Factorization in Call-by-Name and Call-by-Value Calculi via Linear Logic

Claudia Faggian, Giulio Guerrieri

► To cite this version:

Claudia Faggian, Giulio Guerrieri. Factorization in Call-by-Name and Call-by-Value Calculi via Linear Logic. FoSSaCS 2021: International Conference on Foundations of Software Science and Computation Structures, Mar 2021, Luxembourg city, Luxembourg. 10.1007/978-3-030-71995-1_11 . hal-03455783

HAL Id: hal-03455783

<https://hal.science/hal-03455783>

Submitted on 29 Nov 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Factorization in Call-by-Name and Call-by-Value Calculi via Linear Logic

Claudia Faggian¹ and Giulio Guerrieri²

¹ Université de Paris, IRIF, CNRS, F-75013 Paris, France

² University of Bath, Department of Computer Science, Bath, UK

Abstract. In each variant of the λ -calculus, factorization and normalization are two key-properties that show how results are computed. Instead of proving factorization/normalization for the call-by-name (CbN) and call-by-value (CbV) variants separately, we prove them only once, for the bang calculus (an extension of the λ -calculus inspired by linear logic and subsuming CbN and CbV), and then we transfer the result via translations, obtaining factorization/normalization for CbN and CbV. The approach is robust: it still holds when extending the calculi with operators and extra rules to model some additional computational features.

1 Introduction

The λ -calculus is the model of computation underlying functional programming languages and proof assistants. Actually there are many λ -calculi, depending on the *evaluation mechanism* (for instance, call-by-name and call-by-value—CbN and CbV for short) and *computational features* that the calculus aims to model.

In λ -calculi, a rewriting relation formalizes computational steps in program execution, and normal forms are the results of computations. In each calculus, a key question is to define a *normalizing strategy*: How to compute a result? Is there a reduction strategy which is guaranteed to output a result, if any exists?

Proving that a calculus admits a normalizing strategy is complex, and many techniques have been developed. A well-known method first proves *factorization* [5,31,19,3]. Given a calculus with a rewriting relation $\rightarrow$, a strategy $\xrightarrow{\text{L}} \subseteq \rightarrow$ *factorizes* if $\rightarrow^* \subseteq \xrightarrow{\text{L}}^* \cdot \overleftarrow{\xrightarrow{\text{L}}}^*$ ($\overleftarrow{\xrightarrow{\text{L}}}$ is the dual of $\xrightarrow{\text{L}}$), *i.e.* any reduction sequence can be rearranged so as to perform first $\xrightarrow{\text{L}}$ -steps and then the other steps. If, moreover, the strategy satisfies some “good properties”, we can conclude that the strategy is normalizing. Factorization is important also because it is commonly used as a building block in the proof of other properties of the *how-to-compute* kind. For instance, *standardization*, which generalizes factorization: every reduction sequences can be rearranged according to a predefined order between redexes.

Two for One. Quoting Levy [20]: *the existence of two separate paradigms* (CbN and CbV) is troubling because to prove a certain property—such as factorization or normalization—for both systems *we always need to do it twice*.

The *first aim* of our paper is to develop a technique for deriving factorization for both the CbN [5] and CbV [27] λ -calculi as corollaries of a *single* factorization theorem, and similarly for normalization. A key tool in our study is the *bang calculus* [12,16], a calculus inspired by linear logic in which CbN and CbV embed.

The Bang Calculus. The bang calculus is a variant of the λ -calculus where an operator $!$ plays the role of a marker for non-linear management: duplicability and discardability. The bang calculus is nothing but Simpson’s linear λ -calculus [30] without linear abstraction, or the untyped version of the implicative fragment of Levy’s Call-by-Push-Value [20], as first observed by Ehrhard [11].

The motivation to study the bang calculus is to have a general framework where both CbN and CbV λ -calculi can be simulated, via two distinct *translations* inspired by Girard’s embeddings [15] of the intuitionistic arrow into linear logic. So, a certain property can be studied in the bang calculus and then automatically transferred to the CbN and CbV settings by translating back.

This approach has so far mainly been exploited semantically [21,11,12,16,10,8], but can be used also to study operational properties [16,29,14]. In this paper, we push forward this operational direction.

The Least-Level Strategy. We study a strategy from the literature of linear logic [9], namely *least-level reduction* $\xrightarrow{\text{L}}$, which fires a redex at minimal level—the *level* of a redex r is the number of $!$ under which the redex appears.

We prove that the least-level reduction factorizes and normalizes in the bang calculus, and then we transfer the same results to CbN and CbV λ -calculi (for suitable definitions of least-level in CbN and CbV), by exploiting properties of their translations into the bang calculus. A single proof suffices. It is two-for-one! Or even better, three-for-one.

The rewriting study of the least level strategy in the bang calculus is based on simple techniques for factorization and normalization we developed recently with Accattoli [3], which simplify and generalize Takahashi’s method [31].

Subtleties of the Embeddings. Transferring factorization and normalization results via translation is highly non-trivial, *e.g.* in CPS translations [27]. This applies also to transferring least-level factorizations from the bang calculus to the CbN and CbV λ -calculi. To transfer the property smoothly, the translations should preserve levels and normal forms, which is delicate, in particular for CbV. The embedding of CbV into the bang calculus defined in [16,29] does not preserve levels and normal forms (see Remark 16). As a consequence, the CbV translation studied in [16,29] cannot be used to derive least-level factorization or *any* normalization result in a CbV setting from the corresponding result in the bang calculus.

Here we adopt the refined CbV embedding of Bucciarelli et al. [8] which does preserve levels and normal forms. While the preservation of normal forms is already stressed in [8], the preservation of levels is proved here for the first time, and it is based on non-trivial properties of the embedding.

Beyond pure. Our *second aim* is to show that the developed technique for the joined factorization and normalization of CbN and CbV via the bang calculus is *robust*. We do so, by studying extensions of all three calculi with operators (or, in general, with extra rules) which model some additional computational features, such as non-deterministic or probabilistic choice. We then show that the technique scales up smoothly, under mild assumptions on the extension.

A Motivating Example. Let us illustrate our approach on a simple case, which we will use as running example. De’ Liguoro’ and Piperno’s CbN non-deterministic λ -calculus $\Lambda_{\oplus}^{\text{cbn}}$ [23] extends the CbN λ -calculus with an operator $\oplus$ whose reduction models *non-deterministic choice*: $\oplus(t, s)$ rewrites to either t or s . It admits a standardization result, from which it follows that the leftmost-outermost reduction strategy (noted $\xrightarrow{\text{LO}}_{\beta \oplus}$) is *complete*: if t has a *normal form* u then $t \xrightarrow{\text{LO}}_{\beta \oplus}^* u$. In [22], de’ Liguoro considers also a CbV variant $\Lambda_{\oplus}^{\text{cbv}}$, extending with an operator $\oplus$ the CbV λ -calculus. One may prove standardization and completeness—again—from scratch, even though the proofs are similar.

The approach we propose here is to work in the bang calculus enriched with the operator $\oplus$, it is denoted by $\Lambda_{\oplus}$. We show that the calculus satisfies *least-level factorization* from which it follows that the least-level strategy is *complete*, *i.e.* if t has a *normal form* u , then $t \xrightarrow{\text{L}}_{\beta, \oplus}^* u$. The translation then guarantees that analogous results hold also in $\Lambda_{\oplus}^{\text{cbn}}$ and $\Lambda_{\oplus}^{\text{cbv}}$.

The Importance of Being Modular. The bang calculus with operators is actually a general formalism for several calculi, one calculus for each kind of computational feature modeled by operators. Concretely, the reduction $\rightarrow$ consists of $\rightarrow_{\beta_1}$ (which subsumes CbN $\rightarrow_{\beta}$ and CbV $\rightarrow_{\beta_v}$) and other reduction rules $\rightarrow_{\rho}$.

We decompose the proof of factorization of $\rightarrow$ in modules, by using the *modular approach* recently introduced by the authors together with Accattoli [4].

The key module is the least-level factorization of $\rightarrow_{\beta_1}$, because it is where the higher-order comes into play—this is done, once for all. Then, we consider a generic reduction rule $\rightarrow_{\rho}$ to add to $\rightarrow_{\beta_1}$. Our general result is that if $\rightarrow_{\rho}$ has ‘good properties’ and interacts well with $\rightarrow_{\beta_1}$ (which amounts to an easy test, combinatorial in nature), then we have least-level factorization for $\rightarrow_{\beta_1} \cup \rightarrow_{\rho}$.

Putting all together, when $\rightarrow_{\rho}$ is instantiated to a concrete reduction (such as $\rightarrow_{\oplus}$), the user of our method only has to verify a simple test (namely Proposition 34), to conclude that $\rightarrow_{\beta_1} \cup \rightarrow_{\oplus}$ has least-level factorization. In particular factorization for $\rightarrow_{\beta_1}$ is a ready-to-use black box the user need not to worry about—our proof is robust enough to hold whatever the other rules are. Finally, the embedding automatically give least-level factorization for the corresponding CbV and CbN calculi. In Section 7, we illustrate our method on this example.

Subtleties of the Modular Extensions. In order to adopt the modular approach presented in [4] we need to deal with an important difficulty which appears when dealing with normalizing strategies and that it is not studied in [4].

A normalizing strategies select the redex to fire usually through a property such as being a *least level* redex or being the *leftmost-outermost* (shortened to LO) redex—normalizing strategies are *positional*.

The problem is that the—in general— if $\rightarrow = \rightarrow_\beta \cup \rightarrow_\rho$, then $\overrightarrow{\quad}$ reduction is not the union of $\overrightarrow{\quad}_{\text{LO}\beta}$ and $\overrightarrow{\quad}_{\text{LO}\rho}$. I.e., the normalizing strategy of the compound system is not obtained putting together the normalizing strategies of the components. Let us explain the issue on our running example $\rightarrow_{\beta\oplus}$, in the familiar case of leftmost-outermost reduction.

