familiarity in EXP 1: again, with Figure 7 suggesting that CDFs for Paris are generally more precise.

That the expected relationship between imprecision and familiarity or predictability emerges can also be seen as providing further indirect evidence as to the solidity of the proposed elicitation method.

4.5. Matching probabilities and the α-maxmin EU mixture coefficient

Recall that EXP 1 contained a supplementary treatment in which the MPs were elicited for the Paris events E_{t_i} and $E^c_{t_i}$ for which probability intervals had been elicited (Table 2). Henceforth, we denote the MP of an event E by $MP(E)$. Under SEU, $MP(E_{t_i}) = 1 - MP(E^c_{t_i}) = p(E_{t_i})$, the subjective probability of E_{t_i}, for all E_{t_i}. So, as is well-known, comparing $MP(E_{t_i})$ and $1 - MP(E^c_{t_i})$ provides an indication into the violation of SEU. Under the α-maxmin EU model (1), we have the following equations:

$$MP(E_{t_i}) = \alpha \underline{p}(E_{t_i}) + (1-\alpha)\overline{p}(E_{t_i}) \tag{9}$$

$$1 - MP(E^c_{t_i}) = \alpha \overline{p}(E_{t_i}) + (1-\alpha)\underline{p}(E_{t_i}) \tag{10}$$

Drawing on the elicited MPs and our elicitations of upper and lower probabilities, the equations (9) and (10) can be used to elicit the mixture coefficient α in the Hurwicz α-maxmin EU model. Under analysis using the raw data, the median α across subjects is 0.80 (Table 24, Appendix B.3). We also perform a Bayesian estimation of the α in tandem with the lower and upper CDFs, combining equations (9) and (10) and the MP data with our upper and lower CDF elicitations (see Appendix B.2, Tables 15 and 16). Figure 8 plots the distribution over α resulting from this estimation. The Bayesian mean for α is at 0.81, which is broadly consistent with the finding from the raw data. As discussed at more length in Section 5, this is, to our knowledge, the first direct choice-based elicitation of the α in the α-maxmin EU model that fully controls for the set of priors by eliciting the relevant information about them without making any assumption about their shape.

5. Discussion

The proposed method elicits non-degenerate and reasonable upper and lower CDFs. Our elicitations also show that imprecision—a gap between upper and lower probabilities—is widespread, with few subjects having precise probabilities for all events. Moreover, they bring out some determinants of imprecision. For some sources, the width of probability intervals may vary according to the event elicited; moreover, average imprecision decreases with the familiarity of the source of uncertainty, as one might expect. Finally, we draw on our probability interval

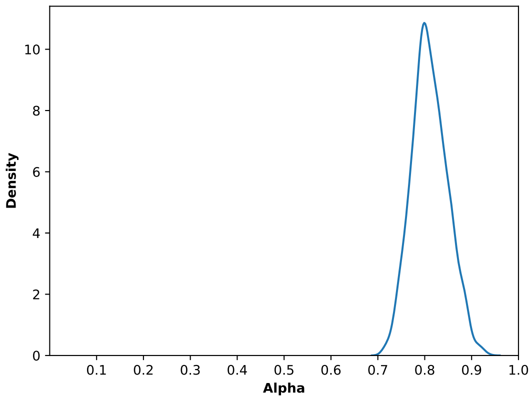


Figure 8: PDF of α from the Bayesian estimation (EXP 1, Paris treatment)

elicitation to elicit the mixture coefficient in the Hurwicz α-maxmin EU model—the first such elicitation, to our knowledge, to fully control for beliefs.

We now discuss the robustness of our procedure, some related literature, and some directions for future development.

Robustness Our approach has been presented in terms of the popular Hurwicz α-maxmin EU decision model (Section 2), which is doubtless one of the most general decision models in which the 'belief component' of the representation is just a set of priors. However, many of the central elements of the approach generalise to other models, including extensions building on sets of priors but weakening the linearity of the Hurwicz function form (1) to account for probability weighting, for instance (see Appendix A.3 for details). First of all, the notion of MPI remains well-defined for all such extensions, and the decision maker's subjective probability interval is always a MPI. Though MPIs are not guaranteed to be unique for every conceivable extension of this sort, they are essentially unique for a family of reasonable extensions (Appendix A.3). Second, the 2D choice list incentivization mechanism only relies on the weak Lower Stochastic Dominance property of preferences (Definition 1, Section 2.5). Apart from the maxmax-EU model (i.e. (1) with $\alpha = 0$), which is very rarely found in subjects, this property is satisfied by any reasonable decision model generalising α-maxmin EU to allow for nonlinear dependence of preferences on upper and lower probabilities (Appendix A.3). In this sense, the 2D choice list incentivization mechanism is widely valid. Finally, whilst the binary-choice procedure relies on the strongest assumption made in Section 2—that $\alpha > \frac{1}{2}$—there is independent evidence that this holds for most of our subjects (Appendix A.3). As noted in Section 3.4, it is the 2D choice list confirmation task that counts for incentivizing subjects' choices, the binary-choice procedure playing the role of an aid to completing it.

The 2D choice list mechanism is incentive compatible whenever subjects treat the branches in the choice list in isolation from each other (Section 2.5); if this isolation assumption is violated, strategic choice may occur. Our implementation was designed to favour such isolation, notably via the realisation of 2D choice lists by a single scrollbar with two cursors (Section 3.4 and Appendix C). Visually very different from Figure 1, this presentation promotes considering each branch in isolation and is less suggestive of strategic opportunities of changing the choice lists by 'moving around the 2D space'. Notwithstanding this, the extent to which isolation holds in our experiments is ultimately an empirical question, and we treat it as such. On this front, the proposed elicitation method has the advantage that strategic reasoning leads to easily recognizable choice patterns. As discussed in Appendix A.3.2, for a subject represented by (1) with $\alpha \in (0, 1)$ (and any set of priors), her optimal response to the 2D choice list task when reasoning strategically is one of the points $[0, 0], [0, 1], [1, 1]$: i.e., one of the vertices of the space of interval-valued urns in Figure 1. Examining the concentration of responses at the vertices thus provides insight into the extent of strategic reasoning in our subject pool. Our data suggests that it is very limited: in EXP 1, only one subject (out of 80) reported vertex points for more than half of the elicited events,[12] whereas for no subject in EXP 2 were more than half of the elicited points among the vertices (Table 9, Appendix B.1; see Appendix A.3.2 for further details).

Related literature Our elicitation method relates to existing experimental and theoretical literature on multiple prior models, and the α-maxmin EU model in particular. Part of this literature is concerned with testing such models, or comparing them to others (e.g. Hey et al., 2010; Baillon and Bleichrodt, 2015); by contrast, the aim here is to elicit probability intervals in the context of a fairly general multiple prior model. Similarly, there is a literature studying willingness to bet on objectively-given probability intervals based on interval-valued urns (e.g. Baillon et al. (2012); Chew et al. (2017)) using matching probabilities or certainty equivalents. The present paper, by contrast, uses such urns in the context of the different task of eliciting subjective probability intervals.

On the theory side, the challenge of incentive-compatible elicitation of multiple prior beliefs under α-maxmin EU is related to identification issues with this model, arising from the fact that different pairs of mixture coefficient α and sets of priors can represent the same preferences over (Savage or Anscombe-Aumann) acts. Proposed approaches to this challenge include pinning down the set of priors using 'unambiguous preferences' (Ghirardato et al., 2004), though this has problems in finite state spaces (Eichberger et al., 2011), or enrichening the state space to include an infinite product structure and invoking symmetry axioms (Klibanoff et al., 2021). Another line of attack concentrates on special cases of the α-maxmin EU model, notably involving some form of probabilistic sophistication, i.e. the assumption that there are precise probabilistic beliefs which completely determine the contributions of events to preferences (Machina and Schmeidler, 1992; Chew and Sagi, 2006). Working with a rich state space à la Savage, Gul and Pesendorfer (2014, 2015) obtain a unique identification of α and the set of priors whenever the latter is generated as the set of extensions of a precise probability measure on a subalgebra of events. (Grant et al. (2019) have extended this approach beyond the assumption of linearity

[12] He / she reported 3 points out of 4 as $[0, 1]$, for both sources.

in upper and lower probabilities built into α-maxmin EU; see Appendix A.) Chateauneuf et al. (2007) obtain a unique identification of α and the set of priors whenever the latter is generated from a precise probability measure via ε-contamination, i.e. mixture with the set of all probability measures. As stated in the Introduction, we specifically avoid the sort of probabilistic sophistication assumption behind these approaches, motivated by the observation that such assumptions are inadmissible precisely in those situations where multiple prior beliefs are most relevant. Indeed, our procedure takes a different approach, based on interval lotteries, with no need for specific richness assumptions on the state space, probabilistic sophistication, or any other (non-standard) assumptions on the set of priors. Hill (2021) provides an axiomatic foundation for α-maxmin EU and extensions by generalising the elicitation principle used in the present paper.

On the experimental front, there is a small literature dealing with incentive-compatible elicitation of multiple priors. One family of approaches purport to elicit multiple priors as the support of second-order beliefs, represented as a measure over the space of probability measures. Beyond the assumption of second-order beliefs, which is foreign to the original multiple prior models (Gilboa and Schmeidler, 1989; Bewley, 2002; Ghirardato et al., 2004), these often make further assumptions about the role of these second-order beliefs in choice. For instance, Qiu and Weitzel (2016) elicit subjects' distributions over the matching probabilities of other participants in the experiment, and purport to deduce subjects' own second-order beliefs from these, relying on the assumption that a subject's opinions about others' matching probabilities coincides with the uncertainty surrounding her own assessment. In a theoretical paper, Karni (2020) develops an ingenious incentive-compatible mechanism for eliciting second-order beliefs and the associated set of priors (as the support), relying on a three-period setup. The mechanism assumes that the subject's second-order beliefs coincide with her beliefs about what she will believe in the interim period. As made clear above, our method relies on no assumptions beyond the α-maxmin EU model (or appropriate weakenings thereof), and in particular there is no role for second-order beliefs or assumptions on how they relate to other beliefs.

Another family of approaches draws on the theoretical literature discussed above, and in particular on the probabilistically sophisticated special case studied by Chateauneuf et al. (2007), where the subject's set of priors is generated as the ε-contamination of a single probability measure with the space of all priors.[13] Dimmock et al. (2015); Baillon et al. (2018b,a) use elicitation of standard MPs (in the case of the last paper, certainty equivalents) to estimate 'ambiguity indices', from which one can back out the mixture coefficient α and the parameters of the ε-contaminated set of priors. As noted previously, multiple prior decision models come to the fore in situations where preferences cannot be reasonably assumed to be generated from precise probabilities, and our elicitation technique was specifically designed to be independent of the assumption of probabilistic sophistication for this reason. Moreover, our data provides

[13]Formally, the assumption is that the set of priors $\mathcal{C} = \{(1-\varepsilon)p + \varepsilon\Delta\}$, where Δ is the space of all probability measures, p is an element of Δ and $\varepsilon \in [0, 1]$.

empirical insight into the aforementioned probabilistic sophistication assumption. In particular, this assumption implies that the imprecision (in the sense of Section 4.3) is the same for all events.[14] As noted in Section 4.3 (see also Table 13, Appendix B.1.1), our observations reject this equality for the sources in EXP 2, though not for the sources in EXP 1. This suggests that there are sources for which their method's underlying assumption does not hold. That said, it may be viable on some sources; indeed, our data indicate that the Paris source in EXP 1 may be one such source. And in fact, we can estimate the ambiguity indices used in the aforementioned papers on the basis of the data from our study (EXP 1, Paris treatment) under their assumption about the set of priors,[15] and find, for instance, that they yield the value 0.82 for the mixture coefficient α—which, reassuringly, is close to the Bayesian and raw estimates reported in Section 4.5. So not only is our elicitation method more robust, insofar as it applies in situations where the assumptions underlying their approach do not hold, it can evaluate precisely in which cases they do hold; in those cases, their approach, implemented on our data, gives the same result as our 'direct' elicitation.

Going beyond the lab, there is a large and growing literature on elicitation of multiple priors or imprecise probabilities in a range of disciplines, from economics to climate science. All such elicitation exercises of which we are aware use stated probability intervals, and as such are not incentive compatible. For instance, Giustinelli et al. (2021) elicit beliefs on dementia and long-term care decisions in a large-scale representative survey (over 1000 subjects), allowing stated probabilities to be interval-valued. Consistently with our results (Section 4.3), they find widespread imprecision. They argue forcefully for the importance of probability-interval elicitation for reducing survey bias and understanding attitudes to and behavior in the face of high-uncertainty events, such as whether one will develop dementia and whether one should insure against it. In another approach, in another domain, Kriegler et al. (2009) elicit beliefs of selected scientists (around 50 subjects) concerning climate tipping points, allowing participants to state probability intervals for these (notoriously uncertain) events. Such expert elicitations, which involve often time-consuming and individualised sessions with selected experts, have emerged as a central tool for managing complex uncertainties (Morgan, 2014). Though they have traditionally aimed at eliciting precise probabilities, Kriegler et al. (2009) shows that imprecision is widespread for some events, and hence once again argue for the relevance of probability-interval elicitation.

[14]If the set of priors is as defined in footnote 13, then, for any E, $(1-\varepsilon)p(E) \in [0, 1-\varepsilon]$, so the probability interval for event E is $[(1-\varepsilon)p(E), (1-\varepsilon)p(E)+\varepsilon]$, and hence the event has imprecision ε.

[15]Specifically, Baillon et al. (2018b) propose the average of $1 - MP(E) - MP(E^c)$ over a selection of events as their measure of the 'ambiguity aversion index' b. The average for the events elicited here can be deduced directly from Table 25 (Appendix B.4), as around 0.16. On the other hand, under (1) with the specified form for the set of priors (see footnote 13), their 'a-insensitivity index' $a = \varepsilon$. Under such sets of priors, as noted in footnote 14, every E has imprecision ε. The average imprecision, as measured by the Imprecision Index (Table 11), thus gives an estimate of their a: it is around 0.25. The mixture coefficient α is related to these indices by $\alpha = \frac{1}{2}\left(\frac{b}{a}+1\right)$ (Baillon et al., 2018b,a), yielding the value in the text.

Future Directions Two leitmotivs emerge from the literature review. On the one hand, our results are consistent with existing studies suggesting that imprecision is widespread for some events. However, based as they are on an incentive-compatible, choice-based and theoretically robust elicitation method, our results are less open to criticisms of existing studies pointing to a lack of incentive compatibility or the reliance on a specific model. On the other hand, as we saw on the ε-contamination example, our method can be used to evaluate the assumptions behind—and hence the effectiveness of—existing methods.

The latter point suggests one direction for future research. As noted, stated probability intervals are typically used in large-scale surveys (such as Giustinelli et al. 2021), but how close, or far, are subjects' stated probability intervals from their actual multiple prior beliefs? Our method can be used to provide insight into this question, for instance by eliciting probability intervals with both the proposed method and stated procedures for the same subjects and on the same set of events, and comparing the results. Although the 'test' events should involve uncertainty that resolves in a reasonable timescale for payment of the incentives—so the method can be applied in an incentivised fashion—the conclusions of such a comparison may be extrapolated to situations where incentive payement is infeasible, for instance if the events of interest are too far in the future (e.g. number of global pandemics before 2100) or counterfactual (e.g. how many pandemics would there be if 60% of original wild habitats had been protected). As such, our method can be used to corroborate, refine, correct and chose between existing stated approaches.

