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► **To cite this version:**

Hervé Hocquard, Seog-Jin Kim, Théo Pierron. Coloring squares of graphs with mad constraints. Discrete Applied Mathematics, 2019, 271, pp.64 - 73. 10.1016/j.dam.2019.08.011 . hal-03488602

**HAL Id: hal-03488602**

**<https://hal.science/hal-03488602>**

Submitted on 21 Dec 2021

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# Coloring squares of graphs with mad constraints<sup>☆</sup>

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## Abstract

A proper vertex  $k$ -coloring of a graph  $G = (V, E)$  is an assignment  $c : V \rightarrow \{1, 2, \dots, k\}$  of colors to the vertices of the graph such that no two adjacent vertices are associated with the same color. The square  $G^2$  of a graph  $G$  is the graph defined by  $V(G) = V(G^2)$  and  $uv \in E(G^2)$  if and only if the distance between  $u$  and  $v$  is at most two. We denote by  $\chi(G^2)$  the chromatic number of  $G^2$ , which is the least integer  $k$  such that a  $k$ -coloring of  $G^2$  exists. By definition, at least  $\Delta(G) + 1$  colors are needed for this goal, where  $\Delta(G)$  denotes the maximum degree of the graph  $G$ . In this paper, we prove that the square of every graph  $G$  with  $\text{mad}(G) < 4$  and  $\Delta(G) \geq 8$  is  $(3\Delta(G) + 1)$ -choosable and even correspondence-colorable. Furthermore, we show a family of 2-degenerate graphs  $G$  with  $\text{mad}(G) < 4$ , arbitrarily large maximum degree, and  $\chi(G^2) \geq \frac{5\Delta(G)}{2}$ , improving a result of Kim and Park.

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## 1. Introduction

A proper vertex  $k$ -coloring of a graph  $G = (V, E)$  is an assignment  $c : V \rightarrow \{1, 2, \dots, k\}$  of colors to the vertices of the graph such that no two adjacent vertices are associated with the same color. The square  $G^2$  of a graph  $G$  is the graph defined by  $V(G) = V(G^2)$  and  $uv \in E(G^2)$  if and only if the distance between  $u$  and  $v$  is at most two. We denote by  $\chi(G^2)$  the chromatic number of  $G^2$ , which is the least integer  $k$  such that a  $k$ -coloring of  $G^2$  exists. In other words, it is a stronger variant of graph coloring where every two vertices within distance two have to receive different colors. By definition, at least  $\Delta(G) + 1$  colors are needed for this goal, where  $\Delta(G)$  denotes the maximum degree of the graph  $G$ . Indeed, if we consider a vertex of maximal degree and its neighbors, they form a set of  $\Delta(G) + 1$  vertices, any two of which are adjacent or have a common neighbor. Hence, at least  $\Delta(G) + 1$  colors are needed to color properly  $G^2$ . This subject was initiated by Kramer and Kramer in [10] and was intensively studied afterwards especially for planar graphs. In 1977, Wegner proposed [13] the following conjecture.

**Conjecture 1** ([13]). *If  $G$  is a planar graph, then:*

- $\chi(G^2) \leq 7$  if  $\Delta(G) = 3$
- $\chi(G^2) \leq \Delta(G) + 5$  if  $4 \leq \Delta(G) \leq 7$
- $\chi(G^2) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$  if  $\Delta(G) \geq 8$ .

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<sup>☆</sup>This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (NRF-2018R1A2B6003412).

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Towards this conjecture, the case of subcubic graphs was settled by Thomassen [11]. Moreover, the last item is proved asymptotically for large  $\Delta$  in [1].

When considering coloring problems on planar graphs, a natural extension is to consider other classes of sparse graphs. In this paper, we are interested in classes defined using the *maximum average degree*, a parameter measuring the sparseness of a graph that we define hereafter. Let  $\text{ad}(H) = \frac{2|E(H)|}{|V(H)|}$  be the *average degree* of a graph  $H$ . The *maximum average degree*  $\text{mad}(G)$  is the maximum value of  $\text{ad}(H)$  when  $H$  is a subgraph of  $G$ . For more details on this invariant see e.g. [6, 12].

Hosseini, Dolama and Sopena in [8] first made the link between the maximum average degree and the chromatic number of the square of a graph. They proved the following result.

**Theorem 2** ([8]). *Let  $G$  be a graph with  $\text{mad}(G) < \frac{16}{7}$ . Then,  $\chi(G^2) = \Delta(G) + 1$ .*

Recently, the following problem was considered in [5] and received some attentions.

**Problem 3** ([5]). For each integer  $k \geq 2$ , what is  $\max\{\chi(G^2) \mid \text{mad}(G) < 2k\}$ ?

For  $k = 2$ , Charpentier [5] conjectured that  $\chi(G^2) \leq 2\Delta(G)$  if  $\text{mad}(G) < 4$ , but it was disproved in [9] where a graph  $G$  such that  $\chi(G^2) = 2\Delta(G) + 2$  and  $\text{mad}(G) < 4$  is constructed. Charpentier [5] proved that for sufficiently large  $\Delta(G)$ ,  $\chi(G^2) \leq 3\Delta(G) + 3$  if  $\text{mad}(G) < 4$ . Thus the results in [5] and [9] implies that

$$2\Delta(G) + 2 \leq \max\{\chi(G^2) \mid \text{mad}(G) < 4\} \leq 3\Delta(G) + 3. \quad (1)$$

In this paper, we study Problem 3 and show that there exists a family of graphs  $G$  with  $\text{mad}(G) < 4$  and arbitrarily large maximum degree such that  $\chi(G^2) \geq \frac{5\Delta(G)}{2}$  (Theorem 17). We also show that  $\chi(G^2) \leq 3\Delta(G) + 1$  if  $\text{mad}(G) < 4$  and  $\Delta(G) \geq 8$  (Theorem 9). Note that the upper bounds  $\chi(G^2) \leq 3\Delta(G) + 1$  are tight for  $\Delta(G) \leq 4$ . These results improve the bounds on (1) to

$$\frac{5\Delta(G)}{2} \leq \max\{\chi(G^2) \mid \text{mad}(G) < 4\} \leq 3\Delta(G) + 1. \quad (2)$$

We also prove upper bounds of  $\chi(G^2)$  for arbitrarily integer  $k \geq 3$  and  $\text{mad}(G) < 2k$ . Charpentier proved [5] that roughly  $(2k - 1)\Delta$  colors are sufficient to color the square of every graph  $G$  with  $\text{mad}(G) < 2k$  and  $\Delta(G) = \Delta$ . For completeness, we give a proof of this result in Section 2. However, we use another method called *ghost discharging*, that we present in Section 2.

