Existence of multi-solitary waves with logarithmic relative distances for the NLS equation

Tiên Vinh Nguyễn

EXISTENCE OF MULTI-SOLITARY WAVES WITH LOGARITHMIC RELATIVE DISTANCES FOR THE NLS EQUATION

EXISTENCE D'ONDES SOLITAIRES MULTIPLES AVEC DISTANCES RELATIVES LOGARITHMIQUES DE SCHRÖDINGER NON LINÉAIRES

NGUYỄN TIẾN VINH

Abstract. We construct 2-solitary wave solutions with logarithmic distance to the nonlinear Schrödinger equation,

$$\mathrm{i}\,\partial_t u + \Delta u + |u|^{p-1}u = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d,$$

in mass-subcritical cases $1 < p < 1 + \frac{4}{d}$ and mass-supercritical cases $1 + \frac{4}{d} < p < \frac{d+2}{d-2}$, i.e. solutions $u(t)$ satisfying

$$\left\| u(t) - \mathrm{e}^{\mathrm{i}\gamma(t)} \sum_{k=1}^{2} Q(\cdot - x_k(t)) \right\|_{H^1} \to 0$$

and

$$|x_1(t) - x_2(t)| \sim 2\log t, \qquad \text{as} \quad t \to +\infty,$$

where Q is the ground state. The logarithmic distance is related to strong interactions between solitary waves.

In the integrable case ($d = 1$ and $p = 3$), the existence of such solutions is known by inverse scattering (E. Olmedilla, Multiple pole solutions of the nonlinear Schrödinger equation, Physica D 25 (1987) 330–346; T. Zakharov, A.B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP 34 (1972) 62–69). The mass-critical case $p = 1 + \frac{4}{d}$ exhibits a specific behavior related to blow-up, previously studied in Y. Martel, P. Raphaël, Strongly interacting blow up bubbles for the mass critical NLS (to appear in Ann. Sci. École Norm. Sup.).

Résumé. On construit des solutions au problème de la propagation de deux ondes solitaires avec distance logarithmique de Schrödinger non linéaire,

$$\mathrm{i}\,\partial_t u + \Delta u + |u|^{p-1}u = 0, \qquad t \in \mathbb{R}, x \in \mathbb{R}^d,$$

dans le cas d'une masse souscritique $1 < p < 1+\frac{4}{d}$ et d'une masse surcritique $1+\frac{4}{d} < p < \frac{d+2}{d-2}$, autrement dit, $u(t)$, qui satisfait

$$\left\| u(t) - \mathrm{e}^{\mathrm{i}\gamma(t)} \sum_{k=1}^{2} Q(\cdot - x_k(t)) \right\|_{H^1} \to 0$$

et

$$|x_1(t) - x_2(t)| \sim 2\log(t) \quad \text{quand} \quad t \to +\infty,$$

où Q est l'état fondamental. La distance logarithmique est liée à l'interaction forte entre ondes solitaires.

Dans le cas intégrable ($d = 1$ et $p = 3$), l'existence d'une telle solution est connue par la méthode dite d'*inverse scaterring* (E. Olmedilla. Multiple pole solutions of the nonlinear Schrödinger equation, Physica D 25 (1987) 330–346 ; T. Zakharov, A.B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP 34 (1972) 62–69). Le cas d'une masse critique $p = 1 + \frac{4}{d}$ introduit un comportement spécifique lié à l'explosion, qui a été étudié précédemment par Y. Martel et P. Raphaël (Strongly interacting blow up bubbles for the mass critical NLS (à paraître dans Ann. Sci. École Norm. Sup.).

1. Introduction

We consider the nonlinear Schrödinger equation in $\mathbb{R}^d$, for any $d \geq 1$:

$$\begin{cases} \mathrm{i}\,\partial_t u = -\Delta u - |u|^{p-1}u, \quad (t,x) \in [0,T) \times \mathbb{R}^d \\ u(0,x) = u_0, \quad u_0 \in H^1 : \mathbb{R}^d \to \mathbb{C}. \end{cases} \tag{NLS}$$

It is well known (see, e.g., [2], [10]) that the NLS equation is locally well-posed in $H^1(\mathbb{R}^d)$ for $1 < p < \frac{d+2}{d-2}$ ($p > 1$ if $d = 1, 2$): for any $u_0 \in H^1(\mathbb{R}^d)$, there exist $T^\star > 0$ and a unique maximal solution $u \in \mathcal{C}([0,T^\star), H^1(\mathbb{R}^d))$ of (NLS). Moreover, the following blow-up criterion holds

$$T^\star < +\infty \quad \text{implies} \quad \lim_{t \uparrow T^\star} \|\nabla u(t)\|_{L^2} = +\infty. \tag{1.1}$$

Recall that the solution u satisfies the following three conservation laws:

- mass,

$$\int_{\mathbb{R}^d} |u(t,x)|^2 \mathrm{d}x = \int_{\mathbb{R}^d} |u_0(x)|^2 \tag{1.2}$$

- energy,

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 - \frac{1}{p+1}\int_{\mathbb{R}^d} |u(t,x)|^{p+1}\mathrm{d}x = E(u_0) \tag{1.3}$$

- momentum,

$$M(u(t)) = \operatorname{Im}\int_{\mathbb{R}^d} \nabla u(t,x)\bar{u}(t,x)\mathrm{d}x = M(u_0) \tag{1.4}$$

for all $t \in [0, T^\star)$. Recall also that (NLS) admits the following symmetries: the transformation of initial data implies the corresponding transformation of solution:

- scaling, $\lambda > 0, \lambda^{\frac{2}{p-1}} u_0(\lambda x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$;
- space translation, $x_0 \in \mathbb{R}^d, u_0(x + x_0) \mapsto u(t, x + x_0)$;
- time translation, $t_0 \in \mathbb{R}, u_{t_0}(x) \mapsto u(t + t_0, x)$;
- space rotation, $A \in SO(d), u_0(A \cdot x_0) \mapsto u(t, A \cdot x_0)$;
- phase, $\gamma \in \mathbb{R}, u_0(x)\mathrm{e}^{\mathrm{i}\gamma} \mapsto u(t,x)\mathrm{e}^{\mathrm{i}\gamma}$;
- Galilean: $\beta \in \mathbb{R}^d, u_0(x)\mathrm{e}^{\mathrm{i}\beta x} \mapsto u(t, x - \beta t)\mathrm{e}^{\mathrm{i}\beta(x - \frac{\beta}{2}t)}$.

As a consequence of (1.2), (1.3), and the Gagliardo–Nirenberg inequality, all solutions to (NLS) are global in the L^2 subcritical case ($1 < p < 1 + \frac{4}{d}$). In contrast, blow-up solutions exist in the L^2 critical case ($p = 1 + \frac{4}{d}$) and the L^2 supercritical case ($1 + \frac{4}{d} < p < \frac{d+2}{d-2}$). See, e.g., [2].

This article is concerned with the construction of special solutions to the NLS equation involving solitary wave solutions (or solitons). We recall the expression of the (standing) solitary waves

$$u(t,x) = \mathrm{e}^{\mathrm{i}\lambda_0^2 t} Q_{\lambda_0}(x) \quad \text{with} \quad Q_{\lambda_0}(x) = \lambda_0^{\frac{2}{p-1}} Q(\lambda_0 x)$$

where $\lambda_0 > 0$ and Q is the ground state, *i.e.* the unique radial positive solution to

$$\Delta Q - Q + Q^p = 0, \quad Q > 0, \quad Q \in H^1(\mathbb{R}^d). \tag{1.5}$$

The whole family of ground-state solitary waves is obtained using the above symmetries. For more properties of Q, see, for example, [2] and [28]. Recall that in the L^2 subcritical case, the solitary waves are stable ([2, 31]), and that in the L^2 critical and L^2 supercritical cases, the solitary waves are unstable [2, 12].

1.1. **Motivation.** So far, the problem of multi-solitary wave solutions to (NLS) has been studied intensively in the integrable case, i.e. for $d = 1$ and $p = 3$, as well as for some nearly integrable models; see [9, 11, 16, 19, 32, 33]. In particular, it is known from the inverse scattering theory that there are three possible 2-soliton behaviors in the integrable case:

(a) relative distance between solitons of order t, [33];
(b) logarithmic relative distance with symmetric solitons (*double-pole* solutions), [19, 33];
(c) finite relative distance periodic in time, [32, 33].

Note that (a) corresponds to a free Galilean motion, while (b) and (c) correspond to a non-free Galilean motion. Remarkably, these solutions admit a pure 2-soliton behavior both for $t \to +\infty$ and $t \to -\infty$. They describe perfectly elastic interactions between solitary waves in the integrable case.

In the non-integrable cases, the problem is much less comprehended, except for multi-solitary waves with free Galilean motion (a) in one direction of time; see Remark 1 below for a precise statement. In the present paper, we raise the question of other possible behaviors of multi-solitons. In other words, we ask whether the above non-generic dynamics (b), (c) of the integrable case persist for non-integrable models. Previous works, see, *e.g.*, [8, 11, 16], study formally the dynamics of interacting pulses for several integrable or non-integrable models, and predict the persistence of the logarithmic regime. Indeed, the 2-soliton dynamics is related in some sense to the simple differential equation $\ddot{z}(t) = -\mathrm{e}^{-2z(t)}$, where $z(t)$ is half of the distance between the solitons, and for which $\log t$ is a special solution. The main point of the present work is to justify that 2-solitons with logarithmic relative distance engage in universal behavior in both subcritical and supercritical NLS equations in the presence of symmetry, thus proving rigorously the persistence of behavior (b) in the non-integrable case.

1.2. **Main result.** In this article, we prove the following general existence result.

Main Theorem (multi-solitary waves with logarithmic distance)**.** *Let $d \geq 1$. Let*

$$1 < p < \frac{d+2}{d-2} \quad (p > 1 \text{ for } d = 1, 2) \quad \textit{and} \quad p \neq 1 + \frac{4}{d}.$$

There exists an H^1 solution $u(t)$ to (NLS) *on* $[0, +\infty)$ *which decomposes asymptotically into two solitary waves, for all $t > 0$,*

$$\left\| u(t) - \mathrm{e}^{\mathrm{i}\gamma(t)} \sum_{k=1}^{2} Q(\cdot - x_k(t)) \right\|_{H^1(\mathbb{R}^d)} \lesssim \frac{1}{t} \tag{1.6}$$

where $x_1(t) = -x_2(t)$ and

$$|x_1(t) - x_2(t)| = 2(1 + o(1)) \log t, \quad \textit{as} \quad t \to +\infty. \tag{1.7}$$

The Main Theorem holds for any space dimension and any $\dot{H}^1$ subcritical nonlinearity, except the mass critical power $p = 1 + \frac{4}{d}$. Indeed, the critical nonlinearity exhibits a different phenomenon of strong interactions due to blow-up, previously studied in [24]; see Remark 2.

Note that the result should holds with a similar proof for any number $K \geq 2$ of solitons located on a regular polygon of size $\log t$. By scaling invariance, we can replace Q in (1.6) by Q_{λ_0} for any $\lambda_0 > 0$. We observe that, in the result, solitons need to have the same sign, the same scaling and the same phase; in fact, the solution is symmetric by $\tau : x \mapsto -x$. Moreover, the solution is also symmetric by the reflection across the axis passing by the center of the two solitons. Remark that the situation is the same with the multi-solitons constructed in [19, 33] for the integrable case.

Remark 1. For the NLS equation, multiple solitary wave solutions with weak interactions, *i.e.* relative distance between solitons of order t, have been constructed in various settings, both in stable and unstable contexts, see, in particular, [4, 20, 25]. A typical result of weakly interacting dynamics is the existence of multi-solitary wave solutions to (NLS) satisfying, as $t \to +\infty$,

$$\left\| u(t) - \sum_{k=1}^{K} \mathrm{e}^{-\mathrm{i}\Gamma_k(t,x)} Q_{\lambda_k}(\cdot - \nu_k t) \right\|_{H^1(\mathbb{R}^d)} \lesssim \mathrm{e}^{-ct}, \quad c > 0, \tag{1.8}$$

for any given set of parameters $\{\nu_k, \lambda_k\}_k \in \mathbb{R}^d \times (0, \infty)$, provided that the following decoupling condition holds: $\nu_k \neq \nu_{k'}$ if $k \neq k'$.

Remark 2. For the L^2 critical case ($p = 1 + \frac{4}{d}$), the existence of bounded multi-solitary wave solutions with logarithmic distances as (1.6)–(1.7) is ruled out. Indeed, for such solutions, one would have

$$\int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 \mathrm{d}x \sim \log^2(t) \tag{1.9}$$

which is in contradiction with the virial identity

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{\mathbb{R}^d} |x|^2 |u|^2 = 16 E(u_0).$$

In fact, in the mass-critical case, the scaling instability directions are excited by the nonlinear interactions, which leads to the infinite-time concentration, as shown by Theorem 1 in [24]: for the L^2 critical two-dimensional case, there exists a global (for $t \geq 0$) solution $u(t)$ that decomposes asymptotically into a sum of solitary waves

$$\left\| u(t) - \mathrm{e}^{\mathrm{i}\gamma(t)} \sum_{k=1}^{K} \frac{1}{\lambda(t)} Q\left(\frac{\cdot - x_k(t)}{\lambda(t)} \right) \right\|_{H^1(\mathbb{R}^d)} \to 0, \quad \lambda(t) = \frac{1 + o(1)}{\log t} \quad \text{as} \quad t \to +\infty, \tag{1.10}$$

where the translation parameters $x_k(t)$ converge to the vertices of a K-sided regular polygon and the solution blows up in infinite time with the rate

$$\|\nabla u(t)\|_{L^2} \sim |\log t| \quad \text{as} \quad t \to +\infty.$$

The regime justified in the present paper is thus different from the one in [24] since, for the critical case, the interactions primarily affect the scaling parameter, leading to blow-up. This notable difference with the sub- and supercritical cases shows that a formal approach may not be sufficient to correctly address such subtle regimes.

We also refer to [14, 15, 17, 18, 22] for previous works on other nonlinear equations where a refined analysis of interactions between solitons is a key point.

Remark 3. We expect solutions in Main Theorem to be unstable, even in L^2 subcritical cases, since generic perturbation can give collision or on the contrary weak interaction. Recall that the appearance of the log regime is closely related to the equation

$$\ddot{z}(t) = -\mathrm{e}^{-2z(t)}$$

where $\log t$ is a solution with initial conditions $z(1) = 0$, $\dot{z}(1) = 1$. From the theory of perturbation, for $z(t) = \log t + \epsilon v_1 + ...$ with initial conditions $z(1) = \epsilon$, $\dot{z}(1) = 1$, one has, at the linear level,

$$\ddot{v}_1 = \frac{2v_1}{t^2}, \quad v_1(1) = 1, \quad \dot{v}_1(1) = 0,$$

whose solution is $\frac{1}{3}t^2 + \frac{2}{3}\frac{1}{t}$, so that we see that the $\log t$ solution is an unstable state as $t \to +\infty$.

Remark 4. We believe that our approach is general. In particular, the strategy of this article can be applied to construct multi-solitary waves with logarithmic relative distance for more general nonlinearity $f(s)$

$$\mathrm{i}\,\partial_t u + \Delta u + f(|u|^2)u = 0$$

where $f(s)$ satisfies standard conditions for the existence of solitary waves (see [23]). Moreover, combining the construction in this paper and the construction of multi-soliton solutions with weak interactions in [4], [20], we prove the existence of multi-solitons, with both solitons distant as Ct and solitons distant as $C\log t$.

Remark 5. One can give a more precise asymptotic description of the distance (1.7) between solitons

$$|x_1(t) - x_2(t)| = 2\log t - \frac{d-1}{2}\log(\log t) - C + O(\log^{-\frac{1}{2}}(t)) \quad \text{ as } t \to +\infty$$

where $C > 0$ a constant depending only on d and p (see (3.22)).

The article is organized as follows. Sections 2, 3, and 4 concern the proof of the Main Theorem in L^2 subcritical cases with $p > 2$. In Section 2, we consider an approximate solution (an ansatz solution) to (NLS) made of two symmetric bubbles and extract the formal evolution system of the geometrical parameters of the bubbles (scaling, position, phase). The key observation is that this system contains forcing terms due to the nonlinear interactions of the waves, and has a special solution corresponding at the main order to the regime of Main Theorem. Here, in contrast with free Galilean motion, the construction of a non-free Galilean motion as (1.7) requires a refined control of strong interactions between the solitary waves to bound the error terms. In Section 3, we consider, using modulation, particular backwards solutions to (NLS) related to the special regime of Main Theorem and prove backward uniform estimates by energy method. In Section 4, we use compactness arguments on a suitable sequence of such backward solutions to finish the proof. Sections 5 deals with the case $1 < p \leq 2$; in this case, there are some extra technical difficulties, even if the strategy of the proof is similar: the interaction becomes stronger, we have to add extra terms in the approximate solution and due to lost of regularity, we have to use some truncations. Finally, the algebraic computations in the proof for L^2 subcritical cases are still valid in L^2 supercritical cases. Section 6 presents additional arguments and modifications needed for L^2 supercritical cases.

1.3. **Notation.** The L^2 scalar product of two complex valued functions $f, g \in L^2(\mathbb{R}^d)$ is denoted by

$$\langle f, g\rangle = \operatorname{Re}\left(\int_{\mathbb{R}^d} f(x)\,\overline{g}(x)\,\mathrm{d}x\right).$$

We denote by $Q(x) := q(|x|)$ the unique radial positive ground state of (NLS):

$$q'' + \frac{d-1}{r}q' - q + q^p = 0, \quad q'(0) = 0, \quad \lim_{r\to+\infty} q(r) = 0. \tag{1.11}$$

It is well known and easily checked by ODE arguments that, for some constant $c_Q > 0$,

$$\text{for all } r > 1, \quad \left|q(r) - c_Q r^{-\frac{d-1}{2}}\mathrm{e}^{-r}\right| + \left|q'(r) + c_Q r^{-\frac{d-1}{2}}\mathrm{e}^{-r}\right| \lesssim r^{-\frac{d-1}{2}-1}\mathrm{e}^{-r}. \tag{1.12}$$

We set

$$I_Q = \int Q^p(x)\, \mathrm{e}^{-x_1} \mathrm{d}x, \quad x = (x_1, ..., x_d).$$

We denote by $\mathcal{Y}$ the set of smooth functions f such that

$$\text{for all } p \in \mathbb{N}, \text{ there exists } q \in \mathbb{N}, \text{ s.t. for all } x \in \mathbb{R}^d, \quad |f^{(p)}(x)| \lesssim |x|^q \mathrm{e}^{-|x|}. \tag{1.13}$$

Let Λ be the generator of L^2-scaling corresponding to (NLS):

$$\Lambda f = \frac{2}{p-1} f + x \cdot \nabla f.$$

The linearization of (NLS) around Q involves the following Schrödinger operators:

$$L_+ := -\Delta + 1 - pQ^{p-1}, \qquad L_- := -\Delta + 1 - Q^{p-1}.$$

From [30], recall the generalized null space relations in subcritical and super-critical cases:

$$\begin{aligned} &L_- Q = 0, \quad L_+(\Lambda Q) = -2Q, \\ &L_+(\nabla Q) = 0, \quad L_-(xQ) = -2\nabla Q. \end{aligned} \tag{1.14}$$

We recall the coercivity property in L^2 subcritical (see [20], [26], [30], [31]): there exists $\mu > 0$ such that for all $\eta \in H^1$,

$$\langle L_+ \operatorname{Re}\eta, \operatorname{Re}\eta\rangle + \langle L_- \operatorname{Im}\eta, \operatorname{Im}\eta\rangle \geq \mu \|\eta\|_{H^1}^2 - \frac{1}{\mu}\left(\langle \eta, Q\rangle^2 + |\langle \eta, xQ\rangle|^2 + \langle \eta, i\Lambda Q\rangle^2\right). \tag{1.15}$$

In L^2 supercritical (but H^1 subcritical), we do not have the same situation since the negative direction can not be controlled by the scaling parameter. We consider the operator

$$\mathcal{L}v = \mathrm{i}L_+ v_1 - L_- v_2 \ \text{ with } v = v_1 + iv_2.$$

The spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ satisfies

$$\sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, e_0\}.$$

It is easy to see that $\mathrm{i}Q, \nabla Q$ are independent and belong to the kernel of $\mathcal{L}$. In [4], [6], [7], [13], it is proved that there exist two eigenfunctions $Y^{\pm}$ (normalized by $||Y^{\pm}||_{L^2} = 1$) associated with eigenvalues $\pm e_0$

$$\mathcal{L}(Y^{\pm}) = \pm e_0 Y^{\pm} \tag{1.16}$$

and $Y^+ = \overline{Y^-}$ belong to $\mathcal{Y}$; in other words, $\operatorname{Re} Y^+, \operatorname{Im} Y^+ \in \mathcal{Y}$. Moreover, there holds a property of positivity based on $Y^{\pm}$: there exists $\mu > 0$ such that, for all $\eta \in H^1$,

$$\begin{aligned} \langle L_+ \operatorname{Re}\eta, \operatorname{Re}\eta\rangle + \langle L_- \operatorname{Im}\eta, \operatorname{Im}\eta\rangle &\geq \mu \|\eta\|_{H^1}^2 \\ &\quad - \frac{1}{\mu}\left(\langle \eta, \mathrm{i}Y^+\rangle^2 + \langle \eta, \mathrm{i}Y^-\rangle^2 + |\langle \eta, xQ\rangle|^2 + \langle \eta, \mathrm{i}\Lambda Q\rangle^2\right). \end{aligned} \tag{1.17}$$

2. Approximate solution for $p > 2$

2.1. **System of modulation equations.** Let $p > 2$. Consider a time-dependent $\mathcal{C}^1$ function of parameters $\vec{q}$ of the form

$$\vec{q} = (\lambda, z, \gamma, v) \in (0, +\infty) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d,$$

with $|v| \ll 1$ and $|z| \gg 1$. We renormalize the flow by considering

$$u(t,x) = \frac{\mathrm{e}^{\mathrm{i}\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)} w(s,y), \quad \mathrm{d}t = \lambda^2(s)\mathrm{d}s, \quad y = \frac{x}{\lambda(s)}, \tag{2.1}$$

so that

$$\mathrm{i}\partial_t u + \Delta u + |u|^{p-1}u = \frac{\mathrm{e}^{\mathrm{i}\gamma}}{\lambda^{2+\frac{2}{p-1}}}\left[\mathrm{i}\dot{w} + \Delta w - w + |w|^{p-1}w - \mathrm{i}\frac{\dot{\lambda}}{\lambda}\Lambda w + (1-\dot{\gamma})w\right] \tag{2.2}$$

($\dot{w}$ denotes derivation with respect to s). We introduce the following $\vec{q}$-modulated ground-state solitary waves, for $k \in \{1,2\}$,

$$P_k(s,y) = \mathrm{e}^{\mathrm{i}\Gamma_k(s,y-z_k(s))}Q(y-z_k(s)) = \mathrm{e}^{\mathrm{i}v_k(s)(y-z_k(s))}Q(y-z_k(s)), \tag{2.3}$$

where we set

$$v_1(s) = -v_2(s) = \frac{1}{2}v(s), \quad z_1(s) = -z_2(s) = \frac{1}{2}z(s), \quad \Gamma_k(s,y) = v_k(s)\cdot y, \tag{2.4}$$

Let

$$\mathbf{P}(s,y) = \mathbf{P}(y;(z(s),v(s))) = \sum_{k=1}^{2} P_k(s,y). \tag{2.5}$$

Then, $\mathbf{P}$ is an approximate solution to the rescaled equation in the following sense.

Lemma 6 (Leading order approximate flow)**.** *Let the vectors of modulation equations be*

$$\vec{m}_k = \begin{pmatrix} \frac{\dot{\lambda}}{\lambda} \\ \dot{z}_k - 2v_k + \frac{\dot{\lambda}}{\lambda}z_k \\ \dot{\gamma} - 1 + |v_k|^2 - \frac{\dot{\lambda}}{\lambda}(v_k\cdot z_k) - (v_k\cdot\dot{z}_k) \\ \dot{v}_k - \frac{\dot{\lambda}}{\lambda}v_k \end{pmatrix}, \qquad \vec{\mathrm{M}}V = \begin{pmatrix} -\mathrm{i}\Lambda V \\ -\mathrm{i}\nabla V \\ -V \\ -yV \end{pmatrix}. \tag{2.6}$$

Then the error $\mathcal{E}_{\mathbf{P}}$ on the re-normalized flow (2.2) *at* $\mathbf{P}$,

$$\mathcal{E}_{\mathbf{P}} = \mathrm{i}\dot{\mathbf{P}} + \Delta\mathbf{P} - \mathbf{P} + |\mathbf{P}|^{p-1}\mathbf{P} - \mathrm{i}\frac{\dot{\lambda}}{\lambda}\Lambda\mathbf{P} + (1-\dot{\gamma})\mathbf{P} \tag{2.7}$$

decomposes as

$$\mathcal{E}_{\mathbf{P}} = [\mathrm{e}^{\mathrm{i}\Gamma_1}\vec{m}_1\cdot\vec{\mathrm{M}}Q](y-z_1(s)) + [\mathrm{e}^{\mathrm{i}\Gamma_2}\vec{m}_2\cdot\vec{\mathrm{M}}Q](y-z_2(s)) + G \tag{2.8}$$

where the interaction term $G = |\mathbf{P}|^{p-1}\mathbf{P} - |P_1|^{p-1}P_1 - |P_2|^{p-1}P_2$ satisfies

$$\|G\|_{L^\infty} \lesssim |z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}, \quad \|\nabla G\|_{L^\infty} \lesssim |z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}. \tag{2.9}$$

Proof of Lemma 6. Firstly, we compute $\mathcal{E}_{P_k} = \mathrm{i}\dot{P}_k + \Delta P_k - P_k + |P_k|^{p-1}P_k - \mathrm{i}\frac{\dot{\lambda}}{\lambda}\Lambda P_k + (1-\dot{\gamma})P_k$. Let $y_{z_k} = y - z_k$, by computations

$$\begin{aligned}
\mathrm{i}\dot{P}_k &= \Big[-(\dot{v}_k\cdot y_{z_k})Q(y_{z_k}) + (v_k\cdot\dot{z}_k)Q(y_{z_k}) - \mathrm{i}\dot{z}_k\cdot\nabla Q(y_{z_k})\Big]\mathrm{e}^{\mathrm{i}v_k\cdot y_{z_k}} \\
\nabla P_k &= \Big[\nabla Q(y_{z_k}) + \mathrm{i}v_k Q(y_{z_k})\Big]\mathrm{e}^{\mathrm{i}v_k\cdot y_{z_k}} \\
\Delta P_k &= \Big[\Delta Q(y_{z_k}) + 2\mathrm{i}v_k\cdot\nabla Q(y_{z_k}) - v_k^2 Q(y_{z_k})\Big]\mathrm{e}^{\mathrm{i}v_k\cdot y_{z_k}} \\
\Lambda P_k &= \Big[\frac{2}{p-1}Q(y_{z_k}) + y\cdot[\nabla Q(y_{z_k}) + \mathrm{i}v_k Q(y_{z_k})]\Big]\mathrm{e}^{\mathrm{i}v_k\cdot y_{z_k}} \\
&= \Big[\Lambda Q(y_{z_k}) + \mathrm{i}v_k\cdot y_{z_k}Q(y_{z_k}) + \mathrm{i}v_k\cdot z_k Q(y_{z_k}) + z_k\cdot\nabla Q(y_{z_k})\Big]\mathrm{e}^{\mathrm{i}v_k\cdot y_{z_k}}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}\mathcal{E}_{P_k} =\Big[&-\mathrm{i}\frac{\dot\lambda}{\lambda}\Lambda Q(y_{z_k}) - \mathrm{i}(\dot z_k - 2v_k + z_k\frac{\dot\lambda}{\lambda})\cdot\nabla Q(y_{z_k})\\ &-(\dot\gamma - 1 - v_k\cdot\dot z_k + |v_k|^2 - v_k\cdot z_k\frac{\dot\lambda}{\lambda})Q(y_{z_k})\\ &-(\dot v_k - v_k\frac{\dot\lambda}{\lambda})\cdot y_{z_k}Q(y_{z_k}) + \Delta Q(y_{z_k}) - Q(y_{z_k}) + |Q(y_{z_k})|^{p-1}Q(y_{z_k})\Big]\mathrm{e}^{\mathrm{i}\Gamma_k(s,y-z_k)}.\end{aligned}$$

Since $\Delta Q - Q + |Q|^{p-1}Q = 0$, we have

$$\mathcal{E}_{P_k} = [\mathrm{e}^{\mathrm{i}\Gamma_k}\vec{m}_k\cdot\vec{\mathrm{M}}Q](y - z_k(s)). \tag{2.10}$$

Returning to the error on the renormalized flow, we obtain

$$\mathcal{E}_{\mathbf{P}} = \mathcal{E}_{P_1} + \mathcal{E}_{P_2} + |\mathbf{P}|^{p-1}\mathbf{P} - \sum_{k=1}^{2}|P_k|^{p-1}P_k. \tag{2.11}$$

Next, we estimate the interaction term $G = |\mathbf{P}|^{p-1}\mathbf{P} - |P_1|^{p-1}P_1 - |P_2|^{p-1}P_2$. Clearly,

$$|G| \lesssim |P_1|^{p-1}|P_2| + |P_2|^{p-1}|P_1|.$$

We observe that, for $z = z_1 - z_2$, by (1.12),

$$Q(y)Q(y-z) \lesssim (1+|y|)^{-\frac{d-1}{2}}(1+|y-z|)^{-\frac{d-1}{2}}\mathrm{e}^{-|y|}\mathrm{e}^{-|z|+|y|} \lesssim |z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|} \tag{2.12}$$

which yields

$$|P_1|^{p-1}|P_2| \lesssim |P_1|\,|P_2||P_1|^{p-2} \lesssim |z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}|P_1|^{p-2}.$$

Thus,

$$|G(s,y)| \lesssim |z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\sum_{k=1}^{2}Q^{p-2}(y - z_k(s)) \tag{2.13}$$

and since $p > 2$, we get

$$||G||_{L^\infty} \lesssim |z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}. \tag{2.14}$$

Similarly, by (1.12) and as $|v| \ll 1$,

$$||\nabla G||_{L^\infty} \lesssim |z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}.$$

□

2.2. **Nonlinear forcing.** For the next parts of the article, we will need the first-order and the second-order approximations of $F(u) = |u|^{p-1}u$, where $u = a + \mathrm{i}b$. We consider the expansion for $|u| \ll 1$

$$F(1+u) = 1 + pa + \mathrm{i}b + \frac{p(p-1)}{2}a^2 + \frac{p-1}{2}b^2 + (p-1)\mathrm{i}ab + O(|u|^k) \tag{2.15}$$

for any $2 < k \le 3$, from which we can deduce formally

$$F'(\mathbf{P}).\epsilon = \frac{p+1}{2}|\mathbf{P}|^{p-1}\epsilon + \frac{p-1}{2}|\mathbf{P}|^{p-3}\mathbf{P}^2\bar\epsilon \tag{2.16}$$

and

$$\frac{\bar\epsilon\cdot F''(\mathbf{P})\cdot\epsilon}{2} = \frac{p-1}{2}\epsilon^2\bar{\mathbf{P}}|\mathbf{P}|^{p-3} + (p-1)|\epsilon|^2\mathbf{P}|\mathbf{P}|^{p-3} + (p-1)\left(\frac{p}{2} - \frac{3}{2}\right)\left(\mathrm{Re}\,(\epsilon\bar{\mathbf{P}})\right)^2\mathbf{P}|\mathbf{P}|^{p-5}.$$

In the case $p>2$, set

$$2^+=\min(3,\frac{p+2}{2})\cdot$$

Remark that $2^+<2^*$ when $p>2$ (where $2^*=\frac{2d}{d-2}$ is the critical exponent of the Sobolev injection). Then, from (2.15), we have

$$F(\mathbf{P}+\epsilon)=F(\mathbf{P})+F'(\mathbf{P})\cdot\epsilon+O(|\epsilon|^p)+O\left(\left|\frac{\epsilon}{\mathbf{P}}\right|^2|\mathbf{P}|^p\right) \tag{2.17}$$

and

$$F(\mathbf{P}+\epsilon)=F(\mathbf{P})+F'(\mathbf{P})\cdot\epsilon+\frac{\bar{\epsilon}\cdot F''(\mathbf{P})\cdot\epsilon}{2}+O(|\epsilon|^p)+O\left(\left|\frac{\epsilon}{\mathbf{P}}\right|^{2^+}|\mathbf{P}|^p\right) \tag{2.18}$$

(note that for $\left|\frac{\epsilon}{P}\right|\gg 1$ we have $F(\mathbf{P}+\epsilon)\sim F(\epsilon)$).

