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POLYNOMIAL STABILITY OF A TRANSMISSION PROBLEM INVOLVING TIMOSHENKO SYSTEMS WITH FRACTIONAL KELVIN-VOIGT DAMPING

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Abstract. In this work, we study the stability of a one-dimensional Timoshenko system with localized internal fractional kelvin-Voigt damping in a bounded domain. First, we reformulate the system into an augmented model and using a general criteria of Arendt-Batty we prove the strong stability. Next, we investigate three cases: the first one when the damping is localized in the bending moment, the second case when the damping is localized in the shear stress, we prove that the energy of the system decays polynomially with rate t^{-1} in both cases. In the third case, the fractional Kelvin-Voigt is acting on the shear stress and the bending moment simultaneously. We show that the system is polynomially stable with decay rate of type $t^{\frac{-4}{2-\alpha}}$. The method is based on the frequency domain approach combined with multiplier technique.

Contents

1. Introduction

We consider the Timoshenko beams subject to a feedback control combines the fractional and the Kelvin-Voigt type. The fractional derivative here is of type Caputo and it is defined by:

$$[D^{\alpha,\eta}\omega](x,t) = \partial_t^{\alpha,\eta}\omega(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)}\frac{d\omega}{ds}(x,s)ds, \tag{1.1}$$

where $\alpha \in (0,1)$ is the order of the derivative, t is the time variable, $\eta \geq 0$ and γ denotes the Gamma function. Fractional calculus includes various extensions of the usual definition of derivative from integer to real order. For mathematical description of the fractional derivative see [11]. Now, we mention some recent results treated the stabilization of beams subject to fractional or Kelvin-Voigt damping. In [4] Contreras and Rivera considered the Timoshenko beam with localized Kelvin-Voigt dissipation distributed over two components: one of them with constitutive law of the type C^1 and the other with discontinuous law. The third component is simply elastic, where the viscosity is not effective. They showed that the system is exponentially stable if and only if the component with discontinuous constitutive law is not in the center of the beam. When the discontinuous component is in the middle, the solution decays polynomially. In [9] Oquendo and Roberto da Luz investigated the asymptotic behavior of the solutions of a Timoshenko beam with a fractional damping. The damping acts only in one of the equations and depends on a parameter $\theta \in [0,1]$. Timoshenko systems with frictional or Kelvin-Voigt dampings are particular cases of this model. They proved that, for regular initial data, the semigroup of this system decays polynomially with rates that depend on θ and some relations between the structural parameters of the system. Moreover, they showed that the decay rates obtained are optimal and the only possibility to obtain exponential decay is when $\theta = 0$ and the wave propagation speeds of the equations coincide. Zhao et al. [5] considered the following Timoshenko beam with Kelvin-Voigt damping:

$$\left\{\begin{array}{ll}\rho_1 u_{tt} - \left[k_1\left(u_x + y\right)_x + D_1(u_{xt} - y_t)\right]_x = 0, & (x,t) \in (0,L) \times \mathbb{R}_+,\\ \rho_2 y_{tt} - \left(k_2 y_x + D_2 y_{xt}\right)_x + k_1\left(u_x + y\right)_x + D_1(u_{xt} - y_t) = 0, & (x,t) \in (0,L) \times \mathbb{R}_+.\end{array}\right. \tag{1.2}$$

They proved that the energy of the system (1.2) subjected to Dirichlet-Neumann boundary conditions is exponentially stable when coefficient functions $D_1,\ D_2 \in C^{1,1}([0,L])$ and satisfy $D_1 \leq cD_2\ (c > 0)$. Next, Malacarne and Rivera [7] considered the Timoshenko system (1.2) under Dirichlet-Neumann boundary conditions. They showed that the system is analytic if and only if the damping is present in both the shear stress and the bending moment. Otherwise, the solution decays polynomially no matter where the damping is effective and that rate is optimal. Later, Tian and Zhang [12] considered the Timoshenko system under fully Dirichlet boundary conditions with locally or globally distributed Kelvin-Voigt damping when coefficient functions $D_1,\ D_2 \in C([0,L])$. When the Kelvin-Voigt is globally distributed, they showed that the corresponding semigroup is analytic. Then, for their system with local Kelvin-Voigt damping, they analyzed the exponential and polynomial stability according to the properties of coefficient functions D_1, D_2. Next, Ghader and Wehbe [13] studied the stabilization of the following Timoshenko system with only one locally or globally distributed Kelvin-Voigt damping:

$$\left\{\begin{array}{ll}\rho_1 u_{tt} - k_1\left(u_x + y\right)_x = 0, & (x,t) \in (0,L) \times \mathbb{R}_+,\\ \rho_2 y_{tt} - \left(k_2 y_x + D y_{xt}\right)_x + k_1\left(u_x + y\right) = 0, & (x,t) \in (0,L) \times \mathbb{R}_+.\end{array}\right. \tag{1.3}$$

They established that the energy of the system (1.3) under fully Dirichlet or mixed boundary conditions decays polynomially.

To the best of our knowledge, the stabilization of Timoshenko system with one or two fractional Kelvin-Voigt damping has never been looked into yet. In particular, in the case, where only the first equation (equation of shear force) is effectively damped. In the present paper, we investigate the stability of the following Timoshenko system with fractional Kelvin-Voigt damping:

$$\left\{\begin{array}{ll}\rho_1 u_{tt} - \left[k_1\left(u_x + y\right) + D_1(x)\partial_t^{\alpha,\eta}(u_x + y)\right]_x = 0, & (x,t) \in (0,L) \times \mathbb{R}_+,\\ \rho_2 y_{tt} - \left[k_2 y_x + D_2(x)\partial_t^{\alpha,\eta} y_x\right]_x + k_1\left(u_x + y\right) + D_1(x)\partial_t^{\alpha,\eta}(u_x + y) = 0, & (x,t) \in (0,L) \times \mathbb{R}_+,\end{array}\right. \tag{1.4}$$

subject to the following initial condidtions:

$$\begin{array}{lll} u(x,0) = u_0(x), & u_t(x,0) = u_1(x), & x \in (0,L),\\ y(x,0) = y_0(x), & y_t(x,0) = y_1(x), & x \in (0,L).\end{array} \tag{1.5}$$

and the following boundary conditions:

$$u(0,t) = y(0,t) = u(L,t) = y(L,t) = 0, \quad t \in \mathbb{R}_+, \tag{1.6}$$

or

$$u(0,t) = y_x(0,t) = u(L,t) = y_x(L,t) = 0, \quad t \in \mathbb{R}_+. \tag{1.7}$$

The coefficients $\rho_1,\ \rho_2,\ k_1$, and k_2 are positive constants, η is non-negative and α in $(0,1)$. We assume that there exists $0 \leq a_1 < b_1 \leq L$, $0 \leq a_2 < b_2 \leq L$ and two positive constants d_1 and d_2, such that

$$D_1(x) = \left\{\begin{array}{ll} d_1, & x \in (a_1,b_1),\\ 0, & x \in (0,a_1) \cup (b_1,L),\end{array}\right. \quad \text{and} \quad D_2(x) = \left\{\begin{array}{ll} d_2, & x \in (a_2,b_2),\\ 0, & x \in (0,a_2) \cup (b_2,L).\end{array}\right.$$

We shall study the direct stability when $D_1 \neq 0$ and $D_2 \neq 0$. As well as the indirect stability when $D_1 = 0$ or $D_2 = 0$. In other words, we shall study the stability of system (1.4)-(1.5) with boundary conditions (1.6) or (1.7) by distinguishing between three cases. In the first one, we consider a fully damped system *i.e.* the two equations are effectively damped. However, in the other two cases, we assume that the system is partially damped *i.e.* only one of these equations is effectively damped. To this end, we introduce the following assumptions on the damping coefficients $D_j(x); j = 1, 2$:

$$(\mathrm{A}_1) \qquad D_1 \neq 0 \quad \text{and} \quad D_2 \neq 0,$$

or

$$(\mathrm{A}_2) \qquad D_1 = 0 \quad \text{and} \quad D_2 \neq 0,$$

or

$$(\mathrm{A}_3) \qquad D_1 \neq 0 \quad \text{and} \quad D_2 = 0.$$

The remaining of this paper is organized as follows: In subsection 2.1, we reformulate problem (1.4) into an augmented model and we prove the well-posedness of the problem by semigroup approach. In subsection 2.2, we show that our system is strongly stable in the sense that its energy converges to zero as t goes to infinity provided that any one of assumptions (A1), (A2) or (A3) holds. For this aim, a general criteria of Arendt-Batty is used. Moreover, using a frequency domain approach combining with a specific multiplier method, in section 3 (respectively in section 4), we prove that the energy of our system decays polynomially to zero like as t^{-1} when the fractional Kelvin-Voigt damping is acting only on the bending moment equation (respectively only on the shear force equation). Finally, in the last section 5, we study the polynomial stability of our system when the fractional Kelvin-Voigt damping is present in both shear stress and bending moment equations. We establish a polynomial energy decay rate of type $t^{\frac{-4}{2-\alpha}}$.

2. Well-Posedness and Strong Stability

2.1. Augmented model and well-posedness. For well-posedness of System (1.4)-(1.5) to either boundary conditions (1.6) or (1.7), we recall Theorem 2 stated in [8].

Theorem 2.1. Let $\alpha \in (0,1)$, $\eta \geq 0$ and $\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}$ be the function defined almost everywhere on $\mathbb{R}$. The relationship between the input V and the output O of the following system

$$\omega_t(x,\xi,t) + \left(\xi^2 + \eta\right)\omega(x,\xi,t) - \mu(\xi)V(x,t) = 0, \quad (x,\xi,t) \in (0,L)\times\mathbb{R}\times\mathbb{R}_+, \tag{2.1}$$

$$\omega(x,\xi,0) = 0, \quad (x,\xi) \in (0,L)\times\mathbb{R}, \tag{2.2}$$

$$O(x,t) - \kappa(\alpha)\int_{\mathbb{R}} \mu(\xi)\omega(x,\xi,t)d\xi = 0, \quad (x,t) \in (0,L)\times\mathbb{R}_+, \tag{2.3}$$

is given by

$$O = I^{1-\alpha,\eta}V, \tag{2.4}$$

where

$$[I^{\alpha,\eta}V](x,t) = \int_0^t \frac{(t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}}{\Gamma(\alpha)}V(x,\tau)d\tau \quad \text{and} \quad \kappa(\alpha) = \frac{\sin(\alpha\pi)}{\pi}.$$

□

Proof.
Step1. Assume that $\eta = 0$. From (2.1)-(2.2), we deduce that

$$\omega(x,\xi,t) = \int_0^t \mu(\xi)e^{-\xi^2(t-\tau)}V(x,\tau)d\tau. \tag{2.5}$$

Hence by using (2.3), we have

$$O(x,t) = \kappa(\alpha)\int_0^t \left(\int_{\mathbb{R}} |\xi|^{2\alpha-1}e^{-\xi^2(t-\tau)}d\xi\right)V(x,\tau)d\tau = \frac{\sin(\alpha\pi)}{\pi}\int_0^t \left(2\int_0^\infty |\xi|^{2\alpha-1}e^{-\xi^2(t-\tau)}d\xi\right)V(x,\tau)d\tau.$$

It follows that

$$O(x,t) = \frac{\sin(\alpha\pi)}{\pi}\int_0^t (t-\tau)^{-\alpha}\Gamma(\alpha)V(x,\tau)d\tau.$$

Using the fact that $\frac{\sin(\alpha\pi)}{\pi} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$ in the above equation, we obtain (2.4) in the particular case $\eta = 0$.

Step2. Assume that $\eta > 0$. By using the following change of function

$$\omega(x,\xi,t) = e^{-\eta t}\psi(x,\xi,t)$$

in (2.1)-(2.3), we obtain

$$\psi_t(x,\xi,t) + \xi^2\psi(x,\xi,t) - e^{\eta t}V(x,t)\mu(\xi) = 0, \quad \eta \geq 0,\ t > 0, \tag{2.6}$$

$$\psi(x,\xi,0) = 0, \tag{2.7}$$

$$O(x,t) - \kappa(\alpha)e^{-\eta t}\int_{\mathbb{R}}\mu(\xi)\psi(x,\xi,t)d\xi = 0. \tag{2.8}$$

Hence, by using Step1, (2.6)-(2.8) yield the desired result:

$$O(x,t) = e^{-\eta t}\int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}e^{\eta\tau}V(x,\tau)d\tau.$$

The proof is thus completed. □

Corollary 2.2. System (1.4)-(1.5) with boundary conditions (1.6) or (1.7) may be recast into the following augmented model

$$\rho_1 u_{tt} - \left(k_1(u_x+y) + \kappa(\alpha)\sqrt{D_1(x)}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi,t)d\xi\right)_x = 0,\ (x,t)\in(0,L)\times\mathbb{R}_+, \tag{2.9}$$

$$\begin{aligned}\rho_2 y_{tt} - \left(k_2 y_x + \kappa(\alpha)\sqrt{D_2(x)}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi,t)d\xi\right)_x&\\ +k_1(u_x+y) + \kappa(\alpha)\sqrt{D_1(x)}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi,t)d\xi &= 0,\ (x,t)\in(0,L)\times\mathbb{R}_+\end{aligned} \tag{2.10}$$

$$\omega^1_t(x,\xi,t) + (\xi^2+\eta)\omega^1(x,\xi,t) - \sqrt{D_1(x)}(u_{xt}+y_t)\mu(\xi) = 0,\ (x,\xi,t)\in(0,L)\times\mathbb{R}\times\mathbb{R}_+, \tag{2.11}$$

$$\omega^2_t(x,\xi,t) + (\xi^2+\eta)\omega^2(x,\xi,t) - \sqrt{D_2(x)}y_{tx}\mu(\xi) = 0,\ (x,\xi,t)\in(0,L)\times\mathbb{R}\times\mathbb{R}_+, \tag{2.12}$$

System (2.9)-(2.12) is subject to the following initial conditions:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x\in(0,L), \tag{2.13}$$

$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad x\in(0,L), \tag{2.14}$$

$$\omega^1(x,\xi,0) = \omega^2(x,\xi,0) = 0, \quad (x,\xi)\in(0,L)\times\mathbb{R}, \tag{2.15}$$

with fully Dirichlet boundary conditions

$$u(0) = u(L) = y(0) = y(L) = 0, \tag{2.16}$$

or with Dirichlet-Neumann boundary conditions

$$u(0) = u(L) = y_x(0) = y_x(L) = 0, \quad \omega^2(0,\xi,t) = \omega^2(L,\xi,t) = 0. \tag{2.17}$$

Proof. Considering the inputs $V_1(x,t) = \sqrt{D_1(x)}(u_{xt}(x,t)+y_t(x,t))$ and $V_2(x,t) = \sqrt{D_2(x)}y_{xt}(x,t)$ respectively in Theorem 2.1, then using (1.1), we get the following outputs

$$\begin{aligned}O_1(x,t) = \sqrt{D_1(x)}I^{1-\alpha,\eta}(u_{tx}+y_t)(x,t) &= \frac{\sqrt{D_1(x)}}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}e^{-\eta(t-\tau)}\partial_\tau(u_x+y)(x,\tau)d\tau\\ &= \sqrt{D_1(x)}\partial_t^{\alpha,\eta}(u_x+y)(x,t),\end{aligned} \tag{2.18}$$

and

$$O_2(x,t)=\sqrt{D_2(x)}I^{1-\alpha,\eta}y_{tx}(x,t) = \frac{\sqrt{D_2(x)}}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}e^{-\eta(t-\tau)}\partial_\tau y_x(x,\tau)d\tau = \sqrt{D_1(x)}\partial_t^{\alpha,\eta}y_x(x,t). \tag{2.19}$$

Consequently, we obtain the following two systems

$$\begin{cases} \omega_t^1(x,\xi,t)+\left(\xi^2+\eta\right)\omega^1(x,\xi,t)-\mu(\xi)\sqrt{D_1(x)}(u_{xt}+y_t)(x,t)=0, & (x,\xi,t)\in(0,L)\times\mathbb{R}\times\mathbb{R}_+,\\ \omega^1(x,\xi,0)=0, & (x,\xi)\in(0,L)\times\mathbb{R},\\ \sqrt{D_1(x)}\partial_t^{\alpha,\eta}(u_x+y)(x,t)-\kappa(\alpha)\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi,t)d\xi=0, & (x,t)\in(0,L)\times\mathbb{R}_+, \end{cases} \tag{2.20}$$

and

$$\begin{cases} \omega_t^2(x,\xi,t)+\left(\xi^2+\eta\right)\omega^2(x,\xi,t)-\mu(\xi)\sqrt{D_2(x)}y_{xt}(x,t)=0, & (x,\xi,t)\in(0,L)\times\mathbb{R}\times\mathbb{R}_+,\\ \omega^2(x,\xi,0)=0, & (x,\xi)\in(0,L)\times\mathbb{R},\\ \sqrt{D_2(x)}\partial_t^{\alpha,\eta}y_x(x,t)-\kappa(\alpha)\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi,t)d\xi=0, & (x,t)\in(0,L)\times\mathbb{R}_+. \end{cases} \tag{2.21}$$

From sytem (2.20) and system (2.21), we deduce that system (1.4)-(1.5) to either boundary conditions (1.6) or (1.7) can be recast into the augmented model (2.9)-(2.15) to either boundary conditions (2.16) or (2.17). The proof is thus complete. □

Now, let $(u,u_t,y,y_t,\omega^1,\omega^2)$ be a regular solution for the system (2.9)-(2.15) to either the boundary conditions (2.16) or (2.17), its associated energy is given by

$$E(t)=\frac{1}{2}\int_0^L\left(\rho_1|u_t|^2+\rho_2|y_t|^2+k_1|u_x+y|^2+k_2|y_x|^2\right)dx+\frac{\kappa(\alpha)}{2}\int_0^L\int_{\mathbb{R}}\left(\left|\omega^1\right|^2+\left|\omega^2\right|^2\right)d\xi dx.$$

Lemma 2.3. System (2.9)-(2.15) subject to either the boundary conditions (2.16) or (2.17) is dissipative in the sense that it is energy is non-increasing function with respect to t and satisfies

$$E'(t)=-\kappa(\alpha)\int_0^L\int_{\mathbb{R}}\left(\xi^2+\eta\right)\left(\left|\omega^1\right|^2+\left|\omega^2\right|^2\right)d\xi dx\leq 0. \tag{2.22}$$