In Section 3, we give the proof of upper bounds of  $\chi(G^2)$  for  $\text{mad}(G) < 4$ , and in Section 4, we present a generic construction that allows to extend the lower bound obtained in [9] for graphs with  $\text{mad} < 4$ .

## 2. Generic Upper Bound

In this section, we include a proof of the following result for completeness.

**Theorem 4** ([5]). *Let  $k$  be an integer and  $G$  be a graph with  $\text{mad}(G) < 2k$ . Then*

$$\chi(G^2) \leq \max\{(2k - 1)\Delta(G) - k^2 + k + 1, (2k - 2)\Delta(G) + 2k^3 + k^2 + 2, (k - 1)\Delta(G) + k^4 + 2k^3 + 2\}$$

In the following, we give two improvements: first, we rewrite the original proof using only degeneracy. This allows to directly extend Theorem 4 to generalized notions of coloring such as list-coloring, or correspondence coloring [7]. Moreover, the original proof uses discharging. We give a shorter proof using a variant of discharging relying on the notion of *ghost vertices* defined below. This allows to fix some errors and inaccuracies of the original proof. We actually prove the following.

**Theorem 5.** *Let  $k$  be an integer and  $G$  be a graph with  $\text{mad}(G) < 2k$ . Then  $G^2$  is  $f(k, \Delta)$ -degenerate, where  $f(k, \Delta) = \max\{(2k - 1)\Delta(G) - k^2 + k, (2k - 2)\Delta(G) + 2k^3 + k^2 + 1, (k - 1)\Delta(G) + k^4 + 2k^3 + 1\}$ .*

To prove this result, we use the discharging method. This method was introduced in [14] to study the Four Color Conjecture. It has been used to prove many results on sparse graphs (for example planar, or with bounded  $\text{mad}$ ), culminating with the Four Color Theorem from [2, 3]. This method leads to two-step proofs. In a first step, we prove that if  $G$  is a minimum counterexample to the theorem, it cannot contain some patterns. Then, we prove that every graph from a given class should contain at least one of these patterns. Put together, these assertions prove that every graph from the given class satisfies the theorem.

We thus assume that the theorem is false by taking a graph  $G$  with  $\text{mad}(G) < 2k$  and maximum degree  $\Delta$ , such that  $G^2$  is not  $f(k, \Delta)$ -degenerate, with minimum number of edges. In Subsection 2.1, we give some configurations and show they are not contained in  $G$  (such a configuration is called reducible). Then, in Subsection 2.2, we use the ghost vertices method to reach a contradiction.

### 2.1. Reducible configurations

Given a vertex  $v \in V(G)$ , we denote by  $d(v)$  its degree in  $G$ , and by  $D(v)$  the number of  $(k + 1)^+$ -vertices adjacent to  $v$  in  $G$ .

**Proposition 6.** *The graph  $G$  does not contain a  $k^-$ -vertex  $u$  adjacent to a vertex  $v$  with  $D(v) \leq k$ .*

*Proof.* Assume that  $G$  contains such a configuration. By minimality,  $(G \setminus uv)^2$  is  $f(k, \Delta)$ -degenerate. Take  $\sigma$  an ordering witnessing this degeneracy, and remove  $u, v$  and every  $k^-$ -vertex of  $G$  from  $\sigma$ .

We prove that  $v$  has at most  $f(k, \Delta)$  neighbors in  $G^2$  that remains in  $\sigma$ . Then, since each  $k^-$ -vertex is adjacent to at most  $k\Delta < f(k, \Delta)$  vertices in  $G^2$ , we obtain that  $G^2$  is  $f(k, \Delta)$ -degenerate, a contradiction.

By hypothesis,  $D(v) \leq k$ . Thus, the number of vertices appearing before  $v$  in  $\sigma$  is at most

$$D\Delta + (\Delta - D)(k - 1) \leq k\Delta + (\Delta - k)(k - 1) = (2k - 1)\Delta - k^2 + k \leq f(k, \Delta) \quad \square$$

**Proposition 7.** *The graph  $G$  does not contain a  $k^-$ -vertex  $u$  with a neighbor  $v$  satisfying:*

- $k < D(v) < 2k$
- $v$  has at most  $k - 1$  neighbors  $w$  with  $D(w) \geq \frac{2k^2}{D(v) - k}$ .

*Proof.* Assume that  $G$  contains such a configuration. Again, consider an ordering  $\sigma$  witnessing that  $(G \setminus uv)^2$  is  $f(k, \Delta)$ -degenerate, and remove  $u, v$  and every  $k^-$ -vertex of  $G$  from  $\sigma$ . Denote by  $h$  the number of neighbors  $w$  of  $v$  satisfying  $D(w) \geq \frac{2k^2}{D(v) - k}$ . By hypothesis,  $h < k$ .

Again, since a  $k^-$ -vertex has at most  $k\Delta$  neighbors in  $G^2$  and  $k\Delta \leq f(k, \Delta)$ , it is sufficient to prove that  $v$  has at most  $f(k, \Delta)$  neighbors in  $G^2$  that remain in  $\sigma$ . The number of such vertices is at most

$$h\Delta + (D(v) - h)\frac{2k^2}{D(v) - k} + (\Delta - D(v))(k - 1) = (k + h - 1)\Delta - D(v)(k - 1) + 2k^2 + \frac{2k^2(k - h)}{D(v) - k}$$

Since  $h < k$ , this is a decreasing function of  $D(v)$ . Hence it is at most

$$(k + h - 1)\Delta + k^2 + 1 + 2k^2(k - h)$$

- If  $\Delta \geq 2k^2$ , this is increasing in  $h$ , and thus at most

$$(2k - 2)\Delta + 3k^2 + 1 \leq f(k, \Delta)$$

- Otherwise, it is decreasing in  $h$ , thus at most

$$(k - 1)\Delta + 2k^3 + k^2 + 1 < f(k, \Delta) \quad \square$$

To state the last reducible configuration, we introduce the notion of light vertex. If  $k < D < 2k$ , a vertex  $v$  is  $D$ -light if

- either  $k + 1 \leq D(v) < k + \frac{Dk}{2D-2k}$  and  $v$  has at most  $k - 1$  neighbors  $w$  with  $D(w) \geq \frac{k^2 D}{(D-k)(D(v)-k)}$ .
- or  $k + \frac{Dk}{2D-2k} \leq D(v) < \frac{Dk}{D-k}$  and  $v$  has less than  $D(v) - \frac{(D(v)-2k)D}{2k-D}$  neighbors  $w$  with  $D(w) \geq 2k$ .