Lemma 7 (Nonlinear interaction estimates)**.** *For $|z|\gg 1, |v|\ll 1$, let*

$$H(z)=p\left[\int_{y\cdot\frac{z}{|z|}>-\frac{|z|}{2}}Q^{p-1}(y)\nabla Q(y)Q(y+z)\mathrm{d}y+\int_{y\cdot\frac{z}{|z|}<-\frac{|z|}{2}}Q^{p-1}(y+z)\nabla Q(y)Q(y)\mathrm{d}y\right]. \tag{2.19}$$

Then the following estimates hold:

$$\left|\langle G,\mathrm{e}^{\mathrm{i}\Gamma_1(y-z_1(s))}\nabla Q(y-z_1(s))\rangle-H(z)\right|\lesssim(|v|^2|z|^2+|v|^2)|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}+|z|^{-\frac{3(d-1)}{4}}\mathrm{e}^{-\frac{3}{2}|z|} \tag{2.20}$$

and

$$\left|H(z)-C_p\frac{z}{|z|}|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\right|\lesssim|z|^{-\frac{d-1}{2}-1}\mathrm{e}^{-|z|} \tag{2.21}$$

where $C_p>0$.

Remark 8. The estimate (2.21) on the leading order of the core part $H(z)$ of the projection $\langle G,[\mathrm{e}^{\mathrm{i}\Gamma_1}\nabla Q](y-z_1(s))\rangle$ is valid not only in the case $p>2$ but also in the case $1<p\le 2$.

Proof of Lemma 7. **Step 1.** Nonlinear interaction estimates. We prove the estimate (2.21) and in this step we will have $p>1$. Consider

$$H(z)=p\int_{y\cdot\frac{z}{|z|}<-\frac{|z|}{2}}Q^{p-1}(y+z)\nabla Q(y)Q(y)\mathrm{d}y+p\int_{y\cdot\frac{z}{|z|}>-\frac{|z|}{2}}Q^{p-1}(y)\nabla Q(y)Q(y+z)\mathrm{d}y\cdot$$

Recall that

$$Q(y)Q(y+z)\lesssim|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}$$
$$Q(y)|\nabla Q(y+z)|\lesssim|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}$$

then with $p>2$, we have

$$\left|\int_{y\cdot\frac{z}{|z|}<-\frac{|z|}{2}}Q^{p-1}(y+z)\nabla Q(y)Q(y)\mathrm{d}y\right|\lesssim\mathrm{e}^{-\min(p-1,\frac{3}{2})|z|}$$

and with $1<p\le 2$, from the decay property of Q, we have for $\delta=\frac{p-1}{2}$

$$\left|\int_{y\cdot\frac{z}{|z|}<-\frac{|z|}{2}}Q^{p-1}(y+z)\nabla Q(y)Q(y)\mathrm{d}y\right|\lesssim\mathrm{e}^{-(p-1)|z|}\left|Q\left(\frac{|z|}{2}\right)\right|^{3-p-\delta}\int Q^{\delta}(y)\mathrm{d}y$$
$$\lesssim\mathrm{e}^{-\frac{p+3}{4}|z|}.$$

We claim that

$$\left| \int_{y\cdot \frac{z}{|z|} > -\frac{|z|}{2}} Q^{p-1}(y)\nabla Q(y)Q(y+z)\mathrm{d}y - c_Q|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\int Q^{p-1}(y)\nabla Q(y)\mathrm{e}^{-y\cdot\frac{z}{|z|}}\mathrm{d}y \right| \lesssim |z|^{-1-\frac{d-1}{2}}\mathrm{e}^{-|z|}. \tag{2.22}$$

Indeed, let $0<\theta<1$ such that $p\theta>1$. For $|y|\geq \theta|z|$, we have:

$$\begin{aligned}\left| \int_{\substack{|y|\geq\theta|z|\\ y\cdot \frac{z}{|z|} > -\frac{|z|}{2}}} Q^{p-1}(y)\nabla Q(y)Q(y+z)\mathrm{d}y \right| &\lesssim \mathrm{e}^{-p\theta|z|}\left|\int Q(y+z)\mathrm{d}y\right| \\ &\lesssim \mathrm{e}^{-p\theta|z|}.\end{aligned}$$

For $|y|<\theta|z|$, as $Q(x)=q(|x|)$ and $|q(r)-c_Q r^{-\frac{d-1}{2}}\mathrm{e}^{-r}|\lesssim r^{-\frac{d-1}{2}-1}\mathrm{e}^{-r}$, we have:

$$\begin{aligned}\left| Q(y+z) - c_Q|y+z|^{-\frac{d-1}{2}}\mathrm{e}^{-|y+z|}\right| &\lesssim |y+z|^{-1-\frac{d-1}{2}}\mathrm{e}^{-|y+z|} \\ &\leq |1-\theta||z|^{-1-\frac{d-1}{2}}\mathrm{e}^{-|z|}\mathrm{e}^{|y|}.\end{aligned}$$

Thus we get:

$$\left| \int_{\substack{|y|<\theta|z|\\ y\cdot \frac{z}{|z|} > -\frac{|z|}{2}}} Q^{p-1}(y)\nabla Q(y)\nabla Q(y+z)\mathrm{d}y - c_Q\int_{\substack{|y|<\theta|z|\\ y\cdot \frac{z}{|z|} > -\frac{|z|}{2}}} Q^{p-1}(y)\nabla Q(y)|y+z|^{-\frac{d-1}{2}}\mathrm{e}^{-|y+z|}\mathrm{d}y\right| \lesssim |z|^{-1-\frac{d-1}{2}}\mathrm{e}^{-|z|}$$

since $\int Q^{p-1}(y)|\nabla Q(y)|\mathrm{e}^{|y|}\mathrm{d}y<+\infty$. On the other hand, $|y|<\theta|z|$ implies

$$\left| |y+z|^{-k} - |z|^{-k}\right| \lesssim |z|^{-1-k}|y|$$

for any $k>0$ and

$$\left|\frac{y+z}{|y+z|} - \frac{z}{|z|}\right| \lesssim |z|^{-1}|y|.$$

Moreover

$$\left| |y+z| - |z| - y\cdot\frac{z}{|z|}\right| \lesssim |z|^{-1}|y|^2$$

then

$$\left|\mathrm{e}^{-|y+z|} - \mathrm{e}^{-|z|-y\cdot\frac{z}{|z|}}\right| \lesssim |z|^{-1}|y|^2\mathrm{e}^{-|z|}\mathrm{e}^{|y|}.$$

Thus we obtain that

$$\left| |y+z|^{-\frac{d-1}{2}}\mathrm{e}^{-|y+z|} - |z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|-y\cdot\frac{z}{|z|}}\right| \lesssim (1+|y|^2)|z|^{-1-\frac{d-1}{2}}\mathrm{e}^{-|z|}\mathrm{e}^{|y|}.$$

Therefore, we have

$$\left| \int_{\substack{|y|<\theta|z|\\ y\cdot \frac{z}{|z|} > -\frac{|z|}{2}}} Q^{p-1}(y)\nabla Q(y)|y+z|^{-\frac{d-1}{2}}\mathrm{e}^{-|y+z|}\mathrm{d}y \right.$$
$$\left. - c_Q|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\int_{\substack{|y|<\theta|z|\\ y\cdot \frac{z}{|z|} > -\frac{|z|}{2}}} Q^{p-1}(y)\nabla Q(y)\mathrm{e}^{-y\cdot\frac{z}{|z|}}\mathrm{d}y\right| \lesssim |z|^{-1-\frac{d-1}{2}}\mathrm{e}^{-|z|}.$$

Next we observe that

$$|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\int_{\substack{|y|\geq\theta|z|\\ y\cdot\frac{z}{|z|}>-\frac{|z|}{2}}} Q^{p-1}(y)\nabla Q(y)\mathrm{e}^{-y\cdot\frac{z}{|z|}}\mathrm{d}y \lesssim \mathrm{e}^{-p\theta|z|}$$

and by (1.12)

$$\left|\int_{y\cdot\frac{z}{|z|}<-\frac{|z|}{2}} Q^{p-1}(y)\nabla Q(y)\mathrm{e}^{-y\cdot\frac{z}{|z|}}\mathrm{d}y\right| \lesssim \mathrm{e}^{-\frac{p-1}{4}|z|},$$

which finishes the proof of (2.22). Finally, in order to obtain (2.21) with $C_p = c_Q I_Q$, we use an integration by parts

$$p\int Q^{p-1}(y)\nabla Q(y)\mathrm{e}^{-y\cdot\frac{z}{|z|}}\mathrm{d}y = \frac{z}{|z|}\int Q^{p}(y)\mathrm{e}^{-y\cdot\frac{z}{|z|}}\mathrm{d}y$$

and remark from the parity of the integral that

$$\int Q^{p}(y)\mathrm{e}^{-y\cdot\frac{z}{|z|}}\mathrm{d}y = \int Q^{p}(y)\mathrm{e}^{-y_1}\mathrm{d}y = I_Q.$$

Step 2 Error bound. Recall the interaction term

$$G = |\mathbf{P}|^{p-1}\mathbf{P} - |P_1|^{p-1}P_1 - |P_2|^{p-1}P_2.$$

From (2.15), we have the following estimates: if $y\cdot\frac{z}{|z|}>0$, then $|P_1|>|P_2|$

$$\left|G - \frac{p+1}{2}|P_1|^{p-1}P_2 - \frac{p-1}{2}|P_1|^{p-3}P_1^2\overline{P_2}\right| \lesssim |P_2|^2|P_1|^{p-2} \tag{2.23}$$

and, if $y\cdot\frac{z}{|z|}<0$, then $|P_2|>|P_1|$

$$\left|G - \frac{p+1}{2}|P_2|^{p-1}P_1 - \frac{p-1}{2}|P_2|^{p-3}P_2^2\overline{P_1}\right| \lesssim |P_1|^2|P_2|^{p-2}. \tag{2.24}$$

We combine (2.23) and (2.24) to obtain, for all y,

$$\begin{aligned}\Bigg|G - \left[\frac{p+1}{2}|P_1|^{p-1}P_2 + \frac{p-1}{2}|P_1|^{p-3}P_1^2\overline{P_2}\right]\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}>0} - \Bigg[\frac{p+1}{2}|P_2|^{p-1}P_1 \\ + \frac{p-1}{2}|P_2|^{p-3}P_2^2\overline{P_1}\Bigg]\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}<0}\Bigg| \lesssim \min(|P_1|^2,|P_2|^2)\max(|P_1|^{p-2},|P_2|^{p-2}).\end{aligned} \tag{2.25}$$

step 3 Projection estimates. Since $\min(|P_1|^2,|P_2|^2)\leq|P_2|^{\frac{3}{2}}|P_1|^{\frac{1}{2}}$ and $\max(|P_1|^{p-2},|P_2|^{p-2})\leq |P_1|^{p-2}+|P_2|^{p-2}$, we have

$$\begin{aligned}&\int Q^{\frac{3}{2}}(y-z)|\nabla Q(y)|Q^{\frac{1}{2}}(y)(Q^{p-2}(y)+Q^{p-2}(y+z))\mathrm{d}y\\ &\lesssim |z|^{-\frac{3(d-1)}{4}}\mathrm{e}^{-\frac{3}{2}|z|}\int(Q^{p-2}(y)+Q^{p-2}(y+z))\mathrm{d}y \lesssim |z|^{-\frac{3(d-1)}{4}}\mathrm{e}^{-\frac{3}{2}|z|}\end{aligned}$$

so we deduce from the error bound (2.25)

$$\Big|\langle G, [\mathrm{e}^{\mathrm{i}\Gamma_1}\nabla Q](y-z_1(s))\rangle - \Big\langle \Big[\frac{p+1}{2}|P_1|^{p-1}P_2 + \frac{p-1}{2}|P_1|^{p-3}P_1^2\overline{P_2}\Big]\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}>0}$$
$$+\Big[\frac{p+1}{2}|P_2|^{p-1}P_1 + \frac{p-1}{2}|P_2|^{p-3}P_2^2\overline{P_1}\Big]\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}<0}, [\mathrm{e}^{\mathrm{i}\Gamma_1}\nabla Q](y-z_1(s))\Big\rangle\Big| \lesssim |z|^{-\frac{3(d-1)}{4}}\mathrm{e}^{-\frac{3}{2}|z|}. \tag{2.26}$$

Using a change of variables, we have

$$\begin{aligned}&\langle |P_1|^{p-1}P_2\mathbb{1}_{y\cdot\frac{z}{|z|}>0}, [\mathrm{e}^{\mathrm{i}\Gamma_1}\nabla Q](y-z_1(s))\rangle\\ =&\,\mathrm{Re}\int_{y\cdot\frac{z}{|z|}>-\frac{|z|}{2}} Q^{p-1}(y)\nabla Q(y)Q(y-z_2+z_1)\mathrm{e}^{\mathrm{i}v_2\cdot(y-z_2+z_1)-\mathrm{i}v_1\cdot y}\mathrm{d}y\\ =&\int_{y\cdot\frac{z}{|z|}>-\frac{|z|}{2}} Q^{p-1}(y)\nabla Q(y)Q(y+z)\cos(v_2\cdot(y+z)-v_1\cdot y)\mathrm{d}y\end{aligned}$$

with $z(s)=z_1(s)-z_2(s)$. Note that

$$|\cos(v_2\cdot(y+z)-v_1\cdot y)-1| \lesssim |v|^2|z|^2+|v|^2|y|^2$$

as the same method to prove (2.22), we get

$$\Big|\langle |P_1|^{p-1}P_2\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}>0}, [\mathrm{e}^{\mathrm{i}\Gamma_1}\nabla Q](y-z_1(s))\rangle - \int_{y\cdot\frac{z}{|z|}>-\frac{|z|}{2}} Q^{p-1}(y)\nabla Q(y)Q(y+z)\mathrm{d}y\Big|$$
$$\lesssim (|v|^2|z|^2+|v|^2)|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}. \tag{2.27}$$

Similarly, for the other projections, we have

$$\Big|\langle |P_1|^{p-3}P_1^2\overline{P_2}\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}>0}, [\mathrm{e}^{\mathrm{i}\Gamma_1}\nabla Q](y-z_1(s))\rangle - \int_{y\cdot\frac{z}{|z|}>-\frac{|z|}{2}} Q^{p-1}(y)\nabla Q(y)Q(y+z)\mathrm{d}y\Big|$$
$$\lesssim (|v|^2|z|^2+|v|^2)|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|} \tag{2.28}$$

$$\Big|\langle |P_1|^{p-3}P_1^2\overline{P_2}\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}<0}, [\mathrm{e}^{\mathrm{i}\Gamma_1}\nabla Q](y-z_1(s))\rangle - \int_{y\cdot\frac{z}{|z|}<-\frac{|z|}{2}} Q^{p-1}(y+z)\nabla Q(y)Q(y)\mathrm{d}y\Big|$$
$$\lesssim (|v|^2|z|^2+|v|^2)|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|} \tag{2.29}$$

and finally

$$\begin{aligned}&\langle |P_2|^{p-1}P_1\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}<0}, [\mathrm{e}^{\mathrm{i}\Gamma_1}\nabla Q](y-z_1(s))\rangle\\ &\qquad= \mathrm{Re}\int_{y\cdot\frac{z}{|z|}<0} Q^{p-1}(y-z_2(s))Q(y-z_1(s))\nabla Q(y-z_1(s))\mathrm{d}y\\ &\qquad= \int_{y\cdot\frac{z}{|z|}<-\frac{|z|}{2}} Q^{p-1}(y+z)\nabla Q(y)Q(y)\mathrm{d}y.\end{aligned} \tag{2.30}$$

From (2.26)–(2.30), we obtain the desired result (2.20). □

2.3. **Formal resolution and estimates of leading order.** From Lemma 6, we derive a simplified modulation system with forcing term and we determine one of its approximate solution that is relevant for the regime of the Main Theorem. Formally, we have the following bounds (making this rigorous will be the goal of the bootstrap estimates in Sect. 3.2)

$$|\vec{m}_1| \lesssim |z|^{-\frac{d-1}{2}} \mathrm{e}^{-|z|}, \tag{2.31}$$

from which we derive a simplified system ($\vec{m}_k$ is defined in (2.6)):

$$\left|\frac{\dot\lambda}{\lambda}\right| + \left|\dot z - 2v + \frac{\dot\lambda}{\lambda} z\right| \lesssim |z|^{-\frac{d-1}{2}} \mathrm{e}^{-|z|}. \tag{2.32}$$

Furthermore, since we expect the interaction to be strong enough such that it will affect the main order of the modulation equations, so by projecting $\mathcal{E}_{\mathbf{P}}$ onto the direction $\mathrm{e}^{\mathrm{i}\Gamma_1}\nabla Q(y - z_1(s))$, we obtain formally that

$$c_2 \dot v_1 \approx -\langle G, \mathrm{e}^{\mathrm{i}\Gamma_1}\nabla Q(y - z_1(s))\rangle \approx -H(z)$$

with $c_2 = \langle -yQ, \nabla Q\rangle > 0$. This remark suggests us to fix

$$\dot v = -\frac{2p}{c_2}\left[\int Q^{p-1}(y)\nabla Q(y) Q(y+z)\mathrm{d}y + \int Q^{p-1}(y+z)\nabla Q(y)Q(y)\mathrm{d}y\right] = -\frac{2}{c_2}H(z) \tag{2.33}$$

so $v(s)$ is completely determined by $z(s)$ and initial data v^{in}. In consequence, there are only three free parameters left (λ, z, γ) corresponding to the scaling, translation, and phase parameters, which we will modulate to obtain orthogonality conditions (as shown below in Lemma 9). We use (2.21) to estimate the main order of $\dot v$

$$\left|\dot v + c\frac{z}{|z|}|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\right| \lesssim |z|^{-\frac{d-1}{2}-1}\mathrm{e}^{-|z|} \tag{2.34}$$

with

$$c = \frac{2\,C_p}{c_2} = \frac{2\,c_Q I_Q}{c_2} > 0. \tag{2.35}$$

It can be checked that for some real functions $z_{\mathrm{mod}}(s)$, $\lambda_{\mathrm{mod}}(s)$, $v_{\mathrm{mod}}(s)$ such that

$$\lambda^{-1}_{\mathrm{mod}}(s) = 1, \quad v_{\mathrm{mod}}(s) = s^{-1}, \quad z_{\mathrm{mod}}^{-\frac{d-1}{2}}\mathrm{e}^{-z_{\mathrm{mod}}} = \frac{s^{-2}}{c} \tag{2.36}$$

then we have the asymptotics as $s \to +\infty$

$$\begin{aligned} &z_{\mathrm{mod}}(s) \sim 2\log(s), \qquad \dot v_{\mathrm{mod}}(s) = -c z_{\mathrm{mod}}^{-\frac{d-1}{2}}(s)\mathrm{e}^{-z_{\mathrm{mod}}(s)},\\ &|\dot z_{\mathrm{mod}}(s) - 2v_{\mathrm{mod}}(s)| \lesssim s^{-1}\log^{-1}(s), \quad |\dot v_{\mathrm{mod}}(s)| \lesssim s^{-2}. \end{aligned} \tag{2.37}$$

Indeed, obviously, $\dot v_{\mathrm{mod}}(s) = -s^{-2} = -c z_{\mathrm{mod}}^{-\frac{d-1}{2}}(s)\mathrm{e}^{-z_{\mathrm{mod}}(s)}$ and by differentiating the equation of z_{mod}, we get

$$-\dot z_{\mathrm{mod}} z_{\mathrm{mod}}^{-\frac{d-1}{2}}\mathrm{e}^{-z_{mod}} - \frac{d-1}{2}\dot z_{\mathrm{mod}} z_{\mathrm{mod}}^{-\frac{d-1}{2}-1}\mathrm{e}^{-z_{\mathrm{mod}}} = -2\frac{s^{-3}}{c}$$

(in the case $d-1=0$, $-\dot z_{\mathrm{mod}}\mathrm{e}^{-z_{mod}} = -2\frac{s^{-3}}{c}$) so $|\dot z_{\mathrm{mod}} - 2s^{-1}| \lesssim s^{-1}\log^{-1}(s)$ thus we can deduce $|\dot z_{\mathrm{mod}}(s) - 2v_{\mathrm{mod}}(s)| \lesssim s^{-1}\log^{-1}(s)$. The above estimates suggest that (2.36) is close to the first-order asymptotics as $s \to +\infty$ for some particular solutions to (2.32) and matches the regime in the Main Theorem.

3. Modulation and backward uniform estimates

Let $(\lambda^{\rm in}, z^{\rm in}, v^{\rm in}) \in (0,+\infty)\times(0,+\infty)\times\mathbb{R}$ to be chosen with $|z^{\rm in}|\gg 1$, $|v^{\rm in}|\ll 1$, $T_{\rm mod}>0$ and $(\vec{e}_1, ..., \vec{e}_d)$ standard basis of $\mathbb{R}^d$. Recall that in this section $p>2$. Let $u(t,x)$ be the backward solution to (NLS) with initial data

$$u(T_{\rm mod},x)=\frac{1}{(\lambda^{\rm in})^{\frac{2}{p-1}}}\mathbf{P}^{\rm in}\left(\frac{x}{\lambda^{\rm in}}\right)\quad\text{where}\quad \mathbf{P}^{\rm in}(y)=\mathbf{P}(y;(z^{\rm in}\vec{e}_1,v^{\rm in}))\tag{3.1}$$

on some time interval including $T_{\rm mod}$. Note that the NLS equation is invariant by rotation and reflection. In particular, if a solution to (NLS) is invariant by the symmetries $\tau: x\mapsto -x$ and $\upsilon:(x_1,x_2,...,x_d)\mapsto(x_1,-x_2,...,-x_d)$ at some time, then it is invariant by the symmetry at any time.

3.1. **Decomposition of $u(t)$.** We will state a standard modulation result with the same idea as in Lemma 3 of [20] or Lemma 2 of [27]. The choice of the special orthogonality conditions (3.5) is related to the generalized null space of the linearized equation around Q in (1.14) and to the coercivity property (1.15)in subcritical cases. See the proof of Lemma 12 for a technical justification of these choices. For $s^{\rm in}\gg 1$ fixed, one has the following.

Lemma 9 (Modulation of the approximate solution)**.** *Let $u(t,x)$ a solution invariant by τ and υ on an interval $[T,T_{\rm mod}]$ satisfying $u(T_{\rm mod},x)\in H^2(\mathbb{R}^d)$ and*

$$\left\|{\rm e}^{-{\rm i}\gamma^{\rm in}}(\lambda^{\rm in})^{\frac{2}{p-1}}u(T_{\rm mod},\lambda^{\rm in}y)-\mathbf{P}(y;(z^{\rm in}\vec{e}_1,v^{\rm in}))\right\|_{H^1}\ll 1$$

for $\mathbf{P}(s,y)=\mathbf{P}(y;(z(s),v(s)))$ as defined in (2.5). Then there exists a unique $\mathcal{C}^1$ function on an open interval $I\ni s^{\rm in}$

$$\vec{q}(s)=(\lambda,z,\gamma,v): I\to(0,+\infty)\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d,$$

with $\vec{q}(s^{\rm in})=(\lambda^{\rm in},z^{\rm in}\vec{e}_1,\gamma^{\rm in},v^{\rm in})$ and a rescaling time function

$$t(s)=T_{\rm mod}-\int_s^{s^{\rm in}}\lambda^2(\tau){\rm d}\tau\tag{3.2}$$

such that $u(t,x)$ decomposes as follows

$$u(t(s),x)=\frac{{\rm e}^{{\rm i}\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)}(\mathbf{P}+\epsilon)(s,y),\quad y=\frac{x}{\lambda(s)}\tag{3.3}$$

where, by setting

$$\epsilon(s,y)=\left[{\rm e}^{{\rm i}\Gamma_1}\eta_1\right](s,y-z_1),\qquad \Gamma_k(s,y)=v_k(s)\cdot y,\tag{3.4}$$

if initially $\langle\eta_1(s^{\rm in}),Q\rangle=\langle\eta_1(s^{\rm in}),yQ\rangle=\langle\eta_1(s^{\rm in}),{\rm i}\Lambda Q\rangle=0$, the decomposition satisfies orthogonality conditions

$$\langle\eta_1(s),Q\rangle=\langle\eta_1(s),yQ\rangle=\langle\eta_1(s),{\rm i}\Lambda Q\rangle=0\tag{3.5}$$

and the extra relation

$$\dot{v}(s)=-\frac{2}{c_2}H(z(s)).\tag{3.6}$$

Moreover, ϵ is also invariant by the symmetry τ and υ.