Moreover, although our method was developed with the aim of demonstrating the possibility of choice-based incentive-compatible probability-interval elicitation, future research could operationalise simpler, parametrised versions, with fewer choice questions, which would be more implementable in field studies. Large-scale surveys often use choice tasks without necessarily incentivising them (e.g. Falk et al., 2018), and questions formulated in terms of bets may trigger different cognitive mechanisms to those formulated in terms of judged probabilities. Our method could thus lay the foundations of a bet-based approach to add to the arsenal of probability-interval elicitation procedures used in practice.

Finally, analogous possibilities exist for expert elicitation exercises, of the sort cited above. Compared to survey studies, these typically involve fewer subjects, with each spending more time; the flip side is that more precision is desired of the elicitation at the individual level. EXP 2 suggests that our method may provide the appropriate individual-level probability-interval elicitation, whilst having theoretically well-founded incentive-compatibility properties. Probability elicitation exercises in decision analysis often use bet-based choice tasks without necessarily incentivising them (e.g. Clemen and Reilly, 2013); again, our method, applied in this context, complements existing stated approaches to eliciting probability intervals.

6. Conclusion

This paper proposes and implements a solution to the open problem of choice-based incentive-compatible elicitation of multiple prior beliefs. It comprises a new preference-based notion—

Matching Probability Intervals—and probability-interval analogues of standard choice lists and bisection elicitation procedures. Theoretically, it operates in the context of the Hurwicz α-maxmin EU model and in the absence of strong assumptions about subjects' sets of priors, most notably any form of probabilistic sophistication.

Our implementation of the elicitation method, in two experiments to elicit subjective upper and lower CDFs over continuous-valued sources of uncertainty, testifies to its feasibility. It finds a predominance of imprecision—a gap between upper and lower probabilities—across our subjects, for all explored sources, showing it to be related to familiarity or predictability. It also allows us to perform what, to our knowledge, is the first elicitation of the mixture coefficient in the α-maxmin EU model that fully controls for beliefs.

A. Theoretical Appendix

In order to bring out the robustness of our proposal, and the assumptions underlying it, we shall at times work with a more general decision model than Hurwicz α-maxmin EU (1). Consider the representation where a bet $(z, E, 0)$ is evaluated according to:

$$W(\underline{p}(E), \overline{p}(E)).u(z) \tag{11}$$

where $\underline{p}(E)$, $\overline{p}(E)$ and u are as in Section 2.3, and W is an 'aggregation function', which is continuous in both coordinates and normalised—$W(x, x) = x$ for all x. α-maxmin EU is the special case where W is linear: $W(x, y) = \alpha x + (1 - \alpha)y$. See Grant et al. (2019) for an axiomatisation of a special case of (11) where the set of priors is generated by a probability measure on a subalgebra, and a thorough discussion of its potential. As in Section 2, we assume the same representation for imprecise risky prospects.

Note that, unlike α-maxmin EU, the general form (11) can accommodate non-linear, Prospect-Theory-style weighting of the lower and upper probabilities, for instance taking $W(x, y) = \alpha w(x) + (1 - \alpha)w(y)$, where w is a Prospect-Theory-style weighting function.

A.1. Proofs

We prove Proposition 1 under representation (11). As noted above, the α-maxmin EU model is a special case.

Proof of Proposition 1. Under (11), it follows from the first preference pattern in Proposition 1 that $W(q, 1 - b) > W(\underline{p}(E), \overline{p}(E))$ for all $q > r$, and similarly for the others. By the continuity of W, it thus follows from the first two preferences that $W(r, 1 - b) = W(\underline{p}(E), \overline{p}(E))$, and from the second pair of preferences that $W(b, 1 - r) = W(1 - \overline{p}(E), 1 - \underline{p}(E))$. It thus follows that $(z, [r, 1 - b], 0) \sim (z, E, 0)$ and $(0, [r, 1 - b], z) \sim (0, E, z)$, so $[r, 1 - b]$ is a MPI for E, as required. $\square$

Note that the converse of Proposition 1 holds—i.e. for any MPI, the preference pattern in the Proposition holds—whenever W is strictly increasing in the first coordinate; this is the case for α-maxmin EU model with $\alpha > 0$.

Proof of Proposition 2. Part a. Plugging in the representations (1) and (2), any $[x, y]$ in the R-B region satisfies:

$$\alpha x + (1 - \alpha)y \geq \alpha\underline{p}(E) + (1 - \alpha)\overline{p}(E)$$
$$\alpha(1 - y) + (1 - \alpha)(1 - x) \geq \alpha(1 - \overline{p}(E)) + (1 - \alpha)(1 - \underline{p}(E))$$

By basic algebra (add α times the first inequality to $(1 - \alpha)$ times the second), one obtains $(\alpha^2 - (1-\alpha)^2)x \geq (\alpha^2 - (1-\alpha)^2)\underline{p}(E)$, whence it follows, since $\alpha > \frac{1}{2}$, that $x \geq \underline{p}(E)$. Similarly, one obtains $((1-\alpha)^2-\alpha^2)y \geq ((1-\alpha)^2-\alpha^2)\overline{p}(E)$, whence, since $\alpha > \frac{1}{2}$, $y \leq \overline{p}(E)$. A similar argument establishes the result for points in W.

Part b. follows directly from the fact that, under 2, whenever $x \leq x'$ and $y \leq y'$, then $(z, [x, y], 0) \preceq (z, [x', y'], 0)$ and $(0, [x, y], z) \succeq (0, [x', y'], z)$. □

We state for completeness the result on the uniqueness of the MPI.

Proposition 3. *For any decision maker represented according to (1) with $\alpha \neq \frac{1}{2}$, and for any event E, there is a unique MPI for E.*

Proof. Existence is immediate from Eqs. (3) and (4). Uniqueness is immediate from the linearity of the indifference curves in $\mathcal{I}$-space (see Figure 1). □

A.2. Binary-choice procedure

A.2.1. Introduction and setup

Our binary-choice procedure is fully described in Figures 10–13. Figure 10 sets out the general structure (and stopping rules). At each step of the procedure, preferences are elicited for a single probability interval $[\underline{p}_i, \overline{p}_i]$: i.e. preferences between the bet on the event and the IL $(z, [\underline{p}_i, \overline{p}_i], 0)$, and between the bet on the complement event and the complementary IL $(0, [\underline{p}_i, \overline{p}_i], z)$. The heart of the procedure, detailed in Figures 11–13, involves specification of the next probability interval proposed for elicitation on the basis of the preferences concerning the previous intervals. We first set out the notation used in the presentation of these parts of the procedure, before explaining informally its main steps. Throughout, we adopt the Euclidean topology on $\mathcal{I} \subseteq \mathbb{R}^2$, and let $d(\bullet, \bullet)$ be the Euclidean distance.

The procedure draws on two formal elements. The first is the assignment of interval-valued urns—or equivalently probability intervals—to preference-defined regions, discussed in Section 2.6. Recall from Section 2.1 that an interval-valued urn $[p, q]$, i.e. with a minimum proportion p of red balls and a minimum proportion $1-q$ of blue balls, corresponds to a probability interval; we shall present the procedure in terms of the latter here. For every event E_i and urn $[p, q]$, the preferences in the choices between the bet on E_i and that on a red ball being drawn from the urn, and between the bet on E_i^c and that on blue from the urn suffice to situate $[p, q]$ in one of the four regions, $R-B, W, R, B$ defined in Table 1 (Section 2.6). For instance, in Figure 9, which we shall use to illustrate the procedure, the probability intervals already elicited are the dots coloured white, red, blue and red-blue according to the (preference-based) region they belong to.

The second element is a 'polar'-style coordinate system for the set of probability intervals $\mathcal{I}$, under which, informally, $(m, \alpha) \in [0, 0.5] \times [0, 1]$ is the probability interval that is α along the piecewise-linear line that goes through the probability intervals $[0, 0]$, $[1, 1]$, and $[m, 1-m]$ (corresponding to the urn with at least proportion m of red balls and at least proportion m of blue balls). The thick grey line in Figure 9 is one such line. Formally, $\sigma : \mathcal{I} \to [0, 0.5] \times [0, 1]$ is defined by:

$$\sigma([p,q]) = \begin{cases} (\frac{p}{p+q}, \frac{p+q}{2}) & p \leq 1-q, p+q \in (0,2) \\ (\frac{1-q}{2-p-q}, \frac{p+q}{2}) & p > 1-q, p+q \in (0,2) \\ (0,0) & p = q = 0 \\ (0,1) & p = q = 1 \end{cases} \tag{12}$$

It is straightforward to check that σ is a well-defined function on $\mathcal{I}$. Every point except for $[0,0],[1,1]$ corresponds to a unique line (parametrised by m) and 'distance' along that line (parametrised by α). $[0,0]$ (respectively $[1,1]$) corresponds to a single α, namely 0 (resp. 1), though it lies on all such lines; we set the corresponding $m = 0$ by convention. For information, the inverse map is given by:

$$\sigma^{-1}(m,\alpha) = \begin{cases} [2\alpha m, 2\alpha(1-m)] & \alpha \leq \frac{1}{2} \\ [(2-2\alpha)m + (2\alpha-1), (2-2\alpha)(1-m) + (2\alpha-1)] & \alpha > \frac{1}{2} \end{cases} \tag{13}$$

We write $\sigma_1([p,q])$ (respectively $\sigma_2([p,q])$) for the first (resp. second coordinate) of $\sigma([p,q])$. Since this is a simple change of coordinates, we shall write $(m,\alpha) \in B$ as short for $\sigma^{-1}(m,\alpha) \in B$, and similarly for other cases.

A.2.2. Presentation of main steps

As discussed in Section 2.6 (Proposition 2), elicited points in the R-B and W regions determine an area in $\mathcal{I}$ 'between the R-B and the W points' to which the MPI must belong. The general aim of the procedure is thus to find progressively 'closer' points in R-B and W, hence reducing the size of this area. This motivates the two main steps in the determination of the next probability interval to be presented for elicitation, $[p_{i+1}, q_{i+1}]$, on the basis of the previously elicited point $[p_i, q_i]$.

On the one hand, if $[p_i, q_i]$ is in the R-B region (respectively, the W region), then by Proposition 2 a. (Section 2.6), the MPI will be North-West of $[p_i, q_i]$ (resp. South-East of $[p_i, q_i]$) in Figure 1—i.e. $\underline{p} \leq p_i$ and $\overline{p} \geq q_i$ (resp. $\underline{p} \geq p_i$ and $\overline{p} \leq q_i$), where the MPI is $[\underline{p}, \overline{p}]$. In such cases, the procedure proposes a $[p_{i+1}, q_{i+1}]$ North-West (resp. South-East) of $[p_i, p_i]$. This exemplified by the $[p_{i+1}, q_{i+1}]$ proposed for point X in Figure 9. The precise proposal for $[p_{i+1}, q_{i+1}]$ depends on whether there is a point in W (resp. R-B); technicalities aside, this is the general strategy of the cases in lines 20-23 and 36-39 of the procedure (Figures 12-13). If the point $[p_{i+1}, q_{i+1}]$ turns out to be in R-B or W, this will further restrict the area where the MPI can lie.

On the other hand, if $[p_i, q_i]$ is in the R or B regions, then Proposition 2 a. does not apply; as discussed in Section 2.6, the aim in such cases is to find a point in the R-B or W regions, to continue reducing the area containing the MPI. The procedure draws on two observations. First, as mentioned above, any point $[p_i, q_i]$ can be equivalently written in another coordinate system, specifying the line it sits on—parametrised by $m = \sigma_1([p_i, q_i])$—and how 'far' along the line it

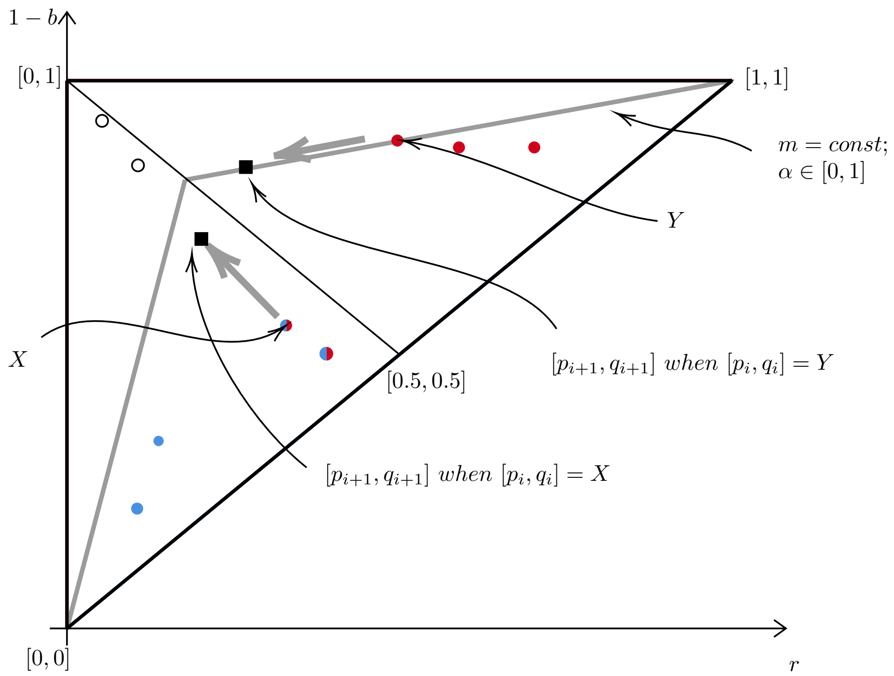


Figure 9: Binary Choice Procedure.

is—parametrised by $\alpha = \sigma_2([p_i, q_i])$. Second, for $[p_i, q_i]$ in R (respectively B), by Proposition 2 b., all points North-East (resp. South-West) of $[p_i, q_i]$ are also in R (resp. B). So the only points in R-B and W on the line $m = \sigma_1([p_i, q_i])$ corresponding to the point $[p_i, q_i]$ must be South-West of $[p_i, q_i]$, i.e. with lower α (resp. North-East, i.e. with higher α). Accordingly, the procedure proposes a point $[p_{i+1}, q_{i+1}]$ on the line $m = \sigma_1([p_i, q_i])$ but shifted in the appropriate direction, as illustrated by the $[p_{i+1}, q_{i+1}]$ proposed for point Y (lying in the R region) in Figure 9. Technicalities aside, this is general strategy for Case 1 (lines 1-17) and the cases in lines 24-34 and lines 40-44 of the procedure (Figures 11–13). Among these cases, all retain the same m (grey line in Figure 9) except those considered in lines 12-17. These treat cases where no point in R-B or W has yet been found; the procedure in these cases increases m during the search, hence looking closer to the 45° line (ie. the line of $[p, q]$ with $p = q$). We used a procedure with this in-built precision bias to favour Bayesian replies (i.e. precise probabilities); in the light of it, our finding of widespread imprecision (Section 4.3) is all the more remarkable.

A.2.3. Convergence

Except for extreme cases, the procedure tends to the MPI.