We may then state our last reducible configuration.

**Proposition 8.** *The graph  $G$  does not contain a vertex  $u$  with  $k < D(u) < 2k$ , no  $k^-$ -neighbor and adjacent to a  $D(u)$ -light vertex  $v$ .*

*Proof.* Assume that  $G$  contains such a configuration. Again, consider an ordering  $\sigma$  witnessing that  $(G \setminus uv)^2$  is  $f(k, \Delta)$ -degenerate, and remove  $u, v$  and every  $k^-$ -neighbor of  $v$  from  $\sigma$ . We consider the ordering  $\sigma'$  obtaining by appending  $v$ , then  $u$ , then the removed  $k^-$ -vertices to  $\sigma$ .

Again, since a  $k^-$ -vertex has at most  $k\Delta$  neighbors in  $G^2$  and  $k\Delta \leq f(k, \Delta)$ , it is sufficient to prove that  $u$  and  $v$  have at most  $f(k, \Delta)$  neighbors in  $G^2$  that appear previously in  $\sigma'$ .

We first count the  $(k + 1)^+$ -neighbors of  $u$  in  $G^2$ : there are  $v$ , the  $(k + 1)^+$ -neighbors of  $v$ , and the neighbors of the  $D(u) - 1$  neighbors of  $u$ . Thus, there are at most

$$1 + D(v) + (D(u) - 1)\Delta \leq 1 + \frac{D(u)k}{D(u) - k} + (2k - 2)\Delta$$

neighbors of  $u$ . This is a decreasing function of  $D(u)$ , hence it is at most

$$(2k - 2)\Delta + k^2 + k + 1 \leq f(k, \Delta)$$

For  $v$ , we consider two cases according to the definition of  $D(u)$ -light vertex.

- Assume that  $k + 1 \leq D(v) < k + \frac{D(u)k}{2D(u)-2k}$  and  $v$  has  $h$  neighbors  $w$  with  $D(w) \geq \frac{k^2 D(u)}{(D(u)-k)(D(v)-k)}$ .

Then, in  $G^2$ , the number of  $(k + 1)^+$ -neighbors  $v$  besides  $u$  is at most:

$$\begin{aligned} & (\Delta - D(v))(k - 1) + h\Delta + (D(v) - h) \frac{k^2 D(u)}{(D(u) - k)(D(v) - k)} \\ &= (k + h - 1)\Delta - D(v)(k - 1) + \frac{k^2 D(u)}{D(u) - k} + \frac{(k - h)k^2 D(u)}{(D(u) - k)(D(v) - k)}. \end{aligned}$$

Since  $h < k$ , this is a decreasing function of  $D(v)$ , hence at most

$$(k + h - 1)\Delta - (k + 1)(k - 1) + (k - h + 1)k^2 + \frac{(k - h + 1)k^3}{D(u) - k}$$

This is decreasing in  $D(u)$ , hence at most

$$(k + h - 1)\Delta - (k + 1)(k - 1) + (k - h + 1)(k^3 + k^2)$$

– If  $\Delta \geq k^3 + k^2$ , this is an increasing function of  $h$ , hence it is at most

$$(2k - 2)\Delta + k^2 + 1 + 2k^3 \leq f(k, \Delta)$$

– Otherwise, this is a decreasing function of  $h$ , hence it is at most

$$(k - 1)\Delta + k^4 + 2k^3 + 1 \leq f(k, \Delta)$$

- Assume that  $k + \frac{D(u)k}{2D(u)-2k} \leq D(v) < \frac{D(u)k}{D(u)-k}$  and  $v$  has  $h$  neighbors  $w$  with  $D(w) \geq 2k$ , where  $h$  is less than  $D(v) - \frac{(D(v)-2k)D(u)}{2k-D(u)}$ .

First observe that

$$D(v) - \frac{(D(v) - 2k)D(u)}{2k - D(u)} = \frac{2D(u)k - (2D(u) - 2k)D(v)}{2k - D(u)}$$

which is a decreasing function of  $D(v)$ , hence it is at most  $k$  since  $D(v) \geq k + \frac{D(u)k}{2D(u)-2k}$ . Hence  $h \leq k - 1$ .

Consider the  $(k + 1)^+$ -neighbors of  $v$  in  $G^2$  (excepted  $u$ ). There are at most

$$h\Delta + (\Delta - D(v))(k - 1) + (2k - 1)(D(v) - h) = (k + h - 1)\Delta + kD(v) - h(2k - 1)$$

such vertices. This is increasing in  $D(v)$ , hence at most

$$(k + h - 1)\Delta + \frac{k^2 D(u)}{D(u) - k} - k - h(2k - 1)$$

This is decreasing in  $D(u)$ , hence at most

$$(k + h - 1)\Delta + k^2(k + 1) - k - h(2k - 1)$$

– If  $\Delta \geq 2k - 1$ , this is increasing in  $h$ , hence at most

$$(2k - 2)\Delta + k^3 - k^2 - 4k - 1 \leq f(k, \Delta)$$

– Otherwise, this is decreasing in  $h$ , hence at most

$$(k - 1)\Delta + k^3 + k^2 - k \leq f(k, \Delta) \quad \square$$

## 2.2. Ghost vertices

To reach a contradiction, we use the discharging method. Moreover, we consider a so called *Ghost vertices method*, introduced earlier by Bonamy, Bousquet and Hocquard [4].

