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The minimum number of spanning trees in regular multigraphs I: the odd-degree case

Jakub Pekárek* Jean-Sébastien Sereni† Zelealem B. Yilma‡

Abstract

In a recent article, Bogdanowicz determines the minimum number of spanning trees a connected cubic multigraph on a fixed number of vertices can have and identifies the unique graph that attains this minimum value. He conjectures that a generalized form of this construction, which we here call a *padded paddle graph*, would be extremal for d-regular multigraphs where $d \geq 5$ is odd.

We prove that, indeed, the padded paddle minimises the number of spanning trees, but this is true only when the number of vertices, n, is greater than $\frac{9d+6}{8}$. We show that a different graph, which we here call the *padded cycle*, is optimal for $n < \frac{9d+6}{8}$. This fully determines the d-regular multi-graphs minimising the number of spanning trees for odd values of d.

The approach we develop can also be applied to the even-degree case. However, the extremal structures are more irregular, and the slightly more technical analysis is done in a companion article.

1 Introduction

The celebrated Matrix-Tree Theorem establishes a link between the number of spanning trees of a (multi-)graph and linear algebra, thereby providing an efficient way to obtain the number of spanning trees of a given graph through the computation of the determinant of a certain matrix. It tells us little, however, about the extremal values taken by this number over graph classes. Following the extremal graph theory tradition, a number of works pursued this line of research.

In this context, a very natural class of graphs to consider is that of regular graphs. In particular, the question has been well studied for regular simple graphs and there are some results on asymptotic values for the minimum and maximum number of spanning trees a connected d-regular n-vertex simple graph can have.

Let us write $\tau(G)$ for the number of spanning trees of a graph G. Let

$$\delta_d = \liminf_{n\to\infty} (\tau(G))^{1/n} \quad \text{and} \quad \eta_d = \limsup_{n\to\infty} (\tau(G))^{1/n}$$

*Computer Science Institute, Charles University, Prague, Czech Republic. Email: pekarej@iuuk.mff.cuni.cz.

†Service public français de la recherche, Centre national de la recherche scientifique (ICube, CSTB), Strasbourg, France. Email: sereni@kam.mff.cuni.cz.

‡Carnegie Mellon University in Qatar. Email: zyilma@qatar.cmu.edu. Supported by the Qatar Foundation through Carnegie Mellon Qatar's Seed Research program.

where the infimum and supremum are taken over all connected d-regular n-vertex simple graphs. McKay [10] showed that

$$\eta_d = \frac{(d-1)^{d-1}}{(d(d-2))^{d/2-1}}$$

while Alon [1] proved that

$$\sqrt{2} \leq \delta_d \leq \Big((d+1)^{d-2}(d-1)\Big)^{1/(d+1)}. \tag{1}$$

For the case $d = 3$, Kostochka [8] showed, in a strong sense, that $\delta_d = 2^{3/4}$, by proving that $\tau(G) \geq 2^{3(n+2)/4}$ for all cubic simple graphs G on $n \geq 5$ vertices (this result was shown to hold for the class of 2-connected cubic graphs in earlier work by Valdes [13]). The value $\delta_3 = 2^{3/4}$ matches the upper bound given by (1). To our knowledge, the exact value of δ_d is not yet determined for $d \geq 4$ and, as underlined by Alon [1], it should indeed be a difficult question.

Alon [1] also proposed to study the question on multigraphs, noticing that loops should not be allowed — for otherwise, for odd d, there is always a d-regular n-vertex multigraph with a unique spanning tree. Let δ'_d be defined as δ_d except that the infimum is taken over all connected d-regular n-vertex multigraphs (without loops). Alon sketched a neat argument proving that δ'_d has order exactly $\sqrt{d}$, where the lower bound is obtained by a slight modification of his Theorem 1.1. Further, he explained that the conclusion of van der Waerden's conjecture, which had been already established by then [3, 4] (nowadays the reader can also consult Gurvits's proof [6] for an elementary and totally different argument) implies that any d-regular n-vertex loopless multigraph actually contains $(\Omega(\sqrt{d}))^n$ linear forests — that is, forests such that each connected component is a path.

We shall provide an exact formula for δ'_d for all odd values of d, and actually even the exact minimum value of τ over the class of connected d-regular n-vertex loopless multigraphs for odd values of d and all possible values of n. In addition, we explicitly provide all graphs attaining this minimum value. Besides Alon's article [1], our work is motivated by and extends a recent result by Bogdanowicz [2], who considered the number of spanning trees of cubic multigraphs. There, it was shown that if G is a connected cubic multigraph on $n \geq 6$ vertices (where, by necessity, n is even), then the number $\tau(G)$ of spanning trees of G is at least $5^2 \cdot 2^{(n-3)/2}$. It was also shown that the unique extremal graph is a path of length $n-3$ with edges of alternating multiplicities 1 and $d-1$ and a pendant triangle at each end. Bogdanowicz [2] further conjectured that for all odd $d \geq 5$ and even $n \geq 6$, a generalization of this construction, which we call the *padded paddle* (see Figure 1), would have the fewest spanning trees of all connected d-regular multigraphs.

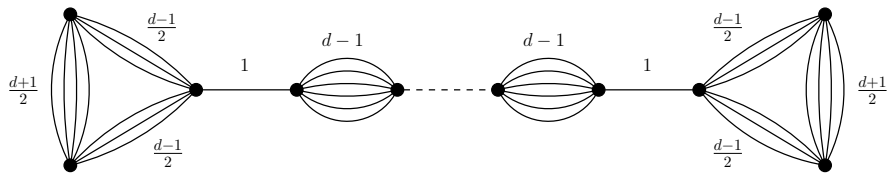


Figure 1: The padded paddle graph

Interestingly, the conjecture is not entirely correct. The padded paddle does minimise the number of spanning trees, but only if the number of vertices is above a certain threshold. Among

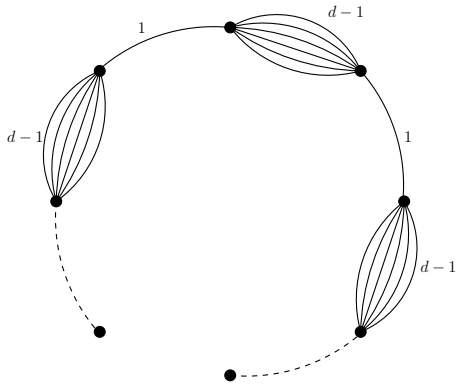


Figure 2: The padded cycle graph

graphs with few vertices (compared to the degree d), we find that a different graph, which we call the *padded cycle* (see Figure 2), is optimal. The existence of these two different regimes is worth noting, and the boundary makes inductive approaches more difficult.

