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GENERALIZED INVERSES AND THE MAXIMAL RADIUS OF REGULARITY OF A FREDHOLM OPERATOR

Catalin BADEA

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Operators possessing analytic generalized inverses satisfying the resolvent identity are studied. Several characterizations and necessary conditions are obtained. The maximal radius of regularity for a Fredholm operator T is computed in terms of the spectral radius of a generalized inverse of T . This provides a partial answer to a conjecture of J. Zemánek.

1 Introduction

Let X be a Banach space. We will denote by $C(X)$ the set of all closed operators with a dense domain and by $B(X)$ the algebra of bounded operators from X into itself. For an operator $T \in C(X)$, we denote by $D(T)$, $N(T)$ and $R(T)$ the domain, the kernel and, respectively, the range of T . The identity operator will be denoted by I . We introduce the following definition for the generalized inverse of T (relative inverse or pseudo-inverse are names also used in the literature).

DEFINITION 1.1 *The operator $T \in C(X)$ possess a generalized inverse if there exists an operator $S \in B(X)$ such that $R(S) \subseteq D(T)$ and*

1. $TST = T$ on $D(T)$.
2. $STS = S$ on X .
3. ST is continuous .

In this case we will say that S is the generalized inverse of T .

The following are some simple remarks about generalized inverses.

REMARKS 1.2 (cf. [3])

1. Using the closed-graph theorem, we get that the operator TS is continuous.
2. The operator TS is a projection onto $R(T)$ such that $N(TS) = N(S)$ and $R(TS) = R(T)$.

3. The condition 3 in the above definition shows that the operator ST can be extended to a bounded projection onto $\overline{R(S)}$ such that $N(ST) = N(T)$ and $R(ST) = \overline{R(S)}$.
4. The operator T possess a generalized inverse if and only if $N(T)$ and $R(T)$ have topological complements in X .

In what follows we will denote $P = TS$ and $Q = ST$.

The following is an open problem : Suppose that $U \subset \text{reg}(T)$, that is, for any $\lambda \in U$, $T - \lambda I$ possesses an analytic generalized inverse in a suitable neighborhood V_λ of λ , where U is an open, connected subset of $\mathbf{C}$. Does a generalized resolvent of T on U always exist ? A generalized resolvent of T on U is an operator-valued function $Rg(T, \lambda)$ on U such that $Rg(T, \lambda)$ is a generalized inverse of $T - \lambda I$ for all $\lambda \in U$ and $Rg(T, \lambda)$ satisfies the resolvent identity. This open problem is mentioned in [15], [12], [13], [16], [14]. Note also that the corresponding problem for left or right invertible operators is also open (cf. [2], [6], [17]).

The paper is organized as follows. In the next section we recall some known results and we prove several characterizations for generalized resolvents. These results will be used in the proof of the main results of this paper (section 3). A consequence of one of it gives a formula for the maximal radius of regularity of a Fredholm operator which answers, in a particular case, a question of Zemánek [17].

2 Generalized inverses and generalized resolvents

We start by recalling some notation and definitions.

DEFINITION 2.1 *Let U be a subset of $\mathbf{C}$. We will say that $T \in C(X)$ possess a generalized inverse on U if $T - \lambda I$ possess a generalized inverse for every $\lambda \in U$.*

We will denote by $\text{reg}(T)$ the set of all complex numbers λ for which T possess an *analytic* generalized inverse in a neighborhood of λ . Then

$$\sigma_g(T) = \mathbf{C} \setminus \text{reg}(T)$$

will denote the *generalized spectrum* of T . Several properties of the classical spectrum $\sigma(T)$ remain true in the case of the generalized one (cf. [11, 12]).

DEFINITION 2.2 *An operator $T \in C(X)$ is called regular if T possess a generalized inverse and $N(T^n) \subset R(T)$, for all $n \geq 0$.*

It is easy to see that the condition “ $N(T^n) \subset R(T)$, for all $n \geq 0$ ” is equivalent to the following one : “ $N(T) \subset R(T^m)$, for all $m \geq 0$ ”.

The following result gives a characterization of $\text{reg}(T)$ in terms of regular operators.

THEOREM 2.3 ((cf. [11])) *For an operator $T \in C(X)$, we have $\lambda_0 \in \text{reg}(T)$ if and only if $T - \lambda_0 I$ is regular.*

In Section 3, we will need the following result for regular operators. It was proved in [10] for bounded operators on a Hilbert space but the proof there remains valid for this more general case.