Example 1. Let us first consider head reduction with respect to β (written $\overrightarrow{\quad}_{\text{h}\beta}$) and with respect to $\beta\oplus$ (written $\overrightarrow{\quad}_{\text{h}\beta\oplus}$). Consider the term $s = (\text{II})(x \oplus y)$, where $\text{I} = \lambda x.x$. The subterm II (which is a β -redex) is in head position whenever we consider the reduction $\rightarrow_\beta$ or its extension $\rightarrow_{\beta\oplus}$. So $s \xrightarrow{\text{h}\beta} \text{I}(x \oplus y)$ and $s \xrightarrow{\text{h}\beta\oplus} \text{I}(x \oplus y)$. Conversely, given $t = (x \oplus y)(\text{II})$ the head position is occupied by $(x \oplus y)$, which is a $\oplus$ -redex, but not a β -redex. Therefore, (II) is not the head-redex in t , neither for β nor for $\beta\oplus$. Otherwise stated:

$$\overrightarrow{\quad}_{\text{h}\beta\oplus} = \overrightarrow{\quad}_{\text{h}\beta} \cup \overrightarrow{\quad}_{\text{h}\oplus}.$$

In contrast, if we consider leftmost-outermost reduction $\overrightarrow{\quad}_{\text{LO}}$, which reduces a redex in the leftmost-outermost position, it is easy to see that

$$\overrightarrow{\quad}_{\text{LO}\beta\oplus} \neq \overrightarrow{\quad}_{\text{LO}\beta} \cup \overrightarrow{\quad}_{\text{LO}\oplus}.$$

Consider again the term $t = (x \oplus y)(\text{II})$. Since $(x \oplus y)$ is not a β -redex, (II) is the leftmost redex for $\rightarrow_\beta$. Instead, (II) is not the LO-redex for $\rightarrow_{\beta\oplus}$ (here the leftmost redex is $(x \oplus y)$). So $t \xrightarrow{\text{LO}\beta} (x \oplus y)\text{I}$ but $t \not\xrightarrow{\text{LO}\beta\oplus} (x \oplus y)\text{I}$.

The least-level factorization for $\rightarrow_{\beta_1}$, $\rightarrow_\beta$, and $\rightarrow_{\beta_v}$ we prove here is robust enough to make it ready to be used as a module in a larger proof, where it may combine with operators and other rules. The key point is to define the least-level reduction from the very beginning as a reduction firing a redex at minimal level with respect to a general set of redexes (containing β_1 , β or β_v , respectively), so that it is “ready” to be extended with other reduction rules (see Section 4).

Proofs. All proofs are available in <https://www.irif.fr/~giuliog/fact.pdf>

2 Background in Abstract Rewriting

An (*abstract*) *rewriting system*, [32, Ch. 2] is a pair $(A, \rightarrow)$ consisting of a set A and a binary relation $\rightarrow \subseteq A \times A$ (called reduction) whose pairs are written $t \rightarrow s$ and called *steps*. A $\rightarrow$ -*sequence* from t is a sequence of $\rightarrow$ -steps. As usual, $\rightarrow^*$ (resp. $\rightarrow^=$) denotes the transitive-reflexive (resp. reflexive) closure of $\rightarrow$.

A relation $\rightarrow$ is *confluent* if $s^* \leftarrow r \rightarrow^* t$ implies $s \rightarrow^* u^* \leftarrow t$ for some u . We say that u is $\rightarrow$ -*normal* (or a $\rightarrow$ -normal form) if there is no t such that $u \rightarrow t$.

In general, a term may or may not reduce to a normal form. If it does, not all reduction sequences necessarily lead to normal form. A term is *weakly* or *strongly normalizing*, depending on if it may or must reduce to normal form. If a term t is strongly normalizing, any choice of steps will eventually lead to a normal form. However, if t is weakly normalizing, how do we compute a normal form? This is the problem tackled by *normalization*: by repeatedly performing *only specific steps*, a normal form will be computed, provided that t can reduce to any.

A *strategy* $\rightarrow_{\bullet} \subseteq \rightarrow$ is a way to control that in a term there are different possible choices of reduction. A *normalizing strategy* for $\rightarrow$, is a reduction strategy which, given a term t , is guaranteed to reach its $\rightarrow$ -normal form, if any exists (a key tool to show that certain terms are not $\rightarrow$ -normalizable).

Definition 2 (Normalizing and complete strategy). A reduction $\rightarrow_{\bullet} \subseteq \rightarrow$ is a strategy for $\rightarrow$ if it has the same normal forms as $\rightarrow$. A strategy $\rightarrow_{\bullet}$ for $\rightarrow$ is:

- complete if $t \rightarrow_{\bullet}^* u$ whenever $t \rightarrow^* u$ with $u \rightarrow$ -normal;
- normalizing if every maximal $\rightarrow_{\bullet}$ -sequence from t ends in a normal form, whenever $t \rightarrow^* u$ for some $\rightarrow$ -normal form u .

Note that if the strategy $\rightarrow_{\bullet}$ is complete and *deterministic* (i.e. for every $t \in A$, $t \rightarrow_{\bullet} s$ for at most one $s \in A$), then $\rightarrow_{\bullet}$ is a normalizing strategy for $\rightarrow$.

Definition 3 (Factorization). Let $(A, \rightarrow)$ be a rewriting system with $\rightarrow = \rightarrow_{\bullet} \cup \rightarrow_{\dagger}$. The relation $\rightarrow$ satisfies **e-factorization**, written $\mathbf{Fact}(\rightarrow_{\bullet}, \rightarrow_{\dagger})$, if

$$\mathbf{Fact}(\rightarrow_{\bullet}, \rightarrow_{\dagger}) : (\rightarrow_{\bullet} \cup \rightarrow_{\dagger})^* \subseteq \rightarrow_{\bullet}^* \cdot \rightarrow_{\dagger}^* \quad (\mathbf{Factorization})$$

Proving Normalization. Factorization provides a simple technique to establish that a strategy is normalizing.

Lemma 4 (Normalization [3]). Let $\rightarrow = \rightarrow_{\bullet} \cup \rightarrow_{\dagger}$, and $\rightarrow_{\bullet}$ be a strategy for $\rightarrow$.

The strategy $\rightarrow_{\bullet}$ is complete for $\rightarrow$ if the following hold:

1. Persistence: If $t \rightarrow_{\dagger} t'$ then t' is not normal.
2. Factorization: $t \rightarrow^* u$ implies $t \rightarrow_{\bullet}^* \cdot \rightarrow_{\dagger}^* u$.

The strategy $\rightarrow_{\bullet}$ is normalizing for $\rightarrow$ if it is complete and:

3. Uniformity: all weakly $\rightarrow_{\dagger}$ -normalizing terms are strongly $\rightarrow_{\bullet}$ -normalizing.

A sufficient condition for uniform normalization and confluence is the following:

Property 5 (Newman [25]) A reduction is quasi-diamond if $(t_1 \leftarrow t \rightarrow t_2)$ implies $(t_1 = t_2 \text{ or } t_1 \rightarrow u \leftarrow t_2 \text{ for some } u)$. If $\rightarrow$ is quasi-diamond then $\rightarrow$ is uniformly normalizing and confluent.

Proving Factorization. Hindley[17] first noted that a local property implies factorization. Let $\rightarrow = \rightarrow_e \cup \rightarrow_i$. We say that $\rightarrow_i$ *strongly postpones* after $\rightarrow_e$, if

$$\text{SP}(\rightarrow_e, \rightarrow_i) : \quad \rightarrow_i \cdot \rightarrow_e \subseteq \rightarrow_e^* \cdot \rightarrow_i^= \quad (\text{Strong Postponement})$$

Lemma 6 (Hindley [17]). $\text{SP}(\rightarrow_e, \rightarrow_i)$ *implies* $\text{Fact}(\rightarrow_e, \rightarrow_i)$.

Strong postponement can rarely be used *directly*, because several interesting reductions—including β -reduction—do not satisfy it. However, it is at the heart of Takahashi’s method [31] to prove head factorization of $\rightarrow_\beta$, via the following immediate property that can be used also to prove other factorizations (see [3]).

Property 7 (Characterization of factorization) *Factorization* $\text{Fact}(\rightarrow_e, \rightarrow_i)$ *holds if and only if there is a reduction* $\rightarrow_i$ *such that* $\rightarrow_i^* = \rightarrow_i^=$ *and* $\text{SP}(\rightarrow_e, \rightarrow_i)$.

The core of Takahashi’s method [31] is to introduce a relation $\Rightarrow_i$, called *internal parallel reduction*, which verifies the hypotheses above. We will follow a similar path in Section 6.1, to prove *least-level* factorization.

Compound systems: proving factorization in a modular way. In this paper, we will consider compound systems that are obtained by extending the λ -calculus with extra rules to model advanced features.

In an abstract setting, let us consider a rewrite system $(A, \rightarrow)$ where $\rightarrow = \rightarrow_\xi \cup \rightarrow_\rho$. Under which condition $\rightarrow$ admits factorization, assuming that both $\rightarrow_\xi$ and $\rightarrow_\rho$ do? To deal with this question, a technique for proving factorization for *compound systems* in a *modular* way has been introduced in [4]. The approach can be seen as an analogous for factorization of the classical technique for confluence based on Hindley-Rosen lemma [5]: if $\rightarrow_\xi, \rightarrow_\rho$ are e-factorizing reductions, their union $\rightarrow_\xi \cup \rightarrow_\rho$ also is, provided that two *local* conditions of commutation hold.

Lemma 8 (Modular factorization [4]). *Let* $\rightarrow_\xi = \rightarrow_e^\xi \cup \rightarrow_i^\xi$ *and* $\rightarrow_\rho = \rightarrow_e^\rho \cup \rightarrow_i^\rho$ *be e-factorizing relations. Let* $\rightarrow_e := \rightarrow_e^\xi \cup \rightarrow_e^\rho$, *and* $\rightarrow_i := \rightarrow_i^\xi \cup \rightarrow_i^\rho$. *The union* $\rightarrow_\xi \cup \rightarrow_\rho$ *fulfills factorization* $\text{Fact}(\rightarrow_e, \rightarrow_i)$ *if the following swaps hold*

$$\rightarrow_i^\xi \cdot \rightarrow_e^\rho \subseteq \rightarrow_e^\rho \cdot \rightarrow_i^{\xi*} \quad \text{and} \quad \rightarrow_i^\rho \cdot \rightarrow_e^\xi \subseteq \rightarrow_e^\xi \cdot \rightarrow_i^{\rho*} \quad (\text{Linear Swaps})$$

The subtlety here is to set $\rightarrow_e^\xi$ and $\rightarrow_e^\rho$ so that $\rightarrow_e = \rightarrow_e^\xi \cup \rightarrow_e^\rho$. As already shown in Section 1, when dealing with normalizing strategies one needs extra care.