We begin by giving a weight  $\omega(v) = d(v) - 2k$  to each vertex of  $G$ . We then design some rules in order to redistribute the weights on  $G$  so that the final weights  $\omega'$  satisfy:

- $\omega'(v) \geq 0$  if  $d(v) > k$ .
- $\omega'(v) \geq d(v) + D(v) - 2k$  if  $d(v) \leq k$ .

In this case, we say that  $v$  is *happy*. We first prove that we reach a contradiction if every vertex is happy. Let  $H$  be the subgraph of  $G$  induced by the  $(k+1)^+$ -vertices. Observe that

$$\sum_{u \in G \setminus H} D(u) = |E(H, G \setminus H)| = \sum_{u \in H} (d(u) - D(u))$$

Thus, we have

$$\begin{aligned} \sum_{u \in H} (D(u) - 2k) &= \sum_{u \in H} (d(u) - 2k) - \sum_{u \in H} (d(u) - D(u)) \\ &= \sum_{u \in G} (d(u) - 2k) - \sum_{u \in G \setminus H} (d(u) - 2k) - \sum_{u \in H} (d(u) - D(u)) \\ &= \sum_{u \in G} \omega'(u) - \sum_{u \in G \setminus H} (d(u) - 2k) - \sum_{u \in H} (d(u) - D(u)) \\ &= \sum_{u \in H} \omega'(u) + \sum_{u \in G \setminus H} (\omega'(u) - d(u) + 2k) - \sum_{u \in H} (d(u) - D(u)) \\ &= \sum_{u \in H} \omega'(u) + \sum_{u \in G \setminus H} (\omega'(u) - d(u) + 2k - D(u)) \end{aligned}$$

Each term of the two last sums is non-negative, hence we obtain that  $\text{mad}(G) \geq \text{ad}(H) \geq 2k$ , a contradiction. This thus ends the proof of Theorem 5.

We consider three discharging rules that we apply in order:

- $R_0$ : Every vertex in  $H$  gives 1 to each of its neighbors outside  $H$ .
- $R_1$ : Every vertex  $u$  with  $D(u) \geq 2k + 1$  gives equitably all its weight to its neighbors  $v$  in  $H$  with  $D(v) < 2k$ .
- $R_2$ : Every vertex with positive weight gives equitably all its weight to its neighbors in  $H$  with negative weight.

We now prove that every vertex is happy. First note that due to  $R_0$ , every vertex  $v$  in  $G \setminus H$  receives a weight of  $D(v)$ , and is not affected by  $R_1$  and  $R_2$ . Its final weight is then at least  $d(v) - 2k + D(v)$ , hence it is happy.

We may thus only consider vertices in  $H$ . Let  $u$  be such a vertex. We separate several cases depending on  $D(u)$ . Observe that after  $R_0$ ,  $u$  has weight  $D(u) - 2k$ . We now prove that  $u$  ends up with non-negative weight after  $R_1$  and  $R_2$ . Observe that if, after applying  $R_0$  or both  $R_0, R_1$  a vertex ends with non-negative weight, then it still has non-negative weight after applying the remaining rules.

- Assume that  $D(u) \leq k$ . Then since  $u \in H$ , we have  $d(u) \geq k + 1$ , so  $u$  has a  $k^-$ -neighbor in  $G$ . This is impossible by Proposition 6.
- Assume that  $D(u) \geq 2k$ . Then  $u$  has positive weight after  $R_0$  and  $u$  is happy.
- Assume that  $k < D(u) < 2k$  and  $u$  has a  $k^-$ -neighbor in  $G$ . Then by Proposition 7,  $u$  has at least  $k$  neighbors  $v$  with  $D(v) \geq \frac{2k^2}{D(u) - k}$ .

Observe that since  $D(u) < 2k$ , we have  $D(v) > 2k$ , hence  $w$  gives weight to  $u$  by  $R_1$ . The amount of such weight is at least

$$\frac{D(v) - 2k}{D(v)} = 1 - \frac{2k}{D(v)} \geq 2 - \frac{D(u)}{k}$$

since the middle term is increasing in  $D(v)$ . Since there are at least  $k$  such vertices  $w$ ,  $u$  receives at least  $2k - D(u)$  and thus ends up with non-negative weight after  $R_1$ . Therefore,  $u$  is happy.

- Finally, assume that  $k < D(u) < 2k$  and  $u$  has no  $k^-$ -neighbor in  $G$ . Let  $v$  be a neighbor of  $u$  in  $H$ . We prove that  $v$  gives at least  $\frac{2k}{D(u)} - 1$  to  $u$  by  $R_1$  or  $R_2$ . If true, this would imply that  $u$  receives at least  $2k - D(u)$  and thus ends up with non-negative weight. We separate several cases:

- Assume that  $D(v) \geq \frac{D(u)k}{D(u)-k}$ . Then since  $D(u) < 2k$ , we have  $D(v) > 2k$ , hence  $v$  gives weight to  $u$  by  $R_1$ . The amount given is at least

$$\frac{D(v) - 2k}{D(v)} = 1 - \frac{2k}{D(v)} \geq 1 - \frac{2k(D(u) - k)}{D(u)k} = \frac{2k}{D(u)} - 1$$

as requested.

- Assume that  $k + \frac{D(u)k}{2D(u)-2k} \leq D(v) < \frac{D(u)k}{D(u)-k}$ . Then, by Proposition 8,  $v$  has at least  $D(v) - \frac{(D(v)-2k)D(u)}{2k-D(u)}$  neighbors  $w$  with  $D(w) \geq 2k$ .