THEOREM 2.4 ((cf. [10])) *Let $T \in C(X)$ be a regular operator and $S \in B(X)$ a generalized inverse of T . Then, for every $i \geq 1$, we have*

$$T^i S^i T^j = \begin{cases} T^i S^{i-j} & : 0 \leq j \leq i \\ T^j & : i \leq j \end{cases}$$

and

$$T^j S^i T^i = \begin{cases} S^{i-j} T^i & : 0 \leq j \leq i \\ T^j & : i \leq j \end{cases}$$

In particular we have $T^n S^n T^n = T^n$ for all $n \geq 1$.

Let $U \subset \mathbf{C}$ be an open set. We will say that the operator-valued function f defined on U satisfies the resolvent identity on U if

$$f(\lambda) - f(\mu) = (\lambda - \mu)f(\lambda)f(\mu)$$

for all λ and μ belonging to the same connected component of U .

DEFINITION 2.5 *Let $U \subset \mathbf{C}$ be an open set. The operator $T \in C(X)$ is said to possess a generalized resolvent on U if T possess a generalized inverse on U satisfying the resolvent identity in U .*

Note that a generalized resolvent $Rg(T, \lambda)$ of T in a connected U is analytic on U . According to [7, page 184], a function satisfying the resolvent identity on an open set is locally analytic.

The following result characterizes generalized resolvents (on connected, open sets) among the analytic generalized inverses. It generalizes a result from [17] and will be used in the next section in the proof of the main result.

THEOREM 2.6 *Let U be an open, connected subset of $\mathbf{C}$, $0 \in U$. Let $T \in C(X)$ possessing an analytic generalized inverse $R(\lambda) = \sum_{n=0}^{\infty} \lambda^n T_n$ on U . Denote $P(\lambda) = (T - \lambda I)R(\lambda)$ and $Q(\lambda) = R(\lambda)(T - \lambda I)$ for $\lambda \in U$. The following conditions are equivalent :*

- (i) $T_n = T_0^{n+1}$ for all $n \geq 1$.
- (ii) $N(P(\lambda)) = N(TT_0)$ and $R(Q(\lambda)) = R(T_0T)$ for all $\lambda \in U$.
- (iii) There exist two closed subspaces Z and W of X such that $N(P(\lambda)) = Z$ and $R(Q(\lambda)) = W$ for all $\lambda \in U$.
- (iv) $R(\lambda)$ is a generalized resolvent of T on U .
- (v) $R(\lambda) - R(0) = \lambda R(\lambda)R(0)$, for all $\lambda \in U$.

PROOF. (i) $\Rightarrow$ (ii) : Since $R(\lambda)$ is a generalized inverse of T on U it follows, in particular, that T_0 is a generalized inverse of T . Since $T_n = T_0^{n+1}$ for all $n \geq 1$, we have

$$Q(\lambda) = R(\lambda)(T - \lambda I) = T_0T - \sum_{n=1}^{\infty} \lambda^n T_0^n (I - T_0T)$$

and

$$P(\lambda) = (T - \lambda I)R(\lambda) = TT_0 - (I - TT_0) \sum_{n=1}^{\infty} \lambda^n T_0^n$$

for every $\lambda \in U$. We prove first the equality $N(P(\lambda)) = N(T_0)$. By the Remark 1.2, (2), we will have $N(P(\lambda)) = N(TT_0)$. Let $u \in N(P(\lambda))$. Then

$$0 = P(\lambda)u = TT_0u - (I - TT_0) \sum_{n=1}^{\infty} \lambda^n T_0^n u.$$

Therefore

$$TT_0u = (I - TT_0) \sum_{n=1}^{\infty} \lambda^n T_0^n u.$$

Applying T_0 to both sides we get $T_0u = 0$ (we have used the equality $T_0TT_0 = T_0$). Therefore $N(P(\lambda)) \subseteq N(T_0)$. Now let $u \in N(T_0)$. For every $\lambda \in U$, we have

$$P(\lambda)u = TT_0u - (I - TT_0) \sum_{n=1}^{\infty} \lambda^n T_0^n u = 0.$$

Hence $N(P(\lambda)) = N(TT_0)$.

