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ON THE CONVERGENCE OF THE CRANK-NICOLSON METHOD FOR THE LOGARITHMIC SCHröDINGER EQUATION

PANAGIOTIS PARASCHIS† AND GEORGIOS E. ZOURARIS†

Abstract. We consider an initial and Dirichlet boundary value problem for a logarithmic Schrödinger equation over a two dimensional rectangular domain. We construct approximations of the solution to the problem using a standard second order finite difference method for space discretization and the Crank-Nicolson method for time discretization, with or without regularizing the logarithmic term. We develop a convergence analysis yielding a new almost second order a priori error estimates in the discrete $L^\infty_t(L^2_x)$ norm, and we show results from numerical experiments exposing the efficiency of the method proposed. It is the first time in the literature where an error estimate for a numerical method applied to the logarithmic Schrödinger equation is provided, without regularizing its nonlinear term.

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1. Introduction

Let $T > 0$, $\Omega = [a_1, a_2] \times [b_1, b_2] \subset \mathbb{R}^2$, $D := [0, T] \times \Omega$ and $u : D \to \mathbb{C}$ be the solution of the following model initial and Dirichlet boundary value problem for the logarithmic Schrödinger equation:

\begin{align}
\frac{du}{dt} &= i\Delta u + i\lambda g(u) + f \quad \text{on} \quad (0, T) \times \operatorname{int}(\Omega), \\
u(0, x) &= u_0(x) \quad \forall \ x \in \operatorname{int}(\Omega), \\
u(t, x) &= 0 \quad \forall \ (t, x) \in (0, T) \times \partial\Omega,
\end{align}

where $\lambda \in \mathbb{R}\setminus\{0\}$, $f \in C(D, \mathbb{C})$, $u_0 \in C(\Omega, \mathbb{C})$ with $u_0|_{\partial\Omega} = 0$, and $g \in C(\mathbb{C}, \mathbb{C})$ given by

\begin{align}
g(w) = \begin{cases} 0, & w = 0, \\ w \ln(|w|), & w \neq 0,
\end{cases} \quad \forall \ w \in \mathbb{C}.
\end{align}

It is well-known (see, e.g., Lemma 1.1.1 in [12]) that $g$ satisfies the Cazenave-Haraux (CH) property:

\begin{align}
|\operatorname{Im}[(g(z) - g(w))(\overline{z} - \overline{w})]| \leq |z - w|^2 \quad \forall \ z, w \in \mathbb{C},
\end{align}

which is the basic tool to ensure uniqueness of the solution $u$ (see, e.g., [12], [9], [16]). Here, we consider a two space dimensional logarithmic Schrödinger equation, because the corresponding numerical approximation problem is computationally more demanding and challenging than that for an one space dimensional one. Our decision to discretize the problem above in space by applying a finite difference method over an orthogonal grid is the reason why we pose the problem over the rectangle $\Omega$. Also, this choice gives us the opportunity to observe the tensorization property (see e.g., [7], [10]) of the logarithmic Schrödinger equation, where, when $f$ vanishes, the tensor product of two solutions to the one dimensional logarithmic Schrödinger equation consists a solution to the two dimensional one. Different types of boundary condition can be imposed on $\partial\Omega$ and we have

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shown our preference to the homogeneous Dirichlet boundary conditions since they can be easily included in a finite difference method.

The logarithmic Schrödinger equation has been introduced in [7] within the area of the nonlinear wave mechanics, and since then it appears as a model in various science topics such as nuclear physics, quantum mechanics, quantum optics and geophysics (see, e.g. [19], [26], [8], [13], [15]). In the physical case the logarithmic Schrödinger equation is homogeneous. Here, we have included a non homogeneous term $f$ to (1.1) in order to show that its presence does not influence the results we obtain and in order to have higher flexibility in the construction of artificial problems with known regular exact solution, which could be useful in testing the implementation and the asymptotic behaviour of our numerical method. That allows us to adopt the formal assumption that the problem above admits a unique solution which is sufficiently smooth for our purposes. At this point, we would like to note that according to the bibliography it is not certain that high regularity of the initial condition is always propagated (see Section 2.3 in [9] and Remark 2.1 in [3]).


Recently the problem of constructing approximations of the solution to the logarithmic Schrödinger equation has been addressed in [5] and [4], where a numerical method is applied along with the substitution of $g(z)$ by the regular function $g_{\varepsilon}(z) = z \ln(\varepsilon + |z|)$ where $\varepsilon$ is a positive parameter close to zero. However, the convergence results presented are pessimistic, because the parameter $\varepsilon$ influences the error bounds obtained in an irregular way. In particular, for a linearly implicit finite difference method formulated in [5], it is shown that the exponential constant, which is an outcome of the application of the discrete Gronwall argument, is of the form $\exp(CT \ln(\varepsilon)^2)$ for the error estimate in the discrete $L^\infty(L^2_x)$ norm (see Theorem 3.1 in [5]). According to the convergence analysis developed, an $O(\varepsilon)$ error bound, is achieved by adopting an $O(\sqrt{\varepsilon} \exp(-CT |\ln(z)|^2))$ size for the time step and the space width. However, for $\varepsilon = 10^{-2}$ and $CT = 1$, we obtain $\sqrt{\varepsilon} \exp(-CT |\ln(z)|^2) \approx 6 \times 10^{-11}$, which seems to be an unrealistic size for the time-step and the mesh-width to expect an accuracy of about two digits. For a time-discrete Lie-Trotter splitting method proposed in [4], the convergence analysis arrives at an $O(\varepsilon + |\ln(z)| \tau^2)$ or $O(\varepsilon + \varepsilon^{-1} \tau)$ error bound in the discrete in time $L^\infty(L^2_x)$ norm, depending on the time regularity of the solution, where $\tau$ is the time step (see Theorem 1 and Remark 3 in [4]). Thus, under the optimal choice $\varepsilon = \tau^2$, one concludes, in both cases, an $O(\tau^2)$ error bound. Analogous results are obtained in [13] for a fractional logarithmic Schrödinger equation. It is evident, that further investigation is required to arrive at a better understanding of how the choice of the method interacts with the regularization of the nonlinear term in the error estimation process.

The bibliography cultivates the impression that the presence of the logarithmic function in the nonlinear term $g$ is a source of singularity resulting in difficulties in the numerical handling of the problem, which is not confirming by results from numerical experiments (see, e.g. Section 6). We would like to stress that in a computer implementation of a numerical method approximating the solution to the logarithmic Schrödinger equation, the user has to define function $g$ exactly as it is described in (1.4), otherwise an overflow signal will possibly pop-up, which could be considered as a false signal for the existence of a singularity. Our opinion is that a numerical method approximating the solution to the logarithmic Schrödinger equation must have its stability properties aligned to the (CH) property (1.5) of the logarithmic term.
In the paper at hand, we apply the Crank-Nicolson method for time-discretization along with a second order finite difference method for space-discretization skipping the use of the finite element method as an attempt to avoid the numerical integration of the logarithmic term. Since the Crank-Nicolson method is a $B$-stable Runge-Kutta method (see, e.g., [17]), we are able to build-up a stability argument based on the (CH) property (1.5) that keeps the Gronwall exponential term free of $\varepsilon$. The convergence analysis shows that the method can achieve an almost second order convergence in the discrete $L^2(\Omega)$ norm, with or without using $g_\varepsilon$ as a regularization of $g$. In particular, we show that there exists a constant $C > 0$, independent of $\varepsilon$, $\tau$, $h_1$ and $h_2$, such that

$$\max_{0 \leq n \leq N} \left( \|U^n - u^n\|_{0, h} + \tau \frac{1}{2} |\ln(\tau)|^{-\frac{1}{2}} |U^n - u^n|_{1, h} \right) \leq C \left[ \tau^2 + \tau^2 |\ln(\tau)| + h_1^2 + h_2^2 \right]$$

if $\varepsilon = 0$,

and

$$\max_{0 \leq n \leq N} \left( \|U^n - u^n\|_{0, h} + \tau \frac{1}{2} |\ln(\tau)|^{-\frac{1}{2}} |U^n - u^n|_{1, h} \right) \leq C \left[ \varepsilon + \tau^2 + \tau^2 |\ln(\tau)| + h_1^2 + h_2^2 \right]$$

if $\varepsilon > 0$,

where $\| \cdot \|_{0, h}$ is a discrete $L^2(\Omega)$ norm, $| \cdot |_{1, h}$ is a discrete $H^1(\Omega)$ norm $\tau$ is the time-step, $h_1$ and $h_2$ are the mesh-length of a uniform partition of $[a_1, a_2]$ and $[b_1, b_2]$, respectively. Thus, $\varepsilon$ does not influence the choice of $\tau$, $h_1$, $h_2$, and the method can achieve its maximum rate of convergence by setting $\varepsilon = \tau^\beta$ with $\beta \geq 2$. Our results show that, given that the solution to the problem is regular enough, it is possible to derive second order error estimates for a numerical method applied to the problem at hand without suffering from the regularization parameter $\varepsilon$ (cf. Remarks 4 and 5 in [4]). We would like to stress that the results we obtain are not directly applied in the case of real evolution equations with logarithmic nonlinearity, since the left hand side in (1.5) becomes zero on real line (see, e.g., [22]).