3 λ -calculi: CbN, CbV, and bang

We present here a generic syntax for λ -calculi, possibly containing operators. All the variants of the λ -calculus we shall study use this language. We assume some familiarity with the λ -calculus, and refer to [5,18] for details.

Given a countable set $\mathbf{Var}$ of variables, denoted by $x, y, z, \dots$, *terms* and *values* (whose sets are denoted by $\Lambda_{\mathcal{O}}$ and $\mathbf{Val}$, respectively) are defined as follows:

$$t, s, r ::= v \mid ts \mid \mathbf{o}(t_1, \dots, t_k) \quad \text{Terms: } \Lambda_{\mathcal{O}} \quad v ::= x \mid \lambda x.t \quad \text{Values: } \mathbf{Val}$$

where $\mathbf{o}$ ranges over a set $\mathcal{O}$ of function symbols called *operators*, each one with its own arity $k \in \mathbb{N}$. If the operators are $\mathbf{o}_1, \dots, \mathbf{o}_n$, the set of terms is indicated as $\Lambda_{\mathbf{o}_1 \dots \mathbf{o}_n}$. When the set $\mathcal{O}$ of operators is empty, the calculus is called *pure*, and the sets of terms is denoted by Λ . Otherwise, the calculus is *applied*.

Terms are identified up to renaming of bound variables, where abstraction is the only binder. We denote by $t\{s/x\}$ the capture-avoiding substitution of s for the free occurrences of x in t . *Contexts* (with exactly one hole $\langle \cdot \rangle$) are generated by the grammar below, and $\mathbf{c}\langle t \rangle$ stands for the term obtained from the context $\mathbf{c}$ by replacing the hole with the term t (possibly capturing free variables).

$$\mathbf{c} ::= \langle \cdot \rangle \mid t\mathbf{c} \mid \mathbf{c}t \mid \lambda x.\mathbf{c} \mid \mathbf{o}(t_1, \dots, \mathbf{c}, \dots, t_k) \quad \text{Contexts: } \mathcal{C}$$

Let ρ be a binary relation on $\Lambda_{\mathcal{O}}$; we call it ρ -rule and denote it also by $\mapsto_{\rho}$, writing $t \mapsto_{\rho} t'$ rather than $(t, t') \in \rho$. A ρ -reduction step $\rightarrow_{\rho}$ is the contextual closure of ρ . Explicitly, $t \rightarrow_{\rho} t'$ holds if $t = \mathbf{c}\langle r \rangle$ and $t' = \mathbf{c}\langle r' \rangle$ for some context $\mathbf{c}$ with $r \mapsto_{\rho} r'$. The term r is called a ρ -redex. The set of ρ -redexes is denoted by $\mathcal{R}_{\rho}$.

Given a set of rules $\mathbf{Rules}$, the relation $\rightarrow = \bigcup_{\rho} \rightarrow_{\rho}$ ($\rho \in \mathbf{Rules}$) can equivalently be defined as the contextual closure of $\mapsto = \bigcup_{\rho} \mapsto_{\rho}$.

3.1 Call-by-Name and Call-by-Value λ -calculi

Pure CbN and Pure CbV λ -calculi. The *pure call-by-name* (CbN for short) λ -calculus [5,18] is $(\Lambda, \rightarrow_{\beta})$, the set of terms Λ together with the β -reduction $\rightarrow_{\beta}$, defined as the contextual closure of the usual β -rule, which we recall in (1) below.

The *pure call-by-value* (CbV for short) λ -calculus [27] is the set Λ endowed with the reduction $\rightarrow_{\beta_v}$, defined as the contextual closure of the β_v -rule in (2).

$$\text{CbN: } (\lambda x.t)s \mapsto_{\beta} t\{s/x\} \quad (1) \quad \text{CbV: } (\lambda x.t)v \mapsto_{\beta_v} t\{v/x\} \quad \text{with } v \in \mathbf{Val} \quad (2)$$

CbN and CbV λ -calculi. A CbN (resp. CbV) λ -calculus is the set of terms endowed with a reduction $\rightarrow$ which extends $\rightarrow_{\beta}$ (resp. $\rightarrow_{\beta_v}$).

In particular, the *applied* setting with operators (when $\mathcal{O} \neq \emptyset$) models in the λ -calculus richer computational features, allowing $\mathbf{o}$ -reductions as the contextual closure of $\mathbf{o}$ -rules of the form $\mathbf{o}(t_1, \dots, t_k) \mapsto_{\mathbf{o}} s$.

Example 9 (Non-deterministic λ -calculus). Let $\mathcal{O} = \{\oplus\}$ where $\oplus$ is a binary operator; let $\rightarrow_{\oplus}$ be the contextual closure of the (non-deterministic) rule below:

$$\oplus(t_1, t_2) \mapsto_{\oplus} t_1 \quad \text{and} \quad \oplus(t_1, t_2) \mapsto_{\oplus} t_2$$

The *non-deterministic CbN λ -calculus* $\Lambda_{\oplus}^{\text{cbn}} = (\Lambda_{\oplus}, \rightarrow_{\beta\oplus})$ is the set $\Lambda_{\oplus}$ with the reduction $\rightarrow_{\beta\oplus} = \rightarrow_{\beta} \cup \rightarrow_{\oplus}$. The *non-deterministic CbV λ -calculus* $\Lambda_{\oplus}^{\text{cbv}} = (\Lambda_{\oplus}, \rightarrow_{\beta_v\oplus})$ is the set $\Lambda_{\oplus}$ with the reduction $\rightarrow_{\beta_v\oplus} = \rightarrow_{\beta_v} \cup \rightarrow_{\oplus}$.

3.2 Bang calculi

The bang calculus [12,16] is a variant of the λ -calculus inspired by linear logic. An operator $!$ plays the role of a marker for duplicability and discardability. Here we allow also the presence of operators other than $!$, ranging over a set $\mathcal{O}$. So, terms and contexts of the bang calculus (denoted by capital letters) are:

$$\begin{aligned} T, S, R &::= x \mid \lambda x.T \mid TS \mid !T \mid \mathbf{o}(T_1, \dots, T_k) & \text{Terms: } \Lambda_{!}\mathcal{O} \\ \mathbf{C} &::= \langle \cdot \rangle \mid \lambda x.\mathbf{C} \mid T\mathbf{C} \mid \mathbf{C}T \mid !\mathbf{C} \mid \mathbf{o}(T_1, \dots, \mathbf{C}, \dots, T_k) & \text{Contexts: } \mathcal{C}_! \end{aligned}$$

Terms of the form $!T$ are called *boxes* and their set is denoted by $\Lambda_{!}\mathcal{O}$. When there are no operators other than $!$ (*i.e.* $\mathcal{O} = \emptyset$), the sets of terms, boxes and contexts are denoted by $\Lambda_!$, $!\Lambda_!$ and $\mathcal{C}_!$, respectively. This syntax can be expressed in the one of Section 3, where $!$ is an unary operator called *bang*.

The pure bang calculus. The *pure* bang calculus $(\Lambda_!, \rightarrow_{\beta_!})$ is the set of terms $\Lambda_!$ endowed with reduction $\rightarrow_{\beta_!}$, the closure under contexts in $\mathcal{C}_!$ of the $\beta_!$ -rule:

$$(\lambda x.T)!S \mapsto_{\beta_!} T\{S/x\} \quad (3)$$

Intuitively, in the bang calculus the bang-operator $!$ marks the only terms that can be erased and duplicated. Indeed, a β -like *redex* $(\lambda x.T)S$ can be fired by $\mapsto_{\beta_!}$ only when its argument S is a box, *i.e.* $S = !R$: if it is so, the content R of the box S (and not S itself) replaces any free occurrence of x in T .³

A proof of confluence of $\beta_!$ -reduction $\rightarrow_{\beta_!}$ is in [16].

Notation 10 *We use the following notations to denote some notable terms.*

$$\iota := \lambda x.x \quad \delta := \lambda x.xx \quad I := \lambda x.!x \quad \Delta := \lambda x.x!x.$$

Remark 11 (Notable terms). The term $I = \lambda x.!x$ plays the role of the identity in the bang calculus: $I!T \rightarrow_{\beta_!} !(x\{T/x\}) = !T$ for any term T . Instead, the term $\iota = \lambda x.x$, when applied to a box $!T$, opens the box, *i.e.* returns its content T : $\iota!T \rightarrow_{\beta_!} x\{T/x\} = T$. Finally, $\Delta! \Delta \rightarrow_{\beta_!} \Delta! \Delta \rightarrow_{\beta_!} \dots$ is a diverging term.

³ Syntax and reduction rule of the bang calculus follow [16], which is slightly different from [12]. Unlike [16] (but akin to [29]), here we do not use ι (aka **der**) as a primitive, since ι and its associated rule $\mapsto_d$ can be simulated, see Remark 11 and (4).

A bang calculus. A *bang calculus* $(\Lambda_{!O}, \rightarrow)$ is the set $\Lambda_{!O}$ of terms endowed with a reduction $\rightarrow$ which extends $\rightarrow_{\beta_l}$. In this paper we shall consider calculi where $\rightarrow$ contains $\rightarrow_{\beta_l}$ and **o**-reductions $\rightarrow_{\mathbf{o}}$ ($\mathbf{o} \in O$) defined from **o**-rules of the form $\mathbf{o}(T_1, \dots, T_k) \mapsto_{\mathbf{o}} S$, and possibly other rules. So, $\rightarrow = \bigcup_{\rho} \rightarrow_{\rho}$ ($\rho \in \text{Rules}$), with $\text{Rules} \supseteq \{! \beta, \mathbf{o} \mid \mathbf{o} \in O\}$. We set $\rightarrow_O = \bigcup_{\mathbf{o} \in O} \rightarrow_{\mathbf{o}}$.

3.3 CbN and CbV translations into the bang calculus

Our motivation to study the bang calculus is to have a general framework where both CbN [5] and CbV [27] λ -calculi can be embedded, via two distinct translations. Here we show how these translations work. We extend the simulation results in [16,29,8] for the pure case to the case with operators (Proposition 13).

Following [8], the CbV translation defined here differs from [16,29] in the application case. Section 5 will show why this optimization is crucial.