Proof. Let $\left(u,u_t,y,y_t,\omega^1,\omega^2\right)$ be a regular solution of (2.9)-(2.15), thus multiplying (2.9) and (2.10) by $\overline{u}_t$ and $\overline{y}_t$, respectively, integrating over $(0,L)$, adding the resulting equations, then using the boundary conditions (2.16) or (2.17), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L\left(\rho_1|u_t|^2+\rho_2|y_t|^2+k_1|u_x+y|^2+k_2|y_x|^2\right)dx\\ &+\Re\left(\kappa(\alpha)\int_0^L\sqrt{D_1(x)}(\overline{u}_{tx}+\overline{y}_t)\int_{\mathbb{R}}\mu(\xi)\omega^1d\xi dx\right)+\Re\left(\kappa(\alpha)\int_0^L\sqrt{D_2(x)}\overline{y}_{tx}\int_{\mathbb{R}}\mu(\xi)\omega^2d\xi dx\right)=0. \end{aligned} \tag{2.23}$$

Next, multiplying (2.11) and (2.12) by $\kappa(\alpha)\overline{\omega^1}$ and $\kappa(\alpha)\overline{\omega^2}$, respectively, integrating in $(0,L)\times\mathbb{R}$, then using the boundary condition (2.16) or (2.17), we get

$$\begin{aligned} &\frac{\kappa(\alpha)}{2}\frac{d}{dt}\int_0^L\int_{\mathbb{R}}|\omega^1(x,\xi)|^2d\xi dx+\kappa(\alpha)\int_0^L\int_{\mathbb{R}}(\xi^2+\eta)|\omega^1(x,\xi)|^2d\xi dx\\ &=\Re\left(\kappa(\alpha)\int_0^L\sqrt{D_1(x)}(u_{xt}+y_t)\int_{\mathbb{R}}\mu(\xi)\overline{\omega^1}(x,\xi)d\xi dx\right), \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} &\frac{\kappa(\alpha)}{2}\frac{d}{dt}\int_0^L\int_{\mathbb{R}}|\omega^2(x,\xi)|^2d\xi dx+\kappa(\alpha)\int_0^L\int_{\mathbb{R}}(\xi^2+\eta)|\omega^2(x,\xi)|^2d\xi dx\\ &=\Re\left(\kappa(\alpha)\int_0^L\sqrt{D_2(x)}y_{tx}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2}(x,\xi)d\xi dx\right). \end{aligned} \tag{2.25}$$

Finally, by adding (2.24), (2.25) and (2.23), we obtain (2.22). Since $\alpha \in (0,1)$, then $\kappa(\alpha) > 0$, and therefore $E' \leq 0$. The proof is thus complete. □

Now, let us define the energy spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ by:

$$\mathcal{H}_1 = \left(H_0^1(0,L) \times L^2(0,L)\right)^2 \times \mathcal{W}^2,$$

$$\mathcal{H}_2 = H_0^1(0,L) \times L^2(0,L) \times H_*^1(0,L) \times L^2(0,L) \times \mathcal{W} \times \mathcal{W}^*,$$

where

$$\mathcal{W} = L^2((0,L) \times \mathbb{R}),$$

$$\mathcal{W}^* = \left\{ f \in L^2((0,L) \times \mathbb{R}) \mid f(0,\xi) = f(L,\xi) = 0 \right\},$$

and

$$H_*^1(0,L) = \left\{ f \in H^1(0,L) \mid \int_0^L f dx = 0 \right\}.$$

It is easy to check that the space H_*^1 is Hilbert spaces over $\mathbb{C}$ equipped with the norm

$$\|u\|^2_{H_*^1(0,L)} = \|u_x\|^2,$$

where $\|\cdot\|$ denotes the usual norm of $L^2(0,L)$. Both energy spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are equipped with the inner product defined by:

$$\begin{aligned}\left\langle U, \widetilde{U} \right\rangle_{\mathcal{H}_j} =& \rho_1 \int_0^L v\overline{\widetilde{v}} dx + \rho_2 \int_0^L z\overline{\widetilde{z}} dx + k_1 \int_0^L (u_x + y)\overline{(\widetilde{u}_x + \widetilde{y})} dx \\ &+ k_2 \int_0^L y_x \overline{\widetilde{y}_x} dx + \kappa(\alpha) \int_0^L \int_{\mathbb{R}} \left(\omega^1 \overline{\widetilde{\omega^1}} + \omega^2 \overline{\widetilde{\omega^2}} \right) d\xi dx,\end{aligned}$$

for all $U = (u, v, y, z, \omega^1, \omega^2)$ and $\widetilde{U} = \left(\widetilde{u}, \widetilde{v}, \widetilde{y}, \widetilde{z}, \widetilde{\omega^1}, \widetilde{\omega^2}\right)$ in $\mathcal{H}_j$, $j = 1, 2$. We use $\|U\|_{\mathcal{H}_j}$ to denote the corresponding norm. We now define the following unbounded linear operators $\mathcal{A}_j$ on $\mathcal{H}_j$ $(j = 1, 2)$ by

$$D(\mathcal{A}_1) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega^1, \omega^2) \in \mathcal{H} |\ v,\ z \in H_0^1(0,L), \\ \left(k_1(u_x + y) + \kappa(\alpha)\sqrt{D_1(x)} \displaystyle\int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi) d\xi \right)_x \in L^2(0,L), \\ \left(k_2 y_x + \kappa(\alpha)\sqrt{D_2(x)} \displaystyle\int_{\mathbb{R}} \mu(\xi)\omega^2(x,\xi) d\xi \right)_x \in L^2(0,L), \\ -(\xi^2 + \eta)\omega^1(x,\xi) + \sqrt{D_1(x)}(v_x + z)\mu(\xi), \quad |\xi|\omega^1 \in \mathcal{W}, \\ -(\xi^2 + \eta)\omega^2(x,\xi) + \sqrt{D_2(x)} z_x \mu(\xi), \quad |\xi|\omega^2 \in \mathcal{W} \end{array} \right\},$$

$$D(\mathcal{A}_2) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega^1, \omega^2) \in \mathcal{H} |\ v \in H_0^1(0,L),\ z \in H_*^1(0,L), \\ \left(k_1(u_x + y) + \kappa(\alpha)\sqrt{D_1(x)} \displaystyle\int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi) d\xi \right)_x \in L^2(0,L), \\ \left(k_2 y_x + \kappa(\alpha)\sqrt{D_2(x)} \displaystyle\int_{\mathbb{R}} \mu(\xi)\omega^2(x,\xi) d\xi \right)_x \in L^2(0,L), \\ -(\xi^2 + \eta)\omega^1(x,\xi) + \sqrt{D_1(x)}(v_x + z)\mu(\xi), \quad |\xi|\omega^1 \in \mathcal{W}, \\ -(\xi^2 + \eta)\omega^2(x,\xi) + \sqrt{D_2(x)} z_x \mu(\xi), \quad |\xi|\omega^2 \in \mathcal{W}^*, \\ y_x(0,t) = y_x(L,t) = 0 \end{array} \right\},$$

and for all $U = (u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$, for $j = 1, 2$,

$$\mathcal{A}_j U^\top = \begin{pmatrix} v \\ \frac{1}{\rho_1}\left(\kappa_1 (u_x + y) + \kappa(\alpha)\sqrt{D_1(x)}\int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi)d\xi\right)_x \\ z \\ \frac{1}{\rho_2}\left(\kappa_2 y_x + \kappa(\alpha)\sqrt{D_2(x)}\int_{\mathbb{R}} \mu(\xi)\omega^2(x,\xi)d\xi\right)_x - \frac{\kappa_1}{\rho_2}(u_x + y) - \frac{\kappa(\alpha)}{\rho_2}\sqrt{D_1(x)}\int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi)d\xi \\ -(\xi^2+\eta)\omega^1(x,\xi) + \sqrt{D_1(x)}(v_x + z)\mu(\xi) \\ -(\xi^2+\eta)\omega^2(x,\xi) + \sqrt{D_2(x)} z_x \mu(\xi) \end{pmatrix}.$$

Remark 2.4. *The condition $|\xi|\omega^j \in \mathcal{W}$, (or in $\mathcal{W}^*$), is imposed to make that*

$$\int_{\mathbb{R}} (\xi^2+\eta)|\omega^j(x,\xi)|^2 d\xi \in L^2(0,L) \quad \textit{and} \quad \sqrt{D_j(x)}\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}}\omega^j(x,\xi)d\xi \in L^2(0,L), (j=1,2).$$

Thus, the Timoshenko system (2.9)-(2.15) is transformed into a first order evolution equation on the Hilbert space $\mathcal{H}_j$

$$\begin{cases} U_t(x,t) = \mathcal{A}_j U(x,t), \\ U(x,0) = U_0(x), \end{cases} \tag{2.26}$$

where

$$U_0(x) = (u_0(x), u_1(x), y_0(x), y_1(x), 0, 0)$$

with $j = 1, 2$ corresponding to the boundary conditions (2.16) and (2.17), respectively.

Lemma 2.5. *Let $0 < \alpha < 1$, $\eta \geq 0$, then the following integrals are well-defined:*

$$I(\eta,\alpha) = \kappa(\alpha)\int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{1+\xi^2+\eta}d\xi \quad \textit{and} \quad \tilde{I}_i(f_i,\eta,\alpha) = \kappa(\alpha)\int_{\mathbb{R}} \frac{\xi^{\frac{2\alpha-1}{2}} f_i}{1+\xi^2+\eta}d\xi dx, \quad \textit{for} \quad i=5 \textit{ or } i=6.$$

Proof. First, $I(\eta,\alpha)$ can be written as

$$I(\eta,\alpha) = 2\frac{\kappa(\alpha)}{(1+\eta)}\int_0^{+\infty} \frac{\xi^{2\alpha-1}}{1+\frac{\xi^2}{1+\eta}}d\xi.$$

Thus $I(\eta,\alpha)$ may be simplified by defining a new variable $y = 1 + \frac{\xi^2}{1+\eta}$. Substituting ξ by $(y-1)^{\frac{1}{2}}(1+\eta)^{\frac{1}{2}}$, we get

$$I(\eta,\alpha) = \frac{\kappa(\alpha)}{(1+\eta)^{1-\alpha}}\int_1^{+\infty} \frac{1}{y(y-1)^{1-\alpha}}dy.$$

Using the fact that $0 < \alpha < 1$, it is easy to see that $y^{-1}(y-1)^{\alpha-1} \in L^1(1,\infty)$, therefore $I(\eta,\alpha)$ is well defined. On the other hand, using the cauchy-schwarz inequality, we obtain

$$\tilde{I}_i(f_i,\eta,\alpha) \leq 2\kappa(\alpha)\left(\int_0^{+\infty} \frac{|\xi|^{2\alpha-1}}{1+\xi^2+\eta}d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} |f_i|^2 d\xi\right)^{\frac{1}{2}} \leq \sqrt{2\kappa(\alpha)I(\eta,\alpha)}\left(\int_{\mathbb{R}} |f_i|^2 d\xi\right)^{\frac{1}{2}}.$$

Since $I(\eta,\alpha)$ is well-defined and $f_i \in \mathcal{W}$ (or $f_5 \in \mathcal{W}$ and $f_6 \in \mathcal{W}^\star$), then $\tilde{I}_i(f_i,\eta,\alpha)$ is well-defined. The proof si thus complete. □

Proposition 2.6. *The unbounded linear operator $\mathcal{A}_j$ is m-dissipative in the energy space $\mathcal{H}_j$, $j = 1, 2$.*

Proof. First, for all $U=(u,v,y,z,\omega^1,\omega^2)\in D(\mathcal{A}_j)$, one has

$$\Re\left\langle \mathcal{A}_j U, U\right\rangle_{\mathcal{H}_j} = -\kappa(\alpha)\int_0^L\int_{\mathbb{R}}\left(\xi^2+\eta\right)\left(\left|\omega^1(x,\xi)\right|^2+\left|\omega^2(x,\xi)\right|^2\right)d\xi dx\leq 0,$$

which implies that $\mathcal{A}_j$ is dissipative. Here $\Re$ is used to denote the real part of a complex number. We next prove the maximality of $\mathcal{A}_j$. Indeed, for $F=(f_1,f_2,f_3,f_4,f_5,f_6)\in\mathcal{H}_j$, we prove the existence of $U=(u,v,y,z,\omega^1,\omega^2)\in D(\mathcal{A}_j)$, unique solution of the equation

$$(I-\mathcal{A}_j)U=F.$$

Equivalently, one must consider the system given by

$$u-v = f_1, \tag{2.27}$$

$$\rho_1 v-\left(\kappa_1\left(u_x+y\right)+\kappa(\alpha)\sqrt{D_1(x)}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right)_x = \rho_1 f_2, \tag{2.28}$$

$$y-z = f_3, \tag{2.29}$$

$$\begin{aligned}\rho_2 z-\left(\kappa_2 y_x+\kappa(\alpha)\sqrt{D_2(x)}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right)_x+\kappa_1(u_x+y)&\\ +\kappa(\alpha)\sqrt{D_1(x)}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi &= \rho_2 f_4,\end{aligned} \tag{2.30}$$

$$(1+\xi^2+\eta)\omega^1(x,\xi)-\sqrt{D_1(x)}(v_x+z)\mu(\xi) = f_5(x,\xi), \tag{2.31}$$

$$(1+\xi^2+\eta)\omega^2(x,\xi)-\sqrt{D_2(x)}z_x\mu(\xi) = f_6(x,\xi). \tag{2.32}$$

From (2.27), (2.29), (2.31), (2.32) and the fact that $\eta\geqslant 0$, we get

$$v=u-f_1 \qquad \text{and} \qquad z=y-f_3, \tag{2.33}$$

$$\omega^1(x,\xi)=\frac{f_5(x,\xi)}{1+\xi^2+\eta}+\frac{\sqrt{D_1(x)}\mu(\xi)u_x}{1+\xi^2+\eta}-\frac{\sqrt{D_1(x)}\mu(\xi)(f_1)_x}{1+\xi^2+\eta}+\frac{\sqrt{D_1(x)}\mu(\xi)y}{1+\xi^2+\eta}-\frac{\sqrt{D_1(x)}\mu(\xi)f_3}{1+\xi^2+\eta}, \tag{2.34}$$

$$\omega^2(x,\xi)=\frac{f_6(x,\xi)}{1+\xi^2+\eta}+\frac{\sqrt{D_2(x)}\mu(\xi)y_x}{1+\xi^2+\eta}-\frac{\sqrt{D_2(x)}\mu(\xi)(f_3)_x}{1+\xi^2+\eta}. \tag{2.35}$$

Inserting (2.33), (2.34) and (2.35) in (2.28) and in (2.30) respectively, we get

$$\rho_1 u-\left(\kappa_1\left(u_x+y\right)+D_1u_xI+D_1yI-D_1(f_1)_xI-D_1f_3I+\sqrt{D_1}\tilde{I}_5\right)_x=\rho_1\left(f_1+f_2\right), \tag{2.36}$$

$$\begin{aligned}\rho_2 y-\left(\kappa_2 y_x+D_2y_xI-D_2(f_3)_xI+\sqrt{D_2}\tilde{I}_6\right)_x+\kappa_1(u_x+y)+D_1u_xI+D_1yI&=\rho_2\left(f_3+f_4\right)\\ &+D_1(f_1)_xI+D_1f_3I-\sqrt{D_1}\tilde{I}_5,\end{aligned} \tag{2.37}$$

with the following boundary conditions

$$u(0)=u(L)=y(0)=y(L)=0, \tag{2.38}$$

or

$$u(0)=u(L)=y_x(0)=y_x(L)=0 \tag{2.39}$$

where $I=I(\eta,\alpha)$ and $\tilde{I}_i=\tilde{I}_i(f_i,\eta,\alpha)$ for $i=5,6$, defined in Lemma 2.5. So, let $(\varphi,\psi)\in\mathcal{V}_j(0,L)$, where $\mathcal{V}_1(0,L)=H_0^1(0,L)\times H_0^1(0,L)$ and $\mathcal{V}_2(0,L)=H_0^1(0,L)\times H_*^1(0,L)$. Multiplying Equations (2.36) and (2.37) by $\overline{\varphi}$ and $\overline{\psi}$ respectively, and integrating over $(0,L)$, then we obtain the following variational Problem:

$$\begin{aligned}
&\int_0^L \left(\rho_1 u\overline{\varphi} + \rho_2 y\overline{\psi} + k_1\left(u_x+y\right)\overline{\left(\varphi_x+\psi\right)} + k_2 y_x\overline{\psi}_x\right)dx + I(\eta,\alpha)\int_0^L D_1(x)(u_x+y)(\overline{\varphi_x+\psi})dx\\
&+I(\eta,\alpha)\int_0^L D_2(x)y_x\overline{\psi_x}dx = \int_0^L \left(\rho_1\left(f_1+f_2\right)\overline{\varphi} + \rho_2\left(f_3+f_4\right)\overline{\psi}\right)dx - \int_0^L \sqrt{D_1}(x)(\overline{\varphi_x+\psi})\tilde{I}_5 dx\\
&+I(\eta,\alpha)\int_0^L D_1(x)(f_1)_x(\overline{\varphi_x+\psi})dx + I(\eta,\alpha)\int_0^L D_1(x)f_3(\overline{\varphi_x+\psi})dx\\
&-\int_0^L \sqrt{D_2}(x)\overline{\psi_x}\tilde{I}_6 dx + I(\eta,\alpha)\int_0^L D_2(x)(f_3)_x\overline{\psi_x}dx, \quad \forall\,(\varphi,\psi)\in\mathcal{V}_j(0,L),\ j=1,2.
\end{aligned} \tag{2.40}$$

Using the fact that $I(\eta,\alpha)>0$, we get that the left hand side of (2.40) is a bilinear continuous coercive form on $\mathcal{V}_j(0,L)\times\mathcal{V}_j(0,L)$, and the right hand side of (2.40) is a linear continuous form on $\mathcal{V}_j(0,L)$. Then, using the Lax-Milligram theorem, we deduce that there exists $(u,y)\in\mathcal{V}_j(0,L)$ unique solution of the variational Problem (2.40). So, defining

$$v := u - f_1 \qquad \text{and} \qquad z := y - f_3, \tag{2.41}$$

$$\omega^1(x,\xi) := \frac{f_5(x,\xi)}{1+\xi^2+\eta} + \frac{\sqrt{D_1}\mu(\xi)u_x}{1+\xi^2+\eta} - \frac{\sqrt{D_1}\mu(\xi)(f_1)_x}{1+\xi^2+\eta} + \frac{\sqrt{D_1(x)}\mu(\xi)y}{1+\xi^2+\eta} - \frac{\sqrt{D_1(x)}\mu(\xi)f_3}{1+\xi^2+\eta} \tag{2.42}$$

and

$$\omega^2(x,\xi) := \frac{f_6(x,\xi)}{1+\xi^2+\eta} + \frac{\sqrt{D_2}\mu(\xi)y_x}{1+\xi^2+\eta} - \frac{\sqrt{D_2}\mu(\xi)(f_3)_x}{1+\xi^2+\eta}. \tag{2.43}$$