Proposition 4. *Suppose preferences are represented according to (1) with $1 > \alpha > \frac{1}{2}$, let E be an event, and let $[\underline{p}_n, \overline{p}_n]$ be the result of the procedure in Figures 11–13 (with initial values set as in Figure 10) applied for n steps. Then $[\underline{p}_n, \overline{p}_n] \to [\underline{p}(E), \overline{p}(E)]$ as $n \to \infty$. Moreover, the procedure also converges in this sense when preferences are represented according to (1) with $\alpha = 1$, $\underline{p}(E) \neq 0$ and $\overline{p}(E) \neq 1$.*

Proof. We provide the main steps of the proof here; they rely on technical Lemmas 1–4, which are detailed in Appendix D. We adopt the notation and initial values from Figure 10; in particular, let El_n be the set of elicited points after n steps. As discussed in Section 2.4, the MPI is $[\underline{p}(E), \overline{p}(E)]$. Moreover, by Proposition 2, at stage n, the MPI is contained in

$$\Phi_n = \left\{ [p,q] \in \mathcal{I} : \begin{array}{c} \max\{p' : [p',q'] \in El_n \cap W\} \leq p \leq \min\{p'' : [p'',q''] \in El_n \cap R-B\}, \\ \max\{q'' : [p'',q''] \in El_n \cap R-B\} \leq q \leq \min\{q' : [p',q'] \in El_n \cap W\} \end{array} \right\} \tag{14}$$

where the maximum of an empty set is taken to be 0 and the minimum 1.

We reason referring to the cases in the procedure (Figures 11–13). At the beginning of the procedure, it is in Case 1 ($El_0 \cap W = El_0 \cap R - B = \emptyset$). By lines 13-16, if no point in W or R-B is found, the points elicited by the procedure will reach the space of precise probabilities (i.e. points $[p,q]$ with $p = q$), where it will follow a standard bisection procedure. All such points have σ_1-value of 0.5. It follows from Lemma 1 that if the MPI is not precise, then a point will be found in R-B, so the procedure moves to Case 2. On the other hand, if the MPI is precise, then, by Lemma 1 and the bisection character of the procedure on the space of precise probabilities, the points elicited in the procedure will converge to it as required.

Now consider cases where the procedure arrives to Case 2 or 3, i.e. it finds a point in R-B or W. By Lemma 3, $\sigma_1([p_n, q_n]) \to \sigma_1([\underline{p}(E),\overline{p}(E)])$ as $n \to \infty$. We distinguish three cases.

- $\sigma_1([\underline{p}(E),\overline{p}(E)]) > 0$ and $\sigma_1([p_n, q_n]) \neq \sigma_1([\underline{p}(E),\overline{p}(E)])$ for all n. By Proposition 2 and the definition of σ (and in particular the slopes of the lines $\sigma_1([p,q]) = m$ for $m > 0$), it follows straightforwardly that $\min_{[p,q]\in El_n} d([\underline{p}(E), \overline{p}(E)], [p,q]) \to 0$ as $n \to 0$, whence $[\underline{p}_n, \overline{p}_n] \to [\underline{p}(E), \overline{p}(E)]$ as required.

- $\sigma_1([\underline{p}(E),\overline{p}(E)]) > 0$ and $\sigma_1([p_i, q_i]) = \sigma_1([\underline{p}(E),\overline{p}(E)])$ for some i. By Lemma 1 and Case 2 (lines 24-33) and Case 3 (lines 40-43), the procedure will, from i onwards, only pass through points with same σ_1-value $\sigma_1([\underline{p}(E),\overline{p}(E)])$, where it will only find points in R and B. Moreover, it follows a bisection-style procedure on the line $\sigma_1([p,q]) = \sigma_1([\underline{p}(E),\overline{p}(E)])$. It follows from standard arguments, Lemma 1 and representation (1) that this procedure converges to $[\underline{p}(E), \overline{p}(E)]$ as required.

- $\sigma_1([\underline{p}(E),\overline{p}(E)]) = 0$ and $\alpha < 1$ in the representation (1). Suppose $\underline{p}(E) = 0$; the other case ($\underline{p}(E) \neq 0$ and so $\overline{p}(E) = 1$) is treated similarly. By Lemma 1, $[p_n, q_n]$ contains a subsequence of points in R-B, with σ_1-value tending to 0. Since $\alpha < 1$, by representation (1), for every $q < \overline{p}(E)$, there exists $p > 0$ such that $(z, [p,q], 0) \prec (z, E, 0)$, and hence such that $[p,q]$ is not in R-B. Moreover, by the representation and Lower Stochastic Dominance, for every $q > \overline{p}(E)$ and p, $(0, [p,q], z) \prec (0, \left[\underline{p}(E), \overline{p}(E)\right], z) \sim (0, E, z)$, so such $[p,q]$ are not in R-B. It follows that the subsequence of $[p_n, q_n]$ consisting of points in R-B converges to $\left[\underline{p}(E), \overline{p}(E)\right]$, so $[p_n, q_n] \to \left[\underline{p}(E), \overline{p}(E)\right]$ as required.

$\square$

Procedure Binary Choice Procedure: structure and stopping rules

Set $[p_1, q_1] = [0.3, 0.7]$, $i = 1$ and $El = \emptyset$; `/*` $[p_i, q_i]$ `is the last interval for which preferences were elicited, and` $El \not\ni [p_i, q_i]$ `is the set of intervals for which preferences have been elicited previously. So` $El \cup \{[p_i, q_i]\}$ `is the set of all elicited intervals, including the one under consideration. */`

Set $rb_{El} = \arg\max_{[p,q] \in El \cap RB} q$, $w_{El} = \arg\min_{[p,q] \in El \cap W} q$, $r_{El} = \arg\min_{[p,q] \in El \cap R} q$, $b_{El} = \arg\max_{[p,q] \in El \cap B} q$; `/*` rb_{El} `is the highest elicited point in R-B,` w_{El} `is the lowest elicited point in` W`, etc. */`

while $|El| < 12$ **do**
    while $d(rb_{El}, w_{El}) \geq 0.15$ **do**
        Elicit preferences for $[p_i, q_i]$;
        Execute algorithm 1 ;
        Add $[p_i, q_i]$ to El ;
        if $d(rb_{El}, w_{El}) < 0.15$ **then**
            return $[p_i, q_i]$;
            Stop

if $El \cap W = \emptyset$ and $El \cap R - B = \emptyset$ **then**
    return $\frac{b_{El} + r_{El}}{2}$

if $El \cap R - B \neq \emptyset$ and $El \cap W = \emptyset$ **then**
    return rb_{El}

if $El \cap W \neq \emptyset$ and $El \cap R - B = \emptyset$ **then**
    return w_{El}

if $El \cap R - B \neq \emptyset$ and $El \cap W \neq \emptyset$ **then**
    return $\frac{rb_{El} + w_{El}}{2}$

Figure 10: Binary choice procedure: structure

Algorithm 1: Determination of Next Binary Choice

Case 1

if $(El \cup \{[p_i, q_i]\}) \cap R - B = (El \cup \{[p_i, q_i]\}) \cap W = \emptyset$ **then**

  if *there is no* $[p, q] \in El$ *with* $\sigma_1([p, q]) = m$, *where* $m = \sigma_1([p_i, q_i])$ **then**

    if $[p_i, q_i] \in R$ **then**

      $[p_{i+1}, q_{i+1}] = \sigma^{-1}(m, \max\{\alpha_0 | (m_0, \alpha_0) \in El \cap B\})$;

    if $[p_i, q_i] \in B$ **then**

      $[p_{i+1}, q_{i+1}] = \sigma^{-1}(m, \min\{\alpha_0 | (m_0, \alpha_0) \in El \cap R\})$;

  if *there exists* $[p, q] \in El$ *with* $\sigma_1([p, q]) = m$, *where* $m = \sigma_1([p_i, q_i])$, *but there exists no* $[p, q] \in El$ *with* $[p, q] \in B$ *and* $\sigma_1([p, q]) = m$ **then**

    $[p_{i+1}, q_{i+1}] = \sigma^{-1}(m, \frac{\sigma_2([p_i, q_i])}{2})$;

  if *there exists* $[p, q] \in El$ *with* $\sigma_1([p, q]) = m$, *where* $m = \sigma_1([p_i, q_i])$, *but there exists no* $[p, q] \in El$ *with* $[p, q] \in R$ *and* $\sigma_1([p, q]) = m$ **then**

    $[p_{i+1}, q_{i+1}] = \sigma^{-1}(m, \frac{1+\sigma_2([p_i, q_i])}{2})$;

  if *there exists* $[p, q], [p', q'] \in El$ *with* $[p, q] \in B$, $[p', q'] \in R$ *and* $\sigma_1([p, q]) = \sigma_1([p', q']) = \sigma_1([p_i, q_i])$=$m$ **then**

    if $[p_i, q_i] \in R$ **then**

$$[p_{i+1}, q_{i+1}] = \begin{cases} ((m + 0.5)/2, (\sigma_2([p_i, q_i] + \alpha')/2) & m \leq 0.44 \\ (0.5, (\sigma_2([p_i, q_i] + \alpha')/2) & m > 0.44 \end{cases}$$

      where $\alpha' = \max\{\alpha_0 | (m, \alpha_0) \in El \cap B\}$;

    if $[p_i, q_i] \in B$ **then**

$$[p_{i+1}, q_{i+1}] = \begin{cases} ((m + 0.5)/2, (\sigma_2([p_i, q_i] + \alpha'')/2) & m \leq 0.44 \\ (0.5, (\sigma_2([p_i, q_i] + \alpha'')/2) & m > 0.44 \end{cases}$$

      where $\alpha'' = \min\{\alpha_0 | (m, \alpha_0) \in R\}$;

end

Figure 11: Determination of Next Binary Choice: Part 1

Notation: σ defined in (12) and (13).

Case 2

if *either* $(El \cup \{[p_i, q_i]\}) \cap R - B \neq \emptyset$ or $El \cup \{[p_i, q_i]\} \cap W \neq \emptyset$, but not both **then**

if $[p_i, q_i] \in R - B$ **then**

$$[p_{i+1}, q_{i+1}] = \begin{cases} \left[\frac{p_i + (p_i + q_i - 1)}{2}, \frac{q_i + 1}{2}\right] & p_i + q_i > 1 \\ \left[\frac{p_i}{2}, \frac{q_i + (p_i + q_i)}{2}\right] & p_i + q_i \leq 1 \end{cases}$$

if $[p_i, q_i] \in W$ **then**

$$[p_{i+1}, q_{i+1}] = \left[\frac{2p_i + q_i}{2}, \frac{p_i + 2q_i}{2}\right]$$

if $[p_i, q_i] \in R$ **then**

if $El \cap R - B \neq \emptyset$ **then**

$$[p_{i+1}, q_{i+1}] = \sigma^{-1}\left(\sigma_1([p_i, q_i])), \frac{\sigma_2([p_i, q_i]) + \alpha'}{2}\right)$$

where

$$\alpha' = \begin{cases} \frac{q'}{2(1 - \sigma_1([p_i, q_i])} & q' \leq 1 - \sigma_1([p_i, q_i]) \\ \frac{q' - 1}{2\sigma_1([p_i, q_i])} + 1 & q' > 1 - \sigma_1([p_i, q_i]) \end{cases}$$

with $[p', q'] = \arg\max_{[p,q] \in El \cap R - B} q$.

if $El \cap W \neq \emptyset$ **then**

$$[p_{i+1}, q_{i+1}] = \sigma^{-1}\left(\sigma_1([p_i, q_i])), \frac{\sigma_2([p_i, q_i]) + \alpha''}{2}\right)$$

where

$$\alpha'' = \begin{cases} \frac{p''}{2\sigma_1([p_i, q_i])} & p'' \leq \sigma_1([p_i, q_i]) \\ \frac{p'' + 1 - 2\sigma_1([p_i, q_i])}{2(1 - \sigma_1([p_i, q_i]))} & p'' > \sigma_1([p_i, q_i]) \end{cases}$$

with $[p'', q''] = \arg\max_{[p,q] \in El \cap W} p$.

if $[p_i, q_i] \in B$ **then**

if $El \cap R - B \neq \emptyset$ **then**

$$[p_{i+1}, q_{i+1}] = \sigma^{-1}\left(\sigma_1([p_i, q_i])), \frac{\sigma_2([p_i, q_i]) + \alpha''}{2}\right)$$

where

$$\alpha'' = \begin{cases} \frac{p''}{2\sigma_1([p_i, q_i])} & p'' \leq \sigma_1([p_i, q_i]) \\ \frac{p'' + 1 - 2\sigma_1([p_i, q_i])}{2(1 - \sigma_1([p_i, q_i]))} & p'' > \sigma_1([p_i, q_i]) \end{cases}$$

with $[p'', q''] = \arg\min_{[p,q] \in El \cap R - B} p$.

if $El \cap W \neq \emptyset$ **then**

$$[p_{i+1}, q_{i+1}] = \sigma^{-1}\left(\sigma_1([p_i, q_i])), \frac{\sigma_2([p_i, q_i]) + \alpha'}{2}\right)$$

where

$$\alpha' = \begin{cases} \frac{q'}{2(1 - \sigma_1([p_i, q_i])} & q' \leq 1 - \sigma_1([p_i, q_i]) \\ \frac{q' - 1}{2\sigma_1([p_i, q_i])} + 1 & q' > 1 - \sigma_1([p_i, q_i]) \end{cases}$$

with $[p', q'] = \arg\min_{[p,q] \in El \cap W} q$.

end

Figure 12: Determination of Next Binary Choice: Part 2

Case 3

if $(El \cup \{[p_i, q_i]\}) \cap R - B \neq \emptyset$ and $(El \cup \{[p_i, q_i]\}) \cap W \neq \emptyset$ **then**

if $[p_i, q_i] \in R - B$ **then**

$$[p_{i+1}, q_{i+1}] = \left[\frac{p_i + p''}{2}, \frac{q_i + q''}{2}\right]$$

with $[p'', q''] = \arg\max_{[p,q] \in El \cap W} p$.

if $[p_i, q_i] \in W$ **then**

$$[p_{i+1}, q_{i+1}] = \left[\frac{p_i + p'}{2}, \frac{q_i + q'}{2}\right]$$

with $[p', q'] = \arg\min_{[p,q] \in El \cap RB} p$.

if $[p_i, q_i] \in B$ **then**

$$[p_{i+1}, q_{i+1}] = \sigma^{-1}\left(\sigma_1([p_i, q_i]), \frac{\sigma_2([p_i, q_i]) + \max(\alpha', \alpha'')}{2}\right)$$

where

$$\alpha' = \begin{cases} \frac{q'}{2(1-\sigma_1([p_i,q_i]))} & q' \leq 1 - \sigma_1([p_i, q_i]) \\ \frac{q'-1}{2\sigma_1([p_i,q_i])} + 1 & q' > 1 - \sigma_1([p_i, q_i]) \end{cases}$$

with $[p', q'] = \arg\max_{[p,q] \in El \cap R-B} q$ and

$$\alpha'' = \begin{cases} \frac{p''}{2\sigma_1([p_i,q_i])} & p'' \leq \sigma_1([p_i, q_i]) \\ \frac{p''+1-2\sigma_1([p_i,q_i])}{2(1-\sigma_1([p_i,q_i]))} & p'' > \sigma_1([p_i, q_i]) \end{cases}$$

with $[p'', q''] = \arg\min_{[p,q] \in El \cap W} q$.

if $[p_i, q_i] \in R$ **then**

$$[p_{i+1}, q_{i+1}] = \sigma^{-1}\left(\sigma_1([p_i, q_i]), \frac{\sigma_2([p_i, q_i]) + min(\alpha', \alpha'')}{2}\right)$$

where

$$\alpha' = \begin{cases} \frac{q''}{2(1-\sigma_1([p_i,q_i]))} & q' \leq 1 - \sigma_1([p_i, q_i]) \\ \frac{q''-1}{2\sigma_1([p_i,q_i])} + 1 & q' > 1 - \sigma_1([p_i, q_i]) \end{cases}$$

with $[p'', q''] = \arg\min_{[p,q] \in El \cap W} q$ and

$$\alpha'' = \begin{cases} \frac{p'}{2\sigma_1([p_i,q_i])} & p' \leq \sigma_1([p_i, q_i]) \\ \frac{p'+1-2\sigma_1([p_i,q_i])}{2(1-\sigma_1([p_i,q_i]))} & p'' > \sigma_1([p_i, q_i]) \end{cases}$$

with $[p', q'] = \arg\max_{[p,q] \in El \cap R-B} q$.

end

Figure 13: Determination of Next Binary Choice: Part 3

A.3. Robustness of the elicitation method

As stated in Section 2, the proposed elicitation method has three novel elements. The first is the notion of MPI, and the observation that they yield the probability intervals generated by the subjects' set of priors. The second is the incentivisation mechanism, based on the 2D choice list set out in Section 2.5. As for elicitation of subjective probabilities (e.g. choice-list methods for eliciting MPs), this is already sufficient to provide an elicitation mechanism for subjects' sets of priors. However, the proposal also includes a chained binary-choice procedure, in the style of the 'bisection' or 'staircase' method for MPs or certainty equivalents, to aid the subject find the MPI (Section 3.4). We now discuss to what extent the proposed elements apply beyond the typical α-maxmin EU representation with $\alpha > \frac{1}{2}$ on which we have focused in Section 2. Whilst we concentrate below on extensions to models of the more general form (11), note that the method also applies under other multiple-prior decision models, most notably multiple-prior minimax (expected) regret.[16] We also further analyse the incentive compatibility properties of the 2D choice list incentivisation mechanism.