We prove now that $R(Q(\lambda)) = R(T_0T)$. Let $u = Q(\lambda)u \in R(Q(\lambda))$ for a fixed $\lambda \in U$. Then

$$\begin{aligned} u &= T_0Tu - \sum_{n=1}^{\infty} \lambda^n T_0^n (I - T_0T)u \\ &= T_0Tu - \sum_{n=1}^{\infty} \lambda^n T_0^n (I - T_0T) \left[T_0 \left(Tu - \sum_{n=1}^{\infty} \lambda^n T_0^{n-1} (I - T_0T) \right) u \right] \\ &= T_0Tu. \end{aligned}$$

Hence $R(Q(\lambda)) \subseteq R(T_0T)$.

Conversely, if $u = T_0Tu \in R(T_0T)$, then

$$\begin{aligned} Q(\lambda)u &= T_0Tu - \sum_{n=1}^{\infty} \lambda^n T_0^n (I - T_0T)u \\ &= u - \sum_{n=1}^{\infty} \lambda^n T_0^n (I - T_0T)T_0Tu = u \in R(Q(\lambda)). \end{aligned}$$

Therefore $R(Q(\lambda)) = R(T_0T)$ for every $\lambda \in U$.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (iv) : Let $\lambda, \mu \in U$. Then

$$(\lambda - \mu)R(\lambda)R(\mu) = R(\lambda)(\lambda - \mu)R(\mu)$$

$$= R(\lambda) [(T - \mu I) - (T - \lambda I)] R(\mu) = R(\lambda)P(\mu) - Q(\lambda)R(\mu).$$

Suppose now that $N(P(\lambda)) = Z$ for all $\lambda \in U$. Then we have $R(\lambda)(I - P(\mu)) = 0$ and so $R(\lambda) = R(\lambda)P(\mu)$ for all $\lambda, \mu \in U$.

Supposing also that $R(Q(\lambda)) = W, \lambda \in U$, we obtain $(I - Q(\lambda))R(\mu) = 0$ for all $\lambda, \mu \in U$. Therefore

$$(\lambda - \mu)R(\lambda)R(\mu) = R(\lambda)P(\mu) - Q(\lambda)R(\mu) = R(\lambda) - R(\mu).$$

(iv) $\Rightarrow$ (v) is clear.

(v) $\Rightarrow$ (i) : Suppose that $R(\lambda) - R(0) = \lambda R(\lambda)R(0)$. Then, for every $\lambda \in U$, we have

$$\sum_{n=0}^{\infty} \lambda^n T_n - T_0 = \lambda \sum_{n=0}^{\infty} \lambda^n T_n T_0.$$

Therefore

$$\sum_{n=1}^{\infty} \lambda^n T_n - \sum_{n=0}^{\infty} \lambda^{n+1} T_n T_0 = 0$$

and thus

$$\sum_{n=0}^{\infty} (T_{n+1} - T_n T_0) \lambda^{n+1} = 0.$$

Hence $T_{n+1} - T_n T_0 = 0$ for all $n \geq 0$, yielding $T_n = T_0^{n+1}$ for all $n \geq 0$. $\diamond$

The next result is a characterization of generalized resolvents, without assuming the power-series development around 0. We will apply this criterion in Example 3.6.

THEOREM 2.7 *Let $T \in C(X)$ and let U be an open, connected subset of $\mathbf{C}$. The following conditions are equivalent :*

(i) *There exist two families of projections $P(\lambda)$ and $Q(\lambda)$, $\lambda \in U$, continuous in λ , such that*

$$R(P(\lambda)) = R(T - \lambda I), N(Q(\lambda)) = N(T - \lambda I); \lambda \in U$$

and

$$P(\lambda)P(\mu) = P(\lambda), Q(\lambda)Q(\mu) = Q(\mu); \lambda, \mu \in U.$$

(ii) *There exists a generalized resolvent of T on U .*

(iii) *There exists an analytic generalized inverse $R(\lambda)$ of T in U , satisfying $R'(\lambda) = R(\lambda)^2$, for all $\lambda \in U$.*

PROOF. (i) $\Rightarrow$ (ii) : Let $u \in X$ and $\lambda \in U$. Then $P(\lambda)u \in R(T - \lambda I)$. Therefore, there exists $v \in D(T)$ such that $P(\lambda)u = (T - \lambda I)v$. Set $Rg(T, \lambda)u = Q(\lambda)v$. Firstly, we show that $Rg(T, \lambda)$ is well-defined. Indeed, if $w \in D(T)$ is such that $(T - \lambda I)w = P(\lambda)u = (T - \lambda I)v$, then

$$v - w \in N(T - \lambda I) = N(Q(\lambda)).$$

Therefore $Q(\lambda)(v - w) = 0$ and thus $Q(\lambda)v = Q(\lambda)w$. Hence $Rg(T, \lambda)$ does not depend on the choice of v .