Observe that  $D(v) \geq 2k$ , hence  $v$  gives weight to  $u$  by  $R_1$ . Note that  $v$  does not give any weight to neighbors  $w$  with  $D(w) \geq 2k$ , hence  $v$  distributes its weight among at most  $\frac{(D(v)-2k)D(u)}{2k-D(u)}$  vertices. Thus  $u$  receives at least

$$\frac{(D(v) - 2k)(2k - D(u))}{(D(v) - 2k)D(u)} = \frac{2k}{D(u)} - 1$$

- Assume that  $k + 1 \leq D(v) < k + \frac{D(u)k}{2D(u)-2k}$ . Then by Proposition 8,  $v$  has at least  $k$  neighbors  $w$  with  $D(w) \geq \frac{k^2 D(u)}{(D(u)-k)(D(v)-k)}$ . Observe that in this case,  $D(w) \geq 2k + 1$  and  $D(v) < 2k$ , hence  $w$  gives weight to  $v$  by  $R_1$ . The transferred amount is at least

$$\frac{D(w) - 2k}{D(w)} = 1 - \frac{2k}{D(w)} \geq 1 - \frac{2(D(u) - k)(D(v) - k)}{kD(u)}$$

Thus, the weight of  $v$  after  $R_1$  is at least

$$D(v) - 2k + k \left( 1 - \frac{2(D(u) - k)(D(v) - k)}{kD(u)} \right) = (D(v) - k) \left( \frac{2k}{D(u)} - 1 \right)$$

This is non-negative, hence either  $u$  has non-negative weight after  $R_1$ , or it receives weight from  $v$  by  $R_2$ . In this case, observe that  $v$  has at least  $k$  neighbors with non-negative charge, hence the transferred weight is at least

$$\frac{D(v) - k}{D(v) - k} \left( \frac{2k}{D(u)} - 1 \right) = \frac{2k}{D(u)} - 1$$

Therefore,  $u$  ends up happy, and we obtain the required contradiction. This ends the proof of Theorem 5.

### 3. Upper bound when $\text{mad} < 4$

In this section, we prove the following result.

**Theorem 9.** *Let  $G$  be a graph with  $\text{mad}(G) < 4$  and  $\Delta \geq 8$ . Then  $\chi(G^2) \leq 3\Delta(G) + 1$ .*

Observe that this improves Theorem 4 when  $8 \leq \Delta \leq 21$ . To prove Theorem 9, we actually prove that, for every  $\Delta \geq 8$ , if  $G$  is a graph with  $\text{mad}(G) < 4$  and  $\Delta(G) \leq \Delta$ , then  $G^2$  is  $3\Delta$ -degenerate. This implies Theorem 9, as well as its generalizations for list and correspondence coloring.

By contradiction, take a graph  $G$  with  $\text{mad}(G) < 4$  and  $\Delta(G) \geq \Delta$ , and assume that  $G^2$  is not  $3\Delta$ -degenerate. Moreover, assume that  $G$  has minimum number of edges among all the graphs having this property. We say that an ordering of the vertices of  $G$  is *good* if every vertex appears after at most  $3\Delta$  of its neighbors in  $G^2$ .

We again use the discharging method. In Subsection 3.1, we prove that  $G$  does not contain some configurations. Then, in Subsection 3.2, we obtain a contradiction using some weight transfer argument.

#### 3.1. Reducible configurations

To introduce the configurations, we need some terminology.

**Definition 10.** Let  $v$  be a  $d$ -vertex of  $G$ , with  $d_i$  neighbors of degree  $i$  ( $i = 2, 3$ ). If  $d \geq 4$ , we say that:

- $v$  is *nice* if  $d - d_2 \geq 8$ .
- $v$  is *good* if  $d - d_2 \geq 6$ .
- $v$  is *weakly good* if  $d - d_2 = 5$ .
- $v$  is *weakly bad of type 1* if  $d - d_2 = 4$  and  $d_3 = 0$ , and *weakly bad of type 2* if  $d - d_2 = 4$  and  $d_3 = 1$ .
- $v$  is *bad* if  $d - d_2 = 3$ .

According to this definition, we first prove the following classification of the vertices of  $G$ .

**Proposition 11.** *Every  $4^+$ -vertex of  $G$  is bad, weakly bad, weakly good or good.*

*Proof.* Assume there is a  $4^+$ -vertex  $v$  of  $G$  which is not bad, weakly bad, weakly good nor good. This implies that either  $d(v) - d_2(v) \leq 2$  or  $d(v) - d_2(v) = 4$  and  $d_3(v) \geq 2$ .

In the first case, since  $d(v) \geq 4$ ,  $v$  has a 2-neighbor  $w$ . By minimality, take  $\sigma$  a good ordering for  $(G \setminus vw)^2$ . Let  $\sigma'$  be the ordering obtained by removing  $v$  and its 2-neighbors from  $\sigma$ , and adding them (in this order) at the end of  $\sigma$ . We show that  $\sigma'$  is a good ordering.

Note that  $v$  has at most  $2\Delta + \Delta - 2 = 3\Delta - 2$  neighbors appearing before it in  $\sigma'$ . Its 2-neighbors are preceded by at most  $2\Delta$  neighbors in  $\sigma'$ . Thus  $\sigma'$  is a good ordering for  $G$ .

In the second case, let  $w_1, w_2$  be two 3-neighbors of  $v$ . By minimality, take a good ordering  $\sigma$  of  $(G \setminus vw_1)^2$ . Let  $\sigma'$  be obtained by removing  $v, w_1, w_2$  and the 2-neighbors of  $v$  from  $\sigma$  and adding them at the end of  $\sigma$ . Note that  $v$  appears after  $2\Delta + \Delta - 4 + 4 = 3\Delta$  of its neighbors. Similarly,  $w_1, w_2$  appear after  $2\Delta + 4$  of their neighbors. Finally, the 2-neighbors of  $v$  have at most  $2\Delta$  neighbors in  $G^2$ , hence previously in  $\sigma'$ . The ordering  $\sigma'$  is then good for  $G$ , a contradiction.  $\square$

We may now introduce the reducible configurations we consider. We roughly show that vertices with small  $d - d_2$  are not close in  $G$ . We study the neighborhood of the vertices of each type, beginning with the  $3^-$ -vertices.