Let $PP_{d,n}$ and $PC_{d,n}$ be the padded paddle and padded cycle graphs, respectively, on n vertices. Routine calculations show that

$$\tau(PP_{d,n}) = \frac{(3d+1)^2}{16}(d-1)^{n/2-1} = \frac{(3d+1)^2}{16(d-1)}(d-1)^{n/2}$$

and

$$\tau(PC_{d,n}) = \frac{n}{2}(d-1)^{n/2} + \frac{n}{2}(d-1)^{n/2-1} = \frac{nd}{2(d-1)}(d-1)^{n/2}.$$

It should be noted here that a misprint appears in the expression for $\tau(PP_{d,n})$ given by Bogdanowicz [2, Theorem 4], which should rather have been the expression above. (Maybe this misprint is what caused the formulation of the incomplete conjecture, the incorrect expression being smaller than both $\tau(PP_{d,n})$ and $\tau(PC_{d,n})$.)

Letting $\alpha = \alpha_d = \frac{(3d+1)^2}{16(d-1)}$ and $\beta(n) = \beta_d(n) = \frac{nd}{2(d-1)}$, we state our main result as follows.

Theorem 1. *Let G be a connected d-regular n-vertex multigraph where $d \geq 3$ is odd and $n \geq 4$. Then,*

$$\tau(G) \geq \min\{\alpha, \beta(n)\} \cdot (d-1)^{n/2}.$$

Furthermore, the only graphs attaining the minimum are the padded cycle graph if $n < \frac{9d+6}{8}$ and the padded paddle graph if $n > \frac{9d+6}{8}$.

Note that for $n = 2$ the only possible multigraph G is a single edge of multiplicity d. It can be understood as a degenerate case of the padded cycle, satisfying the statement as well, with $\tau(G) = \beta(n) \cdot (d-1)^{n/2} = d$.

An analogue result of Theorem 1 classifying the behaviour of optimal structures for even values of d is established in a companion article [12].

The rest of the article is organized as follows. In Section 2, we introduce some notation and preliminary results (many of which are already known) that we will call upon in our proof. Section 3 contains the main elements of our proof of Theorem 1. We finish with some concluding remarks in Section 4. As is apparent from the introduction, the graphs we consider can contain

multiedges but not loops, and the terms "graph" and "multigraph" are interchangeable in what follows. We use the term *simple graph* if we want to forbid both multiedges and loops.

2 Preliminaries

Having in mind a companion article [12], we point out here that all the results presented in this section need no parity assumption on the degree: they can also be used to study d-regular multigraphs for even values of d. We thus make no assumption on the parity of d in this section, and only assume that $d \geq 3$.

Given a multigraph G and any two vertices $i, j \in V(G)$, we define $w_G(i,j)$ to be the number of edges of G between i and j. We write $i \sim_G j$ if $w_G(i,j) \geq 1$, that is, if i and j are adjacent in G. If $w_G(i,j) = 1$ then we may speak unambiguously of *the edge ij of G*. For convenience, if f is an edge of G then we define the *multiplicity of f* (in G) to be the number of edges of G with the same end-vertices as f.

Let $\mathcal{L}(G)$ be the Laplacian matrix of the graph G; that is,

$$\mathcal{L}(G)_{ij} = \begin{cases} \deg_G(i) & \text{if } i = j \\ -w_G(i,j) & \text{if } i \neq j \end{cases}$$

for all $i, j \in V(G)$. The Matrix-Tree Theorem, which applies to multigraphs also, gives the relationship between Laplacian matrices and the number of spanning trees.

Theorem 2 (Kirchhoff's Matrix-Tree Theorem)**.** *For every graph G, every cofactor of its Laplacian $\mathcal{L}(G)$ equals $\tau(G)$.*

The Matrix-Tree Theorem allows us to apply linear algebra tools and helps establish the next statement. For any graph G, let $2G$ be the graph obtained from G by doubling the multiplicity of each edge.

Proposition 3. *Let G, H_1, H_2 be connected graphs on the same vertex set. If $2G = H_1 + H_2$, then $\tau(G) \geq \tau(H_1)$ or $\tau(G) \geq \tau(H_2)$ with at least one strict inequality unless $G = H_1 = H_2$.*

Proof. The matrix form of the Brunn-Minkowski Inequality (see e.g. [7, p. 510]) states that if A and B are $m \times m$ positive definite matrices, then

$$\det(A+B)^{1/m} \geq \det(A)^{1/m} + \det(B)^{1/m},$$

with equality only if $A = cB$ for some $c > 0$. If we take A and B to be the $(n-1) \times (n-1)$ matrices obtained from $\mathcal{L}(H_1)$ and $\mathcal{L}(H_2)$ by deleting their first rows and first columns, thus creating positive definite matrices, we obtain (using also Theorem 2)

$$\left(2^{n-1}\tau(G)\right)^{1/(n-1)} = 2\tau(G)^{1/(n-1)} \geq \tau(H_1)^{1/(n-1)} + \tau(H_2)^{1/(n-1)},$$

and the result follows. □

We will often apply the following consequence of Proposition 3.

Corollary 4. *Suppose G is a connected d-regular graph on n vertices that contains an even cycle with at least 4 vertices. Let M_1 and M_2 be the complementary perfect matchings of the even cycle. If G minimises the number of spanning trees over all connected d-regular graphs on n vertices, then at least one of the graphs $H_1 = G - M_1 + M_2$ and $H_2 = G + M_1 - M_2$ is not connected.*

The next statement gives a lower bound on the number of spanning trees in an almost d-regular graph.

Proposition 5. *Let H be a connected multigraph on n vertices with $nd/2-1$ edges and maximum degree d. Then $\tau(H) \geq (d-1)^{n/2}$ with equality only if H is a path graph with edges of alternating multiplicities $d-1$ and 1.*

Proof. The degree condition and the number of edges imply that almost every vertex in H has degree exactly d. Let x be a vertex of degree at most $d-1$. Let $B_1, \ldots, B_r$ be the maximal 2-connected blocks of H with $x \in B_1$. Let n_i be the number of vertices in B_i. Observe that $\sum_{i=1}^{r} n_i = n + r - 1$.