A.3.1. Matching Probability Intervals

Under the general preferences of the form (11), the equations (5) for the MPI can be rewritten in the obvious way.[17] Clearly, the notion of MPI is well defined, and the subjective probability interval is an MPI. The form of W can however affect the uniqueness of the MPI. More precisely, it is guaranteed to be unique whenever there is a unique solution to the equations, and this only occurs if W satisfies the following 'single-crossing property': every pair of red-and-blue indifference curves in Figure 1 cross at most once.[18] Whether this is the case, and how often it is not, will depend on the functional form of W. We thus consider what form of uniqueness holds for reasonable W.

For instance, the MPI is clearly unique when W is linear and non-symmetric[19]—and hence for α-maxmin EU whenever $\alpha \neq \frac{1}{2}$. A more general interesting case is when W incorporates probability weighting, e.g. is of the form $W(x, y) = \alpha w(x) + (1-\alpha) w(y)$ for a weighting

[16] This model evaluates the choice of act f from a menu M according to $-\max_{p\in\mathcal{C}} \mathbb{E}_p \left(\max_{g\in M} u(g(s)) - u(f(s))\right)$, where $\mathbb{E}_p$ is the expectation with respect to probability measure p and $\mathcal{C}$ is the set of priors (e.g. Berger, 1985; Stoye, 2011). It is straightforward to show that for the choices used by our method—namely binary choices between bets on independent events, in the sense that the joint (multi-prior) distribution over the pair of relevant events is a 'type-1 product' (Walley, 1991, Sect. 9.3.5) of the multiple priors beliefs about each—preferences under this rule correspond to preferences under maxmin-EU (i.e. (1) with $\alpha = 1$) with the same set of priors.

[17] Explicitly:

$$W(\underline{p}, \overline{p}) = W(\underline{p}(E), \overline{p}(E)) \tag{15}$$

$$W(1-\overline{p}, 1-\underline{p}) = W(1-\overline{p}(E), 1-\underline{p}(E)) \tag{16}$$

[18] Technically, for every $A, B \in \mathbb{R}$, $|\{[x, y] \in \mathcal{I} : W(x, y) = A, W(1-y, 1-x) = B\}| \leq 1$.

[19] I.e. it is not the case that $W(x, y) = W(y, x)$ for all x, y.

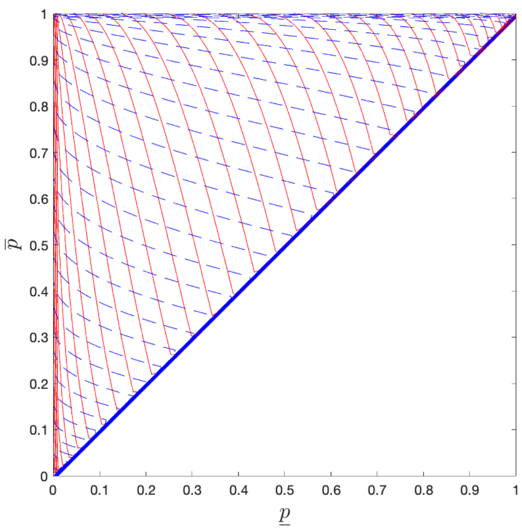


Figure 14: Indifference curves in probability interval space $\mathcal{I}$ under (11) with $W(x,y) = \alpha w(x) + (1-\alpha)w(y)$.

Red lines: indifference curves for IL $(z, [p,q], 0)$: i.e. curves of the form $\alpha w(x) + (1-\alpha)w(y) = C$.

Blue lines: indifference curves for IL $(0, [p,q], z)$: i.e. curves of the form $\alpha w(1-y) + (1-\alpha)w(1-x) = D$.

Parametrisation: Prelec weighting function $w(x) = \left(e^{-(-ln(x))^{\alpha}}\right)^{\beta}$ with $\alpha = 0.54$ and $\beta = 0.85$ (Abdellaoui et al., 2011); $\alpha = 0.8$.

function w. Note that this form can incorporate findings on probability weighting for (two-outcome) lotteries, via w. For such W, if w takes the quasi-linear form often used in literature (Chateauneuf et al., 2007; Wakker, 2010), then MPIs can be shown to remain unique (by a similar reasoning to that for the non-weighted case). Moreover, even for non-linear weighting functions, calculation of relevant cases suggests that MPIs are typically unique. As an example, Figure 14 plots red and blue indifference curves for the specified form of W with w being the popular Prelec weighting function with the parameters found by Abdellaoui et al. (2011) for a Paris temperature source—i.e. one that is similar to the source we used in EXP 1—and an α of 0.8—i.e. close to the value we found for α (Section 4.5). Clearly, red and blue indifference curves typically only cross (at most) once, as required for uniqueness of MPI. Even in the cases where there are multiple MPIs, there will be at most two, with one close to the horizontal or vertical boundary.

In summary, even for reasonable extensions beyond α-maxmin EU, MPIs are well-defined, and the subject's probability interval is always a MPI. Moreover, there is reason to believe that uniqueness continues to hold largely, and where it does not, there is at most one other possible candidate MPI. Note that even in cases of non-uniqueness, the analysis of the 2D choice list incentivisation mechanism is unaffected, and every MPI remains a weakly dominant strategy. So it will yield a candidate probability interval.

A.3.2. 2D Choice List

The 2D choice list set out in Section 2.5 is incentive compatible under the α-maxmin EU model (1) whenever subjects treat the two branches of the choice list in isolation from each other. We first consider the consequences of violation of this isolation assumption, before turning to robustness to generalisation of the decision model.

Robustness to violations of isolation Suppose that the isolation assumption discussed in Section 5 does not hold, and the subject reasons strategically across the two branches of the 2D choice list. Then the choice of MPI is conceptualised as the choice of a (second-order) lottery assigning a probability to playing a bet for or against E or to playing specific ILs according to the mechanism. Assuming the α-maximin EU model (1) at both levels, the subject evaluates each such second-order lottery using the expectation over the values of the bets and ILs. Let $[\underline{p}(E), \overline{p}(E)] = [\underline{p}, \overline{p}]$. For any reported point $[\underline{q}, \overline{q}]$ in this task, by the incentive mechanism defined in Section 2.5:

- the probability of receiving the bet on E is $\frac{\underline{q}}{\overline{q}+1-\underline{q}}$
- the probability of receiving the IL on red is $\frac{\overline{q}-\underline{q}}{\overline{q}+1-\underline{q}}$
- the probability of receiving the bet on E^c is $\frac{1-\overline{q}}{\overline{q}+1-\underline{q}}$
- the probability of receiving the IL on red is $\frac{\overline{q}-\underline{q}}{\overline{q}+1-\underline{q}}$

Using these and the evaluations of the bets and the ILs according to (1) and (2) (with $[\underline{p}, \overline{p}]$), one obtains the following form for the utility of reporting $[\underline{q}, \overline{q}]$ when the true beliefs are $[\underline{p}, \overline{p}]$:

$$\frac{(1-\overline{q})\left(\alpha(1-\overline{p}) - (\alpha-1)\left(1-\underline{p}\right)\right)}{\overline{q}-\underline{q}+1} + \frac{\left(\overline{q}-\underline{q}\right)\left(\frac{\underline{q}}{2} + \frac{\alpha(1-\overline{q})}{2} - \frac{(\alpha-1)\left(1-\underline{q}\right)}{2} - \frac{1}{2}\right)}{\overline{q}-\underline{q}+1}$$
$$+\frac{\underline{q}\left(\alpha\,\underline{p} - \overline{p}\,(\alpha-1)\right)}{\overline{q}-\underline{q}+1} + \frac{\left(\overline{q}-\underline{q}\right)\left(\frac{\overline{q}}{2} + \frac{\alpha\,\underline{q}}{2} - \frac{\overline{q}\,(\alpha-1)}{2}\right)}{\overline{q}-\underline{q}+1}$$

Finding the optimum numerically for a grid of values of $\underline{p}, \overline{p}, \alpha \in [0,1]$ using Matlab, we find that, for every $(\underline{p}, \overline{p}, \alpha)$ (with $\overline{p} \geq \underline{p}$) except for $\underline{p} = 0,\ \overline{p} = 1,\ \alpha = 0$, and those with $\underline{p} = 0.5,\ \alpha = 1$ or $\overline{p} = 0.5,\ \alpha = 1$, the maximum is attained at one or several of the 'vertices' of the triangle in Figure 1, i.e. $[0,0], [0,1], [1,1]$. For $\underline{p} = 0,\ \overline{p} = 1,\ \alpha = 0$ and $\underline{p} = 0.5\ , \alpha = 1$ or $\overline{p} = 0.5\ , \alpha = 1$ with $\underline{p} \neq \overline{p}$, the maximum is attained at all points on one of the boundaries of the triangle, i.e. $\{[0,y] : y \in [0,1]\}, \{[x,1] : x \in [0,1]\}, \{[x,y] : x \in [0,1], y = x\}$. When $\underline{p} = 0.5, \overline{p} = 0.5, \alpha = 1$, the utility above is constant, so all points maximise it.

It follows that, for any subject with $\alpha \in (0,1)$ who violates the isolation assumption and responds to the choice list strategically, every response will be at a vertex of the space $\mathcal{I}$. Our elicitation of α suggests that the vast majority of subjects have α in this range. Even for subjects with $\alpha = 0$ or 1 reasoning strategically, they will have more than one response in the interior of $\mathcal{I}$ if $\alpha = 1$ and they assign precise probability of 0.5 to several elicited events—which, in our experiment involving nested events, would correspond to a peculiar (bimodal) distribution across the variable of interest (temperature, marks). As is clear from Table 9 (Appendix B.1), no subjects give vertex responses for all elicited events, with only one subject (across both experiments) giving vertex responses for over half of the elicited events. Moreover, the vast majority of subjects (73 out of 80 in EXP 1; 51 out of 52 in EXP 2) gave more than one response in the interior of $\mathcal{I}$. The data thus clearly suggests that strategic reasoning is very limited in our sample.

Robustness to generalizations of the decision model As suggested in Section 2.5, the incentivization mechanism implemented by the 2D choice list relies solely on the weak Lower Stochastic Dominance property (Definition 1). Formulated in terms of Eq. (11), this is just the assumption that W is strictly increasing in the first coordinate—or, in terms of preferences, decision makers are sensitive to the lower winning probability. The only reasonable model in the family of form (11) violating this property is the maxmax EU—α-maxmin EU with $\alpha = 0$. Since there is basically no evidence for a significant number of subjects with such preferences, the incentive compatibility of the 2D choice list discussed in Section 2.5 generalizes widely.

A.3.3. Binary-choice procedure

The binary-choice procedure (Section 2.6 and Appendix A.2) is based on the division of $\mathcal{I}$ into regions, displayed in Table 1, and Proposition 2, in particular part a. dictating 'where' the MPI is relative to points in two of the regions (the W and R-B regions). For decision makers represented according to the α-maxmin EU model (1), Proposition 2 a. only holds if $\alpha > \frac{1}{2}$.[20] When $\alpha < \frac{1}{2}$, the opposite of the statement in the Proposition holds: the MPI is North-West of the elicited point (on Figure 1) not when the latter is in R-B, but when it is in W (and similarly for South-East). So the algorithm applied to such decision makers would 'move' in the wrong direction: instead of looking South-East for the MPI after finding a point in R-B, it would look North-West, for instance. Note that, even if the algorithm does not work properly for such decision makers, the 2D choice list incentivisation mechanism is still valid, and hence they would, in principle, correct any issues at the 2D choice list confirmation stage. To gain some insight into the extent of procedure misfunction due to $\alpha < \frac{1}{2}$, we can look at the evidence on the value of α for our subjects, as well as some statistics on the functioning of the procedure.

We find little evidence for widespread $\alpha < \frac{1}{2}$ among our subjects. First of all, the elicitation of α reported in Section 4.5 finds median and 25 percentile values significantly above $\frac{1}{2}$ (Table 24), indicating that less than 25% of subjects have $\alpha < \frac{1}{2}$. Moreover, under the α-maxmin EU model, the sum of the MP of an event and that of its complement is less than (respectively, greater than) one precisely when $\alpha > \frac{1}{2}$ (resp. $\alpha < \frac{1}{2}$; see Appendix B.4), indicating that we

[20] It also holds under the probability weighting specification of (11) mentioned in Section A.3.1 when $\alpha > \frac{1}{2}$.

can use our matching probability data to check for the sign of $\alpha - \frac{1}{2}$. Table 25 (Appendix B.4) displays the descriptive statistics on this sum for the Paris treatment where MPs were elicited, confirming again that $\alpha > \frac{1}{2}$ for over 75% of subjects.

As concerns its functioning, since the procedure 'moves' in the wrong direction for subjects with $\alpha < \frac{1}{2}$, no such subjects will pass through both points in W and points in R-B during the procedure. However, 383 applications of the procedure out of 704 in EXP 1 passed through points in W and R-B (300 out of 606 in EXP 2). Whilst there were nevertheless applications of the procedure which passed through points in R-B but not W (152 in EXP 1, 77 in EXP 2) and in W but not R-B (114 in EXP 1, 105 in EXP 2), these would be expected if the procedure functioned correctly and the probability intervals were large (respectively small). The evidence thus does not support a hypothesis involving misfunctioning of the procedure over explanations, such as this, relating to proper functioning and the character of the elicited intervals.

B. Supplementary Statistics

B.1. Descriptive Statistics

Tables 5–8 report the basic descriptive statistics on the upper and lower elicited probabilities after the 'confirmation' 2D choice list, and before the confirmation screen but after the binary-choice procedure, respectively.

Elicited points on a vertex Table 9 reports counts of the number of subjects with a given number of elicited points at the vertex of the space $\mathcal{I}$ of partially known urns (and corresponding probability intervals) in Figure 1.