We would like to fill out our references list, by mentioning the work [21] for the use of the Crank-Nicolson method to approximate the solution to the logarithmic Schrödinger equation with artificial boundary conditions without providing rigorous error estimates for the numerical approximation error, and the works [23] and [25] for the numerical approximation of the time independent logarithmic Schrödinger equation.

We close the Introduction by giving a brief overview of the paper. In Section 2, we introduce notation, we provide some auxiliary results, we formulate the numerical method and we discuss the existence and uniqueness of the numerical approximations. In Section 3, we estimate the consistency error. Section 4 is dedicated to the convergence analysis of the proposed method and in Section 5 we discuss the application of the fixed point iterations method for computing the numerical approximation. Finally, we expose results from numerical experiments in Section 6.

2. The Crank-Nicolson Finite Difference Method

2.1. Notation and Preliminaries. Let $\mathbb{N}$ be the set of all positive integers. For given $N \in \mathbb{N}$, we consider a uniform partition of the time interval $[0, T]$ with time-step $\tau := \frac{T}{N}$, nodes $t_n := n \tau$ for $n = 0, \ldots, N$, and intermediate nodes $t^{n+\frac{1}{2}} = t_n + \frac{\tau}{2}$ for $n = 0, \ldots, N-1$. Also, for given $J_1, J_2 \in \mathbb{N}$, we consider a uniform partition of $[a_1, a_2]$ with mesh-width $h_1 := \frac{a_2 - a_1}{J_1 - 1}$ and nodes $x_{i, j} := a_1 + i h_1$ for $i = 0, \ldots, J_1 - 1$, and a uniform partition of $[b_1, b_2]$ with mesh-width $h_2 := \frac{b_2 - b_1}{J_2 - 1}$ and nodes $x_{2, j} := b_1 + j h_2$ for $j = 0, \ldots, J_2 - 1$. To simplify the notation, we set $\mathbb{I} := \{(i, j): \ i = 0, \ldots, J_1 + 1, \ j = 0, \ldots, J_2 + 1\}$, $\mathbb{I}^0 := \{(i, j): \ i = 1, \ldots, J_1, \ j = 1, \ldots, J_2\}$, $\partial \Omega := \mathbb{I} \backslash \mathbb{I}^0$. Then, we introduce the discrete matrix space $X_h := \{(V_{\alpha})_{\alpha \in \mathbb{I}^0} \in C([J_1+2]\times[J_2+2]): \ V_{\alpha} = 0 \forall \alpha \in \partial \Omega\}$, a discrete Laplacian operator $\Delta_h: X_h \to X_h$ by

$$(\Delta_h V)_{(i,j)} := \frac{V_{(i-1,j)-2V_{(i,j)-1V_{(i+1,j)-1V_{(i+1,j+1)}}} + \frac{V_{(i,j+1)-2V_{(i+1,j)-1V_{(i+1,j+1)}}}}{h_2^2}}{\forall \ (i, j) \in \mathbb{I}^0, \ \forall V \in X_h}$$

and the operator $l_h: C(\mathbb{D}, \mathbb{C}) \to X_h$ by $(l_h(z))_{\alpha} := z(x_{1, \alpha_1}, x_{2, \alpha_2})$ for all $\alpha \in \mathbb{I}^0$ and $z \in C(\mathbb{D}, \mathbb{C})$. Finally, we simplify the notation, by setting $u^n := l_h(u(t_n, \cdot))$ for $n = 0, \ldots, N$, and by defining, for any $W \in X_h$ and $g \in C(\mathbb{C}, \mathbb{C})$, the vector $g(W) \in X_h$ by $(g(W))_{\alpha} := g(W_{\alpha})$ for all $\alpha \in \mathbb{I}^0$. 

3
We provide $X_n$ with the discrete inner product $(\cdot, \cdot)_{0,n}$ given by $(V, Z)_{0,n} := h_1 h_2 \sum_{\alpha \in \Gamma} V_\alpha Z_\alpha$ for $V, Z \in X_n$, and we shall denote by $\| \cdot \|_{0,n}$ its induced norm, i.e. $\| v \|_{0,n} := ((V, V)_{0,n})^{1/2}$ for $V \in X_n$. Also, we equip $X_n$ with a discrete $L^\infty$ norm $\| \cdot \|_{\infty,n}$ defined by $\| W \|_{\infty,n} := \max_{\alpha \in \Gamma} |W_\alpha|$ for $W \in X_n$ and with a discrete $H^1$-type norm $\| \cdot \|_{1,n}$ given by

$$
\| V \|_{1,n} := \left[ h_1 h_2 \sum_{i=1}^{j_1} \left| \frac{V(i+1) - V(i)}{h_1} \right|^2 + h_1 h_2 \sum_{i=1}^{j_2} \left| \frac{V(i+1) - V(i)}{h_2} \right|^2 \right]^{1/2} \quad \forall V \in X_n.
$$

Later, we will make use of the, easy to verify, discrete integration by part result $X$ for the symmetry of $\Delta_n$

$$
(\Delta_n V, V)_{0,n} = -\| V \|_{1,n}^2 \quad \forall V \in X_n
$$

of the symmetry of $\Delta_n$

$$
(\Delta_n V, V)_{0,n} = (V, \Delta_n V)_{0,n} \quad \forall V \in X_n
$$

and of the following inverse inequality

$$
\| V \|_{\infty,n} \leq \frac{1}{\sqrt{h_1 h_2}} \| V \|_{0,n} \quad \forall V \in X_n.
$$

Below, we show some basic properties of the regularized function $g_\varepsilon$.

**Lemma 2.1.** For $\varepsilon \geq 0$, it holds that

$$
\sup_{z \in \mathbb{C}} |g(z) - g_\varepsilon(z)| \leq \varepsilon.
$$

**Proof.** When $\varepsilon = 0$, (2.4) holds trivially. Now, let $\varepsilon > 0$ and $z \in \mathbb{C}\setminus\{0\}$. Using the mean value theorem there exists $\xi \in (|z|, \varepsilon + |z|)$ such that $\ln(\varepsilon + |z|) - \ln(|z|) = \frac{\varepsilon |z|}{\xi}$. Thus, we obtain

$$
|g(z) - g_\varepsilon(z)| = |z| \ln(\varepsilon + |z|) - \ln(|z|)| = \frac{|z| \varepsilon}{\xi} \leq \varepsilon.
$$

Observing that $g(0) = g_\varepsilon(0) = 0$ and using (2.5), we, easily, arrive at (2.4). \qed

**Lemma 2.2.** For $c > \varepsilon$ and $\varepsilon (0, \frac{1}{2c})$, we have

$$
|g_\varepsilon(z) - g_\varepsilon(w)| \leq 2|\ln(\varepsilon)||z - w| \quad \forall z, w \in \Gamma_c,
$$

where $\Gamma_c := \{ v \in \mathbb{C} : |v| \leq c \}$.