First, it is easy to see that $(v,z)\in\mathcal{V}_j(0,L)$, $j=1,2$. Next, using Equation (2.42), Lemma 2.5 and the fact that $\eta\geq 0$, $\alpha\in(0,1)$, $f_5\in\mathcal{W}$, we get

$$\begin{aligned}
\int_0^L\int_{\mathbb{R}} |\omega^1(x,\xi)|^2 d\xi dx &\leq 5\int_0^L\int_{\mathbb{R}} \frac{|f_5(x,\xi)|^2}{(1+\xi^2+\eta)^2}d\xi dx + 5d_1\int_{\mathbb{R}}\frac{|\xi|^{2\alpha-1}}{(1+\xi^2+\eta)^2}d\xi\int_0^L(|u_x|^2+|(f_1)_x|^2+|y|^2+|f_3|^2)dx\\
&\leq 5\int_0^L\int_{\mathbb{R}}\frac{|f_5(x,\xi)|^2}{(1+\eta)^2}d\xi dx + 5d_1\int_{\mathbb{R}}\frac{|\xi|^{2\alpha-1}}{(1+\xi^2+\eta)}d\xi\int_0^L(|u_x|^2+|(f_1)_x|^2+|y|^2+|f_3|^2)dx\\
&\leq \frac{5}{(1+\eta)^2}\int_0^L\int_{\mathbb{R}}|f_5(x,\xi)|^2 d\xi dx + \frac{5d_1}{\kappa(\alpha)}I(\eta,\alpha)\int_0^L(|u_x|^2+|(f_1)_x|^2+|y|^2+|f_3|^2)dx<\infty.
\end{aligned}$$

It follows that $\omega^1(x,\xi)\in\mathcal{W}$. On the other hand, using equation (2.42), we get

$$\begin{aligned}
\int_0^L\int_{\mathbb{R}}|\xi\omega^1(x,\xi)|^2 d\xi dx &\leq 5\int_0^L\int_{\mathbb{R}}\frac{\xi^2|f_5(x,\xi)|^2}{(1+\xi^2+\eta)^2}d\xi dx\\
&\quad + 10d_1\int_0^{+\infty}\frac{|\xi|^{2\alpha+1}}{(1+\xi^2+\eta)^2}d\xi\int_0^L(|u_x|^2+|(f_1)_x|^2+|y|^2+|f_3|^2)dx.
\end{aligned} \tag{2.44}$$

It is easy to see that

$$\frac{\xi^{2\alpha+1}}{(1+\xi^2+\eta)^2}\underset{0}{\sim}\frac{\xi^{2\alpha+1}}{(1+\eta)^2} \quad\text{and}\quad \frac{\xi^{2\alpha+1}}{(1+\xi^2+\eta)^2}\underset{+\infty}{\sim}\frac{1}{\xi^{3-2\alpha}},$$

and

$$\max_{\xi\in\mathbb{R}}\frac{\xi^2}{(1+\xi^2+\eta)^2} = \frac{1}{4(1+\eta)}<\frac{1}{4}.$$

It follows from equation (2.44) and the fact that $0<\alpha<1$,

$$\int_0^L\int_{\mathbb{R}}|\xi\omega^1(x,\xi)|^2 d\xi dx \leq \frac{5}{4}\int_0^L\int_{\mathbb{R}}|f_5(x,\xi)|^2 d\xi dx + c\int_0^L(|u_x|^2+|(f_1)_x|^2+|y|^2+|f_3|^2)dx<\infty.$$

This implies that $|\xi|\omega^1(x,\xi) \in \mathcal{W}$. Similarly, we prove that $\omega^2(x,\xi)$, and $|\xi|\omega^2(x,\xi)$ belong to $\mathcal{W}$ if the boundary conditions (2.38) is considered or belong to $\mathcal{W}^*$ if the boundary conditions (2.39) is considered. It follows, from (2.42) and (2.43), that

$$-(\xi^2+\eta)\omega^1(x,\xi) + \sqrt{D_1}(v_x+z)\mu(\xi) = \omega^1 - f_5(x,\xi) \in \mathcal{W},$$

$$-(\xi^2+\eta)\omega^2(x,\xi) + \sqrt{D_2}z_x\mu(\xi) = \omega^2 - f_6(x,\xi) \in \mathcal{W} \text{ (or in } \mathcal{W}^\star).$$

On the other hand, taking $\varphi \in C_c^\infty(0,L), \psi \equiv 0$ in (2.40), integrating and using (2.41)-(2.43), we deduce that

$$\int_0^L \left[\rho_1 v - \left(\kappa_1 (u_x+y) + \kappa(\alpha)\sqrt{D_1(x)}\int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi)d\xi\right)_x\right]\overline{\varphi}dx = \rho_1\int_0^L f_2\overline{\varphi}dx, \quad \forall \varphi \in C_c^\infty(0,L).$$

This implies that

$$\left(\kappa_1 (u_x+y) + \kappa(\alpha)\sqrt{D_1(x)}\int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi)d\xi\right)_x = \rho_1(v-f_2) \in L^2(0,L).$$

Similarly, by taking $\varphi \equiv 0$ and $\psi \in C_c^\infty(0,L)$ in (2.40), integrating and using (2.41)-(2.43), we obtain equation (2.30) and consequently

$$\left(k_2 y_x + \kappa(\alpha)\sqrt{D_2}\int_{\mathbb{R}} \mu(\xi)\omega^2(x,\xi)d\xi\right)_x \in L^2(0,L).$$

Therefore, $U = (u,v,y,z,\omega^1,\omega^2) \in D(\mathcal{A}_j)$ is solution of $(I-\mathcal{A}_j)U = F$. To conclude, we need to show the uniqueness of such a solution. So, let $U = (u,v,y,z,\omega^1,\omega^2) \in D(\mathcal{A}_j)$ be a solution of equation $(I-\mathcal{A}_j)U = F$ with $F = 0$, then we directly deduce that $U = 0$. The proof is thus complete. □

Thanks to Lumer-Phillips theorem (see [10]), we deduce that $\mathcal{A}_j$ generates a C_0-semigroup of contraction $e^{t\mathcal{A}_j}$ in $\mathcal{H}_j$, $j = 1,2$. Then the solution of the evolution Problem (2.26) admits the following representation

$$U(t) = e^{t\mathcal{A}_j}U_0, \ t \geq 0.$$

and therefore, Problem (2.26) is well-posed and we have the following result.

Theorem 2.7. For all $U_0 \in \mathcal{H}_j$, Problem (2.26) admits a unique weak solution

$$U \in C(\mathbb{R}_+;\mathcal{H}_j).$$

Moreover, if $U_0 \in D(\mathcal{A}_j)$, then Problem (2.26) admits a unique strong solution

$$U \in C(\mathbb{R}_+; D(\mathcal{A}_j)) \cap C^1(\mathbb{R}_+;\mathcal{H}_j).$$

□

Remark 2.8. *All previous results still valid even when the Timoshenko System* (1.4) *is considered with only one fractional Kelvin-Voigt damping i.e.* $D_1 \equiv 0$ *or* $D_2 \equiv 0$.

Now, we are able to study the strong stability of system (1.4).

2.2. **Strong stability.** In this part, we study the strong stability of system (2.9)-(2.15) either in the boundary conditions (2.16) or (2.17), in which we distinguish between three cases. In the first case, we consider a fully dissipative system *i.e.* the two equations are effectively damped. However, in the other two cases, we assume that the system is partially dissipative *i.e.* only one equation is effectively damped. For this aim, we use a general criteria of Arendt-Batty [1] (see Theorem A.2 in the appendix) to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}_j}$ associated to the Timoshenko System (2.9)-(2.15). Our main result is the following theorem.

Theorem 2.9. Assume that either (A_1), (A_2) or (A_3) holds. Then, the C_0-semigroup $e^{t\mathcal{A}_j}$ is strongly stable in the energy space $\mathcal{H}_j$ in the sense that,

$$\lim_{t\to+\infty} \left\|e^{t\mathcal{A}_j}U_0\right\|_{\mathcal{H}_j} = 0, \quad \forall U_0 \in \mathcal{H}_j, \quad j = 1,2.$$

For brevity, we will show the proof of Theorem 2.9 under assumption (A_3) only, while the proof under assumptions (A_1) and (A_2) are left to the reader. System (2.9)-(2.12) becomes

$$\rho_1 u_{tt}-\left(k_1\left(u_x+y\right)+\kappa(\alpha)\sqrt{D_1(x)}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi,t)d\xi\right)_x = 0,\ (x,t)\in(0,L)\times\mathbb{R}_+, \tag{2.45}$$

$$\rho_2 y_{tt}-k_2y_{xx}+k_1\left(u_x+y\right)+\kappa(\alpha)\sqrt{D_1(x)}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi,t)d\xi = 0,\ (x,t)\in(0,L)\times\mathbb{R}_+, \tag{2.46}$$

$$\omega^1_t(x,\xi,t)+(\xi^2+\eta)\omega^1(x,\xi,t)-\sqrt{D_1(x)}(u_{xt}+y_t)\mu(\xi) = 0,\ (x,\xi,t)\in(0,L)\times\mathbb{R}\times\mathbb{R}_+, \tag{2.47}$$

subject to the following initial conditions:

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in(0,L), \tag{2.48}$$

$$y(x,0)=y_0(x),\quad y_t(x,0)=y_1(x),\quad x\in(0,L), \tag{2.49}$$

$$\omega^1(x,\xi,0)=0,\quad (x,\xi)\in(0,L)\times\mathbb{R}, \tag{2.50}$$

with fully Dirichlet boundary conditions

$$u(0)=u(L)=y(0)=y(L)=0, \tag{2.51}$$

or with Dirichlet-Neumann boundary conditions

$$u(0)=u(L)=y_x(0)=y_x(L)=0. \tag{2.52}$$

The argument for Theorem 2.9 relies on the subsequent lemmas.

Lemma 2.10. *Assume that assumption (A_3) holds. Then, we have*

$$\ker\left(i\lambda I-\mathcal{A}_j\right)=\{0\},\quad \forall\lambda\in\mathbb{R},\quad j=1,2.$$

Proof. Let $U=\left(u,v,y,z,\omega^1\right)\in D\left(\mathcal{A}_j\right)$ and $\lambda\in\mathbb{R}$ such that

$$\mathcal{A}_jU=i\lambda U.$$

Equivalently, we have

$$v=i\lambda u, \tag{2.53}$$

$$\left(k_1(u_x+y)+\kappa(\alpha)\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right)_x=i\rho_1\lambda v, \tag{2.54}$$

$$z=i\lambda y, \tag{2.55}$$

$$k_2y_{xx}-k_1(u_x+y)-\kappa(\alpha)\sqrt{D_1(x)}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi=i\rho_2\lambda z, \tag{2.56}$$

$$-(\xi^2+\eta)\omega^1(x,\xi)+\sqrt{D_1}(v_x+z)\mu(\xi)=i\lambda\omega^1(x,\xi). \tag{2.57}$$

With the following boundary conditions

$$u(0)=u(L)=y(0)=y(L)=0,\quad \text{if}\quad j=1 \tag{2.58}$$

or

$$u(0)=y_x(0)=u(L)=y_x(L)=0,\quad \text{if}\quad j=2. \tag{2.59}$$

First, a straightforward computation gives

$$0=\Re\left\langle i\lambda U,U\right\rangle_{\mathcal{H}_j}=\Re\left\langle \mathcal{A}_jU,U\right\rangle_{\mathcal{H}_j}=-\kappa(\alpha)\int_0^L\int_{\mathbb{R}}\left(\xi^2+\eta\right)\left|\omega^1(x,\xi)\right|^2d\xi dx,$$

consequently, since we deduce that

$$\omega^1(x,\xi)=0\quad \text{a.e. in}\quad (0,L)\times\mathbb{R}. \tag{2.60}$$

Combining (2.60) with (2.53)-(2.57) and using the definition of the function $D_1(x)$, we get

$$\lambda(u_x + y) = 0, \quad \text{over} \quad (a_1, b_1), \tag{2.61}$$

$$k_1(u_x + y)_x + \rho_1\lambda^2 u = 0, \quad \text{over} \quad (0, L), \tag{2.62}$$

and

$$k_2 y_{xx} - k_1(u_x + y) + \rho_2\lambda^2 y = 0, \ \text{over} \ (0, L). \tag{2.63}$$

Here we will distinguish two cases.
Case1. If $\lambda = 0$:
From equations (2.53) and (2.55), we get

$$v = z = 0 \quad \text{on} \quad (0, L).$$

1. If $j = 1$, using equations (2.62), (2.63) and the boundary conditions in (2.58) we can write u and y as

$$u = -\frac{a}{6}x^3 + \frac{aL}{4}x^2 - \frac{aL^2}{12}x \quad \text{and} \quad y = \frac{a}{2}x^2 - \frac{aL}{2}x \tag{2.64}$$

   where a is a constant number to be determined. Now using (2.64) in (2.63) we get $a(k_2 + k_1\frac{L^2}{12}) = 0$. Since $k_1, k_2 > 0$, we deduce that $a = 0$. Then we get $u = y = 0$. Hence, $U = 0$ over $(0, L)$. In this case, the proof is complete.
2. If $j = 2$, from (2.62), (2.63), the boundary conditions in (2.59) and the fact that $y \in H^1_*(0, L)$ (*i.e.*, $\int_0^L y dx = 0$), we get

$$u = y = 0, \quad \text{over} \quad (0, L),$$

   therefore, $U = 0$, also in this case the proof is complete.

Case2. If $\lambda \neq 0$:
From equation (2.61), we get

$$(u_x + y) = 0 \quad \text{over} \quad (a_1, b_1). \tag{2.65}$$

By using (2.65), (2.62) and the boundary conditions (2.58) or (2.59), we get

$$u = y = 0 \quad \text{over} \quad (a_1, b_1). \tag{2.66}$$

Combining equations (2.62), (2.63) and (2.66), we get the following system

$$\begin{cases} k_1(u_x + y)_x + \rho_1\lambda^2 u = 0, \ \text{over} \ (0, L), \\ k_2 y_{xx} - k_1(u_x + y) + \rho_2\lambda^2 y = 0, \ \text{over} \ (0, L), \\ u = y = 0, \ \text{over} \ (a_1, b_1). \end{cases} \tag{2.67}$$

According to the unique continuation theorem we get $U = 0$ over $(0, L)$. The proof is thus completed. □

Lemma 2.11. *Assume that $\eta = 0$ and assumption (A_3) holds. Then, the operator $-\mathcal{A}_j$ is not invertible, and consequently, $0 \in \sigma(\mathcal{A}_j)$, $j = 1, 2$.*

Proof. Let $F = \left(\sin(\frac{\pi x}{L}), 0, 0, 0, 0\right) \in \mathcal{H}_j$, and assume that there exists $U = (u, v, y, z, \omega^1) \in D(\mathcal{A}_j)$ such that

$$-\mathcal{A}_j U = F.$$

It follows that

$$v = -\sin(\frac{\pi x}{L}) \ \text{in} \ (0, L) \quad \text{and} \quad \xi^2\omega^1 + \frac{\pi}{L}\sqrt{D_1}\cos(\frac{\pi x}{L})\mu(\xi) = 0.$$

Hence, we deduce that $\omega^1(x, \xi) = -\frac{\pi}{L}\xi^{\frac{2\alpha-5}{2}}\sqrt{D_1}\cos(\frac{\pi x}{L}) \notin \mathcal{W}$, which contradicts the fact that $U \in D(\mathcal{A}_j)$. Consequently, the operator $-\mathcal{A}_j$ is not invertible, as claimed. The proof is thus complete. □

Lemma 2.12. *Let $0<\alpha<1$, $\eta\geq 0$ and $f_5(x,\xi)\in\mathcal{W}$. Assume that ($\eta>0$ and $\lambda\in\mathbb{R}$) or ($\eta=0$ and $\lambda\in\mathbb{R}^*$), then the following integrals:*

$$\mathtt{I}_1(\lambda,\eta,\alpha)=i\lambda\kappa(\alpha)\int_{\mathbb{R}}\frac{\mu^2(\xi)}{i\lambda+\xi^2+\eta}d\xi,\quad \mathtt{I}_2(\lambda,\eta,\alpha)=\kappa(\alpha)\int_{\mathbb{R}}\frac{\mu(\xi)^2}{i\lambda+\xi^2+\eta}d\xi$$

and

$$\mathtt{I}_3(f_5,\lambda,\eta,\alpha)=\kappa(\alpha)\int_{\mathbb{R}}\frac{\mu(\xi)f_5(x,\xi)}{i\lambda+\xi^2+\eta}d\xi$$

are well defined.