**Proposition 12.** *In  $G$ , no  $3^-$ -vertex is adjacent to a  $3^-$ -vertex.*

*Proof.* Let  $u, v$  be adjacent  $3^-$ -vertices of  $G$ . By minimality, let  $\sigma$  be a good ordering for  $(G \setminus uv)^2$ . Remove  $u$  and  $v$  from  $\sigma$  and add them at the end of  $\sigma$ . In the obtained coloring  $\sigma'$ , both  $u$  and  $v$  are preceded by at most  $2\Delta + 2$  neighbors. Since  $\Delta > 2$ ,  $\sigma'$  is a good ordering for  $G^2$ , a contradiction.  $\square$

**Proposition 13.** *In  $G$ , every  $4^+$ -neighbor of a bad vertex is not bad.*

*Proof.* Let  $u, v$  be adjacent bad vertices of  $G$ . Let  $w$  be a 2-neighbor of  $v$ . By minimality, take a good ordering  $\sigma$  of  $(G \setminus vw)^2$ . We remove  $v$  and the 2-neighbors of  $u$  and  $v$  from  $\sigma$  and add them in this order at the end of  $\sigma$ . In the obtained coloring  $\sigma'$ , the vertex  $v$  appears after at most  $3\Delta$  of its neighbors. Moreover, each of the (at most)  $2\Delta - 6$  uncolored 2-vertices has at most  $\Delta + 4$  neighbors in  $\sigma$ , hence appears after at most  $3\Delta - 2$  neighbors in  $\sigma'$ . Hence  $\sigma'$  is a good ordering for  $G^2$ , a contradiction.  $\square$

**Proposition 14.** *Let  $v$  be a bad neighbor in  $G$  of a weakly bad vertex  $u$ . Then  $v$  has at least two nice neighbors.*

*Proof.* Assume that  $v$  has a neighbor  $w$  such that  $w$  is not nice and  $w \neq u$ . Since  $v$  is bad, it has a neighbor  $x$  of degree 2. By minimality, we take a good ordering  $\sigma$  of  $(G \setminus vx)^2$ . We remove  $v$  and the 2-vertices incident to  $v, w$  from  $\sigma$  and add them in this order at the end of  $\sigma$ .

In the obtained ordering  $\sigma'$ , the vertex  $v$  has at most  $2\Delta + 1 + d(w) - d_2(w)$  neighbors before it. Since  $w$  is not nice, this is bounded by  $2\Delta + 8$  and by  $3\Delta$  since  $\Delta \geq 8$ . Moreover, each 2-vertex has at most  $2\Delta$  neighbors, hence  $\sigma'$  is a good ordering for  $G'$ , a contradiction.  $\square$

**Proposition 15.** *In  $G$ , each weakly bad vertex of type 2 has at least one good neighbor.*

*Proof.* Let  $u$  be a weakly bad vertex of type 2 without nice neighbor. Let  $v_1, v_2, v_3$  be the neighbors of  $u$  that are not good and let  $w$  be the 3-neighbor of  $u$ . By minimality, take a good ordering  $\sigma$  of  $(G \setminus uw)^2$ . We define an ordering  $\sigma'$  by removing  $u, w$  and the 2-vertices adjacent to  $u, v_1, v_2, v_3$  from  $\sigma$  and adding them in this order at the end of  $\sigma$ .

The number of neighbors of  $u$  preceding it in  $\sigma'$  is at most  $\Delta - 2 + d(v_1) - d_2(v_1) + d(v_2) - d_2(v_2) + d(v_3) - d_2(v_3) \leq \Delta + 13$ . Since  $\Delta \geq 8$ , this is bounded by  $3\Delta$ .

The vertex  $w$  has degree 3, hence has at most  $3\Delta$  neighbors in  $G^2$ . Finally, the remaining 2-vertices have at most  $2\Delta$  neighbors. Therefore,  $\sigma'$  is a good ordering for  $G^2$ , a contradiction.  $\square$

**Proposition 16.** *In  $G$ , each weakly good vertex has at most three neighbors that are 3-vertices or bad vertices with at most one nice neighbor.*

*Proof.* Let  $u$  be a weakly good vertex of  $G$  with at least four neighbors  $v_1, \dots, v_4$  that have degree 3 or are bad vertices with at most one nice neighbor.

If  $v_1$  has degree 3, we take a good ordering  $\sigma$  of  $(G \setminus uv_1)^2$  by minimality. Otherwise,  $v_1$  is a bad vertex so it has a 2-neighbor  $w$ . In this case, we take  $\sigma$  as a good ordering of  $(G \setminus v_1w)^2$ .

In both cases, we denote by  $\sigma'$  the ordering obtained by removing  $u, v_1, \dots, v_4$  and their 2-neighbors from  $\sigma$ .

To construct a good ordering for  $G^2$ , we first consider the bad vertices among  $v_1, \dots, v_4$ . Assume that  $v_i$  is bad for some  $i = 1, \dots, 4$  and denote by  $x$  one of its non-nice neighbors. We

remove the 2-neighbors of  $x$  from  $\sigma'$  and add  $v_i$  at the end of  $\sigma'$ . Note that  $v_i$  has at most  $2\Delta + 1 + d(x) - d_2(x) \leq 2\Delta + 8$  appearing in  $\sigma'$ , which is less than  $3\Delta$  since  $\Delta \geq 8$ .

We then add  $u$  at the end of  $\sigma'$ . It is still a good ordering since  $u$  has at most  $2\Delta + 7 \leq 3\Delta$  neighbors in  $\sigma'$ . We then add the remaining vertices  $v_i$  (of degree 3) to the end of  $\sigma'$ . Note that they have at most  $2\Delta + 5$  neighbors in  $\sigma'$ .

Finally, we add all the remaining 2-vertices at the end of  $\sigma'$ . Then  $\sigma'$  is a good coloring for  $G^2$ , a contradiction.  $\square$

### 3.2. Discharging part

We may now reach a contradiction. We give an initial weight  $\omega(v) = d(v) - 4$  to each vertex  $v$  of  $G$ . Since  $\text{mad}(G) < 4$ , the total weight is negative.

Observe that the ghost method we use in Section 2 seems not to be useful there. Indeed, we could have used  $2^-$ -vertices as ghosts. In this case, we should have designed discharging rules such that the following assertions hold:

- If  $v$  is a  $3^+$ -vertex, then  $v$  ends up with non-negative weight.
- If  $v$  is a 2-vertex, then  $v$  ends up with weight at least  $d(v) - 4 + d_{3^+}(v)$ .

Since  $2^-$ -vertices are not adjacent by Proposition 12, the last constraint can be rewritten as:  $2^-$ -vertices have to end with non-negative weight. Thus, we basically end up with what we actually have to prove. We now introduce some discharging rules.