Pick vertices $s_i, t_i \in B_i$ where $s_1 = x$, s_i is an articulation point for all $i \geq 2$, and $t_i \neq s_i$ is arbitrary. Let $s_i = y_{i,1}, y_{i,2}, \ldots, y_{i,n_i} = t_i$ be an *st-labeling* (also known as a *bipolar orientation*) of B_i. That is, for each $j \in \{2, \ldots, n_i - 1\}$, the vertex $y_{i,j}$ has at least one neighbour in B_i that comes before it and at least one neighbour in B_i that comes after it in the ordering. It is well known that there is such an ordering for any pair of vertices (s,t) in a 2-connected graph [9].

We can build a spanning tree of B_i by selecting, for each vertex $y_{i,j}$ with $j \leq n_i - 1$, one of its incident edges leading to a vertex that comes after $y_{i,j}$ in the ordering in B_i. For $j \in \{1, \ldots, n_i - 1\}$, let $d_{i,j}$ be the number of edges in B_i between $y_{i,j}$ and $\{y_{i,s} : s > j\}$. Observe that $\tau(B_i) \geq \prod_{j=1}^{n_i-1} d_{i,j}$ with equality only if B_i has exactly two vertices (otherwise, B_i contains a cycle and other spanning trees exist — where some vertex $y_{i,j}$ has more than one neighbour $y_{i,s}$ with $s > j$). Then,

$$\tau(H) = \prod_{i=1}^{r} \tau(B_i) \geq \prod_{i=1}^{r} \prod_{j=1}^{n_i-1} d_{i,j}. \tag{2}$$

The product of the rightmost side of (2) contains $\sum_{i=1}^{r}(n_i - 1) = n - 1$ terms. Note also that the number of edges in B_i is exactly $\sum_{j=1}^{n_i-1} d_{i,j}$, and hence the number of edges in H is $\sum_{i=1}^{r} \sum_{j=1}^{n_i-1} d_{i,j} = dn/2 - 1$. As $1 \leq d_{i,j} \leq d-1$ for all i, j, the product is minimised when $n/2$ of the terms are $d-1$ and the remaining $n/2 - 1$ terms are 1. Therefore, $\tau(H) \geq (d-1)^{n/2}$.

Note also that if the block tree of H is not a path, it will have at least three leaves. This would imply the existence of at least one terminal vertex t_{i,n_i} of degree d. This means that B_i has at least three vertices and the above inequality is again strict. Therefore, equality is attained only if H is the path graph as specified. □

It is well known that if H is a graph and f is an edge in H, then

$$\tau(H) = \tau(H - f) + \tau(H/f),$$

where H/f is the graph obtained by contracting the edge f (that is, deleting all edges between the endpoints of f and then identifying the endpoints). As, in general, both deletion and contraction

of an edge in a regular graph result in a graph which is not regular, we regain regularity by employing a 'lifting' operation similar to one used by Ok and Thomassen [11]. Let x, y_1 and y_2 be three distinct vertices in a graph H, and suppose that f_i is an edge in H between x and y_i, for $i \in \{1, 2\}$. *Lifting* f_1 and f_2 means deleting the two edges f_1 and f_2 and adding an edge between y_1 and y_2. If x is a vertex of degree $2m$ in H, a *complete lift* of x is the process of first performing a sequence of m lifts of pairs of edges incident with x and then deleting the vertex x (which is, by then, isolated), thereby producing a multigraph H_x. Observe that if there exists a vertex y such that $w_H(x, y) \geq m + 1$, then it is not possible to perform a complete lift of x since it will not be possible to pair up the edges incident with x such that edges in a same pair span three different vertices, as required in our definition of lift. Conversely, it is possible to perform a complete lift of x as soon as $w_H(x, y) \leq m$ for all vertices y. It is possible to produce a connected multigraph H_x via a complete lift of x if, in addition, H is connected and $H - x$ has at most $m + 1$ components.

Theorem 6 (Ok and Thomassen [11])**.** *Let H be a graph with a vertex x of degree $2m$. Let H_x be a graph obtained from H by a complete lift of x. Then*

$$\tau(H) \geq c_m \, \tau(H_x),$$

where

$$c_m = \min_{d_1, d_2, \ldots, d_k} \min_{X} \frac{\prod_{i=1}^{k} d_i}{\tau(X)},$$

where the minimum is taken over all sequences of positive integers $d_1, \ldots, d_k$ with varying length k such that $\sum_{i=1}^{k} d_i = 2m$, and over all connected k-vertex graphs X with degree sequence $d'_1, d'_2, \ldots, d'_k$ such that $d'_i \leq d_i$ for each $i \in \{1, \ldots, k\}$.

Ok and Thomassen [11] have determined the values of c_m for $m \in \{1, \ldots, 4\}$. We have calculated a few more values using exhaustive computer search. The values, and the graphs attaining them, are given in Table 1. (For $m = 5$, the diamond graph, the graph obtained from K_4 by removing one edge, also attains the value $c_5 = 9/2$.)

m	1	2	3	4	5	6	7	8	9
c_m	1	2	8/3	18/5	9/2	81/16	6	48/7	375/49
$\frac{2(m+1)}{3}$	4/3	2	8/3	10/3	4/3	14/3	16/3	6	20/3
extremal graphs									

Table 1: Values of the Ok-Thomassen c_m term, the lower bound from Proposition 9, and graphs attaining the c_m values.

We need, however, to obtain a general lower bound on c_m for all values of m. While in the definition of c_m the minimum is taken over all graphs X with degree sequence $d'_i \leq d_i$ (and, therefore, potentially fewer than m edges), it is enough to consider only graphs with exactly m

edges, as we formalise and show next. This observation will allow us to combine Theorem 6 with a (direct extension to multigraphs of a) theorem of Grone and Merris [5].

Proposition 7. *Using the notation of Theorem 6,*

$$c_m = \min_{d_1,d_2,\ldots,d_k} \min_X \frac{\prod_{i=1}^k d_i}{\tau(X)},$$

where the minimum is taken over all sequences of positive integers $d_1, \ldots, d_k$ with varying length k such that $\sum_{i=1}^k d_i = 2m$, and over all connected k-vertex graphs X with degree sequence $d_1, d_2, \ldots, d_k$.