Proof. The integrals $\mathtt{I}_1$ and $\mathtt{I}_2$ can be written in the following form

$$\mathtt{I}_1(\lambda,\eta,\alpha)=\lambda^2\mathtt{I}_4(\lambda,\eta,\alpha)+i\lambda\mathtt{I}_5(\lambda,\eta,\alpha)\quad\text{and}\quad \mathtt{I}_2(\lambda,\eta,\alpha)=-i\lambda\mathtt{I}_4(\lambda,\eta,\alpha)+\mathtt{I}_5(\lambda,\eta,\alpha),$$

where

$$\mathtt{I}_4(\lambda,\eta,\alpha)=\kappa(\alpha)\int_{\mathbb{R}}\frac{\mu(\xi)^2}{\lambda^2+(\xi^2+\eta)^2}d\xi\quad\text{and}\quad \mathtt{I}_5(\lambda,\eta,\alpha)=\kappa(\alpha)\int_{\mathbb{R}}\frac{\mu(\xi)^2(\xi^2+\eta)}{\lambda^2+(\xi^2+\eta)^2}d\xi.$$

We need to prove that $\mathtt{I}_4$ and $\mathtt{I}_5$ are well defined. First, we have

$$\mathtt{I}_4(\lambda,\eta,\alpha)=2\kappa(\alpha)\int_0^{+\infty}\frac{\xi^{2\alpha-1}}{\lambda^2+(\xi^2+\eta)^2}d\xi=2\kappa(\alpha)\int_0^1\frac{\xi^{2\alpha-1}}{\lambda^2+(\xi^2+\eta)^2}d\xi+2\kappa(\alpha)\int_1^{+\infty}\frac{\xi^{2\alpha-1}}{\lambda^2+(\xi^2+\eta)^2}d\xi.$$

Hence, in the both cases where ($\eta>0$ and $\lambda\in\mathbb{R}$) or ($\eta=0$ and $\lambda\in\mathbb{R}^*$), we have

$$\frac{\xi^{2\alpha-1}}{\lambda^2+(\xi^2+\eta)^2}\underset{0}{\sim}\frac{\xi^{2\alpha-1}}{\lambda^2+\eta^2}\quad\text{and}\quad\frac{\xi^{2\alpha-1}}{\lambda^2+(\xi^2+\eta)^2}\underset{+\infty}{\sim}\frac{1}{\xi^{5-2\alpha}}.$$

Since $0<\alpha<1$, then $\mathtt{I}_4(\lambda,\eta,\alpha)$ is well-defined. Next, we have

$$\mathtt{I}_5(\lambda,\eta,\alpha)=2\kappa(\alpha)\int_0^{+\infty}\frac{\xi^{2\alpha-1}(\xi^2+\eta)}{\lambda^2+(\xi^2+\eta)^2}d\xi=2\kappa(\alpha)\int_0^1\frac{\xi^{2\alpha-1}(\xi^2+\eta)}{\lambda^2+(\xi^2+\eta)^2}d\xi+2\kappa(\alpha)\int_1^{+\infty}\frac{\xi^{2\alpha-1}(\xi^2+\eta)}{\lambda^2+(\xi^2+\eta)^2}d\xi.$$

Similar to $\mathtt{I}_4$ in the both cases where ($\eta>0$ and $\lambda\in\mathbb{R}$) or ($\eta=0$ and $\lambda\in\mathbb{R}^*$), we have

$$\frac{\xi^{2\alpha-1}(\xi^2+\eta)}{\lambda^2+(\xi^2+\eta)^2}\underset{0}{\sim}\frac{\xi^{2\alpha-1}(\xi^2+\eta)}{\lambda^2+\eta^2}\quad\text{and}\quad\frac{\xi^{2\alpha-1}(\xi^2+\eta)}{\lambda^2+(\xi^2+\eta)^2}\underset{+\infty}{\sim}\frac{1}{\xi^{3-2\alpha}}.$$

Since $0<\alpha<1$, then $\mathtt{I}_5(\lambda,\eta,\alpha)$ is well-defined. For $\mathtt{I}_3$, using Cauchy-Schwarz inequality and the fact that $f_5(x,\xi)\in\mathcal{W}$ and that $\mathtt{I}_4<\infty$, we get

$$\begin{aligned}\int_0^L\left|\mathtt{I}_3(f_5,\lambda,\eta,\alpha)\right|^2dx&=\kappa(\alpha)^2\int_0^L\left|\int_{\mathbb{R}}\frac{\xi^{\frac{2\alpha-1}{2}}f_5(x,\xi)}{i\lambda+\xi^2+\eta}d\xi\right|^2dx\\&\leq\kappa(\alpha)^2\left(\int_{\mathbb{R}}\frac{\xi^{2\alpha-1}}{\lambda^2+(\xi^2+\eta)^2}d\xi\right)\int_0^L\int_{\mathbb{R}}\left|f_5(x,\xi)\right|^2d\xi dx<\infty.\end{aligned}$$

The proof is thus complete. $\square$

Lemma 2.13. *Assume that assumption (A_3) holds and assume that either $(\eta,\lambda)\in\mathbb{R}_+^*\times\mathbb{R}$ or $\eta=0$ and $\lambda\in\mathbb{R}^*$. Then, $i\lambda I-\mathcal{A}_j$ is surjective, $j=1,2$.*

Proof. Let $F=(f_1,f_2,f_3,f_4,f_5)\in\mathcal{H}_j$, we must prove that there exists $U=(u,v,y,z,\omega^1)\in D(\mathcal{A}_j)$ such that

$$(i\lambda U-\mathcal{A}_j)U=F.$$

Equivalently, we have

$$i\lambda u - v = f_1, \tag{2.68}$$

$$i\lambda y - z = f_3, \tag{2.69}$$

$$\lambda^2 u + \frac{1}{\rho_1}\left(k_1(u_x+y) + \kappa(\alpha)\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right)_x = -f_2 - i\lambda f_1, \tag{2.70}$$

$$\lambda^2 y + \frac{1}{\rho_2}k_2 y_{xx} - \frac{k_1}{\rho_2}(u_x+y) - \frac{\kappa(\alpha)}{\rho_2}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi = -f_4 - i\lambda f_3, \tag{2.71}$$

$$\omega^1(x,\xi) = \frac{f_5(x,\xi)}{i\lambda+\xi^2+\eta} + \frac{\sqrt{D_1}\mu(\xi)i\lambda u_x}{i\lambda+\xi^2+\eta} - \frac{\sqrt{D_1}\mu(\xi)(f_1)_x}{i\lambda+\xi^2+\eta} + \frac{\sqrt{D_1(x)}\mu(\xi)i\lambda y}{i\lambda+\xi^2+\eta} - \frac{\sqrt{D_1(x)}\mu(\xi)f_3}{i\lambda+\xi^2+\eta}. \tag{2.72}$$

System (2.68)-(2.72) considered with fully Dirichlet boundary conditions (2.51) or with Dirichlet-Neumann boundary conditions (2.52). Now, inserting (2.72) in (2.70) and in (2.71), respectively, we get

$$\rho_1\lambda^2 u + \left(\kappa_1(u_x+y) + D_1(u_x+y)I_1 - D_1((f_1)_x+f_3)I_2 + \sqrt{D_1}I_3\right)_x = F_1, \tag{2.73}$$

$$\rho_2\lambda^2 y + \kappa_2 y_{xx} - \kappa_1(u_x+y) - D_1(u_x+y)I_1 = F_2, \tag{2.74}$$

where

$$F_1 = -\rho_1(f_2 + i\lambda f_1),$$
$$F_2 = -\rho_2(f_4+i\lambda f_3) - D_1((f_1)_x+f_3)I_2(\lambda,\eta,\alpha) + \sqrt{D_1}I_3(\lambda,\eta,\alpha),$$

and $\mathtt{I}_1 := \mathtt{I}_1(\lambda,\eta,\alpha)$, $\mathtt{I}_2 := \mathtt{I}_2(\lambda,\eta,\alpha)$ and $\mathtt{I}_3 := \mathtt{I}_3(f_5,\lambda,\eta,\alpha)$ are defined in Lemma 2.12. System (2.73)-(2.74) considered with fully Dirichlet boundary conditions (2.51) or with Dirichlet-Neumann boundary conditions (2.52). Using Lemma 2.12, we get $\mathtt{I}_1(\lambda,\eta,\alpha)$, $\mathtt{I}_2(\lambda,\eta,\alpha)$ and $\mathtt{I}_3(f_5,\lambda,\eta,\alpha)$ are well-defined, and $\Re(\mathtt{I}_1(\lambda,\eta,\alpha)) > 0$.

Now, we distinguish two cases:

Case 1. $\eta > 0$ and $\lambda = 0$, then system (2.73)-(2.74) becomes

$$\begin{cases} -\left(\kappa_1(u_x+y) + \kappa(\alpha)\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\left(\frac{f_5(x,\xi)}{\xi^2+\eta} - \frac{\sqrt{D_1}\mu(\xi)((f_1)_x+f_3)}{\xi^2+\eta}\right)d\xi\right)_x = \rho_1 f_2, \\ -\kappa_2 y_{xx} + \kappa_1(u_x+y) = \rho_2 f_4 - \kappa(\alpha)\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\left(\frac{f_5(x,\xi)}{\xi^2+\eta} - \frac{\sqrt{D_1}\mu(\xi)((f_1)_x+f_3)}{\xi^2+\eta}\right)d\xi \end{cases}$$

with fully Dirichlet boundary conditions (2.51) or with Dirichlet-Neumann boundary conditions (2.52). By applying Lax-Milligram theorem and using Lemma 2.12 it is easy to see that the above system has a unique strong solution $(u,y) \in \mathcal{V}_j(0,L)$, $j = 1,2$. In this case the proof is complete.

Case 2. $\eta \geq 0$ and $\lambda \in \mathbb{R}^*$, then system (2.73)-(2.74) becomes

$$\begin{cases} \rho_1\lambda^2 u + (\kappa_1(u_x+y) + D_1(u_x+y)I_1(\lambda,\eta,\alpha))_x = G_1, \\ \rho_2\lambda^2 y + \kappa_2 y_{xx} - \kappa_1(u_x+y) - D_1(u_x+y)I_1(\lambda,\eta,\alpha) = G_2, \end{cases} \tag{2.75}$$

such that

$$G_1 = -\rho_1(f_2+i\lambda f_1) + \left(D_1((f_1)_x+f_3)I_2(\lambda,\eta,\alpha) - \sqrt{D_1}I_3(f_5,\lambda,\eta,\alpha)\right)_x,$$
$$G_2 = -\rho_2(f_4+i\lambda f_3) - D_1((f_1)_x+f_3)I_2(\lambda,\eta,\alpha) + \sqrt{D_1}I_3(f_5\lambda,\eta,\alpha)$$

with fully Dirichlet boundary conditions (2.51) or with Dirichlet-Neumann boundary conditions (2.52). Now define the linear unbounded operators $\mathcal{L}_1 : \mathcal{V}_1(0,L) = H_0^1(0,L)\times H_0^1(0,L) \to H^{-1}(0,L)\times H^{-1}(0,L)$ and $\mathcal{L}_2 : \mathcal{V}_2(0,L) = H_0^1(0,L)\times H_\star^1(0,L) \to H^{-1}(0,L)\times \left(H_\star^1(0,L)\right)'$ by

$$\mathcal{L}_j(u,y) = \left(-\frac{1}{\rho_1}(k_1(u_x+y) + D_1(u_x+y)\mathtt{I}_1(\lambda,\eta,\alpha))_x, -\frac{k_2}{\rho_2}y_{xx} + \frac{k_1}{\rho_2}(u_x+y) + \frac{1}{\rho_2}D_1(u_x+y)\mathtt{I}_1(\lambda,\eta,\alpha)\right).$$

Let $\mathcal{U}=(u,y)$ and $\mathcal{F}=(G_1,G_2)$, then we transform System (2.75) with fully Dirichlet boundary conditions (2.51) or with Dirichlet-Neumann boundary conditions (2.52). into the following form:

$$(\lambda^2\mathcal{I}-\mathcal{L}_j)\mathcal{U}=\mathcal{F}. \tag{2.76}$$

Since $\mathcal{L}_1$ is an isomorphism from $H_0^1(0,L)\times H_0^1(0,L)$ onto $H^{-1}(0,L)\times H^{-1}(0,L)$ and $\mathcal{L}_2$ is an isomorphism from $H_0^1(0,L)\times H_\star^1(0,L)$ onto $H^{-1}(0,L)\times\left(H_\star^1(0,L)\right)'$ and $\mathcal{I}$ is a compact operator from $H_0^1(0,L)\times H_0^1(0,L)$ onto $H^{-1}(0,L)\times H^{-1}(0,L)$ and from $H_0^1(0,L)\times H_\star^1(0,L)$ onto $H^{-1}(0,L)\times\left(H_\star^1(0,L)\right)'$), then, using Fredholm's Alternative theorem, problem (2.76) admits a unique solution in $H^{-1}(0,L)\times H^{-1}(0,L)$ (when $j=1$), and in $H_0^1(0,L)\times H_\star^1(0,L)$ (when $j=2$) if and only if $\lambda^2\mathcal{I}-\mathcal{L}_j$ is injective. For that purpose, let $\mathcal{U}_h=(u_h,y_h)\in\ker\left(\lambda^2\mathcal{I}-\mathcal{L}_j\right)$. Then, if we set

$$v_h=i\lambda u_h,\quad z_h=i\lambda y_h,\quad\text{and}\quad \omega_h^1=\frac{i\lambda\sqrt{D_1}\mu(\xi)}{i\lambda+\xi^2+\eta}(u_x+y),$$

we deduce that $U_h=\left(u_h,v_h,y_h,z_h,\omega_h^1\right)\in D(\mathcal{A}_j)$ is solution of

$$(i\lambda-\mathcal{A}_j)\,U_h=0,\quad j=1,2.$$

It follows from Lemma 2.10, that $u_h=v_h=y_h=z_h=\omega_h^1=0$. This implies that equation (2.76) admits a unique solution (u,y) in $H^{-1}(0,L)\times H^{-1}(0,L)$ (when $j=1$), and in $H_0^1(0,L)\times H_\star^1(0,L)$ (when $j=2$) and, we have

$$\left(\kappa_1\left(u_x+y\right)+D_1(u_x+y)I_1\left(\lambda,\eta,\alpha\right)\right)_x\in L^2(0,L),$$
$$\kappa_2 y_{xx}-\kappa_1(u_x+y)-D_1(u_x+y)I_1\left(\lambda,\eta,\alpha\right)\in L^2(0,L).$$

Now, define $v:=i\lambda u-f_1,\ z:=i\lambda y-f_3$ and

$$\omega^1(x,\xi)=\frac{f_5(x,\xi)}{i\lambda+\xi^2+\eta}+\frac{\sqrt{D_1(x)}\mu(\xi)}{i\lambda+\xi^2+\eta}(v_x+z). \tag{2.77}$$

It is easy to see that $\omega^1(x,\xi)$ and $|\xi|\omega^1(x,\xi)\in\mathcal{W}$. This implies that $U=(u,v,y,z,\omega^1)\in D(\mathcal{A}_j)$ is the unique solution of equation $(i\lambda I-\mathcal{A}_j)U=F$, $j=1,2$, and the proof is thus complete. ☐

We are now in a position to conclude the proof of Theorem 2.9.

Proof of Theorem 2.9. Using Lemma 2.10, we directly deduce that $\mathcal{A}_j$ has non pure imaginary eigenvalues. According to Lemmas 2.10, 2.11 and 2.13 and with the help of the closed graph theorem of Banach, we deduce that $\sigma(\mathcal{A})\cap\, i\mathbb{R}=\{\phi\}$ if $\eta>0$ and $\sigma(\mathcal{A})\cap\, i\mathbb{R}=\{0\}$ if $\eta=0$. Thus, we get the conclusion by applying Theorem A.2 of Arendt and Batty. ☐

In the following sections, we aim to establish the polynomial stability of System (2.9)-(2.15) in three cases: In the first one, the damping is effective only in the bending moment. However, in the second one, the damping is effective only in the shear stress. Finally, in the third case, the damping is present in both the shear stress and the bending moment. For this purpose, we will use a frequency domain approach method, namely we will use Theorem A.3.

3. Polynomial stability when the damping is effective in the bending moment.

In this section, we study the polynomial stability of System (2.9)-(2.15) either in the boundary conditions (2.16) or (2.17) in the case $\eta>0$, when the fractional Kelvin-Voigt damping is acting only on the bending moment equation, i.e assumption (A_2) holds. In this case, System (2.9)-(2.12) becomes

$$\rho_1 u_{tt}-k_1\left(u_x+y\right)_x = 0,\ (x,t)\in(0,L)\times\mathbb{R}_+, \tag{3.1}$$

$$\rho_2 y_{tt}-\left(k_2 y_x+\kappa(\alpha)\sqrt{D_2(x)}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi,t)d\xi\right)_x+k_1\left(u_x+y\right) = 0,\ (x,t)\in(0,L)\times\mathbb{R}_+, \tag{3.2}$$

$$\omega_t^2(x,\xi,t)+(\xi^2+\eta)\omega^2(x,\xi,t)-\sqrt{D_2(x)}y_{tx}\mu(\xi) = 0,\ (x,\xi,t)\in(0,L)\times\mathbb{R}\times\mathbb{R}_+, \tag{3.3}$$

subject to the following initial conditions

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in(0,L), \tag{3.4}$$

$$y(x,0)=y_0(x),\quad y_t(x,0)=y_1(x),\quad x\in(0,L), \tag{3.5}$$

$$\omega^2(x,\xi,0)=0,\quad (x,\xi)\in(0,L)\times\mathbb{R}, \tag{3.6}$$

with fully Dirichlet boundary conditions

$$u(0)=u(L)=y(0)=y(L)=0, \tag{3.7}$$

or with Dirichlet-Neumann boundary conditions

$$u(0)=u(L)=y_x(0)=y_x(L)=0,\quad \omega^2(0,\xi,t)=\omega^2(L,\xi,t)=0. \tag{3.8}$$

Our main result in this section is the following theorem.

Theorem 3.1. *Assume that $\eta>0$ and assumption (A_2) holds. Then, for $j=1,2$, there exists $c>0$ such that, for every $U_0\in D(\mathcal{A}_j)$, the following energy estimation holds:*

$$E(t)\leq \frac{c}{t}\|U_0\|^2_{D(\mathcal{A}_j)},\qquad t>0. \tag{3.9}$$

According to Theorem A.3 and by taking $\ell=2$, the polynomial energy decay (3.9) holds if the following conditions

$$i\mathbb{R}\subset\rho(\mathcal{A}_j) \tag{H$_1$}$$

and

$$\sup_{\lambda\in\mathbb{R}}\|(i\lambda I-\mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)}=O\left(|\lambda|^2\right), \tag{H$_2$}$$

are satisfied. Since condition (H_1) is already proved in Theorem 2.9 in the case $\eta>0$. We will prove condition (H_2) by an argument of contradiction. For this purpose, suppose that (H_2) is false, then there exists

$$\left\{(\lambda_n,U_n:=(u_n,v_n,y_n,z_n,\omega_n^2)^\top)\right\}\subset\mathbb{R}\times D(\mathcal{A}_j)$$

with

$$|\lambda_n|\to+\infty\quad\text{and}\quad \|U_n\|_{\mathcal{H}_j}=\|(u_n,v_n,y_n,z_n,\omega^2_n)\|_{\mathcal{H}_j}=1 \tag{3.10}$$

such that

$$\lambda_n^2\,(i\lambda_nI-\mathcal{A}_j)\,U_n=F_n:=(f_{1,n},f_{2,n},f_{3,n},f_{4,n},f_{5,n})^\top\to0\ \text{ in }\ \mathcal{H}_j. \tag{3.11}$$

For simplicity, we drop the index n. Equivalently, from (3.11), we have

$$i\lambda u-v = \lambda^{-2}f_1\quad\text{in } H_0^1(0,L), \tag{3.12}$$

$$i\lambda v-\frac{k_1}{\rho_1}(u_x+y)_x = \lambda^{-2}f_2\quad\text{in } L^2(0,L), \tag{3.13}$$

$$i\lambda y-z = \lambda^{-2}f_3\quad\text{in } \mathcal{O}_j(0,L), \tag{3.14}$$

$$i\lambda z-\frac{k_2}{\rho_2}\left(y_x+\frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right)_x+\frac{k_1}{\rho_2}(u_x+y) = \lambda^{-2}f_4\quad\text{in } L^2(0,L), \tag{3.15}$$

$$\left(i\lambda+\xi^2+\eta\right)\omega^2-i\lambda\sqrt{D_2}y_x\mu(\xi) = \lambda^{-2}\left(f_5-\sqrt{D_2}(f_3)_x\mu(\xi)\right)\quad\text{in } \mathcal{W}_j, \tag{3.16}$$

where

$$\mathcal{O}_j(0,L)=\begin{cases}H_0^1(0,L), & \text{if } j=1,\\ H_*^1(0,L), & \text{if } j=2.\end{cases}\quad\text{and}\quad \mathcal{W}_j=\begin{cases}\mathcal{W}, & \text{if } j=1,\\ \mathcal{W}^*, & \text{if } j=2.\end{cases}$$

By inserting (3.12) in (3.13) and (3.14) in (3.15), we deduce that

$$\lambda^2 u + \frac{k_1}{\rho_1}(u_x + y)_x = -\left(\lambda^{-2} f_2 + i\lambda^{-1} f_1\right), \tag{3.17}$$

$$\lambda^2 y + \frac{k_2}{\rho_2}\left(y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}} \mu(\xi)\omega^2(x,\xi)d\xi\right)_x - \frac{k_1}{\rho_2}(u_x + y) = -\left(\lambda^{-2} f_4 + i\lambda^{-1} f_3\right). \tag{3.18}$$

From the above system, $\|U\|_{\mathcal{H}_j} = 1$ and $\|F\|_{\mathcal{H}_j} = o(1)$, we remark that

$$\begin{cases} \|u\| = O\left(|\lambda|^{-1}\right), \ \|y\| = O\left(|\lambda|^{-1}\right), \ \|u_{xx}\| = O\left(|\lambda|\right), \\ \left\|\left(y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}} \mu(\xi)\omega^2(x,\xi)d\xi\right)_x\right\| = O\left(|\lambda|\right). \end{cases} \tag{3.19}$$

Our main goal is to find a contradiction with (3.10) such as $\|U_n\|_{\mathcal{H}_j} = o(1)$. For clarity, we divide the proof into several Lemmas.