We show that $Rg(T, \lambda)$ is a generalized inverse of $T - \lambda I$. For all $u \in X$, we have $(I - Q(\lambda))u \in N(T - \lambda I) \subset D(T)$. Also $(T - \lambda I)(I - Q(\lambda)) = 0$ and $Q(\lambda)(D(T)) \subset D(T)$. We obtain $T - \lambda I = (T - \lambda I)Q(\lambda)$ and $R(Rg(T, \lambda)) \subset D(T)$. Using the definition of $Rg(T, \lambda)$, we have

$$(T - \lambda I)Rg(T, \lambda)(T - \lambda I)u = (T - \lambda I)Q(\lambda)u = (T - \lambda I)u$$

for all $u \in D(T)$. Thus

$$(T - \lambda I)Rg(T, \lambda)(T - \lambda I) = T - \lambda I$$

on $D(T)$.

On the other hand, if $u \in X$ and $v \in D(T)$ are such that $P(\lambda)u = (T - \lambda I)v$, then

$$\begin{aligned} Rg(T, \lambda)(T - \lambda I)Rg(T, \lambda)u &= Rg(T, \lambda)(T - \lambda I)Q(\lambda)v \\ &= Rg(T, \lambda)(T - \lambda I)v = Q(\lambda)v = Rg(T, \lambda)u. \end{aligned}$$

Hence

$$Rg(T, \lambda)(T - \lambda I)Rg(T, \lambda) = Rg(T, \lambda)$$

on X .

Now we have to show that $Rg(T, \lambda) \in B(X)$. Because linearity is clear, and $D(Rg(T, \lambda)) = X$, it is sufficient to show that the operator $Rg(T, \lambda)$ is closed. Let $u_n \rightarrow u$ and $Rg(T, \lambda)u_n \rightarrow w$ as $n \rightarrow +\infty$. Then there exist $v_n \in D(T)$ such that

$$P(\lambda)u_n = (T - \lambda I)v_n = (T - \lambda I)Q(\lambda)v_n \rightarrow P(\lambda)u.$$

On the other hand,

$$Rg(T, \lambda)u_n = Q(\lambda)v_n \rightarrow w = Q(\lambda)w,$$

since $R(Q(\lambda))$ is closed. Thus $Q(\lambda)v_n \rightarrow w$ and $(T - \lambda I)Q(\lambda)v_n \rightarrow P(\lambda)u$. Since T is closed, we obtain that $w \in D(T)$ and $P(\lambda)u = (T - \lambda I)w$. Hence $Rg(T, \lambda)u = Q(\lambda)w = w$, showing that the operator $Rg(T, \lambda)$ is closed.

It is not complicated to see, using the definition of $Rg(T, \lambda)$, that

$$(T - \lambda I)Rg(T, \lambda) = P(\lambda)$$

and

$$Rg(T, \lambda)(T - \lambda I) = Q(\lambda)$$

for all $\lambda \in U$. We now show that $Rg(T, \lambda)$ satisfies the resolvent identity in U . For $\lambda, \mu \in U$, we have

$$\begin{aligned} (\lambda - \mu)Rg(T, \lambda)Rg(T, \mu) &= Rg(T, \lambda) [(T - \mu I) - (T - \lambda I)] Rg(T, \mu) \\ &= Rg(T, \lambda)P(\mu) - Q(\lambda)Rg(T, \mu) \\ &= Rg(T, \lambda)P(\lambda)P(\mu) - Q(\lambda)Q(\mu)Rg(T, \mu) \\ &= Rg(T, \lambda)P(\lambda) - Q(\mu)Rg(T, \mu) \\ &= Rg(T, \lambda) - Rg(T, \mu). \end{aligned}$$

Therefore $Rg(T, \lambda)$ verifies the resolvent identity in U and (ii) is proved.