**Proof.** Let $z, w \in \Gamma_c$ with $|w| \leq |z|$. Applying the mean value theorem, we conclude that $\ln(\varepsilon + |z|) - \ln(\varepsilon + |w|) = \xi^{-1}(|z| - |w|)$, where $\xi \in [\varepsilon + |w|, \varepsilon + |z|]$. Now, we use the latter equality to obtain

$$
\begin{align*}
|g_\varepsilon(z) - g_\varepsilon(w)| & \leq |z - w| \left| \ln(\varepsilon + |z|) + |w| \ln(\varepsilon + |z|) - \ln(\varepsilon + |w|) \right| \\
& \leq |z - w| \max \{|\ln(\varepsilon)|, \ln(\varepsilon + c)\} + \frac{|w|}{\xi} ||z| - |w|| \\
& \leq |z - w| \left[ \max \{|\ln(\varepsilon)|, \ln(2c)\} + \frac{|w|}{\xi} \right] \\
& \leq |z - w| \ln(\varepsilon) + \frac{|w|}{\xi + |w|} \\
& \leq |z - w| |\ln(\varepsilon)| + 1 \\
& \leq 2|z - w| |\ln(\varepsilon)|,
\end{align*}
$$

and, thus, (2.6) follows. Since (2.6) is symmetric with respect to $z$ and $w$, it holds also when $|w| > |z|$. \qed
2.2. The $(\varepsilon\text{CNFD})$ method. For $\varepsilon \geq 0$, the $\varepsilon$-Crank-Nicolson finite difference $(\varepsilon\text{CNFD})$ method is implicit, requiring, at every time step, the solution of a nonlinear system of algebraic equation and its structure is as follows:

Step 1: Set
\begin{equation}
U^0 := l_0[u_0] \in X_0.
\end{equation}

Step 2: For $n = 0, \ldots, N - 1$, find $U^{n+1} \in X_n$ such that
\begin{equation}
\frac{U^{n+1} - U^n}{\tau} = i \Delta_n \left( \frac{U^{n+1} + U^n}{2} \right) + i \lambda g_{\varepsilon} \left( \frac{U^{n+1} + U^n}{2} \right) + l_n \left[ f(t^{n+\frac{1}{2}}, \cdot) \right],
\end{equation}
where $g_{\varepsilon} : \mathbb{C} \to \mathbb{C}$ (see, e.g., [3, 4]) is a regularization of $g$ given by $g_{\varepsilon}(z) := z \ln(\varepsilon + |z|)$ for $z \in \mathbb{C}$.

Remark 2.1. When $\varepsilon = 0$, the $(\varepsilon\text{CNFD})$ method is the usual Crank-Nicolson method (cf., e.g., [1]).

Remark 2.2. In the homogeneous case, i.e. when $f \equiv 0$, it is well-known that the solution $u$ has the following conservation property
\begin{equation}
\int_\Omega a(t, x)^2 \, dx = \int_\Omega |u_0(x)|^2 \, dx \quad \forall \ t \in [0, T],
\end{equation}
which is entitled as ‘conservation of the mass’ or ‘conservation of the charge’ (see, e.g., [3, 4]).

2.3. Existence and uniqueness of the $(\varepsilon\text{CNFD})$ approximations. First, we investigate the existence of the $(\varepsilon\text{CNFD})$ approximations, by employing the following lemma (see [4]).

Lemma 2.3. Let $(\mathbb{C}, \mathcal{H}, (\cdot, \cdot)_\mathcal{H})$ be a complex finite dimensional inner product space, $\cdot, \cdot_\mathcal{H}$ be the associated norm and $\mu : \mathcal{H} \to \mathcal{H}$ be a continuous operator. If there exists a positive constant $\sigma > 0$ such that $\text{Re}(\mu(z)) \geq 0$ for all $z \in \mathcal{H}$ with $|z|_\mathcal{H} = \sigma$, then there exists $w \in \mathcal{H}$ such that $\mu(w) = 0$ and $\|w\|_\mathcal{H} \leq \sigma$.

Proposition 2.1. For $\varepsilon \geq 0$, there exist $(U^n)_{n=1}^N \in X_n$ satisfying (2.8).

Proof. First, we observe that $U^0 \in X_0$ exists by construction. Let us assume that, for some $\kappa \in \{0, \ldots, N - 1\}$, the approximation $U^\kappa$ exists, and let us define a continuous nonlinear operator $\mu_\kappa : X_\kappa \to X_\kappa$ by
\[ \mu_\kappa(V) := V - i \frac{\varepsilon}{2} \Delta_\kappa(V) - i \frac{\lambda}{2} \lambda g_{\varepsilon}(V) + Z_\kappa \quad \forall \ V \in X_\kappa, \]
where $Z_\kappa := -U^{\kappa-\frac{1}{2}} l_n \left[ f(t^{\kappa+\frac{1}{2}}, \cdot) \right]$. It is easily seen that the existence of an approximation $U^{\kappa+1} \in X_\kappa$ is equivalent to $\mu_\kappa \left( \frac{U^{\kappa+1} + U^{\kappa}}{2} \right) = 0$, and thus, it is sufficient to show the existence of a root for $\mu_\kappa$.

Let $V \in X_\kappa$ with $\|V\|_{0,\kappa} = \sigma$. Using (2.1) and the Cauchy-Schwarz inequality, we obtain
\begin{align*}
\text{Re}(\mu_\kappa(V)) &= \text{Re} \left[ \|V\|_{0,\kappa}^2 + i \frac{\lambda}{2} \lambda g_{\varepsilon}(V) \cdot (Z_\kappa, V)_{0,\kappa} \right] - \|Z_\kappa\|_{0,\kappa} \text{Im} \left[ (Z_\kappa, V)_{0,\kappa} \right] \\
&= \|V\|_{0,\kappa}^2 + \frac{\lambda}{2} \lambda |g_{\varepsilon}(V)|_{0,\kappa} - \|Z_\kappa\|_{0,\kappa} \text{Re} \left[ (Z_\kappa, V)_{0,\kappa} \right] \\
&\geq \|V\|_{0,\kappa}^2 + \|Z_\kappa\|_{0,\kappa} (\sigma - |Z_\kappa|_{0,\kappa}) \\
&\geq \sigma (\sigma - |Z_\kappa|_{0,\kappa}).
\end{align*}

Adopting the choice $\sigma = |Z_\kappa|_{0,\kappa}$, we obtain $\text{Re}(\mu_\kappa(V)) \geq 0$. Then, under the light of Lemma 2.3 with $\mathcal{H} = X_\kappa$ and $(\cdot, \cdot)_{\mathcal{H}} = (\cdot, \cdot)_{0,\kappa}$, there exists $W \in X_\kappa$ such that $\|W\|_{0,\kappa} \leq \sigma$ and $\mu_\kappa(W) = 0$, which ends the proof. \qed
Working towards to ensure the uniqueness of the (εCNFD) approximations, we will make use of the following lemma.

**Lemma 2.4.** For $\varepsilon \geq 0$, it holds that
\[
\text{Im} \left[ (g_\varepsilon (V) - g_\varepsilon (W), V - W)_{\partial \Omega} \right] \leq \| V - W \|_{0, H}^2 \quad \forall \ V, W \in X_\varepsilon.
\]

**Proof.** Recalling that (see, e.g., Lemma 1.1.1 in [12] and Lemma 2.4 in [5])
\[
\| V - W \|_{0, H}^2 \leq C \| V - W \|_{0, H}
\]
the inequality (2.10), easily, follows. □

Below, assuming that $\tau$ is small enough, we provide the uniqueness of the (εCNFD) approximations.

**Proposition 2.2.** For $\varepsilon \geq 0$ and $\tau \in \left(0, \frac{1}{2|\lambda|}\right)$, there exists unique $(U^n)_{n=1}^N \subset X_\varepsilon$ satisfying (2.8).