Proof. Consider a positive integer m, a sequence $D = d_1, \ldots, d_k$ for some positive integer k such that $\sum_{i=1}^k d_i = 2m$ and a connected graph X with degree sequence $D' = d'_1, \ldots, d'_k$ where $d'_i \leq d_i$ for each $i \in \{1, \ldots, k\}$. Our goal is to show that if $D \neq D'$ then c_m is not attained by D and X.

As long as there are at least two indices $i \neq j$ such that $d'_i < d_i$ and $d'_j < d_j$, we can add a new edge between the two corresponding vertices of X to form a new connected graph X'. The number of spanning trees of X' is larger than that of X, and thus X' along with the sequence D show that c_m is not attained by D and X. We may thus assume that there exists a unique index $i \in \{1, \ldots, k\}$ such that $d'_i < d_i$. It then follows that $d'_i \leq d_i - 2$ as both $\sum_{i=1}^k d_i$ and $\sum_{i=1}^k d'_i$ are even.

Consider the sequence $(s_j)_{1 \leq j \leq k+1}$ defined by

$$s_j = \begin{cases} d_j & \text{if } j \neq i, \\ d_i - 1 & \text{if } j = i, \\ 1 & \text{if } j = k+1, \end{cases}$$

which satisfies that $\sum_{j=1}^{k+1} s_j = \sum_{j=1}^k d_j = 2m$. Note that

$$\prod_{j=1}^{k+1} s_j = \prod_{j=1}^k d_j \cdot \left(1 - \frac{1}{d_i}\right) < \prod_{j=1}^k d_j.$$

Let X' be the connected graph obtained from X by adding a new vertex of degree 1 joined to the vertex with degree d'_i. Then X' has degree sequence $s'_1, \ldots, s'_{k+1}$, where $s'_j = d'_j$ if $j \neq i$, while $s'_i = d'_i + 1$ and $s'_{k+1} = 1$. Consequently $s'_j \leq s_j$ for each $j \in \{1, \ldots, k+1\}$. Moreover, $\tau(X') = \tau(X)$, and therefore c_m is not attained by D and X, which concludes the proof. □

Knowing that X has exactly m edges allows us to use the following result of Grone and Merris [5], which was originally stated for simple graphs but the proof of which, using linear algebra, applies equally to multigraphs.

Theorem 8 (Grone and Merris [5])**.** *If the degree sequence of a graph X is $d_1, d_2, \ldots, d_k$, then*

$$\tau(X) \leq \left(\frac{k}{k-1}\right)^{k-1} \frac{\prod d_i}{\sum d_i}.$$

Theorem 8 and Table 1 give us the following bounds.

Proposition 9. *For all positive integers m we have $c_m > 2m/e$, where e is the base of the natural logarithm. In addition, $c_m \geq 2(m+1)/3$ for all $m \geq 2$ with strict inequality if $m \geq 4$.*

Proof. The first part of the statement directly follows from Proposition 7 and Theorem 8 by using the inequality $(1 + \frac{1}{k-1})^{k-1} < e$, valid for each positive integer k.

The second inequality follows from the first by observing that $c_m > 2m/e \geq 2(m+1)/3$ for $m \geq \lceil e/(3-e) \rceil = 10$. For the remaining cases, namely $m \in \{2, \ldots, 9\}$, direct computations of c_m and $\frac{2(m+1)}{3}$ are provided in Table 1, which concludes the proof. □

We finish this section by showing that one of the features of the padded paddle, the pendant triangle, must arise whenever there are more than d edges between three vertices. To that end, given a subset $X \subseteq V(G)$, we let ∂X be the number of edges with exactly one endpoint in X.

Lemma 10. *If G is a connected d-regular multigraph minimising the number of spanning trees and $\partial\{u, v, w\} \leq d - 2$, then, without loss of generality, $N_G(v) \cup N_G(w) \subset \{u, v, w\}$.*

Proof. We are going to build a connected d-regular n-vertex multigraph G' that has fewer spanning trees than G unless the conclusion of the statement holds in G. To this end, let H be the subgraph of G induced by $\{u, v, w\}$, and note that H is necessarily a triangle. Let G_H be obtained from G by replacing u, v, w by a single vertex x joined to each vertex having a neighbour in $\{u, v, w\}$ in G (with multiplicities). Note that $\tau(G) \geq \tau(G_H)\tau(H)$.

Moreover, the degree d' of x in G_H is $\partial\{u, v, w\}$, which must have the same parity as d. Since $d' \leq d - 2$, we deduce that $d - d'$ is a positive and even integer. We can thus create a connected d-regular n-vertex multigraph G' from G_H by adding two vertices y and z such that $w_{G'}(x, y) = w_{G'}(x, z) = (d - d')/2$ and $w_{G'}(y, z) = (d + d')/2$. In particular, $T = \{x, y, z\}$ induces a pendant triangle in G', since $N_{G'}(y) \cup N_{G'}(z) \subseteq \{x, y, z\}$. Moreover, $G' - y - z$ is isomorphic to G_H. It follows that $\tau(G') = \tau(G_H)\tau(T)$, as every spanning tree of G' decomposes into a spanning tree of T and a spanning tree of $G_H = G' - y - z$.

We now observe that $\tau(T) \leq \tau(H)$ with equality if and only if H and T are isomorphic. Indeed, suppose that $w_G(u, v) = a$, $w_G(u, w) = b$ and $w_G(v, w) = c$, with $1 \leq a \leq b \leq c$. Then $\tau(H) = ab + ac + bc = (s^2 - a^2 - b^2 - c^2)/2$ where $s = a + b + c = \frac{1}{2}(3d - d') \geq d + 1$. With s fixed and the degree conditions $a + b, a + c, b + c \leq d$, the quantity $\tau(H)$ is minimised when, up to symmetry, $a' = b' = s - d$ and $c' = 2d - s$, that is, when H and T are isomorphic — and, in particular, all edges of G in $\partial\{u, v, w\}$ are incident to the same vertex of H. The conclusion follows. □

3 Proof of Theorem 1

We prove Theorem 1 by induction on the number n of vertices.