(ii) $\Rightarrow$ (i) : Take $P(\lambda) = (T - \lambda I)Rg(T, \lambda)$ and $Q(\lambda) = Rg(T, \lambda)(T - \lambda I)$ for all $\lambda \in U$.

(ii) $\Leftrightarrow$ (iii) : This equivalence follows from an easy computation and using the fact that generalized resolvents are analytic [7]. $\diamond$

REMARK 2.8 The projections $P(\lambda)$ and $Q(\lambda)$ obtained in (i) of the previous theorem are analytic in U . Indeed, we have $P(\lambda) = (T - \lambda I)Rg(T, \lambda)$ and $Q(\lambda) = Rg(T, \lambda)(T - \lambda I)$. Since $Rg(T, \lambda)$ satisfies the resolvent identity, $P(\lambda)$ and $Q(\lambda)$ are analytic in U .

3 The maximal radius of regularity of a Fredholm operator

Let $T \in C(X)$. The operator T is said to be a *Fredholm* operator if $R(T)$ is closed and $\max\{\dim N(T), \text{codim } R(T)\} < \infty$. The Fredholm domain $\rho_e(T)$ of T is defined by

$$\rho_e(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ Fredholm}\}.$$

The set $\rho_e^r(T)$ defined by

$$\rho_e^r(T) = \rho_e(T) \cap \text{reg}(T)$$

is an open set. Using [13, Corollaire 2.3], we have that $\rho_e^r(T)$ is the set of all $\lambda \in \rho_e(T)$ such that the application $z \rightarrow \dim N(T - zI)$ is constant in a neighborhood of λ . Using the continuity of the index, the same set $\rho_e^r(T)$ coincides with the set of all $\lambda \in \rho_e(T)$ such that the application $z \rightarrow \text{codim } R(T - zI)$ is constant in a neighborhood of λ . Therefore

$$0 \in \rho_e^r(T) \implies \text{dist}(0, \mathbf{C} \setminus \rho_e^r(T)) = \text{dist}(0, \sigma_g(T)).$$

It is this distance that we call the maximal radius of regularity of T (if $0 \in \rho_e^r(T)$).

Let $\gamma(T)$ be the reduced minimum modulus of T :

$$\gamma(T) = \inf\{\|Tx\| : x \in D(T) ; \text{dist}(x, N(T)) = 1\}.$$

It was proved in 1975 by K.H. Förster and M.A. Kaashoek [4] that

$$(*) \quad 0 \in \rho_e(T) \implies \text{dist}(0, \sigma_g(T) \setminus \{0\}) = \lim_{n \rightarrow \infty} \gamma(T^n)^{1/n}$$

and

$$(**) \quad 0 \in \rho_e^r(T) \implies \text{dist}(0, \sigma_g(T)) = \lim_{n \rightarrow \infty} \gamma(T^n)^{1/n}.$$

Recently, J. Zemánek [17] conjectured a different representation for the distance of 0 to the left spectrum of T if T is assumed to be left invertible. Instead of the reduced minimum modulus, his representation is now in terms of the spectral radius of left inverses of T . Note that the conjecture in [17] is stated in the more general framework of Banach algebras. To be more specific, let $\rho_\ell(T)$ be the set of all $\lambda \in \mathbf{C}$ such that $T - \lambda I$ is left

invertible and let $\sigma_\ell(T) = \mathbf{C} \setminus \rho_\ell(T)$ be the left spectrum of T . The *conjecture* in [17] for bounded linear operators $T \in B(X)$ can be stated as follows :

$$(***) \quad 0 \in \rho_\ell(T) \implies \text{dist}(0, \sigma_\ell(T)) = \sup\left\{\frac{1}{r(S)} : ST = I\right\},$$

where $r(S)$ is the spectral radius of S .

One-sided invertible operators are particular cases of operators with generalized inverses. Therefore, we can consider the analogue of the conjecture of Zemánek for $\text{dist}(0, \sigma_g(T))$. The main results we will prove here are two analogues of the Förster-Kaashoek's results for Zemánek's conjecture. This implies (Corollary 3.4) a positive answer for $(***)$ under the additional assumption of the Fredholmness of T .

We start with an alternative version of $(**)$.

THEOREM 3.1 *Let $T \in C(X)$ be a linear operator such that $0 \in \rho_e^r(T)$. Then*

$$\text{dist}(0, \sigma_g(T)) = \sup\left\{\frac{1}{r(S)} : TST = T\right\},$$

where $r(S)$ is the spectral radius of S .