Monotonicity Tables 10a and 10b report the descriptive statistics for the individuallevel Kendall τ_b, calculated over the events in each source.

B.1.1. Imprecision

Table 11 presents the descriptive statistics for the Imprecision Index, whereas Table 12 displays counts of the number of subjects with various numbers of precise elicited points, as well as differences before the 2D choice list confirmation stage of the experiment as opposed to after. Table 13 presents the results of ANOVAs of the imprecision concerning an event against the event, for each source, where the null hypothesis is that imprecision is invariant across events.

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
-2	80.0	0.29	0.17	-0.01	0.15	0.29	0.40	0.7
2	80.0	0.38	0.22	0.00	0.20	0.35	0.50	1.0
5	80.0	0.48	0.23	0.00	0.35	0.46	0.66	1.0
8	80.0	0.57	0.24	0.05	0.42	0.59	0.75	1.0

(a) Lower probabilities Paris

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
-2	80.0	0.55	0.21	0.09	0.40	0.55	0.67	1.0
2	80.0	0.65	0.19	0.23	0.51	0.65	0.80	1.0
5	80.0	0.74	0.17	0.25	0.62	0.76	0.88	1.0
8	80.0	0.82	0.14	0.50	0.75	0.85	0.94	1.0

(b) Upper probabilities Paris

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
15	80.0	0.31	0.22	0.00	0.14	0.26	0.45	0.95
18	80.0	0.35	0.26	0.00	0.14	0.32	0.47	1.00
20	80.0	0.41	0.27	-0.01	0.20	0.40	0.61	1.00
22	80.0	0.43	0.26	-0.01	0.20	0.39	0.61	1.00

(c) Lower probabilities Sydney

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
15	80.0	0.58	0.27	0.01	0.37	0.56	0.84	0.99
18	80.0	0.66	0.24	0.03	0.50	0.69	0.88	1.00
20	80.0	0.71	0.23	0.01	0.60	0.76	0.89	1.00
22	80.0	0.73	0.23	0.00	0.58	0.80	0.92	1.00

(d) Upper probabilities Sydney

Table 5: Descriptive Statistics: Elicited lower and upper probabilities after 2D choice list, EXP 1

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
7	52.0	0.05	0.07	-0.01	0.00	0.04	0.08	0.35
10	52.0	0.15	0.12	0.00	0.06	0.12	0.19	0.50
12	52.0	0.24	0.15	0.00	0.14	0.20	0.31	0.63
15	52.0	0.40	0.19	0.08	0.26	0.38	0.55	0.73
17	52.0	0.60	0.16	0.18	0.54	0.64	0.71	0.86

(a) Lower probabilities Maths

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
7	52.0	0.12	0.14	0.00	0.02	0.08	0.16	0.60
10	52.0	0.23	0.17	0.00	0.10	0.20	0.30	0.65
12	52.0	0.35	0.18	0.04	0.22	0.32	0.48	0.72
15	52.0	0.54	0.20	0.08	0.40	0.52	0.70	0.87
17	52.0	0.75	0.15	0.22	0.65	0.78	0.86	1.01

(b) Upper probabilities Maths

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
7	52.0	0.12	0.08	0.01	0.06	0.11	0.17	0.36
10	52.0	0.22	0.12	0.02	0.14	0.20	0.29	0.50
12	52.0	0.33	0.13	0.14	0.22	0.32	0.40	0.60
15	52.0	0.54	0.14	0.19	0.46	0.56	0.65	0.83
17	52.0	0.71	0.13	0.25	0.65	0.74	0.83	0.90

(c) Lower probabilities Contraction

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
7	52.0	0.22	0.14	0.02	0.11	0.20	0.31	0.56
10	52.0	0.37	0.14	0.06	0.30	0.34	0.46	0.65
12	52.0	0.51	0.14	0.20	0.40	0.50	0.64	0.74
15	52.0	0.74	0.11	0.40	0.67	0.77	0.82	0.90
17	52.0	0.86	0.07	0.60	0.82	0.86	0.91	1.00

(d) Upper probabilities Contraction

Table 6: Descriptive Statistics: Elicited lower and upper probabilities after 2D choice list, EXP 2

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
-2	80.0	0.28	0.15	0.0	0.15	0.30	0.39	0.64
2	80.0	0.36	0.21	0.0	0.20.	0.33	0.45	1.00
5	80.0	0.48	0.23	0.0	0.35	0.46	0.67	1.00
8	80.0	0.54	0.25	0.05	0.35	0.55	0.74	1.00

(a) Lower probabilities Paris

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
-2	80.0	0.55	0.21	0.0	0.39	0.56	0.68	0.99
2	80.0	0.64	0.18	0.23	0.52	0.67	0.78	1.00
5	80.0	0.75	0.17	0.25	0.61	0.79	0.89	1.00
8	80.0	0.81	0.14	0.50	0.73	0.85	0.92	1.00

(b) Upper probabilities Paris

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
15	80.0	0.28	0.21	0.0	0.12	0.26	0.38	0.95
18	80.0	0.33	0.27	0.0	0.11	0.30	0.45	1.00
20	80.0	0.43	0.28	0.0	0.19	0.42	0.61	1.00
22	80.0	0.42	0.26	0.0	0.21	0.35	0.61	1.00

(c) Lower probabilities Sydney

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
15	80.0	0.60	0.27	0.00	0.42	0.61	0.83	0.99
18	80.0	0.67	0.25	0.03	0.50	0.66	0.89	1.00
20	80.0	0.73	0.23	0.00	0.60	0.79	0.89	1.00
22	80.0	0.74	0.22	0.00	0.62	0.80	0.91	1.00

(d) Upper probabilities Sydney

Table 7: Descriptive Statistics: Elicited lower and upper probabilities after binary-choice procedure and before 2D choice list, EXP 1

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
7	52.0	0.11	0.16	0.00	0.01	0.06	0.14	1.00
10	52.0	0.18	0.14	0.00	0.08	0.14	0.30	0.50
12	52.0	0.25	0.15	0.00	0.15	0.22	0.32	0.60
15	52.0	0.38	0.22	0.04	0.19	0.33	0.53	1.00
17	52.0	0.54	0.18	0.07	0.43	0.60	0.66	0.86

(a) Lower probabilities Maths

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
7	52.0	0.18	0.23	0.00	0.03	0.08	0.28	1.00
10	52.0	0.27	0.22	0.00	0.10	0.23	0.44	0.70
12	52.0	0.40	0.23	0.00	0.22	0.38	0.61	0.78
15	52.0	0.56	0.21	0.08	0.39	0.58	0.71	1.00
17	52.0	0.73	0.16	0.22	0.62	0.76	0.85	0.99

(b) Upper probabilities Maths

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
7	52.0	0.15	0.09	0.01	0.08	0.15	0.20	0.40
10	52.0	0.26	0.12	0.00	0.18	0.23	0.35	0.48
12	52.0	0.32	0.16	0.04	0.21	0.30	0.40	0.71
15	52.0	0.53	0.19	0.16	0.40	0.55	0.66	1.00
17	52.0	0.67	0.20	0.07	0.62	0.68	0.83	1.00

(c) Lower probabilities Contraction

Event E_t for $t =$	count	mean	std	min	25%	50%	75%	max
7	52.0	0.30	0.23	0.03	0.12	0.24	0.41	0.85
10	52.0	0.41	0.18	0.00	0.31	0.35	0.56	0.86
12	52.0	0.54	0.16	0.20	0.44	0.55	0.65	0.87
15	52.0	0.74	0.14	0.37	0.66	0.74	0.84	1.00
17	52.0	0.85	0.09	0.50	0.82	0.86	0.91	1.00

(d) Upper probabilities Contraction

Table 8: Descriptive Statistics: Elicited lower and upper probabilities after binary-choice procedure and before 2D choice list, EXP 2

Point	Paris					Sydney				
	# subjects					# subjects				
	0	1	2	3	4	0	1	2	3	4
$[0,0]$	80	0	0	0	0	80	0	0	0	0
$[0,1]$	80	0	0	0	0	80	0	0	0	0
$[1,1]$	79	0	0	1	0	78	1	0	1	0
$[0,0]$, $[0,1]$ or $[1,1]$	79	0	0	1	0	78	1	0	1	0

(a) EXP 1

Point	Maths						Contraction					
	# subjects						# subjects					
	0	1	2	3	4	5	0	1	2	3	4	5
$[0,0]$	46	5	1	0	0	0	52	0	0	0	0	0
$[0,1]$	52	0	0	0	0	0	52	0	0	0	0	0
$[1,1]$	52	0	0	0	0	0	52	0	0	0	0	0
$[0,0]$, $[0,1]$ or $[1,1]$	46	5	1	0	0	0	52	0	0	0	0	0

(b) EXP 2

Table 9: For each type of point, the table indicates the number of subjects with the specified number of elicited points being of this type.

	MP Lower	MP Upper	Paris Lower	Paris Upper	Sydney Lower	Sydney Upper
count	74	78	79	78	78	78
mean	0.62	0.66	0.56	0.56	0.27	0.41
std	0.46	0.38	0.45	0.47	0.59	0.50
min	-0.91	-0.91	-0.91	-0.91	-1.00	-1.00
25%	0.55	0.55	0.33	0.33	-0.14	0.00
50%	0.71	0.69	0.67	0.67	0.33	0.55
75%	0.91	0.91	1.00	1.00	0.67	0.91
max	1.00	1.00	1.00	1.00	1.00	1.00

(a) EXP 1

	Contraction Lower	Contraction Upper	Maths Lower	Maths Upper
count	52	52	52	52
mean	0.99	0.99	0.98	1.00
std	0.02	0.03	0.07	0.01
min	0.95	0.80	0.53	0.95
25%	1.00	1.00	1.00	1.00
50%	1.00	1.00	1.00	1.00
75%	1.00	1.00	1.00	1.00
max	1.00	1.00	1.00	1.00

(b) EXP 2

Table 10: Individual-level Kendall τ_b descriptive statistics

Note τ_b is not defined for some subjects in EXP 1 (because of too many ties), and they were dropped.

	EXP 1		EXP 2	
	Paris	Sydney	Maths	Contraction
count	80	80	52	52
mean	0.25	0.29	0.13	0.19
std	0.17	0.20	0.09	0.11
min	0	0	0	0.01
25%	0.10	0.10	0.06	0.11
50%	0.23	0.28	0.11	0.17
75%	0.35	0.43	0.19	0.23
max	0.82	0.75	0.45	0.53

Table 11: Imprecision Index (Eq. (8)) descriptive statistics; EXP 1 and EXP 2.

	EXP 1				EXP 2			
# Precise events	Paris		Sydney		Maths		Contraction	
	2D C.L.	B-C Proc	2D C.L.	B-C Proc	2D C.L.	B-C Proc	2D C.L.	B-C Proc
0	51	48	48	44	20	12	31	19
1	14	14	18	23	14	14	12	20
2	7	11	8	8	12	10	5	3
3	6	0	2	4	3	10	3	4
4	2	1	4	1	0	2	1	7
5	-	-	-	-	3	4	0	2
Total	80	80	80	80	52	52	52	52

Table 12: Number of subjects with given number of precise events, per source. Data given after the 2D choice list confirmation screen (2D C.L.) and after the binary-choice procedure but before the confirmation screen (B-C Proc).

	Source	F	p-value
EXP 1	Paris	0.1048	0.957
	Sydney	0.4769	0.698
EXP 2	Contraction	4.0352	0.003
	Maths	5.863	0.00015

Table 13: One-sided ANOVAs of the imprecision related to an event (dependent variable) on the event (factor), for each source. (H_0: the imprecision is identical across all events in the source.)

B.2. Bayesian estimation

B.2.1. Statistical approach

Estimation of upper and lower CDFs in EXP 1 and EXP 2 Recall that T denotes the space of possible values of the variables of interest (minimum temperatures in EXP 1, grades in EXP 2). For each source, we estimate general models of the form:

$$\begin{cases} \underline{p}(E) = \underline{f}(E) + \underline{\epsilon} \\ \overline{p}(E) = \overline{f}(E) + \overline{\epsilon} \end{cases} \tag{17}$$

where $\underline{p}(E)$ (resp. $\overline{p}(E)$) are the elicited lower (resp. upper) probabilities of the cumulative event E (Section 2.2), $\underline{f}$ and $\overline{f}$ are CDFs over T from specified two-parameter families, with parameters $\underline{a}, \underline{b}$ (resp. $\overline{a}, \overline{b}$), and $\underline{\epsilon}$ and $\overline{\epsilon}$ are zero-mean normal distributions with variance $\underline{\sigma}^2$ and $\overline{\sigma}^2$ respectively.

For each equation, the parameter space is $\Theta \subseteq \mathbb{R}^3$, with a typical point $(\underline{a}, \underline{b}, \underline{\sigma})$ (resp. $(\overline{a}, \overline{b}, \overline{\sigma})$) specifying an $\underline{f}$ (resp. $\overline{f}$) and the variance of the relevant error term. We specify the following priors over the hyperparameters : a, b, σ are realisations from $A \sim N(\mu_a, \sigma_a^2)$, $B \sim N(\mu_b, \sigma_b^2)$ and $\Sigma = \sigma_\sigma \mid Y \mid$ with $Y \sim N(0, 1)$.

We use a MCMC-like approach to estimate the posterior distributions of these distributions through the use of the Python package PyMC3, and more specifically, the No-U-Turn Sampler algorithm (NUTS) (Hoffman and Gelman, 2014).

The likelihood of observations $x_1, ..., x_n$ pertaining to $t_1, \ldots, t_n$ (e.g. elicited lower probabilities for cumulative events $E_{t_i} = \{t \in T : t \leq t_i\}$) given the point $(a, b, \sigma) \in \Theta$ is:

$$L(a, b, \sigma | x_1, \ldots, x_n) \quad = \quad \prod_{i \in \{1, \ldots, n\}} \varphi \left(\frac{x_i - f_{(a,b)}(\{t \leq t_i\})}{\sigma} \right)$$

where $f_{(a,b)}$ is the CDF with parameters a, b and φ is the density of the normal distribution.

	Temperature (EXP 1)	Grade (EXP 2)
Family 1	Truncated Normal $\mathcal{N}(a,b)$	Truncated Normal $\mathcal{N}(a,b)$
Family 2	Beta $B(a,b)$	Beta $B(a,b)$
Support	[min of min stated temperature, max of max stated temperature]	[0,20]

Table 14: Families of distributions over T (temperature; mark)

Note the minima and maxima in the first column are taken across all subjects' responses (Section 3.3).

Hence the likelihood of hyperparameters $\mu_a, \sigma_a^2, \mu_b, \sigma_b^2, \mu_\sigma, \sigma_\sigma^2$ given observations $x_1 \dots x_n$ is :

$$
\begin{aligned}
& L(\mu_a, \sigma_a^2, \mu_b, \sigma_b^2, \mu_\sigma, \sigma_\sigma^2 | x_1, \dots, x_n) \\
= \quad & \int_{(a,b,\sigma)\in\Theta} L(a,b,\sigma|x_1,\dots,x_n) dp(a,b,\sigma|\mu_a, \sigma_a^2, \mu_b, \sigma_b^2, \mu_\sigma, \sigma_\sigma^2)
\end{aligned}
$$

$L(\mu_{\underline{a}}, \sigma_{\underline{a}}^2, \mu_{\underline{b}}, \sigma_{\underline{b}}^2, \mu_{\underline{\sigma}}, \sigma_{\underline{\sigma}}^2 | \underline{x_1}, \dots, \underline{x_n})$ and $L(\mu_{\overline{a}}, \sigma_{\overline{a}}^2, \mu_{\overline{b}}, \sigma_{\overline{b}}^2, \mu_{\overline{\sigma}}, \sigma_{\overline{\sigma}}^2 | \overline{x_1}, \dots, \overline{x_n})$ are used by the NUTS algorithm to estimate the posterior distributions of A, B and Σ, where $\underline{x_1}, \dots, \underline{x_n}, \overline{x_1}, \dots, \overline{x_n}$ are the elicited lower and upper probabilities respectively, under the parametric families for f given in Table 14.