Lemma 3.2. Let $\alpha \in (0,1)$, $\eta > 0$ and $\lambda \in \mathbb{R}$, then

$$A_1 := A_1(\lambda,\eta,\alpha) = \int_{\mathbb{R}} \frac{|\xi|^{\alpha+\frac{1}{2}}}{(|\lambda| + \xi^2 + \eta)^2} d\xi = c_1(|\lambda| + \eta)^{\frac{\alpha}{2} - \frac{5}{4}},$$

$$A_2 := A_2(\lambda,\eta) = \left(\int_{\mathbb{R}} \frac{1}{(|\lambda| + \xi^2 + \eta)^2} d\xi\right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}} \frac{1}{(|\lambda| + \eta)^{\frac{3}{4}}}$$

and

$$A_3 := A_3(\lambda,\eta) = \left(\int_{\mathbb{R}} \frac{\xi^2}{(|\lambda| + \xi^2 + \eta)^4} d\xi\right)^{\frac{1}{2}} = \frac{\sqrt{\pi}}{4} \frac{1}{(|\lambda| + \eta)^{\frac{5}{4}}},$$

where $c_1 = \displaystyle\int_1^{\infty} \frac{(y-1)^{\frac{\alpha}{2} - \frac{1}{4}}}{y^2} dy$.

Proof. A_1 can be written as

$$A_1(\lambda,\eta,\alpha) = \frac{2}{(\lambda+\eta)^2} \int_0^{+\infty} \frac{\xi^{\alpha+\frac{1}{2}}}{(1 + \frac{\xi^2}{\lambda+\eta})^2} d\xi.$$

Next, performing the change of variable $y = 1 + \frac{\xi^2}{\lambda+\eta}$ and substituting ξ by $(y-1)^{\frac{1}{2}}(\lambda+\eta)^{\frac{1}{2}}$, we get

$$A_1(\lambda,\eta,\alpha) = (\lambda+\eta)^{\frac{\alpha}{2} - \frac{5}{4}} \int_1^{+\infty} \frac{(y-1)^{\frac{\alpha}{2} - \frac{1}{4}}}{y^2} dy.$$

Using the fact that $\alpha \in]0,1[$, it is easy to see that $y^{-2}(y-1)^{\frac{\alpha}{2} - \frac{1}{4}} \in L^1(1,+\infty)$. Hence, the last integral in the above equation is well defined. Now, A_2 can be written as

$$(A_2(\lambda,\eta))^2 = \frac{2}{(\lambda+\eta)^2} \int_0^{\infty} \frac{1}{(1 + (\frac{\xi}{\sqrt{\lambda+\eta}})^2)^2} d\xi = \frac{2}{(\lambda+\eta)^{\frac{3}{2}}} \int_0^{\infty} \frac{1}{(1+s^2)^2} = \frac{2}{(\lambda+\eta)^{\frac{3}{2}}} \times \frac{\pi}{4}.$$

Therefore, $A_2 = \sqrt{\frac{\pi}{2}} \frac{1}{(\lambda+\eta)^{\frac{3}{4}}}$. Finally, A_3 can be written as

$$(A_3(\lambda,\eta))^2 = \frac{2}{(\lambda+\eta)^4} \int_0^{\infty} \frac{\xi^2}{(1 + (\frac{\xi}{\sqrt{\lambda+\eta}})^2)^4} d\xi = \frac{2}{(\lambda+\eta)^{\frac{5}{2}}} \times \frac{\pi}{32}.$$

Then $A_3(\lambda,\eta) = \frac{\sqrt{\pi}}{4} \frac{1}{(\lambda+\eta)^{\frac{5}{4}}}$. The proof has been completed. □

Lemma 3.3. Assume that $\eta > 0$ and assumption (A_2) holds. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^2) \in D(\mathcal{A}_j)$ of system (3.12)-(3.16) satisfies the following asymptotic behavior estimation:

$$\int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) \left|\omega^2(x,\xi)\right|^2 d\xi dx = o\left(\lambda^{-2}\right). \tag{3.20}$$

Proof. Taking the inner product of F with U in $\mathcal{H}_j$, then using the fact that U is uniformly bounded in $\mathcal{H}_j$, we get

$$\kappa(\alpha)\int_0^L \int_{\mathbb{R}}(\xi^2+\eta)\left|\omega^2(x,\xi)\right|^2 d\xi dx = -\Re\left(\langle \mathcal{A}_j U, U\rangle_{\mathcal{H}_j}\right) = \Re\left(\langle\, i\lambda U - \mathcal{A}_j U, U\rangle_{\mathcal{H}_j}\right) = o\left(\lambda^{-2}\right).$$

□

Lemma 3.4. Assume that $\eta > 0$ and assumption (A_2) holds. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^2) \in D\left(\mathcal{A}_j\right)$ of system (3.12)-(3.16) satisfies the following asymptotic behavior estimation:

$$\int_{a_2}^{b_2} |y_x|^2\, dx = o\left(\lambda^{-(3+\alpha)}\right). \tag{3.21}$$

Proof. From (3.16), we get

$$|\lambda|\sqrt{D_2(x)}|\xi|^{\frac{2\alpha-1}{2}}|y_x| \leq (|\lambda|+\xi^2+\eta)|\omega^2(x,\xi)| + |\lambda|^{-2}|f_5(x,\xi)| + |\lambda|^{-2}\sqrt{D_2(x)}|\xi|^{\frac{2\alpha-1}{2}}|(f_3)_x|.$$

Multiplying the above inequality by $(\lambda+\xi^2+\eta)^{-2}|\xi|$, integrating over $\mathbb{R}$ with respect to the variable ξ, we obtain

$$\begin{aligned} |\lambda|\sqrt{D_2}|y_x|\int_{\mathbb{R}}\frac{|\xi|^{\frac{2\alpha+1}{2}}}{(\lambda+\xi^2+\eta)^2}d\xi \ &\leq\ \int_{\mathbb{R}}\frac{|\xi\omega^2(x,\xi)|}{(\lambda+\xi^2+\eta)}d\xi + |\lambda|^{-2}\int_{\mathbb{R}}\frac{|\xi f_5(x,\xi)|}{(\lambda+\xi^2+\eta)^2}d\xi \\ &+\ |\lambda|^{-2}\sqrt{D_2}|(f_3)_x|\int_{\mathbb{R}}\frac{|\xi|^{\frac{2\alpha+1}{2}}}{(\lambda+\xi^2+\eta)^2}d\xi. \end{aligned} \tag{3.22}$$

Next, applying the Cauchy-Schwarz inequality to (3.22), we obtain

$$A_1\sqrt{D_2}|\lambda y_x| \leq A_2\left(\int_{\mathbb{R}}|\xi\omega^2(x,\xi)|^2 d\xi\right)^{\frac{1}{2}} + |\lambda|^{-2}A_3\left(\int_{\mathbb{R}}|f_5(x,\xi)|^2 d\xi\right)^{\frac{1}{2}} + |\lambda|^{-2}A_1\sqrt{D_2}|(f_3)_x|, \tag{3.23}$$

where A_1, A_2 and A_3 are defined in Lemma 3.2. Using Young's inequality and the definition of the function $D_2(x)$ in (3.23), we arrive at

$$d_2\int_{a_2}^{b_2}|\lambda|^2|y_x|^2 dx \leq 3\frac{A_2^2}{A_1^2}\int_0^L\int_{\mathbb{R}}|\xi\omega^2(x,\xi)|^2 d\xi dx + 3|\lambda|^{-4}\frac{A_3^2}{A_1^2}\int_0^L\int_{\mathbb{R}}|f_5(x,\xi)|^2 d\xi dx + 3|\lambda|^{-4}d_2\int_{a_2}^{b_2}|(f_3)_x|^2 dx.$$

It follows from Lemma 3.2 that

$$d_2\int_{a_2}^{b_2}|\lambda|^2|y_x|^2 dx \leq \frac{1}{c_1(|\lambda|+\eta)^{\alpha-1}}\frac{o(1)}{\lambda^2} + \frac{1}{c_1(|\lambda|+\eta)^{\alpha}}\frac{o(1)}{\lambda^4} + \frac{o(1)}{\lambda^4}.$$

Since $\alpha \in (0,1)$, we have $\min(1+\alpha, 4+\alpha, 4) = 1+\alpha$. Hence from the above equation, we get

$$\int_{a_2}^{b_2}|\lambda|^2|y_x|^2 dx = \frac{o(1)}{\lambda^{1+\alpha}},$$

and so,

$$\int_{a_2}^{b_2}|y_x|^2 dx = \frac{o(1)}{\lambda^{3+\alpha}}.$$

The proof is thus completed. □

Lemma 3.5. Assume that $\eta > 0$ and assumption (A_2) holds. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^2) \in D\left(\mathcal{A}_j\right)$ of system (3.12)-(3.16) satisfies the following asymptotic behavior estimation:

$$\int_{a_2}^{b_2}\left|y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2 dx = o(\lambda^{-2}).$$

Proof. Using the fact that $|P+Q|^2 \leq 2P^2+2Q^2$, we obtain

$$\begin{aligned} \int_{a_2}^{b_2}\left|y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2 dx \ &\leq 2\int_{a_2}^{b_2}|y_x|^2 dx + 2d_2\frac{\kappa(\alpha)^2}{k_2^2}\int_{a_2}^{b_2}\left(\int_{\mathbb{R}}\frac{\mu(\xi)\sqrt{\xi^2+\eta}}{\sqrt{\xi^2+\eta}}\omega^2(x,\xi)d\xi\right)^2 dx \\ &\leq 2\int_{a_2}^{b_2}|y_x|^2 dx + c_2\int_{a_2}^{b_2}\int_{\mathbb{R}}(\xi^2+\eta)|\omega^2(x,\xi)|^2 d\xi dx, \end{aligned}$$

where $c_2 = 2d_2 \dfrac{\kappa(\alpha)^2}{k_2^2} A_4(\alpha,\eta)$ and $A_4(\alpha,\eta) = \displaystyle\int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{|\xi|^2+\eta} d\xi$. We have

$$\frac{|\xi|^{2\alpha-1}}{|\xi|^2+\eta} \underset{0}{\sim} \frac{|\xi|^{2\alpha-1}}{\eta} \qquad \text{and} \qquad \frac{|\xi|^{2\alpha-1}}{|\xi|^2+\eta} \underset{+\infty}{\sim} \frac{1}{|\xi|^{3-2\alpha}},$$

since $0<\alpha<1$ and $\eta>0$, then A_4 is well defined. Using (3.20) and (3.21), we get our desired result. □

Lemma 3.6. Assume that $\eta > 0$ and assumption (A_2) holds. Let $\epsilon < \frac{b_2-a_2}{4}$. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^2) \in D(\mathcal{A}_j)$ of system (3.12)-(3.16) satisfies the following asymptotic behavior estimation:

$$\int_{a_2+\epsilon}^{b_2-\epsilon} |\lambda y|^2 dx = \frac{O(1)}{\lambda^2}. \tag{3.24}$$

Proof. We define the function $\theta \in C_0^\infty(0,L)$ such that $0 \le \theta(x) \le 1$, for all $x \in (0,L)$ and

$$\theta(x) = \begin{cases} 1 & \text{if} \quad x \in (a_2+\epsilon, b_2-\epsilon), \\ 0 & \text{if} \quad x \in (0,L) \setminus (a_2,b_2). \end{cases} \tag{3.25}$$

First, multiplying equation (3.18) by $\theta\overline{y}$, integrating over $(0,L)$, then using the fact that λy, y are uniformly bounded in $L^2(0,L)$, $\|f_4\| = o(1)$ and $\|f_3\| = o(1)$, we obtain

$$\int_0^L \theta|\lambda y|^2 dx + \frac{k_2}{\rho_2}\int_0^L \left(y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}} \mu(\xi)\omega^2(x,\xi)d\xi \right)_x (\theta\overline{y})\, dx - \frac{k_1}{\rho_2}\int_0^L \theta(u_x+y)\overline{y} dx = o(\lambda^{-2}). \tag{3.26}$$

Using integration by parts and the definition of $\theta(x)$, we get

$$\begin{aligned}\frac{k_2}{\rho_2}\int_0^L \left(y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}} \mu(\xi)\omega^2(x,\xi)d\xi \right)_x (\theta\overline{y})\, dx = &-\frac{k_2}{\rho_2}\int_0^L \theta \left(y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}} \mu(\xi)\omega^2(x,\xi)d\xi \right) \overline{y}_x dx \\ &-\frac{k_2}{\rho_2}\int_0^L \theta' \left(y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}} \mu(\xi)\omega^2(x,\xi)d\xi \right) \overline{y} dx.\end{aligned}$$

Now, using (3.21), Lemma 3.5, the definition of $D_2(x)$ and the fact that λy is bounded in $L^2(0,L)$, we get

$$\frac{k_2}{\rho_2}\int_0^L \left(y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}} \mu(\xi)\omega^2(x,\xi)d\xi \right)_x (\theta\overline{y})\, dx = \frac{o(1)}{\lambda^{\frac{5+\alpha}{2}}} + \frac{o(1)}{\lambda^2}. \tag{3.27}$$

On the other hand, using Young's inequality, we get

$$\frac{k_1}{\rho_2}|u_x+y|\,|\theta||y| \le \frac{k_1^2}{2\varepsilon\rho_2^2}\theta\frac{|u_x+y|^2}{\lambda^2} + \frac{\varepsilon\theta}{2}|\lambda y|^2.$$

Consequently, we obtain

$$\frac{k_1}{\rho_2}\int_0^L |\theta||u_x+y||y| dx \le \frac{k_1^2}{2\varepsilon\rho_2^2}\int_0^L \theta\frac{|u_x+y|^2}{\lambda^2}dx + \frac{\varepsilon}{2}\int_0^L \theta|\lambda y|^2 dx. \tag{3.28}$$

Inserting (3.27) and (3.28) in (3.26) and using the fact that u_x+y is uniformly bounded in $L^2(0,L)$, we arrive at

$$(1-\frac{\varepsilon}{2})\int_0^L \theta|\lambda y|^2 dx \le \frac{o(1)}{\lambda^{\frac{5+\alpha}{2}}} + \frac{o(1)}{\lambda^2} + \frac{O(1)}{\lambda^2}. \tag{3.29}$$

Then, for ε small enough, we have

$$0 < \int_0^L \theta|\lambda y|^2 dx \le \frac{O(1)}{\lambda^2}. \tag{3.30}$$

From the above estimation and the definition of $\theta(x)$, we obtain (3.24). □

Lemma 3.7. Assume that $\eta > 0$ and assumption (A_2) holds. Let $0 < \epsilon < \frac{b_2-a_2}{4}$. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^2) \in D(\mathcal{A}_j)$ of system (3.12)-(3.16) satisfies the following asymptotic behavior estimation:

$$\int_{a_2+\epsilon}^{b_2-\epsilon} |u_x|^2 dx = o(1). \tag{3.31}$$

Proof. First, multiplying (3.18) by $-\theta\overline{u}_x$, integration by parts, using the fact that λu is bounded in $L^2(0,L)$, and $\|f_4\|=o(1), \|f_3\|=o(1), \|(f_3)_x\|=o(1)$, we obtain

$$-\int_0^L \theta\lambda^2 y\overline{u}_x dx - \frac{k_2}{\rho_2}\int_0^L \left(y_x+\frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right)_x(\theta\overline{u}_x)\,dx + \frac{k_1}{\rho_2}\int_0^L\theta(u_x+y)\overline{u}_x dx = o(\lambda^{-2}).$$

Consequently, we obtain

$$\begin{aligned}\frac{k_1}{\rho_2}\int_0^L\theta|u_x|^2dx &= -\frac{k_1}{\rho_2}\int_0^L\theta y\overline{u}_x dx+\int_0^L\theta\lambda^2y\overline{u}_xdx\\ &+\frac{k_2}{\rho_2}\int_0^L\left(y_x+\frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right)_x(\theta\overline{u}_x)\,dx+\frac{o(1)}{\lambda^2}.\end{aligned}\tag{3.32}$$

Now, using the integration by parts, the fact that λu is uniformly bounded in $L^2(0,L)$, (3.24) and (3.21), we obtain

$$\int_0^L\theta\lambda^2y\overline{u}_xdx=-\int_0^L\theta\lambda y_x\lambda\overline{u}dx-\int_0^L\theta'\lambda y\lambda\overline{u}dx=\frac{o(1)}{\lambda^{\frac{1+\alpha}{2}}}+\frac{O(1)}{\lambda}.\tag{3.33}$$

Using integration by parts, the fact that $\dfrac{\|u_{xx}\|}{\lambda}=O(1)$, u_x is uniformly bounded in $L^2(0,L)$ and Lemma (3.5), we find

$$\begin{aligned}&\frac{k_2}{\rho_2}\int_0^L\left(y_x+\frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right)_x(\theta\overline{u}_x)\,dx=\\ &-\frac{k_2}{\rho_2}\int_0^L\theta\lambda\left(y_x+\frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right)\frac{1}{\lambda}\overline{u}_{xx}dx\\ &-\frac{k_2}{\rho_2}\int_0^L\theta'\left(y_x+\frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right)\overline{u}_xdx\\ &\qquad=o(1)+\frac{o(1)}{\lambda}.\end{aligned}\tag{3.34}$$

Next, using (3.24) and the fact that u_x is bounded, we see that

$$\frac{k_1}{\rho_2}\int_0^L\theta y\overline{u}_xdx=\frac{O(1)}{\lambda^2}.\tag{3.35}$$