PROOF. Let $S \in B(X)$ be a generalized inverse of T . Then, using Theorem 2.6, we get $T^n S^n T^n = T^n$, for all $n \geq 1$. By [4, Lemma 4], we have $\gamma(T^n) \geq 1/(\|S^n\|)$ and therefore

$$\lim_{n \rightarrow \infty} \gamma(T^n)^{1/n} \geq \frac{1}{r(S)}.$$

Using [4, Theorem 5] we get the inequality

$$\text{dist}(0, \sigma_g(T)) \geq \sup\left\{\frac{1}{r(S)} : TST = T\right\}.$$

In order to prove the other inequality, set $d = \text{dist}(0, \sigma_g(T))$. Then

$$B(0, d) = \{\lambda \in \mathbf{C} : |\lambda| < d\} \subseteq \rho_e^r(T).$$

Suppose that $d < \infty$. Let ε be a positive number and put

$$K = \overline{B}\left(0, \frac{d}{1+\varepsilon}\right) = \{\lambda \in \mathbf{C} : |\lambda| \leq \frac{d}{1+\varepsilon}\}.$$

Then $K \subset \rho_e^r(T)$ is compact. Using [13, Theorem 3.1], there is a generalized resolvent $Rg(T, \lambda)$ for T on K . Then, for all $\lambda \in K$,

$$Rg(T, \lambda) = \sum_{n=0}^{\infty} \lambda^n T_n, T_n \in B(X), R(T_n) \subset D(T).$$

By Cauchy's integral formula, we have

$$T_n = \frac{1}{2\pi i} \int_{|\lambda|=\frac{d}{1+\varepsilon}} \lambda^{-(n+1)} Rg(T, \lambda) d\lambda$$

for all $n \geq 0$. Denoting $M = \max\{\|Rg(T, \lambda)\| : \lambda \in K\}$, we obtain

$$\|T_n\| \leq M \left(\frac{1 + \varepsilon}{d} \right)^{n+1}, n \geq 0.$$

Using now Theorem 2.6, we get

$$\|T_0^{n+1}\| \leq M \left(\frac{1 + \varepsilon}{d} \right)^{n+1}, n \geq 0,$$

which implies $r(T_0) \leq (1 + \varepsilon)/d$. Since $TT_0T = T$, we have

$$\sup\left\{\frac{1}{r(S)} : TST = T\right\} \geq \frac{1}{r(T_0)} \geq \frac{d}{1 + \varepsilon}.$$

Since $\varepsilon > 0$ was arbitrarily chosen, we have

$$\sup\left\{\frac{1}{r(S)} : TST = T\right\} \geq d.$$

Suppose now that $d = \infty$. Let ε be a positive number and $K = \{\lambda \in \mathbf{C} : |\lambda| \leq \frac{1}{\varepsilon}\}$. Then $K \subset \rho_e^r(T)$ is compact. Using similar arguments and notation, T possess in K a generalized resolvent $Rg(T, \lambda)$ and $r(T_0) \leq \varepsilon$. Hence

$$\sup\left\{\frac{1}{r(S)} : TST = T\right\} \geq \frac{1}{r(T_0)} \geq \frac{1}{\varepsilon}.$$

Since this holds for every $\varepsilon > 0$, we have

$$\sup\left\{\frac{1}{r(S)} : TST = T\right\} = \infty = d.$$

The proof is now complete. $\diamond$

We recall now the Kato decomposition for Fredholm operators (cf. for instance [4]). Namely, X decomposes into two T -invariant closed subspaces X_0 and X_1 and, if T_i is the restriction of T to X_i , $i = 0, 1$, then :

1. $\dim X_1 < \infty$;
2. $T = T_0 \oplus T_1$;
3. T_0 is regular ;
4. $X_1 \subset N(T^k)$ for some k .

The following result is a counterpart of (*).

