A switching observer for a class of non-uniformly observable systems via singular time-rescaling

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***Abstract*—We present a switching observer for a class of non-uniformly observable systems that are affine in the unmeasured states and nonlinear in the measured output. Using a singular time-rescaling, the dynamics of the estimation error is transformed into that of a bimodal switched linear system. Sufficient conditions that guarantee the observer's uniform asymptotic stability are provided; these are stated in terms of persistency of excitation and dwell-time of the (output-dependent) time-scaling function evaluated along the trajectories of the system. Unlike most results on switching observers, our approach does not rely on the solution of a set of linear matrix inequalities to compute the observer gain. In addition, we compare our scheme against the Kalman observer in a particular, but meaningful, case-study of observer-based control in automotive systems.**

***Index Terms*—Observers, non-uniformly observable systems, switched systems, non-strict Lyapunov functions, dwell-time, persistency of excitation, time scaling, Kalman observer.**

I. INTRODUCTION

Since the seminal work of Luenberger, for continuous time-invariant linear systems, the observer design problem has been extensively studied in the literature and various approaches for different classes of nonlinear systems have been proposed —see, *e.g.*, [1]–[9] and references therein. In this note we address the observer design problem for non-uniformly observable systems, affine in the unmeasured states and nonlinear in the measured output. These are systems of the form

$$\dot{z} = \gamma(y)\big[Az + d(y)\big] + b(y)u \qquad (1a)$$
$$y = Cz, \qquad (1b)$$

where $z \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}$ is the output, the pair (A, C) is observable, and $\gamma : \mathbb{R} \to \mathbb{R}$ is a smooth function; we do not exclude output values $y \in \mathbb{R}$ such that $\gamma(y) = 0$. Although systems (1) may appear restrictive at first sight, it is important to emphasize that many systems of the more general form

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x),$$

are equivalent to (1) up to a coordinate transformation, $z = \Phi(x)$ where $x \in \mathbb{R}^n$. Conditions for the existence of such a transformation are given in [10], [11], and [12]. Beyond academic interest, however, our motivation to study systems of the form (1) comes from the fact that they appear in automotive control [13], mobile robotics [14], and chemical engineering [15], to mention a few applications —see also Section III.

Most observer-based schemes in the literature rely on the assumption that the system is uniformly observable. Yet, this is frequently untrue in many concrete control engineering problems [15], [16]. Indeed, in many cases $\gamma(y(t))$ may be equal (or tend) to zero, which leads to loss of observability. For instance, for initial states $z_0 \in \mathbb{R}^n$ such that $\gamma(Cz_0) = 0$, the trajectories $z(t)$ generated via (1a) with $u \equiv 0$ satisfy $\gamma(y(t)) \equiv 0$ and are thus indistinguishable. For a more detailed description of the observability properties of this system, see [12, Prop. 1]. In the observer design for systems of the form (1), we allow for points such that $\gamma(y) = 0$; this makes our main results more interesting for applications, but the respective proofs technically more involved. In our approach, the lack of uniform observability is overcome by constructing an n-dimensional *switching* observer, and imposing both *persistency of excitation* and zero-crossing *dwell-time* on the output.

Many other switching observers have been proposed in the literature, but mostly for switched linear systems [17]–[20]. They have also been used in several concrete applications, such as cellular processes [21], catalyst converters [22], mechanical systems [23], [24], and cyber-physical systems [25]. These approaches, however, do not provide a constructive procedure to design the observer gain. Instead, they rely on the existence of a solution for a set of linear matrix inequalities, from which the observer gain can be computed. Fundamentally different to designing observers *for* switched systems, in this note, switching is merely a technical mean to achieve the goal of designing an observer for non-uniformly observable systems.

Another technical building block is an output-dependent change of time-scale. In that regard, we borrow inspiration from the time-rescaling approach proposed in [10] and [11] —*cf.* [26], for the construction of observers for systems that cannot be linearized using a coordinate change only. As in the earliest work on time-rescaling [27], the time-scaling function considered in [10], [11], and [26] is regular and this restricts the construction of the proposed observers to uniformly observable systems. In this note, we use a singular time-scaling function (possibly not possessing a smooth inverse), thereby generalizing the observer design methodology of [11] to the realm of non-uniformly observable systems.