Now, inserting (3.33), (3.34) and (3.35) in (3.32), it leads to

$$\frac{k_1}{\rho_2}\int_0^L\theta|u_x|^2dx=o(1).\tag{3.36}$$

From the above estimation and the definition of $\theta(x)$, we obtain (3.31). ☐

Lemma 3.8. Assume that $\eta>0$ and assumption (A$_2$) holds. Let $\epsilon<\frac{b_2-a_2}{4}$. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^2)\in D(\mathcal{A}_j)$ of system (3.12)-(3.16) satisfies the following asymptotic behavior estimation:

$$\int_{a_2+\epsilon}^{b_2-\epsilon}|\lambda u|^2dx=o(1).\tag{3.37}$$

Proof. First, multiplying equation (3.17) by $\theta\overline{u}$, integrating over $(0,L)$, using the fact that λu and u are uniformely bounded in $L^2(0,L)$, and the fact that $\|f_1\|=o(1),\|f_2\|=o(1)$, we remark that

$$\int_0^L\theta|\lambda u|^2dx+\frac{k_1}{\rho_1}\int_0^L(u_x+y)_x(\theta\overline{u})dx=o(\lambda^{-2}).\tag{3.38}$$

Using integration by parts, we obtain

$$\frac{k_1}{\rho_1}\int_0^L(u_x+y)_x(\theta\overline{u})dx=-\frac{k_1}{\rho_1}\int_0^L\theta(u_x+y)\overline{u}_xdx-\frac{k_1}{\rho_1}\int_0^L\theta'(u_x+y)\overline{u}dx,$$

inserting the above equation in (3.38), we get

$$\begin{aligned}\int_0^L \theta|\lambda u|^2 dx &= \frac{k_1}{\rho_1}\int_0^L \theta(u_x+y)\overline{u}_x dx + \frac{k_1}{\rho_1}\int_0^L \theta'(u_x+y)\overline{u} dx + o(\lambda^{-2})\\ &= \frac{k_1}{\rho_1}\left(\int_0^L \theta|u_x|^2 dx + \int_0^L \theta y\overline{u}_x dx + \int_0^L \theta' u_x\overline{u} dx + \int_0^L \theta' y\overline{u} dx\right) + o(\lambda^{-2}).\end{aligned} \tag{3.39}$$

Now, using definition of $\theta(x)$, (3.24), (3.31) and the fact that u bounded in $L^2(0,L)$, we get (3.37). □

From what precedes, Lemmas 3.3-3.8, we deduce that

$$\|U\|_{\mathcal{H}_j} = o(1), \quad \text{over} \quad (a_2+\epsilon, b_2-\epsilon).$$

Lemma 3.9. Assume that $\eta > 0$ and assumption (A_2) holds. Let $\phi \in C^1([0,L])$ and $\phi(0) = \phi(L) = 0$ be a given function. Then, for $j = 1,2$, the solution $(u,v,y,z,\omega^2) \in D(\mathcal{A}_j)$ of system (3.12)-(3.16) satisfies the following behavior estimation:

$$\int_0^L \phi'\left(\rho_1|\lambda u|^2 + k_1|u_x|^2 + \rho_2|\lambda y|^2 + k_2\left|y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2(x)}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2\right)dx = o(1).$$

Proof. First, multiplying equation (3.17) by $2\rho_1\phi\overline{u}_x$, integrating over $(0,L)$, taking the real part, the fact that u_x is uniformly bounded in $L^2(0,L)$, $\|f_1\| = o(1)$ and $\|f_2\| = o(1)$, we obtain

$$\rho_1\int_0^L \phi\frac{d}{dx}|\lambda u|^2 dx + k_1\int_0^L \phi\frac{d}{dx}|u_x|^2 dx + \Re\left\{2k_1\int_0^L \phi y_x\overline{u}_x dx\right\} = o(\lambda^{-1}). \tag{3.40}$$

Now, multiplying equation (3.18) by $2\rho_2\phi\left(\overline{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2(x)}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)$, integrating over $(0,L)$, taking the real part, then using the fact that $\left(y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2(x)}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right)$ is uniformly bounded in $L^2(0,L)$, $\|y\| = O(|\lambda|^{-1})$, $\|f_3\| = o(1)$ and $\|f_4\| = o(1)$, we obtain

$$\begin{aligned}&\Re\left\{2\rho_2\lambda^2\int_0^L \phi y\left(\overline{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)dx\right\}\\ &+k_2\int_0^L \phi\frac{d}{dx}\left|y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2 dx\\ &-\Re\left\{2k_1\int_0^L \phi u_x\left(\overline{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)dx\right\}\\ &\underbrace{-\Re\left\{2k_1\int_0^L \phi y\left(\overline{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)dx\right\}}_{=o(1)}\\ &=\underbrace{\Re\left\{2\rho_2\int_0^L \phi\left(-\lambda^{-2}f_4 - i\lambda^{-1}f_3\right)\left(\overline{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)dx\right\}}_{=o(\lambda^{-1})}.\end{aligned} \tag{3.41}$$

Moreover, by using the definition of $D_2(x)$ and Cauchy-Schwarz inequality, the fact that $0 < \alpha < 1$ and $\eta > 0$ and by using Lemma 3.3, Lemma 3.6 and the fact that u_x is uniformly bounded in $L^2(0,L)$, we obtain

$$\left\{\begin{aligned}&\Re\left\{2\rho_2\lambda^2\int_0^L \phi y\left(\overline{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)dx\right\} = \rho_2\int_0^L \phi\frac{d}{dx}|\lambda y|^2 dx + o(\lambda^{-1}),\\ &-\Re\left\{2k_1\int_0^L \phi u_x\left(\overline{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)dx\right\} = -\Re\left\{2k_1\int_0^L \phi u_x\overline{y}_x dx\right\} + o(\lambda^{-1}).\end{aligned}\right.$$

Inserting the above equations in (3.41), we obtain

$$\rho_2\int_0^L \phi\frac{d}{dx}|\lambda y|^2dx+k_2\int_0^L \phi\frac{d}{dx}\left|y_x+\frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2dx-\Re\left\{2k_1\int_0^L\phi u_x y_x dx\right\}=o(1). \tag{3.42}$$

Adding (3.40) and (3.42), then using integration by parts, we obtain (3.9). The proof is thus complete. ☐

Lemma 3.10. Assume that $\eta>0$ and assumption (A$_2$) holds. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^2)\in D(\mathcal{A}_j)$ of system (3.12)-(3.16) satisfies the following asymptotic behavior estimation:

$$\|U\|_{\mathcal{H}_j}=o(1). \tag{3.43}$$

Proof.

Let $a_2+\epsilon<a_2+2\epsilon<b_2-\epsilon$ and define the cut-off functions $\theta_1,\theta_2\in C_0^\infty([0,L])$ by

$$\theta_1(x)=\begin{cases}1 & \text{if} \quad x\in(0,a_2+\epsilon),\\ 0 & \text{if} \quad x\in(a_2+2\epsilon,L),\\ \in[0,1] & \text{elsewhere}\end{cases}$$

and

$$\theta_2(x)=\begin{cases}1 & \text{if} \quad x\in(b_2-\epsilon,L),\\ 0 & \text{if} \quad x\in(0,b_2-2\epsilon),\\ \in[0,1] & \text{elsewhere.}\end{cases}$$

Take $\phi=x\theta_1$ in Lemma 3.9, then use the fact that $||U||_{\mathcal{H}_j}=o(1)$ in $(a_2+\epsilon,b_2-\epsilon)$ and $a_2+\epsilon<a_2+2\epsilon<b_2-\epsilon$, we have

$$\begin{aligned}&\int_0^L x\theta_1'\left(\rho_1|v|^2+k_1|u_x|^2+\rho_2|z|^2+k_2\left|y_x+\frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2\right)dx\\ &+\int_0^L\theta_1\left(\rho_1|v|^2+k_1|u_x|^2+\rho_2|z|^2+k_2\left|y_x+\frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2\right)dx=o(1),\end{aligned} \tag{3.44}$$

therefore, we get

$$\int_0^{a_2+\epsilon}\left(\rho_1|v|^2+k_1|u_x|^2+\rho_2|z|^2+k_2\left|y_x+\frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2\right)dx=o(1). \tag{3.45}$$

Moreover, using estimation (3.20), the definition of D_2 and (3.45), we observe that

$$\begin{aligned}\int_0^{a_2+\epsilon}|y_x|^2dx&\le 2\int_0^{a_2+\epsilon}\left|y_x+\frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2dx\\ &+2\frac{d_2\kappa(\alpha)^2}{k_2^2}\int_{a_2}^{a_2+\epsilon}\int_{\mathbb{R}}|\xi|^{2\alpha-1}|\omega^2(x,\xi)|^2d\xi dx\\ &=o(1)\end{aligned} \tag{3.46}$$

using (3.45) and (3.46), we get

$$\|U\|_{\mathcal{H}_j}=o(1) \text{ on } (0,a_2+\epsilon).$$

Similarly, by taking $\phi=(x-L)\theta_2$, we can prove $\|U\|_{\mathcal{H}_j}=o(1)$ on $(b_2-\epsilon,L)$. Therefore,

$$\|U\|_{\mathcal{H}_j}=o(1) \text{ on } (0,L).$$

Thus, the proof is complete. ☐

Proof of Theorem 3.1. For $j=1,2$, from Lemma 3.10, we get that $\|U\|_{\mathcal{H}_j}=o(1)$, which contradicts (3.10). This implies that

$$\sup_{\lambda\in\mathbb{R}}\left\|(i\lambda I-\mathcal{A}_j)^{-1}\right\|_{\mathcal{L}(\mathcal{H}_j)}=O(\lambda^2).$$

The result follows from Theorem A.3. ☐

4. Polynomial stability when the damping is effective in shear force.

In this section, we study the polynomial stability of system (2.45)-(2.50) with the boundary conditions (2.51) or (2.52) in the case $\eta > 0$, when the fractional Kelvin-Voigt damping is acting only on the shear force equation, i.e assumption (A_3) holds. Our main result in this section is the following theorem.

Theorem 4.1. *Assume that $\eta > 0$ and assumption (A_3) holds. Then, for $j = 1, 2$, there exists $c > 0$ such that, for every $U_0 \in D(\mathcal{A}_j)$, the following energy estimation holds:*

$$E(t) \leq \frac{c}{t} \|U_0\|^2_{D(\mathcal{A}_j)}, \qquad t > 0. \tag{4.1}$$

According to Theorem A.3 and by taking $\ell = 2$, the polynomial energy decay (4.1) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}_j) \tag{H$_3$}$$

and

$$\sup_{\lambda\in\mathbb{R}} \|(i\lambda I - \mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)} = O\left(|\lambda|^2\right), \tag{H$_4$}$$

are satisfied. Since condition (H_3) is already proved in Theorem 2.9 in the case $\eta > 0$. We will prove condition (H_4) by an argument of contradiction. For this purpose, suppose that (H_4) is false, then there exists

$$\left\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n, \omega^1_n)^\top)\right\} \subset \mathbb{R} \times D(\mathcal{A}_j)$$

with

$$|\lambda_n| \to +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}_j} = \|(u_n, v_n, y_n, z_n, \omega^1_n)\|_{\mathcal{H}_j} = 1 \tag{4.2}$$

such that

$$\lambda_n^2 (i\lambda_n I - \mathcal{A}_j) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n})^\top \to 0 \ \text{ in } \ \mathcal{H}_j. \tag{4.3}$$

For simplicity, we drop the index n. Equivalently, from (4.3), we have

$$i\lambda u - v = \lambda^{-2} f_1 \quad \text{in } H^1_0(0, L), \tag{4.4}$$

$$i\lambda v - \frac{k_1}{\rho_1}\left((u_x + y) + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi)d\xi\right)_x = \lambda^{-2} f_2 \quad \text{in } L^2(0, L), \tag{4.5}$$

$$i\lambda y - z = \lambda^{-2} f_3 \quad \text{in } \mathcal{O}_j(0, L), \tag{4.6}$$

$$i\lambda z - \frac{k_2}{\rho_2} y_{xx} + \frac{k_1}{\rho_2}(u_x + y) + \frac{\kappa(\alpha)}{\rho_2}\sqrt{D_1}\int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi)d\xi = \lambda^{-2} f_4 \quad \text{in } L^2(0, L), \tag{4.7}$$

$$\left(i\lambda + \xi^2 + \eta\right)\omega^1 - i\lambda\sqrt{D_1}(u_x + y)\mu(\xi) = \lambda^{-2}\left[f_5 - \sqrt{D_1}\mu(\xi)\left((f_1)_x + f_3\right)\right] \quad \text{in } \mathcal{W}. \tag{4.8}$$

where

$$\mathcal{O}_j(0, L) = \begin{cases} H^1_0(0, L), & \text{if } j = 1, \\ H^1_*(0, L), & \text{if } j = 2. \end{cases}$$

By inserting (4.4) in (4.5) and (4.6) in (4.7), we obtain

$$\lambda^2 u + \frac{k_1}{\rho_1}\left((u_x + y) + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi)d\xi\right)_x = -[\lambda^{-2} f_2 + i\lambda^{-1} f_1], \tag{4.9}$$

$$\lambda^2 y + \frac{k_2}{\rho_2} y_{xx} - \frac{k_1}{\rho_2}(u_x + y) - \frac{\kappa(\alpha)}{\rho_2}\sqrt{D_1}\int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi)d\xi = -[\lambda^{-2} f_4 + i\lambda^{-1} f_3]. \tag{4.10}$$

From the above system, $\|U\|_{\mathcal{H}_j}=1$ and $\|F\|_{\mathcal{H}_j}=o(1)$, we remark that

$$\left\{\begin{array}{l}\|u\|=O\left(|\lambda|^{-1}\right),\ \|y\|=O\left(|\lambda|^{-1}\right),\ \|y_{xx}\|=O\left(|\lambda|\right),\\ \left\|\left((u_x+y)+\dfrac{\kappa(\alpha)}{k_2}\sqrt{D_2}\displaystyle\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right)_x\right\|=O\left(|\lambda|\right).\end{array}\right. \tag{4.11}$$

Our main goal is to find a contradiction with (4.2) such as $\|U_n\|_{\mathcal{H}_j}=o(1)$. For clarity, we divide the proof into several Lemmas.

Lemma 4.2. Assume that $\eta>0$ and assumption (A_3) holds. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^1)\in D\left(\mathcal{A}_j\right)$ of system (4.4)-(4.8) satisfies the following asymptotic behavior estimation:

$$\int_0^L\int_{\mathbb{R}}(\xi^2+\eta)\left|\omega^1(x,\xi)\right|^2d\xi dx=o\left(\lambda^{-2}\right). \tag{4.12}$$

Proof. Taking the inner product of F with U in $\mathcal{H}_j$, then using the fact that U is uniformly bounded in $\mathcal{H}_j$, we get

$$\kappa(\alpha)\int_0^L\int_{\mathbb{R}}(\xi^2+\eta)\left|\omega^1(x,\xi)\right|^2d\xi dx=-\Re\left(\langle\mathcal{A}_jU,U\rangle_{\mathcal{H}_j}\right)=\Re\left(\langle\, i\lambda U-\mathcal{A}_jU,U\rangle_{\mathcal{H}_j}\right)=o\left(\lambda^{-2}\right).$$

□

Lemma 4.3. Assume that $\eta>0$ and assumption (A_3) holds. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^1)\in D\left(\mathcal{A}_j\right)$ of system (4.4)-(4.8) satisfies the following asymptotic behavior estimation:

$$\int_{a_1}^{b_1}\left|u_x+y\right|^2dx=o\left(\lambda^{-(3+\alpha)}\right). \tag{4.13}$$

Proof. From (4.8), we get

$$|\lambda|\sqrt{D_1(x)}|\xi|^{\frac{2\alpha-1}{2}}|u_x+y|\leq(|\lambda|+\xi^2+\eta)|\omega^1(x,\xi)|+|\lambda|^{-2}|f_5(x,\xi)|+|\lambda|^{-2}\sqrt{D_1(x)}|\xi|^{\frac{2\alpha-1}{2}}|(f_1)_x+f_3|.$$

Multiplying the above equation by $(\lambda+\xi^2+\eta)^{-2}|\xi|$, integrating over $\mathbb{R}$ and proceeding in a similar way as in Lemma 3.4 (Section 3), we get our desired estimation (4.13). Thus, the proof is complete. □

Lemma 4.4. Assume that $\eta>0$ and assumption (A_3) holds. Then, for $j=1,2$, the solution $U\in D\left(\mathcal{A}_j\right)$ of system (4.4)-(4.8) satisfies the following asymptotic behavior estimation:

$$\int_{a_1}^{b_1}\left|(u_x+y)+\frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right|^2dx=o(\lambda^{-2}).$$

Proof. using the fact that $|P+Q|^2\leq 2P^2+2Q^2$, we obtain

$$\begin{aligned}\int_{a_1}^{b_1}\left|(u_x+y)+\frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right|^2dx&\leq 2\int_{a_1}^{b_1}\left|u_x+y\right|^2dx+2d_1\frac{\kappa(\alpha)^2}{k_1^2}\int_{a_1}^{b_1}\left(\int_{\mathbb{R}}\frac{\mu(\xi)\sqrt{\xi^2+\eta}}{\sqrt{\xi^2+\eta}}\omega^1(x,\xi)d\xi\right)^2dx\\&\leq 2\int_{a_1}^{b_1}\left|u_x+y\right|^2dx+c_3\int_{a_1}^{b_1}\int_{\mathbb{R}}(\xi^2+\eta)|\omega^1(x,\xi)|^2d\xi dx\end{aligned}$$

where $c_3=2d_1\dfrac{\kappa(\alpha)^2}{k_1^2}A_4(\alpha,\eta)$ and $A_4(\alpha,\eta)$ is defined in Lemma 3.5. Using (4.12) and (4.13), we get our desired result. Hence, the proof is complete. □

Lemma 4.5. Assume that $\eta>0$ and assumption (A_3) holds. Let $\epsilon<\frac{b_1-a_1}{4}$. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^1)\in D\left(\mathcal{A}_j\right)$ of system (4.4)-(4.8) satisfies the following asymptotic behavior estimation:

$$\int_{a_1+\epsilon}^{b_1-\epsilon}|\lambda u|^2dx=\frac{o(1)}{\lambda^2}. \tag{4.14}$$

Proof. We define the function $\theta_3\in C_0^\infty(0,L)$ such that $0\leq\theta_3(x)\leq 1$, for all $x\in(0,L)$, by

$$\theta_3(x)=\left\{\begin{array}{lll}1 & \text{if} & x\in(a_1+\epsilon,b_1-\epsilon),\\ 0 & \text{if} & x\in(0,L)\setminus(a_1,b_1).\end{array}\right. \tag{4.15}$$