**Proof.** Proposition 2.1 yields the existence of $(U^n)_{n=1}^N \subset X_\varepsilon$ satisfying (2.8). Let $\kappa \in \{0, \ldots, N - 1\}$ and $W \in X_\varepsilon$ such that
\[
W - U^n = \iota \Delta u \left[ \frac{W + U^n}{2} \right] + i \lambda g_\varepsilon \left[ \frac{W + U^n}{2} \right] + \iota n \left[ f \left( t^{n+\frac{1}{2}}, \cdot \right) \right].
\]
Subtracting (2.12) from (2.8) (with $n = \kappa$), we obtain
\[
\frac{U^\kappa + W}{\tau} = \iota \Delta u \left[ \frac{U^\kappa + W}{2} \right] + i \lambda g_\varepsilon \left[ \frac{U^\kappa + W}{2} \right] - g_\varepsilon \left[ \frac{W + U^n}{2} \right].
\]
Taking the $(\cdot, \cdot)_{\partial \Omega}$-inner product of both sides of (2.13) with $(U^\kappa + W)$, using (2.1) and keeping the real parts, we obtain
\[
\| U^\kappa + W \|_{0, \Omega}^2 = -2 \lambda \tau \text{Im} \left[ \frac{U^\kappa + W}{2} \right] \left( g_\varepsilon \left( \frac{U^\kappa + W}{2} \right) - g_\varepsilon \left( \frac{W + U^n}{2} \right) \right),
\]
which, after applying (2.10), yields
\[
\| U^\kappa + W \|_{0, \Omega}^2 \leq 2 \tau | \lambda | \| U^\kappa + W \|_{0, \Omega}^2.
\]
Using the assumption that $2 \tau | \lambda | < 1$, from (2.14), obviously, follows that $W = U^\kappa + 1$. □

3. **Consistency**

3.1. **Time-discretization consistency error.** For $n = 0, \ldots, N - 1$ and $\varepsilon > 0$, we define $r_\varepsilon^n \in X_\varepsilon$ by
\[
r_\varepsilon^n = \frac{u^{n+1} - u^n}{\tau} = \iota n \left[ \frac{u(t_{n+\frac{1}{2}}) - u(t_n)}{2} \right] + \iota \lambda g_\varepsilon \left( \frac{u(t_{n+\frac{1}{2}})}{2} \right) + f(t^{n+\frac{1}{2}}, \cdot) + r_\varepsilon^n.
\]
Using (1.1) and (3.1), we obtain
\[
r_\varepsilon^n = \xi^n + \rho_\varepsilon^n, \quad n = 0, \ldots, N - 1,
\]
where
\[
\xi^n := \iota n \left[ \frac{u(t_{n+\frac{1}{2}}) - u(t_n)}{2} \right] - i \iota \iota n \left[ \frac{u(t_{n+\frac{1}{2}}) - u(t_n)}{2} \right] \Delta u(t^{n+\frac{1}{2}}, \cdot) + f(t^{n+\frac{1}{2}}, \cdot) + r_\varepsilon^n.
\]
and
\[
\rho_\varepsilon^n := i \lambda g_\varepsilon \left( \frac{u(t_{n+\frac{1}{2}})}{2} \right) - g_\varepsilon (u(t^{n+\frac{1}{2}}, \cdot)).
\]
Applying the Taylor formula, we, easily, obtain the following estimate
\[
|\xi^n|_{m, \Omega} \leq C \tau^2 \left( \max_D \left| \Delta u_{n|t} \right| + \max_D \left| u_{n|t} \right| \right), \quad n = 0, \ldots, N - 1.
\]
In the sequel, in order to estimate $\rho_\varepsilon^n$, we consider the following cases with respect to $\varepsilon$.

**Case 1:** $\varepsilon = 0$. 

First, we split the error term \( \rho_0^n \) into three parts as follows
\[
\left| \rho_0^n \right|_{\infty,H} \leq \left| \int_0^1 g \left( u(\frac{t_{n+1}}{2}) + u(t_n) \right) dt \right|_{\infty,H} + \left| \int_0^1 g_{t^n} \left( u(\frac{t_{n+1}}{2}) + u(t_n) \right) dt \right|_{\infty,H} + \left| \int_0^1 g_\frac{\Delta t}{2} \left( u(\frac{t_{n+1}}{2}) + u(t_n) \right) dt \right|_{\infty,H},
\]

Then, we use (2.4) (with \( \varepsilon = \tau^2 \), (2.6) (with \( \varepsilon = \tau^2 \) and \( \tau \in \left( 0, \frac{1}{2\varepsilon} \right) \)), and the Taylor formula to get
\[
\left| \rho_0^n \right|_{\infty,H} \leq C \left[ \tau^2 + \left| \ln(\tau) \right| \left| \frac{u(t_{n+1}) + u(t_n)}{2} - u(\frac{t_{n+1}}{2}) \right|_{\infty,H} \right] \leq C \left[ \tau^2 + \tau^2 \left| \ln(\tau) \right| \max_D |u|_{t\|} \right], \quad n = 0, \ldots, N - 1.
\]

Finally, from (3.2), (3.3) and (3.4), we obtain
\[
\max_{0 \leq n \leq N-1} \left| \rho_0^n \right|_{\infty,H} \leq C_{\tau^2} \left[ \tau^2 \left( 1 + \max_D |\Delta u|_{t\|} + \max_D |u|_{t\|} \right) + \tau^2 \left| \ln(\tau) \right| \max_D |u|_{t\|} \right].
\]

**Case 2: \( \varepsilon > 0 \).**

First, we split the error term \( \rho_\varepsilon^n \) into two parts as follows
\[
\left| \rho_\varepsilon^n \right|_{\infty,H} \leq \left| \int_0^1 g \left( u(\frac{t_{n+1}}{2}) + u(t_n) \right) dt \right|_{\infty,H} + \left| \int_0^1 g_{t^n} \left( u(\frac{t_{n+1}}{2}) + u(t_n) \right) dt \right|_{\infty,H} + \left| \int_0^1 g_\frac{\Delta t}{2} \left( u(\frac{t_{n+1}}{2}) + u(t_n) \right) dt \right|_{\infty,H}, \quad n = 0, \ldots, N - 1.
\]

Then, we use (2.4), (2.6) (with \( \varepsilon = \tau^2 \) and \( \tau \in \left( 0, \frac{1}{2\varepsilon} \right) \)), and the Taylor formula to get
\[
\left| \rho_\varepsilon^n \right|_{\infty,H} \leq C \left[ \varepsilon + \left| \ln(\varepsilon) \right| \left| \frac{u(t_{n+1}) + u(t_n)}{2} - u(\frac{t_{n+1}}{2}) \right|_{\infty,H} \right] \leq C \left[ \varepsilon + \tau^2 \left| \ln(\varepsilon) \right| \max_D |u|_{t\|} \right], \quad n = 0, \ldots, N - 1.
\]

Thus, (3.2), (3.3) and (3.6) yield
\[
\max_{0 \leq n \leq N-1} \left| \rho_\varepsilon^n \right|_{\infty,H} \leq C_{\tau^2} \left[ \varepsilon + \tau^2 \left( \max_D |\Delta u|_{t\|} + \max_D |u|_{t\|} \right) + \tau^2 \left| \ln(\varepsilon) \right| \max_D |u|_{t\|} \right].
\]

**3.2. Space discretization consistency error.** For \( \varepsilon \geq 0 \) and \( n = 0, \ldots, N - 1 \), let \( s_\varepsilon^n \in X_n \) be given by
\[
\frac{u_{n+1}^n - u_n^n}{\tau} = i \Delta u_{n+1} + i \Delta u_n + i \lambda g_\varepsilon \left( \frac{u_{n+1}^n + u_n^n}{2} \right) + h_\varepsilon \left( f(\frac{t_{n+1}}{2}) \right) + s_\varepsilon^n.
\]
Then, subtracting (3.8) from (3.1), we obtain
\[
s_\varepsilon^n - s_\varepsilon^n = \frac{i}{2} \left( \int_0^1 \left[ \Delta u(t_{n+1}, \cdot) + \Delta u(t_n, \cdot) \right] - \left[ \Delta u(t_{n+1}) + \Delta u(t_n) \right] \right), \quad n = 0, \ldots, N - 1.
\]
After using the Taylor formula with respect to the space variables (see, e.g. [20]), we conclude
\[
\max_{0 \leq n \leq N-1} \left| s_\varepsilon^n - s_\varepsilon^n \right|_{\infty,H} \leq C_{\tau^2} \left( h_1^2 \max_D |\partial_{x_1}^2 u| + h_2^2 \max_D |\partial_{x_2}^2 u| \right).
\]