Lemma 11 (Base cases)**.** *Theorem 1 holds for all odd d and $n \in \{4, 6\}$. More precisely,*

1. *For all odd $d \geq 3$, the padded cycle has the fewest spanning trees of all connected d-regular multigraphs on 4 vertices.*

2. *For all odd $d \geq 5$, the padded cycle has the fewest spanning trees of all connected d-regular multigraphs on 6 vertices.*

3. *The padded paddle has the fewest spanning trees of all connected cubic multigraphs on 6 vertices.*

Proof. By Proposition 3, any graph that minimises the number of spanning trees cannot be the convex combination of other graphs. In particular, Corollary 4 implies that if such a graph contains C_4 as a subgraph, then there cannot be a path, edge-disjoint from the cycle, that connects opposite vertices in the cycle. This implies, for one, that the underlying simple graph cannot contain the diamond graph. Note also that every vertex in the underlying simple graph must have degree at least 2.

For $n = 4$, it follows that the underlying simple graph must be C_4. Then, by Corollary 4, the multigraph minimising the number of spanning trees is the padded cycle.

For $n = 6$, we will show first that the only graphs that are not convex combinations of other graphs are the four graphs in Figure 3, which we call $PC_{d,6}, Gb_{d,6}, Gc_{d,6}, PP_{d,6}$. We reach this conclusion by looking at potential underlying simple graphs and considering some cases.

Case 1: The underlying simple graph does not contain C_4.

The connected C_4-free 6-vertex graphs with minimum degree at least two are C_6, two triangles connected by an edge, and the graph obtained from C_6 by adding a chord between two vertices at distance 2. If the underlying simple graph is C_6, then Corollary 4 implies the multigraph has to be $PC_{d,6}$. For two triangles connected by an edge, keeping in mind that d is odd, it is straightforward to see that $PP_{d,6}$ and $Gc_{d,6}$ are the extremal configurations. The third option, C_6 plus a chord forming a C_3 and a C_5, cannot be the underlying simple graph of a regular multigraph (the triple forming the triangle and the complementary triple should have induced graphs with an equal number of edges but that is not the case).

Case 2: The underlying simple graph contains C_4.

Suppose that $v_1v_2v_3v_4$ is a 4-cycle (necessarily chordless). We consider how to extend the graph creating neither a new path between v_1 and v_3 nor one between v_2 and v_4.

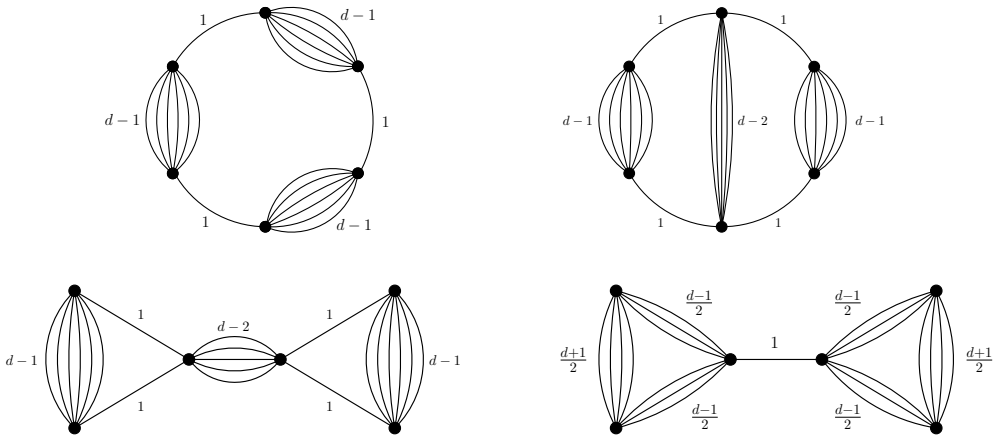


Figure 3: The graphs $PC_{d,6},\ Gb_{d,6},\ Gc_{d,6},\ PP_{d,6}$.

Case 2a: v_5 and v_6 are adjacent.

Recall that the minimum degree in the underlying simple graph is at least 2. Now, if either v_5 or v_6 has two or more neighbours among v_1, v_2, v_3, v_4, it is not possible to avoid creating a diamond or a path joining opposite vertices of the 4-cycle. Let us, therefore, suppose that both v_5 and v_6 have exactly one neighbour each among v_1, v_2, v_3, v_4. If they have a common neighbour, say v_1, then the resulting fish graph cannot be extended to a regular multigraph (v_1 is forced to have degree d inside the C_4 because v_2, v_3 and v_4 do). Otherwise, the simple graph must be the domino graph (C_6 with a chord forming two copies of C_4). Applying Corollary 4 to the two copies of C_4, we see that $Gb_{d,6}$ is the only candidate with this underlying simple graph.

Case 2b: v_5 and v_6 are not adjacent.

Then both v_5 and v_6 have exactly two neighbours each among v_1, v_2, v_3, v_4. The only way to avoid creating a diamond or a path edge-disjoint from the 4-cycle and connecting two opposite vertices on it is, without loss of generality, to have edges v_1v_5, v_2v_5, v_3v_6, and v_4v_6 (thus forming the co-domino graph). But this graph contains a C_6 and the graph remains connected after the removal of either maximum matching of the C_6, contradicting Corollary 4.

Therefore, we need only compare the number of spanning trees of $PC_{d,6}, Gb_{d,6}, Gc_{d,6}$, and $PP_{d,6}$. We have

$$\begin{aligned}
\tau(PC_{d,6}) &= 3d^3 - 6d^2 + 3d, \\
\tau(Gb_{d,6}) &= 4d^3 - 8d^2 + 3d, \\
\tau(Gc_{d,6}) &= 4d^3 - 12d^2 + 9d - 2, \text{ and} \\
\tau(PP_{d,6}) &= (9d^4 - 12d^3 - 2d^2 + 4d + 1)/16,
\end{aligned}$$

with $PC_{d,6}$ attaining the minimum for $d \geq 5$. The graphs $Gc_{d,6}$ and $PP_{d,6}$, which are identical for $d = 3$, are optimal for that case. □

Now fix $n \geq 8$ and suppose that Theorem 1 holds for all d-regular multigraphs on at most $n-2$ vertices. Let G be a connected d-regular n-vertex multigraph minimising the number of spanning trees. We will prove statements on the structure of G which will eventually show that G must be isomorphic to $PC_{d,n}$ or $PP_{d,n}$ as required. We will start by showing that the existence of multiple bridges implies the presence of a structure common to both the padded paddle and the padded cycle, namely, a path with edges of alternating multiplicity 1 and $d-1$.