Likelihood estimation of α in EXP 1 (Paris treatment) For the Bayesian estimation of the mixture coefficient α in the α-maxmin EU model, we supplement the general model (17) with the following equations

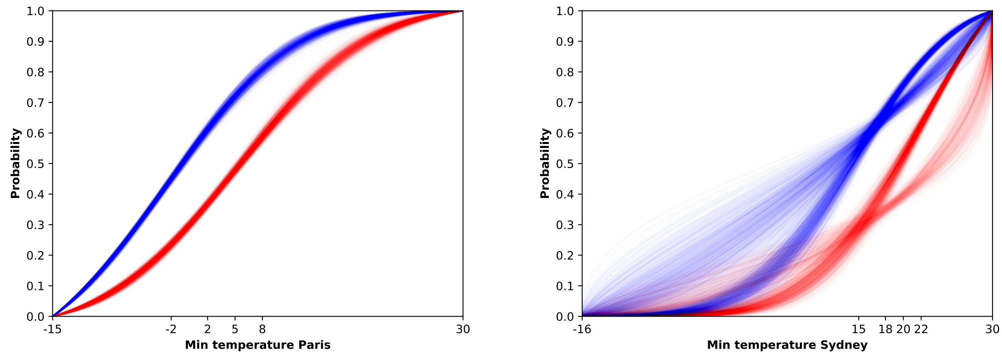
$$
\begin{cases}
MP(E) = \alpha \underline{p}(E) + (1-\alpha)\overline{p}(E) + \epsilon_{\underline{\alpha}} \\
1 - MP(E^c) = \alpha \overline{p}(E) + (1-\alpha)\underline{p}(E) + \epsilon_{\overline{\alpha}}
\end{cases}
\tag{18}
$$

which are discussed in Section 4.5. We assume that α follows a beta distribution $B(a_\alpha, b_\alpha)$, and the $\epsilon_{\overline{\alpha}}$ and $\epsilon_{\underline{\alpha}}$ are zero-mean normal distributions, with the hyperparameters independent and normally distributed, as above, with variances $\sigma_{\overline{\alpha}}^2$ and $\sigma_{\underline{\alpha}}^2$.

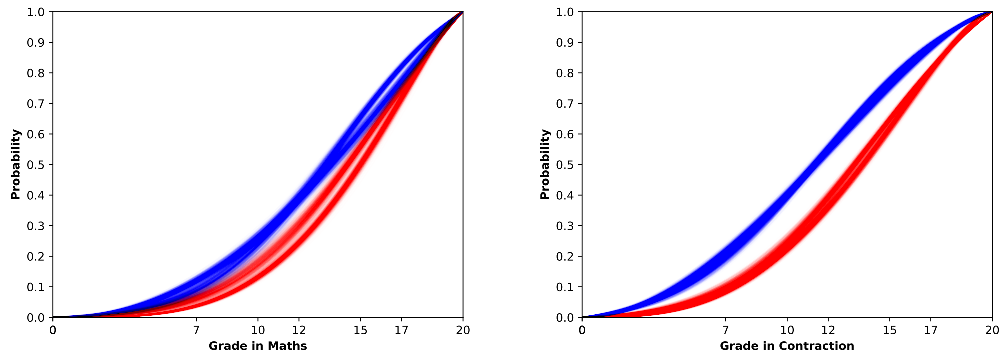
The MPs have been elicited for the Paris treatment in EXP 1. The hyperparameters concerning the upper and lower CDFs discussed above and those for α were estimated under the model composed of (17) and (18) using the NUTS algorithm, with the procedure set out above.

B.2.2. Analysis

Figure 15 displays the upper and lower distributions under the parametric families not shown in Figure 4. Tables 15-22 give statistics on the distribution over parameters under the estimated hyperparameters.



(a) EXP 1



(b) EXP 2

Figure 15: Bayesian estimation of lower and upper CDFs: plots of 1000 samples from MCMC (Truncated Normal distribution for EXP 1; Beta distribution for EXP 2)

	mean	sd	mc_error	hpd_2.5	hpd_97.5	n_eff	Rhat
$\underline{a}$	5.25	0.31	0.01	4.65	5.87	1009.11	1.0
$\overline{a}$	-2.57	0.37	0.01	-3.27	-1.87	542.99	1.0
a_α	3.42	1.62	0.06	0.53	6.38	630.90	1.0
b_α	1.80	1.04	0.05	0.10	3.80	429.61	1.0
$\underline{b}$	11.35	0.75	0.02	9.71	12.66	1214.33	1.0
$\overline{b}$	11.00	0.64	0.03	9.78	12.17	614.95	1.0
$\underline{\sigma}$	0.22	0.01	0.00	0.20	0.23	1043.67	1.0
$\overline{\sigma}$	0.18	0.01	0.00	0.17	0.19	1055.83	1.0
$\sigma_{\overline{\alpha}}$	0.21	0.01	0.00	0.20	0.23	930.43	1.0
$\sigma_{\underline{\alpha}}$	0.19	0.01	0.00	0.17	0.20	909.78	1.0
α	0.81	0.04	0.00	0.74	0.88	754.54	1.0

Table 15: Statistics for parameters under Bayesian estimation; Paris (EXP 1); Normal parametrisation

Note mc_error: Monte Carlo procedure standard error; hdp_2.5 / hdp_97.5: Highest posterior density 2.5 and 97.5 percentiles; n_eff : count of iteration in the MCMC procedure.

	mean	sd	mc_error	hpd_2.5	hpd_97.5	n_eff	Rhat
$\underline{a}$	1.43	0.23	0.01	1.04	1.91	569.30	1.0
$\underline{b}$	1.64	0.29	0.01	1.13	2.24	571.99	1.0
$\overline{a}$	1.07	0.16	0.01	0.73	1.39	541.73	1.0
$\overline{b}$	2.46	0.35	0.01	1.76	3.17	522.49	1.0
a_α	4.32	1.76	0.06	1.06	7.87	812.98	1.0
b_α	1.90	1.08	0.04	0.18	3.92	606.36	1.0
$\underline{\sigma}$	0.22	0.01	0.00	0.20	0.23	1163.74	1.0
$\overline{\sigma}$	0.18	0.01	0.00	0.17	0.19	1239.11	1.0
$\sigma_{\overline{\alpha}}$	0.21	0.01	0.00	0.20	0.23	1134.15	1.0
$\sigma_{\underline{\alpha}}$	0.19	0.01	0.00	0.17	0.20	1409.46	1.0
α	0.81	0.04	0.00	0.74	0.88	1079.99	1.0

Table 16: Statistics for parameters under Bayesian estimation; Paris (EXP 1); Beta parametrisation

Note mc_error: Monte Carlo procedure standard error; hdp_2.5 / hdp_97.5: Highest posterior density 2.5 and 97.5 percentiles; n_eff : count of iteration in the MCMC procedure.

	mean	sd	mc_error	hpd_2.5	hpd_97.5	n_eff	Rhat
$\underline{a}$	22.03	0.45	0.01	21.11	22.80	1130.92	1.0
$\overline{a}$	14.66	0.46	0.01	13.78	15.48	876.71	1.0
$\underline{b}$	9.62	0.95	0.03	7.88	11.67	1018.57	1.0
$\overline{b}$	9.04	0.85	0.02	7.42	10.78	882.98	1.0
$\underline{\sigma}$	0.26	0.01	0.00	0.23	0.28	933.24	1.0
$\overline{\sigma}$	0.25	0.01	0.00	0.23	0.27	831.75	1.0

Table 17: Statistics for parameters under Bayesian estimation; Sydney (EXP 1); Normal parametrisation

Note mc_error: Monte Carlo procedure standard error; hdp_2.5 / hdp_97.5: Highest posterior density 2.5 and 97.5 percentiles; n_eff : count of iteration in the MCMC procedure.

	mean	sd	mc_error	hpd_2.5	hpd_97.5	n_eff	Rhat
$\underline{a}$	1.12	0.55	0.04	0.31	2.21	1.91	1.57
$\overline{a}$	0.14	0.32	0.03	-0.27	0.67	1.07	4.19
$\underline{b}$	1.32	0.48	0.02	0.49	2.24	320.38	1.00
$\overline{b}$	0.94	0.24	0.01	0.50	1.37	321.66	1.00
$\underline{\sigma}$	0.25	0.01	0.00	0.23	0.27	522.03	1.00
$\overline{\sigma}$	0.24	0.01	0.00	0.22	0.26	671.56	1.00

Table 18: Statistics for parameters under Bayesian estimation; Sydney (EXP 1); Beta parametrisation

Note mc_error: Monte Carlo procedure standard error; hdp_2.5 / hdp_97.5: Highest posterior density 2.5 and 97.5 percentiles; n_eff : count of iteration in the MCMC procedure.

	mean	sd	mc_error	hpd_2.5	hpd_97.5	n_eff	Rhat
$\underline{a}$	15.88	0.17	0.01	15.57	16.22	788.64	1.0
$\overline{a}$	5.40	0.28	0.01	4.86	5.95	955.04	1.0
$\underline{b}$	13.97	0.17	0.00	13.65	14.33	1218.24	1.0
$\overline{b}$	5.03	0.25	0.01	4.58	5.53	928.02	1.0
$\underline{\sigma}$	0.14	0.01	0.00	0.13	0.16	1158.52	1.0
$\overline{\sigma}$	0.17	0.01	0.00	0.16	0.19	958.54	1.0

Table 19: Statistics for parameters under Bayesian estimation; Maths (EXP 2); Normal parametrisation

Note mc_error: Monte Carlo procedure standard error; hdp_2.5 / hdp_97.5: Highest posterior density 2.5 and 97.5 percentiles; n_eff : count of iteration in the MCMC procedure.

	mean	sd	mc_error	hpd_2.5	hpd_97.5	n_eff	Rhat
$\underline{a}$	3.76	0.09	0.0	3.58	3.93	978.44	1.0
$\overline{a}$	1.46	0.05	0.0	1.36	1.55	900.67	1.0
$\underline{b}$	2.27	0.09	0.0	2.10	2.43	1049.31	1.0
$\overline{b}$	1.22	0.05	0.0	1.12	1.32	1031.36	1.0
$\underline{\sigma}$	0.16	0.01	0.0	0.14	0.17	1311.22	1.0
$\overline{\sigma}$	0.19	0.01	0.0	0.17	0.20	1247.27	1.0

Table 20: Statistics for parameters under Bayesian estimation; Maths (EXP 2); Beta parametrisation

Note mc_error: Monte Carlo procedure standard error; hdp_2.5 / hdp_97.5: Highest posterior density 2.5 and 97.5 percentiles; n_eff : count of iteration in the MCMC procedure.

B.3. Matching Probability data and analysis of α

Table 23 provides descriptive statistics on the elicited MPs. Table 24 provide descriptive statistics on the α estimated from the raw data (from equations (9) and (10)). These equations cannot be applied to estimate α whenever the upper and lower probabilities of an event coincide, i.e. $\underline{p}(E) = \overline{p}(E)$; Table 24 performs the estimates using all events for which the equations can be applied—and hence only removes the two subjects for which the upper and lower probabilities coincide for all events (Table 12).

B.4. Elicitation-free check of $\alpha > \frac{1}{2}$

Under the α-maxmin EU model (1), it follows from Eqs. 9 and 10 that

$$MP(E) + MP(E^c) = 1 + (\overline{p}(E) - \underline{p}(E)).(1 - 2\alpha)$$

Since $\overline{p}(E) - \underline{p}(E) \geq 0$ by definition, it follows that, whenever there is imprecision, $MP(E) + MP(E^c) < 1$ if and only if $\alpha > \frac{1}{2}$.

Table 25 displays the descriptive statistics for the sum $MP(E) + MP(E^c)$ for the Paris source in EXP1. It is clear that the vast majority of subjects have a sum of MPs less than 1 indicating an α greater than 0.5. Indeed, over 80% of subjects have sum of MPs less than or equal to 1.

	mean	sd	mc_error	hpd_2.5	hpd_97.5	n_eff	Rhat
$\underline{a}$	14.17	0.13	0.00	13.93	14.42	1090.28	1.0
$\overline{a}$	5.32	0.22	0.01	4.90	5.78	1068.01	1.0
$\underline{b}$	11.61	0.13	0.00	11.37	11.87	1167.95	1.0
$\overline{b}$	5.39	0.21	0.01	4.99	5.78	1209.38	1.0
$\underline{\sigma}$	0.12	0.01	0.00	0.11	0.13	1501.55	1.0
$\overline{\sigma}$	0.12	0.01	0.00	0.11	0.13	1144.85	1.0

Table 21: Statistics for parameters under Bayesian estimation; Contraction (EXP 2); Normal parametrisation

Note mc_error: Monte Carlo procedure standard error; hdp_2.5 / hdp_97.5: Highest posterior density 2.5 and 97.5 percentiles; n_eff : count of iteration in the MCMC procedure.

	mean	sd	mc_error	hpd_2.5	hpd_97.5	n_eff	Rhat
$\underline{a}$	3.06	0.09	0.0	2.90	3.23	888.13	1.00
$\overline{a}$	1.57	0.05	0.0	1.48	1.67	937.24	1.00
$\underline{b}$	1.96	0.07	0.0	1.82	2.11	478.84	1.01
$\overline{b}$	1.56	0.06	0.0	1.45	1.66	460.19	1.00
$\underline{\sigma}$	0.14	0.01	0.0	0.13	0.15	966.76	1.00
$\overline{\sigma}$	0.13	0.01	0.0	0.12	0.15	894.44	1.00

Table 22: Statistics for parameters under Bayesian estimation; Contraction (EXP 2); Beta parametrisation

Note mc_error: Monte Carlo procedure standard error; hdp_2.5 / hdp_97.5: Highest posterior density 2.5 and 97.5 percentiles; n_eff : count of iteration in the MCMC procedure.

$MP(E_t)$ Event $t =$	count	mean	std	min	25%	50%	75%	max
-2	80	0.35	0.21	0.02	0.17	0.37	0.47	1.00
2	80	0.44	0.20	0.02	0.27	0.47	0.57	0.97
5	80	0.54	0.23	0.02	0.37	0.55	0.68	0.97
8	80	0.60	0.21	0.17	0.47	0.57	0.76	0.97

$1 - MP(E_t^c)$ Event $t =$	count	mean	std	min	25%	50%	75%	max
-2	80	0.50	0.19	0.03	0.38	0.48	0.63	0.98
2	80	0.59	0.19	0.23	0.48	0.57	0.74	0.98
5	80	0.71	0.20	0.23	0.53	0.73	0.92	0.98
8	80	0.77	0.17	0.43	0.63	0.80	0.93	0.98

Table 23: Descriptive statistics for $MP(E_i)$ and $1 - MP(E_i^c)$ in Paris treatment, EXP 1

	α
count	78
mean	0.97
std	0.66
min	-0.32
25%	0.62
50%	0.80
75%	1.17
max	3.84

Table 24: Descriptive statistics for α, estimated from raw data according to Eqns (9) and (10). Estimation conducted across all subjects such that, for any least one event E, $\underline{p}(E) \neq \overline{p}(E)$.