CbN and *CbV translations* are two maps $(\cdot)^n: \Lambda_O \rightarrow \Lambda_{!O}$ and $(\cdot)^v: \Lambda_O \rightarrow \Lambda_{!O}$, respectively, translating terms of the λ -calculus into terms of the bang calculus:

$$\begin{aligned} x^n &:= x & (\lambda x.t)^n &:= \lambda x.t^n & (\mathbf{o}(t_1, \dots, t_k))^n &:= \mathbf{o}(t_1^n, \dots, t_k^n) & (ts)^n &:= t^n !s^n; \\ x^v &:= !x & (\lambda x.t)^v &:= !(\lambda x.t^v) & (\mathbf{o}(t_1, \dots, t_k))^v &:= \mathbf{o}(t_1^v, \dots, t_k^v) & (ts)^v &:= \begin{cases} T s^v & \text{if } t^v = !T \\ (\iota t^v) s^v & \text{otherwise.} \end{cases} \end{aligned}$$

Example 12. Consider the λ -term $\omega := \delta\delta$: then, $\delta^n = \Delta$, $\delta^v = !\Delta$ and $\omega^n = \Delta !\Delta = \omega^v$ (δ and Δ are defined in Notation 10). The λ -term ω is diverging in CbN and CbV λ -calculi, and so is $\omega^n = \omega^v$ in the bang calculus, see Remark 11.

For any term $t \in \Lambda_O$, t^n and t^v are just different decorations of t by means of the bang-operator $!$ (recall that $\iota = \lambda x.x$). The translation $(\cdot)^n$ puts the argument of any application into a box: in CbN any term is duplicable or discardable. On the other hand, only *values* (i.e. abstractions and variables) are translated by $(\cdot)^v$ into boxes, as they are the only terms duplicable or discardable in CbV.

As in [16,29], we prove that the CbN translation $(\cdot)^n$ (resp. CbV translation $(\cdot)^v$) from the pure CbN (resp. CbV) λ -calculus into the bang calculus is *sound* and *complete*: it maps β -reductions (resp. β_v -reductions) of the λ -calculus into β_l -reductions of the bang calculus, and conversely β_l -reductions — when restricted to the image of the translation — into β -reductions (resp. β_v -reductions). The same holds if we consider any **o**-reduction for operators.

In the simulation, $\rightarrow_d$ denotes the contextual closure of the rule:

$$\iota !T \mapsto_d T \quad (\text{this is nothing but } (\lambda x.x)!T \mapsto_{\beta_l} T) \quad (4)$$

Clearly, $\rightarrow_d \subseteq \rightarrow_{\beta_l}$ (Remark 11). We write $T \rightarrow_d S$ if $T \rightarrow_d^* S$ and S is **d**-normal.

Proposition 13 (Simulation of CbN and CbV). *Let $t \in \Lambda_O$ and $\mathbf{o} \in O$.*

1. CbN soundness: *If $t \rightarrow_{\beta} t'$ then $t^n \rightarrow_{\beta_l} t'^n$. If $t \rightarrow_{\mathbf{o}} t'$ then $t^n \rightarrow_{\mathbf{o}} t'^n$.*
 CbN completeness: *If $t^n \rightarrow_{\beta_l} S$ then $S = t'^n$ and $t \rightarrow_{\beta} t'$, for some $t' \in \Lambda_O$.
 If $t^n \rightarrow_{\mathbf{o}} S$ then $S = t'^n$ and $t \rightarrow_{\mathbf{o}} t'$, for some $t' \in \Lambda_O$.*

2. CbV soundness: If $t \rightarrow_{\beta_v} t'$ then $t^v \rightarrow_{\beta_! \rightarrow_d} t'^v$ with t'^v *d-normal*. If $t \rightarrow_{\mathbf{o}} t'$ then $t^v \rightarrow_{\mathbf{o} \rightarrow_d} t'^v$ with t'^v *d-normal*.
 CbV completeness: If $t^v \rightarrow_{\beta_! \rightarrow_d} S$ then $t^v \rightarrow_{\beta_! \rightarrow_d} S$ with $S = t'^v$ and $t \rightarrow_{\beta_v} t'$, for some $t' \in \Lambda_{\mathcal{O}}$. If $t^v \rightarrow_{\mathbf{o} \rightarrow_d} S$ then $t^v \rightarrow_{\mathbf{o} \rightarrow_d} S$ with $S = t'^v$ and $t \rightarrow_{\mathbf{o}} t'$, for some $t' \in \Lambda_{\mathcal{O}}$.

Example 14. Let $t = ((\lambda z.z)x)y$ and $t' = xy$. Then $t \rightarrow_{\beta} t'$ while $t^n = ((\lambda z.z)!x)!y \rightarrow_{\beta_!} x!y = t'^n$; and $t \rightarrow_{\beta_v} t'$ while $t^v = (\iota((\lambda z.z)!x))!y \rightarrow_{\beta_!} (\iota!x)!y \rightarrow_d x!y = t'^v$.

4 The least-level strategy

The bang calculus $\Lambda_!$ has a natural normalizing strategy, issued by linear logic (where it was first used in [9]), namely the *least-level reduction*. It reduces only redexes at *least level*, where the *level* of a redex R in a term T is the number of boxes $!$ in which R is nested.

Least-level reduction is easily extended to a general bang calculus $(\Lambda_{! \mathcal{O}}, \rightarrow)$. The level of a redex R is then the number of boxes $!$ and operators $\mathbf{o}$ in which R is nested; intuitively, least-level reduction fires a redex which is *minimally nested*.

Below, we formalize the reduction in a way that is independent of the specific shape of the redexes, and even of specific definition of level one chooses. The interest of least-level reduction is in the properties it satisfies. All our developments will rely on such properties, rather than the specific definition of least level.

In this section, $\rightarrow = \bigcup_{\rho} \rightarrow_{\rho}$, for $\rho \in \text{Rules}$ a set of rules. We write $\mathcal{R} = \bigcup_{\rho} \mathcal{R}_{\rho}$ for the set of *all* redexes.

4.1 Least-level reduction in bang calculi

The *level* of an occurrence of redex R in a term T is a measure of its depth. Formally, we indicate the *occurrence of a subterm* R in T with the context $\mathbf{C}$ such that $\mathbf{C}\langle R \rangle = T$. Its level then corresponds to the *level* $\ell(\mathbf{C})$ of the hole in $\mathbf{C}$. The definition of *level* in a bang calculus $\Lambda_{! \mathcal{O}}$ is formalized as follows.

$$\begin{aligned} \ell(\langle \cdot \rangle) &= 0 & \ell(\lambda x. \mathbf{C}) &= \ell(\mathbf{C}) & \ell(\mathbf{C}T) &= \ell(\mathbf{C}) & \ell(T\mathbf{C}) &= \ell(\mathbf{C}) \\ \ell(!\mathbf{C}) &= \ell(\mathbf{C}) + 1 & \ell(\mathbf{o}(\dots, \mathbf{C}, \dots)) &= \ell(\mathbf{C}) + 1 \end{aligned} \quad (5)$$

Note that the level increases by 1 in the scope of $!$, and of any operator $\mathbf{o} \in \mathcal{O}$.

A reduction step $T \rightarrow_{\rho} S$ is *at level* k if it fires a ρ -redex at level k ; it is *least-level* if it reduces a redex whose level is minimal.

The *least level* $\ell\ell(T)$ of a term T expresses the minimal level of any occurrence of redexes in T ; if no redex is in T , we set $\ell\ell(T) = \infty$. Formally

Definition 15 (Least-level reduction). Let $\rightarrow = \bigcup_{\rho} \rightarrow_{\rho}$ ($\rho \in \text{Rules}$) and $\mathcal{R} = \bigcup_{\rho} \mathcal{R}_{\rho}$ the set of redexes. Given a function $\ell(-)$ from contexts into $\mathbb{N}$:

– The least level of a term T is defined as

$$\ell(T) := \inf\{\ell(\mathbf{C}) \mid T = \mathbf{C}\langle R \rangle \text{ for some } R \in \mathcal{R}\} \in (\mathbb{N} \cup \{\infty\}).^4 \quad (6)$$

- A ρ -reduction step $T \rightarrow_\rho S$ is:
 1. at level k , written $T \rightarrow_{\rho:k} S$, if $T := \mathbf{C}\langle R \rangle \rightarrow_\rho \mathbf{C}\langle R' \rangle =: S$ and $\ell(\mathbf{C}) = k$.
 2. least-level, written $T \xrightarrow{\rho}_\rho S$, if $T \rightarrow_{\rho:k} S$ and $k = \ell(T)$.
 3. internal, written $T \xrightarrow{\rho}_\rho S$, if $T \rightarrow_{\rho:k} S$ and $k > \ell(T)$.
- Least-level reduction is $\xrightarrow{\rho} = \bigcup_\rho \xrightarrow{\rho}_\rho$ ($\rho \in \text{Rules}$).
- Internal reduction is $\xrightarrow{\rho} = \bigcup_\rho \xrightarrow{\rho}_\rho$ ($\rho \in \text{Rules}$).

Note that $\rightarrow = \xrightarrow{\rho} \cup \xrightarrow{\rho}$. Note also that the definition of least level of a term depends on the set $\mathcal{R} = \bigcup_\rho \mathcal{R}_\rho$ of redexes associated with $\rightarrow$.⁵

Normal Forms. It is immediate that $\xrightarrow{\rho} \subseteq \rightarrow$ is a *strategy* for $\rightarrow$. Indeed, $\xrightarrow{\rho}$ and $\rightarrow$ have the *same normal forms* because if M has a $\rightarrow$ -redex, it has a redex at least-level, i.e. it has a $\xrightarrow{\rho}$ -redex.

Remark 16 (Least level of normal forms). Note that $\ell(T) = \infty$ if and only if T is $\rightarrow$ -normal, because $\ell(\mathbf{C}) \in \mathbb{N}$ for all contexts $\mathbf{C}$.

A good least-level reduction. The beauty of least-level reduction for the bang calculus, is that it satisfies some elegant properties, which allow for neat proofs, in particular monotonicity and internal invariance (in Definition 17). The developments in the rest of the paper rely on such properties, and in fact will apply to any calculus whose reduction $\rightarrow$ has the properties described below.

Definition 17 (Good least-level). A reduction $\rightarrow$ has a good least-level if:

1. Monotonicity: $T \rightarrow S$ implies $\ell(T) \leq \ell(S)$.
2. Internal invariance: $T \xrightarrow{\rho}_\rho S$ implies $\ell(T) = \ell(S)$.