We first apply the following rule: each vertex gives 1 to its neighbors of degree 2 and  $\frac{1}{3}$  to its neighbors of degree 3. Observe that nice vertices are all good. We may then state our other rules:

1. Every nice vertex gives  $\frac{1}{2}$  to its bad neighbors.
2. Every  $4^+$ -vertex which is not nice gives  $\frac{1}{3}$  to each bad neighbor having at most one nice neighbor.
3. Every good vertex gives  $\frac{1}{3}$  to its weakly bad neighbors of type 2.

We now show that every vertex of  $G$  ends up with non-negative weight, which is a contradiction with the hypothesis  $\text{mad}(G) < 4$ . We separate several cases according to the type of vertices we consider.

**$3^-$ -vertices.** By the first rule, each 2-vertex  $v$  of  $G$  receives 1 from each of its neighbors. Moreover,  $v$  does not lose any weight, thus its final weight is  $\omega'(v) = 2 - 4 + 2 \times 1 = 0$ .

Similarly, each 3-vertex ends up with non-negative weight since it does not lose weight and each of its neighbors gives it  $\frac{1}{3}$  by the first rule. So  $\omega'(v) = 3 - 4 + 3 \times \frac{1}{3} = 0$ .

**Bad vertices.** Let  $v$  be a bad vertex of  $G$ . After applying the first rule,  $v$  has weight  $-1$ . Recall that bad vertices are not good, and no neighbor of  $v$  is bad by Proposition 13, so  $v$  does not lose some additional weight.

Due to Rule 1, if  $v$  has at least two nice neighbors, then  $v$  ends up with  $\omega'(v) = -1 + 2 \times \frac{1}{2} = 0$ . Otherwise, Rule 2 applies, and  $v$  receives  $3 \times \frac{1}{3}$  from its  $4^+$ -neighbors. Thus  $\omega'(v) \geq 0$ .

**Weakly bad vertices.** Let  $v$  be a weakly bad vertex of  $G$ . Recall that  $v$  is not good. Moreover, if  $v$  has a bad neighbor  $w$ , then Proposition 14 ensures that  $w$  has two nice neighbors, so  $v$  does not lose any weight during the second phase.

Thus, if  $v$  has type 1, then it ends up with no weight after the first phase so its final weight is  $\omega'(v) = 0$ .

Otherwise,  $v$  has type 2, so it has weight  $-\frac{1}{3}$  after the first phase. By Proposition 15, it has a good neighbor, so it receives  $\frac{1}{3}$  by Rule 3, and ends up with weight 0.

**Weakly good vertices.** Let  $v$  be a weakly good vertex of  $G$ . After giving weight to 2-vertices,  $v$  ends up with weight 1. Note that  $v$  is not good, so  $v$  only loses weight for each vertex of degree 3 or to bad neighbors with at most one nice neighbor. By Proposition 16,  $v$  has at most three such neighbors, so  $v$  ends up with non-negative weight.

**Good vertices.** Let  $v$  be a good vertex of  $G$  of degree  $d$  with  $d_2$  neighbors of degree 2. If  $v$  is not nice, it loses  $\frac{1}{3}$  for at most  $d - d_2$  neighbors, hence its final weight is at most  $d - 4 - d_2 - \frac{d-d_2}{3} = \frac{2}{3}(d - d_2) - 4 \geq 0$  since  $d - d_2 \geq 6$ .

Otherwise,  $v$  loses  $\frac{1}{2}$  for at most  $d - d_2$  neighbors, so its final weight is at most  $d - 4 - d_2 - \frac{d-d_2}{2} = \frac{d-d_2}{2} - 4 \geq 0$  since  $d - d_2 \geq 8$ .

By Proposition 11, every vertex has been considered by one of the previous arguments. Therefore, every vertex ends up with non-negative weight, which concludes.

#### 4. Lower Bound

In this section, we investigate the lower bounds for  $\chi(G^2)$  when  $G$  is a graph with  $\text{mad}(G) < 4$ . We first consider graphs with small  $\Delta$ , here  $\Delta \leq 5$ .

##### 4.1. Small $\Delta$

For  $\Delta = 1$ ,  $G$  is a matching, hence  $G^2$  is 2-colorable, which is tight when  $G = P_2$ .

For  $\Delta = 2$ ,  $G$  is a path or a cycle, hence  $G^2$  is 4-degenerated and 5-colorable. This is tight, as shown by  $C_5$ .

For  $\Delta = 3$ , the Petersen graph needs 10 colors since it has diameter two. This achieves the upper bound  $3\Delta + 1$  for  $\Delta = 3$ .

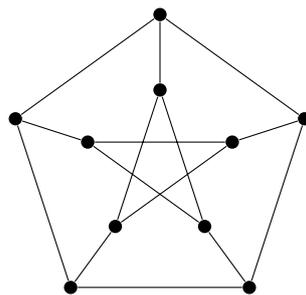


Figure 1:  $\chi(G^2) = 10$ ,  $\text{mad} < 4$ ,  $\Delta = 3$

For  $\Delta = 4$ , the following graph also has diameter two and thus needs 13 colors, also achieving the bound  $3\Delta + 1$ .

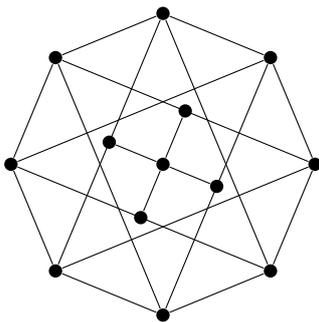


Figure 2:  $\chi(G^2) = 13$ ,  $\text{mad} < 4$ ,  $\Delta = 4$

Finally, for  $\Delta = 5$ , the following graph needs 15 colors (the black and red vertices induce a clique in the square). This graph is build from a Petersen graph adding five vertices of degree 3 linked by paths of length 2. Note that this graph has  $\text{mad} 4$ . However, removing the red part leads to a graph of  $\text{mad}$  less than 4 that needs 14 colors.