Proof of Lemma 9. **Step 1** Orthogonality conditions. We show that the orthogonality conditions (3.5) and the extra relation (3.6) are equivalent to solve a system of ODEs. Remark that we can go easily from the rescaled time s to t and conversely

$$s = s(t) = s^{\mathrm{in}} - \int_t^{T_{\mathrm{mod}}} \frac{\mathrm{d}\tau}{\lambda^2(\tau)} \tag{3.7}$$

with $T_{\mathrm{mod}} = t(s^{\mathrm{in}})$. Denote

$$\mathbf{P}(s,y) = \left[\mathrm{e}^{\mathrm{i}\Gamma_1}\mathbf{P}_1\right](s, y - z_1), \qquad \mathcal{E}_{\mathbf{P}}(s,y) = \left[\mathrm{e}^{\mathrm{i}\Gamma_1}\mathcal{E}_{\mathbf{P}_1}\right](s, y - z_1)$$

$$G(s,y) = [\mathrm{e}^{\mathrm{i}\Gamma_1} G_1](s, y - z_1)$$

where $G = |\mathbf{P}|^{p-1}\mathbf{P} - |P_1|^{p-1}P_1 - |P_2|^{p-1}P_2$. Let $w = \mathbf{P} + \epsilon$ as in (2.1). It follows from the equation of w (2.2) and the equation of $\mathbf{P}$ (2.7) that

$$\mathrm{i}\dot{\epsilon} + \Delta\epsilon - \epsilon + \left(|\mathbf{P} + \epsilon|^{p-1}(\mathbf{P} + \epsilon) - |\mathbf{P}|^{p-1}\mathbf{P}\right) - \mathrm{i}\frac{\dot{\lambda}}{\lambda}\Lambda\epsilon + (1 - \dot{\gamma})\epsilon + \mathcal{E}_{\mathbf{P}} = 0. \tag{3.8}$$

We rewrite the equation of ϵ into the following equation for η_1 (see also the proof of Lemma 6)

$$\mathrm{i}\dot{\eta}_1 + \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1 + \mathcal{E}_{\mathbf{P}_1} = 0. \tag{3.9}$$

Thus, for $A(y), B(y) \in \mathcal{Y}$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\langle \eta_1, A + \mathrm{i}B\rangle = -\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1)& \\ + \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1 + \mathcal{E}_{\mathbf{P}_1}, \mathrm{i}A - B\rangle.&\end{aligned}$$

Choose $A = Q, B = 0$ and $A = yQ, B = 0$ and $A = 0, B = \Lambda Q$ then the conditions

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle \eta_1(s), Q\rangle = \frac{\mathrm{d}}{\mathrm{d}s}\langle \eta_1(s), yQ\rangle = \frac{\mathrm{d}}{\mathrm{d}s}\langle \eta_1(s), \mathrm{i}\Lambda Q\rangle = 0$$

are equivalent to

$$\begin{cases} \left\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1 + \mathcal{E}_{\mathbf{P}_1}, iQ \right\rangle = 0 \\ \left\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1 + \mathcal{E}_{\mathbf{P}_1}, \mathrm{i}yQ \right\rangle = 0 \\ \left\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1 + \mathcal{E}_{\mathbf{P}_1}, -\Lambda Q \right\rangle = 0. \end{cases}$$

We claim that the above system is equivalent to an autonomous system of ordinary differential equations on $(\theta(s), z(s), \gamma(s), v(s), t(s))$ where $\theta(s) = \ln(\lambda(s))$. Indeed, remark that

$$\epsilon(s,y) = \mathrm{e}^{\frac{2}{p-1}\theta(s)} u(t(s), \mathrm{e}^{\theta(s)}y) - \mathbf{P}(y; (z(s), v(s))) \tag{3.10}$$

and the expression of $\mathcal{E}_{\mathbf{P}_1}$ (from (2.7)–(2.8))

$$\mathcal{E}_{\mathbf{P}_1} = [\vec{m}_1 \cdot \vec{\mathrm{M}}Q](y) + [\mathrm{e}^{\mathrm{i}(\Gamma_2(y+z) - \Gamma_1(y))}\vec{m}_2 \cdot \vec{\mathrm{M}}Q](y + z) + G_1$$

then we get

$$\begin{cases} \langle \vec{m}_1 \cdot \vec{\mathrm{M}} Q, \mathrm{i}Q\rangle + \langle \mathrm{e}^{\mathrm{i}(\Gamma_2(y+z)-\Gamma_1(y))}\vec{m}_2 \cdot \vec{\mathrm{M}} Q(y+z), \mathrm{i}Q\rangle + \langle \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1, \mathrm{i}Q\rangle = \mathcal{F}_1(\theta,z,\gamma,v,t) \\ \langle \vec{m}_1 \cdot \vec{\mathrm{M}} Q, \mathrm{i}yQ\rangle + \langle \mathrm{e}^{\mathrm{i}(\Gamma_2(y+z)-\Gamma_1(y))}\vec{m}_2 \cdot \vec{\mathrm{M}} Q(y+z), \mathrm{i}yQ\rangle + \langle \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1, \mathrm{i}yQ\rangle \\ \qquad\qquad\qquad\qquad = \mathcal{F}_2(\theta,z,\gamma,v,t) \\ \langle \vec{m}_1 \cdot \vec{\mathrm{M}} Q, -\Lambda Q\rangle + \langle \mathrm{e}^{\mathrm{i}(\Gamma_2(y+z)-\Gamma_1(y))}\vec{m}_2 \cdot \vec{\mathrm{M}} Q(y+z), -\Lambda Q\rangle + \langle \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1, -\Lambda Q\rangle \\ \qquad\qquad\qquad\qquad = \mathcal{F}_3(\theta,z,\gamma,v,t) \end{cases} \tag{3.11}$$

with

$$\begin{aligned} \mathcal{F}_1(\theta,z,\gamma,v,t) &= -\left\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + G_1, \mathrm{i}Q\right\rangle \\ \mathcal{F}_2(\theta,z,\gamma,v,t) &= -\left\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + G_1, \mathrm{i}yQ\right\rangle \\ \mathcal{F}_3(\theta,z,\gamma,v,t) &= -\left\langle \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + G_1, -\Lambda Q\right\rangle . \end{aligned}$$

Note that $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are $\mathcal{C}^1$ functions. Indeed, if we replace η_1 by the expression (3.10) and its definition, it is clear that any term not containing u is continuously differentiable. For terms concerning $u(t,x)$, by integration by parts and chain rule, we show how to prove that typical terms, integrals of the form

$$\frac{\mathrm{d}}{\mathrm{dt}} \operatorname{Re}\left(\int u(t,x)A(x)\mathrm{d}x\right), \quad \frac{\mathrm{d}}{\mathrm{dt}} \operatorname{Re}\left(\int |u(t,x)|^{p-1}u(t,x)A(x)\mathrm{d}x\right)$$

for $A(x)$ some complex functions such that $\operatorname{Re} A(x), \operatorname{Im} A(x) \in \mathcal{Y}$, are continuous. We have

$$\frac{\mathrm{d}}{\mathrm{dt}} \operatorname{Re}\left(\int u(t,x)A(x)\mathrm{d}x\right) = -\operatorname{Im}\left(\int u(t,x)\Delta A(x)\mathrm{d}x\right) - \operatorname{Im}\left(\int |u(t,x)|^{p-1}u(t,x)A(x)\mathrm{d}x\right) \tag{3.12}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \operatorname{Re}\left(\int |u(t,x)|^{p-1}u(t,x)A(x)\mathrm{d}x\right) &= p\operatorname{Re}\left(\int \partial_t u(t,x)|u(t,x)|^{p-1}A(x)\mathrm{d}x\right) = \\ -\,p\operatorname{Im}\left(\int \Delta u(t,x)|u(t,x)|^{p-1}A(x)\mathrm{d}x\right) &- p\operatorname{Im}\left(\int |u(t,x)|^{2p-2}u(t,x)A(x)\mathrm{d}x\right). \end{aligned} \tag{3.13}$$

Recall the persistence of H^2 regularity for the NLS equation (see Theorem 5.3.1 in [2]), since $u(T_{\mathrm{mod}},x) \in H^2(\mathbb{R}^d)$ then $u \in \mathcal{C}^1([0,T_{\mathrm{mod}}],L^2(\mathbb{R}^d)) \cap \mathcal{C}([0,T_{\mathrm{mod}}],H^2(\mathbb{R}^d))$. By Sobolev's injection ($\frac{d+6}{d-2} < \frac{2\,d}{d-4}$), we have $u \in \mathcal{C}([0,T_{\mathrm{mod}}],L^{2p-1}(\mathbb{R}^d))$ and thus the right-hand sides of (3.12), (3.13) are well-defined and continuous. Therefore, in particular, since initially

$$\langle \eta_1(s^{\mathrm{in}}), Q\rangle = \langle \eta_1(s^{\mathrm{in}}), yQ\rangle = \langle \eta_1(s^{\mathrm{in}}), \mathrm{i}\Lambda Q\rangle = 0,$$

the decomposition $(\vec{q},\epsilon)$ will satisfy (3.5)if (3.11) holds.

step 2 System of ODEs. We show the existence of the decomposition $(\vec{q},\epsilon)$ for $u(t)$ and a rescaling time $t(s)$ by solving the following system on $(\theta, z, \gamma, v, t)$

$$\begin{cases} \langle \vec{m}_1 \cdot \vec{\mathrm{M}}Q, \mathrm{i}Q\rangle + \langle \mathrm{e}^{\mathrm{i}(\Gamma_2(y+z)-\Gamma_1(y))}\vec{m}_2 \cdot \vec{\mathrm{M}}Q(y+z), \mathrm{i}Q\rangle + \langle \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1, \mathrm{i}Q\rangle = \mathcal{F}_1(\theta, z, \gamma, v, t) \\ \langle \vec{m}_1 \cdot \vec{\mathrm{M}}Q, \mathrm{i}yQ\rangle + \langle \mathrm{e}^{\mathrm{i}(\Gamma_2(y+z)-\Gamma_1(y))}\vec{m}_2 \cdot \vec{\mathrm{M}}Q(y+z), \mathrm{i}yQ\rangle + \langle \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1, \mathrm{i}yQ\rangle \\ \qquad = \mathcal{F}_2(\theta, z, \gamma, v, t) \\ \langle \vec{m}_1 \cdot \vec{\mathrm{M}}Q, -\Lambda Q\rangle + \langle \mathrm{e}^{\mathrm{i}(\Gamma_2(y+z)-\Gamma_1(y))}\vec{m}_2 \cdot \vec{\mathrm{M}}Q(y+z), -\Lambda Q\rangle + \langle \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1, -\Lambda Q\rangle \\ \qquad = \mathcal{F}_3(\theta, z, \gamma, v, t) \\ \dot{v} = -\frac{2}{c_2}H(z) \\ \dot{t}(s) = \lambda^2(s). \end{cases} \tag{3.14}$$

On the one hand, we calculate

$$\langle \vec{m}_1 \cdot \vec{\mathrm{M}}Q, \mathrm{i}Q\rangle = (\tfrac{\dot{\lambda}}{\lambda})\langle -\mathrm{i}\Lambda Q, \mathrm{i}Q\rangle = -c_1(\tfrac{\dot{\lambda}}{\lambda})$$

$$\langle \vec{m}_1 \cdot \vec{\mathrm{M}}Q, \mathrm{i}yQ\rangle = (\dot{z} - 2v + \tfrac{\dot{\lambda}}{\lambda}z)\langle -\mathrm{i}\nabla Q, \mathrm{i}yQ\rangle = c_2(\dot{z} - 2v + \tfrac{\dot{\lambda}}{\lambda}z)$$

$$\langle \vec{m}_1 \cdot \vec{\mathrm{M}}Q, -\Lambda Q\rangle = c_1(\dot{\gamma} - 1 + |v|^2 - \tfrac{\dot{\lambda}}{\lambda}(v\cdot z) - (v \cdot \dot{z}))$$

with $c_1 = \langle \Lambda Q, Q\rangle, c_2 = \langle -\nabla Q, yQ\rangle$ non-zero. On the other hand, there exist a matrix $\mathcal{M}(\theta, z, \gamma, v, t) = (m_{ij})_{5\times 5}$ and $\vec{\mathcal{G}}(\theta, z, \gamma, v, t)$ such that

$$\begin{pmatrix} \langle \mathrm{e}^{\mathrm{i}(\Gamma_2(y+z)-\Gamma_1(y))}\vec{m}_2 \cdot \vec{\mathrm{M}}Q(y+z), \mathrm{i}Q\rangle + \langle \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1, \mathrm{i}Q\rangle \\ \langle \mathrm{e}^{\mathrm{i}(\Gamma_2(y+z)-\Gamma_1(y))}\vec{m}_2 \cdot \vec{\mathrm{M}}Q(y+z), \mathrm{i}yQ\rangle + \langle \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1, \mathrm{i}yQ\rangle \\ \langle \mathrm{e}^{\mathrm{i}(\Gamma_2(y+z)-\Gamma_1(y))}\vec{m}_2 \cdot \vec{\mathrm{M}}Q(y+z), -\Lambda Q\rangle + \langle \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1, -\Lambda Q\rangle \\ 0 \\ 0 \end{pmatrix}$$
$$= (\dot{\theta}, \dot{z}, \dot{\gamma}, \dot{v}, \dot{t})\mathcal{M}(\theta, z, \gamma, v, t) + \vec{\mathcal{G}}(\theta, z, \gamma, v, t) \tag{3.15}$$

where all entries of $\mathcal{M}(\theta, z, \gamma, v, t)$ are small $|m_{ij}| \ll 1$ as $z^{\mathrm{in}} \gg 1$ and $||\epsilon(s^{\mathrm{in}})||_{H^1} \ll 1$ (from hypothesis). Then the system (3.14) can be rewritten as an autonomous system

$$(\dot{\theta}, \dot{z}, \dot{\gamma}, \dot{v}, \dot{t})\mathcal{A}(\theta, z, \gamma, v, t) + (\dot{\theta}, \dot{z}, \dot{\gamma}, \dot{v}, \dot{t})\mathcal{M}(\theta, z, \gamma, v, t) = \vec{\mathcal{H}}(\theta, z, \gamma, v, t) \tag{3.16}$$

where

$$\vec{\mathcal{H}}(\theta, z, \gamma, v, t) = \begin{pmatrix} \mathcal{F}_1(\theta, z, \gamma, v, t) \\ \mathcal{F}_2(\theta, z, \gamma, v, t) + 2c_2 v \\ \mathcal{F}_3(\theta, z, \gamma, v, t) + c_1 - c_1|v|^2 \\ -\frac{2}{c_2}H(z) \\ \mathrm{e}^{2\theta} \end{pmatrix} - \vec{\mathcal{G}}(\theta, z, \gamma, v, t)$$

and the matrix $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} -c_1 & c_2 z & c_1(v\cdot z) & 0 & 0 \\ 0 & c_2 & c_1 v & 0 & 0 \\ 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, the perturbed matrix $(\mathcal{A} + \mathcal{M})(\theta, z, \gamma, v, t)$ is invertible ($\det \mathcal{A} = -c_1^2 c_2 < 0$). As same as the way to deal with $\mathcal{F}$, one can check that $\mathcal{M}$, $\vec{\mathcal{G}}$ are continuously differentiable, thus so are the entries of $(\mathcal{A}+\mathcal{M})^{-1}$ and $\vec{\mathcal{H}}$. Therefore,

$$\mathcal{R}(\theta,z,\gamma,v,t)=[(\mathcal{A}+\mathcal{M})^{-1}\cdot\vec{\mathcal{H}}](\theta,z,\gamma,v,t)$$

satisfies the hypothesis of the Cauchy–Lipschitz theorem and the system of ODEs

$$(\dot{\theta},\dot{z},\dot{\gamma},\dot{v},\dot{t})=\mathcal{R}(\theta,z,\gamma,v,t) \tag{3.17}$$

admits a unique solution $(\theta(s),z(s),\gamma(s),v(s),t(s))$ to the initial value problem. We obtain the decomposition $(\lambda(s),z(s),\gamma(s),v(s))$ of $u(t)$ and the renormalization of time $t(s)$.

□

Observe from (3.1) that the initial data

$$\begin{aligned} w(s^{\text{in}})=\mathbf{P}^{\text{in}}(y;(z^{\text{in}}\vec{e_1},v^{\text{in}})),\quad \lambda(s^{\text{in}})=\lambda^{\text{in}},\quad \gamma(s^{\text{in}})=0,\\ z(s^{\text{in}})=z^{\text{in}}\vec{e_1},\quad v(s^{\text{in}})=v^{\text{in}},\quad \epsilon(s^{\text{in}})\equiv 0 \end{aligned} \tag{3.18}$$

and $u(T_{\text{mod}},x)$ satisfy the hypothesis of Lemma 9.

Proposition 10 (Uniform backwards estimates for $p>2$)**.** *There exists $s_0>10$ satisfying the following condition: for all $s^{\text{in}}>s_0$, there is a choice of initial parameters $(\lambda^{\text{in}},z^{\text{in}},v^{\text{in}})$ with*

$$\begin{aligned} &\left|c^{-\frac{1}{2}}(z^{\text{in}})^{\frac{d-1}{4}}\mathrm{e}^{\frac{1}{2}z^{\text{in}}}-s^{\text{in}}\right|<s^{\text{in}}\log^{-\frac{1}{2}}(s^{\text{in}}),\quad z^{\text{in}}>0,\\ &\lambda^{\text{in}}=1,\quad v^{\text{in}}=c^{\frac{1}{2}}(z^{\text{in}})^{-\frac{d-1}{4}}\mathrm{e}^{-\frac{1}{2}z^{\text{in}}}\cdot\vec{e_1}, \end{aligned} \tag{3.19}$$

such that the solution u to (NLS) *corresponding to* (3.1) *exists. Moreover, the decomposition of u given by Lemma 9 on the rescaled interval of time $[s_0,s^{\text{in}}]$*

$$u(s,x)=\frac{\mathrm{e}^{\mathrm{i}\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)}(\mathbf{P}+\epsilon)(s,y),\quad y=\frac{x}{\lambda(s)},\quad \mathrm{d}t=\lambda^2(s)\,\mathrm{d}s$$

verifies the uniform estimates for all $s\in[s_0,s^{\text{in}}]$

$$\begin{aligned} &\left|\,|z(s)|-2\log(s)\right|\lesssim\log(\log(s)),\quad \left|\lambda^{-1}(s)-1\right|\lesssim s^{-1},\\ &|v(s)|\lesssim s^{-1},\quad \|\epsilon(s)\|_{H^1}\lesssim s^{-1},\quad \left||z(s)|^{\frac{d-1}{2}}\mathrm{e}^{|z(s)|}-cs^2\right|\lesssim s^2\log^{-\frac{1}{2}}(s). \end{aligned} \tag{3.20}$$

Remark 11. The key point in Proposition 10 is that s_0 and the constants in (3.20) are independent of s^{in} as $s^{\text{in}}\to+\infty$. Observe that the estimates (3.20) match the discussion in Sect. 2.3. The decomposition in Lemma 9 is only local, but the estimates in (3.20) guarantee the global existence of the decomposition. The choice of v^{in} is direct while the choice of z^{in} is based on a contradiction argument and a topological constraint.

The next subsection is devoted to the proof of Proposition 10 containing several technical steps. The proof relies on a bootstrap argument, integration of the differential system of geometrical parameters and energy estimates. Pick a smooth function $\tilde{\chi}:[0,+\infty)\to[0,\infty)$, non increasing, with $\tilde{\chi}\equiv 1$ on $[0,\frac{1}{10}]$, $\tilde{\chi}\equiv 0$ on $[\frac{1}{8},+\infty)$. We define the localized momentum:

$$\mathcal{M}_k(s,\epsilon)=\operatorname{Im}\int(\nabla\epsilon\,\bar{\epsilon})\chi_k=\operatorname{Im}\int(\nabla\eta_k\bar{\eta}_k)\chi \tag{3.21}$$

for $\chi_k(s,y)=\tilde{\chi}\left(\log^{-1}(s)|y-z_k(s)|\right)$ and $\chi=\tilde{\chi}\left(|\log^{-1}(s)y|\right)$

3.2. **Proof of Proposition 10.**

3.2.1. ***Bootstrap bounds***. We shall consider the following bootstrap estimates

$$\begin{aligned} &\left|c^{-\frac{1}{2}}|z|^{\frac{d-1}{4}}\mathrm{e}^{\frac{1}{2}|z|}-s\right|\leq s\log^{-\frac{1}{2}}(s),\\ &\|\epsilon(s)\|_{H^1}\leq C^* s^{-1}\end{aligned}\tag{3.22}$$

with $C^*>1$ a constant to be chosen large enough. Note that the estimate on z and the estimate (2.34) of $\dot{v}$ imply that, for s large

$$\left||z|-2\log(s)\right|\lesssim\log(\log(s)),\quad\left||\dot{v}|-s^{-2}\right|\lesssim s^{-2}\log^{-\frac{1}{2}}(s),\quad\left||v|-s^{-1}\right|\lesssim s^{-1}\log^{-\frac{1}{2}}(s)\tag{3.23}$$

where the last inequality is obtained by integrating the second one with the choice of initial data v^{in} in (3.19). Next, we define

$$s^*=\inf\{\tau\in[s_0,s^{\mathrm{in}}];\ (3.22)\text{ holds on }[\tau,s^{\mathrm{in}}]\}.\tag{3.24}$$

3.2.2. ***Control of the modulation equations***. Denote $\vec{m}_k^*$ the system $\vec{m}_k$ without equation $\dot{z}_k-2v_k+\frac{\dot{\lambda}}{\lambda}z_k$ and M^* the vector M without the direction $-\mathrm{i}\nabla V$.

Lemma 12 (Pointwise control of the modulation equations and the error)**.** *The following estimates hold on* $[s^*,s^{\mathrm{in}}]$.

$$|\vec{m}_k^*(s)|\lesssim(C^*)^2s^{-2}.\tag{3.25}$$

$$|\langle\eta_1(s),\mathrm{i}\nabla Q\rangle|\lesssim(C^*)^2s^{-1}\log^{-1}(s),\tag{3.26}$$

$$|\dot{z}-2v|\lesssim s^{-1}\log^{-\frac{3}{4}}(s).\tag{3.27}$$

Moreover, for all $s\in[s^*,s^{\mathrm{in}}]$*, for all* $y\in\mathbb{R}^2$*,*

$$|\mathcal{E}_{\mathbf{P}}(s,y)|\lesssim s^{-1}\log^{-\frac{3}{4}}(s)\sum_{k=1}^{2}Q(y-z_k(s))+|G(s,y)|.\tag{3.28}$$

Proof of Lemma 12. Since $\epsilon(s^{\mathrm{in}})\equiv 0$, we may define

$$s^{**}=\inf\{s\in[s^*,s^{\mathrm{in}}];\ |\langle\eta_1(\tau),\mathrm{i}\nabla Q\rangle|\leq C^{**}\tau^{-1}\log^{-1}(\tau)\text{ holds on }[s,s^{\mathrm{in}}]\},$$

for some constant $C^{**}>0$ to be chosen large enough. We work on the interval $[s^{**},s^{\mathrm{in}}]$. Recall equation for η_1 (3.9) as below

$$\mathrm{i}\dot{\eta}_1+\Delta\eta_1-\eta_1+(|\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1)-|\mathbf{P}_1|^{p-1}\mathbf{P}_1)+\vec{m}_1\cdot\vec{\mathrm{M}}\eta_1+\mathcal{E}_{\mathbf{P}_1}=0.$$

Let $A(y)$ and $B(y)$ be two real-valued functions in $\mathcal{Y}$. We claim the following estimate on $[s^{**},s^{\mathrm{in}}]$

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\langle\eta_1,A+\mathrm{i}B\rangle-\Big[\langle\eta_1,\mathrm{i}L_-A-L_+B\rangle-\langle\vec{m}_1\cdot\vec{\mathrm{M}}Q,\mathrm{i}A-B\rangle\Big]\right|\lesssim(C^*)^2s^{-2}+s^{-1}|\vec{m}_1|.\tag{3.29}$$

We compute from (3.9),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\langle\eta_1,A+\mathrm{i}B\rangle&=\langle\dot{\eta}_1,A+\mathrm{i}B\rangle=\langle\mathrm{i}\dot{\eta}_1,\mathrm{i}A-B\rangle\\ &=\langle-\Delta\eta_1+\eta_1-(\frac{p+1}{2}Q^{p-1}\eta_1+\frac{p-1}{2}Q^{p-1}\overline{\eta}_1),\mathrm{i}A-B\rangle\\ &-\langle|\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1)-|\mathbf{P}_1|^{p-1}\mathbf{P}_1-\frac{p+1}{2}Q^{p-1}\eta_1-\frac{p-1}{2}Q^{p-1}\overline{\eta}_1,\mathrm{i}A-B\rangle\\ &-\langle\vec{m}_1\cdot\vec{\mathrm{d}}\eta_1,\mathrm{i}A-B\rangle-\langle\mathcal{E}_{\mathbf{P}_1},\mathrm{i}A-B\rangle.\end{aligned}$$

First, since A and B are real-valued, we have

$$\langle -\Delta\eta_1 + \eta_1 - (\frac{p+1}{2}Q^{p-1}\eta_1 + \frac{p-1}{2}Q^{p-1}\overline{\eta}_1), \mathrm{i}A - B\rangle = \langle \eta_1, \mathrm{i}L_- A - L_+ B\rangle.$$

Second, recall the expression of $\mathbf{P}_1$

$$\mathbf{P}_1 = Q(y) + \mathrm{e}^{\mathrm{i}(\Gamma_2(y-(z_2-z_1))-\Gamma_1(y))}Q(y-(z_2-z_1)).$$

By the expansion in (2.17), we can deduce the first order and the error of

$$\begin{aligned}&|\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 - \frac{p+1}{2}Q^{p-1}\eta_1 - \frac{p-1}{2}Q^{p-1}\overline{\eta}_1\\ &= \frac{p+1}{2}(|\mathbf{P}_1|^{p-1} - Q^{p-1})\eta_1 + \frac{p-1}{2}(|\mathbf{P}_1|^{p-3}\mathbf{P}_1^2 - Q^{p-1})\overline{\eta}_1 + O(\left|\frac{\eta_1}{\mathbf{P}_1}\right|^2 |\mathbf{P}_1|^p) + O(|\eta_1|^p).\end{aligned}$$

By (3.22)–(3.23), for some $q > 0$,

$$\begin{aligned}&|\langle(|\mathbf{P}_1|^{p-1} - Q^{p-1})\eta_1, (\mathrm{i}A-B)\rangle| + |\langle(|\mathbf{P}_1|^{p-3}\mathbf{P}_1^2 - Q^{p-1})\overline{\eta}_1, (\mathrm{i}A-B)\rangle|\\ &\lesssim |z|^q\mathrm{e}^{-|z|}\|\eta_1\|_{L^2} \lesssim C^* s^{-3}\log^q(s).\end{aligned}$$

Using the Cauchy–Schwarz and Gagliardo–Nirenberg inequalities (as $p > 2$),

$$\left\langle \left|\frac{\eta_1}{\mathbf{P}_1}\right|^2 |\mathbf{P}_1|^p, (\mathrm{i}A-B)\right\rangle \lesssim ||\epsilon||_{L^2}^2 \lesssim (C^*)^2 s^{-2},$$

$$\langle |\eta_1|^p, (\mathrm{i}A-B)\rangle \lesssim ||\epsilon||_{H^1}^p \lesssim (C^*)^2 s^{-2}.$$

Therefore,

$$\left|\langle |\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 - \frac{p+1}{2}Q^{p-1}\eta_1 - \frac{p-1}{2}Q^{p-1}\overline{\eta}_1, \mathrm{i}A-B\rangle\right| \lesssim (C^*)^2 s^{-2}. \tag{3.30}$$

Next, using (3.22)–(3.23), we obtain

$$|\langle \vec{m}_1 \cdot \vec{\mathrm{d}}\eta_1, \mathrm{i}A-B\rangle| \lesssim C^* s^{-1}|\vec{m}_1(s)|.$$

Finally, we need to prove following estimate

$$\left|\langle \mathcal{E}_{\mathbf{P}_1}, \mathrm{i}A-B\rangle - \langle \vec{m}_1 \cdot \vec{\mathrm{M}}Q, \mathrm{i}A-B\rangle\right| \lesssim s^{-2} + s^{-1}|\vec{m}_1|. \tag{3.31}$$

Indeed, recall that we have

$$\mathcal{E}_{\mathbf{P}_1} = [\vec{m}_1 \cdot \vec{\mathrm{M}}Q](y) + [\mathrm{e}^{\mathrm{i}(\Gamma_2(y-(z_2-z_1))-\Gamma_1(y))}\vec{m}_2 \cdot \vec{\mathrm{M}}Q](y-(z_2-z_1)) + G_1.$$

From (2.14) and (3.22)–(3.23),

$$|\langle G_1, \mathrm{i}A-B\rangle| \lesssim \|G\|_{L^\infty} \lesssim |z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|} \lesssim s^{-2}.$$

Since $A, B \in \mathcal{Y}$, we have

$$|\langle \mathrm{e}^{\mathrm{i}(\Gamma_2(y-(z_2-z_1))-\Gamma_1(y))}(\vec{m}_2 \cdot \vec{\mathrm{M}}Q(\cdot-(z_2-z_1))), \mathrm{i}A-B\rangle| \lesssim s^{-1}|\vec{m}_1|,$$

so the proof of (3.31) is complete.

We now use (3.29) to control the modulation vector $\vec{m}_1$. Note that η_1 satisfies the orthogonality

conditions (3.5).
$\underline{\langle \eta_1, Q\rangle = 0}$. Let $A = Q$ and $B = 0$. Since $L_-Q = 0$ and $\langle \vec{m}_1 \cdot \vec{\mathrm{M}}Q, \mathrm{i}Q\rangle = -c_1(\frac{\dot\lambda}{\lambda})$, we obtain

$$\left|\frac{\dot\lambda}{\lambda}\right| \lesssim (C^*)^2 s^{-2} + s^{-1}|\vec{m}_1|. \tag{3.32}$$

$\underline{\langle \eta_1, \mathrm{i}\Lambda Q\rangle = 0}$. Let $A = 0$ and $B = \Lambda Q$. Since $L_+(\Lambda Q) = -2Q$, $\langle \eta_1, Q\rangle = 0$ and $\langle \vec{m}_1 \cdot \vec{\mathrm{M}}Q, -\Lambda Q\rangle = c_1(\dot\gamma - 1 + |v|^2 - \frac{\dot\lambda}{\lambda}(v\cdot z) - (v \cdot \dot z))$, we obtain

$$\left|\dot\gamma - 1 + |v|^2 - \frac{\dot\lambda}{\lambda}(v\cdot z) - (v\cdot \dot z)\right| \lesssim (C^*)^2 s^{-2} + s^{-1}|\vec{m}_1|. \tag{3.33}$$

$\underline{\langle \eta_1, yQ\rangle = 0}$. Let $A = yQ$ and $B = 0$. Since $L_-(yQ) = -2\nabla Q$, $|\langle \eta_1, \mathrm{i}\nabla Q\rangle| \lesssim C^{**}s^{-1}\log^{-1}(s)$ and $\langle \vec{m}_1 \cdot \vec{\mathrm{M}}Q, \mathrm{i}yQ\rangle = c_2(\dot z - 2v + \frac{\dot\lambda}{\lambda}z)$, we obtain

$$\left|\dot z - 2v + \frac{\dot\lambda}{\lambda}z\right| \lesssim C^{**}s^{-1}\log^{-1}(s) + (C^*)^2 s^{-2} + s^{-1}|\vec{m}_1|. \tag{3.34}$$

By (3.23) and (3.32),

$$\left|\dot v - \frac{\dot\lambda}{\lambda}v\right| \lesssim |\dot v| + \left|\frac{\dot\lambda}{\lambda}\right||v| \lesssim s^{-2}. \tag{3.35}$$

Combining Eqs. (3.32) to (3.35), we have proved, for all $s \in [s^{**}, s^{\mathrm{in}}]$,

$$|\vec{m}_1^*(s)| \lesssim (C^*)^2 s^{-2} \tag{3.36}$$

and

$$|\dot z - 2v| \lesssim s^{-1}\log^{-\frac{3}{4}}(s). \tag{3.37}$$

Now we turn to the study of localized momentum $\mathcal{M}_k$:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{M}_1 = \operatorname{Im}\int (\nabla\eta_1\,\bar\eta_1)\dot\chi + \langle \mathrm{i}\dot\eta_1, 2\mathrm{i}\nabla\eta_1 + \eta_1\nabla\mathrm{i}\rangle.$$

We claim that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{M}_1 = \left\langle \frac{\bar\eta_1\cdot F''(\mathbf{P}_1)\cdot\eta_1}{2}, \nabla Q\right\rangle + \left(\dot z_1 - 2v_1 + \frac{\dot\lambda}{\lambda}z_1\right)\langle \mathrm{i}\nabla Q, \nabla\eta_1\rangle + O(\log^{-1}(s)\|\eta_1\|_{H^1}^2). \tag{3.38}$$

Note that, by direct computations,

$$|\dot\chi| \lesssim |s^{-1}\log^{-2}(s)y\tilde\chi'(\log^{-1}(s)y)| \lesssim s^{-1}\log^{-1}(s)$$

and so, by (3.22)–(3.23),

$$\left|\operatorname{Im}\int(\nabla\eta_1\,\bar\eta_1)\dot\chi\right| \lesssim s^{-1}\log^{-1}(s)\|\eta_1\|_{H^1}^2 \lesssim s^{-3}\log^{-\frac{1}{2}}(s).$$

Now, we use the equation (3.9) of η_1

$$\begin{aligned}\mathrm{i}\dot\eta_1 + \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1) + \vec{m}_1^*\cdot\vec{\mathrm{M}}^*\eta_1 - (\dot z_1 - 2v_1 + \frac{\dot\lambda}{\lambda}z_1)\cdot\mathrm{i}\nabla\eta_1 \\ + \vec{m}_1^*\cdot\vec{\mathrm{M}}^*Q - (\dot z_1 - 2v_1 + \frac{\dot\lambda}{\lambda}z_1)\cdot\mathrm{i}\nabla Q + [\mathrm{e}^{\mathrm{i}(\Gamma_2(y-z)-\Gamma_1(y))}\vec{m}_2\cdot\vec{\mathrm{M}}Q](y-z) + G_1 = 0\end{aligned}$$

to estimate $\langle \mathrm{i}\dot{\eta}_1, 2\mathrm{i}\nabla\eta_1 + \eta_1\nabla \mathrm{i}\rangle$. By integration by parts, we check the following

$$\langle \Delta\eta_1, 2\mathrm{i}\nabla\eta_1 + \eta_1\nabla \mathrm{i}\rangle = -2\langle \nabla\eta_1\cdot\nabla\chi, \nabla\eta_1\rangle + \frac{1}{2}\int |\eta_1|^2\nabla(\Delta\chi).$$