First, multiplying (4.9) by $\theta_3\overline{u}$, integrating over $(0,L)$, using the fact that λu and u are uniformly bounded in $L^2(0,L)$ and $\|f_1\| = o(1), \|f_2\| = o(1)$, we obtain

$$\int_0^L \theta_3|\lambda u|^2 dx + \frac{k_1}{\rho_1}\int_0^L \left[(u_x+y) + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right]_x (\theta_3\overline{u})dx = o(\lambda^{-2}). \tag{4.16}$$

Using integration by parts on the second term of the left hand side of (4.16), we get

$$\begin{aligned}\frac{k_1}{\rho_1}\int_0^L \left[(u_x+y) + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right]_x (\theta_3\overline{u})dx &= -\frac{k_1}{\rho_1}\int_0^L \theta_3\left[(u_x+y) + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right]\overline{u}_x dx\\ &\quad -\frac{k_1}{\rho_1}\int_0^L \theta_3'\left[(u_x+y) + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right]\overline{u}dx.\end{aligned}$$

Now, using (4.13), Lemma 4.4 and the fact that λy is bounded in $L^2(0,L)$, we get

$$\begin{aligned}&\frac{k_1}{\rho_1}\int_0^L \theta_3\left[(u_x+y) + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right]\overline{u}_x dx\\ &\qquad = \frac{k_1}{\rho_1}\int_0^L \theta_3\left[(u_x+y) + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right](\overline{u}_x+\overline{y})dx\\ &\qquad -\frac{k_1}{\rho_1}\int_0^L \theta_3\frac{1}{\lambda}\left[(u_x+y) + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right](\lambda\overline{y})dx\\ &\qquad = \frac{o(1)}{\lambda^{\frac{5+\alpha}{2}}} + \frac{o(1)}{\lambda^2}.\end{aligned} \tag{4.17}$$

Thanks to Lemma 4.4 and the fact that λu is uniformly bounded in $L^2(0,L)$, we have

$$\frac{k_1}{\rho_1}\int_0^L \theta_3'\frac{1}{\lambda}\left[(u_x+y) + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right]\lambda\overline{u}dx = \frac{o(1)}{\lambda^2}. \tag{4.18}$$

Now, inserting (4.17) and (4.18) in (4.16), we get

$$\int_0^L \theta_3|\lambda u|^2 dx = \frac{o(1)}{\lambda^2}. \tag{4.19}$$

From the above estimation and the definition of $\theta_3(x)$, we obtain (4.14). Thus, the proof is complete. □

Lemma 4.6. Assume that $\eta > 0$ and assumption (A_3) holds. Let $\epsilon < \frac{b_1-a_1}{4}$, then, for $j=1,2$, the solution $(u,v,y,z,\omega^1) \in D(\mathcal{A}_j)$ of system (4.4)-(4.8) satisfies the following asymptotic behavior estimation:

$$\int_{a_1+\epsilon}^{b_1-\epsilon} |\lambda y|^2 dx = o(1). \tag{4.20}$$

Proof. First, multiplying (4.10) by $\theta_3(\overline{u_x+y})$, integrating over $(0,L)$, using integration by parts, the fact that $\lambda u, \lambda y$ are uniformly bounded in $L^2(0,L)$ and $\|f_3\| = o(1), \|f_4\| = o(1)$, we obtain

$$\begin{aligned}&\int_0^L \theta_3\lambda^2 y(\overline{u_x+y})dx + \frac{k_2}{\rho_2}\int_0^L \theta_3 y_{xx}(\overline{u_x+y})dx - \frac{k_1}{\rho_2}\int_0^L \theta_3|u_x+y|^2 dx\\ &-\frac{\kappa(\alpha)}{\rho_2}\int_0^L \theta_3\sqrt{D_1(x)}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi(\overline{u_x+y})dx = o(\lambda^{-2}).\end{aligned} \tag{4.21}$$

Using (4.12), (4.13) and the fact that $\frac{1}{\lambda}\|y_{xx}\| = O(1)$, we get

$$\begin{cases}\displaystyle\frac{k_2}{\rho_2}\int_0^L \theta_3\frac{1}{\lambda}y_{xx}\lambda(\overline{u_x+y})dx = \frac{o(1)}{\lambda^{\frac{1+\alpha}{2}}},\\ \displaystyle\frac{k_1}{\rho_2}\int_0^L \theta_3|u_x+y|^2 dx = \frac{o(1)}{\lambda^{3+\alpha}},\\ \displaystyle\frac{\kappa(\alpha)}{\rho_2}\int_0^L \theta_3\sqrt{D_1(x)}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi(\overline{u_x+y})dx = \frac{o(1)}{\lambda^{\frac{5+\alpha}{2}}}.\end{cases} \tag{4.22}$$

Inserting (4.22) in (4.21) and since $0<\alpha<1$, we get

$$\int_0^L \theta_3\lambda^2 y\overline{u}_x dx + \int_0^L \theta_3|\lambda y|^2 dx = \frac{o(1)}{\lambda^{\frac{1+\alpha}{2}}}. \tag{4.23}$$

Using the integration by parts, we get

$$\int_0^L \theta_3\lambda^2 y\overline{u}_x dx = -\int_0^L \theta_3'\lambda y\lambda\overline{u}dx - \int_0^L \theta_3\lambda y_x\lambda\overline{u}dx. \tag{4.24}$$

Now, using (4.14) and the fact that λy and y_x are uniformly bounded in $L^2(0,L)$, we get

$$\int_0^L \theta_3\lambda^2 y\overline{u}_x dx = o(1). \tag{4.25}$$

Inserting (4.25) in (4.23), we get

$$\int_0^L \theta_3|\lambda y|^2 dx = o(1).$$

From the above estimation and the definition of $\theta_3(x)$, we obtain (4.20). Thus, the proof is complete. □

Lemma 4.7. Assume that $\eta>0$ and assumption (A_3) holds. Let $\epsilon<\frac{b_1-a_1}{4}$. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^1)\in D(\mathcal{A}_j)$ of system (4.4)-(4.8) satisfies the following asymptotic behavior estimation:

$$\int_{a_1+\epsilon}^{b_1-\epsilon} |y_x|^2 dx = o(1). \tag{4.26}$$

Proof. First, multiplying (4.10) by $\theta_3\overline{y}$, integrating over $(0,L)$, using integration by parts, using the fact that λy is bounded in $L^2(0,L)$, $\|f_3\|=o(1)$ and $\|f_4\|=o(1)$, we obtain

(4.27)

$$\int_0^L \theta_3\lambda^2|y|^2dx + \frac{k_2}{\rho_2}\int_0^L \theta_3 y_{xx}\overline{y}dx - \frac{k_1}{\rho_2}\int_0^L \theta_3(u_x+y)\overline{y}dx - \frac{\kappa(\alpha)}{\rho_2}\int_0^L \theta_3\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\overline{y}dx = o(\lambda^{-2}).$$

Using integration by parts for the second term of the left hand side, we get

$$\frac{k_2}{\rho_2}\int_0^L \theta_3 y_{xx}\overline{y}dx = -\frac{k_2}{\rho_2}\int_0^L \theta_3' y_x\overline{y}dx - \frac{k_2}{\rho_2}\int_0^L \theta_3|y_x|^2dx. \tag{4.28}$$

Inserting (4.28) in (4.27), using (4.20), (4.12) and the fact that y_x and u_x+y are uniformly bounded in $L^2(0,L)$, we get

$$\frac{k_2}{\rho_2}\int_0^L \theta_3|y_x|^2dx = o(1). \tag{4.29}$$

From the above estimation and the definition of $\theta_3(x)$, we obtain (4.26). Thus, the proof is complete. □

From what precedes, from Lemmas 4.2-4.7, we deduce that

$$\|U\|_{\mathcal{H}_j} = o(1), \quad \text{over} \quad (a_1-\epsilon, b_1-\epsilon).$$

Lemma 4.8. Assume that $\eta>0$ and assumption (A_3) holds. Let $h\in C^1([0,L])$ and $h(0)=h(L)=0$ be a given function. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^1)\in D(\mathcal{A}_j)$ of system (4.4)-(4.8) satisfies the following asymptotic behavior estimation:

$$\int_0^L h'\left(\rho_1|\lambda u|^2 + k_2|y_x|^2 + \rho_2|\lambda y|^2 + k_1\left|u_x + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right|^2\right)dx = o(1).$$

Proof. Let $S := u_x + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi$, from Lemma 4.2, the definition of $D_1(x)$ and the fact that u_x is uniformly bounded in $L^2(0,L)$, we get S is uniformly bounded in $L^2(0,L)$. First, multiplying (4.9) by

$2\rho_1 h\overline{S}$, integrating over $(0,L)$, taking the real part, then using the fact that $\|f_1\| = o(1)$ and $\|f_2\| = o(1)$, we obtain

$$\Re\left\{\rho_1 \int_0^L 2h\lambda^2 u\overline{S}dx\right\} + \Re\left\{k_1 \int_0^L hy_x\overline{S}dx\right\} + k_1 \int_0^L h\left(|S|^2\right)_x dx = \underbrace{\Re\left\{-2\rho_1 \int_0^L h\left(\lambda^{-2}f_2 + i\lambda^{-1}f_1\right)\overline{S}dx\right\}}_{o(\lambda^{-1})}. \tag{4.30}$$

Moreover, from the definition of S and the definition of $D_1(x)$ and from using Cauchy-Schwarz inequality, the fact that $0 < \alpha < 1$ and $\eta > 0$, Lemma 4.2, Lemma 4.5 and the fact that y_x is uniformly bounded in $L^2(0,L)$, we obtain

$$\begin{cases} \Re\left\{2k_1 \int_0^L hy_x\overline{S}dx\right\} = \Re\left\{2k_1 \int_0^L hy_x\overline{u_x}dx\right\} + \underbrace{\Re\left\{2\kappa(\alpha)d_1 \int_{a_1}^{b_1} hy_x \int_{\mathbb{R}} \mu(\xi)\overline{\omega^1(x,\xi)}d\xi dx\right\}}_{=o(\lambda^{-1})}, \\ \Re\left\{\rho_1 \int_0^L 2h\lambda^2 u\overline{S}dx\right\} = \rho_1 \int_0^L h\left(|\lambda u|^2\right)_x dx + \underbrace{\Re\left\{2\rho_1 \frac{\kappa(\alpha)}{k_1} d_1 \int_{a_1}^{b_1} h\lambda^2 u \int_{\mathbb{R}} \mu(\xi)\overline{\omega^1(x,\xi)}d\xi dx\right\}}_{=o(\lambda^{-1})}. \end{cases}$$

Inserting the above estimations in (4.30), we get

$$\rho_1 \int_0^L h\left(|\lambda u|^2\right)_x dx + k_1 \int_0^L h\left(|S|^2\right)_x dx + \Re\left\{2k_1 \int_0^L hy_x\overline{u_x}dx\right\} = o(\lambda^{-1}). \tag{4.31}$$

Now, multiplying (4.10) by $2\rho_2 h\overline{y_x}$, integrating over $(0,L)$, taking the real part, then using Lemma 4.2, the fact that y_x is uniformly bounded in $L^2(0,L)$, $\|y\| = O(|\lambda|^{-1})$, $\|f_3\| = o(1)$ and $\|f_4\| = o(1)$, we obtain

$$\rho_2 \int_0^L h\left(|\lambda y|^2\right)_x dx + k_2 \int_0^L h\left(|y_x|^2\right)_x dx - \Re\left\{2k_1 \int_0^L hu_x\overline{y_x}\right\}dx - \underbrace{\Re\left\{k_1 \int_0^L 2hy\overline{y_x}\right\}dx}_{=o(1)}$$
$$- \underbrace{\Re\left\{2\kappa(\alpha)d_1 \int_{a_1}^{b_1} h\overline{y_x} \int_{\mathbb{R}} \mu(\xi)\overline{\omega^1(x,\xi)}d\xi dx\right\}}_{=o(\lambda^{-1})} = \underbrace{\Re\left\{-2\rho_2 \int_0^L h\left(\lambda^{-2}f_4 + i\lambda^{-1}f_4\right)\overline{y_x}dx\right\}}_{=o(\lambda^{-1})}. \tag{4.32}$$

Adding (4.31) and (4.32), then using integration by parts, we obtain

$$\int_0^L h'\left(\rho_1|\lambda u|^2 + k_2|y_x|^2 + \rho_2|\lambda y|^2 + k_1\left|u_x + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi)d\xi\right|^2\right)dx = o(1). \tag{4.33}$$

The proof is thus complete. ☐

Lemma 4.9. Assume that $\eta > 0$ and assumption (A_3) holds. Then, for $j = 1,2$, the solution $(u,v,y,z,\omega^1) \in D(\mathcal{A}_j)$ of system (4.4)-(4.8) satisfies the following asymptotic behavior estimation:

$$\|U\|_{\mathcal{H}_j} = o(1). \tag{4.34}$$

Proof.
Let $a_1 + \epsilon < a_1 + 2\epsilon < b_1 - \epsilon$ and define the cut-off functions $\theta_4, \theta_5 \in C_0^\infty([0,L])$ by

$$\theta_4(x) = \begin{cases} 1 & \text{if} \quad x \in (0, a_1+\epsilon), \\ 0 & \text{if} \quad x \in (a_1+2\epsilon, L), \\ \in [0,1] & \text{elsewhere} \end{cases}$$

and

$$\theta_5(x) = \begin{cases} 1 & \text{if} \quad x \in (b_1 - \epsilon, L), \\ 0 & \text{if} \quad x \in (0, b_1 - 2\epsilon), \\ \in [0,1] & \text{elsewhere.} \end{cases}$$

First, by taking $\phi = x\theta_4$ in Lemma 4.8 and proceeding in a similar way as in Lemma 3.10 we get $\|U\|_{\mathcal{H}_j} = o(1)$ on $(0, a_1 + \epsilon)$. Moreover, by taking $\phi = (x - L)\theta_5$, we can prove $\|U\|_{\mathcal{H}_j} = o(1)$ on $(b_1 - \epsilon, L)$. Thus, the proof is complete. ☐

Proof of Theorem 4.1. From Lemma 4.9 we get that $\|U\|_{\mathcal{H}_j} = o(1)$, which contradicts (4.2). This implies that

$$\sup_{\lambda\in\mathbb{R}} \left\|(i\lambda I - \mathcal{A}_j)^{-1}\right\|_{\mathcal{L}(\mathcal{H}_j)} = O\left(\lambda^2\right).$$

The result follows from Theorem A.3. ☐

5. Polynomial stability when the damping is effective in shear force and bending moment.

In this section, we study the polynomial stability of system (2.9)-(2.15) in the case $\eta > 0$, when the fractional Kelvin-Voigt damping is present in both shear stress and bending moment equations and the support of D_1 and D_2 intersect, i.e assumption (A_1) holds and $0 < a_1 < a_2 < b_1 < b_2 < L$. Our main result in this section is the following theorem.

Theorem 5.1. *Assume that $\eta > 0$ and assumption (A_1) holds. Then, for $j = 1, 2$, there exists $c > 0$ such that, for every $U_0 \in D(\mathcal{A}_j)$, the following energy estimation holds:*

$$E(t) \leq \frac{c}{t^{\frac{4}{2-\alpha}}} \|U_0\|^2_{D(\mathcal{A}_j)}, \qquad t > 0. \tag{5.1}$$

According to Theorem A.3 and by taking $\ell = 1 - \frac{\alpha}{2}$, the polynomial energy decay (5.1) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}_j) \tag{H$_5$}$$

and

$$\sup_{\lambda\in\mathbb{R}} \|(i\lambda I - \mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)} = O\left(|\lambda|^{1-\frac{\alpha}{2}}\right), \tag{H$_6$}$$

are satisfied. In the case $\eta > 0$, according to Theorem 2.9, condition (H_5) is proved. Now, we will prove condition (H_6) by an argument of contradiction. For this purpose, suppose that (H_6) is false, then there exists

$$\left\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n, \omega_n^1, \omega_n^2)^\top)\right\} \subset \mathbb{R} \times D(\mathcal{A}_j)$$

with

$$|\lambda_n| \to +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}_j} = \|(u_n, v_n, y_n, z_n, \omega_n^1, \omega_n^2)\|_{\mathcal{H}_j} = 1 \tag{5.2}$$

such that

$$\lambda_n^{1-\frac{\alpha}{2}} (i\lambda_n I - \mathcal{A}) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}, f_{6,n})^\top \to 0 \ \text{ in } \ \mathcal{H}_j. \tag{5.3}$$

For simplicity, we drop the index n. Equivalently, from (5.3), we have

$$i\lambda u - v = \lambda^{\frac{\alpha}{2}-1} f_1 \quad \text{in } H_0^1(0,L), \tag{5.4}$$

$$i\lambda v - \frac{k_1}{\rho_1}(S_{d_1})_x = \lambda^{\frac{\alpha}{2}-1} f_2 \quad \text{in } L^2(0,L), \tag{5.5}$$

$$i\lambda y - z = \lambda^{\frac{\alpha}{2}-1} f_3 \quad \text{in } \mathcal{O}_j(0,L), \tag{5.6}$$

$$\begin{aligned} i\lambda z - \frac{k_2}{\rho_2}(S_{d_2})_x + \frac{k_1}{\rho_2}(u_x+y) & \\ + \frac{\kappa(\alpha)\sqrt{D_1}}{\rho_2}\int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi)d\xi &= \lambda^{\frac{\alpha}{2}-1} f_4 \quad \text{in } L^2(0,L), \end{aligned} \tag{5.7}$$

$$\left(i\lambda + \xi^2 + \eta\right)\omega^1 - i\lambda\sqrt{D_1}(u_x+y)\mu(\xi) = \lambda^{\frac{\alpha}{2}-1}\left[f_5 - \sqrt{D_1}\mu(\xi)\left((f_1)_x + f_3\right)\right] \quad \text{in } \mathcal{W}, \tag{5.8}$$

$$\left(i\lambda + \xi^2 + \eta\right)\omega^2 - i\lambda\sqrt{D_2}y_x\mu(\xi) = \lambda^{\frac{\alpha}{2}-1}\left[f_6 - \sqrt{D_2}(f_3)_x\mu(\xi)\right] \quad \text{in } \mathcal{W}_j, \tag{5.9}$$

where

$$\mathcal{O}_j(0,L) = \begin{cases} H_0^1(0,L), & \text{if } j=1, \\ H_*^1(0,L), & \text{if } j=2, \end{cases} \qquad \mathcal{W}_j = \begin{cases} \mathcal{W}, & \text{if } j=1, \\ \mathcal{W}^*, & \text{if } j=2, \end{cases}$$

$S_{d_1} = \left((u_x+y) + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right)$ and $S_{d_2} = \left(y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right)$.
Here we will check the condition (H_6) by finding a contradiction with (5.2) by showing $\|U\|_{\mathcal{H}_j} = o(1)$. For clarity, we divide the proof into several Lemmas.