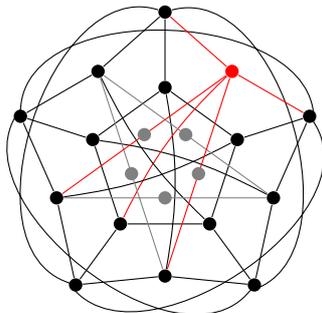


Figure 3:  $\chi(G^2) = 14$ ,  $\text{mad} < 4$ ,  $\Delta = 5$

#### 4.2. Large $\Delta$

We now give a construction improving the result of [9] when  $\text{mad}(G) < 4$ , even when  $G$  is 2-degenerate. We actually prove the following result.

**Theorem 17.** *There exists a family of 2-degenerate graphs  $G$  with  $\text{mad}(G) < 4$ , arbitrarily large maximum degree, and  $\chi(G^2) \geq \frac{5\Delta(G)}{2}$ .*

Let  $t$  be an integer. We define  $G_t$  as the graph obtained from  $K_5$  by applying successively the two following operations:

- Replacing each edge  $e$  by a copy of  $K_{2,t}$  by identifying the endpoints of the edge with the two vertices in the same partition. We denote by  $V_e$  the  $t$  vertices added while replacing  $e$ .
- For each pair of non-incident edges  $e, f$ , we add a path over two edges between each pair of vertices in  $V_e \times V_f$ .

For  $t > 2$ , observe that  $\Delta(G_t) = 4t$  and  $G_t$  is 2-degenerated (consider the vertices by reversing their order of creation). Thus  $\text{mad}(G_t) < 4$ .

Moreover, the vertices in  $\cup_{e \in E(K_5)} V_e$  induce a clique of size  $10t$  in  $G_t^2$ . Therefore, we have  $\chi(G_t^2) \geq 10t = \frac{5\Delta(G_t)}{2}$ .

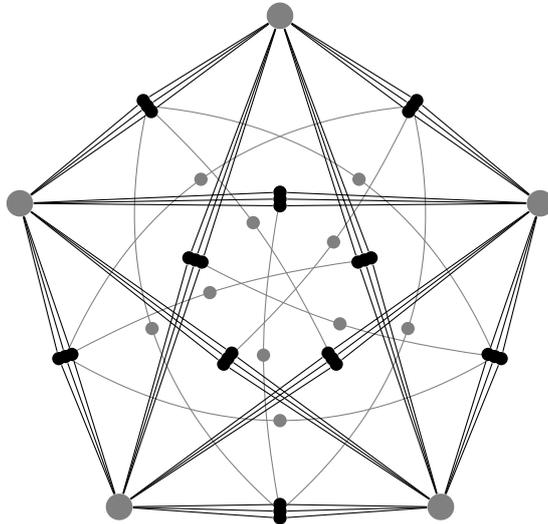


Figure 4: The graph  $G_t$ , black vertices induce a clique in  $G_t^2$

Observe that a similar construction can be done starting from any cliques  $K_n$ . For  $n = 6$ , this gives the same lower bound. However, when  $n \geq 7$ , the clique number of  $G_t^2$  is  $\frac{tn(n-1)}{2}$  while  $\Delta(G_t) = t(\frac{n(n-1)}{2} - 2n + 3)$ , which gives a worse lower bound.

## 5. Conclusion

In this paper we investigate lower and upper bounds for square coloring of graphs with maximum average degree bounded, especially with  $\text{mad} < 4$ . Reducing the gap between the lower bounds and the upper bounds in (2) is an interesting problem. So we have the following question.

**Question 18.** Is there an integer  $D$  such that every graph  $G$  with  $\Delta(G) \geq D$  and  $\text{mad}(G) < 4$  has  $\chi(G^2) \leq \frac{5\Delta(G)}{2}$ ?

Note that the constructions in Theorem 17 are actually 2-degenerate. So we propose the following question.

**Question 19.** Is there an integer  $D$  such that every graph  $G$  with  $\Delta(G) \geq D$  has  $\chi(G^2) \leq \frac{5\Delta(G)}{2}$  if  $G$  is 2-degenerate?

Moreover, while this lower bound cannot be strengthened using larger cliques, there may be a way of generalizing the given construction. Indeed, instead of considering a clique and replacing edges by a bipartite graph  $K_{2,p}$ , consider an hypergraph on  $kr$  vertices where all the hyperedges of size  $k$  are present, and replace each hyperedge by a bipartite graph  $K_{k,p}$  (the construction for Theorem 17 is the case  $k = 2$ ). Denote by  $V_e$  the vertices added while applying this construction

to the hyperedge  $e$  and by  $G$  the obtained graph. The problem is then to add paths of length 2 between  $V_e$  and  $V_f$  for every pair  $(e, f)$  of non incident hyperedges. Given a set of  $k$  pairwise non-incident edges  $\{e_1, \dots, e_k\}$ , we can add  $p^2$  vertices of degree  $k$  to  $G$  such that  $V_{e_1} \cup \dots \cup V_{e_k}$  induces a clique in  $G^2$ . However, if this is done for every set of  $k$  pairwise non-incident edges, the degree of vertices in each  $V_e$  is too large to obtain a good bound.

Thus, we need to find a suitable packing of the hyperedges of the considered hypergraph. In other terms, we have to solve the following problem:

**Question 20.** Given an integer  $k$ , is there an integer  $r$  and a family  $\mathcal{S}$  of sets such that the following holds?

1. Each set of  $\mathcal{S}$  is a set of  $r$  pairwise disjoint  $k$ -subsets of  $\llbracket 1, rk \rrbracket$ .
2. If  $S, T$  are two  $k$ -subsets of  $\llbracket 1, rk \rrbracket$ , there exists an element of  $\mathcal{S}$  containing both  $S$  and  $T$ .
3. If  $S$  is a  $k$ -subset of  $\llbracket 1, rk \rrbracket$ ,  $S$  is contained in at most  $\frac{1}{k-1} \binom{k(r-1)}{k}$  elements of  $\mathcal{S}$ .

Solving this problem with  $r = k$  would yield a bound of the same order than in [9]. However, we believe that the parameter  $r$  can be optimized (as done in Section 4, with  $k = 2$  and  $r = 3$ ) to obtain much better values. Note that for our purposes, the bound of Item 3 can be weakened up to an additive constant, or even to  $\frac{1}{k-1} \binom{k(r-1)}{k} (1 + o_r(1))$  (with possibly some consequences on the resulting lower bound).

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