We have

$$|\langle \nabla\eta_1\cdot\nabla\chi, \nabla\eta_1\rangle| \lesssim \log^{-1}(s)\|\eta_1\|^2_{H^1}$$

and as $|\nabla(\Delta\chi)| \lesssim \log^{-3}(s)$ we obtain

$$\left|\int |\eta_1|^2\nabla(\Delta\chi)\right| \lesssim \log^{-3}(s)\|\eta_1\|^2_{H^1}.$$

In conclusion for term $\Delta\eta_1$ in the equation of η_1, we get

$$|\langle \Delta\eta_1, 2\mathrm{i}\nabla\eta_1 + \eta_1\nabla \mathrm{i}\rangle| \lesssim \log^{-1}(s)\|\eta_1\|^2_{H^1}.$$

For the term η_1, we simply verify by integration by parts that

$$\langle \eta_1, 2\mathrm{i}\nabla\eta_1 + \eta_1\nabla \mathrm{i}\rangle = 0.$$

From (3.36) and (3.37), we also have that

$$\left|\langle \vec{m}_1\cdot\vec{\mathrm{M}}\eta_1, 2\mathrm{i}\nabla\eta_1 + \eta_1\nabla \mathrm{i}\rangle\right| \lesssim s^{-1}\log^{-\frac{1}{2}}(s)||\eta_1||^2_{H^1},$$

$$\left|\langle \vec{m}_1^*\cdot\vec{\mathrm{M}}^*Q, 2\mathrm{i}\nabla\eta_1 + \eta_1\nabla \mathrm{i}\rangle\right| \lesssim (C^*)^2 s^{-2}||\eta_1||_{H^1} \lesssim s^{-\frac{5}{2}},$$

$$\left|\langle [\vec{m}_2\cdot\vec{\mathrm{M}}Q](\cdot - z), 2\mathrm{i}\nabla\eta_1 + \eta_1\nabla \mathrm{i}\rangle\right| \lesssim s^{-1}\log^{-\frac{1}{2}}(s)\mathrm{e}^{-\frac{7}{8}z}||\eta_1||_{H^1} \lesssim s^{-3},$$

and

$$\left|\langle G_1, 2\mathrm{i}\nabla\eta_1 + \eta_1\nabla \mathrm{i}\rangle\right| \lesssim \|G_1\|_{L^\infty}\ \log^{\frac{\mathrm{d}}{2}}(s)\|\epsilon\|_{H^1} \lesssim s^{-\frac{3}{2}}\|\eta_1\|_{H^1}.$$

where we use the Cauchy–Schwarz inequality and the fact that the support of χ is contained in $\{|y| \le \frac{1}{8}\log(s)\}$. Now we will deal with the term $\langle |\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1, 2\mathrm{i}\nabla\eta_1 + \eta_1\nabla \mathrm{i}\rangle$. By (2.17), we consider

$$|\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 = F'(\mathbf{P}_1)\cdot\eta_1 + O(|\eta_1|^p) + O\left(\left|\frac{\eta_1}{\mathbf{P}_1}\right|^2|\mathbf{P}_1|^p\right)$$

and, using the Gagliardo–Nirenberg inequality (note that if $p > 2$ then $3 < 2^*$),

$$\left|\left\langle \left|\frac{\eta_1}{\mathbf{P}_1}\right|^2|\mathbf{P}_1|^p, 2\mathrm{i}\nabla\eta_1 + \eta_1\nabla\chi\right\rangle\right| \lesssim ||\eta_1||^3_{H^1} \lesssim s^{-\frac{5}{2}},$$

$$|\langle |\eta_1|^p, 2\mathrm{i}\nabla\eta_1 + \eta_1\nabla\chi\rangle| \lesssim ||\eta_1||^{p+1}_{H^1} \lesssim s^{-2}\log^{-2}(s).$$

Then, we have

$$|\langle F'(\mathbf{P}_1)\cdot\eta_1, \eta_1\nabla \mathrm{i}\rangle| \lesssim |\nabla\chi|\,||\epsilon||^2_{H^1} \lesssim \log^{-1}(s)||\eta_1||^2_{H^1}.$$

Finally, by integration by parts, we get

$$\langle F'(\mathbf{P}_1)\cdot\eta_1, 2\mathrm{i}\nabla\eta_1\rangle = -2\left\langle \nabla\mathbf{P}_1\chi, \frac{\bar{\eta}_1\cdot F''(\mathbf{P}_1)\cdot\eta_1}{2}\right\rangle - \langle F'(\mathbf{P}_1)\cdot\eta_1, \eta_1\nabla \mathrm{i}\rangle,$$

therefore the collection of above bounds gives

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{M}_1 = 2\langle \nabla \mathbf{P}_1\chi, \frac{\bar{\eta}_1 \cdot F''(\mathbf{P}_1)\cdot \eta_1}{2}\rangle + 2\left(\dot{z}_1 - 2v_1 + \frac{\dot{\lambda}}{\lambda}z_1\right)\langle \mathrm{i}\nabla Q, \mathrm{i}\nabla \eta_1\rangle + O(\log^{-1}(s)\|\eta_1(s)\|_{H^1}^2). \tag{3.39}$$

We finish the proof of (3.38) by showing the following estimate

$$|\langle \nabla \mathbf{P}_1\chi, \bar{\eta}_1\cdot F''(\mathbf{P}_1)\cdot\eta_1\rangle - \langle \nabla Q, \bar{\eta}_1\cdot F''(\mathbf{P}_1)\cdot \eta_1\rangle| \lesssim \left|\int_{|y|<\frac{1}{8}\log s}(\bar{\eta}_1\cdot F''(\mathbf{P}_1)\cdot\eta_1)\nabla Q(\cdot+z)\right| \\ + \left|\int_{|y|>\frac{1}{10}\log s}(\bar{\eta}_1\cdot F''(\mathbf{P}_1)\cdot\eta_1)\nabla Q\right| \lesssim s^{-\frac{1}{20}}||\epsilon||_{H_1}^2,$$

and

$$\left(\dot{z}_1 - 2v_1 + \frac{\dot{\lambda}}{\lambda}z_1\right)|\langle \mathrm{i}\nabla Q, \mathrm{i}\nabla\eta_1\rangle - \langle \mathrm{i}\nabla Q, \nabla \eta_1\rangle| \lesssim s^{-1}\log^{-\frac{1}{2}}(s)\left|\int_{|y|>\frac{1}{10}\log s}\nabla Q \nabla \bar{\eta}_1\right| \\ \lesssim s^{-1-\frac{1}{20}}\log^{-\frac{3}{4}}(s)\|\eta_1\|_{H^1}$$

here we use (1.12). On the other hand, from (3.29), refining up to order s^{-2}, using $L_+(\nabla Q) = 0$ and (2.18), we have that

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle \eta_1, \mathrm{i}\nabla Q\rangle = \left\langle \frac{\bar{\eta}_1\cdot F''(\mathbf{P}_1)\cdot\eta_1}{2}, \nabla Q\right\rangle - c_2\left(\dot{v}_1 - \frac{\dot{\lambda}}{\lambda}v_1\right) \\ - \langle G_1, \nabla Q\rangle + \langle \vec{m}_1 \cdot \vec{\mathrm{M}}\eta_1, \nabla Q\rangle + O(s^{-2^+}).$$

From (2.20) and the choice of v in (3.6), we get

$$\left|c_2\dot{v}_1 + \langle G_1, \nabla Q\rangle - c_2\frac{\dot{\lambda}}{\lambda}v_1\right| \lesssim (|v|^2|z|^2 + |v|^2)|z|^{-\frac{1}{2}}\mathrm{e}^{-|z|} + |z|^{-\frac{3(d-1)}{4}}\mathrm{e}^{-\frac{3}{2}|z|} + |v|\left|\frac{\dot{\lambda}}{\lambda}\right| \lesssim s^{-3},$$

then, from (3.36), we obtain

$$|\langle \vec{m}_1^* \cdot \vec{\mathrm{M}}^*\eta_1, \nabla Q\rangle| \lesssim s^{-2}\|\eta_1\|_{H^1}.$$

Thus, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle \eta_1, \mathrm{i}\nabla Q\rangle = \left\langle \frac{\bar{\eta}_1\cdot F''(\mathbf{P}_1)\cdot\eta_1}{2}, \nabla Q\right\rangle - \left(\dot{z}_1 - 2v_1 + \frac{\dot{\lambda}}{\lambda}z_1\right)\langle \mathrm{i}\nabla\eta_1, \nabla Q\rangle + O(s^{-2^+})\cdot$$

Note that $\langle \mathrm{i}\nabla\eta_1, \nabla Q\rangle = -\langle \mathrm{i}\nabla Q, \nabla \eta_1\rangle$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle \eta_1, \mathrm{i}\nabla Q\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{Im}\int(\nabla\eta_1\bar{\eta}_1)\chi + O(\log^{-1}(s)\|\eta_1\|_{H^1}^2)\cdot$$

This information, combined with $\langle \eta_1(s^{\mathrm{in}}), \mathrm{i}\nabla Q\rangle = 0$ and $\mathcal{M}_1(t^{\mathrm{in}}) = 0$, implies that

$$\left|\langle \eta_1, \mathrm{i}\nabla Q\rangle - \frac{1}{2}\mathcal{M}_1\right| \lesssim (C^*)^2 s^{-1}\log^{-1}(s).$$

From the bootstrap (3.22), we deduce that $|\langle \eta_1, \mathrm{i}\nabla Q\rangle| \lesssim (C^*)^2 s^{-1}\log^{-1}$ so, if we take C^{**} big enough such that $\frac{C^{**}}{2} \gtrsim (C^*)^2$, then $s^{**} = s^*$. Those estimates (3.27) and (3.28) are direct consequences of (3.23), (3.25) and (3.32). □

3.2.3. ***Energy functional.*** Consider the nonlinear energy functional for ϵ

$$\mathbf{H}(s,\epsilon)=\frac{1}{2}\int\left(|\nabla\epsilon|^2+|\epsilon|^2-\frac{2}{p+1}\left(|\mathbf{P}+\epsilon|^{p+1}-|\mathbf{P}|^{p+1}-(p+1)|\mathbf{P}|^{p-1}\operatorname{Re}(\epsilon\overline{\mathbf{P}})\right)\right)$$

and

$$\mathbf{J}=\sum_k J_k,\quad J_k(s,\epsilon)=v_k\cdot\mathcal{M}_k(s,\epsilon).$$

where $\mathcal{M}_k(s,\epsilon)$ the localized moment defined in (3.21). Finally, we set

$$\mathbf{W}(s,\epsilon)=\mathbf{H}(s,\epsilon)-\mathbf{J}(s,\epsilon).$$

The functional $\mathbf{W}$ is coercive in ϵ at the main order and it is an almost conserved quantity for the problem (see [29] for a similar functional).

Proposition 13 (Coercivity and time control of the energy functional)**.** *For all $s\in[s^*,s^{\text{in}}]$,*

$$\mathbf{W}(s,\epsilon(s))\gtrsim\|\epsilon(s)\|_{H^1}^2, \tag{3.40}$$

and

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}[\mathbf{W}(s,\epsilon(s))]\right|\lesssim s^{-2}\|\epsilon(s)\|_{H^1}. \tag{3.41}$$

Proof of Proposition 13. **Step 1.** Coercivity. The proof of the coercivity (3.40) is a standard consequence of the coercivity property (1.15) around one solitary wave with the orthogonality properties (3.5), (3.26), and an elementary localization argument. We refer to the proof of Lemma 4.1 in Appendix B of [23] for a similar proof.

Step 2. Variation of the energy. We estimate the time variation of the functional $\mathbf{H}$ and claim that for all $s\in[s^*,s^{\text{in}}]$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}[\mathbf{H}(s,\epsilon(s))]-\sum_{k=1}^{2}\dot{z}_k\cdot\langle\nabla P_k,\frac{\bar{\epsilon}.F''(\mathbf{P})\cdot\epsilon}{2}\rangle\right|\lesssim s^{-2}\|\epsilon(s)\|_{H^1}+s^{-1}\log^{-\frac{3}{4}}(s)\|\epsilon\|_{H^1}^2. \tag{3.42}$$

The time derivative of $s\mapsto H(s,\epsilon(s))$ splits into two parts

$$\frac{\mathrm{d}}{\mathrm{d}s}[\mathbf{H}(s,\epsilon(s))]=D_s\mathbf{H}(s,\epsilon(s))+\langle D_\epsilon\mathbf{H}(s,\epsilon(s)),\dot{\epsilon}_s\rangle,$$

where D_s denotes differentiation of $\mathbf{H}$ with respect to s and D_ϵ denotes differentiation of $\mathbf{H}$ with respect to ϵ. Firstly we compute:

$$\begin{aligned}D_s\mathbf{H}=&-\operatorname{Re}\int[\dot{\mathbf{P}}(|\mathbf{P}+\epsilon|^{p-1}\overline{(\mathbf{P}+\epsilon)}-|\mathbf{P}|^{p-1}\overline{\mathbf{P}})\\&-\frac{p-1}{2}|\mathbf{P}|^{p-3}(\dot{\mathbf{P}}\overline{\mathbf{P}}+\dot{\overline{\mathbf{P}}}\mathbf{P})\operatorname{Re}(\bar{\epsilon}\mathbf{P})-|\mathbf{P}|^{p-1}\bar{\epsilon}\dot{\mathbf{P}}](y)\,\mathrm{d}y\\=&-\operatorname{Re}\int[\dot{\mathbf{P}}(|\mathbf{P}+\epsilon|^{p-1}\overline{(\mathbf{P}+\epsilon)}-|\mathbf{P}|^{p-1}\overline{\mathbf{P}})\\&-\frac{p-1}{2}|\mathbf{P}|^{p-3}\frac{\epsilon\overline{\mathbf{P}}^2\dot{\mathbf{P}}+\bar{\epsilon}|\mathbf{P}|^2\dot{\mathbf{P}}+\epsilon|\mathbf{P}|^2\dot{\overline{\mathbf{P}}}+\bar{\epsilon}\mathbf{P}^2\dot{\overline{\mathbf{P}}}}{2}-|\mathbf{P}|^{p-1}\bar{\epsilon}\dot{\mathbf{P}}](y)\,\mathrm{d}y\\=&-\langle\dot{\mathbf{P}},|\mathbf{P}+\epsilon|^{p-1}(\mathbf{P}+\epsilon)-|\mathbf{P}|^{p-1}\mathbf{P}-\frac{p+1}{2}\epsilon|\mathbf{P}|^{p-1}-\frac{p-1}{2}\bar{\epsilon}\mathbf{P}^2|\mathbf{P}|^{p-3}\rangle.\end{aligned}$$

We observe that $\dot{P}_k=-\dot{z}_k\cdot\nabla P_k+\mathrm{i}\dot{v}_k\cdot(y-z_k)P_k$. Denote

$$K=|\mathbf{P}+\epsilon|^{p-1}(\mathbf{P}+\epsilon)-|\mathbf{P}|^{p-1}\mathbf{P}-\frac{p+1}{2}\epsilon|\mathbf{P}|^{p-1}-\frac{p-1}{2}\bar{\epsilon}\mathbf{P}^2|\mathbf{P}|^{p-3}$$

then by (2.16), $K = |\mathbf{P}+\epsilon|^{p-1}(\mathbf{P}+\epsilon) - |\mathbf{P}|^{p-1}\mathbf{P} - F'(\mathbf{P})\cdot\epsilon$, we deduce from (2.17) that

$$|K| \lesssim |\epsilon|^2|\mathbf{P}|^{p-2} + |\epsilon|^p$$

so we obtain

$$|\langle \mathrm{i}\dot{v}_k \cdot (y-z_k)P_k, K\rangle| \lesssim (||\epsilon||_{H^1}^2 + ||\epsilon||_{H^1}^p)|\dot{v}| \lesssim s^{-2}||\epsilon||_{H^1}^2.$$

Next, we look more precisely at K

$$K = \frac{\bar{\epsilon}\cdot F''(\mathbf{P})\cdot\epsilon}{2} + O(\left|\frac{\epsilon}{\mathbf{P}}\right|^{2^+}|\mathbf{P}|^p) + O(|\epsilon|^p)$$

as $|\dot{z}_k| \lesssim s^{-1}$ and $p - 2^+ > 0$, we have

$$\left|\left\langle -\dot{z}_k\cdot\nabla P_k, \left|\frac{\epsilon}{\mathbf{P}}\right|^{2^+}|\mathbf{P}|^p\right\rangle\right| \lesssim s^{-1}||\epsilon||_{H^1}^{2^+}$$

and

$$|\langle -\dot{z}_k\cdot\nabla P_k, |\epsilon|^p\rangle| \lesssim s^{-1}||\epsilon||_{H^1}^p.$$

Combining these computations, we get

$$D_s\mathbf{H}(s,\epsilon) = \sum_{k=1}^{2}\langle \dot{z}_k\cdot\nabla P_k, \frac{\bar{\epsilon}.F''(\mathbf{P})\cdot\epsilon}{2}\rangle + O(s^{-1}\|\epsilon\|_{H^1}^{2^+}) + O(s^{-2}\|\epsilon\|_{H^1}^2) + O(s^{-1}\|\epsilon\|_{H^1}^p). \quad (3.43)$$

Secondly, we consider

$$D_\epsilon\mathbf{H}(s,\epsilon) = -\Delta\epsilon + \epsilon - \left(|\mathbf{P}+\epsilon|^{p-1}(\mathbf{P}+\epsilon) - |\mathbf{P}|^{p-1}\mathbf{P}\right)$$

and note that the equation (3.8) of ϵ can be rewritten as

$$\mathrm{i}\dot{\epsilon} - D_\epsilon\mathbf{H}(s,\epsilon) - \mathrm{i}\frac{\dot{\lambda}}{\lambda}\Lambda\epsilon + (1-\dot{\gamma})\epsilon + \mathcal{E}_{\mathbf{P}} = 0$$

so that

$$\begin{aligned}\langle D_\epsilon\mathbf{H}(s,\epsilon), \dot{\epsilon}\rangle &= \langle \mathrm{i}D_\epsilon\mathbf{H}(s,\epsilon), \mathrm{i}\dot{\epsilon}\rangle \\ &= \frac{\dot{\lambda}}{\lambda}\langle D_\epsilon\mathbf{H}(s,\epsilon), \Lambda\epsilon\rangle - (1-\dot{\gamma})\langle \mathrm{i}D_\epsilon\mathbf{H}(s,\epsilon), \epsilon\rangle - \langle \mathrm{i}D_\epsilon\mathbf{H}(s,\epsilon), \mathcal{E}_{\mathbf{P}}\rangle.\end{aligned}$$

On the other hand, from (3.25) and (3.22)–(3.23), we have

$$\left|\frac{\dot{\lambda}}{\lambda}\langle D_\epsilon\mathbf{H}(s,\epsilon), \Lambda\epsilon\rangle\right| \lesssim \left|\frac{\dot{\lambda}}{\lambda}\right|\left(\|\epsilon\|_{H^1}^2 + \|\epsilon\|_{H^1}^{p+1}\right) \lesssim (C^*)^2 s^{-2}\|\epsilon\|_{H^1}^2,$$

$$|(1-\dot{\gamma})\langle \mathrm{i}D_\epsilon\mathbf{H}(s,\epsilon), \epsilon\rangle| \lesssim |1-\dot{\gamma}|\,(\|\epsilon\|_{H^1}^2 + \|\epsilon\|_{H^1}^{p+1}) \lesssim (C^*)^2 s^{-2}\|\epsilon\|_{H^1}^2.$$

For the last term, we rewrite

$$\begin{aligned}\langle \mathrm{i}D_\epsilon\mathbf{H}(s,\epsilon), \mathcal{E}_{\mathbf{P}}\rangle =& \langle -\mathrm{i}\Delta\epsilon + \mathrm{i}\epsilon - \mathrm{i}\left(|\mathbf{P}+\epsilon|^{p-1}(\mathbf{P}+\epsilon) - |\mathbf{P}|^{p-1}\mathbf{P}\right), \\ & [\mathrm{e}^{\mathrm{i}\Gamma_1}\vec{m}_1\cdot\vec{\mathrm{M}}Q](y-z_1(s)) + [\mathrm{e}^{\mathrm{i}\Gamma_2}\vec{m}_2\cdot\vec{\mathrm{M}}Q](y-z_2(s)) + G\rangle.\end{aligned}$$

Recall that with $\eta_1=\eta_1^1+\mathrm{i}\eta_1^2$ for η_1^1,η_1^2 real, from the expression of operators L_+ and L_-

$$\begin{aligned}
I_1 =&\langle -\mathrm{i}\Delta\epsilon+\mathrm{i}\epsilon-\mathrm{i}\left(|\mathbf{P}+\epsilon|^{p-1}(\mathbf{P}+\epsilon)-|\mathbf{P}|^{p-1}\mathbf{P}\right),[\mathrm{e}^{\mathrm{i}\Gamma_1}\vec{m}_1\cdot\vec{\mathrm{M}}Q](y-z_1(s))\rangle\\
=&\langle -\mathrm{i}\Delta\eta_1+\mathrm{i}\eta_1-\mathrm{i}(|\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1)-|\mathbf{P}_1|^{p-1}\mathbf{P}_1),\vec{m}_1\cdot\vec{\mathrm{M}}Q\rangle\\
=&\langle \mathrm{i}L_+\eta_1^1-L_-\eta_1^2,\vec{m}_1\cdot\vec{\mathrm{M}}Q\rangle\\
&-\left\langle \mathrm{i}\left(|\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1)-|\mathbf{P}_1|^{p-1}\mathbf{P}_1-\frac{p+1}{2}Q^{p-1}\eta_1-\frac{p-1}{2}Q^{p-1}\overline{\eta}_1\right),\vec{m}_1\cdot\vec{\mathrm{M}}Q\right\rangle\\
=&-\frac{\dot\lambda}{\lambda}\langle\eta_1,-2Q\rangle+(\dot v-\frac{\dot\lambda}{\lambda}v)\langle\eta_1,-2\mathrm{i}\nabla Q\rangle\\
&-\left\langle \mathrm{i}\left(|\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1)-|\mathbf{P}_1|^{p-1}\mathbf{P}_1-\frac{p+1}{2}Q^{p-1}\eta_1-\frac{p-1}{2}Q^{p-1}\overline{\eta}_1\right),\vec{m}_1\cdot\vec{\mathrm{M}}Q\right\rangle.
\end{aligned}$$

By orthogonality of η_1 (3.5), (3.26) and the estimate (3.25), (3.30), we get

$$|I_1|=O((C^*)^2s^{-3}\log^{-\frac34}(s)).$$

By symmetry, we have the same estimate for I_2. Finally, from (2.13) and (3.22), we have $||G||_{H^1}\lesssim s^{-2}$, so using an integration by parts and the Cauchy–Schwarz inequality

$$|\langle -\mathrm{i}\Delta\epsilon+\mathrm{i}\epsilon-\mathrm{i}\left(|\mathbf{P}+\epsilon|^{p-1}(\mathbf{P}+\epsilon)-|\mathbf{P}|^{p-1}\mathbf{P}\right),G\rangle|\lesssim s^{-2}||\epsilon||_{H^1}. \tag{3.44}$$

The collection of above estimates finishes the proof of (3.42).

Step 3. Variation of the localized momentum. We now claim: for all $s\in[s^*,s^{\mathrm{in}}]$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}[\mathbf{J}(s,\epsilon(s))]-\sum_{k=1}^{2}2v_k\cdot\langle\nabla P_k,\frac{\bar\epsilon.F''(\mathbf{P})\cdot\epsilon}{2}\rangle\right|\lesssim s^{-2}\log^{-\frac34}(s)\|\epsilon(s)\|_{H^1}. \tag{3.45}$$

Indeed, we compute, for any k,

$$\frac{\mathrm{d}}{\mathrm{d}s}[J_k(s,\epsilon(s))]=\dot v_k\cdot\,\mathrm{Im}\int(\nabla\epsilon\,\bar\epsilon)\chi_k+v_k\cdot\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Im}\int(\nabla\epsilon\,\bar\epsilon)\chi_k.$$

By (3.22) and (3.23), we have

$$\left|\dot v_k\cdot\,\mathrm{Im}\int(\nabla\epsilon\,\bar\epsilon)\chi_k\right|\lesssim s^{-2}\|\epsilon\|_{H^1}^2.$$

Recall from (3.38) that

$$\begin{aligned}
|v_k|&\left|\frac{\mathrm{d}}{\mathrm{d}s}[\mathcal{M}_k(s,\epsilon(s))]-2\cdot\langle\nabla P_k\frac{\bar\epsilon\cdot F''(\mathbf{P})\cdot\epsilon}{2}\rangle-2\left(\dot z_k-2v_k+\frac{\dot\lambda}{\lambda}z_k\right)\langle\mathrm{i}\nabla Q,\nabla\eta_k\rangle,\right|\\
&\lesssim s^{-1}\log^{-1}(s)\|\epsilon(s)\|_{H^1}^2.
\end{aligned}$$

From (3.27),

$$|v_k|\left|\left(\dot z_k-2v_k+\frac{\dot\lambda}{\lambda}z_k\right)\langle\mathrm{i}\nabla Q,\nabla\eta_k\rangle\right|\lesssim s^{-2}\log^{-\frac34}(s)\|\epsilon\|_{H^1}$$

so we get (3.45).

step 4 Conclusion. Recall that, by (3.27), $|\dot z_k-2v_k|\lesssim s^{-1}\log^{-\frac34}(s)$ so

$$\left|(\dot z_k-2v_k)\cdot\langle\nabla P_k,\frac{\bar\epsilon\cdot F''(\mathbf{P})\cdot\epsilon}{2}\rangle\right|\lesssim s^{-1}\log^{-\frac34}(s)\|\epsilon\|_{H^1}^2,$$

and (3.41) now follows from (3.42), (3.45). This concludes the proof of Proposition 13. □

3.2.4. ***End of the bootstrap argument***. We close the bootstrap estimates (3.22).