Lemma 12. *Suppose uv and xy are two distinct bridges (i.e., cutedges of multiplicity one) in G, such that $v \neq y$ and with u and x in the same component H of $G - uv - xy$ (possibly $u = x$). Then, H is a path graph between u and x with edges of alternating multiplicities $d-1$ and 1.*

Proof. Let C_1, C_2, H be the three components of $G - uv - xy$ with $v \in C_1$, $y \in C_2$, and $u, x \in H$. Let n' be the number of vertices in H. Note that n' is necessarily even, since d is odd and dn' equals twice the number of edges induced by H plus the two bridges. Let G' be the d-regular n-vertex multigraph obtained from the disjoint union of C_1 and C_2 by adding a path of length n' with edges of alternating multiplicities $d-1$ and 1 between v and y.

Since H is an n'-vertex graph with $n'd/2 - 1$ edges and maximum degree d, Proposition 5 implies that $\tau(H) \geq (d-1)^{n'/2}$ with equality only if H is isomorphic to the alternating path specified in the statement. It follows that

$$\tau(G) = \tau(H)\tau(C_1)\tau(C_2) \geq (d-1)^{n'/2}\tau(C_1)\tau(C_2) = \tau(G'),$$

with strict inequality unless H is the alternating path as described. □

One immediate consequence of Lemma 12 is the following.

Corollary 13. *If $u \sim_G v$ and $w_G(u,v) \leq d-2$ for two vertices $u, v \in V(G)$, then $G-u-v$ has at most $d - w_G(u,v)$ components. Moreover, for every vertex x, the graph $G-x$ has at most $(d+1)/2$ components.*

We proceed to show that every edge of G has one of the extreme multiplicities 1 or $d-1$ unless it is contained in a pendant triangle. For each of the three cases we then show necessary structural properties fully characterizing the only two possible structures for G.

Lemma 14. *If $u \sim_G v$, then one of the following holds:*

1. *$w_G(u,v) \in \{1, d-1\}$; or*
2. *u and v are part of a pendant triangle.*

Proof. The statement being trivial for $d = 3$, we assume that $d \geq 5$. Recall that, by Lemma 10, if u and v have a common neighbour z such that $\{u, v, z\}$ induces at least $d+1$ edges in G, then the triple $\{u, v, z\}$ forms a pendant triangle.

Let us then set $m = w_G(u,v)$ and suppose, contrary to the statement, that $2 \leq m \leq d-2$ and yet there is no vertex z such that $\{u, v, z\}$ induces at least $d+1$ edges.

Let G' be obtained from G by first deleting all edges between u and v, and next identifying the vertices u and v into a new vertex x, which has thus degree $2(d-m)$ in G'. Observe that $\tau(G) \geq m \cdot \tau(G')$. Note that there is no vertex z in G' with $w_{G'}(x,z) \geq d-m+1$. In addition, Corollary 13 implies that $G-u-v = G'-x$ has at most $d-m$ components. As observed earlier, it follows that it is possible to produce a connected graph by performing a complete lift of x in G'.

Let G'_x be a connected graph obtained from G' by performing a complete lift of x. By Theorem 6 and Proposition 9,

$$\tau(G) \geq m \cdot \tau(G') \geq m \cdot c_{d-m} \cdot \tau(G'_x) \geq \frac{2m(d-m+1)}{3}\tau(G'_x) \geq \frac{4}{3}(d-1)\tau(G'_x).$$

By Proposition 9, and recalling that $m \in \{2, \ldots, d-2\}$ with d being odd, there is equality if and only if $d = 5$ and $m = 2$.
Now, by our inductive hypothesis, $\tau(G'_x) \geq \min\{\alpha, \beta(n-2)\} \cdot (d-1)^{n/2-1}$. So, if $d \geq 7$ then

$$\tau(G) > \min\left\{\frac{4}{3}\alpha, \frac{4}{3}\beta(n-2)\right\} \cdot (d-1)^{n/2} \geq \min\{\alpha, \beta(n)\} \cdot (d-1)^{n/2}.$$

If $d = 5$, we notice that $\frac{4}{3}\beta_5(n-2) \geq \beta_5(n) > \alpha_5$ as $n \geq 8$. Therefore,

$$\tau(G) \geq \min\left\{\frac{4}{3}\alpha, \frac{4}{3}\beta(n-2)\right\} \cdot (d-1)^{n/2} > \alpha \cdot (d-1)^{n/2}.$$

□

We now consider edges of multiplicity 1 and $d-1$, starting with $d-1$.

Lemma 15. *If $w_G(u,v) = d-1$ for two vertices $u, v \in V(G)$, then one of the following holds:*

1. *$G - u - v$ is disconnected;*
2. *u and v are part of a pendant triangle; or*
3. *G is the padded cycle.*

Proof. Suppose that $G - u - v$ is connected. If u and v have a common neighbour w, then the triangle $\{u, v, w\}$ is a pendant triangle. So suppose that $a \sim_G u$ and $b \sim_G v$ with $v \neq a \neq b \neq u$. We need to show that G is the padded cycle graph. Let $G' = G - u - v + f$ where f is an edge joining a and b. Note that G' is a connected d-regular graph with $n-2$ vertices, to which the induction hypothesis thus applies, and $G' - f = G - u - v$ is a connected graph with $n-2$ vertices, maximum degree d and $nd/2 - (d+1) = (n-2)d/2 - 1$ edges, to which Proposition 5 thus applies. We make the following observations:

1. $\tau(G' - f) \geq (d-1)^{n/2-1}$ (by Proposition 5);
2. $\tau(G' - f) + \tau(G'/f) = \tau(G') \geq \min\{\alpha, \beta(n-2)\}(d-1)^{n/2-1}$; and
3. $\tau(G) = (d-1)\tau(G'/f) + (2d-1)\tau(G' - f) = (d-1)\tau(G') + d\,\tau(G' - f)$.

Plugging 1 and 2 into the last equality given by 3, we obtain

$$\begin{aligned}
\tau(G) &\geq \min\{\alpha, \beta(n-2)\}(d-1)^{n/2} + \frac{d}{d-1}(d-1)^{n/2} \\
&= \min\left\{\alpha + \frac{d}{d-1},\, \beta(n-2) + \frac{d}{d-1}\right\}(d-1)^{n/2} \\
&\geq \min\{\alpha, \beta(n)\}\,(d-1)^{n/2}.
\end{aligned}$$

We observe that equality holds only if all inequalities written are equalities, which cannot hold unless $\tau(G' - f) = (d-1)^{n/2-1}$. By Proposition 5, this happens only if $G' - f$ is a path with edges of alternating multiplicities $d-1$ and 1. This implies that G must be the padded cycle graph, as required.