Now, it may be argued that constructing an observer for (1) is a simple task that has been thoroughly addressed in the literature. For instance, for the discretized version of this

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system, an observer can be designed using a Kalman filter [28, § 6.4]. In a continuous-time setting, system (1) fits into the more general class of state-affine systems

$$\dot{z} = A(t,u,y)z + B(t,u,y), \qquad y = Cz,$$

studied, *e.g.*, in [5], [29], [30]. Nevertheless, such approaches present several drawbacks. Using a Kalman-like construction as in [5], [29] involves solving a Riccati ODE of dimension $n(n+1)/2$ to determine the observer gain, which increases the complexity and the computation time necessary to implement the observer. Moreover, finding explicit conditions that guarantee the convergence of the observer is in general non trivial. In [30], for instance, weak sufficient conditions in terms of persistency of excitation of the output trajectories, $y(t)$, are given, but which are fairly difficult to verify.

To the best of our knowledge, we present the first switching observer that does not rely on solving a set of linear matrix inequalities to compute the observer gain. As a matter of fact, in contrast with other papers not relying on uniform observability, such as [30], we give an explicit expression for the observer gain that guarantees the asymptotic stability for the observer. Moreover, it is one of the few articles, along with [25], where a switching observer for a non-switched system is proposed. This note completes and builds upon the preliminary results presented in [12].

II. Observer design

Since system (1) is affine in the unmeasured variables, a relatively simple manner to define a state observer is to mimic the Luenberger type, so let the state estimate $\hat{z}$ be defined by

$$\dot{\hat{z}} = \gamma(y)\big[A\hat{z} + d(y) + K(y)(y - C\hat{z})\big] + b(y)u, \tag{2}$$

where the column vector $K(\,\cdot\,)$ depends on the system's output. The difficulty of such design resides in defining the observer gain $K(y)$ such that the estimation error, $e := \hat{z} - z$, tends to zero in spite of the loss of observability at the instants t such that $\gamma(y(t)) = 0$.

Furthermore, beyond mere convergence of the estimation error, the problem that we address is that of designing $K(\cdot)$ such that the origin, for the estimation error dynamics,

$$\dot{e} = \gamma(y)\big[A - K(y)C\big]e, \tag{3}$$

is globally asymptotically stable, uniformly in y —*cf.* [30].

To that end, we proceed according to the following reasoning. Let us assume temporarily that the function γ is sign-definite, that is, either $\gamma(y) > 0$ or $\gamma(y) < 0$ for all $y \in \mathbb{R}$. Akin to [11], in such case, we may define a new time variable τ, such that

$$\frac{d\tau}{dt} = \begin{cases} \gamma(y(t)) & \text{if } \gamma(y(t)) > 0 \\ -\gamma(y(t)) & \text{if } \gamma(y(t)) < 0. \end{cases} \tag{4}$$

Then, in the new time scale defined by τ, the estimation error dynamics becomes

$$\frac{de}{d\tau} = \begin{cases} \big[A - K(y)C\big]e & \text{if } \gamma(y) > 0 \\ -\big[A - K(y)C\big]e & \text{if } \gamma(y) < 0 \end{cases} \tag{5}$$

so, to ensure the stability of the origin for (5), depending on whether $\gamma(y) > 0$ or $\gamma(y) < 0$, the gain $K(y)$ can be defined as a constant matrix K such that $A - KC$ or $-[A - KC]$ is Hurwitz.

Assuming that $\gamma(y)$ is sign-definite rules out the existence of the singular set $\{\gamma(y) = 0\}$, where the system looses its observability [12, Prop. 1]. Nevertheless, this assumption is clearly over-conservative and precludes the use of the proposed observer (2) for some concrete examples of control systems of the form (1) —*cf.* [13]. Therefore, as in [12], in this note we consider the scenario in which $\gamma(y(t))$ may vanish or change sign at some time-instances t. Note that such scenario imposes the technical difficulty of overcoming the lack of observability for values of $y(t) \in \mathbb{R}$ such that $\gamma(y(t)) = 0$. Roughly speaking, our main result establishes that the sign-indefiniteness of $\gamma(y)$ may be overcome by using a switching-based observer design that we describe next.