Step 1. Closing the estimate in ϵ. By (3.41)in Proposition 13 and then (3.22)–(3.23), we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}[\mathbf{W}(s,\epsilon(s))]\right| \lesssim s^{-2}\|\epsilon\|_{H^1} \lesssim C^* s^{-3}.$$

Thus, by integration on $[s, s^{\mathrm{in}}]$ for any $s \in [s^*, s^{\mathrm{in}}]$, using $\epsilon(s^{\mathrm{in}}) = 0$ (see (3.18)), we obtain

$$|\mathbf{W}(s,\epsilon(s))| \lesssim C^* s^{-2}.$$

By (3.40), in Proposition 13 we get

$$\|\epsilon(s)\|_{H^1}^2 \leq C_0 C^* s^{-2}.$$

Therefore, for C^* large enough such that $C_0 C^* \leq \frac{(C^*)^2}{4}$, we have $\|\epsilon\|_{H^1} \leq \frac{C^*}{2} s^{-1}$, which strictly improves the estimate on $||\epsilon||_{H^1}$ in (3.22).

step 2 Closing the parameter z. Now, we need to finish the bootstrap argument for $z(s)$. Note that

$$\left|\dot{v} + c\frac{z}{|z|}|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\right| \lesssim s^{-2}\log^{-1}(s)$$

$$|\dot{z} - 2v| \lesssim s^{-1}\log^{-\frac{3}{4}}(s)$$

thus we deduce

$$\left|\dot{v}\cdot\frac{z}{|z|} + c|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\right| \lesssim s^{-2}\log^{-1}(s)$$

$$\left|\dot{z}\cdot\frac{z}{|z|} - 2v\cdot\frac{z}{|z|}\right| \lesssim s^{-1}\log^{-\frac{3}{4}}(s).$$

We get

$$\left|2\left(v\cdot\frac{z}{|z|}\right)\left(\dot{v}\cdot\frac{z}{|z|}\right) + c\,\dot{z}\cdot\frac{z}{|z|}|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\right| \lesssim s^{-3}\log^{-\frac{3}{4}}(s)$$

since $|v| \lesssim s^{-1}, |\dot{v}| \lesssim s^{-2}$. Therefore, by the explicit choice of initial data

$$v(s^{\mathrm{in}}) = \sqrt{c}(z^{\mathrm{in}})^{-\frac{d-1}{4}}\mathrm{e}^{-\frac{1}{2}z^{\mathrm{in}}}\vec{e}_1, \quad z(s^{\mathrm{in}}) = z^{\mathrm{in}}\vec{e}_1,$$

we integrate on $[s, s^{\mathrm{in}}]$ for any $s \in [s^*, s^{\mathrm{in}})$, if $d-1>0$

$$\left|\left(v\cdot\frac{z}{|z|}\right)^2 - c|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\right| \lesssim s^{-2}\log^{-\frac{3}{4}}(s) + \int_s^{s^{\mathrm{in}}} |\dot{z}||z|^{-\frac{d-1}{2}-1}\mathrm{e}^{-|z|} \lesssim s^{-2}\log^{-\frac{3}{4}}(s),$$

if $d-1=0$, $\left|2(v\cdot\frac{z}{|z|})(\dot{v}\cdot\frac{z}{|z|}) + c\,\dot{z}\cdot\frac{z}{|z|}\mathrm{e}^{-|z|}\right| \lesssim s^{-3}\log^{-\frac{3}{4}}(s)(s)$ implies also $\left|(v\cdot\frac{z}{|z|})^2 - c\mathrm{e}^{-|z|}\right| \lesssim s^{-2}\log^{-\frac{3}{4}}(s)$. In both cases, combining with (3.27), we get

$$\left|(v\cdot\frac{z}{|z|}) - \sqrt{c}|z|^{-\frac{d-1}{4}}\mathrm{e}^{-\frac{1}{2}|z|}\right| + \left|(\dot{z}\cdot\frac{z}{|z|}) - 2(v\cdot\frac{z}{|z|})\right| \lesssim s^{-1}\log^{-\frac{3}{4}}(s)$$

so $\left|(\dot{z}\cdot\frac{z}{|z|}) - 2\sqrt{c}|z|^{-\frac{d-1}{4}}\mathrm{e}^{-\frac{1}{2}|z|}\right| \lesssim s^{-1}\log^{-\frac{3}{4}}(s)$. Next, note that if $d-1>0$

$$\frac{\mathrm{d}}{\mathrm{d}s}(|z|^{\frac{d-1}{4}}\mathrm{e}^{\frac{1}{2}|z|}) = \frac{1}{2}\dot{z}\cdot\frac{z}{|z|}|z|^{\frac{d-1}{4}}\mathrm{e}^{\frac{1}{2}|z|} + \frac{d-1}{4}\dot{z}\cdot\frac{z}{|z|}|z|^{\frac{d-1}{4}-1}\mathrm{e}^{\frac{1}{2}|z|}$$

and if $d-1=0$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{\frac{1}{2}|z|}) = \frac{1}{2}\dot{z}\cdot\frac{z}{|z|}\mathrm{e}^{\frac{1}{2}|z|}$$

thus

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\left(|z|^{\frac{d-1}{4}}\mathrm{e}^{\frac{1}{2}|z|}\right) - c^{\frac{1}{2}}\right| \lesssim \log^{-\frac{3}{4}}(s) + \frac{d-1}{4}|\dot{z}||z|^{\frac{d-1}{4}-1}\mathrm{e}^{\frac{1}{2}|z|} \lesssim \log^{-\frac{3}{4}}(s) \tag{3.46}$$

here we use $|z| \lesssim \log^{-1}(s)$ and $|\dot{z}| \lesssim s^{-1}$. Next, we need to adjust the initial choice of z^{in} through a topological argument (see [4] for a similar argument). We define ζ and ξ the following two functions on $[s^*, s^{\text{in}}]$

$$\zeta(s) = c^{-\frac{1}{2}}|z|^{\frac{d-1}{4}}\mathrm{e}^{\frac{1}{2}|z|}, \quad \xi(s) = (\zeta(s)-s)^2 s^{-2}\log(s). \tag{3.47}$$

Then, (3.46) writes

$$|\dot{\zeta}(s) - 1| \lesssim \log^{-\frac{3}{4}}(s). \tag{3.48}$$

According to (3.22), our objective is to prove that there exists a suitable choice of

$$\zeta(s^{\text{in}}) = \zeta^{\text{in}} \in [s^{\text{in}} - s^{\text{in}}\log^{-\frac{1}{2}}(s^{\text{in}}), s^{\text{in}} + s^{\text{in}}\log^{-\frac{1}{2}}(s^{\text{in}})],$$

so that $s^* = s_0$. Assume for the sake of contradiction that for all $\zeta^\sharp \in [-1,1]$, the choice

$$\zeta^{\text{in}} = s^{\text{in}} + \zeta^\sharp s^{\text{in}}\log^{-\frac{1}{2}}(s^{\text{in}})$$

leads to $s^* = s^*(\zeta^\sharp) \in (s_0, s^{\text{in}})$. Since all estimates in (3.22) except the one on $z(s)$ have been strictly improved on $[s^*, s^{\text{in}}]$, it follows from $s^*(\zeta^\sharp) \in (s_0, s^{\text{in}}]$ and continuity that

$$|\zeta(s^*(\zeta^\sharp)) - s^*| = s^*\log^{-\frac{1}{2}} s^* \quad \text{i.e.} \quad \zeta(s^*(\zeta^\sharp)) = s^* \pm s^*\log^{-\frac{1}{2}} s^*.$$

We need a transversality condition to reach a contradiction. We compute:

$$\dot{\xi}(s) = 2(\zeta(s)-s)(\dot{\zeta}(s)-1)s^{-2}\log(s) - (\zeta(s)-s)^2(2s^{-3}\log(s) - s^{-3}). \tag{3.49}$$

At $s = s^*$, this gives

$$|\dot{\xi}(s^*) + 2(s^*)^{-1}| \lesssim (s^*)^{-1}\log^{-\frac{1}{4}}(s^*).$$

Thus, for s_0 large enough,

$$\dot{\xi}(s^*) < -(s^*)^{-1}. \tag{3.50}$$

A consequence of the transversality property (3.50) is the continuity of the function $\zeta^\sharp \in [-1,1] \mapsto s^*(\zeta^\sharp)$. Indeed, let $\epsilon > 0$, then there exists $\delta > 0$ such that $\xi(s^*(\zeta^\sharp) - \epsilon) > 1 + \delta$ and $\xi(s^*(\zeta^\sharp) + \epsilon) < 1 - \delta$. Moreover, by definition of $s^*(\zeta^\sharp)$ (choosing δ small enough) for all $s \in [s^*(\zeta^\sharp) + \epsilon, s^{\text{in}}]$, we have $\xi(s) < 1 - \delta$. But from the continuity of the flow, there exists $\iota > 0$ such that for all $|\tilde{\zeta}^\sharp - \zeta^\sharp| < \iota$

$$\forall s \in [s^*(\zeta^\sharp) - \epsilon, s^{\text{in}}], \quad |\tilde{\xi}(s) - \xi(s)| \leq \delta/2$$

so we obtain that $s^*(\zeta^\sharp) - \epsilon \leq s^*(\tilde{\zeta}^\sharp) \leq s^*(\xi^\sharp) + \epsilon$ and the continuity of $s^*(\zeta^\sharp)$, as expected. Thus we deduce the continuity of the function Φ defined by

$$\mathrm{i} : \zeta^\sharp \in [-1,1] \mapsto (\zeta(s^*) - s^*)(s^*)^{-1}\log^{\frac{1}{2}}(s^*) \in \{-1, 1\}.$$

Moreover, for $\zeta^\sharp = -1$ and $\zeta^\sharp = 1$, $\xi(s^{\text{in}}) = 1$ in these two cases; from (3.49) we have that $\dot{\xi}(s^{\text{in}}) < 0$, thus $s^* = s^{\text{in}}$. Therefore, $\Phi(-1) = -1$ and $\Phi(1) = 1$, but this is a contradiction with continuity.

In conclusion, there exists at least a choice of

$$\zeta(s^{\text{in}}) = \zeta^{\text{in}} \in (s^{\text{in}} - s^{\text{in}}\log^{-\frac{1}{2}}(s^{\text{in}}), s^{\text{in}} + s^{\text{in}}\log^{-\frac{1}{2}}(s^{\text{in}}))$$

such that $s^* = s_0$. This concludes our bootstrap argument.

Step 3. Estimate on the parameter λ. From (3.25), we obtain

$$\left|\frac{\dot{\lambda}}{\lambda}\right| \lesssim s^{-2}.$$

By integration on $[s, s^{\text{in}}]$, for any $s \in [s_0, s^{\text{in}}]$, using the value $\lambda(s^{\text{in}}) = \lambda^{\text{in}} = 1$ (see (3.19)), we have

$$|\log(\lambda(s))| \lesssim s^{-1},$$

and thus

$$|\lambda(s) - 1| \lesssim s^{-1}$$

or in other words

$$\left|\lambda^{-1}(s) - 1\right| \lesssim s^{-1}. \tag{3.51}$$

□

4. Compactness arguments

4.1. **Construction of a sequence of backwards solutions.**

Lemma 14. *There exist $t_0 > 1$ and a sequence of solutions $u_n \in \mathcal{C}([t_0, T_n], H^1)$ of* (NLS)*, where*

$$T_n \to +\infty \quad \text{as } n \to +\infty, \tag{4.1}$$

satisfying the following estimates, for all $t \in [t_0, T_n]$,

$$\begin{gathered} \left| |z_n(t)| - 2\log t \right| \lesssim \log(\log t), \quad \left|\lambda_n^{-1}(t) - 1\right| \lesssim t^{-1}, \\ |v_n(t)| \lesssim t^{-1}, \quad \|\epsilon_n(t)\|_{H^1} \lesssim t^{-1}, \quad \left| |z_n(t)|^{\frac{d-1}{2}} \mathrm{e}^{|z_n(t)|} - ct^2 \right| \lesssim t^2 \log^{-\frac{1}{2}}(t), \end{gathered} \tag{4.2}$$

where $(\lambda_n, z_n, \gamma_n, v_n)$ are the parameters of the decomposition of u_n, i.e.

$$u_n(t,x) = \frac{\mathrm{e}^{\mathrm{i}\gamma_n(t)}}{\lambda_n^{\frac{2}{p-1}}(t)} \left(\sum_{k=1}^{2} \left[\mathrm{e}^{\mathrm{i}\Gamma_{k,n}} Q\right] \left(\frac{x}{\lambda_n(t)} + \frac{(-1)^k}{2} z_n(t) \right) + \epsilon_n\left(t, \frac{x}{\lambda_n(t)}\right) \right), \tag{4.3}$$

with $\Gamma_{k,n}(t,x) = \frac{(-1)^{k+1}}{2} v_n(t) \cdot \frac{x}{\lambda_n(t)}$.

Proof of Lemma 14. Applying Proposition 10 with $s^{\text{in}} = n$ for any large n, there exists a solution $u_n(t)$ of (NLS) defined on the time interval $[0, T_n]$ where

$$T_n = \int_{s_0}^{n} \lambda_n^2(s) \mathrm{d}s.$$

and whose decomposition satisfies the uniform estimates (3.20). First, we see that $T_n \to +\infty$ as $n \to +\infty$, which follows directly from the estimate on $\lambda_n(s)$. From the definition of the rescaled time s (see (3.2)), for any $s \in [s_0, n]$, we have

$$t(s) = \int_{s_0}^{s} \lambda_n^2(s')\, \mathrm{d}s' \quad \text{where} \quad |\lambda_n^2(s) - 1| \lesssim s^{-1}.$$

Fix $t_0 = \bar{s}_0$ with $\bar{s}_0 > s_0$ large enough and independent of n such that, for all s with $n \geq s > \bar{s}_0$,

$$\frac{1}{2} s \leq \int_{s_0}^{s} \lambda_n^2(s')\, \mathrm{d}s' = s\left(1 + O(s^{-1})\right) \leq \frac{3}{2} s$$

then, for all $t \in [t_0, T_n]$,

$$t(s) = s\left(1 + O(s^{-1})\right) \geq \frac{1}{2}s$$

and

$$s = t\left(1 + O(t^{-1})\right) \cdot$$

Thus, we get from (3.20)

$$\begin{gathered}
\left| |z_n(s)| - 2\log(s) \right| \lesssim \log(\log(s)) \Leftrightarrow \left| |z_n(s(t))| - 2\log(t) \right| \lesssim \log(\log(t)) \\
\left|\lambda_n^{-1}(s) - 1\right| \lesssim s^{-1} \Leftrightarrow \left|\lambda_n^{-1}(s(t)) - 1\right| \lesssim t^{-1} \\
\|\epsilon_n(s)\|_{H^1} \lesssim s^{-1} \Leftrightarrow \|\epsilon_n(s(t))\|_{H^1} \lesssim t^{-1} \\
|v_n(s)| \lesssim s^{-1} \Leftrightarrow |v_n(s(t))| \lesssim t^{-1}.
\end{gathered}$$

□

4.2. **Compactness argument.** Next, we claim a strong compactness result in $L^2(\mathbb{R}^d)$.

Lemma 15. *There exist $u_0 \in H^1(\mathbb{R}^d)$ and a sub-sequence, still denoted u_n, such that*

$$u_n(t_0) \rightharpoonup u_0 \ \textit{weakly in } H^1(\mathbb{R}^d)$$

$$u_n(t_0) \to u_0 \ \textit{in } H^\sigma(\mathbb{R}^d), \ \textit{for } 0 \leq \sigma < 1$$

as $n \to \infty$.

Proof of Lemma 15. By interpolation, it is enough to prove that the sub-sequence $u_n(t_0) \xrightarrow{L^2} u_0$ as $n \to \infty$. First, we claim the following: $\forall \delta_1 > 0, \delta_1 \ll 1$, $\exists n_0 \gg 1$, $\exists K_1 = K_1(\delta_1) > 0$ such that $\forall n \geq n_0$

$$\int_{|x|>K_1} |u_n(t_0, x)|^2 \mathrm{d}x < \delta_1. \tag{4.4}$$

Indeed, denote $x_n(t) = z_n(t)\lambda_n(t)$ and

$$\tilde{R}_n(t,x) = \mathrm{e}^{\mathrm{i}\gamma_n(t)} \sum_{k=1}^{2} \left[\mathrm{e}^{\mathrm{i}\Gamma_{k,n}} Q_{\lambda_n^{-1}(t)}\right] \left(x + \frac{(-1)^k}{2} x_n(t)\right)$$

$$R_n(t,x) = \mathrm{e}^{\mathrm{i}\gamma_n(t)} \sum_{k=1}^{2} Q\left(x + \frac{(-1)^k}{2} x_n(t)\right)$$

then we have

$$\begin{aligned}
||u_n(t) - R_n(t)||_{H^1} &\leq ||\epsilon_n(t)||_{H^1} + 2||\tilde{R}_n(t) - R_n(t)||_{H^1} \\
&\lesssim ||\epsilon_n(t)||_{H^1} + \left|\lambda_n^{-1}(t) - 1\right| + |v_n(t)| \lesssim t^{-1}. \quad (4.5)
\end{aligned}$$

We get a direct consequence of the above estimate

$$||u_n(t)||_{H^1} < C \tag{4.6}$$

for all $t \in [t_0, T_n]$ since $||R_n(t)||_{H^1} \leq 2||Q||_{H^1}$. Furthermore, for fixed δ_1, there exists $t_1 > t_0$ such that

$$||u_n(t_1) - R_n(t_1)||_{H^1} \lesssim (t_1)^{-1} < \sqrt{\delta_1}$$

for n large enough that $T_n > t_1$; in others words, we have

$$\int |u_n(t_1, x) - R_n(t_1, x)|^2 \mathrm{d}x < \delta_1.$$

Besides, $|x_n(t_1) - 2\log(t_1)| \lesssim \log(\log t_1)$ then for $K_2 \gg 1$ large enough, we have

$$\int_{|x|>K_2} |R_n(t_1,x)|^2 \mathrm{d}x < \delta_1.$$

Consider now a $\mathcal{C}^1$ cut-off function $g : \mathbb{R} \to [0,1]$ such that : $g \equiv 0$ on $(-\infty, 1]$, $0 < g' < 2$ on $(1,2)$ and $g \equiv 1$ on $[2, +\infty)$. Since $||u_n(t)||_{H^1} < C$ bounded in H^1 independently of n and $t \in [t_0, T_n]$, we can choose $\gamma_1 > 0$ independent of n such that

$$\gamma_1 \geq \frac{2}{\delta_1}(t_1 - t_0)\, C^2.$$

We have by direct calculations, for $t \in [t_0, T_n]$

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\int |u_n(t,x)|^2 g\left(\frac{|x|-K_2}{\gamma_1}\right)\mathrm{d}x\right| &= \left|\frac{1}{\gamma_1}\,\mathrm{Im}\int u\left(\nabla\bar{u}\cdot\frac{x}{|x|}\right) g'\left(\frac{|x|-K_2}{\gamma_1}\right)\right| \\ &\leq \frac{2}{\gamma_1}\sup_{T_n\geq t\geq t_0} ||u_n(t)||_{H^1}^2 \leq \frac{\delta_1}{t_1 - t_0}.\end{aligned}$$

By integration from t_0 to t_1

$$\begin{aligned}&\int |u_n(t_0,x)|^2 g\left(\frac{|x|-K_2}{\gamma_1}\right)\mathrm{d}x - \int |u_n(t_1,x)|^2 g\left(\frac{|x|-K_2}{\gamma_1}\right)\mathrm{d}x \\ &\qquad\leq \int_{t_0}^{t_1}\left|\frac{\mathrm{d}}{\mathrm{d}t}\int |u_n(t,x)|^2 g\left(\frac{|x|-K_2}{\gamma_1}\right)\mathrm{d}x\right| \leq \delta_1.\end{aligned}$$

From the properties of g we conclude:

$$\begin{aligned}&\int_{|x|>2\gamma_1+K_2} |u_n(t_0,x)|^2\mathrm{d}x \leq \int |u_n(t_0,x)|^2 g\left(\frac{|x|-K_2}{\gamma_1}\right)\mathrm{d}x \\ &\leq \int |u_n(t_1,x)|^2 g\left(\frac{|x|-K_2}{\gamma_1}\right)\mathrm{d}x + \delta_1 \leq \int_{|x|>K_2} |u_n(t_1,x)|^2\mathrm{d}x + \delta_1 \leq 5\delta_1.\end{aligned}$$

Thus (4.4) is proved. As $||u_n(t_0)||_{H^1} < C$, there exists a subsequence of (u_n) (still denoted by (u_n)) and $u_0 \in H^1$ such that

$$\begin{aligned}&u_n(t_0) \rightharpoonup u_0 \quad \text{weakly in } H^1(\mathbb{R}^2), \\ &u_n(t_0) \to u_0 \quad\ \ \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^d), \text{ as } n \to +\infty\end{aligned}$$

and by (4.4), we conclude that $u_n(t_0) \xrightarrow{L^2} u_0$ as required. □

Let us finish the proof of the Main Theorem in subcritical cases with $p > 2$. We consider u the solution to (NLS) corresponding to $u(t_0) = u_0$. By continuous dependence of the solution upon the initial data (see [2] and [3]), for all $0 \leq \sigma < 1$, for all $t \in [t_0, +\infty)$,

$$u_n(t) \to u(t) \quad \text{in } H^\sigma(\mathbb{R}^d).$$

Moreover, the decomposition $(\vec{q}, \epsilon)$ of u satisfies, for all $t \geq t_0$,

$$\vec{q}_n(t) \to \vec{q}(t), \quad \epsilon_n(t) \to \epsilon(t) \text{ in } H^\sigma, \quad \epsilon_n(t) \rightharpoonup \epsilon(t) \text{ in } H^1 \tag{4.7}$$

(see, e.g., [26], Claim p. 598). In particular, for all $t \in [t_0, +\infty)$, $u(t)$ decomposes as

$$u(t,x) = \frac{\mathrm{e}^{\mathrm{i}\gamma(t)}}{\lambda^{\frac{2}{p-1}}(t)}\left(\sum_{k=1}^{2}\left[\mathrm{e}^{\mathrm{i}\Gamma_k}Q\right]\left(\frac{x + \frac{(-1)^k}{2}\lambda(t)z(t)}{\lambda(t)}\right) + \epsilon\left(t, \frac{x}{\lambda(t)}\right)\right), \tag{4.8}$$

where $\Gamma_k(t,y)=\frac{(-1)^{k+1}}{2}v(t)\cdot y$, and it follows from the uniform estimates (4.2) that

$$\begin{gathered} \left|\,|z(t)|-2\log t\right|\lesssim\log(\log t),\quad \left|\lambda^{-1}(t)-1\right|\lesssim t^{-1},\\ |v(t)|\lesssim t^{-1},\quad \|\epsilon(t)\|_{H^1}\lesssim t^{-1},\quad \left||z(t)|^{\frac{d-1}{2}}\mathrm{e}^{|z(t)|}-ct^2\right|\lesssim t^2\log^{-\frac12}(t). \end{gathered}\tag{4.9}$$

We obtain $|x_1(t)-x_2(t)|=\lambda(t)|z(t)|\to 2(1+o(1))\log t$; more precisely

$$\left|\,|x_1(t)-x_2(t)|-2\log(t)\right|\lesssim\log(\log(t))$$

and the following estimate

$$\left\|u(t)-\mathrm{e}^{\mathrm{i}\gamma(t)}\sum_{k=1}^{2}Q\left(x-x_k(t)\right)\right\|_{H^1}\lesssim||\epsilon(t)||_{H^1}+|\lambda^{-1}(t)-1|+|v(t)|\lesssim t^{-1}.\tag{4.10}$$

5. Sub-critical cases with $1<p\le 2$

In this section, we show the difficulties occurring and sketch the proof of the Main Theorem in the case $1<p\le 2$. In this case, let

$$2^+=\min(2^*,\frac{p+3}{2}).$$

Note that $p-2^+>-1$. From (2.15), we deduce the following Taylor expansions:

$$F(\mathbf{P}+\epsilon)=F(\mathbf{P})+F'(\mathbf{P})\cdot\epsilon+O(|\epsilon|^p)\tag{5.1}$$

$$F(\mathbf{P}+\epsilon)=F(\mathbf{P})+F'(\mathbf{P})\cdot\epsilon+O\left(\left|\frac{\epsilon}{\mathbf{P}}\right|^2|\mathbf{P}|^p\right)\tag{5.2}$$

(since $|\epsilon|>\frac{|\mathbf{P}|}{2}$ then $|\epsilon|^p\lesssim|\frac{\epsilon}{\mathbf{P}}|^2|\mathbf{P}|^p$ and $|\epsilon|\le\frac{|\mathbf{P}|}{2}$ then $|\frac{\epsilon}{\mathbf{P}}|^2|\mathbf{P}|^p\lesssim|\epsilon|^p$) and

$$F(\mathbf{P}+\epsilon)=F(\mathbf{P})+F'(\mathbf{P})\cdot\epsilon+\frac{\bar{\epsilon}\cdot F''(\mathbf{P})\cdot\epsilon}{2}+O\left(\left|\frac{\epsilon}{\mathbf{P}}\right|^{2^+}|\mathbf{P}|^p\right).\tag{5.3}$$

In the following remark, we identify new problems compared with the case $p>2$.

Remark 16. Let us try to control the nonlinear interaction term

$$G(y;(z(s),v(s)))=|\mathbf{P}|^{p-1}\mathbf{P}-|P_1|^{p-1}P_1-|P_2|^{p-1}P_2.$$

Since $|P_1|>|P_2|$ for $y\cdot\frac{z}{|z|}>0$ and $|P_2|>|P_1|$ for $y\cdot\frac{z}{|z|}<0$, one has, by (2.15),

$$\begin{aligned} |G(y;(z(s),v(s)))|&=\left||P_1+P_2|^{p-1}(P_1+P_2)-|P_1|^{p-1}P_1-|P_2|^{p-1}P_2\right|\\ &\lesssim|P_1|^{p-1}|P_2|\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}>0}+|P_2|^{p-1}|P_1|\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}<0}. \end{aligned}\tag{5.4}$$

Using the asymptotic behavior of Q, on the half space $\{y\cdot\frac{z}{|z|}>0\}$,

$$\begin{aligned} |P_1|^{p-1}|P_2|\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}>0}&\lesssim|P_1P_2|^{p-1}|P_2|^{2-p}\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}>0}\lesssim|z|^{-\frac{(p-1)(d-1)}{2}}\mathrm{e}^{-(p-1)|z|}|P_2|^{2-p}\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}>0}\\ &\lesssim|z|^{-\frac{(p-1)(d-1)}{2}}\mathrm{e}^{-(p-1)|z|}\left|\frac{z}{2}\right|^{-\frac{(2-p)(d-1)}{2}}\mathrm{e}^{-\frac{2-p}{2}|z|}\lesssim|z|^{-\frac{d-1}{2}}\mathrm{e}^{-\frac{p}{2}|z|}. \end{aligned}\tag{5.5}$$

By symmetry, we have the same estimate on the other half space $\{y\cdot\frac{z}{|z|}<0\}$ and thus

$$\|G\|_{L^\infty}\lesssim|z|^{-\frac{d-1}{2}}\mathrm{e}^{-\frac{p}{2}|z|}\sim s^{-p}\tag{5.6}$$

(to be compared with (2.9)). Now for the projection of interaction, we recall that its core part (as identified in the proof of Lemma 7 and in step 4 of Proposition 17) is given by

$$H(z) = p\int_{y\cdot\frac{z}{|z|}>-\frac{|z|}{2}} Q^{p-1}(y)\,\nabla Q(y)\,Q(y+z)\,\mathrm{d}y + p\int_{y\cdot\frac{z}{|z|}<-\frac{|z|}{2}} Q^{p-1}(y+z)\,\nabla Q(y)\,Q(y)\,\mathrm{d}y$$

and the following estimate of $H(z)$ is still valid for $1<p\leq 2$ (see Lemma 7)

$$\left|H(z) - c_Q I_Q \frac{z}{|z|}|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\right| \lesssim |z|^{-1-\frac{d-1}{2}}\mathrm{e}^{-|z|}. \tag{5.7}$$

In summary, the projection $\langle G, \mathrm{e}^{\mathrm{i}\Gamma_1}\nabla Q(y-z_1(s))\rangle$ and thus $\dot{v}$ are still of order s^{-2}; however, the interaction G is of order $s^{-p}\gg s^{-2}$ in L^∞ norm. Therefore, there still exist some terms in the interaction that perturb our regime and prevent us from closing the bootstrap arguments (for example, (3.44)).

In view of the above remark, we look for a refined approximate solution $\mathbf{P}$ of the form

$$\begin{aligned}\mathbf{P}(s,y) = \mathbf{P}(y;(z(s),v(s))) &= \sum_{k=1}^{2}\mathrm{e}^{\mathrm{i}v_k(s)(y-z_k(s))}Q(y-z_k(s)) + W(y;(z(s),v(s)))\\ &= \sum_{k=1}^{2}P_k(s,y) + W(y;(z(s),v(s))),\end{aligned} \tag{5.8}$$

where $W(y;(z(s),v(s)))$ is to be determined.

Proposition 17 (Expansion of the refined approximate solution)**.** *There exists a series of $(J+1)$ functions $R_j(y;(z(s),v(s)))$ that are invariant by τ and υ such that by setting*

$$W(y;(z(s),v(s))) = \sum_{j=0}^{J} R_j(y;(z(s),v(s))),$$

the error $\mathcal{E}_{\mathbf{P}}$ defined as in (2.7) *admits the decomposition*

$$\mathcal{E}_{\mathbf{P}} = [\mathrm{e}^{\mathrm{i}\Gamma_1}\vec{m}_1\cdot\vec{\mathrm{M}}Q](y-z_1(s)) + [\mathrm{e}^{\mathrm{i}\Gamma_2}\vec{m}_2\cdot\vec{\mathrm{M}}Q](y-z_2(s)) + G_0, \tag{5.9}$$

where under the bootstrap assumptions (3.22) *and the pointwise control of the modulation equations* (3.25)–(3.27)

$$|z|\lesssim\log(s),\quad |\dot{z}|\lesssim s^{-1},\quad |v|\lesssim s^{-1},\quad |\dot{v}|\lesssim s^{-2},\quad \left|\frac{\dot{\lambda}}{\lambda}\right|\lesssim (C^*)^2 s^{-2},\quad |\dot{\gamma}-1|\lesssim (C^*)^2 s^{-2},$$

the corrected interaction term G_0 satisfies

$$\|G_0\|_{L^2}\lesssim s^{-2},\quad \|\nabla G_0\|_{L^2}\lesssim s^{-2}. \tag{5.10}$$

Moreover, G_0 is symmetric and

$$\left|\langle G_0, \mathrm{e}^{\mathrm{i}\Gamma_1(y-z_1(s))}\nabla Q(y-z_1(s))\rangle - C_p\frac{z}{|z|}|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\right| \lesssim s^{-2}\log^{-1}(s) \tag{5.11}$$

with $C_p>0$.

Remark 18. In fact, before the pointwise control of the modulation equations in Lemma 12, we bound $\|G_0\|_{L^2}, \|\nabla G_0\|_{L^2}$ by z, v and $s^{-p}|\vec{m}_1|$, then once we have the control on $\vec{m}_1$, we will obtain (5.10).

Proof of Proposition 17. **Step 1.** Properties of the Helmholtz operators. We recall well-known properties of $(-\Delta+1)u_s(y) = f_s(y)$ in $\mathbb{R}^d$. The operator $(-\Delta+1)^{-1}$ is continuous from L^2 to H^1, in particular

$$\|u\|_{H^1} \le \|f\|_{L^2}.$$

It is self-adjoint

$$\langle u, (-\Delta+1)g\rangle = \langle(-\Delta+1)u, g\rangle = \langle f, g\rangle, \tag{5.12}$$

invariant by τ, υ and $(-\Delta+1)\dot{u}_s(y) = \dot{f}_s(y)$ ($\dot{f}$ denotes the derivative with respect to time s). Moreover, by theory of elliptic equation (see, e.g., [1]), we have an explicit kernel representation E_d for $(-\Delta+1)^{-1}$ as follows

$$E_d(x) = -(2\pi)^{-\frac{d}{2}}\left(\frac{1}{|x|}\right)^{\frac{d}{2}-1}\mathcal{K}_{\frac{d}{2}-1}(|x|)$$

$$u(x) = \int_{\mathbb{R}^d} E_d(x-y)f(y)\mathrm{d}y \tag{5.13}$$

where $\mathcal{K}_\alpha$ is a modified Bessel function of second kind that is decreasing exponentially when $|x| \to +\infty$. This is a convolution of type $L^1 \star L^\infty$, so we deduce that

$$\|u\|_{L^\infty} \lesssim \|f\|_{L^\infty}. \tag{5.14}$$

Next, we claim the exponential decay property: assume that a regular function f is exponentially decreasing in the direction e_j, $\mathrm{e}^{\delta|y_j|}|f(y)| \le C$ with $0<\delta<1$, then so is the solution u of $(-\Delta+1)^{-1}$.
Indeed, we consider

$$\begin{aligned}\mathrm{e}^{\delta|x_j|}|u(x)| &= \mathrm{e}^{\delta|x_j|}\left|\int_{\mathbb{R}^d} E_d(x-y)f(y)\mathrm{d}y\right| \\ &\lesssim C\left|\int_{\mathbb{R}^d}\left(\frac{1}{|x-y|}\right)^{\frac{d}{2}-1}\mathrm{e}^{-|x-y|}\mathrm{e}^{\delta(|x_j|-|y_j|)}\mathrm{d}y\right| \lesssim C\left\|\left(\frac{1}{|x|}\right)^{\frac{d}{2}-1}\mathrm{e}^{-(1-\delta)|x|}\right\|_{L^1} \lesssim C.\end{aligned}$$

Step 2. Iteration of R_j. We introduce a suitable smooth cut-off function that localizes the points whose distances to center of two solitons are smaller than $|z|$. Denote $\psi_0 : \mathbb{R} \to [0,1]$ such that

$$0 \le \psi_0' \le C, \quad \psi_0 \equiv 0 \text{ on } (-\infty,-1], \quad \psi_0 \equiv 1 \text{ on } [0,+\infty)$$

and

$$\psi(y; z(s)) = \psi_0\left(|z(s)| - \left|y + \frac{z(s)}{2}\right|\right)\psi_0\left(|z(s)| - \left|y - \frac{z(s)}{2}\right|\right).$$

Recall the definition of G

$$G(y;(z(s),v(s))) = |P_1+P_2|^{p-1}(P_1+P_2) - |P_1|^{p-1}P_1 - |P_2|^{p-1}P_2$$

and denote $\mathfrak{pr}_i$ the projection on the direction ∇Q around each soliton

$$\mathfrak{pr}_i(f) = \frac{\langle f(\cdot), \nabla Q(\cdot + \frac{(-1)^i}{2}z(s))\rangle}{\|\nabla Q(\cdot + \frac{(-1)^i}{2}z(s))\|_{L^2}^2}\nabla Q(\cdot + \frac{(-1)^i}{2}z(s)).$$

Setting

$$A_0(y;(z(s),v(s))) = G(y;(z(s),v(s)))\psi(y;z(s)),$$
$$\tilde{A}_0 = A_0 - \mathfrak{pr}_1(A_0) - \mathfrak{pr}_2(A_0),$$

$$A_1 = |P_1 + P_2 + R_0|^{p-1}(P_1 + P_2 + R_0) - |P_1 + P_2|^{p-1}(P_1 + P_2),$$
$$\tilde{A}_1 = A_1 - \mathfrak{pr}_1(A_1) - \mathfrak{pr}_2(A_1)$$

and for $j \geq 2$

$$A_j = |P_1 + P_2 + \sum_{k=0}^{j-1} R_k|^{p-1}(P_1 + P_2 + \sum_{k=0}^{j-1} R_k) - |P_1 + P_2 + \sum_{k=0}^{j-2} R_k|^{p-1}(P_1 + P_2 + \sum_{k=0}^{j-2} R_k),$$
$$\tilde{A}_j = A_j - \mathfrak{pr}_1(A_j) - \mathfrak{pr}_2(A_j)\cdot$$

Observe that

$$\sum_{j=1}^{J} A_j = |P_1 + P_2 + \sum_{k=0}^{j-1} R_k|^{p-1}(P_1 + P_2 + \sum_{k=0}^{j-1} R_k) - |P_1 + P_2|^{p-1}(P_1 + P_2). \tag{5.15}$$

Then let

$$R_j(y;(z(s),v(s))) = (-\Delta + 1)^{-1}\tilde{A}_j\cdot$$

We will show by induction on j the following properties.