**4. Convergence.**

We are now ready to derive an \( L_\varepsilon^n(L_\varepsilon^n) \) error estimate for the \( (\varepsilon CNFD) \) method in the theorem below.
Theorem 4.1. Let \( c := e + |\tau| + \max_0^m |u| \) and \((U^m)_{m=0}^N\) be \((\epsilon CNFD)\) approximations specified by (2.7)–(2.8). If \( \epsilon = 0 \) and \( \tau \in (0, \frac{1}{2\tau}) \), then there exists a constant \( C_{cnf} > 0 \) independent of \( \tau, h_1 \) and \( h_2 \), such that
\[
\max_0^m \| u^m - U^m \|_{0,h} \leq C_{cnf} \left[ \tau^2 + \tau^2 |\ln(\tau)| + h_1^2 + h_2^2 \right].
\]
If \( \epsilon, \tau \in (0, \frac{1}{2\tau}) \), then there exists a constant \( C_{cnf} > 0 \) independent of \( \tau, h_1, h_2 \) and \( \epsilon \), such that
\[
\max_0^m \| u^m - U^m \|_{0,h} \leq C_{cnf} \left[ \epsilon + \tau^2 + \tau^2 |\ln(\epsilon)| + h_1^2 + h_2^2 \right].
\]

Proof. To simplify the notation, we set \( e^m := u^m - U^m \) for \( m = 0, \ldots, N \). In the sequel, we will use the symbol \( C \) to denote a generic constant that is independent of \( \tau, h_1, h_2 \) and \( \epsilon \), and may changes value from one line to the other.

Subtract (2.8) from (3.8), to obtain the error equation
\[
e^{n+1} - e^n = i \tau \Delta u \left( \frac{e^{n+1} + e^n}{2} \right) + i \tau \lambda \left[ g_\epsilon \left( \frac{U^{n+1} + U^n}{2} \right) - g_\epsilon \left( \frac{u^{n+1} + u^n}{2} \right) \right] + \tau s^n_e
\]
for \( n = 0, \ldots, N-1 \). Take the \((\cdot, \cdot)_{0,h}\)-inner product of both sides of (4.3) with \((e^{n+1} + e^n)\), use (2.1) and keep the real parts of the equality obtained to get
\[
\|e^{n+1}\|_{0,h}^2 - \|e^n\|_{0,h}^2 = K^n_1 + K^n_2, \quad n = 0, \ldots, N-1,
\]
where
\[
K^n_1 := \tau \text{Re} \left[ (s^n_e, e^{n+1} + e^n)_{0,h} \right],
K^n_2 := -2 \lambda \tau \text{Im} \left[ \left( g_\epsilon \left( \frac{u^{n+1} + u^n}{2} \right) - g_\epsilon \left( \frac{U^{n+1} + U^n}{2} \right) \right) \right].
\]

For \( n = 0, \ldots, N-1 \), using the Cauchy-Schwarz inequality and (3.9), we have
\[
K^n_1 \leq \tau \|s^n_e\|_{0,h} \|e^{n+1} + e^n\|_{0,h}
\]
\[
\leq \tau \left( \|s^n_e\|_{0,h}^2 + \|r^n_e\|_{0,h}^2 \right) \|e^{n+1} + e^n\|_{0,h}
\]
\[
\leq C \tau \left( h_1^2 + h_2^2 + \max_{0\leq n\leq N-1} |r^n_e|_{\infty,h} \right) \left( \|e^{n+1}\|_{0,h} + \|e^n\|_{0,h} \right)
\]
and
\[
K^n_2 \leq \frac{|\lambda\tau|}{2} \left( \|e^{n+1}\|_{0,h} + \|e^n\|_{0,h} \right)^2.
\]

Combining (4.4), (4.5) and (4.6), we conclude that
\[
(1 - \frac{|\lambda\tau|}{2}) \|e^{n+1}\|_{0,h} \leq \left( 1 + \frac{|\lambda\tau|}{2} \right) \|e^n\|_{0,h}
\]
\[
+ C \tau \left( h_1^2 + h_2^2 + \max_{0\leq n\leq N-1} |r^n_e|_{\infty,h} \right), \quad n = 0, \ldots, N-1.
\]

Since \( \tau \in (0, \frac{1}{2\tau}) \), we conclude that \( (1 - \frac{|\lambda\tau|}{2})^{-1} < \frac{4}{3} \) and \( (1 + \frac{|\lambda\tau|}{2}) (1 - \frac{|\lambda\tau|}{2})^{-1} \leq (1 + \frac{4}{3} |\lambda| \tau) \). The latter inequalities along with (4.7) easily yield that
\[
\|e^{n+1}\|_{0,h} \leq \left( 1 + \frac{4}{3} |\lambda| \tau \right) \|e^n\|_{0,h} + \tilde{C} \tau \left( h_1^2 + h_2^2 + \max_{0\leq n\leq N-1} |r^n_e|_{\infty,h} \right), \quad n = 0, \ldots, N-1.
\]

Next, we sum with respect to \( n \), from 0 up to \( m-1 \), to obtain
\[
\|e^m\|_{0,h} \leq \|e^0\|_{0,h} + \tilde{C} T \left( h_1^2 + h_2^2 + \max_{0\leq n\leq N-1} |r^n_e|_{\infty,h} \right) + \frac{4}{3} |\lambda| \tau \sum_{n=0}^{m-1} \|e^n\|_{0,h}, \quad m = 1, \ldots, N.
\]

Using that \( e^0 = 0 \) and applying a standard discrete Gronwall argument on (4.8) (see, e.g. Lemma 8.14 in [23]), we arrive at
\[
\max_{0\leq n\leq N} \|e^n\|_{0,h} \leq \tilde{C} T e^{\frac{4|\lambda\tau|}{3}} \left( h_1^2 + h_2^2 + \max_{0\leq n\leq N-1} |r^n_e|_{\infty,h} \right),
\]
which, along with the consistency bounds (3.5) and (3.7), establishes (4.1) and (4.2), respectively.

Remark 4.1. According to the error estimate (4.2), the (εCNFD) method has an almost second order convergence with respect to τ after choosing ε = τ^2. Thus, the use of the ε-regularization g_ε of g does not affect the asymptotic complexity of the (εCNFD) method, which determined by the values of N, J_1 and J_2.

Remark 4.2. The error estimate (4.1) establishes an almost second order convergence of the usual Crank-Nicolson finite differences that corresponds to the value ε = 0. This is an indication that the regularization of the logarithmic term is not necessary for this method.

Let us now provide an L^∞_t(H^1) error estimate for the (εCNFD) method without requiring additional regularity for the solution u to the problem.

Corollary 4.1. Let c := e + |λ| + max |u| and (U^m)^∞ m=0 be (εCNFD) approximations specified by (2.7)-(2.8). If ε = 0, τ ∈ (0, 1/2ε) and

\[ C_{1,ε} h_1^{-2} \tau^2 \left( \left| \tau^2 + \tau^2 \left| \ln(\tau) \right| \right| + h_1^2 + h_2^2 \right) \leq \epsilon, \]

where C_{1,ε} is the constant in (4.1), then there exists a constant C_{1,ε} > 0 independent of τ, h_1 and h_2, such that

\[ \max_{0 \leq m \leq N} |u^m - U^m|_{1,H} \leq C_{1,ε} \tau^{\frac{1}{2}} |\ln(\tau)|^{\frac{1}{2}} \left( \tau^2 + \tau^2 |\ln(\tau)| + h_1^2 + h_2^2 \right). \]

If ε, τ ∈ (0, 1/2ε) and

\[ C_{2,ε} h_1^{-2} \tau^2 \left( \left| \varepsilon + \tau^2 + \tau^2 \right| \left| \ln(\varepsilon) \right| \right| + h_1^2 + h_2^2 \right) \leq \epsilon, \]

where C_{2,ε} is the constant in (4.2) then there exists a constant C_{2,ε} > 0 independent of τ, h_1, h_2 and ε, such that

\[ \max_{0 \leq m \leq N} |u^m - U^m|_{1,H} \leq C_{2,ε} \tau^{\frac{1}{2}} |\ln(\varepsilon)|^{\frac{1}{2}} \left( \varepsilon + \tau^2 + \tau^2 |\ln(\varepsilon)| + h_1^2 + h_2^2 \right). \]

Proof. For simplicity, we keep the notation and the notation convention of the proof of Theorem 4.1.