Choose two constant vectors:

$$K_+ = \begin{bmatrix} k_1^+ & \dots & k_n^+ \end{bmatrix}^\top, \quad K_- = \begin{bmatrix} k_1^- & \dots & k_n^- \end{bmatrix}^\top$$

such that the matrices

$$A_+ := A - K_+C, \tag{6}$$

$$A_- := -A + K_-C \tag{7}$$

are Hurwitz and let us define

$$K(y(t)) = \begin{cases} K_+, & \text{if } \gamma(y(t)) > 0 \\ K_-, & \text{if } \gamma(y(t)) < 0. \end{cases} \tag{8}$$

Then, the dynamics of the estimation error, e, in the τ time-scale, becomes

$$\frac{de}{d\tau} = \begin{cases} \big[A - K_+C\big]e, & \text{if } \gamma(y(t)) > 0 \\ -\big[A - K_-C\big]e, & \text{if } \gamma(y(t)) < 0, \end{cases}$$

which is a switched system. Indeed, in compact form, we have

$$\frac{de}{d\tau} = \mathcal{A}_{\sigma(\tau)}e, \tag{9}$$

where $\sigma : \mathbb{R}_{\geq 0} \to \{+1, -1\}$ is a switching function defined farther below (see Lemma 2), $\mathcal{A}_{+1} := A_+$, and $\mathcal{A}_{-1} := A_-$.

Equation (9) represents a switched system that may be analyzed using, for example, the methods proposed in [31]. To that end, a potential approach is to choose two constant vectors K_+ and K_- such that A_+ and A_- in (6) and (7), respectively, are Hurwitz and, in addition, to choose those vectors in such a way that the switched system (9) admits a common (non-strict) Lyapunov function (valid for both A_+ and A_-), in order to ensure exponential stability of the estimation error dynamics, *albeit*, in the τ time-scale. This brings us to our first statement.

Theorem 1: Let (A, C) be an observable pair, define $Q = C^\top C$ and consider the matrices defined in (6), (7). If K_+ is such that A_+ is Hurwitz, then there exists a unique gain K_- such that A_- is also Hurwitz and the Lyapunov equations

$$A_+^\top P + PA_+ = -Q \tag{10}$$

$$A_-^\top P + PA_- = -Q \tag{11}$$

admit a common solution $P = P^\top > 0$. ◁

The following remarks are in order. Theorem 1 is essentially contained in the preliminary work [12] —in the latter it is not stated that the matrix A_- is also Hurwitz, but an explicit solution for P is given and it is also shown that if the pair (A,C) is in the observable canonical form, then the elements of K_- are given by

$$k_i^- := (-1)^i\, k_i^+ + \big[1-(-1)^i\big]a_i, \qquad \forall\, 1 \le i \le n. \tag{12}$$

We also remark that even though both A_+ and A_- are Hurwitz, the matrix Q in (10)-(11) is only positive semidefinite. As a matter of fact, it is known from [32] that these Lyapunov equations do not admit a common solution P if $n = 2$ and Q is positive definite. Therefore, Theorem 1 is a statement of interest in its own right. The methods proposed in [31] provide a straightforward stability analysis for (9), assuming that one disposes of a (non-strict) Lyapunov function. Nevertheless, for an arbitrary switched system, we are not aware of any statement that asserts the existence of a common Lyapunov function, let alone a general analytical method to construct it. In the particular case of (9), Theorem 1 not only guarantees the existence of this common function for particular values of the gains K_+ and K_-, but its proof —see [12]— provides an analytic expression of P.

Now, although intuitive, the rationale that leads to equation (9) hides several technical difficulties. Firstly, it relies on the ability of defining the new *time* variable τ which, in order to make sense as such, must be well-defined. That is, as a function of t, the new time τ must be continuous, strictly increasing, and radially unbounded. Secondly, strictly speaking, system (3) is non-autonomous as it depends on the output trajectories $y(t,t_0,z_0)$. Thus, rather than defining τ as in (4) we define it, more precisely for each (t_0,z_0), as

$$\tau = \nu(t,t_0,z_0) = \int_{t_0}^{t} \big|\gamma(y(s,t_0,z_0))\big| ds, \tag{13}$$

which satisfies

$$\frac{d\tau}{dt} = \big|\gamma(y(s,t_0,z_0))\big|.$$

This means that τ is not only a function of t, but it is inherently parameterized by the system's initial conditions (t_0,z_0). Therefore, appropriate technical conditions, stated below as Assumption 1, must be imposed in order to establish the uniform global asymptotic stability of the origin for (3).