— R_j is almost orthogonal to $\nabla(Q^p)(\cdot \pm \frac{1}{2}z)$, i.e.

$$\langle R_j(\cdot), \nabla(Q^p)(\cdot \pm \frac{1}{2}z)\rangle \lesssim s^{-3}. \tag{5.16}$$

— The L^∞, H^1 norm of R_j satisfies

$$\|R_{j+1}\|_{L^\infty} \lesssim s^{-(p-1)}\|R_j\|_{L^\infty} \lesssim s^{-p},$$
$$\|R_{j+1}\|_{H^1} \lesssim s^{-(p-1-\kappa)}\|R_j\|_{H^1} \lesssim s^{-p}\log^{dp}(s)$$

with $0 < \kappa \ll 1$ to be determined (see (5.33), (5.34)).

— After a finite number $(J+1)$ of steps, the function R_J satisfies the two following estimates: there is $\epsilon > 0$

$$|Q^{p-1}(y)R_J(y + \frac{z}{2})| + |Q^{p-1}(y)R_J(y + \frac{z}{2})| \lesssim \mathrm{e}^{-\epsilon|y|}s^{-2} \tag{5.17}$$

$$\|R_J\|_{H^1}^p + s^{p(p-1)}\|R_J\|_{H^1} \ll s^{-2} \tag{5.18}$$

independently of z, v ((5.18) means thats there exists $\delta > 0$ such that $\|R_J\|_{H^1}^p + s^{-p(p-1)}\|R_J\|_{H^1} \lesssim s^{-2-\delta}$).

Note that a direct consequence of the above estimates is

$$\begin{aligned}
&\|A_{J+1}\|_{L^2} \\
&= \left\| |P_1 + P_2 + \sum_{j=0}^{J} R_j|^{p-1}(P_1 + P_2 + \sum_{j=0}^{J} R_j) - |P_1 + P_2 + \sum_{j=0}^{J-1} R_j|^{p-1}(P_1 + P_2 + \sum_{j=0}^{J-1} R_j) \right\|_{L^2} \\
&\lesssim \left\| |P_1 + P_2 + \sum_{j=0}^{J-1} R_j|^{p-1}|R_J| + |R_J|^p \right\|_{L^2} \\
&\lesssim \|Q^{p-1}(\cdot)R_J(\cdot + \frac{z}{2})\|_{L^2} + \|R_J\|_{L^2}^p + s^{p(p-1)}\|R_J\|_{L^2} \lesssim s^{-2}
\end{aligned} \tag{5.19}$$

since $\|R_j\|_{L^\infty} \lesssim \|R_0\|_{L^\infty} \lesssim s^{-p}, \forall j = \overline{1, J}$.

Let us begin with R_0. We have that

$$\begin{aligned}|G(y;(z(s),v(s)))| &= \big||P_1+P_2|^{p-1}(P_1+P_2)-|P_1|^{p-1}P_1-|P_2|^{p-1}P_2\big|\\ &\lesssim |P_1|^{p-1}|P_2|\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}>0}+|P_2|^{p-1}|P_1|\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}<0}.\end{aligned}$$

Consider

$$\begin{aligned}|P_1|^{p-1}|P_2|\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}>0} &\lesssim \mathrm{e}^{-(p-1)(z_1-y\cdot\frac{z}{|z|})}\left|\frac{z}{2}\right|^{-\frac{d-1}{2}}\mathrm{e}^{-(y\cdot\frac{z}{|z|}-z_2)}\\ &\lesssim |z|^{-\frac{d-1}{2}}\mathrm{e}^{-\frac{p}{2}|z|}\mathrm{e}^{-(2-p)y\cdot\frac{z}{|z|}}\lesssim s^{-p}\mathrm{e}^{-(2-p)y\cdot\frac{z}{|z|}}|z|^{-\frac{(2-p)(d-1)}{2}},\end{aligned}\tag{5.20}$$

by symmetry, we also have the same estimate on $\{y\cdot\frac{z}{|z|}<0\}$. Thus, from the definition of ψ, we get

$$\|\mathrm{e}^{(2-p)|y\cdot\frac{z}{|z|}|}A_0(y;(z(s),v(s)))\|_{L^\infty}\lesssim s^{-p}|z|^{-\frac{(2-p)(d-1)}{2}}\lesssim s^{-p}\tag{5.21}$$

and

$$\|A_0(y;(z(s),v(s)))\|_{L^2}\lesssim s^{-p}\log^d(s).\tag{5.22}$$

The estimate (5.21) yields

$$\begin{aligned}|A_0(y+\frac{z}{2})| &\lesssim \mathrm{e}^{-(2-p)|y\cdot\frac{z}{|z|}+\frac{z}{2}|}s^{-p}|z|^{-\frac{(2-p)(d-1)}{2}}\\ &\lesssim \mathrm{e}^{(2-p)|y\cdot\frac{z}{|z|}|}\mathrm{e}^{-(2-p)\frac{|z|}{2}}|z|^{-\frac{(2-p)(d-1)}{2}}s^{-p}\lesssim \mathrm{e}^{(2-p)|y\cdot\frac{z}{|z|}|}s^{-2}\end{aligned}$$

so it gives a control on projections of A_0

$$\left|\int_{\mathbb{R}^d}A_0(y+\frac{z}{2})\nabla Q(y)\,\mathrm{d}y\right|\lesssim s^{-2}.\tag{5.23}$$

Therefore, from definition of $\tilde{A}_0$

$$\|\mathrm{e}^{(2-p)|y\cdot\frac{z}{|z|}|}\tilde{A}_0\|_{L^\infty}\lesssim s^{-p}|z|^{-\frac{(2-p)(d-1)}{2}},\quad \|\tilde{A}_0\|_{L^2}\lesssim s^{-p}\log^d(s)\cdot$$

From step 1, we can transfer these properties to $R_0(y;(z(s),v(s)))$

$$\|\mathrm{e}^{(2-p)|y\cdot\frac{z}{|z|}|}R_0(y;(z(s),v(s)))\|_{L^\infty}\lesssim s^{-p}|z|^{-\frac{(2-p)(d-1)}{2}},\tag{5.24}$$

$$\|R_0(y;(z(s),v(s)))\|_{H^1}\lesssim s^{-p}\log^d(s).\tag{5.25}$$

To show the almost orthogonality condition, we note that $(-\Delta+1)\nabla Q=\nabla(Q^p)$, so from the self-adjoint property (5.12) of $(-\Delta+1)$, we have

$$\begin{aligned}\left|\langle R_0,\nabla(Q^p)(\cdot+\frac{z}{2})\rangle\right| &= \left|\langle A_0-\mathfrak{pr}_1(A_0)-\mathfrak{pr}_2(A_0),\nabla Q(\cdot+\frac{z}{2})\rangle\right|\\ &= \left|\langle\mathfrak{pr}_1(A_0),\nabla Q(\cdot+\frac{z}{2})\rangle\right|\lesssim s^{-2}\langle\nabla Q(\cdot-\frac{z}{2}),\nabla Q(\cdot+\frac{z}{2})\rangle\lesssim s^{-3}.\end{aligned}$$

If $\frac{3}{2}<p\le 2$, we see that R_0 satisfies already the conditions (5.17), (5.18) as

$$\begin{aligned}\|R_0\|_{H^1}^p &\lesssim s^{-p^2}\log^{dp}(s)\le s^{-\frac{9}{4}}\log^{dp}(s)\ll s^{-2}\\ s^{p(p-1)}\|R_0\|_{H^1} &\lesssim s^{-\frac{3}{4}}s^{-\frac{3}{2}}\ll s^{-2}\end{aligned}$$

and $|R_0(y+\frac{z}{2})|\lesssim \mathrm{e}^{-(2-p)|y\cdot\frac{z}{|z|}+\frac{z}{2}|}s^{-p}|z|^{-\frac{(2-p)(d-1)}{2}}\lesssim \mathrm{e}^{(2-p)|y\cdot\frac{z}{|z|}|}s^{-2}$ so for $\epsilon=2p-3>0$

$$|Q^{p-1}(y)R_0(y+\frac{z}{2})|\lesssim \mathrm{e}^{(2-p)|y\cdot\frac{z}{|z|}|}Q^{(2-p)}(y)s^{-2}Q^{(2p-3)}(y)\lesssim \mathrm{e}^{-\epsilon|y|}s^{-2}.$$

Thus $J = 0$ and $W = R_0(y; (z(s), v(s)))$ in this case.

If $\frac{4}{3} < p \le \frac{3}{2}$, we consider $A_1(y; (z(s), v(s)))$, by (2.15), we obtain

$$\Big| |P_1 + P_2 + R_0|^{p-1}(P_1 + P_2 + R_0) - |P_1 + P_2|^{p-1}(P_1 + P_2) - \frac{p+1}{2}|P_1 + P_2|^{p-1}R_0 - \frac{p-1}{2}|P_1 + P_2|^{p-3}(P_1 + P_2)^2\overline{R_0}\Big| \lesssim |R_0|^p. \quad (5.26)$$

Next remark that for $1 < p \le 2$, $\big||P_1 + P_2|^{p-1} - |P_1|^{p-1} - |P_2|^{p-1}\big| \lesssim \min(|P_1|^{p-1}, |P_2|^{p-1})$ so the main part of $A_1 = |P_1 + P_2 + R_0|^{p-1}(P_1 + P_2 + R_0) - |P_1 + P_2|^{p-1}(P_1 + P_2)$ can be computed by

$$\begin{aligned} &\Big\|\frac{p+1}{2}|P_1 + P_2|^{p-1}R_0 + \frac{p-1}{2}|P_1 + P_2|^{p-3}(P_1 + P_2)^2\overline{R_0} - p|P_1 + P_2|^{p-1}R_0\Big\|_{L^2} \\ &\qquad\lesssim \Big\|(|v|^2|y|^2 + |v|^2|z|^2)|R_0|(|P_1|^{p-1} + |P_2|^{p-1})\Big\|_{L^2} \ll s^{-2} \end{aligned} \quad (5.27)$$

$$\Big||P_1 + P_2|^{p-1}R_0 - (|P_1|^{p-1} + |P_2|^{p-1})R_0\Big| \lesssim \min(|P_1|^{p-1}, |P_2|^{p-1})|R_0| \quad (5.28)$$

here in (5.27) we use the bootstrap assumptions and the control of modulation equations. Let estimate $R_0(y)Q^{p-1}(y + \frac{z}{2})$, from the decreasing properties of R_0 (5.24), we have

$$\begin{aligned} |R_0(y)Q^{p-1}(y + \frac{z}{2})| &\lesssim \mathrm{e}^{-(2-p)|y\cdot\frac{z}{|z|}|}s^{-p}|z|^{-\frac{(2-p)(d-1)}{2}}\mathrm{e}^{(p-1)|y\cdot\frac{z}{|z|}|}\mathrm{e}^{-(p-1)\frac{|z|}{2}} \\ &\lesssim \mathrm{e}^{-(3-2p)|y\cdot\frac{z}{|z|}|}s^{-(2p-1)}|z|^{-\frac{(3-2p)(d-1)}{2}} \end{aligned} \quad (5.29)$$

so for $\kappa \ll 1$ determined later in (5.33)

$$\|R_0(y)\, Q^{p-1}(y + \frac{z}{2})\|_{L^2} \lesssim s^{-(2p-1-\kappa)}. \quad (5.30)$$

The collection of above estimates gives a bound on norm L^2 and on the decay property of A_1

$$\begin{aligned} \|A_1\|_{L^2} &\lesssim \|R_0\|_{L^2}^p + \|R_0|P_2|^{p-1}\|_{L^2} + \|R_0|P_2|^{p-1}\|_{L^2} \\ &\lesssim s^{-p^2}\log^{dp}(s) + s^{-(2p-1-\kappa)} \le s^{-(2p-1-\kappa)}, \end{aligned}$$

$$\begin{aligned} &\|\mathrm{e}^{(3-2p)|y\cdot\frac{z}{|z|}|}A_1\|_{L^\infty} \\ &\quad\lesssim \|\mathrm{e}^{(3-2p)|y\cdot\frac{z}{|z|}|}|R_0|^p\|_{L^\infty} + \|\mathrm{e}^{(3-2p)|y\cdot\frac{z}{|z|}|}R_0|P_2|^{p-1}\|_{L^\infty} + \|\mathrm{e}^{(3-2p)|y\cdot\frac{z}{|z|}|}R_0|P_2|^{p-1}\|_{L^\infty} \\ &\quad\lesssim s^{-p^2} + s^{-(2p-1)}|z|^{-\frac{(3-2p)(d-1)}{2}} \le s^{-(2p-1)}|z|^{-\frac{(3-2p)(d-1)}{2}} \end{aligned}$$

as the decay $\mathrm{e}^{-(2-p)|y\cdot\frac{z}{|z|}|}$ of R_0 is faster than the one of $\mathrm{e}^{-(3-2p)|y\cdot\frac{z}{|z|}|}$. Finally, we consider

$$\begin{aligned}
&\left|\langle A_1, \nabla Q(y+\frac{z}{2})\rangle - p\left\langle Q^{p-1}(y-\frac{z}{2})R_0 + Q^{p-1}(y+\frac{z}{2})R_0, \nabla Q(y+\frac{z}{2})\right\rangle\right| \\
&\lesssim \left\langle |R_0|^p, \nabla Q(y+\frac{z}{2})\right\rangle + \left\langle \min(|P_1|^{p-1}, |P_2|^{p-1})|R_0|, \nabla Q(y+\frac{z}{2})\right\rangle \\
&\lesssim \left\langle \mathrm{e}^{-(2-p)p|y\cdot\frac{z}{|z|}|} s^{-p^2}|z|^{-\frac{(2-p)p(d-1)}{2}} \mathrm{e}^{(2-p)p|y\cdot\frac{z}{|z|}|}\mathrm{e}^{-(2-p)p\frac{|z|}{2}}, Q^{1-(2-p)p}(y+\frac{z}{2})\right\rangle \\
&\quad + \left\langle s^{-(p-1)}\mathrm{e}^{-(2-p)|y\cdot\frac{z}{|z|}|} s^{-p}|z|^{-\frac{(2-p)(d-1)}{2}} \mathrm{e}^{(2-p)|y\cdot\frac{z}{|z|}|}\mathrm{e}^{-(2-p)\frac{|z|}{2}}, Q^{1-(2-p)}(y+\frac{z}{2})\right\rangle \\
&\lesssim s^{-2p} + s^{-(p+1)} \ll s^{-2}.
\end{aligned}$$

We can deduce from the almost orthogonality (5.16) that

$$\langle A_1, \nabla Q(y \pm \frac{z}{2})\rangle \ll s^{-2}, \tag{5.31}$$

in other words, we have

$$\|\mathfrak{pr}_i(A_1)\|_{L^2} \ll s^{-2}, \qquad \mathrm{i} = 1, 2. \tag{5.32}$$

Therefore, we have the following estimates for $\tilde{A}_1 = A_1 - \mathfrak{pr}_1(A_1) - \mathfrak{pr}_2(A_2)$

$$\|\tilde{A}_1\|_{L^2} \lesssim s^{-(2p-1-\kappa)}, \quad \|\mathrm{e}^{(3-2p)|y\cdot\frac{z}{|z|}|}\tilde{A}_1\|_{L^\infty} \lesssim s^{-(2p-1)}|z|^{-\frac{(3-2p)(d-1)}{2}}$$

and the analogue for R_1

$$\|R_1\|_{H^1} \lesssim s^{-(2p-1-\kappa)}, \quad \|\mathrm{e}^{(3-2p)|y\cdot\frac{z}{|z|}|}R_1\|_{L^\infty} \lesssim s^{-(2p-1)}|z|^{-\frac{(3-2p)(d-1)}{2}}.$$

There exists $0 < \kappa \ll 1$ such that for all $p > \frac{4}{3}$

$$-(2p-1-\kappa)p < -2, \quad -(2p-1-\kappa) - p(p-1) < -2 \tag{5.33}$$

so $\|R_J\|_{H^1}^p + s^{p(p-1)}\|R_J\|_{H^1} \lesssim s^{-(2p-1-\kappa)p} + s^{-(2p-1-\kappa)-p(p-1)} \ll s^{-2}$ and for $\epsilon = 3p-4 > 0$

$$\begin{aligned}
|Q^{p-1}(y)R_1(y+\frac{z}{2})| &\lesssim \mathrm{e}^{-(3-2p)|y\cdot\frac{z}{|z|}+\frac{z}{2}|} s^{-(2p-1)}|z|^{-\frac{(3-2p)(d-1)}{2}} Q^{p-1}(y) \\
&\le \mathrm{e}^{(3-2p)|y\cdot\frac{z}{|z|}|} Q^{(3-2p)}(y) s^{-2} Q^{(3p-4)}(y) \lesssim \mathrm{e}^{-\epsilon|y|} s^{-2}.
\end{aligned}$$

The almost orthogonal property of V_1 is a direct consequence of $\langle \tilde{A}_1(\cdot \pm \frac{z}{2}), \nabla Q\rangle \lesssim s^{-3}$. Thus $J = 1$ and $W = R_0(y; (z(s), v(s))) + R_1(y; (z(s), v(s)))$ in this case.

If $\frac{J+3}{J+2} < p \le \frac{J+2}{J+1}$, we proceed the same way and after $(J+1)$ steps, our process will finish with

$$W = \sum_{j=0}^{J} R_j(y; (z(s), v(s))),$$

$\epsilon = (J+2)p - (J+3) > 0$ and $0 < \kappa \ll 1$ such that for all $\frac{J+2}{J+1} < p \le \frac{J+1}{J}$

$$-((J+1)p - J - \kappa)p < -2, \quad -((J+1)p - J - \kappa) - p(p-1) < -2. \tag{5.34}$$

Step 3. Estimate of G_0. Let $\mathbf{P} = P_1 + P_2 + W$ and put into the definition $\mathcal{E}_{\mathbf{P}}$, it follows from the computations in Lemma 6 that

$$\mathcal{E}_{\mathbf{P}} = [\mathrm{e}^{\mathrm{i}\Gamma_1}\vec{m}_1 \cdot \vec{\mathrm{M}}Q](y - z_1(s)) + [\mathrm{e}^{\mathrm{i}\Gamma_2}\vec{m}_2 \cdot \vec{\mathrm{M}}Q](y - z_2(s)) + |\mathbf{P}|^{p-1}\mathbf{P} - |P_1|^{p-1}P_1 - |P_1|^{p-1}P_1 + \sum_{j=0}^{J}(\Delta - 1)R_j + \sum_{j=0}^{J}[\mathrm{i}\dot{R}_j - \mathrm{i}\frac{\dot{\lambda}}{\lambda}\Lambda R_j + (1-\dot{\gamma})R_j]. \quad (5.35)$$

Note that

$$\sum_{j=1}^{J}(\Delta - 1)R_j = -\sum_{j=1}^{J}\tilde{A}_j = -\sum_{j=1}^{J}A_j + \sum_{j=1}^{J}[\mathfrak{pr}_1(A_j) + \mathfrak{pr}_2(A_j)]$$

thus following (5.9) and (5.15), we have the explicit expression of G_0

$$\begin{aligned} G_0 &= |P_1 + P_2 + \sum_{j=0}^{J} R_j|^{p-1}(P_1 + P_2 + \sum_{j=0}^{J} R_j) - |P_1 + P_2 + \sum_{j=0}^{J-1} R_j|^{p-1}(P_1 + P_2 + \sum_{j=0}^{J-1} R_j) \\ &\quad + \sum_{j=1}^{J}[A_j + (\Delta - 1)R_j] + |P_1 + P_2|^{p-1}(P_1 + P_2) - |P_1|^{p-1}P_1 - |P_2|^{p-1}P_2 + (\Delta - 1)R_0 \\ &\quad + \sum_{j=0}^{J}[\mathrm{i}\dot{R}_j - \mathrm{i}\frac{\dot{\lambda}}{\lambda}\Lambda R_j + (1-\dot{\gamma})R_j] \\ &= |P_1 + P_2 + \sum_{j=0}^{J} R_j|^{p-1}(P_1 + P_2 + \sum_{j=0}^{J} R_j) - |P_1 + P_2 + \sum_{j=0}^{J-1} R_j|^{p-1}(P_1 + P_2 + \sum_{j=0}^{J-1} R_j) \\ &\quad + \sum_{j=1}^{J}[\mathfrak{pr}_1(A_j) + \mathfrak{pr}_2(A_j)] + G + (\Delta - 1)R_0 + \sum_{j=0}^{J}[\mathrm{i}\dot{R}_j - \mathrm{i}\frac{\dot{\lambda}}{\lambda}\Lambda R_j + (1-\dot{\gamma})R_j] \\ &= A_{J+1} + \sum_{j=1}^{J}[\mathfrak{pr}_1(A_j) + \mathfrak{pr}_2(A_j)] + \mathfrak{pr}_1(G\psi) + \mathfrak{pr}_2(G\psi) + G(1-\psi) + \sum_{j=0}^{J}[\mathrm{i}\dot{R}_j - \mathrm{i}\frac{\dot{\lambda}}{\lambda}\Lambda R_j + (1-\dot{\gamma})R_j]. \end{aligned}$$

We bound the first term by (5.19)

$$\|A_{J+1}\|_{L^2} \lesssim s^{-2}.$$

Next, from pointwise control of the modulation equations, we have $\left|\frac{\dot{\lambda}}{\lambda}\right|, |1 - \dot{\gamma}| \lesssim (C^*)^2 s^{-2}$ and $\|R_j\|_{H^1} < \|R_0\|_{H^1} \lesssim s^{-p}\log^d(s)$; therefore,

$$\left\| \sum_{j=0}^{J} \mathrm{i}\frac{\dot{\lambda}}{\lambda}\Lambda R_j - (1-\dot{\gamma})R_j \right\|_{L^2} \ll s^{-2}. \quad (5.36)$$

We recall (5.23) that

$$\|\mathfrak{pr}_1(G\psi)\|_{L^2} + \|\mathfrak{pr}_2(G\psi)\|_{L^2} \lesssim s^{-2}$$

and similarly to (5.32), we have

$$\|\mathfrak{pr}_1(A_j)\|_{L^2} + \|\mathfrak{pr}_2(A_j)\|_{L^2} \ll s^{-2}, \qquad \forall j \geq 1\cdot$$

The term

$$\|G(1-\psi)\|_{L^2} \lesssim |z|^{-\frac{d-1}{2}} \mathrm{e}^{-|z|}(\|P_1\|_{L^2}^{p-1} + \|P_2\|_{L^2}^{p-1}) \lesssim s^{-2}$$

is a consequence of the choice of localized cut-off function ψ and the decay property of Q. For the last term, we have $\dot{R}_j = (-\Delta+1)^{-1}\dot{\tilde{A}}_j$, so

$$\|\dot{R}_j\|_{H^1} \leq \|\dot{\tilde{A}}_j\|_{L^2}.$$

We consider R_0 and A_0; proceeding in the same way as when we controlled G in (5.4), we have that $\dot{G}$ decays more rapidly because of extra terms $\dot{z}$ and $\dot{v}$. In fact, we have

$$|\dot{G}| \leq \left|(\dot{P}_1+\dot{P}_2)|P_1+P_2|^{p-1} - \dot{P}_1|P_1|^{p-1} - \dot{P}_2|P_2|^{p-1}\right|$$
$$+\left|(\dot{P}_1+\dot{P}_2)|P_1+P_2|^{p-2}(P_1+P_2) - \dot{P}_1|P_1|^{p-2}P_1 - \dot{P}_2|P_2|^{p-2}P_2\right|$$

and

$$\dot{P}_k = \dot{z}_k \nabla P_k + \mathrm{i}\,\dot{v}_k(y-z_k)P_k.$$

Then for $|P_1|>|P_2|$, we deduce from the asymptotic behavior of $Q, \nabla Q$ at infinity that

$$\left|(\nabla P_1 - \nabla P_2)|P_1+P_2|^{p-1} - \nabla P_1|P_1|^{p-1} + \nabla P_2|P_2|^{p-1}\right|$$
$$=\left|\nabla P_1|P_1|^{p-1}\left[\left(1-\frac{\nabla P_2}{\nabla P_1}\right)\left|1+\frac{P_2}{P_1}\right|^{p-1} - 1 + \frac{\nabla P_2}{\nabla P_1}\left|\frac{P_2}{P_1}\right|^{p-1}\right]\right| \lesssim |P_1|^{p-1}|P_2|\cdot \mathbb{1}_{y\cdot\frac{z}{|z|}>0}$$

and

$$\left|(\nabla P_1 - \nabla P_2)|P_1+P_2|^{p-2}(P_1+P_2) - \nabla P_1|P_1|^{p-2}P_1 + \nabla P_2|P_2|^{p-2}P_2\right|$$
$$=\left|\nabla P_1|P_1|^{p-2}P_1\left[\left(1-\frac{\nabla P_2}{\nabla P_1}\right)\left|1+\frac{P_2}{P_1}\right|^{p-2}\left(1+\frac{P_2}{P_1}\right) - 1 + \frac{\nabla P_2}{\nabla P_1}\left|\frac{P_2}{P_1}\right|^{p-2}\frac{P_2}{P_1}\right]\right|$$
$$\lesssim |P_1|^{p-1}|P_2|\cdot \mathbb{1}_{y\cdot\frac{z}{|z|}>0}.$$

We do the same way in case $|P_2|>|P_1|$ and for function $(y-z_k)P_k$; thus, we obtain from (5.20) that

$$|\dot{G}| \lesssim |\dot{z}|s^{-p} + |\dot{v}|s^{-p} \lesssim s^{-(p+1)}$$

so $\left\|\dot{A}_0\right\|_{L^2} \lesssim \|\dot{G}\mathrm{i}\|_{L^2} + |\dot{z}|\|G\nabla\mathrm{i}\|_{L^2} \ll s^{-2}$. Next remark that for a function f

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{pr}_i(f)\right| \lesssim \left|\mathfrak{pr}_i(\dot{f})\right| + |\dot{z}||\mathfrak{pr}_i(f)|, \qquad \mathrm{i}=1,2 \tag{5.37}$$

thus $\left\|\dot{\tilde{A}}_0\right\|_{L^2} \ll s^{-2}$, by properties of $(-\Delta+1)^{-1}$, this implies $\|\dot{R}_0\|_{L^2} \ll s^{-2}$. We will prove by induction that

$$\|R_j\|_{L^2}, \forall j \geq 1.$$

For A_j $(j\geq 1)$, we have

$$|\dot{A}_j| \lesssim \left|(\dot{P}_1+\dot{P}_2+\sum_{k=0}^{j-1}\dot{R}_k)|P_1+P_2+\sum_{k=0}^{j-1}R_k|^{p-1} - (\dot{P}_1+\dot{P}_2+\sum_{k=0}^{j-2}\dot{R}_k)|P_1+P_2+\sum_{k=0}^{j-2}R_k|^{p-1}\right|.$$

As $\|\dot{R}_k\|_{L^2} \ll s^{-2}$ for $0 \le k < j$, it is sufficient to prove that

$$\left|(\dot{P}_1+\dot{P}_2)|P_1+P_2+\sum_{k=0}^{j-1}R_k|^{p-1}-(\dot{P}_1+\dot{P}_2)|P_1+P_2+\sum_{k=0}^{j-2}R_k|^{p-1}\right| \ll s^{-2}. \tag{5.38}$$

Let us estimate

$$B_j=\left|(\nabla P_1-\nabla P_2)|P_1+P_2+\sum_{k=0}^{j-1}R_k|^{p-1}-(\nabla P_1-\nabla P_2)|P_1+P_2+\sum_{k=0}^{j-2}R_k|^{p-1}\right|.$$

We have three cases to consider.

At a given point x, if it holds $\max(|P_1|,|P_2|,|V_0|,...,|V_{j-1}|) > \max(|P_1|,|P_2|)$, then

$$B_j \lesssim \sum_{k=0}^{j-1}|V_k|^p \lesssim s^{-p};$$

otherwise, if $\max(|P_1|,|P_2|,|V_0|,...,|V_{j-1}|)=|P_1|$, then, by the first-order Taylor expansion

$$\begin{aligned}B_j=&\Bigg|\nabla P_1|P_1|^{p-1}\Bigg[\frac{1-\nabla P_2/\nabla P_1}{1+P_2/P_1+\sum_{k=0}^{j-1}R_k/P_1}\left(1+\frac{P_2}{P_1}\sum_{k=0}^{j-1}\frac{R_k}{P_1}\right)\left|1+\frac{P_2}{P_1}\sum_{k=0}^{j-1}\frac{R_k}{P_1}\right|^{p-1}\\&-(1-\frac{\nabla P_2}{\nabla P_1})\left|1+\frac{P_2}{P_1}+\sum_{k=0}^{j-2}\frac{R_k}{P_1}\right|^{p-1}\Bigg]\Bigg| \lesssim |P_1|^{p-1}|P_2|\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}>0}+\sum_{k=0}^{j-1}|P_1|^{p-1}|R_k|\\&\lesssim s^{-p},\end{aligned}$$

and similarly for the case $\max(|P_1|,|P_2|,|V_0|,...,|V_{j-1}|)=|P_2|$. Thus, $B_j \lesssim s^{-p}$, from which we deduce (5.38). Recall the estimate for the derivative of a projection (5.37), so we get $\left\|\dot{\tilde{A}}_j\right\|_{L^2} \ll s^{-2}$. In conclusion, we have $\|G_0\|_{L^2} \lesssim s^{-2}$. Similarly, the same estimate holds for ∇G_0, which finishes the proof of (5.10).