$MP(E_t) + MP(E_t^c)$ Event $t =$	count	mean	std	min	25%	50%	75%	max
-2	80	0.84	0.20	0.29	0.71	0.89	0.98	1.31
2	80	0.85	0.20	0.29	0.73	0.89	0.99	1.34
5	80	0.83	0.22	0.24	0.69	0.89	0.99	1.29
8	80	0.83	0.18	0.39	0.69	0.89	0.99	1.26

Table 25: Empirical distribution of average $MP(E) + MP(E^c)$ across all events for which MPs were elicited (those concerning Paris temperature in EXP1).

C. Experimental design and displays

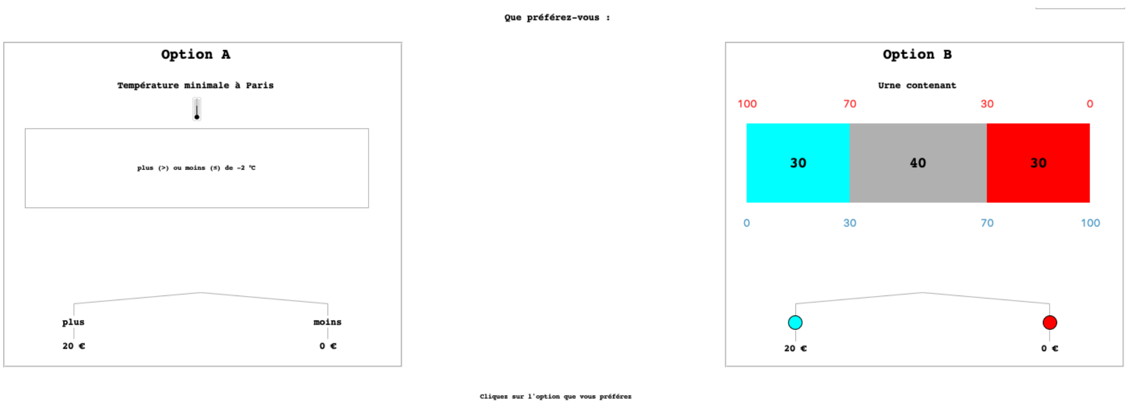
C.1. Probability interval elicitation: displays

Figure 16 shows the display in a typical step of the binary-choice procedure. Specifically, the two figures show the two choice questions making up the step, involving bets on complementary events (temperature below vs above; bet on red vs blue).

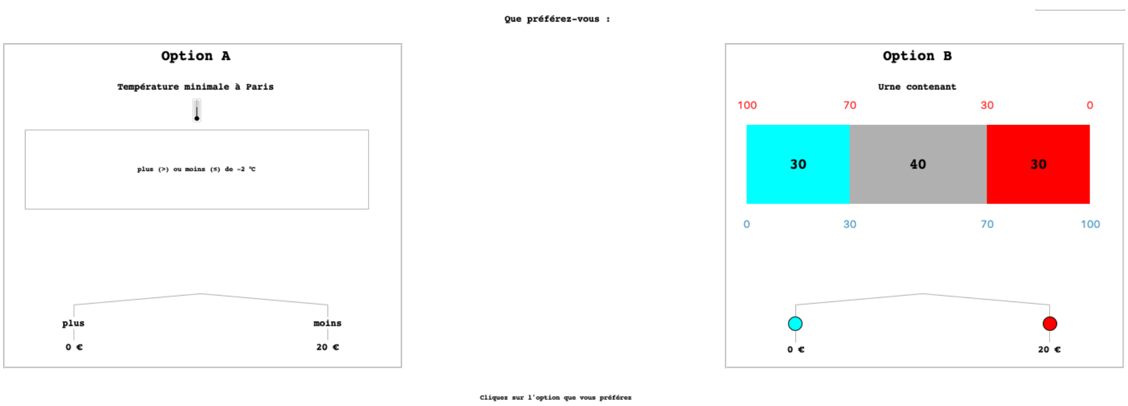
At the end of the binary choice procedure, the two-cursor scrollbar, realising the 2D-choice list described in Section 2.5, is displayed, and the subject is invited to verify all choices, and correct them if required, prior to confirmation. The top pane of Figure 17 shows a typical confirmation screen that appears at the end of the binary-choice procedure, where the retained values for red and blue balls are 66 and 29 respectively. The red lines below then above the bar indicate that, for an urn with at least 29 blue balls and a minimum number of red balls greater than 66, option B (the bet on red from the urn) is preferred over A (the bet on the temperature being less than -2°C), whereas when there are at least 29 blue urns and the minimum number of red balls is less than 66, option A is preferred over B. Similarly, the blue lines indicate how preferences vary over urns with at least 66 red balls, for different minimum numbers of blue balls.

The bottom pane of Figure 17 illustrates a situation where the subject is verifying and correcting preferences as the number of blue balls vary, by moving the blue cursor (which is thus highlighted). The red cursor is kept fixed at its provisional value,[21] and, for each position of the blue cursor, the choice between the bet on the event (temperature greater than 2°) and the bet on the urn with the specified minimum number of blue balls and at least 66 red balls is presented, with the chosen option (as per Section 2.5) indicated. In this example, whilst the provisional values imply a preference for the bet on the event over that on the urn when only

[21] If the subject tries to move the red cursor, the blue cursor returns to its provisional value, and remains fixed there whilst the red cursor is being moved.

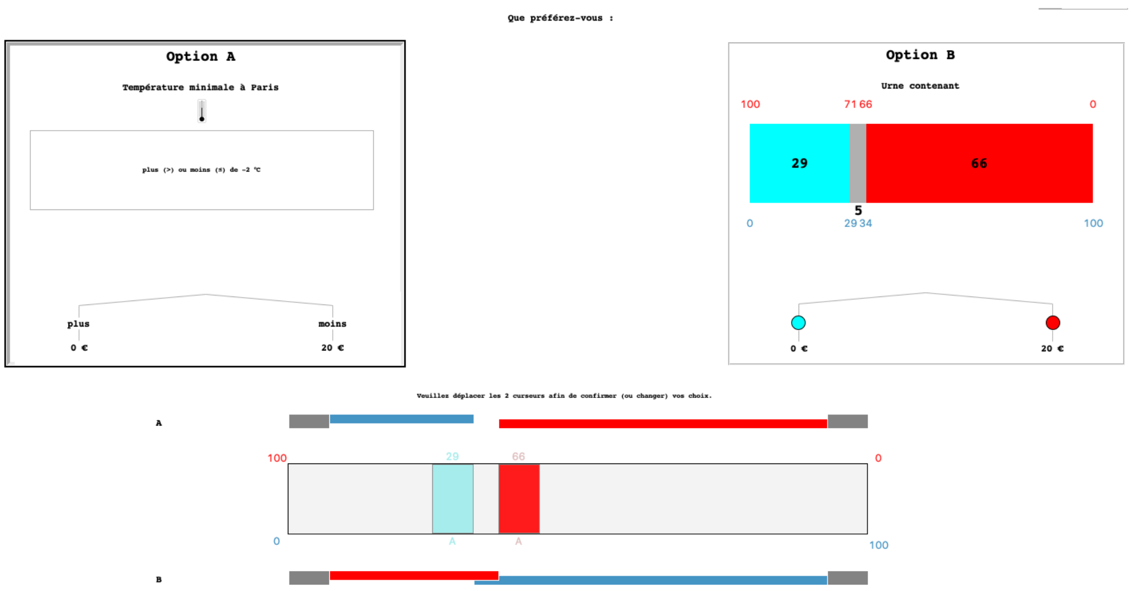


(a) Bet on the complement of the event E_{-2}

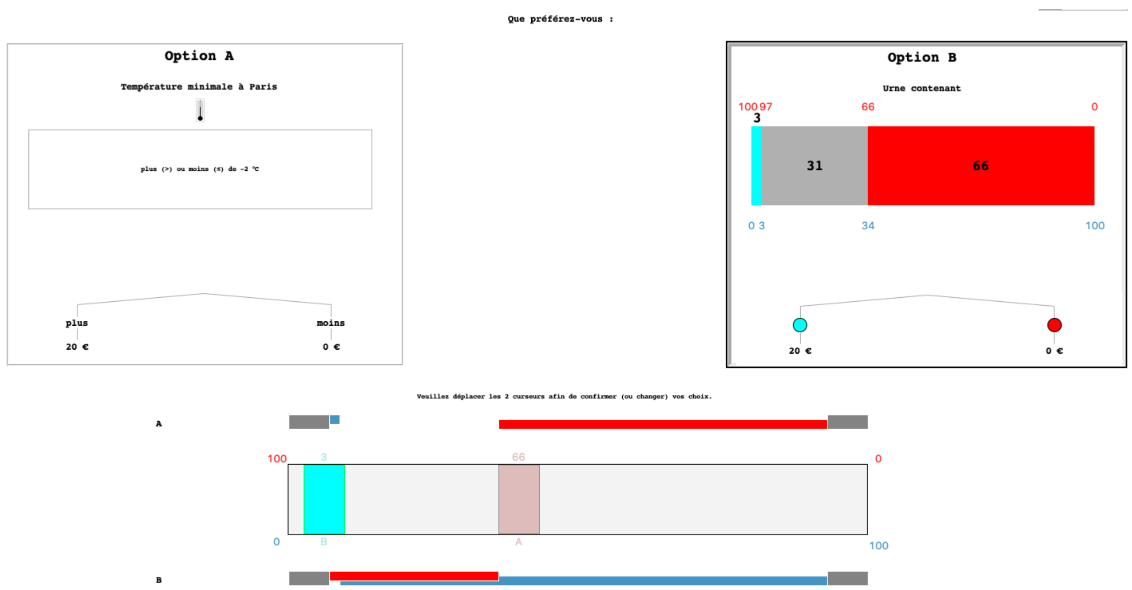


(b) Bet on the event E_{-2}

Figure 16: Displays for a step in the binary-choice procedure



(a) First display



(b) Moving the blue cursor to confirm and correct choices

Figure 17: Two-cursor scrollbar confirmation screen, implementing the 2D choice list

3 balls are guaranteed to be blue, the subject prefers the urn. She may change her choice by clicking on the urn or on the cursor. The bottom pane of Figure 17 shows the display just after she has corrected her choice: the highlighted preferred option is now the bet on the urn, and the blue lines above and below the scrollbar are modified accordingly. To confirm her response for the event, the subject has to scroll the blue cursor across the entire confirmation line, scanning all the choices, and likewise for the red cursor.

In EXP 2, there was a final confirmation screen after the elicitation for all events in a given source, presented in Figure 18. All interval-valued urns corresponding to the choices made and confirmed by the subject for the source are presented on the left. They are graphically depicted on the right: the red line shows the minimum number of red balls for each event (mark, in the case of this source), whereas the blue line plots 100 minus the minimum number of blue balls. To change a choice, a subject can either click on the choice on the right hand plot or on the corresponding urn in the sidebar on the left. By doing so, she returns to the corresponding two-cursor scrollbar confirmation screen, as in Figure 17. She may modify her choices on this screen as described above, and must reconfirm before proceeding.

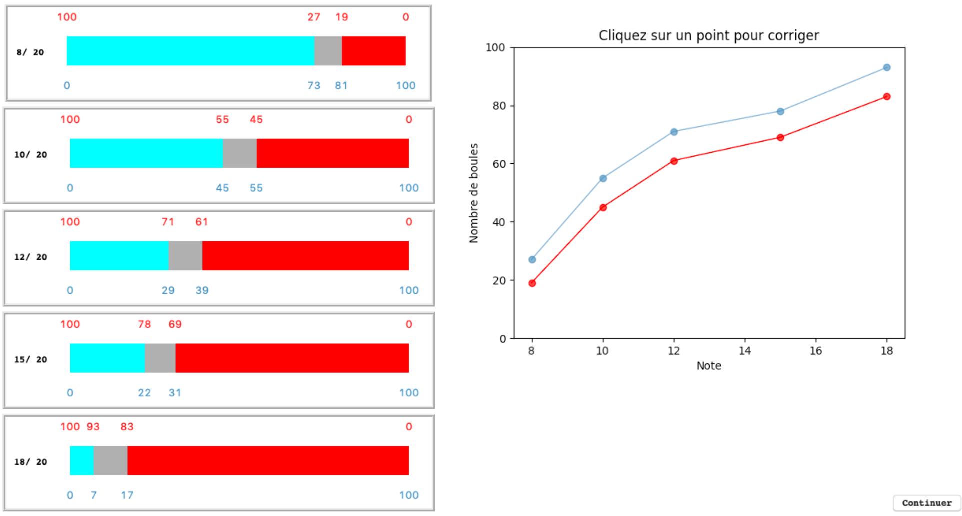
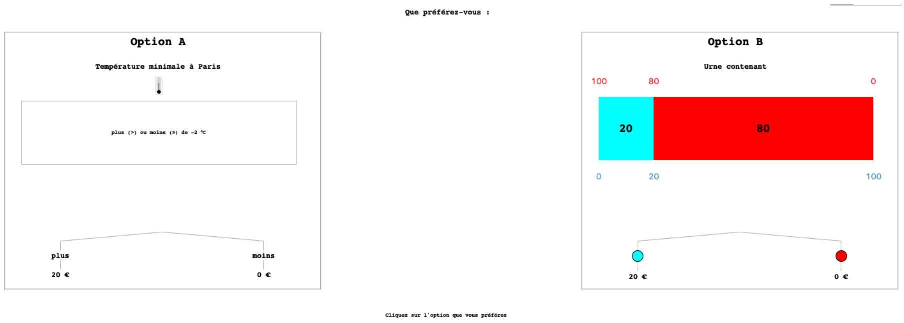


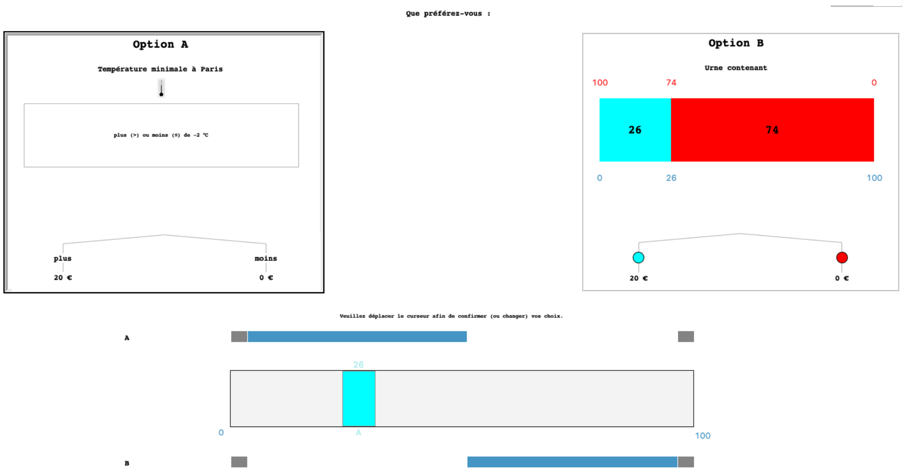
Figure 18: Omnibus confirmation screen

C.2. MP elicitation: displays

Figure 19 shows the displays for a typical choice in the MP elicitation (top pane) and the confirmation screen (bottom pane). These are comparable to the displays for probability interval elicitation, with the exception that a standard single-cursor scrollbar is used for confirmation. For the latter, as for the probability interval confirmation screen, the subject may use the cursor to scan choices and may click on the relevant option to modify her choice. She must scan all choices before confirming.