In writing (13), it is implicitly assumed that the trajectories are forward complete. As a matter of fact, following standard practice in the literature on observer design, it is assumed that each pair of initial conditions $(t_0,z_0) \in \mathbb{R}\times\mathbb{R}^n$ and each input $u(\,\cdot\,)$ generates a unique (smooth) trajectory $z(t,t_0,z_0,u(\,\cdot\,))$ that is uniformly bounded in $t_0 \in \mathbb{R}$. To avoid a cumbersome notation, however, in the sequel we drop the argument u by considering trajectories generated by a fixed, but arbitrary, input.

Assumption 1: The function γ in (1a) and the output trajectories, $y(t,t_0,z_0)$, satisfy the following conditions:

(*persistency of excitation*) there exist $\mu_0 > 0$ and $T_0 > 0$ such that, for all $(t_0,z_0) \in \mathbb{R}\times\mathbb{R}^n$,

$$\int_{t}^{t+T_0} \gamma(y(s,t_0,z_0))^2 ds \ge \mu_0, \quad \forall\, t \ge t_0; \tag{14}$$

(*dwell-time*) there exists $T_D > 0$ such that, for any $(t_0,z_0) \in \mathbb{R}\times\mathbb{R}^n$ and any two instants $t_k \neq t_l$ satisfying $\gamma(y(t_k,t_0,z_0)) = \gamma(y(t_l,t_0,z_0)) = 0$,

$$\big|t_k - t_l\big| \ge T_D. \tag{15}$$

Remark 1: Both conditions, (14) and (15), are required to hold uniformly in the output trajectories, hence, in t_0 and compact sets of initial conditions z_0. The persistency of excitation condition cannot, in general, be verified analytically, but it is common in the literature (even in scenarii involving output feedback control) because of its clear meaning and of its importance as a necessary condition for uniform global asymptotic stability. It holds, for instance, if $t \mapsto \gamma(y(t))$ is oscillatory. Also, we stress the second part of Assumption 1 does not define *dwell-time* —for this see, e.g., [31]—, but T_D gives a lower bound on the time between two consecutive switches of the sign of $\gamma(y(t))$.

Assumption 1 ensures that the change of time (13) is defined via a globally invertible function. More precisely, the following statement holds —the proof is presented in Section IV.

Lemma 1: Consider a continuous function $y \mapsto \gamma(y)$ and an absolutely continuous function $t \mapsto y(t,t_0,z_0)$, bounded uniformly in t_0 and satisfying Assumption 1. Then, for each pair (t_0,z_0), the function $\nu(\cdot,t_0,z_0) : [t_0,+\infty) \mapsto [0,+\infty)$, defined by (13), is continuous, strictly increasing, and radially unbounded. It is thus globally invertible. Moreover, for each (t_0,z_0), $\nu(t,t_0,z_0) \to \infty$ as $t \to \infty$, uniformly in t_0. $\triangleleft$

Furthermore, the dwell-time condition (15) ensures that the time instants at which the observability singularity $\{\gamma(y) = 0\}$ is crossed do not accumulate. This condition is fundamental for the following reasons. For the purpose of implementation, according to (8), the value of the observer gain is assigned by verifying whether the sign of $\rho(t) := \gamma(y(t))$ is positive or negative. In turn, this defines a switching signal in the natural time-scale, *i.e.*, with respect to t. For the purpose of analysis, however, the system's behavior is considered as defined in the transformed time-scale, with respect to τ. The following statement, whose proof is also provided in Section IV, ensures that the dwell-time condition is preserved under the time-scale transformation.