Taking the (\cdot,\cdot)_{0,H}-inner product of both sides of (4.3) with (e^{n+1} - e^n), using (2.1) and (2.2), and, finally, keeping the imaginary parts of the equality obtained we arrive at

\[ |e^{n+1}|_{1,H}^2 - |e^n|_{1,H}^2 = Z_1^n + Z_2^n, \quad n = 0, \ldots, N - 1, \]

where

\[ Z_1^n := 2 \Im \left( (s^n, e^{n+1} - e^n)_{0,H} \right), \]
\[ Z_2^n := 2 \lambda \Re \left( (g_{\varepsilon} \left( \frac{u^{n+1} + u^n}{2} \right) - g_{\varepsilon} \left( \frac{U^{n+1} + U^n}{2} \right), e^{n+1} - e^n)_{0,H} \right). \]

Summing with respect to n, from 0 up to m - 1, and using that e^0 = 0, from (4.14) we obtain

\[ |e^m|_{1,H}^2 = \sum_{n=0}^{m-1} (Z_1^n + Z_2^n), \quad m = 0, \ldots, N - 1, \]

which, obviously, yields the following error bound

\[ \max_{0 \leq m \leq N} |e^m|_{1,H}^2 \leq T \tau^{-1} \max_{0 \leq m \leq N-1} (|Z_1^n| + |Z_2^n|). \]

Assuming that τ ∈ (0, 1/2ε) and ε = 0 or ε ∈ (0, 1/2ε), we combine (2.3) with (4.10) and (4.1), or, with (4.12) and (4.2), to conclude

\[ \max_{0 \leq n \leq N-1} |U^n|_{\infty,H} \leq \max_{0 \leq n \leq N-1} |u^n|_{\infty,H} + \max_{0 \leq n \leq N-1} |U^n - u^n|_{\infty,H} \leq \max_0 |u| + \epsilon < c. \]
Also, using the Cauchy-Schwarz inequality and (3.9), it follows that
\[ |Z^n| \leq 2 |s^n|_{0,0} |e^{n+1} - e^n|_{0,0} \]
\[ \leq 4 \max_{0 \leq n \leq N} |e^n|_{0,0} \left( |s^n|_{0,0} + |r^n|_{0,0} \right) \]
\[ \leq C \max_{0 \leq n \leq N} |e^n|_{0,0} \left( h^2_1 + h^2_2 + \max_{0 \leq n \leq N-1} |r^n|_{\infty,\infty} \right) \]
\[ \leq C \left( h^2_1 + h^2_2 + \max_{0 \leq n \leq N-1} |r^n|_{\infty,\infty} \right)^2, \quad n = 0, \ldots, N-1, \]
which, along with (4.9), yields
\[ (4.17) \quad |Z^n| \leq C \left( h^2_1 + h^2_2 + \max_{0 \leq n \leq N-1} |r^n|_{\infty,\infty} \right)^2, \quad n = 0, \ldots, N-1. \]

Case I. Let us assume that \( \varepsilon = 0 \) and \( \tau \in \left( 0, \frac{1}{2N} \right) \). Using (2.4), (2.6) and (4.16), we get
\[ |Z^n| \leq 4|\lambda| \max_{0 \leq n \leq N} |e^n|_{0,0} \left[ \left| \tau^2 + |\ln(\tau)| \right| \max_{0 \leq n \leq N} |e^n|_{0,0} \right] \]
\[ \leq C \left[ \tau^4 + (1 + |\ln(\tau)|) \max_{0 \leq n \leq N} |e^n|_{0,0} \right] \]
\[ \leq C \left[ \tau^4 + |\ln(\tau)| \max_{0 \leq n \leq N} |e^n|_{0,0} \right] \]
\[ \leq C \left[ \tau^4 + |\ln(\tau)| \max_{0 \leq n \leq N} |e^n|_{0,0} \right], \quad n = 0, \ldots, N-1, \]
which, along with (4.9), yields
\[ (4.18) \quad |Z^n| \leq C \left[ \tau^4 + |\ln(\tau)| \left( h^2_1 + h^2_2 + \max_{0 \leq n \leq N-1} |r^n|_{\infty,\infty} \right) \right]^2, \quad n = 0, \ldots, N-1. \]

Now, we combine (4.17) (with \( \varepsilon = 0 \)), (4.18) and (4.5), to get
\[ (4.19) \quad \max_{0 \leq n \leq N-1} (|Z^n_1| + |Z^n_2|) \leq C |\ln(\tau)| \left[ h^2_1 + h^2_2 + \tau^2 + \tau^2 |\ln(\tau)| \right]^2. \]

Case II. Let us assume that \( \varepsilon, \tau \in \left( 0, \frac{1}{2N} \right) \). Applying the Cauchy-Schwarz inequality, (2.6) and (4.16), we get
\[ |Z^n_2| \leq 4|\lambda| \max_{0 \leq n \leq N} |e^n|_{0,0} \left[ \left| g\left( \frac{u^n + u^n}{2} \right) - g\left( \frac{u^n + u^n}{2} \right) \right|_{0,0} \right] \]
\[ \leq C |\ln(\varepsilon)| \max_{0 \leq n \leq N} |e^n|_{0,0}, \quad n = 0, \ldots, N-1, \]
which, along with (4.9), yields
\[ (4.20) \quad |Z^n_2| \leq C |\ln(\varepsilon)| \left( h^2_1 + h^2_2 + \max_{0 \leq n \leq N-1} |r^n|_{\infty,\infty} \right)^2, \quad n = 0, \ldots, N-1. \]

Hence, under the light of (4.17), (4.20) and (3.7), it follows that
\[ (4.21) \quad \max_{0 \leq n \leq N-1} (|Z^n_1| + |Z^n_2|) \leq C |\ln(\varepsilon)| \left[ h^2_1 + h^2_2 + \varepsilon + \tau^2 + \tau^2 |\ln(\varepsilon)| \right]^2. \]

Finally, we obtain (4.11) as an outcome of (4.15) and (4.19), and (4.13) by combining (4.15) and (4.21).

**Remark 4.3.** After choosing \( h_1 = O(\tau), h_2 = O(\tau) \) and \( \varepsilon = \tau^2 \), from (4.11) and (4.13), we can, easily, conclude an \( O(\tau^{\frac{3}{2}} |\ln(\tau)|^{\frac{3}{2}}) \) error bound of the discrete \( L^\infty(H^1) \) norm.
5. COMPUTING THE ($\varepsilon$CNFD) APPROXIMATIONS

At every time step, the corresponding ($\varepsilon$CNFD) approximation is defined, according to (2.8), as the solution of a nonlinear system of algebraic equations, which one has to approximate in a computer implementation of the method. This can be done by applying the fixed point iterations method, which we discuss and analyze below.