THEOREM 3.2 *Let $T \in C(X)$ be a linear operator such that $0 \in \rho_e(T)$ and $X = X_0 \oplus X_1$ a fixed Kato decomposition with respect to T . Then*

$$\text{dist}(0, \sigma_g(T) \setminus \{0\}) = \sup\left\{\frac{1}{r(S)} : TST = T; SX_0 \subseteq X_0\right\}.$$

PROOF. Since X_1 in Kato decomposition is finite dimensional, $T - \lambda I$ is Fredholm if and only if $T_0 - \lambda I_0$ is Fredholm, where $\lambda \in \mathbf{C}$ and I_0 is the identity operator on X_0 . But $0 \in \rho_e(T)$; thus T_0 is Fredholm. Using (*) and [4, page 125], we have

$$\text{dist}(0, \sigma_g(T) \setminus \{0\}) = \lim_{n \rightarrow \infty} \gamma(T^n)^{1/n} = \lim_{n \rightarrow \infty} \gamma(T_0^n)^{1/n} = \text{dist}(0, \sigma_g(T_0) \setminus \{0\}).$$

Since T_0 is Fredholm, the previous theorem and the previous equations yield

$$\text{dist}(0, \sigma_g(T) \setminus \{0\}) = \sup\left\{\frac{1}{r(S_0)} : T_0 S_0 T_0 = T_0\right\}.$$

Let $d = \sup\left\{\frac{1}{r(S_0)} : T_0 S_0 T_0 = T_0\right\}$ and suppose that $d < \infty$. Let ε be a positive number. Then there exists $S_0 = S_0(\varepsilon) \in B(X_0)$ such that $T_0 S_0 T_0 = T_0$ and

$$d - \varepsilon < \frac{1}{r(S_0)} \leq d.$$

Since $X_1 \subset N(T^k)$ for some k , the operator T_1 is defined on all of the finite dimensional space X_1 and is nilpotent. Let $T_1' = P T_1' P^{-1}$ with

$$T_1' = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then

$$S_1' = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

is a generalized inverse for T_1' , $S_1 = P S_1' P^{-1}$ is a generalized inverse for T_1 and S_1 is nilpotent. Let $S = S_0 \oplus S_1$ with respect to Kato's decomposition. Then $TST = T$ and $SX_0 \subseteq X_0$. In particular, the set in the sup is always nonvoid.

Since $S^n = S_0^n \oplus S_1^n$, for all n , and S_1 is nilpotent, we have $r(S) = r(S_0)$. Then

$$d - \varepsilon < \frac{1}{r(S_0)} = \frac{1}{r(S)} \leq \sup\left\{\frac{1}{r(S)} : TST = T; SX_0 \subseteq X_0\right\}.$$

But this holds for all $\varepsilon > 0$ and thus

$$d \leq \sup\left\{\frac{1}{r(S)} : TST = T; SX_0 \subseteq X_0\right\}.$$

For the other inequality, consider an operator S such that $TST = T$ and $SX_0 \subseteq X_0$. Then the matrix of S , with respect to Kato decomposition, has the following form

$$\begin{pmatrix} S_1 & * \\ 0 & * \end{pmatrix}$$

and thus

$$S^n = \begin{pmatrix} S_1^n & * \\ 0 & * \end{pmatrix}.$$

This implies $\|S^n\| \geq \|S_1^n\|$, so $r(S) \geq r(S_1)$. On the other hand, $TST = T$ implies $T_0S_1T_0 = T_0$. Therefore

$$\frac{1}{r(S)} \leq \frac{1}{r(S_1)} \leq \sup\left\{\frac{1}{r(S_0)} : T_0S_0T_0 = T_0\right\}.$$

Thus $d \geq \sup\left\{\frac{1}{r(S)} : TST = T; SX_0 \subseteq X_0\right\}$.

The case $d = \infty$ can be proved in a similar fashion. $\diamond$

REMARK We do not know if condition $SX_0 \subseteq X_0$ can be removed in the above theorem. It follows from the same Theorem that the sup does not depend upon Kato decomposition.

Set $s(T) = \sup\left\{\frac{1}{r(S)} : TST = T\right\}$.

COROLLARY 3.3 *Let $T \in C(X)$ and suppose that $0 \in \rho_e^r(T)$. Then, for all $n \geq 1$, we have $s(T^n) = s(T)^n$.*

PROOF. Let $n \geq 1$. Using Theorem 3.1 and [4, Theorem 5], we get

$$s(T) = \lim_{k \rightarrow \infty} \gamma(T^k)^{1/k}.$$

Therefore

$$s(T^n) = \lim_{k \rightarrow \infty} (\gamma(T^{kn})^{1/kn})^n = s(T)^n.$$

The proof is complete. $\diamond$

We want to note that C. Schmoegele [16] studied when $\lim_{n \rightarrow \infty} s(T^n)^{1/n}$ can be expressed as a distance from 0 to a modified spectrum. Corollary 3.3 implies his result in our situation.