Lemma 2: Let $\gamma(y)$ and $y(t,t_0,z_0)$ satisfy Assumption 1 and consider the function $\nu(\cdot,t_0,z_0)$ defined in (13). Let $\rho : \mathbb{R} \to \{-1,+1\}$ and $\sigma : \mathbb{R}_{\ge 0} \to \{-1,+1\}$ be two functions related by $\sigma(\tau) = \rho \circ \nu^{-1}(\tau)$. If $t \mapsto \rho(t)$ has a dwell-time, then so does $\tau \mapsto \sigma(\tau)$. $\triangleleft$

Furthermore, by imposing T_D to be independent of the output trajectory that generates each switching sequence, it is ensured that stability and convergence for the switched system in the τ time-scale remain uniform in the natural time-scale. Our main statement, presented below, relies on Theorem 1 and Lemmata 1 and 2; its proof is presented in Section IV.

Theorem 2 (main result): Consider system (1), with (A,C) observable and under Assumption 1. Consider also the observer given by (2) and (8), with K_+ such that $A - K_+C$ is Hurwitz. Then, there exists K_- such that, for the estimation error dynamics (3), the origin is globally asymptotically stable, uniformly in the output trajectories. Furthermore, for a pair (A, C) in observable companion form, the elements of $K_- := [k_1^- \ \cdots \ k_n^-]$ may be taken according to (12). $\triangleleft$

III. Application example

In this section, we illustrate the performance of the proposed observer on a concrete example of automotive control; specifically, concerning the antilock braking system (ABS). The objective is to estimate the unmeasurable tyre *extended braking stiffness* (XBS) —a variable closely related to the adhesion coefficient between the wheel and the road— through measurements of the acceleration of the wheel and the vehicle. The interest in doing so is that using the XBS one may control the ABS so as to maximize tyre-road adherence. For an in-depth discussion on this problem, see *e.g.*, [13] and [16].

The dynamics of the system is described by

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \frac{z_1}{v_x(t)}\left(\begin{bmatrix}0 & \theta_1\\ 0 & \theta_3\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}0\\ \theta_4\end{bmatrix}\right) + \begin{bmatrix}\theta_2\\ 0\end{bmatrix} u, \tag{16a}$$

$$y = z_1, \tag{16b}$$

where the measured output z_1 is the difference between the linear acceleration of the wheel and the longitudinal acceleration of the vehicle, z_2 is the XBS, u is the time derivative of the brake pressure, θ_1 and θ_2 are parameters that depend on the wheel and the brake actuator, θ_3 and θ_4 are parameters that depend on the road conditions (all the parameters θ_i are assumed to be known and constant), and $v_x(t)$ is the longitudinal speed of the vehicle. The latter is assumed to be strictly positive and separated from zero and it is considered as a known external variable. This is important because, then, the system (16) may be considered as in the form (1) with $\gamma(y) := y$. To better see this, replace the original time dt with $dt/v_x(t)$ and the control input u with $uv_x(t)$.

The performance of the switching observer is illustrated in Figure 1. The simulation scenario corresponds to an ABS braking maneuver of a vehicle traveling on dry asphalt with an initial speed of 120 km/h. The system's parameters are $\theta_1 = 562.5$ N·kg^{-1}, $\theta_2 = 4.37$ N·kg^{-1}·bar^{-1}, $\theta_3 = 23.99$, and $\theta_4 = 12.47$. The control input is defined by a hybrid ABS controller [33] that generates an oscillatory output trajectory (see Figure 1a), hence Assumption 1 holds. The estimation results of the switching observer are compared with respect to those obtained via a Kalman observer (see [5, Th. 4]).

Both observers provide a good estimation of the acceleration offset, despite the measurement of the latter being perturbed by a zero-mean white noise with a standard deviation $\sigma = 4.5\,\mathrm{m/s^2}$, typical in an ABS [13]. Concerning the estimation of the XBS, however, the switching observer clearly outperforms the Kalman observer. With the switching observer the estimation of z_2 exhibits a slight deviation from its true value whenever the latter goes from the troughs of the waveform towards zero, whereas with the Kalman observer the estimation exhibits a much larger deviation (starting at the crest of the waveform) and for a longer amount of time. Moreover, we remark that a correct estimation of the time instants in which z_2 crosses zero is of critical importance for the control of the ABS, thus rendering the Kalman observer unsuited for this particular application.