Step 4. Estimate of projection. From step 3, the terms whose norm L^2 is of order s^{-2} are $A_{J+1}, \mathfrak{pr}_1(G\psi), \mathfrak{pr}_2(G\psi), G(1-\psi)$. As $|\langle \mathfrak{pr}_2(G\psi), \mathrm{e}^{\mathrm{i}\Gamma_1(y-z_1(s))}\nabla Q(y-z_1(s))\rangle| \ll s^{-2}$ and similarly to (5.32), we can show that $|\mathfrak{pr}_1(A_{J+1})| \ll s^{-2}$; thus

$$\langle G_0, \mathrm{e}^{\mathrm{i}\Gamma_1(y-z_1(s))}\nabla Q(y-z_1(s))\rangle = \langle G, \mathrm{e}^{\mathrm{i}\Gamma_1(y-z_1(s))}\nabla Q(y-z_1(s))\rangle + o(s^{-2})\cdot$$

For $1<p\le 2$, we also have the analogous estimates of (2.23), (2.24)

$$\begin{aligned}\Bigg||\mathbf{P}|^{p-1}\mathbf{P}-|P_1|^{p-1}P_1-|P_2|^{p-1}P_2-\left[\frac{p+1}{2}|P_1|^{p-1}P_2+\frac{p-1}{2}|P_1|^{p-3}P_1^2\overline{P_2}\right]\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}>0}\\-\left[\frac{p+1}{2}|P_2|^{p-1}P_1+\frac{p-1}{2}|P_2|^{p-3}P_2^2\overline{P_1}\right]\cdot\mathbb{1}_{y\cdot\frac{z}{|z|}<0}\Bigg| \lesssim \min(|P_1|^p,|P_2|^p).\end{aligned} \tag{5.39}$$

We note that, for $\delta=\frac{p-1}{2}>0$,

$$\begin{aligned}\left|\int_{y\cdot\frac{z}{|z|}>-\frac{|z|}{2}}Q^p(y+z)\nabla Q(y)\mathrm{d}y\right| &\lesssim |z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}Q^{(p-1)-\delta}\left(\frac{|z|}{2}\right)\int Q^\delta(y)\mathrm{d}y\\&\lesssim s^{-(p+1-\delta)} \ll s^{-2}\log^{-1}(s),\end{aligned}$$

$$\left|\int_{y\cdot\frac{z}{|z|}<-\frac{|z|}{2}} Q^p(y)\nabla Q(y)\mathrm{d}y\right| \lesssim Q^{(p+1)-\delta}\left(\frac{|z|}{2}\right)\int Q^\delta(y)\mathrm{d}y \lesssim s^{-(p+1-\delta)} \ll s^{-2}\log^{-1}(s)\cdot$$

We repeat the approach in step 3 of Lemma 7 and combine it with (5.7) to conclude that

$$\left|\langle G_0, \mathrm{e}^{\mathrm{i}\Gamma_1(y-z_1(s))}\nabla Q(y-z_1(s))\rangle - C_p\frac{z}{|z|}|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\right| \lesssim |z|^{-1-\frac{d-1}{2}}\mathrm{e}^{-|z|} \lesssim s^{-2}\log^{-1}(s)$$

as required. □

The modulation part remains the same as for $p>2$ (see Lemma 9), except that the extra relation will be

$$\dot{v} = -\frac{2}{c_2}H_0(v,z) \tag{5.40}$$

where

$$\begin{aligned} H_0(v,z) &= \left\langle G_0(y;(v(s),z(s))), \mathrm{e}^{\mathrm{i}\frac{v(s)}{2}(y-\frac{z(s)}{2}(s))}\nabla Q\left(y-\frac{z(s)}{2}\right)\right\rangle \\ &= \langle G_0, \mathrm{e}^{\mathrm{i}\Gamma_1(y-z_1(s))}\nabla Q(y-z_1(s))\rangle. \end{aligned} \tag{5.41}$$

Remark that by (5.11), the main order of $\dot{v}$ still remains

$$\left|\dot{v} + c\frac{z}{|z|}|z|^{-\frac{d-1}{2}}\mathrm{e}^{-|z|}\right| \lesssim |z|^{-\frac{d-1}{2}-1}\mathrm{e}^{-|z|}.$$

We claim the following analogue of Proposition 10in the context $1<p\le 2$ for L^2 subcritical.

Proposition 19 (Uniform backwards estimates for $1<p\le 2$)**.** *There exists $s_0 \gg 1$ satisfying the following condition: for all $s^{\mathrm{in}} > s_0$, there is a choice of initial parameters $(\lambda^{\mathrm{in}}, z^{\mathrm{in}}, v^{\mathrm{in}})$ such that the solution u to* (NLS) *corresponding to* (3.1) *exists. Moreover, the decomposition of u with extra relation* (5.40) *on the rescaled interval of time $[s_0, s^{\mathrm{in}}]$*

$$u(s,x) = \frac{\mathrm{e}^{\mathrm{i}\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)}(\mathbf{P}+\epsilon)(s,y), \quad y = \frac{x}{\lambda(s)}, \quad \mathrm{d}t = \lambda^2(s)\mathrm{d}s$$

verifies the uniform estimates for all $s \in [s_0, s^{\mathrm{in}}]$

$$\begin{aligned} &\big|\,|z(s)| - 2\log(s)\big| \lesssim \log(\log(s)), \quad \left|\lambda^{-1}(s) - 1\right| \lesssim s^{-1}, \\ &|v(s)| \lesssim s^{-1}, \quad \|\epsilon(s)\|_{H^1} \lesssim s^{-1}, \quad \left||z(s)|^{\frac{d-1}{2}}\mathrm{e}^{|z(s)|} - cs^2\right| \lesssim s^2\log^{-\frac{1}{2}}(s). \end{aligned} \tag{5.42}$$

Proof of Proposition 19. We only sketch the proof, since it is very similar to Section 3.2, the main difference is the localization to avoid singularities due to the small power p in Taylor expansions (5.1)–(5.3).

Step 1. Modulation equations. Consider

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\langle \eta_1, A+\mathrm{i}B\rangle &= \langle \eta_1, \mathrm{i}L_-A - L_+B\rangle - \langle \vec{m}_1\cdot\vec{\mathrm{d}}\eta_1, \mathrm{i}A-B\rangle - \langle \mathcal{E}_{\mathbf{P}_1}, \mathrm{i}A-B\rangle \\ &- \langle |\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 - \frac{p+1}{2}Q^{p-1}\eta_1 - \frac{p-1}{2}Q^{p-1}\overline{\eta}_1, \mathrm{i}A-B\rangle \end{aligned} \tag{5.43}$$

where the expression of $\mathbf{P}_1$ is given by

$$\mathbf{P}_1 = Q(y) + \mathrm{e}^{\mathrm{i}(\Gamma_2(y-(z_2-z_1))-\Gamma_1(y))}Q(y-(z_2-z_1)) + \sum_{j=0}^{J}\mathrm{e}^{-\mathrm{i}\Gamma_1(y)}R_j(y+z_1).$$

Let C the set such that $\max\left(|R_0(y+z_1)|, ..., |R_J(y+z_1)|\right) \geq \frac{1}{J+2}Q(y)$ then for $y \in C$

$$|Q(y)| \lesssim \|R_{\rm i}\|_{L^\infty} \leq s^{-p}, \qquad \text{for some i} \in \{0, ...J\}\cdot$$

Since $|A|, |B| \lesssim |x|^q \mathrm{e}^{-|x|}$, from the asymptotic behavior (1.12) of Q, over the set C, we have

$$|A| + |B| \lesssim s^{-p} \log^q s. \tag{5.44}$$

Next, denote

$$\Gamma(s, y) = \Gamma_2(y - (z_2 - z_1)) - \Gamma_1(y) = -\frac{1}{2} iv \cdot (y + z) - \frac{1}{2} \mathrm{i} v \cdot y, \tag{5.45}$$

from the estimates $||z| - 2\log(s)| \lesssim \log(\log(s))$ and $\left||v| - s^{-1}\right| \lesssim s^{-1}\log^{-1}(s)$, there exists a constant c_0 (independent of $s^{\rm in}$) such that if $|y| \leq c_0 s$ then $|\Gamma(s, y)| \leq \frac{\pi}{2}$. Let $D = \{y \in \mathbb{R}^d, |y| > c_0 s\}$, we have for $y \in C^c \cap D^c$

$$\frac{1}{J+2} Q(y) \leq |\mathbf{P}_1(y)| \lesssim 1 \tag{5.46}$$

since $|R_0(y+z_1)|, ..., |R_J(y+z_1)| < \frac{1}{J+2}Q(y)$ and $\operatorname{Re}\left[\mathrm{e}^{\mathrm{i}\Gamma} Q(y+z)\right] > 0$. And we have for $y \in C \cup D$, using $A, B \in \mathcal{Y}$ and (5.44),

$$|A(y)| + |B(y)| \lesssim \min(\mathrm{e}^{-\frac{c_0}{2}s}, s^{-p}\log^q(s)) \lesssim s^{-1^+} \tag{5.47}$$

with $1^+ = \frac{p+1}{2}$. We denote

$$\varphi(s, y) = \mathbb{1}_{D^c}\mathbb{1}_{C^c}. \tag{5.48}$$

A consequence of (5.46) and (5.47) is that

$$|\mathbf{P}_1(y)|^{-m} Q(y)^n \varphi(s, y) \lesssim 1 \quad \text{ for } n \geq m > 0 \tag{5.49}$$

and

$$(|A(y)| + |B(y)|)(1 - \varphi(s, y)) \lesssim s^{-1^+}. \tag{5.50}$$

By the Cauchy–Schwarz and Gagliardo–Nirenberg inequalities,

$$\begin{aligned}
&|\langle |\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 - \frac{p+1}{2}Q^{p-1}\eta_1 - \frac{p-1}{2}Q^{p-1}\overline{\eta}_1, (\mathrm{i}A - B)(1 - \varphi(s, y))\rangle| \\
&\lesssim \langle |\eta_1| + |\eta_1|^p, (\mathrm{i}A + B)(1 - \varphi(s, \cdot))\rangle \lesssim s^{-1^+}(\|\eta_1\|_{H^1} + \|\eta_1\|_{H^1}^p) \lesssim C^* s^{-(1+1^+)}.
\end{aligned}$$

From the expansion in (5.1), we get

$$\begin{aligned}
&\left[|\mathbf{P}_1 + \eta_1|^{p-1}(\mathbf{P}_1 + \eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 - \frac{p+1}{2}Q^{p-1}\eta_1 - \frac{p-1}{2}Q^{p-1}\overline{\eta}_1\right]\varphi(s, y) \\
&= \left[\frac{p+1}{2}(|\mathbf{P}_1|^{p-1} - Q^{p-1})\eta_1 + \frac{p-1}{2}(|\mathbf{P}_1|^{p-3}\mathbf{P}_1^2 - Q^{p-1})\overline{\eta}_1 + O(\left|\frac{\eta_1}{\mathbf{P}_1}\right|^2 |\mathbf{P}_1|^p)\right]\varphi(s, y).
\end{aligned}$$

We control the first two terms as before in the case $p > 2$

$$\begin{aligned}
|\langle(|\mathbf{P}_1|^{p-1} - Q^{p-1})\eta_1, (\mathrm{i}A - B)\varphi(s, \cdot)\rangle| + |\langle(|\mathbf{P}_1|^{p-3}\mathbf{P}_1^2 - Q^{p-1})\overline{\eta}_1, (\mathrm{i}A - B)\varphi(s, \cdot)\rangle| \\
\lesssim C^* s^{-(p+1)}\log^q(s)
\end{aligned}$$

and, for the last term, we use (5.49) to remark that $|\mathbf{P}_1|^{p-2}|\mathrm{i}A - B|\varphi(s, \cdot) \lesssim 1$, then deduce the inequality

$$\left\langle \left|\frac{\eta_1}{\mathbf{P}_1}\right|^2 |\mathbf{P}_1|^p, (\mathrm{i}A - B)\varphi(s, \cdot)\right\rangle \lesssim ||\epsilon||_{L^2}^2 \lesssim (C^*)^2 s^{-2}.$$

To summarize, we have shown that

$$\left\langle |\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1)-|\mathbf{P}_1|^{p-1}\mathbf{P}_1-\frac{p+1}{2}Q^{p-1}\eta_1-\frac{p-1}{2}Q^{p-1}\overline{\eta}_1,\mathrm{i}A-B\right\rangle \lesssim s^{-2}. \tag{5.51}$$

Next, it is obvious that we still have, as before,

$$|\langle \vec{m}_1\cdot\vec{\mathrm{d}}\eta_1,\mathrm{i}A-B\rangle|\lesssim C^*s^{-1}|\vec{m}_1(s)|.$$

To prove the estimate

$$\left|\langle \mathcal{E}_{\mathbf{P}_1},\mathrm{i}A-B\rangle-\langle \vec{m}_1\cdot\vec{\mathrm{M}}Q,\mathrm{i}A-B\rangle\right|\lesssim s^{-2}+s^{-1}|\vec{m}_1|, \tag{5.52}$$

we recall $\mathcal{E}_{\mathbf{P}_1}=[\vec{m}_1\cdot\vec{\mathrm{M}}Q](y)+[\mathrm{e}^{\mathrm{i}(\Gamma_2(y-(z_2-z_1))-\Gamma_1(y))}\vec{m}_2\cdot\vec{\mathrm{M}}Q](y-(z_2-z_1))+\mathrm{e}^{-\mathrm{i}\Gamma_1(y)}G_0(y+z_1)$. From (5.10)

$$|\langle \mathrm{e}^{-\mathrm{i}\Gamma_1(y)}G_0(y+z_1),\mathrm{i}A-B\rangle|\lesssim \|G_0\|_{L^2}\lesssim s^{-2}$$

and finally since $A,B\in\mathcal{Y}$, we have

$$|\langle \mathrm{e}^{\mathrm{i}(\Gamma_2(y-(z_2-z_1))-\Gamma_1(y))}(\vec{m}_2\cdot\vec{\mathrm{M}}Q(\cdot-(z_2-z_1))),\mathrm{i}A-B\rangle|\lesssim s^{-1}|\vec{m}_1|,$$

which yields the estimate (3.29)in the case $1<p\leq 2$. We project η_1 onto three null spaces of the linearized equation around Q and obtain the almost orthogonality for the forth null space by the localized momentum thanks to the special choice of $\dot{v}$ in (5.41) (as in Section 3.2.2). Indeed, proceeding in the same way as in (5.43), taking into account the terms of order s^{-2}, we have that

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle \eta_k,\mathrm{i}\nabla Q\rangle=\left\langle \frac{\bar{\eta}_k\cdot F''(\mathbf{P}_k)\cdot\eta_k}{2},\nabla Q\right\rangle+\left(\dot{z}_k-2\,v_k+\frac{\dot{\lambda}}{\lambda}z_k\right)\langle \mathrm{i}\nabla Q,\nabla\eta_k\rangle+O(C^*s^{-(1+1^+)}).$$

For the estimate of localized momentum $\mathcal{M}_k$: for all $s\in[s^*,s^{\mathrm{in}}]$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{M}_k=\left\langle \frac{\bar{\eta}_k\cdot F''(\mathbf{P}_k)\cdot\eta_k}{2},\nabla Q\right\rangle+\left(\dot{z}_k-2\,v_k+\frac{\dot{\lambda}}{\lambda}z_k\right)\langle \mathrm{i}\nabla Q,\nabla\eta_k\rangle+O(\log^{-1}(s)\|\eta_k\|_{H^1}^2). \tag{5.53}$$

Recall that from the equation of $\mathrm{i}\dot{\eta}_k$ (3.9), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{M}_k=\ &\mathrm{Im}\int(\nabla\eta_k\,\bar{\eta}_k)\dot{\chi}-\langle \Delta\eta_k-\eta_k+(|\mathbf{P}_k+\eta_k|^{p-1}(\mathbf{P}_k+\eta_k)-|\mathbf{P}_k|^{p-1}\mathbf{P}_k)\\&+\vec{m}_k^*\cdot\vec{\mathrm{M}}^*\eta_k-(\dot{z}_k-2\,v_k+\frac{\dot{\lambda}}{\lambda}z_k)\cdot\mathrm{i}\nabla\eta_k+\vec{m}_k^*\cdot\vec{\mathrm{M}}^*Q-(\dot{z}_k-2\,v_k+\frac{\dot{\lambda}}{\lambda}z_k)\cdot\mathrm{i}\nabla Q\\&+[\mathrm{e}^{\mathrm{i}(\Gamma_j(y-z)-\Gamma_k(y))}\vec{m}_j\cdot\vec{\mathrm{M}}Q](y\pm z)+G_k,2\mathrm{i}\nabla\eta_k+\eta_k\nabla\mathrm{i}\rangle.\end{aligned}$$

We proceed the same way as in Lemma 12 for L^2 subcritical cases with $p>2$, except for the term

$$\langle |\mathbf{P}_k+\eta_k|^{p-1}(\mathbf{P}_k+\eta_k)-|\mathbf{P}_k|^{p-1}\mathbf{P}_k,2\mathrm{i}\nabla\eta_k+\eta_k\nabla\chi\rangle\cdot$$

First, by (5.1)

$$|\mathbf{P}_k+\eta_k|^{p-1}(\mathbf{P}_k+\eta_k)-|\mathbf{P}_k|^{p-1}\mathbf{P}_k=F'(\mathbf{P}_k)\cdot\epsilon+O(|\eta_k|^p)$$

and then we have

$$|\langle |\eta_p|^p,2\mathrm{i}\nabla\eta_k+\eta_k\nabla\chi\rangle|\lesssim ||\epsilon||_{H^1}^{p+1}\lesssim s^{-2}\log^{-2}(s).$$

Second, we consider

$$|\langle F'(\mathbf{P}_k)\cdot\eta_k, \eta_k\nabla \mathrm{i}\rangle| \lesssim |\nabla\chi|\,||\eta_k||^2_{H^1} \lesssim \log^{-1}(s)||\epsilon||^2_{H^1}.$$

Finally, by integration by parts, we obtain

$$\langle F'(\mathbf{P}_k)\cdot\eta_k, \mathrm{i}\nabla\eta_k\rangle = -\frac{1}{2}\langle\nabla\mathbf{P}_k\chi, \bar{\eta}_k\cdot F''(\mathbf{P}_k)\cdot\eta_k\rangle - \frac{1}{2}\langle F'(\mathbf{P}_k)\cdot\eta_k, \eta_k\nabla\mathrm{i}\rangle.$$

These estimates yield (5.53), since in the support of χ, we have $|P_k| \gtrsim s^{-\frac{1}{8}} \geq \|V_j\|_{L^\infty}, \forall j = \overline{0, J}$ so $\varphi_k \equiv 1$, then

$$\begin{aligned}
&\big|\langle\nabla\mathbf{P}_k\chi, \bar{\eta}_k\cdot F''(\mathbf{P})\cdot\eta_k\rangle - \langle\varphi_k\nabla P_k, \bar{\eta}_k\cdot F''(\mathbf{P}_k)\cdot\eta_k\rangle\big|\\
&\lesssim \left[\sum_{j\neq k}\left|\int_{|y|<\frac{1}{8}\log s}(\bar{\eta}_k\cdot F''(\mathbf{P}_k)\cdot\eta_k)\nabla Q\right| + \left|\int_{|y|>\frac{1}{10}\log s}\varphi_k(\bar{\eta}_k\cdot F''(\mathbf{P}_k)\cdot\eta_k)\nabla Q(y\pm z)\right|\right]\\
&\lesssim s^{-\frac{p-1}{20}}||\epsilon||^2_{H_1}
\end{aligned}$$

and

$$|\langle\mathrm{i}\nabla Q, \mathrm{i}\nabla\eta_k\rangle - \langle\mathrm{i}\nabla Q, \nabla\eta_k\rangle| \lesssim \left|\int_{|y|>\frac{1}{10}\log s}\nabla Q\nabla\bar{\eta}_k\right| \lesssim s^{-\frac{1}{20}}\|\eta_k\|_{H^1}$$

here we use the property (5.49) of φ_k that $\varphi_k \neq 0$ implies $\left|\frac{\nabla P_j}{\mathbf{P}}\right| \lesssim 1$ and $\left|\frac{\nabla P_k}{\mathbf{P}}\right| \lesssim 1$.

Step 2. Control the energy functional. We still consider the energy functional

$$\begin{aligned}
\mathbf{W}(s,\epsilon) =&\mathbf{H}(s,\epsilon) - \mathbf{J}(s,\epsilon)\\
=&\frac{1}{2}\int\left(|\nabla\epsilon|^2 + |\epsilon|^2 - \frac{2}{p+1}\left(|\mathbf{P}+\epsilon|^{p+1} - |\mathbf{P}|^{p+1} - (p+1)|\mathbf{P}|^{p-1}\operatorname{Re}(\epsilon\overline{\mathbf{P}})\right)\right)\\
&-\sum_{k=1}^{2}v_k\cdot\,\mathrm{Im}\int(\nabla\epsilon\,\bar{\epsilon})\chi_k
\end{aligned}$$

and remark that we still have the coercivity property

$$\mathbf{W}(s,\epsilon(s)) \gtrsim \|\epsilon(s)\|^2_{H^1}$$

(see for example [17], [22]). Define

$$\varphi_1(s,y) = \varphi(s, y - z_1(s)) \tag{5.54}$$

a function localized to the first soliton $\mathbf{P}_1$. Similarly, we can define an analogous function $\varphi_2(s,y)$ localized to the second soliton $\mathbf{P}_2$.

We claim an estimate on the derivative of $\mathbf{H}$ by $\dot{z}_k\cdot\langle\nabla P_k, \frac{\bar{\epsilon}.F''(\mathbf{P})\cdot\epsilon}{2}\rangle$, but now localized by φ_k

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}[\mathbf{H}(s,\epsilon(s))] - \sum_{k=1}^{2}\dot{z}_k\cdot\langle\varphi_k\nabla P_k, \frac{\bar{\epsilon}.F''(\mathbf{P})\cdot\epsilon}{2}\rangle\right| \lesssim s^{-2}\|\epsilon(s)\|_{H^1} + s^{-2}\|\epsilon\|^2_{H^1}. \tag{5.55}$$

Recall that we have

$$\frac{\mathrm{d}}{\mathrm{d}s}[\mathbf{H}(s,\epsilon(s))] = D_s\mathbf{H}(s,\epsilon(s)) + \langle D_\epsilon\mathbf{H}(s,\epsilon(s)), \dot{\epsilon}_s\rangle,$$

and

$$D_s\mathbf{H} = \langle\dot{\mathbf{P}}, K\rangle,\ \langle D_\epsilon\mathbf{H}(s,\epsilon), \dot{\epsilon}\rangle = \frac{\dot{\lambda}}{\lambda}\langle D_\epsilon\mathbf{H}(s,\epsilon), \Lambda\epsilon\rangle - (1-\dot{\gamma})\langle\mathrm{i}D_\epsilon\mathbf{H}(s,\epsilon),\epsilon\rangle - \langle\mathrm{i}D_\epsilon\mathbf{H}(s,\epsilon), \mathcal{E}_{\mathbf{P}}\rangle$$

with $K = |\mathbf{P}+\epsilon|^{p-1}(\mathbf{P}+\epsilon) - |\mathbf{P}|^{p-1}\mathbf{P} - \frac{p+1}{2}\epsilon|\mathbf{P}|^{p-1} - \frac{p-1}{2}\overline{\epsilon}\mathbf{P}^2|\mathbf{P}|^{p-3}$. We observe from (5.47) that for $\dot{P}_k = -\dot{z}_k \cdot \nabla P_k + \mathrm{i}\dot{v}_k \cdot (y - z_k)P_k$, over the set $C \cup D$, $|\dot{P}_k| \lesssim s^{-(1+1^+)}$, then

$$|\langle \dot{P}_k, K(1-\varphi_k)\rangle| \lesssim s^{-(1+1^+)}||\epsilon||_{H^1}.$$

From (5.1), $|K| \lesssim |\epsilon|^2|\mathbf{P}|^{p-2}$, so we obtain

$$|\langle \mathrm{i}\dot{v}_k \cdot (y-z_k)P_k, K\varphi_k\rangle| \lesssim |\dot{v}|\, ||\epsilon||^2_{H^1} \lesssim s^{-2}||\epsilon||^2_{H^1}$$

since $\frac{Q(y-z_k)}{|\mathbf{P}|\varphi_k} \lesssim 1$ by (5.49). Next we look more precisely at K

$$K = \frac{\overline{\epsilon}.F''(\mathbf{P})\cdot\epsilon}{2} + O(\left|\frac{\epsilon}{\mathbf{P}}\right|^{2^+}|\mathbf{P}|^p)$$

since $|\dot{z}_k| \lesssim s^{-1}$ and $p - 2^+ > -1$, we also have

$$\left|\left\langle -\dot{z}_k \cdot \nabla P_k, \left|\frac{\epsilon}{\mathbf{P}}\right|^{2^+}|\mathbf{P}|^p\varphi_k\right\rangle\right| \lesssim s^{-1}||\epsilon||^{2^+}_{H^1}.$$

We deal the first two terms of $\langle D_\epsilon\mathbf{H}(s,\epsilon), \dot{\epsilon}\rangle$ as in the case $p > 2$

$$\left|\frac{\dot{\lambda}}{\lambda}\langle D_\epsilon\mathbf{H}(s,\epsilon), \Lambda\epsilon\rangle\right| \lesssim \left|\frac{\dot{\lambda}}{\lambda}\right|\left(\|\epsilon\|^2_{H^1} + \|\epsilon\|^{p+1}_{H^1}\right) \lesssim (C^*)^2 s^{-2}\|\epsilon\|^2_{H^1},$$

$$|(1-\dot{\gamma})\langle \mathrm{i}D_\epsilon\mathbf{H}(s,\epsilon), \epsilon\rangle| \lesssim |1-\dot{\gamma}|\,(\|\epsilon\|^2_{H^1} + \|\epsilon\|^{p+1}_{H^1}) \lesssim (C^*)^2 s^{-2}\|\epsilon\|^2_{H^1}.$$

Recall that for the last term we have

$$\begin{aligned}\langle \mathrm{i}D_\epsilon\mathbf{H}(s,\epsilon), \mathcal{E}_{\mathbf{P}}\rangle =& \langle -\mathrm{i}\Delta\epsilon + \mathrm{i}\epsilon - \mathrm{i}\left(|\mathbf{P}+\epsilon|^{p-1}(\mathbf{P}+\epsilon) - |\mathbf{P}|^{p-1}\mathbf{P}\right),\\ & [\mathrm{e}^{\mathrm{i}\Gamma_1}\vec{m}_1 \cdot \vec{\mathrm{M}}Q](y - z_1(s)) + [\mathrm{e}^{\mathrm{i}\Gamma_2}\vec{m}_2 \cdot \vec{\mathrm{M}}Q](y - z_2(s)) + G_0\rangle\end{aligned}$$

so from the properties of operators L_+ and L_-

$$\begin{aligned}I_1 =& \langle -\mathrm{i}\Delta\epsilon + \mathrm{i}\epsilon - \mathrm{i}\left(|\mathbf{P}+\epsilon|^{p-1}(\mathbf{P}+\epsilon) - |\mathbf{P}|^{p-1}\mathbf{P}\right), [\mathrm{e}^{\mathrm{i}\Gamma_1}\vec{m}_1 \cdot \vec{\mathrm{M}}Q](y - z_1(s))\rangle\\ =& -\frac{\dot{\lambda}}{\lambda}\langle \eta_1, -2Q\rangle + (\dot{v} - \frac{\dot{\lambda}}{\lambda}v)\langle \eta_1, -2\mathrm{i}\nabla Q\rangle\\ & - \left\langle \mathrm{i}\left(|\mathbf{P}_1+\eta_1|^{p-1}(\mathbf{P}_1+\eta_1) - |\mathbf{P}_1|^{p-1}\mathbf{P}_1 - \frac{p+1}{2}Q^{p-1}\eta_1 - \frac{p-1}{2}Q^{p-1}\overline{\eta}_1\right), \vec{m}_1 \cdot \vec{\mathrm{M}}Q\right\rangle.\end{aligned}$$

By the same way to prove (5.51), combining with the orthogonality of η_1 (3.5), (3.26) and the estimate of modulation equation (3.25), we get

$$|I_1| = O((C^*)^4 s^{-4}) + O((C^*)^2 s^{-4}).$$

Finally, using integration by parts and the Cauchy–Schwarz inequality, from the bound for H^1 norm of G_0 (5.10), we obtain

$$|\langle -\mathrm{i}\Delta\epsilon + \mathrm{i}\epsilon - \mathrm{i}\left(|\mathbf{P}+\epsilon|^{p-1}(\mathbf{P}+\epsilon) - |\mathbf{P}|^{p-1}\mathbf{P}\right), G_0\rangle| \lesssim s^{-2}||\epsilon||_{H^1}.$$

Combining these computations, the proof of (5.55) is finished.We still have the same estimate for the localized momentum J_k: for all $s \in [s^*, s^{\mathrm{in}}]$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}[\mathbf{J}(s,\epsilon(s))] - \sum_{k=1}^{2} 2\,v_k \cdot \langle \varphi_k\nabla P_k, \frac{\overline{\epsilon}.F''(\mathbf{P})\cdot\epsilon}{2}\rangle\right| \lesssim s^{-2}\log^{-\frac{3}{4}}(s)\|\epsilon(s)\|_{H^1}. \tag{5.56}$$

(by using (5.53)). Then we can deduce from the modulation equation $|\dot{z}_k - 2v_k| \lesssim s^{-1}\log^{-\frac{3}{4}}(s)$ that

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}[\mathbf{W}(s,\epsilon(s))]\right| \lesssim s^{-2}\|\epsilon(s)\|_{H^1}.$$

The rest of the proof stays unchanged in comparison to the case $p > 2$ in Section 3.2.4. $\square$