Point 1. states that no step can decrease the least level of a term. Point 2. says that internal steps cannot change the least level of a term. Therefore, only least-level steps may increase the least level. Together, they imply persistence: only least-level steps can approach normal forms.

Property 18 (Persistence) If $\rightarrow$ has a good least-level, then $T \xrightarrow{\rho}_\rho S$ implies S is not $\rightarrow$ -normal.

The pure bang calculus $(\Lambda_!, \rightarrow_{\beta_!})$ has a good least-level; the same holds true when extending the reduction with operators.

Proposition 19 (Good least-level of bang calculi). Given $\Lambda_{!O}$, let $\rightarrow = \rightarrow_{\beta_!} \cup \rightarrow_O$, where each $\mathbf{o} \in O$ has a redex of shape $\mathbf{o}(P_1, \dots, P_k)$. The reduction $\rightarrow$ has a good least-level.

⁴ Recall that $\inf \emptyset = \infty$, when $\emptyset$ is seen as the empty subset of $\mathbb{N}$ with the usual order.

⁵ We should write $\ell_{\mathcal{R}}(T)$, $\xrightarrow{\rho}_{\mathcal{R}}$ and $\xrightarrow{\rho}_{\mathcal{R}}$, but we avoid it for the sake of readability.

4.2 Least-level for a bang calculus: examples.

Let us examine more closely least-level reduction for a bang calculus $(\Lambda_{!O}, \rightarrow)$. For concreteness, we consider $\text{Rules} = \{\beta_!, \mathbf{o} \mid \mathbf{o} \in O\}$, hence the set of redexes is $\mathcal{R} = \mathcal{R}_{\beta_!} \cup \mathcal{R}_O$, where $\mathcal{R}_O$ is a set of terms of shape $\mathbf{o}(P_1, \dots, P_k)$.

We observe that the least level $\ell\ell(T)$ of a term $T \in \Lambda_{!O}$ can easily be defined in a direct way, inductively.

- $\ell\ell(T) = 0$ if $T \in \mathcal{R} = \mathcal{R}_{\beta_!} \cup \mathcal{R}_O$,
- otherwise:

$$\ell\ell(x) = \infty \quad \ell\ell(\lambda x.T) = \ell\ell(T) \quad \ell\ell(!T) = \ell\ell(T) + 1 \quad \ell\ell(TS) = \min\{\ell\ell(T), \ell\ell(S)\}$$

Example 20 (Least level of a term). Let $R \in \mathcal{R}_{\beta_!}$. If $T_0 := R(!R)$, then $\ell\ell(T_0) = 0$. If $T_1 := x!R$ then $\ell\ell(T_1) = 1$. If $T_2 := \mathbf{o}(x, y)!R$ then $\ell\ell(T_2) = 0$, as $\mathbf{o}(x, y) \in \mathcal{R}_O$.

Intuitively, least-level reduction fires a redex that is *minimally nested*, where a redex is any subterm whose form is in $\mathcal{R} = \mathcal{R}_{\beta_!} \cup \mathcal{R}_O$. Note that least-level reduction can choose to fire one among possibly *several* redexes at minimal level.

Example 21. Let us revisit Example 20 with $R = (\lambda x.x)!z \in \mathcal{R}_{\beta_!}$ ($R \mapsto_{\beta_!} z$). Then $T_1 := x!R \xrightarrow{\beta_!} x!z$ but $T_0 := R(!R) \not\xrightarrow{\beta_!} R!z$ and similarly $T_2 := \mathbf{o}(x, y)!R \not\xrightarrow{\beta_!} \mathbf{o}(x, y)!z$. Observe also that $\mathbf{o}(x, \underline{R}) \not\xrightarrow{\beta_!} \mathbf{o}(x, z)$.

Example 22. Let $R = (\lambda x.x)!z$. Two least-level steps are possible in $(\lambda z.R)!R$: $(\lambda z.R)!R \xrightarrow{\beta_!} (\lambda x.x)!R$, and $(\lambda z.\underline{R})!R \xrightarrow{\beta_!} (\lambda z.z)!R$. But $(\lambda z.R)!R \not\xrightarrow{\beta_!} (\lambda z.R)!z$.

4.3 Least-level for CbN and CbV λ -calculi

The definition of least-level reduction in Section 4.1 is independent from the specific notion of level that is chosen, and also from the specific calculus. The idea is that the reduction strategy persistently fires a redex at minimal level, once such a notion is set.

Least-level reduction can indeed be defined also for the CbN and CbV λ -calculi, given an opportune definition of level. In CbN, we count the number of nested arguments and operators containing the occurrence of redex. In CbV, we count the number of nested operators and *unapplied* abstractions containing the redex, where an abstraction is unapplied if it is not the right-hand side of an application. Formally, an occurrence of redex is identified by a context (as explained in Section 4.1), and we define the following $\ell^{\text{CbN}}(\cdot)$ and $\ell^{\text{CbV}}(\cdot)$ functions from $\mathcal{C}$ to $\mathbb{N}$, the *level* in CbN and CbV λ -calculi.

$$\begin{aligned}
 \ell^{\text{CbN}}(\langle \cdot \rangle) &= 0 & \ell^{\text{CbV}}(\langle \cdot \rangle) &= 0 \\
 \ell^{\text{CbN}}(\lambda x.\mathbf{c}) &= \ell^{\text{CbN}}(\mathbf{c}) & \ell^{\text{CbV}}(\lambda x.\mathbf{c}) &= \ell^{\text{CbV}}(\mathbf{c}) + 1 \\
 \ell^{\text{CbN}}(\mathbf{c}t) &= \ell^{\text{CbN}}(\mathbf{c}) & \ell^{\text{CbV}}(\mathbf{c}t) &= \begin{cases} \ell^{\text{CbV}}(\mathbf{c}') & \text{if } \mathbf{c} = \lambda x.\mathbf{c}' \\ \ell^{\text{CbV}}(\mathbf{c}) & \text{otherwise} \end{cases} \\
 \ell^{\text{CbN}}(t\mathbf{c}) &= \ell^{\text{CbN}}(\mathbf{c}) + 1 & \ell^{\text{CbV}}(t\mathbf{c}) &= \ell^{\text{CbV}}(\mathbf{c}) \\
 \ell^{\text{CbN}}(\mathbf{o}(\dots, \mathbf{c}, \dots)) &= \ell^{\text{CbN}}(\mathbf{c}) + 1 & \ell^{\text{CbV}}(\mathbf{o}(\dots, \mathbf{c}, \dots)) &= \ell^{\text{CbV}}(\mathbf{c}) + 1
 \end{aligned}$$

In both CbN and CbV λ -calculi, the *least level* of a term (denoted by $\ell\ell^{\text{CbN}}(\cdot)$ and $\ell\ell^{\text{CbV}}(\cdot)$) and *least-level* and *internal* reductions are given by Definition 15 (replace $\ell(\cdot)$ with $\ell^{\text{CbN}}(\cdot)$ for CbN and $\ell^{\text{CbV}}(\cdot)$ for CbV).

In Section 5 we will see that the definitions of CbN and CbV least level are not arbitrary, but induced by the CbN and CbV translations defined in Section 3.3.

5 Embedding of CbN and CbV by level

Here we refine the analysis of the CbN and CbV translations given in Section 3.3, by showing two new results: translations preserve normal forms (Proposition 23) and least-level (Proposition 26), back and forth. This way, to obtain least-level *factorization* or least-level *normalization* results, it suffices to prove them in the bang calculus. The translation transfers the results into the CbN and CbV λ -calculi (Theorem 27). We use here the expression “translate” in a strong sense: the results for CbN and CbV λ -calculi are obtained from the corresponding results in the bang calculus almost for free, just via CbN and CbV translations.

Preservation of normal forms. The targets of the CbN translation $(\cdot)^n$ and CbV translation $(\cdot)^v$ into the bang calculus can be *characterized syntactically*. A fine analysis of these fragments of the bang calculus (see [?] for details) proves that both CbN and CbV translations preserve normal forms, back and forth.

Proposition 23 (Preservation of normal forms). *Let $t, s \in \Lambda$ and $\mathbf{o} \in \mathcal{O}$.*

1. CbN: t is β -normal iff t^n is β_1 -normal; t is $\mathbf{o}$ -normal iff t^n is $\mathbf{o}$ -normal.
2. CbV: t is β_v -normal iff t^v is β_1 -normal; t is $\mathbf{o}$ -normal iff t^v is $\mathbf{o}$ -normal.

By Remark 16, Proposition 23 can be seen as the fact that CbN and CbV translations preserve the least-level of a term, back and forth, when the least-level is infinite. Actually, this holds more in general for any value of the least-level.

Preservation of levels. We aim to show that least-level steps in CbN and CbV λ -calculi correspond to least-level steps in the bang calculus, back and forth, via CbN and CbV translations respectively (Proposition 26). This result is subtle, one of the main technical contributions of this paper.

First, we extend the definition of translations to contexts. The *CbN* and *CbV translations for contexts* are two functions $(\cdot)^n: \mathcal{C} \rightarrow \mathcal{C}_!$ and $(\cdot)^v: \mathcal{C} \rightarrow \mathcal{C}_!$, respectively, mapping contexts of the λ -calculus into contexts of the bang calculus:

$$\begin{aligned}
 \langle \cdot \rangle^n &= \langle \cdot \rangle & \langle \cdot \rangle^v &= \langle \cdot \rangle \\
 (\lambda x. \mathbf{c})^n &= \lambda x. \mathbf{c}^n & (\lambda x. \mathbf{c})^v &= !(\lambda x. \mathbf{c}^v) \\
 (\mathbf{o}(t_1, \dots, \mathbf{c}, \dots, t_k))^n &= \mathbf{o}(t_1^n, \dots, \mathbf{c}^n, \dots, t_k^n) & (\mathbf{o}(t_1, \dots, \mathbf{c}, \dots, t_k))^v &= \mathbf{o}(t_1^v, \dots, \mathbf{c}^v, \dots, t_k^v) \\
 (\mathbf{c}t)^n &= \mathbf{c}^n ! (t^n) & (\mathbf{c}t)^v &= \begin{cases} \mathbf{C} t^v & \text{if } \mathbf{c}^v = !\mathbf{C} \\ (\iota \mathbf{c}^v) t^v & \text{otherwise} \end{cases} \\
 (t\mathbf{c})^n &= t^n ! (\mathbf{c}^n); & (t\mathbf{c})^v &= \begin{cases} T \mathbf{c}^v & \text{if } t^v = !T \\ (\iota t^v) \mathbf{c}^v & \text{otherwise.} \end{cases}
 \end{aligned}$$

Note that CbN (resp. CbV) level of a context defined in Section 4.3 increases by 1 whenever the CbN (resp. CbV) translation for contexts add !. Thus, CbN and CbV translations preserve, back and forth, the level of a redex and the least-level of a term. Said differently, the level for CbN and CbV is defined in Section 4.3 so as to enable the preservation of level via CbN and CbV translations.