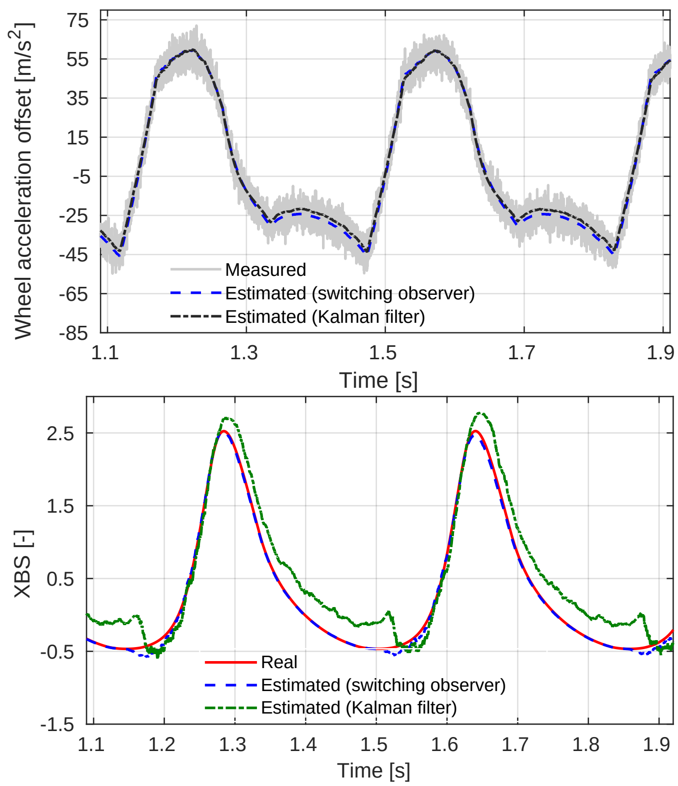

Fig. 1. Comparison between the proposed switching observer and the well-known Kalman observer: application to the antilock braking system. Top: Measurement and estimation of the acceleration offset, z_1. Bottom: Estimation of the tyre extended braking stiffness, z_2.

IV. Proofs

A. Proof of Theorem 1

Existence: Given K_+ such that $A_+ = A - K_+C$ is Hurwitz and $Q = C^\top C$, let P denote the unique solution of (10). Then, $K_- = K_+ - P^{-1}C^\top$ is such that P satisfies (11). The matrix $A_- := -A + K_+C$ is necessarily Hurwitz, because (A_-, C) is observable (see, *e.g.*, [34, Prop. 5.4]).

Uniqueness: Assume, without loss of generality, that the pair (A, C) is in observable companion form. Now, to examine whether there exist other gains K_- such that P satisfies (11), we start by observing that such P necessarily satisfies

$$A_0^\top P + PA_0 = -Q, \quad \text{with} \quad A_0 = \frac{1}{2}\left(A_+ + A_-\right). \tag{17}$$

This is because A_0 belongs to the matrix pencil generated by A_+ and A_-. Premultiplying both sides of the first equation in (17) by P^{-1}, we obtain

$$P^{-1}A_0^\top P + A_0 = -P^{-1}Q. \tag{18}$$

Then, observing that $A_0^\top P = PA_0$ and premultiplying both sides of this expression by P^{-1}, we obtain

$$P^{-1}A_0^\top P = A_0. \tag{19}$$

From (18) and (19) we see that $A_+ + A_- = -P^{-1}Q$, *i.e.*,

$$(K_+ - K_-)C = -P^{-1}Q,$$

whose unique solution is $K_- = K_+ - P^{-1}C^\top$.

B. Proof of Theorem 2

Let $z_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$ be arbitrarily fixed initial conditions generating, through (1a), the state trajectories $z(t,t_0,z_0)$ and, through (1b), the output trajectory $y(t,t_0,z_0) = Cz(t,t_0,z_0)$. Then, consider the estimation error equation

$$\dot{e} = \gamma(y(t,t_0,z_0))\big[A - K(\gamma(y(t,t_0,z_0)))C\big]e \tag{20}$$

with initial conditions (t_0, e_0), where $e_0 = \hat{z}_0 - z_0$.