From the uniform backwards estimates in Proposition 19, since $\|R_j\|_{H^1} \ll s^{-1}$ for $j = \overline{0, J}$, we have that

$$\begin{aligned}\left\|u(t(s),x) - \frac{\mathrm{e}^{\mathrm{i}\gamma(s)}}{\lambda^{\frac{2}{p-1}}(s)}\sum_{k=1}^{2}\left[\mathrm{e}^{\mathrm{i}\Gamma_k}Q\right]\left(\frac{x}{\lambda(s)} + \frac{(-1)^k}{2}z(s)\right)\right\|_{H^1} &\lesssim \|\epsilon(s)\|_{H^1} + \sum_{j=0}^{J}\|R_j(s)\|_{H^1}\\ &\lesssim s^{-1}\end{aligned}$$

then we proceed like in Section 4 to obtain the existence of a solution $u(t)$ satisfying the regime (1.6)in subcritical cases with $1 < p \leq 2$

$$\left\|u(t) - \mathrm{e}^{\mathrm{i}\gamma(t)}\sum_{k=1}^{2}Q(\cdot - x_k(t))\right\|_{H^1} \lesssim \frac{1}{t}.$$

6. Supercritical cases

In this section, we will present the necessary modifications to prove the result in the L^2 supercritical cases $(1 + \frac{4}{d} < p < \frac{d+2}{d-2})$ (see [4]). For $k \in \{1,2\}$, $z_1(s) = -z_2(s) = \frac{1}{2}z(s)$, $v_1(s) = -v_2(s) = \frac{1}{2}v(s)$, denote

$$\begin{aligned} Y_k^{\pm}(s,y) &= \mathrm{e}^{\mathrm{i}\Gamma_k(s,y-z_k(s))}Y^{\pm}(y - z_k(s)) \qquad (6.1)\\ Z_k(s,y) &= \mathrm{e}^{\mathrm{i}\Gamma_k(s,y-z_k(s))}\mathrm{i}\Lambda Q(y - z_k(s))\\ V_k(s,y) &= \mathrm{e}^{\mathrm{i}\Gamma_k(s,y-z_k(s))}\mathrm{i}\nabla Q(y - z_k(s))\\ W_k(s,y) &= \mathrm{e}^{\mathrm{i}\Gamma_k(s,y-z_k(s))}(y - z_k(s))Q(y - z_k(s)).\end{aligned}$$

Let

$$\mathbf{Y}^{\pm}(s,y) = \mathbf{Y}^{\pm}(y;(z(s),v(s))) = \sum_{k=1}^{2}Y_k^{\pm}(s,y), \mathbf{Z}(s,y) = \mathbf{Z}(y;(z(s),v(s))) = \sum_{k=1}^{2}Z_k(s,y),$$

$$\begin{aligned}\mathbf{V}(s,y) &= \mathbf{V}(y;(z(s),v(s))) = V_1(s,y) - V_2(s,y),\\ \mathbf{W}(s,y) &= \mathbf{W}(y;(z(s),v(s))) = W_1(s,y) - W_2(s,y)\cdot\end{aligned}$$

We need some extra parameters to control the instability created by $Y^{\pm}$. Consider a solution to the NLS equation with symmetric initial data like those below: for $\mathfrak{b} = (\mathfrak{b}^+, \mathfrak{b}^-, \mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3) \in \mathbb{R}^5$, $||\mathfrak{b}|| \leq C(s^{\mathrm{in}})^{-\frac{3}{2}}$ (the constant C is independent of s^{in} and given in Lemma 20):

$$u(T_{\mathrm{mod}},x) = \frac{1}{(\lambda^{\mathrm{in}})^{\frac{2}{p-1}}}w(s^{\mathrm{in}},y), \quad y = \frac{x}{\lambda^{\mathrm{in}}} \qquad (6.2)$$

with

$$\begin{aligned} w(s^{\mathrm{in}}) = \mathbf{P}^{\mathrm{in}}(y;(z^{\mathrm{in}}\vec{e}_1,v^{\mathrm{in}})) &+ \mathfrak{b}^+\mathrm{i}\mathbf{Y}^+(y;(z^{\mathrm{in}}\vec{e}_1,v^{\mathrm{in}})) + \mathfrak{b}^-\mathrm{i}\mathbf{Y}^-(y;(z^{\mathrm{in}}\vec{e}_1,v^{\mathrm{in}}))\\ &+ \mathfrak{b}_1\mathbf{Z}(y;(z^{\mathrm{in}}\vec{e}_1,v^{\mathrm{in}})) + \mathfrak{b}_2\mathbf{V}(y;(z^{\mathrm{in}}\vec{e}_1,v^{\mathrm{in}})) + \mathfrak{b}_3\mathbf{W}(y;(z^{\mathrm{in}}\vec{e}_1,v^{\mathrm{in}})). \qquad (6.3)\end{aligned}$$

Then we get

$$\epsilon(s^{\rm in}) = \mathfrak{b}^+ \mathrm{i}\mathbf{Y}^+(y;(z^{\rm in}\vec{e}_1, v^{\rm in})) + \mathfrak{b}^- \mathrm{i}\mathbf{Y}^-(y;(z^{\rm in}\vec{e}_1, v^{\rm in})) + \mathfrak{b}_1\mathbf{Z}(y;(z^{\rm in}\vec{e}_1, v^{\rm in})) + \mathfrak{b}_2\mathbf{V}(y;(z^{\rm in}\vec{e}_1, v^{\rm in})) + \mathfrak{b}_3\mathbf{W}(y;(z^{\rm in}\vec{e}_1, v^{\rm in})).$$

Lemma 20 (Modulated data in direction $Y^{\pm}$)**.** *There exists $C>0$ such that for all $s^{\rm in}\geq s_0$ and for all $a^{\rm in}\in[-(s^{\rm in})^{-\frac{3}{2}}, (s^{\rm in})^{-\frac{3}{2}}]$, there is a unique $\mathfrak{b}$ so that $||\mathfrak{b}||\leq C|a^{\rm in}|$ (C independent of $s^{\rm in}$) and the initial data satisfies*

$$\langle \eta_1(s^{\rm in}), \mathrm{i}Y^-\rangle = a^{\rm in}, \quad \langle \eta_1(s^{\rm in}), \mathrm{i}Y^+\rangle = \langle \eta_1(s^{\rm in}), \mathrm{i}\Lambda Q\rangle = \langle \eta_1(s^{\rm in}), yQ\rangle = \langle \eta_1(s^{\rm in}), \mathrm{i}\nabla Q\rangle = 0 \tag{6.4}$$

with η_1 defined as in (3.4).

Proof of Lemma 20. Let

$$\mathfrak{c} = (\langle \eta_1(s^{\rm in}), \mathrm{i}Y^+\rangle, \langle \eta_1(s^{\rm in}), \mathrm{i}Y^-\rangle, \langle \eta_1(s^{\rm in}), \mathrm{i}\Lambda Q\rangle, \langle \eta_1(s^{\rm in}), \mathrm{i}\nabla Q\rangle, \langle \eta_1(s^{\rm in}), yQ\rangle)\cdot$$

We consider the linear maps

$$\begin{aligned} \Psi :\ \mathbb{R}^5 &\to H^1(\mathbb{R}^d) & \qquad \Phi :\ H^1(\mathbb{R}^d) &\to \mathbb{R}^5 \\ \mathfrak{b} &\mapsto \epsilon(s^{\rm in}) & \epsilon(s^{\rm in}) &\to \mathfrak{c} \end{aligned}$$

and $\Omega = \mathrm{i}\circ\Psi : \mathbb{R}^5\to\mathbb{R}^5$. We compute

$$\Psi(h) = (\mathrm{i}\mathbf{Y}^+(y;(z^{\rm in}\vec{e}_1, v^{\rm in})), \mathrm{i}\mathbf{Y}^-(y;(z^{\rm in}\vec{e}_1, v^{\rm in})), \mathbf{Z}(y;(z^{\rm in}\vec{e}_1, v^{\rm in})), \mathbf{V}(y;(z^{\rm in}\vec{e}_1, v^{\rm in})), \mathbf{W}(y;(z^{\rm in}\vec{e}_1, v^{\rm in})))\cdot h$$

and

$$\Phi(v) = \begin{pmatrix} \int v(y)[\mathrm{e}^{-\mathrm{i}\Gamma_1}\overline{\mathrm{i}Y^+}](y-\frac{1}{2}z^{\rm in}\vec{e}_1)\,\mathrm{d}y \\ \int v(y)[\mathrm{e}^{-\mathrm{i}\Gamma_1}\overline{\mathrm{i}Y^-}](y-\frac{1}{2}z^{\rm in}\vec{e}_1)\,\mathrm{d}y \\ \int v(y)[\mathrm{e}^{-\mathrm{i}\Gamma_1}\overline{\mathrm{i}\Lambda Q}](y-\frac{1}{2}z^{\rm in}\vec{e}_1)\,\mathrm{d}y \\ \int v(y)[\mathrm{e}^{-\mathrm{i}\Gamma_1}\overline{\mathrm{i}\nabla Q}](y-\frac{1}{2}z^{\rm in}\vec{e}_1)\,\mathrm{d}y \\ \int v(y)[\mathrm{e}^{-\mathrm{i}\Gamma_1}\overline{yQ}](y-\frac{1}{2}z^{\rm in}\vec{e}_1)\,\mathrm{d}y \end{pmatrix}$$

then we can deduce that for some complex functions $A(y), B(y)\in\mathbf{Y}$

$$\Omega = \mathrm{i}\circ\Psi = N + O(\left|\langle A(y+z^{\rm in}\vec{e}_1), B(y)\rangle\right|) = N + O(\mathrm{e}^{-|z^{\rm in}|})$$

where

$$N = \begin{pmatrix} \langle \mathrm{i}Y^+, \mathrm{i}Y^+\rangle & \langle \mathrm{i}Y^-, \mathrm{i}Y^+\rangle & \langle \mathrm{i}\Lambda Q, \mathrm{i}Y^+\rangle & \langle \mathrm{i}\nabla Q, \mathrm{i}Y^+\rangle & \langle yQ, \mathrm{i}Y^+\rangle \\ \langle \mathrm{i}Y^+, \mathrm{i}Y^-\rangle & \langle \mathrm{i}Y^-, \mathrm{i}Y^-\rangle & \langle \mathrm{i}\Lambda Q, \mathrm{i}Y^-\rangle & \langle \mathrm{i}\nabla Q, \mathrm{i}Y^-\rangle & \langle yQ, \mathrm{i}Y^-\rangle \\ \langle \mathrm{i}Y^+, \mathrm{i}\Lambda Q\rangle & \langle \mathrm{i}Y^-, \mathrm{i}\Lambda Q\rangle & \langle \mathrm{i}\Lambda Q, \mathrm{i}\Lambda Q\rangle & \langle \mathrm{i}\nabla Q, \mathrm{i}\Lambda Q\rangle & \langle yQ, \mathrm{i}\Lambda Q\rangle \\ \langle \mathrm{i}Y^+, \mathrm{i}\nabla Q\rangle & \langle \mathrm{i}Y^-, \mathrm{i}\nabla Q\rangle & \langle \mathrm{i}\Lambda Q, \mathrm{i}\nabla Q\rangle & \langle \mathrm{i}\nabla Q, \mathrm{i}\nabla Q\rangle & \langle yQ, \mathrm{i}\nabla Q\rangle \\ \langle \mathrm{i}Y^+, yQ\rangle & \langle \mathrm{i}Y^-, yQ\rangle & \langle \mathrm{i}\Lambda Q, yQ\rangle & \langle \mathrm{i}\nabla Q, yQ\rangle & \langle yQ, yQ\rangle \end{pmatrix}$$

and $\Omega(0) = 0$. Remark that N is the Gramian matrix of $\mathrm{i}Y^+, \mathrm{i}Y^-, \mathrm{i}\Lambda Q, \mathrm{i}\nabla Q, yQ$, which are linearly independent since, if for some $m, n, p, q, r \in \mathbb{R}$ (not all zeros),

$$m\,\mathrm{i}Y^+ + n\,\mathrm{i}Y^- + p\,\mathrm{i}\Lambda Q + q\,yQ + r\,\mathrm{i}\nabla Q = 0$$

then $m\,Y^+ + n\,Y^- + p\,\Lambda Q - q\,\mathrm{i}yQ + r\,\nabla Q = 0$. We apply $\mathcal{L}$ to both sides of the equality $(L_+(\Lambda Q) = -2Q, L_-(xQ) = -2\nabla Q, L_+(\nabla Q) = 0)$ and get

$$me_0 Y^- - ne_0 Y^- - 2\,p\,\mathrm{i}\,Q - 2\,q\,\nabla Q = 0$$

so $m = n = p = q = 0$ as $Y^+, Y^-, \mathrm{i}Q, \nabla Q$ are linearly independent thus $r = 0$, a contradiction. Therefore, $\det N \neq 0$ and with $|z^{\mathrm{in}}| \gg 1$, we have that Ω is invertible around 0 and

$$||\Omega^{-1}|| \leq ||\mathrm{Gram}(\mathrm{i}Y^+, \mathrm{i}Y^-, \mathrm{i}\Lambda Q, \mathrm{i}\nabla Q, yQ)|| + 2$$

Therefore, for any $a^{\mathrm{in}} \in [-(s^{\mathrm{in}})^{-\frac{3}{2}}, (s^{\mathrm{in}})^{-\frac{3}{2}}]$, we can choose

$$\mathfrak{b} = \Omega^{-1}((0, a^{\mathrm{in}}, 0, 0, 0)), \qquad ||\mathfrak{b}|| \leq ||\Omega^{-1}||\,|a^{\mathrm{in}}|$$

to conclude the lemma. □

In fact, the coefficients $\mathfrak{b}_1$, $\mathfrak{b}_2$, and $\mathfrak{b}_3$ can be determined explicitly from $\mathfrak{b}^+, \mathfrak{b}^-$ as follows

$$\mathfrak{b}_1 = \frac{1}{\|\Lambda Q\|_{L^2}^2 + \langle \mathrm{e}^{\mathrm{i}\Gamma_0(\cdot)}\mathrm{i}\Lambda Q(\cdot + z^{\mathrm{in}}\vec{e}_1), \mathrm{i}\Lambda Q\rangle}\big(\mathfrak{b}^+\langle \mathrm{i}Y^+, \mathrm{i}\Lambda Q\rangle + \mathfrak{b}^+\langle \mathrm{e}^{\mathrm{i}\Gamma_0(\cdot)}\mathrm{i}Y^+(\cdot + z^{\mathrm{in}}\vec{e}_1), \mathrm{i}\Lambda Q\rangle + \mathfrak{b}^-\langle \mathrm{i}Y^-, \mathrm{i}\Lambda Q\rangle + \mathfrak{b}^-\langle \mathrm{e}^{\mathrm{i}\Gamma_0(\cdot)}\mathrm{i}Y^-(\cdot + z^{\mathrm{in}}\vec{e}_1), \mathrm{i}\Lambda Q\rangle\big)$$

$$\mathfrak{b}_2 = \frac{1}{\|\nabla Q\|_{L^2}^2 - \langle \mathrm{e}^{\mathrm{i}\Gamma_0(\cdot)}[\mathrm{i}\nabla Q](\cdot + z^{\mathrm{in}}\vec{e}_1), \mathrm{i}\nabla Q\rangle}\big(\mathfrak{b}^+\langle \mathrm{i}Y^+, \mathrm{i}\nabla Q\rangle + \mathfrak{b}^+\langle \mathrm{e}^{\mathrm{i}\Gamma_0(\cdot)}\mathrm{i}Y^+(\cdot + z^{\mathrm{in}}\vec{e}_1), \mathrm{i}\nabla Q\rangle + \mathfrak{b}^-\langle \mathrm{i}Y^-, \mathrm{i}\nabla Q\rangle + \mathfrak{b}^-\langle \mathrm{e}^{\mathrm{i}\Gamma_0(\cdot)}\mathrm{i}Y^-(\cdot + z^{\mathrm{in}}\vec{e}_1), \mathrm{i}\nabla Q\rangle\big)$$

$$\mathfrak{b}_3 = \frac{1}{\|yQ\|_{L^2}^2 - \langle \mathrm{e}^{\mathrm{i}\Gamma_0(\cdot)}[yQ](\cdot + z^{\mathrm{in}}\vec{e}_1), yQ\rangle}\big(\mathfrak{b}^+\langle \mathrm{i}Y^+, yQ\rangle + \mathfrak{b}^+\langle \mathrm{e}^{\mathrm{i}\Gamma_0(\cdot)}\mathrm{i}Y^+(\cdot + z^{\mathrm{in}}\vec{e}_1), yQ\rangle + \mathfrak{b}^-\langle \mathrm{i}Y^-, yQ\rangle + \mathfrak{b}^-\langle \mathrm{e}^{\mathrm{i}\Gamma_0(\cdot)}\mathrm{i}Y^-(\cdot + z^{\mathrm{in}}\vec{e}_1), yQ\rangle\big)$$

where $\Gamma_0(y) = -\frac{1}{2}\,\mathrm{i}\,v^{\mathrm{in}} \cdot (y + z^{\mathrm{in}}\vec{e}_1) - \frac{1}{2}\,\mathrm{i}\,v^{\mathrm{in}} \cdot y$. This specific choice is made so that, initially, we have the following orthogonality conditions

$$\langle \eta_1(s^{\mathrm{in}}), \mathrm{i}\Lambda\rangle = \langle \eta_1(s^{\mathrm{in}}), yQ\rangle = 0 \tag{6.5}$$

and $\langle \eta_1(s^{\mathrm{in}}), \mathrm{i}\nabla Q\rangle = 0$. We recall the decomposition of $u(t)$: there exists a $\mathcal{C}^1$ function

$$\vec{q}(t) = (\lambda, z, \gamma, v) : [s_0, s^{\mathrm{in}}] \to (0, +\infty) \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \times \mathbb{R}^d$$

such that we can modulate $u(t)$ on $[s_0, s^{\mathrm{in}}]$ as

$$u(t(s), x) = \frac{\mathrm{e}^{\mathrm{i}\gamma(s)}}{\lambda(s)}(\mathbf{P} + \epsilon)(s, y)$$

and $\langle \eta_1(s), \mathrm{i}\Lambda\rangle = \langle \eta_1(s), yQ\rangle = 0$. Here we obtain only two orthogonality conditions, as the initial data satisfies only two (6.5). The proof of uniform estimates will remain the same, except for some modifications that we will clarify immediately. Denote

$$a^{\pm}(s) = \langle \eta_1(s), \mathrm{i}Y^{\pm}\rangle, \tag{6.6}$$

Lemma 20 allows us to establish a one-to-one mapping between the choice of $(\mathfrak{b}^+, \mathfrak{b}^-)$ and the constraints $a^+(s^{\mathrm{in}}) = 0, a^-(s^{\mathrm{in}}) = a^{\mathrm{in}}$ for any choice of a^{in}. We now define the maximal time interval $[S(a^{\mathrm{in}}), s^{\mathrm{in}}]$ on which (3.22) holds and

$$|a^\pm(s)| \le s^{-\frac{3}{2}} \tag{6.7}$$

for all $s \in [S(a^{\mathrm{in}}), s^{\mathrm{in}}]$. We will prove that there exists a choice of

$$a^{\mathrm{in}} \in [-(s^{\mathrm{in}})^{-\frac{3}{2}}, (s^{\mathrm{in}})^{-\frac{3}{2}}]$$

and z^{in} such that $S(a^{\mathrm{in}}) = s_0$. The first thing changed is that $\epsilon(s^{\mathrm{in}})$ may not be zero, but we still have $\epsilon(s^{\mathrm{in}}) \lesssim ||\mathfrak{b}|| \lesssim (s^{\mathrm{in}})^{-\frac{3}{2}}$. This is enough to conclude that $|\mathbf{W}(s, \epsilon(s))| \lesssim C^* s^{-2}$ from the fact that $\left|\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{W}(s, \epsilon(s))\right| \lesssim C^* s^{-3}$. Next, from $\langle \eta_1(s^{\mathrm{in}}), \mathrm{i}\nabla Q\rangle = 0$, we still deduce that $|\langle \eta_1, \mathrm{i}\nabla Q\rangle| \lesssim (C^*)^2 s^{-1}\log^{-1}(s)$ by considering the localized momentum $\mathcal{M}_k$. The second thing that needs to be modified is the coercivity of $\mathbf{W}$. By (1.17),

$$\mathbf{W}(s, \epsilon(s)) \gtrsim \|\epsilon(s)\|_{H^1}^2 + O(s^{-3})$$

the process in Section 3 is still valid as long as we have (6.7). We claim the following preliminary estimates on the parameters $a^\pm(s)$.

Lemma 21. *For all* $s \in [S(a^{\mathrm{in}}), s^{\mathrm{in}}]$,

$$\left|\frac{\mathrm{d}a^\pm}{\mathrm{d}s}(s) \mp e_0 a^\pm(s)\right| \lesssim ||\epsilon||_{H^1}^2 \tag{6.8}$$

Proof of Lemma 21. Applying the inequality (3.29) with $A = -\operatorname{Im} Y^+, B = \operatorname{Re} Y^+$ and using the equation of $Y^\pm$ (1.16)

$$\Big|\frac{\mathrm{d}}{\mathrm{d}s}\langle \eta_1, \mathrm{i}\operatorname{Re} Y^+ - \operatorname{Im} Y^+\rangle - \big[\langle \eta_1, -\mathrm{i}L_-(\operatorname{Im} Y^+) - L_+(\operatorname{Re} Y^+)\rangle - \langle \vec{m}_1 \cdot \vec{\mathrm{M}}Q, -\mathrm{i}\operatorname{Im} Y^+ - \operatorname{Re} Y^+\rangle\big]\Big| \lesssim (C^*)^2 s^{-2} + s^{-1}|\vec{m}_1| \tag{6.9}$$

so we get

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\langle \eta_1, \mathrm{i}Y^+\rangle - \langle \eta_1, \mathrm{i}\mathcal{L}(Y^+)\rangle\right| \lesssim (C^*)^2 s^{-2} + s^{-1}|\vec{m}_1| + |\langle \vec{m}_1 \cdot \vec{\mathrm{M}}Q, Y^+\rangle|.$$

This implies $\left|\frac{\mathrm{d}a^+}{\mathrm{d}s}(s) - e_0 a^+(s)\right| \lesssim ||\epsilon||_{H^1}^2$. In the same way, we also obtain

$$\left|\frac{\mathrm{d}a^-}{\mathrm{d}s}(s) + e_0 a^-(s)\right| \lesssim ||\epsilon||_{H^1}^2$$

as desired. □

By the same arguments in Section 3, we improve all estimates in the bootstrap bounds, except those of $a^\pm(s)$ and $z(s)$. It seems to us that the reasoning to close the bootstrap bound of $z(s)$ still works; in fact, it does; however, we will control $a^\pm(s)$ through a suitable value of a^{in} also by a topological argument, so we have to choose $(z^{\mathrm{in}}, a^{\mathrm{in}})$ at the same time.

Lemma 22 (Control of $a^+(s)$)**.** *For all* $a^{\mathrm{in}} \in [-(s^{\mathrm{in}})^{-\frac{3}{2}}, (s^{\mathrm{in}})^{-\frac{3}{2}}]$, *the following inequality holds for all* $s \in [S(a^{\mathrm{in}}), s^{\mathrm{in}}]$

$$|a^+(s)| \le \frac{1}{2}s^{-\frac{3}{2}}. \tag{6.10}$$

Proof of Lemma 22. It follows from (3.22), (6.8) and $a^+(s^{\rm in})=0$ that, for all $s\in[S(a^{\rm in}),s^{\rm in}]$,

$$\begin{aligned}|a^+(s)| &\lesssim (C^*)^2\mathrm{e}^{e_0 s}\int_s^{s^{\rm in}}\mathrm{e}^{-e_0\tau}\tau^{-2}\mathrm{d}\tau\\ &= \frac{(C^*)^2}{e_0}\mathrm{e}^{e_0 s}[\mathrm{e}^{-e_0 s}s^{-2}-\mathrm{e}^{-e_0 s^{\rm in}}(s^{\rm in})^{-2}]-2\frac{(C^*)^2}{e_0}\mathrm{e}^{e_0 s}\int_s^{s^{\rm in}}\mathrm{e}^{-e_0\tau}\tau^{-3}\mathrm{d}\tau\\ &\le \frac{(C^*)^2}{e_0}s^{-2}\le\frac{1}{2}s^{-\frac{3}{2}}\end{aligned}$$

for s_0 to be large enough. □

Lemma 23 (Control of $a^-(s)$ and closing the parameter z)**.** *There exist $z^{\rm in}$ and $a^{\rm in}\in[-(s^{\rm in})^{-\frac{3}{2}},(s^{\rm in})^{-\frac{3}{2}}]$ such that $S(a^{\rm in})=s_0$.*

Proof of Lemma 23. We argue by contradiction. Consider $\zeta(s)$, $\xi(s)$ as defined in (3.47) and

$$\mathcal{N}(s)=s^3(a^-(s))^2.$$

Suppose for all $(\zeta^\sharp,a^\sharp)\in\mathbb{D}=[-1,1]\times[-1,1]$, the choice of

$$\zeta^{\rm in}=s^{\rm in}+\zeta^\sharp s^{\rm in}\log^{-\frac{1}{2}}(s^{\rm in}),\quad a^{\rm in}=a^\sharp(s^{\rm in})^{-\frac{3}{2}}$$

gives us $S(a^{\rm in})=S(\zeta^\sharp,a^\sharp)\in(s_0,s^{\rm in})$. Recall that

$$\dot\xi(s)=2(\zeta(s)-s)(\dot\zeta(s)-1)s^{-2}\log(s)-(\zeta(s)-s)^2(2s^{-3}\log(s)-s^{-3}). \tag{6.11}$$

On the other hand, for $s\in(S(\zeta^\sharp,a^\sharp),s^{\rm in}]$, then by (3.22) and (6.8), we have

$$\begin{aligned}\dot{\mathcal{N}}(s)&=s^3(3s^{-1}a^-(s)+2\frac{\mathrm{d}a^-}{\mathrm{d}s}(s))a^-(s)\\ &=s^3(3s^{-1}-2e_0)(a^-(s))^2+O\left(||\epsilon||^2_{H^1}s^3|a^-(s)|\right).\end{aligned}$$

Due to the bound on $||\epsilon||^2_{H^1}$, we obtain

$$\dot{\mathcal{N}}(s)\le s^3(3s^{-1}-2e_0)(a^-(s))^2+C(C^*)^2s^{-\frac{1}{2}}\sqrt{\mathcal{N}(s)}$$

then, for s_0 large enough ($\frac{3}{s_0}<\frac{1}{2}e_0$ and $C(C^*)^2s_0^{-\frac{1}{2}}<\frac{1}{2}e_0$), the estimate becomes

$$\dot{\mathcal{N}}(s)\le-\frac{3}{2}e_0\mathcal{N}(s)+C(C^*)^2s^{-\frac{1}{2}}\sqrt{\mathcal{N}(s)}. \tag{6.12}$$

Denote

$$\begin{aligned}\Psi_1(s)&=(\zeta(s)-s)(s)^{-1}\log^{\frac{1}{2}}(s),\\ \Psi_2(s)&=a^-(s)(s)^{\frac{3}{2}}.\end{aligned}$$

From the definition of $S(a^{\rm in})$ and the continuity of the flow, at the limit $S(\zeta^\sharp,a^\sharp)$, we have one of the following situations:

$$\Psi_1(S(\zeta^\sharp,a^\sharp))=\pm1,\qquad \Psi_2\in[-1,1] \tag{6.13}$$

or

$$\Psi_2(S(\zeta^\sharp,a^\sharp))=\pm1,\qquad \Psi_1\in[-1,1]. \tag{6.14}$$

Remark that, in the first case, we have

$$\dot\xi(S(\zeta^\sharp,a^\sharp))<-(S(\zeta^\sharp,a^\sharp))^{-1}<0$$

and in the second case, we have $\mathcal{N}(S(\zeta^\sharp, a^\sharp)) = 1$

$$\dot{\mathcal{N}}(S(\zeta^\sharp, a^\sharp)) \leq -e_0 < 0.$$

A consequence of the above transversality property is the continuity of the map $(\zeta^\sharp, a^\sharp) \mapsto S((\zeta^\sharp, a^\sharp))$, thus the following map

$$\begin{array}{cccc} \Psi & : & \mathbb{D} & \to & \partial\mathbb{D} \\ & & (\zeta^\sharp, a^\sharp) & \mapsto & (\Psi_1(S(\zeta^\sharp, a^\sharp)), \Psi_2(S(\zeta^\sharp, a^\sharp))), \end{array}$$

where $\partial\mathbb{D}$ is the boundary of $\mathbb{D}$, is also continuous. Note that if $a^\sharp = \pm 1$, then from (6.12), $\dot{\mathcal{N}}(s^{\text{in}}) < 0$, we have $S(\zeta^\sharp, a^\sharp) = s^{\text{in}}$ and if $\zeta^\sharp = \pm 1$, then from (6.11), $\dot{\xi}(s^{\text{in}}) < 0$, we also have $S(\zeta^\sharp, a^\sharp) = s^{\text{in}}$. Thus $\Psi(\zeta^\sharp, a^\sharp) = (\zeta^\sharp, a^\sharp)$ for all $(\zeta^\sharp, a^\sharp) \in \partial\mathbb{D}$, which means that the restriction of Ψ to the boundary of $\mathbb{D}$ is the identity. But the existence of such a map contradicts the Brouwer fixed-point theorem. In conclusion, there exist final data $(z^{\text{in}}, a^{\text{in}})$ such that $S(a^{\text{in}}) = s_0$. □

Finally, we still have the strong compactness result as in Lemma 15

$$u_n(t_0) \rightharpoonup u_0 \text{ weakly in } H^1(\mathbb{R}^d)$$

$$u_n(t_0) \to u_0 \text{ in } H^\sigma(\mathbb{R}^d), \text{ for } 0 \leq \sigma < 1$$

then we also consider u, the solution to the NLS equation corresponding to u_0, by local well-posedness and continuous dependence (in [3]) for L^2 super-critical of (NLS), we have, for all $t \in [t_0, +\infty)$,

$$u_n(t) \to u(t) \quad \text{in } H^\sigma(\mathbb{R}^d), \quad s_{\text{c}} \leq \sigma < 1$$

where s_{c} is the critical exponent $s_{\text{c}} = \frac{\text{d}}{2} - \frac{2}{p-1} < 1$. Thus we can pass to the limit the decomposition $(\vec{q}, \epsilon)$ and get

$$\left\| u(t) - \mathrm{e}^{\mathrm{i}\gamma(t)} \sum_{k=1}^{2} Q\left(x - x_k(t)\right) \right\|_{H^1} \lesssim t^{-1}. \tag{6.15}$$

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CMLS, École polytechnique, CNRS, Université Paris-Saclay, 91128 Palaiseau, France
E-mail address: tien-vinh.nguyen@polytechnique.edu