Lemma 24 (Preservation of level via CbN translation).

1. For contexts: For any context $\mathbf{c} \in \mathcal{C}$, one has $\ell^{\text{CbN}}(\mathbf{c}) = \ell(\mathbf{c}^n)$.
2. For reduction: For any term $t \in \Lambda_{\mathcal{O}}: t \rightarrow_{\beta:k} s$ if and only if $t^n \rightarrow_{\beta_1:k} s^n$; and $t \rightarrow_{\mathbf{o}:k} s$ if and only if $t^n \rightarrow_{\mathbf{o}:k} s^n$, for any $\mathbf{o} \in \mathcal{O}$.
3. For least-level of a term: For any term $t \in \Lambda_{\mathcal{O}}$, one has $\ell\ell^{\text{CbN}}(t) = \ell\ell(t^n)$.

Lemma 25 (Preservation of level via CbV translation).

1. For contexts: For any context $\mathbf{c} \in \mathcal{C}$, one has $\ell^{\text{CbV}}(\mathbf{c}) = \ell(\mathbf{c}^v)$.
2. For reduction: For any term $t \in \Lambda_{\mathcal{O}}: t \rightarrow_{\beta_v:k} s$ if and only if $t^v \rightarrow_{\beta_1:k} \rightarrow_{\mathbf{d}:k}^{\overline{\overline{}}} s^v$; and $t \rightarrow_{\mathbf{o}:k} s$ if and only if $t^v \rightarrow_{\mathbf{o}:k} \rightarrow_{\mathbf{d}:k}^{\overline{\overline{}}} s^v$, for any $\mathbf{o} \in \mathcal{O}$.
3. For least-level of a term: For any term $t \in \Lambda_{\mathcal{O}}$, one has $\ell\ell^{\text{CbV}}(t) = \ell\ell(t^v)$.

From the two lemmas above it follows that CbN and CbV translations preserve least-level and internal reductions, back and forth.

Proposition 26 (Preservation of least-level and internal reductions).

Let t be a λ -term and $\mathbf{o} \in \mathcal{O}$.

1. CbN least-level: $t \xrightarrow{\text{L}}_{\beta} s$ iff $t^n \xrightarrow{\text{L}}_{\beta_1} s^n$; and $t \xrightarrow{\text{L}}_{\mathbf{o}} s$ iff $t^n \xrightarrow{\text{L}}_{\mathbf{o}} s^n$.
2. CbN internal: $t \xrightarrow{\text{L}}_{\beta} s$ iff $t^n \xrightarrow{\text{L}}_{\beta_1} s^n$; and $t \xrightarrow{\text{L}}_{\mathbf{o}} s$ iff $t^n \xrightarrow{\text{L}}_{\mathbf{o}} s^n$.
3. CbV least-level: $t \xrightarrow{\text{L}}_{\beta_v} s$ iff $t^v \xrightarrow{\text{L}}_{\beta_1} \xrightarrow{\text{L}}_{\mathbf{d}}^{\overline{\overline{}}} s^v$; and $t \xrightarrow{\text{L}}_{\mathbf{o}} s$ iff $t^v \xrightarrow{\text{L}}_{\mathbf{o}} \xrightarrow{\text{L}}_{\mathbf{d}}^{\overline{\overline{}}} s^v$.
4. CbV internal: $t \xrightarrow{\text{L}}_{\beta_v} s$ iff $t^v \xrightarrow{\text{L}}_{\beta_1} \xrightarrow{\text{L}}_{\mathbf{d}}^{\overline{\overline{}}} s^v$; and $t \xrightarrow{\text{L}}_{\mathbf{o}} s$ iff $t^v \xrightarrow{\text{L}}_{\mathbf{o}} \xrightarrow{\text{L}}_{\mathbf{d}}^{\overline{\overline{}}} s^v$.

As a consequence, least-level reduction induces factorization in CbN and CbV λ -calculi as soon as it does in the bang calculus. And, by Proposition 23, it is a normalizing strategy in CbN and CbV as soon as it is so in the bang calculus.

Theorem 27 (Factorization and normalization by translation). Let $\Lambda_{\mathcal{O}}^{\text{cbn}} = (\Lambda_{\mathcal{O}}, \rightarrow_{\beta} \cup \rightarrow_{\mathcal{O}})$ and $\Lambda_{\mathcal{O}}^{\text{cbv}} = (\Lambda_{\mathcal{O}}, \rightarrow_{\beta_v} \cup \rightarrow_{\mathcal{O}})$.

1. If $\Lambda_{1\mathcal{O}}$ admits least-level factorization $\text{Fact}(\xrightarrow{\text{L}}, \xrightarrow{\text{L}})$, then so do $\Lambda_{\mathcal{O}}^{\text{cbn}}$ and $\Lambda_{\mathcal{O}}^{\text{cbv}}$.
2. If $\Lambda_{1\mathcal{O}}$ admits least-level normalization, then so do $\Lambda_{\mathcal{O}}^{\text{cbn}}$ and $\Lambda_{\mathcal{O}}^{\text{cbv}}$.

A similar result will hold also when extending the pure calculi with a rule $\mapsto_{\rho}$ other than $\mapsto_{\mathbf{o}}$, as long as the translation preserves redexes.

Remark 28 (Preservation of least-level and of normal forms.). Preservation of normal form and least-level is delicate. For instance, it does not hold with the definition CbV translation $(\cdot)^v$ in [16,29]. There, the translation $t = rs \in \Lambda$ would be $t^v = (\iota!(r^v))s^v$ and then Proposition 23 and Proposition 26 would not hold: $\iota!(r^v)$ is a β_1 -redex in t^v (see Remark 11) and hence t^v would not be normal even though so is t , and $\ell\ell(t^v) = 0$ even though $\ell\ell^{\text{CbV}}(t) \neq 0$. This is why we defined two distinct case when defining $(\cdot)^v$ for applications, akin to [8].

6 Least-level factorization via bang calculus

We have shown that least-level factorization in a bang calculus $\Lambda_{!}\mathcal{O}$ implies least-level factorization in the corresponding CbN and CbV calculi, via forth-and-back translation. The central question now is *how to prove least-level factorization* for a bang calculus: the rest of the paper is devoted to that.

Overview. Let us overview our approach by considering $\mathcal{O} = \{\mathbf{o}\}$, and $\rightarrow = \rightarrow_{\beta_!} \cup \rightarrow_{\mathbf{o}}$. Since by definition $\xrightarrow{\text{L}} = \xrightarrow{\text{L}}_{\beta_!} \cup \xrightarrow{\text{L}}_{\mathbf{o}}$ (and $\xrightarrow{\neg\text{L}} = \xrightarrow{\neg\text{L}}_{\beta_!} \cup \xrightarrow{\neg\text{L}}_{\mathbf{o}}$), Lemma 8 states that we can *decompose* least-level factorization of $\rightarrow$ in three modules:

1. prove L-factorization of $\rightarrow_{\beta_!}$, *i.e.* $\rightarrow_{\beta_!}^* \subseteq \xrightarrow{\text{L}}_{\beta_!}^* \cdot \xrightarrow{\neg\text{L}}_{\beta_!}$
2. prove L-factorization of $\rightarrow_{\mathbf{o}}$, *i.e.* $\rightarrow_{\mathbf{o}}^* \subseteq \xrightarrow{\text{L}}_{\mathbf{o}}^* \cdot \xrightarrow{\neg\text{L}}_{\mathbf{o}}$
3. prove the two linear swaps of Lemma 8.

Please note that the least level for both $\xrightarrow{\text{L}}_{\beta_!}$ and $\xrightarrow{\text{L}}_{\mathbf{o}}$ is defined with respect to the redexes $\mathcal{R} = \mathcal{R}_{\beta_!} \cup \mathcal{R}_{\mathbf{o}}$, so to have $\xrightarrow{\text{L}} = \xrightarrow{\text{L}}_{\beta_!} \cup \xrightarrow{\text{L}}_{\mathbf{o}}$. This addresses the issue we mentioned in Example 1.

Clearly, points 2. and 3. depend on the specific rule $\mapsto_{\mathbf{o}}$. However, the beauty of a modular approach is that point 1. can be established in general: we do not need to know $\mapsto_{\mathbf{o}}$, only the shape of its redexes $\mathcal{R}_{\mathbf{o}}$. In Section 6.1 we provide a general result of L-factorization for $\rightarrow_{\beta_!}$ (Theorem 29). In fact, we shall show a bit more: the way of decomposing the study of factorization that we have sketched, can be applied to study least-level factorization of any reduction $\rightarrow = \rightarrow_{\beta_!} \cup \rightarrow_{\rho}$, as long as $\rightarrow$ has a good least-level.

Once (1.) is established (once and for all), to prove factorization of a reduction $\rightarrow_{\beta_!} \cup \rightarrow_{\mathbf{o}}$ we are only left with (2.) and (3.). In Section 6.3 we show that the proof of the two linear swaps can be reduced to a single, simple test, involving only the $\mapsto_{\mathbf{o}}$ step (Proposition 34). In Section 7, we will illustrate how all elements play together on a concrete case, applying them to non-deterministic λ -calculi.

6.1 Factorization of $\rightarrow_{\beta_!}$ in a bang calculus

We prove that $\rightarrow_{\beta_!}$ -reduction *factorizes* via least-level reduction (Theorem 29). The result holds for a definition of $\xrightarrow{\text{L}}_{\beta_!}$ (as in Section 4) where the set of redexes $\mathcal{R}$ is $\mathcal{R}_{\beta_!} \cup \mathcal{R}_{\mathbf{o}}$ —this generalization has essentially no cost, and allows us to use Theorem 29 as a module in the factorization of a larger reduction.