(a) Binary-choice procedure



(b) Confirmation scrollbar

Figure 19: MP displays

D. Technical Appendix: Lemmas for the proof of Proposition 4

In the following Lemmas, we suppose that preferences are represented according to (1) with $\alpha > \frac{1}{2}$, with E the event of interest with the subjective probability interval $[\underline{p}(E),\overline{p}(E)]$.

Lemma 1. *For every $m \in [0, 0.5]$:*

- *If $\sigma_1\left(\left[\underline{p}(E),\overline{p}(E)\right]\right) < m$, there exists $[p,q] \in R-B$ with $\sigma_1([p,q]) = m$, but no $[p,q] \in W$ with $\sigma_1([p,q]) = m$;*
- *If $\sigma_1\left(\left[\underline{p}(E),\overline{p}(E)\right]\right) > m$, there exists $[p,q] \in W$ with $\sigma_1([p,q]) = m$, but no $[p,q] \in R-B$ with $\sigma_1([p,q]) = m$;*
- *If $\sigma_1\left(\left[\underline{p}(E),\overline{p}(E)\right]\right) = m$, each $[p,q]$ with $\sigma_1([p,q]) = m$ and $[p,q] \neq \left[\underline{p}(E),\overline{p}(E)\right]$ belongs to either R or B.*

Proof. Straightforward to check from the representation (1) and the definition of σ (12). (See also Figures 1 and 9.) □

Lemma 2. *If $\underline{p}(E) = \overline{p}(E)$ and Case 1 arrives at a point $[p_i, q_i]$ with $p_i = q_i$, then the procedure remains in Case 1, and $[\underline{p}_n, \overline{p}_n] \to [\underline{p}(E), \overline{p}(E)]$ as $n \to \infty$.*

Proof. Once the procedure reaches the subspace of precise probabilities, it executes a standard bisection procedure (lines 13–16, Figure 11). □

Lemma 3. *Suppose that the procedure reaches a point $[p_i,q_i]$ in R-B or W. Then the sequence $\sigma_1([p_n, q_n]) \to \sigma_1(\left[\underline{p}(E),\overline{p}(E)\right])$ as $n \to \infty$.*

Proof. Consider a stage i in the procedure where a point has just been found in R-B or W. So the area containing the MPI is Φ_i (Eq. (14)). Let

$$\begin{aligned} m_i^W &= \min \sigma_1(\Phi_i) \\ &= \sigma_1\left([\max\left\{p' : [p',q'] \in El_n \cap W\right\}, \min\left\{q' : [p',q'] \in El_n \cap W\right\}]\right) \\ m_i^{RB} &= \max \sigma_1(\Phi_i) \\ &= \begin{cases} \sigma_1\left([\min\left\{p'' : [p'',q''] \in El_n \cap R-B\right\}, \max\left\{q'' : [p'',q''] \in El_n \cap R-B\right\}]\right) & El_n \cap R-B \neq \emptyset \\ 0.5 & \textit{otherwise} \end{cases} \end{aligned}$$

and $|\Phi_i| = m_i^{R-B} - m_i^W$. The latter is the maximum difference in σ_1 values across all pairs of points in Φ_i. In the first two subcases of Case 3 (lines 35-39), the next probability interval elicited is

$$\begin{aligned} [p_{i+1}, q_{i+1}] = &\frac{1}{2}[\max\left\{p' : [p',q'] \in El_n \cap W\right\}, \min\left\{q' : [p',q'] \in El_n \cap W\right\}] \\ &+ \frac{1}{2}[\min\left\{p'' : [p'',q''] \in El_n \cap R-B\right\}, \max\left\{q'' : [p'',q''] \in El_n \cap R-B\right\}] \end{aligned}$$

In the second subcase of Case 2 (lines 22-23), where a point in W has been found, but no point in R-B, the next probability interval elicited is

$$[p_{i+1}, q_{i+1}] = \frac{1}{2}\left[\frac{1}{2}\left(\begin{array}{c}\min\{p'' : [p'', q''] \in El_n \cap W\} + \\ \max\{q'' : [p'', q''] \in El_n \cap W\}\end{array}\right), \frac{1}{2}\left(\begin{array}{c}\min\{p'' : [p'', q''] \in El_n \cap W\} + \\ \max\{q'' : [p'', q''] \in El_n \cap W\}\end{array}\right)\right]$$
$$+\frac{1}{2}\left[\min\{p'' : [p'', q''] \in El_n \cap W\}, \max\{q'' : [p'', q''] \in El_n \cap W\}\right]$$

where $[\frac{1}{2}(\min\{p'' : [p'', q''] \in El_n \cap R - B\} + \max\{q'' : [p'', q''] \in El_n \cap R - B\}), \frac{1}{2}(\min\{p'' : [p'', q''] \in El_n \cap R$ is the point on the diagonal of precise probabilities (i.e. degenerate probability intervals) that is closest to $[\min\{p'' : [p'', q''] \in El_n \cap W\}, \max\{q'' : [p'', q''] \in El_n \cap W\}]$ (it is on the downwards sloping 45° line from $[\min\{p'' : [p'', q''] \in El_n \cap W\}, \max\{q'' : [p'', q''] \in El_n \cap W\}]$). So this point has σ_1-value 0.5.

In first subcase of Case 2 (lines 20-21), where a point in R-B has been found, but no point in W, the next probability interval elicited, $[p_{i+1}, q_{i+1}]$, is a $\frac{1}{2} - \frac{1}{2}$ mix of $[\min\{p'' : [p'', q''] \in El_n \cap R - B\}, \max\{q'' : [p'', q''] \in El_n \cap R - B\}]$ with

$$\begin{cases} \left[\begin{array}{c}\min\{p'' : [p'', q''] \in El_n \cap R - B\} + \\ \max\{q'' : [p'', q''] \in El_n \cap R - B\} - 1\end{array}, 1\right] & \begin{array}{c}\min\{p'' : [p'', q''] \in El_n \cap R - B\} \\ + \max\{q'' : [p'', q''] \in El_n \cap R - B\}\end{array} > 1 \\ \left[0, \begin{array}{c}\min\{p'' : [p'', q''] \in El_n \cap R - B\} + \\ \max\{q'' : [p'', q''] \in El_n \cap R - B\}\end{array}\right] & \begin{array}{c}\min\{p'' : [p'', q''] \in El_n \cap R - B\} \\ + \max\{q'' : [p'', q''] \in El_n \cap R - B\}\end{array} \leq 1 \end{cases}$$

which is the point on the upper boundary (with either lower bound for the probability interval 0 or upper bound 1) that is on the downwards sloping 45° line from $[\min\{p'' : [p'', q''] \in El_n \cap R - B\}, \max\{q'' : [p'', q''] \in El_n \cap R - B\}]$. This point has σ_1-value 0.

Clearly, in all cases, $m_i^W < \sigma_1([p_{i+1}, q_{i+1}]) < m_i^{RB}$. Moreover, by the rest of the subcases in Cases 2 & 3, if this point is not in R-B or W, all the subsequent points elicited will have the same σ_1-value as $[p_{i+1}, q_{i+1}]$. And whenever a point in R-B is found, the next area containing the MPI, Φ_{i+1}, will have the same minimum σ_1-value m_i^W, but its maximum value will be replaced by $\sigma_1([p_{i+1}, q_{i+1}])$. By Lemma 4, it follows that

$$\begin{aligned} |\Phi_i|.\frac{m_i^W}{m_i^{RB} + m_i^W} &\leq |\Phi_{i+1}| \\ &\leq |\Phi_i|.\frac{1 - m_i^W}{(1 - m_i^{RB}) + (1 - m_i^W)} \end{aligned}$$

Similarly, whenever a point in W is found, the next area containing the MPI, Φ_{i+1}, will have the same maximum σ_1 value m_i^{RB}, but its minimum value will be replaced by $\sigma_1([p_{i+1}, q_{i+1}])$, whence

$$|\Phi_i|.\frac{1-m_i^{RB}}{(1-m_i^{RB})+(1-m_i^{W})} \leq |\Phi_{i+1}|$$
$$\leq |\Phi_i|.\frac{m_i^{RB}}{m_i^{RB}+m_i^{W}}$$

Since, for any $j > i$, $m_j^{RB} \leq m_i^{RB}$ and $m_j^{W} \geq m_i^{W}$, for any such j, $\frac{1-m_j^W}{(1-m_j^{RB})+(1-m_j^W)} \leq \frac{1-m_i^W}{(1-m_i^{RB})+(1-m_i^W)}$ and $\frac{m_j^{RB}}{m_j^{RB}+m_j^W} \leq \frac{m_i^{RB}}{m_i^{RB}+m_i^W}$. So, for any $j = i+k$ with $k \in \mathbb{N}$, $k \geq 1$, $|\Phi_j| \leq \left(\max\left\{\frac{1-m_i^W}{(1-m_i^{RB})+(1-m_i^W)}, \frac{m_i^{RB}}{m_i^{RB}+m_i^W}\right\}\right)^k.|\Phi_i|$. So the sequence $[m_n^W, m_n^{RB}]$ is a bisection-like sequence of decreasing intervals (in the sense of containment), each of which contains $\sigma_1([\underline{p}(E),\overline{p}(E)])$. Moreover, by the previous observation, whenever a point $[p,q]$ is found in W with $\sigma_1([p,q]) > 0$, then the sequence $|\Phi_n| = m_n^{RB} - m_n^W \to 0$ as $n \to \infty$, so $\sigma_1([p_n,q_n]) \to \sigma_1([\underline{p}(E),\overline{p}(E)])$ as required. (Recall that $0.5 \geq m_n^{RB} \geq m_n^W \geq 0$ for all n.)

We now separate two cases, according to whether $\sigma_1\left(\left[\underline{p}(E),\overline{p}(E)\right]\right) = 0$ or not. Suppose first that $\sigma_1\left(\left[\underline{p}(E),\overline{p}(E)\right]\right) = \delta > 0$. We show that the procedure will either arrive at a point with σ_1-value δ, or a point in W. At a stage i in the procedure where no points in W have been found, but a point in R-B has, $m_i^W = 0$ and $0.5 \geq m_i^{R-B} > 0$. At each subsequent stage, by Lemma 1, either i. no point is found in W or R-B; ii. a point is found in W or R-B, and the next such point is in W; iii. a point is found in W or R-B, and the next such point is in R-B. In case ii., the claim is established; in case i., by Lemma 1, the procedure is examining points with σ_1-value δ, and the claim is established. Assume for reductio that at all such stages, the σ_1-value of the explored points is not δ, but no point in W is found—i.e. we are always in case iii. Then, by the previous observations, for every $j = i+k$ with $k \in \mathbb{N}$, $k \geq 1$, $|\Phi_j| \leq \left(\frac{1-m_i^W}{(1-m_i^{RB})+(1-m_i^W)}\right)^k.|\Phi_i| = \left(\frac{1}{2-m_i^{RB}}\right)^k.m_i^{RB}$. Hence $|\Phi_j| = m_j^{RB} \to 0$, contradicting the fact that there are no points with σ_1-value less that δ in R-B. Hence the procedure eventually finds a point in W. By the previous observation it follows that $\sigma_1([p_n,q_n]) \to \sigma_1([\underline{p}(E),\overline{p}(E)])$ as required.

Now consider the case where $\sigma_1\left(\left[\underline{p}(E),\overline{p}(E)\right]\right) = 0$. By Lemma 1, whenever the procedure searches for a point on a line $\sigma_1([p,q]) = m > 0$, it will find a point in R-B. Hence, by the previous argument, it produces a sequence of points $[p_n,q_n]$ in R-B, defining Φ_n and associated $[m_n^W, m_n^{RB}]$, with $m_n^W = 0$ and $m_j^{RB} \to 0$, as required.

□

Lemma 4. *Let $[p_W,q_W]$ be a point in W, with $\sigma_1([p_W,q_W]) = m_W$ and suppose that the line $\sigma_1([p,q]) = m_{R-B}$ contains a point in R-B but not in W. Then, for any point $[p_{R-B},q_{R-B}] \in R-B$ with $\sigma_1([p_{R-B},q_{R-B}]) = m$*

$$\sigma_1([\frac{p_W+p_{R-B}}{2}, \frac{q_W+q_{R-B}}{2}]) \in \left[\frac{2m_W.m_{R-B}}{m_W+m_{R-B}}, \frac{m_W(1-m_{R-B})+m_{R-B}(1-m_w)}{(1-m_{R-B})+(1-m_w)}\right]$$

Moreover, the same holds for a given point $[p_{R-B},q_{R-B}] \in R-B$ and any point $[p_W,q_W] \in W$ on the line $\sigma_1([p,q]) = m_W$.

Proof. We first restrict attention to points $[p, q]$ with $p < 1 - q$ (or, in the polar-style coordinate system, $\alpha < \frac{1}{2}$). For any points $[p_1, q_1]$ and $[p_2, q_2]$, written in polar-style coordinate system as (m_1, α_1) and (m_2, α_2), by (12) and (13), the midpoint (in Cartesian coordinates), $\frac{1}{2}[p_1, q_1] + \frac{1}{2}[p_2, q_2]$ is $\left(\frac{\alpha_1 m_1 + \alpha_2 m_2}{\alpha_1 + \alpha_2}, \frac{\alpha_1 + \alpha_2}{2}\right)$ in the polar system. Written in the polar coordinate system, let $[p_W, q_W]$ be (m_W, α_W); the points on the line $\sigma_1([p, q]) = m_{R-B}$ are (m_{R-B}, α), for varying α. Note that, by Proposition 2, $m_{R-B} > m_W$. It follows from representation 1 that $(z, [p', q'], 0) \prec (z, [p, q], 0)$ whenever $q' < q$ and $p' < p$, whence, since $[p_W, q_W] \in W$, we have that $(z, [p', q'], 0) \prec (z, E, 0)$ for all $q' < q_W$ and $p' < p_W$, so such points are not in R-B. So any point $[p, q]$ on $\sigma_1([p, q]) = m_{R-B}$ which is in R-B is such that $p \geq p_W$. By a similar argument (using the fact that $(0, [p', q'], z) \prec (0, E, z)$ for all $q' > q_W$ and $p' > p_W$), any point $[p, q]$ on $\sigma_1([p, q]) = m_{R-B}$ which is in R-B is such that $q \geq q_W$. So any point $[p, q]$ on $\sigma_1([p, q]) = m_{R-B}$ which is in R-B has $\alpha > \frac{\alpha_W m_W}{m_{R-B}}$ (where, by 13, this is in the α of the point on $\sigma_1([p, q]) = m_{R-B}$ with $p = p_W = 2\alpha_W m_W$); similarly, any such point has $\alpha < \frac{\alpha_W(1-m_W)}{(1-m_{RB})}$. Plugging these bounds into the expression for the midpoint yields the result. Similar calculations yield the same result for the cases of $p > 1 - q$ for some or all of the point considered. Finally, analogous arguments establish the conclusion for $[p_{R-B}, q_{R-B}] \in R-B$ fixed and $[p_W, q_W] \in W$ on the line $\sigma_1([p, q]) = m_W$. $\square$

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