In view of the dwell-time condition in Assumption 1, the time instants $\{t_k\}$ where $\gamma(y(t_k,t_0,z_0)) = 0$ do not accumulate, regardless of (t_0,z_0). Therefore, and since t_0 and z_0 are arbitrarily fixed, for all $t \neq t_k$, we may define, with a mild abuse of notation, $\rho(t) := \operatorname{sgn}\big(\gamma(y(t,t_0,z_0))\big)$, where $\operatorname{sgn}(\cdot)$ takes values in $\{-1,+1\}$. It follows that, for almost all $t \geq t_0$,

$$\frac{\dot{e}}{|\gamma(y(t))|} = \rho(t)\big[A - K(\gamma(y(t)))C\big]e, \tag{21}$$

where, for simplicity, we write $y(t)$ instead of $y(t,t_0,z_0)$. The solutions of (21) coincide with those of (20) for all $t \geq t_0$ and are absolutely continuous. Furthermore, let $\tau = \nu(t,t_0,z_0)$ be defined as in (13), then $\tau_0 := \nu(t_0,t_0,z_0)$ satisfies $\tau_0 = 0$. Moreover, note that in view of Lemma 1, $\nu(\cdot,t_0,z_0)$ is continuous, strictly increasing, and radially unbounded. This, in turn, implies that $t = \nu^{-1}(\cdot,t_0,z_0)$ exists and is continuous on $[0,\infty)$. Let $\bar{y}(\tau,z_0) \equiv y(t,t_0,z_0)$, that is, we denote by $\bar{y}(\tau,z_0)$ the system's output trajectories in the τ time-scale.

It follows that for each e_0 the system

$$\frac{de}{d\tau} = \sigma(\tau)\big[A - K(\gamma(\bar{y}(\tau,z_0)))C\big]e,$$

where σ is defined in Lemma 2, generates solutions $e(\cdot,e_0)$ that are absolutely continuous functions of τ.

Now, by assumption, A_+ in (6) is Hurwitz, so by Theorem 1 we obtain the unique K_- such that A_- is also Hurwitz and, moreover, such that the respective Lyapunov equations (10), (11) admit a common solution $P = P^\top > 0$. Thus, define

$$\mathcal{A}_{\sigma(\tau)} = \begin{cases} A_+ & \text{if } \sigma(\tau) > 0 \\ A_- & \text{if } \sigma(\tau) < 0. \end{cases} \tag{22}$$

It follows, from Assumption 1 and Lemma 1, that the solutions of the estimation error equation (20) coincide with those of the linear switched system described by (9) and (22), with $\tau_0 = 0$.

Furthermore, in view of the dwell-time condition (15) and by virtue of Lemma 2, the switching signal σ has a dwell time. Therefore, from [31, Th. 4], it follows that the origin for the system described by (9) and (22) is globally exponentially stable, uniformly with respect to the switching signals. In view of the continuity and invertibility of τ (see the proof of Lemma 2 below) it follows that $e = 0$ is globally asymptotically stable. Uniformity follows from the fact that the previous developments hold regardless of t_0.

The second statement is a direct consequence of Theorem 1 and so this concludes the proof of Theorem 2.

C. Proofs of Lemmata 1 and 2

Proof of Lemma 1: Let (t_0,z_0) be arbitrarily fixed. Then, with a slight abuse of notation, in the sequel we drop these arguments when clear from the context. From (13) we see that $\nu(t) > 0$ for all $t \geq t_0$ and $\nu(t_0) = 0$. Continuity follows directly from that of γ and absolute continuity of $y(\cdot,t_0,z_0)$. To show that ν is strictly increasing we proceed by contradiction. If ν is not strictly increasing, there exist an interval $[a,b] \subset [t_0,\infty)$ and a constant $c > 0$ such that $\nu(t) = c$ for all $t \in [a,b]$ or, equivalently, $\dot{\nu}(t) = \big|\gamma(y(t))\big| = 0$ on the same interval. Nonetheless, this contradicts the dwell-time condition in Assumption 1 for any $T_D \leq b - a$. Moreover, this holds regardless of $(t_0,z_0) \in \mathbb{R} \times \mathbb{R}^n$.