Let $c = c + |\lambda| + \max_n |u|$, $\varepsilon = \tau^2$ or $\varepsilon = 0$, and $\tau \epsilon \left(0, \frac{1}{c}\right)$. Then, Propositions 2.1 and 2.2 yield the existence and uniqueness of the ($\varepsilon$CNFD) approximations and the error estimates (4.1) (when $\varepsilon = 0$) and (4.2) (when $\varepsilon = \tau^2$) hold. Also, let us assume that

$$\max\{C_{1,\varepsilon w}, C_{2,\varepsilon w}\} \leq \frac{\varepsilon}{2} \left[2\tau^2 + 2\tau^2|\ln(\tau)| + h^2 + h^2\right] \leq c$$

where $C_{1,\varepsilon w}$ and $C_{2,\varepsilon w}$ are the constants in (4.1) and (4.2), respectively. Then, using (2.3) and (4.1) (when $\varepsilon = 0$) or (4.2) (when $\varepsilon = \tau^2$), we, easily, conclude that

$$\max_{0 \leq n \leq N} |U^n|_{\sigma, h} \leq c.$$

Let $n, c \in C^1(\mathbb{R}, \mathbb{R})$ (see [22]) be an odd auxiliary function defined by

$$n_+(s) := \begin{cases} s, & \text{if } s \in [0, c], \\ q(s), & \text{if } s \in (c, 2c], \\ 2c, & \text{if } s > 2c, \end{cases} \quad \forall s \geq 0,$$

where $q \in P^3[c, 2c]$ is a polynomial satisfying: $q(c) = c$, $q'(c) = 1$, $q(2c) = 2c$ and $q'(2c) = 0$. Obviously it holds that $n_+(s) = s$ when $|s| \leq c$, and we can show (see [24]) that

$$\sup_r |n_r| \leq 2c, \quad \sup_r |n_r'| \leq \frac{4}{3}.$$  

We extend $n_+$ on $\mathbb{C}$, by setting $n_+(z) := n_+(\text{Re}(z)) + i n_+(\text{Im}(z))$ for $z \in \mathbb{C}$. Then, in view of (5.2), it holds that

$$|n_+(z)| \leq \sqrt{2} \sup_r |n_r| \leq 2\sqrt{2}c \leq 4c,$$

$$|n_+(z) - n_+(w)| \leq \sup_r |n_r'| |z - w| \leq \frac{4}{3} |z - w| \quad \forall z, w \in \mathbb{C}.$$

Let $n = 0, \ldots, N - 1$. Then, we introduce an operator $\Phi_{\varepsilon,n} : X_n \mapsto X_n$ given by

$$\Phi_{\varepsilon,n}(V) := (\mathcal{J}_n - i \frac{\tau}{2} \Delta_n)^{-1} \left[U^n + i \frac{\tau}{2} g_\varepsilon(n_+(V)) + \frac{\tau}{2} h_n \left[f\left(e^{\lambda \frac{\tau}{2}}\right)\right]\right] \quad \forall V \in X_n,$$

where $\mathcal{J}_n : X_n \mapsto X_n$ is the identity operator. Under the light of (5.1), we have $n_+\left(\frac{U^{n+1} + U^n}{2}\right) = \frac{U^{n+1} + U^n}{2}$ and thus $\frac{U^{n+1} + U^n}{2}$ is a fixed point of $\Phi_{\varepsilon,n}$, i.e. $\frac{U^{n+1} + U^n}{2} = \Phi_{\varepsilon,n}\left(\frac{U^{n+1} + U^n}{2}\right)$.

Let us, first, consider the case $\varepsilon = \tau^2$. Since, it holds that

$$\|\mathcal{J}_n - i \frac{\tau}{2} \Delta_n\|_{0,h} \leq \|V\|_{0,h} \quad \forall V \in X_n,$$

we use (2.6) and (5.3), to obtain

$$\|\Phi_{\varepsilon,n}(V) - \Phi_{\varepsilon,n}(W)\|_{0,h} \leq \frac{\varepsilon |\lambda|}{2} \|\mathcal{J}_n - i \frac{\tau}{2} \Delta_n\|_{0,h} \|g_\varepsilon(n_+(V)) - g_\varepsilon(n_+(W))\|_{0,h}$$

$$\leq \frac{\varepsilon |\lambda|}{2} \|g_\varepsilon(n_+(V)) - g_\varepsilon(n_+(W))\|_{0,h} \leq 2 \tau |\lambda| |\ln(\tau)| |n_+(V) - n_+(W)|_{0,h}$$

$$\leq \frac{2}{\tau} \tau |\lambda| |\ln(\tau)| |V - W|_{0,h} \quad \forall V, W \in X_n.$$

Under the condition $\frac{8}{3} |\tau| |\ln(\tau)| \leq \frac{1}{3}$, (5.5) yields that the operator $\Phi_{\varepsilon,n}$ is a contraction map on $X_n$, which results in the corresponding fixed point iterations method.
Now, we consider the case $\varepsilon = 0$. Using (5.4), (5.5), (5.6) and (5.2), we get
\[
\|\Phi_{0,n}(V) - \Phi_{0,n}(W)\|_{0,h} \leq \frac{\tau|\lambda|}{2} \left\|\left(\mathbb{S}_h - i\frac{\tau}{2} \mathbb{A}_h\right)^{-1}\left[g(n_*(V)) - g(n_*(W))\right]\right\|_{0,h}
\]
\[
\leq \frac{\tau|\lambda|}{2} \left\|g(n_*(V)) - g(n_*(W))\right\|_{0,h}
\]
\[
\leq \frac{\tau|\lambda|}{2} \left[2|D|^{\frac{1}{2}} \tau^2 + \|g_{\tau_2}(n_*(V)) - g_{\tau_2}(n_*(W))\|_{0,h}\right]
\]
\[
\leq \tau|\lambda| \left[2|D|^{\frac{1}{2}} \tau^2 + \frac{\tau}{2} \varepsilon \|\ln(r)\| \|V - W\|_{0,h} \quad \forall V, W \in X_h,
\]
where $|D|$ is the area of $D$. Let us impose the condition $\frac{8|\lambda|}{3} \varepsilon |\ln(r)| \leq \frac{1}{2}$ and define the sequence $(\Psi^\ell)_{\ell=0}^{\infty} \subset X_h$ by $\Psi^0 = U^n$ and $\Psi^{\ell+1} : = \Phi_{0,n}(\Psi^\ell)$ for $\ell \in \mathbb{N}_0$. Using (5.6) and that $\frac{U_{n+1} + U^n}{2}$ is a fixed point of $\Phi_{0,n}$, we obtain
\[
\|\Psi^{\ell+1} - \frac{U_{n+1} + U^n}{2}\|_{0,h} \leq \left|\lambda\right| |D|^{\frac{1}{2}} \tau^3 + \frac{1}{2} \left\|\Psi^\ell - \frac{U_{n+1} + U^n}{2}\right\|_{0,h} \quad \forall \ell \in \mathbb{N}_0,
\]
which, after applying an induction argument, yields
\[
\|\Psi^{\ell+1} - \frac{U_{n+1} + U^n}{2}\|_{0,h} \leq \left|\lambda\right| |D|^{\frac{1}{2}} \tau^3 \left(\sum_{\ell=1}^{\ell} \frac{1}{2^{\ell-1}}\right) + \frac{1}{2} \left\|\Psi^0 - \frac{U_{n+1} + U^n}{2}\right\|_{0,h} \quad \forall \ell \in \mathbb{N}_0.
\]
By stopping the computation when $\frac{1}{2^{\ell-1}} \leq \tau^3$, we have
\[
\|\Psi^{\ell+1} - \frac{U_{n+1} + U^n}{2}\|_{0,h} \leq \left(2 \left|\lambda\right| |D|^{\frac{1}{2}} + \frac{1}{2} \left\|U_{n+1} - U^n\|_{0,h}\right) \tau^3,
\]
and thus we arrive at an $O(\tau^3)$ approximation of the average $\frac{U_{n+1} + U^n}{2}$.

In actual computations, we have observed that the fixed point iterations method (choosing as an initial guess the approximation of the previous time step) converges without using the cut-off function $n_*$ (see Section 6).

6. Numerical Results

The ($\varepsilon$CNFD) method has been implemented in Python 3.7.0 programs, where we solve the nonlinear systems of algebraic equations by applying the iterative fixed point method along with the GMRES method for solving the corresponding linear systems by calling the subroutine gmres of the library scipy.sparse.linalg.