We prove factorization via Takahashi's Parallel Reduction method [31]. We define a reflexive reduction $\Xi_{\text{L}\beta_!}$ (called parallel internal $\beta_!$ -reduction) which satisfies the conditions of Property 7, *i.e.* $\Xi_{\text{L}\beta_!}^* = \xrightarrow{\neg\text{L}}_{\beta_!}^*$ and $\Xi_{\text{L}\beta_!} \cdot \xrightarrow{\text{L}}_{\beta_!} \subseteq \xrightarrow{\text{L}}_{\beta_!}^* \cdot \Xi_{\text{L}\beta_!}$.

The tricky point is to prove $\Xi_{\text{L}\beta_!} \cdot \xrightarrow{\text{L}}_{\beta_!} \subseteq \xrightarrow{\text{L}}_{\beta_!}^* \cdot \Xi_{\text{L}\beta_!}$. We adapt the proof technique in [3]. All details are in [?]. Here we just give the definition of $\Xi_{\text{L}\beta_!}$.

We first introduce $\Rightarrow_{\beta_1:n}$ (the parallel version of $\rightarrow_{\beta_1:n}$), which fires simultaneously a number of β_1 -redexes at level at least n (and $\Rightarrow_{\beta_1:\infty}$ does not reduce any β_1 -redex: $T \Rightarrow_{\beta_1:\infty} S$ implies $T = S$).

$$\frac{}{x \Rightarrow_{\beta_1:\infty} x} \quad \frac{T \Rightarrow_{\beta_1:n} T' \quad \lambda x.T \Rightarrow_{\beta_1:n} \lambda x.T'}{\lambda x.T \Rightarrow_{\beta_1:n} \lambda x.T'} \quad \frac{T \Rightarrow_{\beta_1:m} T' \quad S \Rightarrow_{\beta_1:n} S' \quad TS \Rightarrow_{\beta_1:\min\{m,n\}} T'S'}{TS \Rightarrow_{\beta_1:\min\{m,n\}} T'S'} \quad \frac{T \Rightarrow_{\beta_1:n} T'}{!T \Rightarrow_{\beta_1:n+1} !T'}$$

$$\frac{T \Rightarrow_{\beta_1:n} T' \quad S \Rightarrow_{\beta_1:m} S'}{TS \Rightarrow_{\beta_1:\min\{m,n\}} T'S'} \quad \frac{}{(\lambda x.T)!S \Rightarrow_{\beta_1:0} T'\{S'/x\}}$$

The *parallel internal β_1 -reduction* $\Rightarrow_{\beta_1}$ is the parallel version of $\rightarrow_{\beta_1}$, which fires simultaneously a number of β_1 -redexes that are not at minimal level. Formally,

$$T \Rightarrow_{\beta_1} S \quad \text{if } T \Rightarrow_{\beta_1:n} S \text{ with } n = \infty \text{ or } n > \ell\ell(T).$$

Theorem 29 (Least-level factorization of $\rightarrow_{\beta_1}$). *Assume $\rightarrow = \rightarrow_{\beta_1} \cup \rightarrow_{\rho}$ has good least-level in $\Lambda_1\mathcal{O}$. Then: $T \rightarrow_{\beta_1}^* S$ implies $T \xrightarrow{\beta_1}^* \cdot \xrightarrow{\beta_1}^* S$.*

Corollary 30 (Least-level factorization in the pure bang calculus). *In the pure bang calculus $(\Lambda_1, \rightarrow_{\beta_1})$, if $T \rightarrow_{\beta_1}^* S$ then $T \xrightarrow{\beta_1}^* \cdot \xrightarrow{\beta_1}^* S$.*

6.2 Pure calculi and least-level normalization

Least-level factorization of $\rightarrow_{\beta_1}$ implies in particular least-level factorization for $\rightarrow_{\beta}$ and $\rightarrow_{\beta_v}$. As a consequence, least-level reduction is a normalizing strategy for all three pure calculi: the bang calculus, the CbN, and the CbV λ -calculus.

The pure bang calculus. $\xrightarrow{\beta_1}$ is a *normalizing strategy* for $\rightarrow_{\beta_1}$. Indeed, it satisfies all ingredients in Lemma 4. Since we have least-level factorization (Corollary 30), same normal forms, and *persistence* (Proposition 19), $\xrightarrow{\beta_1}$ is a *complete strategy* for $\rightarrow_{\beta_1}$: If N is β_1 -normal and $M \rightarrow_{\beta_1}^* N$, then $M \xrightarrow{\beta_1}^* N$.

We already observed (Example 22) that the least-level reduction $\xrightarrow{\beta_1}$ is non-deterministic, because several redexes at least level may be available. Such non-determinism is however inessential, because $\xrightarrow{\beta_1}$ is *uniformly normalizing*.

Lemma 31 (Quasi-Diamond). *In $(\Lambda_1, \rightarrow_{\beta_1})$, the reduction $\xrightarrow{\beta_1}$ is quasi-diamond (Property 5), and therefore uniformly normalizing.*

Putting all the ingredients together, we have (by Lemma 4):

Theorem 32 (Least-level normalization). *In the pure bang calculus $\xrightarrow{\beta_1}$ is a normalizing strategy for $\rightarrow_{\beta_1}$.*

Theorem 32 means not only that if T is β_1 -normalizable then T can reach its normal form by just performing least-level steps, but also that performing *whatever* least-level steps eventually leads to the normal form, if any.

Pure CbV and CbN λ -calculi. By forth-and-back translation (Theorem 27) the least-level factorization and normalization results for the pure bang calculus immediately transfers to the CbN and CbV setting.

Theorem 33 (CbV and CbN least-level normalization).

- CbN: In $(\Lambda, \rightarrow_\beta)$, $\xrightarrow{\perp}\beta$ is a normalizing strategy for $\rightarrow_\beta$.
- CbV: In $(\Lambda, \rightarrow_{\beta_v})$, $\xrightarrow{\perp}\beta_v$ is a normalizing strategy for $\rightarrow_{\beta_v}$.

6.3 Least-level Factorization, Modularly.

As anticipated at the beginning of this section, we can use Theorem 29 also as part of the proof of factorization for a more complex calculus. We now introduce one more useful tool: a simple test to establish least-level factorization of a reduction $\rightarrow_{\beta_1} \cup \rightarrow_\rho$ (where $\rightarrow_\rho$ is a new reduction added to $\rightarrow_{\beta_1}$). We shall give an example of its use in Section 7 (see the proof of Theorem 36).

The test embodies Lemma 8, and the fact that we already know (once for all) that $\rightarrow_{\beta_1}$ factorizes via $\xrightarrow{\perp}\beta_1$. It turns out that the proof of the two linear swaps can be reduced to a single, simple test, which only involves the $\mapsto_\rho$ step.

Proposition 34 (Test for modular least-level factorization). *Let $\rightarrow_\rho$ be the contextual closure of a rule $\mapsto_\rho$, and assume $\rightarrow = (\rightarrow_{\beta_1} \cup \rightarrow_\rho)$ has a good least-level. Then $\rightarrow$ factorizes via $\xrightarrow{\perp} = (\xrightarrow{\perp}\beta_1 \cup \xrightarrow{\perp}\rho)$ if the following hold:*

1. L-factorization of $\rightarrow_\rho$: $\rightarrow_\rho^* \subseteq \xrightarrow{\perp}\rho^* \cdot \xrightarrow{\perp}\rho^*$
2. $\mapsto_\rho$ is substitutive: $R \mapsto_\rho R'$ implies $R\{Q/x\} \mapsto_\rho R'\{Q/x\}$.
3. Root linear swap: $\xrightarrow{\perp}\beta_1 \cdot \mapsto_\rho \subseteq \mapsto_\rho \cdot \xrightarrow{\perp}\beta_1^*$.

Note that, as usual, at point (1.) the least level is defined w.r.t. $\mathcal{R} = \mathcal{R}_{\beta_1} \cup \mathcal{R}_\rho$.

7 Case study: non-deterministic λ -calculi

To show how to use our framework, we apply the set of tools which we have developed on our running example. We extend the bang calculus with a non-deterministic operator, then considering $(\Lambda_{! \oplus}, \rightarrow_{\beta_1 \oplus})$ where $\rightarrow_{\beta_1 \oplus} = (\rightarrow_{\beta_1} \cup \rightarrow_\oplus)$, and $\rightarrow_\oplus$ is the contextual closure of the (non-deterministic) rules:

$$\mathbf{o}(P, Q) \mapsto_\oplus P \quad \mathbf{o}(P, Q) \mapsto_\oplus Q \quad (7)$$

First step: non-deterministic bang calculus. We analyze $\Lambda_{! \oplus}$. We use our modular test to prove least-level factorization for $\Lambda_{! \oplus}$: if $\rightarrow_{\beta_1 \oplus}^* U$ then $T \xrightarrow{\perp}\beta_1 \oplus^* \cdot \xrightarrow{\perp}\beta_1 \oplus^* U$. By Lemma 4, an immediate consequence of the factorization result is that the least-level strategy is *complete*, i.e. if U is normal: $T \rightarrow_{\beta_1 \oplus}^* U$ implies $T \xrightarrow{\perp}\beta_1 \oplus^* U$.

Second step: CbN and CbV non-deterministic calculi. By translation, we have for free, that the analogous results hold in $\Lambda_{\oplus}^{\text{cbn}}$ and $\Lambda_{\oplus}^{\text{cbv}}$, as defined in Example 9. So, least-level factorization holds for both calculi, and moreover

- *CbN completeness:* in $\Lambda_{\oplus}^{\text{cbn}}$, if u is normal: $t \rightarrow_{\beta_{\oplus}}^* u$ implies $t \xrightarrow{\text{L}}_{\beta_{\oplus}}^* u$.
- *CbV completeness:* in $\Lambda_{\oplus}^{\text{cbv}}$, if u is normal: $t \rightarrow_{\beta_{v\oplus}}^* u$ implies $t \xrightarrow{\text{L}}_{\beta_{v\oplus}}^* u$.

What do we really need to prove? The only result we need to prove is least-level factorization of $\rightarrow_{\beta_i\oplus}$. Completeness then follows by Lemma 4 and the translations will automatically take care of transferring the results.

To prove factorization of $\rightarrow_{\beta_i\oplus}$, most of the work is done, as L-factorization of $\rightarrow_{\beta_i}$ is already established; we then use our test (Proposition 34) to extend $\rightarrow_{\beta_i}$ with $\rightarrow_{\oplus}$. The only ingredients we need are substitutivity of $\oplus$ (which is an obvious property), and the following easy lemma.