As a side remark, notice also that $\beta(n) = \beta(n-2) + \frac{d}{d-1}$, and hence equality holds in the last inequality only if $\alpha + \frac{d}{d-1} \geq \beta(n)$, that is, $n \leq (9d^2 + 22d + 1)/(8d)$. □

Lemma 16. *If $w_G(u,v) = 1$ for two vertices $u, v \in G$, then one of the following must hold:*

1. *$G - uv$ is disconnected;*
2. *u and v are part of a pendant triangle; or*

3. G is the padded cycle.

Proof. Assume, on the contrary, that $w_G(u,v) = 1$, the graph $G - uv$ is connected, u and v are not part of a pendant triangle and yet G is not the padded cycle. We will show that $\tau(G)$ is too large by writing

$$\tau(G) = \tau(G - uv) + \tau(G/uv)$$

and bounding from below each of the two terms in the right side. In both cases, we will perform a complete lifting operation so we first argue that it is possible to obtain connected graphs at the completion of the respective complete lifting operations. We start by establishing some facts on the multiplicities of edges incident with u or v.

(A). *Every edge of G incident to u or v has multiplicity less than $d-1$.*

Proof. Suppose, on the contrary, that $w_G(u,u') = d-1$, and hence u' and v are the only two neighbours of u. Let v' be the only neighbour of u' different from u. First notice that $v' \neq v$ because u and v are not part of a pendant triangle. Second, $G - u - u'$ must be connected because $G - uv$ is, and hence $G - uv$ contains a path from v to u' that avoids u, and hence contains v'. Consequently, Lemma 15 implies that G is the padded cycle, a contradiction. The same reasoning applies to v by symmetry. ⌟

It turns out that (A) actually leads to a stronger statement.

(B). *Neither u nor v is incident to an edge of multiplicity greater than $\frac{d-1}{2}$.*

Proof. Suppose, on the contrary, that $w_G(u,u') \geq \frac{d+1}{2}$. Since $\frac{d+1}{2} \geq 2$ as $d \geq 3$ and u is not incident to an edge of multiplicity $d-1$ by (A), we deduce from Lemma 14 the existence of a vertex w such that $T = \{u, u', w\}$ induces a pendant triangle. Since $\{u, v\}$ is not part of a pendant triangle, it follows that $w \neq v$ and $N_G(u') \cup N_G(w) \subset T$. In particular, $w_G(u,u') + w_G(u',w) = d = w_G(u',w) + w_G(u,w)$, which implies that $w_G(u,w) = w_G(u,u') \geq \frac{d+1}{2}$. Then the degree of u in G is at least $w_G(u,u') + w_G(u,w) \geq d+1$, a contradiction. ⌟

Let us now consider $\tau(G/uv)$. For simplicity of notation, let $H = G/uv$ be obtained by contracting u and v into a single vertex w. Note that w has degree $2(d-1)$ in H. As observed previously, it is possible to obtain a connected graph by completely lifting the vertex w if $H - w$ has at most d components and w is not incident to an edge of multiplicity at least d. The first condition holds by Corollary 13 as $H - w = G - u - v$ has at most $d-1$ components. Second, w is not incident to an edge of multiplicity greater than $d-1 \geq 2$ in H, for otherwise u or v would be incident to an edge of multiplicity greater than $\frac{d-1}{2}$ in G, contradicting (B).

Let H_w be a connected d-regular $(n-2)$-vertex graph obtained by completely lifting the contracted vertex w in the graph $H = G/uv$. By Theorem 6 and the induction hypothesis, we have

$$\tau(G/uv) = \tau(H) \geq c_{d-1}\,\tau(H_w) \geq c_{d-1}\,\min\{\alpha, \beta(n-2)\} \cdot (d-1)^{n/2-1}. \tag{3}$$

We now turn our attention to $\tau(G - uv)$. In $G - uv$, both u and v have even degree $d-1$. We will first completely lift u in $G - uv$ and then proceed to completely lift the vertex v in the

resulting graph. The assertion (B) guarantees that we can indeed sequentially completely lift the vertices u and v in $G - uv$. It directly follows from the definitions that a graph produced by this process will be d-regular and have $n - 2$ vertices. What remains is to show that this process can lead to a connected graph.

Let G_{uv} be a graph obtained by first completely lifting u in $G - uv$ (yielding a connected graph G'_u) and next completely lifting v in G'_u. Suppose the lifts are performed so that G_{uv} has the smallest possible number of connected components among all graphs constructed in this way. We will show that G_{uv} is connected.

To this end, let E_u be the set of edges created by lifting u, that is, the edges in G'_u but not in G. Similarly, let E_v be the set of edges created by lifting v, that is, the edges in G_{uv} but not in G'_u. We build an auxiliary multigraph L (possibly containing loops) as follows. For each connected component of $G - u - v$ we create an associated vertex in L. For each edge e in $E_u \cup E_v$, we add an edge between the vertices associated to the end-vertices of e in L. (This may create loops if two edges leading to the same connected component of $G - u - v$ were lifted together.) It follows that G_{uv} and L have the same number of connected components. To lighten the writing, we shall canonically identify the edges of L with those in $E_u \cup E_v$.

Because G is connected, each connected component of $G - u - v$ contains a neighbour of u or a neighbour of v. Consequently, these connected components can be partitioned into $\mathcal{C}_u$, $\mathcal{C}_v$, and $\mathcal{C}_{uv}$, depending on whether they have an edge only to u, only to v, or to both u and v, respectively. Furthermore, $\mathcal{C}_{uv}$ is not empty because $G - uv$ is connected; let x be a vertex of L associated to a connected component in $\mathcal{C}_{uv}$.

By Lemma 12, at most one connected component in $\mathcal{C}_u \cup \mathcal{C}_v$ has exactly one edge to $\{u, v\}$. Consequently, at most one vertex of L has degree 1, all the others having degree at least 2. Therefore, every connected component of L contains a cycle (where a loop is considered to be a cycle).