We validate the code by computing the numerical approximation error in the discrete $L^\infty$ norm $E^\infty(N, J_1, J_2) = \max_{0 \leq n \leq N} \|U^n - u^n\|_{0,h}$ and in the $L^\infty$ norm $E^\infty(N, J_1, J_2) = \max_{0 \leq n \leq N} \|U^n - u^n\|_{\infty,h}$, when we are aware of the exact solution to the problem. Also, choosing $\nu \in \mathbb{N}$, a function $f : (0, +\infty) \rightarrow (0, +\infty)^3$ and $(N, J_1, J_2) = f(\nu)$, we compute the experimental order of convergence with respect to $\nu$, which corresponds to given values $\nu_1$ and $\nu_2$ of $\nu$, by using the formula
\[
\ln\left[E(\varepsilon(\nu_1))/E(\varepsilon(\nu_2))\right]/\ln(\nu_2/\nu_1),
\]
where $E = E^0$ or $E^\infty$.

6.1. Example 1. Let $T = 1$, $\Omega = [0,1] \times [0,1]$, $(N, J_1, J_2) = f(\nu) = (\nu, \nu, \nu)$, $\varepsilon \in \{0, \tau^3, \tau^2\}$, $\nu \in \{20, 40, 80, 160, 320\}$ and $\lambda = 1$. The initial value $u_0$ and the load $f$ are such that the function
\[
u \to u_{ex}(t, x) = (1 + i) e^{-t} \sin(3\pi x_1) \sin(3\pi x_2)
\]
to be the exact solution to the problem (1.1) and (1.4). We note that the artificial exact solution $u_{ex}$ has been designed in the way to be zero on the boundary of $\Omega$ and to change sign in the interior of $\Omega$. On Table 1 and Table 2, we show the numerical approximation errors we computed and the corresponding experimental order of convergence. The results, clearly, confirm a second order experimental order of convergence with respect to $\nu$, for all choices for $\varepsilon$ and for both norms. Also, we observe that the smaller value for $\varepsilon$ we choose, the smaller numerical errors we get. The
efficiency of the numerical method for $\varepsilon = 0$ is a strong indication that the use of the regularization $g_\varepsilon$ may not be necessary.

\begin{table}[h]
\centering
\begin{tabular}{cccc}
$\nu$ & $E^\nu(f(\nu))$ & Rate & $E^\infty(f(\nu))$ & Rate \\
\hline
20 & 5.9250(-3) & 1.97 & 1.1840(-2) & 1.88 \\
80 & 4.0169(-4) & 1.97 & 8.0380(-4) & 1.96 \\
320 & 1.0106(-4) & 1.99 & 2.0219(-4) & 1.99 \\
\end{tabular}
\caption{Example 1}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{cccc}
$\nu$ & $E^\nu(f(\nu))$ & Rate & $E^\infty(f(\nu))$ & Rate \\
\hline
20 & 2.2941(-2) & — & 4.3628(-2) & — \\
80 & 4.0169(-4) & 1.97 & 8.0380(-4) & 1.96 \\
320 & 1.0106(-4) & 1.99 & 2.0219(-4) & 1.99 \\
\end{tabular}
\caption{Example 1}
\end{table}

6.2. Example 2. Let $T = 2$, $\Omega = [-12, 12] \times [-12, 12]$, $(N, J_1, J_2) = f(\nu) = (\nu, \nu, \nu)$, $\varepsilon = \tau^2$, $\nu \in \{40, 80, 160, 320\}$, $f = 0$, $\lambda = 2$ and initial condition

$$u_0(x) = \pi^{-\frac{1}{4}} \exp\left(-\frac{1}{2} (x_1^2 + x_2^2) + i (x_1 + x_2)\right) \quad \forall x \in \Omega,$$

which is almost zero on the boundary of $\Omega$. Letting

$$w(t, \xi) := \pi^{-\frac{1}{4}} \exp\left(-\frac{1}{2} (\xi - 2t)^2 + i \xi - i \frac{1}{2} (4 + \ln(\pi)) t\right) \quad \forall t \geq 0, \quad \forall \xi \in \mathbb{R},$$

be a uniformly moving Gausson in one space dimension case (see, e.g., [2], [9], [5]), $u_0$ is the value at $t = 0$ of the function

$$u_\varepsilon(t, x) = w(t, x_1) w(t, x_2) \quad \forall t \in [0, +\infty), \quad \forall x \in \mathbb{R}^2,$$

which is an exact solution to the logarithmic Schrödinger equation (1.1). It is easily seen that $u_\varepsilon$ has almost compact support up to time $t = 5$ and thus we can consider that it is a solution to the problem (1.1)–(1.4) on $\mathcal{D}$. In the numerical experiments we compute the error approximating $u_\varepsilon$ and we post the results on Table 3. We observe, again, a second order experimental convergence and that the method is efficient for $\varepsilon = 0$.

6.3. Example 3. Let $T = 1.5$, $\Omega = [-12, 12] \times [-12, 12]$, $(N, J_1, J_2) = f(\nu) = (\nu, \nu, \nu)$, $\varepsilon = 0$, $\nu = 150$, $f = 0$, $\lambda = 1$ and initial condition $u_0(x) = \tanh(x_1) \tanh(x_2) e^{-|x|^2}$ for $x \in \Omega$ (cf. [5]). In Figure 1 we post snapshots of the absolute value of the numerical solution for various time levels and in Figure 2 we show snapshot of the real and the imaginary part of the numerical approximation for $t = 1.5$. It is clear that the fact that the solution takes zero values is not a source of singularity.
(εCNFD) method with ε = 0
Table 3.

<table>
<thead>
<tr>
<th>ν</th>
<th>$E^0(f(\nu))$</th>
<th>Rate</th>
<th>$E^\infty(f(\nu))$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>3.9987(-2)</td>
<td></td>
<td>4.9215(-1)</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>9.7244(-3)</td>
<td>2.03</td>
<td>1.0367(-1)</td>
<td>2.24</td>
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<tr>
<td>160</td>
<td>2.4242(-3)</td>
<td>2.00</td>
<td>2.5763(-2)</td>
<td>2.00</td>
</tr>
<tr>
<td>320</td>
<td>6.0708(-4)</td>
<td>1.99</td>
<td>6.4455(-3)</td>
<td>1.99</td>
</tr>
</tbody>
</table>

(εCNFD) method with ε = $\tau^2$

<table>
<thead>
<tr>
<th>ν</th>
<th>$E^0(f(\nu))$</th>
<th>Rate</th>
<th>$E^\infty(f(\nu))$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>4.0281(-2)</td>
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<td>4.9741(-1)</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>9.9408(-3)</td>
<td>2.01</td>
<td>1.0538(-1)</td>
<td>2.23</td>
</tr>
<tr>
<td>160</td>
<td>2.4835(-3)</td>
<td>2.00</td>
<td>2.6368(-2)</td>
<td>1.99</td>
</tr>
<tr>
<td>320</td>
<td>6.2222(-4)</td>
<td>1.99</td>
<td>6.5854(-3)</td>
<td>2.00</td>
</tr>
</tbody>
</table>

**Figure 1.** Example 3: Snapshots of the absolute value of the numerical approximations for $t = 0$, $t = 0.5$, $t = 1.0$, $t = 1.5$.

**Figure 2.** Example 3: Snapshots of the real and imaginary part of the numerical approximations for $t = 1.5$.

7. **Conclusions**

We propose the approximation of the solution to the logarithmic Schrödinger equation over a two dimensional rectangular domain by the ($\varepsilon$CNFD) method described in Section 2.2. The $B$-stability property of the Crank-Nicolson method allows us to build-up a stability argument based on the (CH) property (2.11), which leads to an almost second order error estimate in the discrete $L^\infty_t(L^2_x)$ norm, even in the case $\varepsilon = 0$, where no regularization of the logarithmic term $g$ is imposed. In
the convergence analysis we handle the parameter $\varepsilon$ as a discretization parameter by restricting its use to the estimation of the consistency error. That approach results in error estimation constants that are free of $\varepsilon$. Finally, a set of numerical experiments confirms the theoretical findings. Future work could target the construction of higher order numerical methods for the problem at hands, combining higher order $B$-stable Runge-Kutta methods with higher order space discretization techniques.

References

[22] P. Paraschis and G. Zouraris. Backward Euler finite difference approximations of a logarithmic heat equation over a 2D rectangular domain. hal-03220015 (May 6, 2021).