The following consequence of Theorem 3.2 is a partial result for Zemánek's conjecture.

COROLLARY 3.4 *Suppose that $T \in B(X)$ is a left invertible and Fredholm bounded linear operator. Then*

$$\text{dist}(0, \sigma_\ell(T)) = \sup\left\{\frac{1}{r(S)} : ST = I\right\}.$$

PROOF. We have $\sigma_g(T) \subset \sigma_\ell(T)$ and $0 \in \rho_\ell(T) \cap \rho_e(T)$. Therefore

$$\text{dist}(0, \sigma_\ell(T)) \leq \text{dist}(0, \sigma_g(T)) = \sup\left\{\frac{1}{r(S)} : TST = T\right\}$$

(by Theorem 3.2)

$$= \sup\left\{\frac{1}{r(S)} : ST = I\right\}$$

(since T is left invertible)

$$\leq \text{dist}(0, \sigma_\ell(T))$$

(cf. [17]).

The proof is complete. ◇

Now we mention and briefly discuss some open problems.

PROBLEM 3.5 When is the sup attained in the formula

$$\sup\left\{\frac{1}{r(S)} : TST = T\right\} = \text{dist}(0, \sigma_g(T))$$

in Theorem 3.1 ?

The proof of Theorem 3.1 shows that the sup is attained if and only if one can construct a generalized resolvent $Rg(T, \lambda)$ for T defined on the *all* open set $\rho_e^r(T)$, instead of arbitrary compact subsets K . The later is still an open problem.

Note also that the formula of Theorem 3.1 implies the following result : If $T \in C(X)$, $0 \in \rho_e^r(T)$ and there exists a generalized inverse S_0 of T such that $TS_0T = T$ and $\sigma(S_0) = \{0\}$, then $\rho_e^r(T) = \mathbf{C}$. *We think that the converse also holds.* This certainly holds if the answer to Problem 3.5 is positive.

EXAMPLE 3.6 Consider $X = C[0, 1]$ and $T \in C(X)$ given by $T(f) = f'$ and $D(T) = C^1[0, 1]$. Then $\rho_e^r(T) = \mathbf{C}$. Moreover, there exists a generalized resolvent on the whole $\mathbf{C}$. Indeed, if we define

$$Rg(\lambda)(f)(x) = \int_0^x f(t)e^{\lambda(x-t)} dt, \quad f \in X,$$

then we have

$$P(\lambda) = (T - \lambda)Rg(\lambda) = I$$

and

$$Q(\lambda) = Rg(\lambda)(T - \lambda) = I - F(\lambda),$$

where $F(\lambda)(f)(x) = f(0)\exp(\lambda x)$. It is easy to see that both conditions of Theorem 2.7 are satisfied. Thus $Rg(\lambda)$ satisfies the resolvent identity on $\mathbf{C}$. Also, the Volterra operator S_0 defined by

$$S_0(f)(x) = Rg(0)(f)(x) = \int_0^x f(t) dt$$

is a quasi-nilpotent operator. ◇

Note that operators whose Fredholm domain is the whole complex plane were studied by several authors : cf. [8] and the references cited therein. We also want to note that the equality $\rho_e^r(T) = \mathbf{C}$ clearly implies that $\rho_e(T) = \mathbf{C}$. In fact, if $0 \in \rho_e(T)$, and $S_0 \in B(X)$ is a generalized inverse of T , then $\rho_e(T) = \mathbf{C}$ if and only if $\sigma_e(S_0) = \{0\}$, that is S_0 is a Riesz operator. Here σ_e is the essential spectrum. Indeed, we have

$$(I - \lambda S_0)T = (T - \lambda S_0T) = T - \lambda + \lambda(I - S_0T) .$$

Since $\lambda(I - S_0T)$ is of finite rank, $T - \lambda$ is Fredholm if and only if $I - \lambda S_0$ is (cf. [5]). Therefore $\rho_e(T) = \mathbf{C}$ if and only if S_0 is a Riesz operator.

PROBLEM 3.7 If $0 \in \text{reg}(T)$, does it follow that $s(T) = \text{dist}(0, \sigma_g(T))$?

For Hilbert space bounded operators the answer is positive. The details will be published elsewhere.

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