Suppose now that G_{uv}, and hence L, is not connected. Then L contains an edge yz that is not in the same connected component as x and belongs to a cycle. Without loss of generality, assume that $yz \in E_u$. Let uu_1 and uu_2 be the two edges of $G - uv$ that were lifted to create yz. By the definition of x, there exists an edge xx' that belongs to E_u. Similarly, let uu_3 and uu_4 be the two edges of $G - uv$ that were lifted to create xx'.

Now, if we rather lift the pairs (uu_1, uu_3) and (uu_2, uu_4) instead of (uu_1, uu_2) and (uu_3, uu_4), and keep all other lifts the same, we obtain an auxiliary graph L' that has fewer connected components than L. Indeed, the edge yz belongs to a cycle in L, meaning that the two different connected components of x and y in L will become one in L'. This contradicts the definition of G_{uv}, and thus implies that L, and hence G_{uv}, is connected. Therefore, we can apply Theorem 6 to obtain

$$\tau(G - uv) \geq c_{\frac{d-1}{2}}\, \tau(G'_u) \geq \left(c_{\frac{d-1}{2}}\right)^2 \tau(G_{uv}) \geq \left(c_{\frac{d-1}{2}}\right)^2 \min\{\alpha, \beta(n-2)\} \cdot (d-1)^{n/2-1}. \quad (4)$$

Combining (3) and (4), we have

$$\tau(G) = \tau(G - uv) + \tau(G/uv) \geq \left[\left(c_{\frac{d-1}{2}}\right)^2 + c_{d-1}\right] \min\{\alpha, \beta(n-2)\} \cdot (d-1)^{n/2-1}$$

and, noting that $\beta(n) \leq \frac{4}{3}\beta(n-2)$ for $n \geq 8$, we infer that G fails to be optimal if $\left(c_{\frac{d-1}{2}}\right)^2 + c_{d-1} > 4(d-1)/3$. For $d = 3$ and $d = 5$, we have

$$c_1^2 + c_2 = 1^2 + 2 = 3 > 8/3 \qquad \text{and} \qquad c_2^2 + c_4 = 2^2 + 18/5 = 38/5 > 16/3.$$

For $d \geq 7$, we apply Proposition 9 and obtain

$$\left(c_{\frac{d-1}{2}}\right)^2 + c_{d-1} > \left(\frac{2(\frac{d-1}{2})}{e}\right)^2 + \frac{2(d-1)}{e} = \left(\frac{d-1}{e^2} + \frac{2}{e}\right)(d-1) > \frac{4}{3}(d-1).$$

□

Let us now gather the implications of the lemmas above. One consistent theme is that the padded cycle is a candidate to be the graph with the fewest spanning trees. If G is not the padded cycle, then each edge must either disconnect the graph (and have multiplicity 1 or $d-1$) or belong to a pendant triangle. So the structure of G must be a tree with edges of multiplicity 1 or $d-1$ and some pendant triangles. However, by Lemma 12, the tree structure cannot have two adjacent edges of multiplicity 1. Therefore, G must be a long, alternating path with pendant triangles at either end (internal vertices cannot support a pendant triangle). It only remains to show that there is exactly one pendant triangle at each of the two ends of the long path.

Lemma 17. *A vertex in G belongs to at most one pendant triangle.*

Proof. Suppose that $T = \{x, u, v\}$ and $T' = \{x, y, z\}$ induce distinct pendant triangles. Let w be a vertex such that xw is a bridge. Such a vertex must exist because x cannot be incident to an edge of multiplicity $d-1$, and at least one edge incident to x does not belong to a pendant triangle, since d is odd. As G is not a convex combination of other graphs, we may assume that $w_G(x,u) = w_G(x,v) = 1$, up to swapping T and T'.

Let $w_G(x,y) = w_G(x,z) = a$. Then $\tau(G) = (2d-1)(a^2 + 2a(d-a))\tau(G-u-v-y-z)$.

Now consider the graph G' obtained from G by deleting u and v, altering the triangle induced by T' such that $w_{G'}(x,y) = w_{G'}(x,z) = a+1$, deleting the edge xw, and introducing vertices u', v' such that $w_{G'}(x,u') = w_{G'}(x,v') = 1$ and $w_{G'} = (u',v') = d-1$ (in other words, we remove one of the triangles at x and extend the path in $G-u-v-y-z$ by 2). Then $\tau(G') = (d-1)((a+1)^2 + 2(a+1)(d-a-1))\tau(G-u-v-y-z)$. So, we have $\tau(G) - \tau(G') = f(a)\tau(G-u-v-y-z)$ where $f(a) = (2d-1)(a^2+2a(d-a)) - (d-1)((a+1)^2+2(a+1)(d-a-1))$. Simplifying the above, we obtain $f(a) = -da^2 + 2a(d^2+d-1) + (-2d^2+3d-1)$, and so $f(a)$ is a quadratic polynomial which achieves its maximum value at $a = (d^2+d-1)/d$. As $f(1) = 4d-3 > 0$, it follows that $f(a)$ is positive in the interval of interest, that is $a \in [1, (d-3)/2]$. Therefore, $\tau(G) > \tau(G')$, contradicting the optimality of G. □

It follows that if G is not the padded cycle graph, then it must be the padded paddle graph. Comparing the values α_d and $\beta_d(n)$, we see that $\tau(PC_{d,n}) < \tau(PP_{d,n})$ if and only if $n \leq (3d+1)^2/8d$. The parity of d allows us to slightly simplify this expression to the one that appears in the statement of Theorem 1.

4 Remarks

Naturally, Bogdanowicz's conjecture challenges us to ask a similar question for even degree d. As the padded paddle graph does not generalize to cases where d is even (at least not with edge multiplicities 1 and $d-1$), it stands to reason that the padded cycle will be the unique optimal graph (perhaps overtaking another structure as n grows). However, if d is even then it is also possible to consider graphs with an odd number n of vertices, which does imply a change of behaviour: as it happens, the parity renders the padded cycle impossible and a richer variety of optimal structures is to be expected. In a companion article [12], we establish the analogue of Theorem 1 for even values of the degree d. There are indeed more extremal structures, but these occur only for specific small values of d and n. The tools presented in Section 2 still form the base of the argumentation, albeit the richer variety of optimal structures require some more technical work.

One may also consider the question of maximising the number of spanning trees in d-regular multigraphs. In particular, it would be interesting to determine whether, for $n \geq d+1$, the extremal graphs would be simple graphs.

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