Lemma 35 (Roots). *Let $\rho \in \{\beta_i, \oplus\}$. If $T \xrightarrow{\text{L}}_{\rho} P \mapsto_{\oplus} S$ then $T \mapsto_{\oplus} \cdot \xrightarrow{\rho} S$.*

Theorem 36 (Least-level factorization).

1. In $(\Lambda_{\oplus}, \rightarrow)$, $\text{Fact}(\xrightarrow{\text{L}}, \xrightarrow{\text{L}})$ holds for $\rightarrow = \rightarrow_{\oplus} \cup \rightarrow_{\beta_i}$.
2. Least-level factorization holds in $(\Lambda_{\oplus}^{\text{cbn}}, \rightarrow_{\oplus} \cup \rightarrow_{\beta})$, and in $(\Lambda_{\oplus}^{\text{cbv}}, \rightarrow_{\oplus} \cup \rightarrow_{\beta_v})$.

Proof. 1. It is enough to verify the hypotheses of Proposition 34.

2. It follows from Theorem 27 and Theorem 36.1. □

8 Conclusions and Related Work

The combination of translations (Theorem 27), L-factorization for $\rightarrow_{\beta_i}$ (Theorem 29), and modularity (Proposition 34), give us a powerful method to analyze factorization in various λ -calculi that *extend* the pure CbN and CbV calculi. The main novelty is transferring the results from a calculus to another via translations.

We chose to study least-level reduction as a normalizing strategy because it is natural to define in the bang calculus, and it is easier to transfer via translations to CbN and CbV calculi than leftmost-outermost. Since leftmost-outermost is the most common normalizing strategy in CbN, it is worth noticing that least-level normalization implies leftmost-outermost normalization (and vice-versa). This is an easy consequence of the—easy to check—fact that their union is quasi-diamond (and hence, again, uniformly normalizing). A proof of least-level normalization lends a proof also of leftmost-outermost normalization.

Related Work. Many calculi inspired by linear logic subsume CbN and CbV, such as [6,7,28,24] (other than the ones already cited). We chose the bang calculus for its simplicity, which eases the analysis of the CbN and CbV translations.

Least-level reduction is studied for linear-logic-based calculi in [33,2] and for linear logic proof-nets in [9,26]. Least-level factorization and normalization for the pure CbN λ -calculus is studied in [3].

References

- 1.
2. Accattoli, B.: An Abstract Factorization Theorem for Explicit Substitutions. In: 23rd International Conference on Rewriting Techniques and Applications (RTA'12). Leibniz International Proceedings in Informatics (LIPIcs), vol. 15, pp. 6–21. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik (2012). <https://doi.org/10.4230/LIPIcs.RTA.2012.6>
3. Accattoli, B., Faggian, C., Guerrieri, G.: Factorization and normalization, essentially. In: Programming Languages and Systems - 17th Asian Symposium, APLAS 2019. Lecture Notes in Computer Science, vol. 11893, pp. 159–180. Springer (2019). https://doi.org/10.1007/978-3-030-34175-6_9
4. Accattoli, B., Faggian, C., Guerrieri, G.: Factorize factorization. In: Computer Science Logic - 29th EACSL Annual Conference, CSL 2021. LIPIcs, vol. 183. Schloss Dagstuhl (2021), to appear
5. Barendregt, H.P.: The Lambda Calculus: Its Syntax and Semantics, vol. 103. North Holland (1984)
6. Benton, P.N., Bierman, G.M., de Paiva, V., Hyland, M.: A term calculus for intuitionistic linear logic. In: Typed Lambda Calculi and Applications, International Conference on Typed Lambda Calculi and Applications, TLCA '93. Lecture Notes in Computer Science, vol. 664, pp. 75–90. Springer (1993). <https://doi.org/10.1007/BFb0037099>
7. Benton, P.N., Wadler, P.: Linear logic, monads and the lambda calculus. In: Proceedings, 11th Annual IEEE Symposium on Logic in Computer Science, LICS 1996. pp. 420–431. IEEE Computer Society (1996). <https://doi.org/10.1109/LICS.1996.561458>
8. Bucciarelli, A., Kesner, D., Ríos, A., Viso, A.: The bang calculus revisited. In: Functional and Logic Programming - 15th International Symposium, FLOPS 2020. Lecture Notes in Computer Science, vol. 12073, pp. 13–32. Springer (2020). https://doi.org/10.1007/978-3-030-59025-3_2
9. de Carvalho, D., Pagani, M., Tortora de Falco, L.: A semantic measure of the execution time in linear logic. *Theor. Comput. Sci.* **412**(20), 1884–1902 (2011)
10. Chouquet, J., Tasson, C.: Taylor expansion for Call-By-Push-Value. In: 28th EACSL Annual Conference on Computer Science Logic (CSL 2020). Leibniz International Proceedings in Informatics (LIPIcs), vol. 152, pp. 16:1–16:16. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik (2020). <https://doi.org/10.4230/LIPIcs.CSL.2020.16>
11. Ehrhard, T.: Call-by-push-value from a linear logic point of view. In: Programming Languages and Systems - 25th European Symposium on Programming (ESOP 2016). Lecture Notes in Computer Science, vol. 9632, pp. 202–228 (2016)
12. Ehrhard, T., Guerrieri, G.: The bang calculus: an untyped lambda-calculus generalizing call-by-name and call-by-value. In: Proceedings of the 18th International Symposium on Principles and Practice of Declarative Programming (PPDP 2016). pp. 174–187. ACM (2016). <https://doi.org/10.1145/2967973.2968608>
13. Faggian, C., Ronchi Della Rocca, S.: Lambda calculus and probabilistic computation (extended version), available at <http://arxiv.org/abs/1901.02853>
14. Faggian, C., Ronchi Della Rocca, S.: Lambda calculus and probabilistic computation. In: 34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019. pp. 1–13 (2019). <https://doi.org/10.1109/LICS.2019.8785699>

15. Girard, J.: Linear logic. *Theor. Comput. Sci.* **50**, 1–102 (1987). [https://doi.org/10.1016/0304-3975\(87\)90045-4](https://doi.org/10.1016/0304-3975(87)90045-4)
16. Guerrieri, G., Manzonetto, G.: The bang calculus and the two Girard’s translations. In: *Proceedings Joint International Workshop on Linearity & Trends in Linear Logic and Applications (Linearity-TLLA 2018)*. EPTCS, vol. 292, pp. 15–30 (2019)
17. Hindley, J.R.: The Church-Rosser Property and a Result in Combinatory Logic. Ph.D. thesis, University of Newcastle-upon-Tyne (1964)
18. Hindley, J.R., Seldin, J.P.: *Introduction to Combinators and Lambda-Calculus*. Cambridge University Press (1986)
19. Hirokawa, N., Middeldorp, A., Moser, G.: Leftmost Outermost Revisited. In: *26th International Conference on Rewriting Techniques and Applications (RTA 2015)*. *Leibniz International Proceedings in Informatics (LIPIcs)*, vol. 36, pp. 209–222. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik (2015). <https://doi.org/10.4230/LIPIcs.RTA.2015.209>
20. Levy, P.B.: Call-by-push-value: A subsuming paradigm. In: *Typed Lambda Calculi and Applications, 4th International Conference (TLCA’99)*. *Lecture Notes in Computer Science*, vol. 1581, pp. 228–242 (1999). https://doi.org/10.1007/3-540-48959-2_17
21. Levy, P.B.: Call-by-push-value: Decomposing call-by-value and call-by-name. *High. Order Symb. Comput.* **19**(4), 377–414 (2006). <https://doi.org/10.1007/s10990-006-0480-6>
22. de’ Liguoro, U.: Non-deterministic untyped λ -calculus. A study about explicit non determinism in higher-order functional calculi. Ph.D. thesis, Università di Roma La Sapienza (1991), <http://www.di.unito.it/~deligu/papers/UdLTesi.pdf>
23. de’ Liguoro, U., Piperno, A.: Non deterministic extensions of untyped lambda-calculus. *Inf. Comput.* **122**(2), 149–177 (1995)
24. Maraist, J., Odersky, M., Turner, D.N., Wadler, P.: Call-by-name, call-by-value, call-by-need and the linear lambda calculus. *Theor. Comput. Sci.* **228**(1-2), 175–210 (1999). [https://doi.org/10.1016/S0304-3975\(98\)00358-2](https://doi.org/10.1016/S0304-3975(98)00358-2)
25. Newman, M.: On theories with a combinatorial definition of equivalence. *Annals of Mathematics* **43**(2) (1942)
26. Pagani, M., Tranquilli, P.: The conservation theorem for differential nets. *Math. Struct. Comput. Sci.* **27**(6), 939–992 (2017). <https://doi.org/10.1017/S0960129515000456>
27. Plotkin, G.D.: Call-by-name, call-by-value and the lambda-calculus. *Theor. Comput. Sci.* **1**(2), 125–159 (1975)
28. Ronchi Della Rocca, S., Roversi, L.: Lambda calculus and intuitionistic linear logic. *Stud Logica* **59**(3), 417–448 (1997). <https://doi.org/10.1023/A:1005092630115>
29. Santo, J.E., Pinto, L., Uustalu, T.: Modal embeddings and calling paradigms. In: *4th International Conference on Formal Structures for Computation and Deduction, FSCD 2019*. *LIPIcs*, vol. 131, pp. 18:1–18:20. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik (2019). <https://doi.org/10.4230/LIPIcs.FSCD.2019.18>
30. Simpson, A.K.: Reduction in a linear lambda-calculus with applications to operational semantics. In: *Term Rewriting and Applications, 16th International Conference (RTA 2005)*. *Lecture Notes in Computer Science*, vol. 3467, pp. 219–234 (2005). https://doi.org/10.1007/978-3-540-32033-3_17
31. Takahashi, M.: Parallel reductions in lambda-calculus. *Inf. Comput.* **118**(1), 120–127 (1995)
32. Terese: *Term Rewriting Systems*, *Cambridge Tracts in Theoretical Computer Science*, vol. 55. Cambridge University Press (2003)

33. Terui, K.: Light affine lambda calculus and polynomial time strong normalization. *Archive for Mathematical Logic* **46**(3-4), 253–280 (2007)