Now, because y is uniformly bounded and γ is continuous, $\gamma(y(\cdot,t_0,z_0))$ is also bounded, uniformly for all $t_0 \in \mathbb{R}$ and all $|z_0| < r$, and for any $r > 0$. That is, for any $r > 0$ there exists $\gamma_{\max}(r) > 0$ such that $\big|\gamma(y(t,t_0,z_0))\big| \leq \gamma_{\max}(r)$ for all $t \geq t_0$ and all $|z_0| < r$. This implies that

$$\int_{t_0}^{t} \big|\gamma(y(s,t_0,z_0))\big| ds \leq \gamma_{\max}(r)[t - t_0]$$

and, from (13), it follows that $\nu(t) \leq \gamma_{\max}(r)[t-t_0]$. Hence, ν is defined on $[t_0,\infty)$. Furthermore, because $\big|\gamma(y(t))\big| \leq \gamma_{\max}(r)$, we also have

$$\left|\frac{\gamma(y(t))}{\gamma_{\max}(r)}\right| \geq \left[\frac{\gamma(y(t))}{\gamma_{\max}(r)}\right]^2$$

so, multiplying both sides of the expression above by $(\gamma_{\max}(r))^2$ and integrating from t to $t + T_0$, we obtain

$$\int_t^{t+T_0} \gamma_{\max}(r)\big|\gamma(y(s,t_0,z_0))\big| ds \geq \int_t^{t+T_0} \gamma(y(t,t_0,z_0))^2 ds.$$

In turn, from the latter and the persistency-of-excitation condition (14), it follows that, for any $n \in \mathbb{N}$,

$$\int_t^{t+nT_0} \big|\gamma(y(s,t_0,z_0))\big| ds \geq n\frac{\mu_0}{\gamma_{\max}(r)}.$$

Evaluating the limit as $n \to +\infty$ on both sides of the latter inequality and comparing it to (13), we see that

$$\lim_{t\to\infty} \nu(t) \geq \lim_{n\to\infty} \int_{t_0}^{t_0+nT_0} \big|\gamma(y(s,t_0,z_0))\big| ds = +\infty$$

The fact that μ_0 and T_0 are independent of t_0 guarantees that the limit above holds uniformly in t_0.

Proof of Lemma 2: We proceed by contradiction. Assume that $t \mapsto \rho(t)$ has a dwell time, but $\tau \mapsto \sigma(\tau)$ does not. Therefore, for any $\varepsilon > 0$ there exists τ_k and τ_l such that $\tau_k \neq \tau_l$ and $\gamma(y(\tau_k)) = \gamma(y(\tau_l)) = 0$ and $|\tau_k - \tau_l| < \varepsilon$. Next, let $\{\varepsilon_n\}$ be a sequence converging to zero and, for each ε_n, consider a pair (τ_k^n, τ_l^n) as defined previously. That is, let the converging sequence $\{\varepsilon_n\}$ generate a sequence $\{(\tau_k^n,\tau_l^n)\}$. Now, in view of Lemma 1, $t \mapsto \nu(t)$ is continuous, strictly increasing and radially unbounded. Therefore, $\tau \mapsto \nu^{-1}(\tau)$ exists and is continuous on $[0,+\infty)$. Then, set $\delta_n := |\nu^{-1}(\tau_k^n) - \nu^{-1}(\tau_l^n)|$. It follows, from continuity of ν^{-1}, that the sequence $\{\varepsilon_n\}$ generates another converging sequence $\{\delta_n\}$, which may be equivalently written as $\{|t_k^n - t_l^n|\} \to 0$. The latter, however, contradicts the premise that $t \mapsto \rho(t)$ has a dwell time.

V. Conclusions

The design of our switching observer for non-uniformly observable systems relies on a singular time-rescaling approach to transform the estimation error dynamics into a linear system whose dynamics switches between two stable modes. Although it was illustrated, through a concrete practical example, that the approach may supersede Kalman-based designs, several interesting theoretical questions remain open. One pertains to the optimal choice of the observer gains for each stable mode. For instance, one may consider setting K_+ to the value given by the minimum mean-square linear estimator, which is computed using the solution of an algebraic Riccati equation. Nevertheless, the conditions that guarantee both the solvability of the Riccati equation and equations (10) and (11) are not straightforward. Identifying the systems for which this optimal choice generates a common Lyapunov function is an interesting question to study. Furthermore, it seems interesting to explore the links between Theorem 1 and the back-and-forth-nudging technique [35], in which one is confronted to a similar problem of guaranteeing the stability of a system that is iteratively integrated forward and backward in time.

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