Lemma 5.2. Assume that $\eta > 0$ and assumption (A_1) holds. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^1,\omega^2) \in D(\mathcal{A}_j)$ of system (5.4)-(5.9) satisfies the following asymptotic behavior estimations:

$$\int_0^L\int_{\mathbb{R}}(\xi^2+\eta)\left|\omega^1(x,\xi)\right|^2 d\xi dx = o\left(\lambda^{\frac{\alpha}{2}-1}\right) \text{ and } \int_0^L\int_{\mathbb{R}}(\xi^2+\eta)\left|\omega^2(x,\xi)\right|^2 d\xi dx = o\left(\lambda^{\frac{\alpha}{2}-1}\right), \tag{5.10}$$

$$\int_{a_2}^{b_1}|y_x|^2 dx = o\left(\lambda^{-\frac{\alpha}{2}-2}\right) \text{ and } \int_{a_2}^{b_1}|z_x|^2 dx = o\left(\lambda^{-\frac{\alpha}{2}}\right), \tag{5.11}$$

$$\int_{a_2}^{b_1}|u_x+y|^2 dx = o\left(\lambda^{-\frac{\alpha}{2}-2}\right) \text{ and } \int_{a_2}^{b_1}|v_x+z|^2 dx = o\left(\lambda^{-\frac{\alpha}{2}}\right), \tag{5.12}$$

$$\int_{a_2}^{b_1}|S_{d_1}|^2 = o(\lambda^{\frac{\alpha}{2}-1}) \text{ and } \int_{a_2}^{b_1}|S_{d_2}|^2 = o(\lambda^{\frac{\alpha}{2}-1}). \tag{5.13}$$

Proof. First, we proof the second estimation of (5.11). From (5.6), we have

$$z_x = i\lambda y_x - \lambda^{\frac{\alpha}{2}-1}(f_3)_x.$$

It follows that

$$\|z_x\|_{L^2(a_2,b_1)} \leq \|\lambda y_x\|_{L^2(a_2,b_1)} + |\lambda|^{\frac{\alpha}{2}-1}\|(f_3)_x\|_{L^2(a_2,b_1)} \leq \frac{o(1)}{\lambda^{\frac{\alpha}{4}}} + \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}.$$

Since $\alpha \in (0,1)$, we have $\min(\frac{\alpha}{4}, 1-\frac{\alpha}{2}) = \frac{\alpha}{4}$, hence, from the above equation, we get

$$\int_{a_2}^{b_1}|z_x|^2 dx = o(\lambda^{-\frac{\alpha}{2}}).$$

Now, we proof the second estimation of (5.12). From (5.4) and (5.6) we have

$$v_x = i\lambda u_x - \lambda^{\frac{\alpha}{2}-1}(f_1)_x \quad \text{and} \quad z = i\lambda y - \lambda^{\frac{\alpha}{2}-1}f_3.$$

It follows that

$$\|v_x+z\|_{L^2(a_2,b_1)} \leq \|\lambda(u_x+y)\|_{L^2(a_2,b_1)} + |\lambda|^{\frac{\alpha}{2}-1}\|(f_1)_x+f_3\|_{L^2(a_2,b_1)} \leq \frac{o(1)}{\lambda^{\frac{\alpha}{4}}} + \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}.$$

Since $\alpha \in (0,1)$, we get

$$\int_{a_2}^{b_1} |v_x + z|^2 dx = o(\lambda^{-\frac{\alpha}{2}}).$$

Finally, the proof of the remaining estimations can follow using similar computations as in Section 3 (Lemmas 3.3, 3.4 and 3.5) and Section 4 (Lemmas 4.2, 4.3 and 4.4). □

Lemma 5.3. Assume that $\eta > 0$, assumption (A_1) holds. Let $g \in C^1([a_2, b_1])$ such that

$$g(b_1) = -g(a_2) = 1, \max_{x\in(a_2,b_1)} |g(x)| = c_g \text{ and } \max_{x\in(a_2,b_1)} |g'(x)| = c_{g'},$$

where c_g and c_g' are strictly positive constant. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$ of system (5.4)-(5.9) satisfies the following asymptotic behavior estimations:

$$|S_{d_2}(b_1)|^2 + |S_{d_2}(a_2)|^2 \leq \frac{\rho_2}{2k_2}\lambda^{1+\frac{\alpha}{2}} \int_{a_2}^{b_1} |z|^2 dx + o(1), \tag{5.14}$$

and

$$|z(b_1)|^2 + |z(a_2)|^2 \leq (\frac{\rho_2}{2k_2}\lambda^{1-\frac{\alpha}{2}} + 2c_g') \int_{a_2}^{b_1} |z|^2 dx + \frac{o(1)}{\lambda}. \tag{5.15}$$

Proof. First, we will prove equation (5.14). Multiplying (5.7) by $-2\frac{\rho_2}{k_2} g\overline{S_{d_2}}$ and integrating over (a_2, b_1), we get

$$\begin{aligned}|S_{d_2}(b_1)|^2 + |S_{d_2}(a_2)|^2 = &\int_{a_2}^{b_1} g' \, |S_{d_2}|^2 dx + \Re\left(2\frac{\rho_2 i\lambda}{k_2}\int_{a_2}^{b_1} g\, z\, \overline{S_{d_2}} dx\right) + \Re\left(2\frac{k_1}{k_2}\int_{a_2}^{b_1} g\,(u_x+y)\,\overline{S_{d_2}} dx\right) \\ &+\Re\left(2\frac{\kappa(\alpha)}{k_2}\int_{a_2}^{b_1} g\,\sqrt{D_1}\int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi)d\xi\,\overline{S_{d_2}}dx\right) - \Re\left(\frac{2\rho_2}{k_2}\,\lambda^{\frac{\alpha}{2}-1}\int_{a_2}^{b_1} g\, f_4\,\overline{S_{d_2}}dx\right),\end{aligned}$$

consequently,

$$\begin{aligned}|S_{d_2}(b_1)|^2 + |S_{d_2}(a_2)|^2 \leq\; &c_g' \int_{a_2}^{b_1} |S_{d_2}|^2 dx + 2\frac{\rho_2\lambda}{k_2}c_g \int_{a_2}^{b_1} |z|\,|S_{d_2}|\, dx + 2\frac{k_1}{k_2}c_g\int_{a_2}^{b_1} |u_x+y|\,|S_{d_2}|\,dx \\ &+2\frac{\kappa(\alpha)}{k_2}d_1 c_g\int_{a_2}^{b_1}\left|\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right| |S_{d_2}|\,dx + \frac{2\rho_2}{k_2}\,\lambda^{\frac{\alpha}{2}-1}c_g\int_{a_2}^{b_1}|f_4|\,|S_{d_2}|\,dx.\end{aligned} \tag{5.16}$$

Using Young's inequality and second estimation of (5.13), we obtain

$$\frac{2\rho_2\lambda c_g}{k_2}|z|\,|S_{d_2}| \leq \frac{\rho_2\lambda^{1+\frac{\alpha}{2}}}{2k_2}|z|^2 + \frac{2\rho_2\lambda^{1-\frac{\alpha}{2}}c_g^2}{k_2}|S_{d_2}|^2 \leq \frac{\rho_2\lambda^{1+\frac{\alpha}{2}}}{2k_2}|z|^2 + o(1). \tag{5.17}$$

Using Cauchy-Schwarz inequality, first estimations of (5.10) and (5.12), second estimation of (5.13) and using the fact that $||f_4|| = o(1)$, we obtain

$$\begin{cases}\displaystyle\int_{a_2}^{b_1}|u_x+y|\,|S_{d_2}|\,dx = \frac{o(1)}{\lambda^{\frac{3}{2}}}, & \displaystyle\int_{a_2}^{b_1}|S_{d_2}|^2 dx = \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}, \\ \displaystyle\int_{a_2}^{b_1}\left|\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right| |S_{d_2}|\,dx = \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}, & \displaystyle\lambda^{\frac{\alpha}{2}-1}\int_{a_2}^{b_1}|f_4|\,|S_{d_2}|\,dx = \frac{o(1)}{\lambda^{\frac{3}{2}-\frac{3\alpha}{4}}}.\end{cases} \tag{5.18}$$

Inserting (5.17) and (5.18) in (5.16), and since $\alpha \in (0,1)$, we obtain

$$|S_{d_2}(b_1)|^2 + |S_{d_2}(a_2)|^2 \quad\leq\quad \frac{\rho_2\lambda^{1+\frac{\alpha}{2}}}{2k_2}\int_{a_2}^{b_1}|z|^2 dx + o(1). \tag{5.19}$$

Now, we will prove (5.15). From equation (5.6), we have

$$z_x = i\lambda y_x - \lambda^{\frac{\alpha}{2}-1}(f_3)_x. \tag{5.20}$$

Multiplying Equation (5.20) by $2g\overline{z}$ and integrating over (a_2, b_1), then taking the real part, we get

$$|z(b_1)|^2 + |z(a_2)|^2 = \int_{a_2}^{b_1} g'|z|^2 dx + \Re\left\{2i\lambda\int_{a_2}^{b_1} g y_x \overline{z} dx\right\} - \Re\left\{2\lambda^{\frac{\alpha}{2}-1}\int_{a_2}^{b_1} g(f_3)_x\overline{z}dx\right\}.$$

Then,

$$|z(b_1)|^2+|z(a_2)|^2\leq c_{g'}\int_{a_2}^{b_1}|z|^2dx+2c_g\lambda\int_{a_2}^{b_1}|y_x||z|dx+2c_g\lambda^{\frac{\alpha}{2}-1}\int_{a_2}^{b_1}|(f_3)_x||z|dx. \tag{5.21}$$

Using Young's inequality, we have

$$2c_g\lambda|y_x||z|\leq\frac{\rho_2}{2k_2}\lambda^{1-\frac{\alpha}{2}}|z|^2+\frac{2k_2c_g^2}{\rho_2}\lambda^{1+\frac{\alpha}{2}}|y_x|^2\qquad\text{and}\qquad 2c_g\lambda^{\frac{\alpha}{2}-1}|(f_3)_x||z|\leq c_{g'}|z|^2+\frac{c_g^2}{c_{g'}}\lambda^{-2+\alpha}|(f_3)_x|^2.$$

Using the above inequalities, first estimation of (5.11) and the fact that $||(f_3)_x||=o(1)$, then equation (5.21) becomes

$$|z(b_1)|^2+|z(a_2)|^2\leq\left(\frac{\rho_2}{2k_2}\lambda^{1-\frac{\alpha}{2}}+2c_{g'}\right)\int_{a_2}^{b_1}|z|^2dx+\frac{o(1)}{\lambda}.$$

□

Lemma 5.4. Assume that $\eta>0$ and assumption (A$_1$) holds. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^1,\omega^2)\in D(\mathcal{A}_j)$ of system (5.4)-(5.9) satisfies the following asymptotic behavior estimations:

$$\int_{a_2}^{b_1}|z|^2dx=\frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}}\quad\text{and}\quad\int_{a_2}^{b_1}|y|^2dx=\frac{o(1)}{\lambda^{3+\frac{\alpha}{2}}}. \tag{5.22}$$

Proof. Multiplying Equation (5.7) by $-i\lambda^{-1}\overline{z}$ and integrating over (a_2,b_1), then taking the real part, we get

$$\begin{aligned}\int_{a_2}^{b_1}|z|^2dx=&-\frac{k_2}{\rho_2}\lambda^{-1}\Re\left\{i\int_{a_2}^{b_1}(S_{d_2})_x\,\overline{z}dx\right\}+\frac{k_1}{\rho_2}\lambda^{-1}\Re\left\{i\int_{a_2}^{b_1}(u_x+y)\overline{z}dx\right\}\\&+\frac{\kappa(\alpha)}{\rho_2}\lambda^{-1}\Re\left\{i\int_{a_2}^{b_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\,\overline{z}dx\right\}-\lambda^{\frac{\alpha}{2}-2}\Re\left\{i\int_{a_2}^{b_1}f_4\,\overline{z}dx\right\},\end{aligned}$$

consequently,

$$\begin{aligned}\int_{a_2}^{b_1}|z|^2dx\leq&\frac{k_2}{\rho_2}\lambda^{-1}\left|\int_{a_2}^{b_1}(S_{d_2})\,\overline{z}_xdx\right|+\frac{k_2}{\rho_2}\lambda^{-1}\left|S_{d_2}(b_1)\right|\left|z(b_1)\right|+\frac{k_2}{\rho_2}\lambda^{-1}\left|S_{d_2}(a_2)\right|\left|z(a_2)\right|\\&+\frac{k_1}{\rho_2}\lambda^{-1}\left|\int_{a_2}^{b_1}(u_x+y)\overline{z}dx\right|+\frac{\kappa(\alpha)}{\rho_2}\lambda^{-1}\left|\int_{a_2}^{b_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\,\overline{z}dx\right|+\lambda^{\frac{\alpha}{2}-2}\int_{a_2}^{b_1}|f_4||z|dx.\end{aligned} \tag{5.23}$$

Using Cauchy-Schwarz inequality, the fact that z is uniformly bounded in $L^2(0,L)$ and $||f_4||=o(1)$, we get

$$\lambda^{\frac{\alpha}{2}-2}\int_{a_2}^{b_1}|f_4||z|dx=o(\lambda^{\frac{\alpha}{2}-2}). \tag{5.24}$$

Using Cauchy-Schwarz inequality, the second estimations in (5.11) and (5.13), we get

$$\frac{k_2}{\rho_2}\lambda^{-1}\left|\int_{a_2}^{b_1}(S_{d_2})\,\overline{z}_xdx\right|=\frac{o(1)}{\lambda^{\frac{3}{2}}}. \tag{5.25}$$

Using Cauchy-Schwarz inequality, the first estimation in (5.12) and the fact that z is uniformly bounded in $L^2(0,L)$, we get

$$\frac{k_1}{\rho_2}\lambda^{-1}\left|\int_{a_2}^{b_1}(u_x+y)\overline{z}dx\right|=\frac{o(1)}{\lambda^{2+\frac{\alpha}{4}}}. \tag{5.26}$$

Using Young's inequality, Cauchy-Schwarz inequality and the first estimation of (5.10), we get

$$\begin{aligned}
&\frac{\kappa(\alpha)}{\rho_2}\lambda^{-1}\left|\int_{a_2}^{b_1}\sqrt{D_1}\overline{z}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\,dx\right| \\
&\leq \frac{1}{2}\,\lambda^{-1+\frac{\alpha}{2}}\int_{a_2}^{b_1}|z|^2\,dx+\frac{\kappa(\alpha)^2 d_1}{2\rho_2^2}\,\lambda^{-1-\frac{\alpha}{2}}\int_{a_2}^{b_1}\left(\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right)^2 dx \\
&\leq \frac{1}{2}\,\lambda^{-1+\frac{\alpha}{2}}\int_{a_2}^{b_1}|z|^2\,dx+\frac{\kappa(\alpha)^2 d_1}{2\rho_2^2}\,\lambda^{-1-\frac{\alpha}{2}}\int_{a_2}^{b_1}\left(\int_{\mathbb{R}}\frac{\mu(\xi)\sqrt{\xi^2+\eta}}{\sqrt{\xi^2+\eta}}\omega^1(x,\xi)d\xi\right)^2 dx \qquad (5.27)\\
&\leq \frac{1}{2}\,\lambda^{-1+\frac{\alpha}{2}}\int_{a_2}^{b_1}|z|^2\,dx+c_4\,\lambda^{-1-\frac{\alpha}{2}}\int_{a_2}^{b_1}\int_{\mathbb{R}}(\xi^2+\eta)\left|\omega^1(x,\xi)\right|^2 d\xi dx \\
&\leq \frac{1}{2}\,\lambda^{-1+\frac{\alpha}{2}}\int_{a_2}^{b_1}|z|^2\,dx+\frac{o(1)}{\lambda^2},
\end{aligned}$$

where $c_4=\dfrac{\kappa(\alpha)^2 d_1}{2\rho_2^2}\displaystyle\int_{\mathbb{R}}\frac{|\xi|^{2\alpha-1}}{|\xi|^2+\eta}d\xi$. Since $0<\alpha<1$ and $\eta>0$, then c_4 is well defined.
Inserting estimations (5.24)-(5.27) in (5.23), we get

$$(1-\frac{1}{2}\,\lambda^{-1+\frac{\alpha}{2}})\int_{a_2}^{b_1}|z|^2dx\leq\frac{k_2}{2\rho_2}\lambda^{-1+\frac{\alpha}{2}}\left(|z(a_2)|^2+|z(b_1)|^2\right)+\frac{k_2}{2\rho_2}\lambda^{-1-\frac{\alpha}{2}}\left(|S_{d_2}(a_2)|^2+|S_{d_2}(b_1)|^2\right)+\frac{o(1)}{\lambda^{\frac{3}{2}}}+\frac{o(1)}{\lambda^{2-\frac{\alpha}{2}}}. \qquad (5.28)$$

Now, inserting estimations (5.14) and (5.15) in (5.28), we get

$$(\frac{1}{2}-\frac{1}{2}\,\lambda^{-1+\frac{\alpha}{2}}-\frac{k_2}{\rho_2}c_g'\lambda^{-1+\frac{\alpha}{2}})\int_{a_2}^{b_1}|z|^2dx\quad\leq\quad\frac{o(1)}{\lambda^{2-\frac{\alpha}{2}}}+\frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}}+\frac{o(1)}{\lambda^{\frac{3}{2}}}. \qquad (5.29)$$

Since $0<\alpha<1$, then $\min(2-\frac{\alpha}{2},1+\frac{\alpha}{2},\frac{3}{2})=1+\frac{\alpha}{2}$. Consequently,

$$(\frac{1}{2}-\frac{1}{2}\lambda^{-1+\frac{\alpha}{2}}-\frac{k_2}{\rho_2}c_g'\lambda^{-1+\frac{\alpha}{2}})\int_{a_2}^{b_1}|z|^2dx\leq\frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}}. \qquad (5.30)$$

Since $|\lambda|\longrightarrow+\infty$, for λ large enough, we get

$$0<(\frac{1}{2}-\frac{1}{2}\,\lambda^{-1+\frac{\alpha}{2}}-\frac{k_2}{\rho_2}c_g'\lambda^{-1+\frac{\alpha}{2}})\int_{a_2}^{b_1}|z|^2dx\leq\frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}}, \qquad (5.31)$$

hence, the first asymptotic estimate of (5.22) holds. Then inserting the first asymptotic estimate of (5.22) in (5.6), we get the second estimate of (5.22). Thus, the proof is complete. □

Lemma 5.5. Assume that $\eta>0$ and assumption (A_1) holds . Then, for $j=1,2$, the solution $(u,v,y,z,\omega^1,\omega^2)\in D(\mathcal{A}_j)$ of system (5.4)-(5.9) satisfies the following asymptotic behavior estimations:

$$|S_{d_1}(b_1)|^2+|S_{d_1}(a_2)|^2\leq\frac{\rho_1}{2k_1}\lambda^{1+\frac{\alpha}{2}}\int_{a_2}^{b_1}|v|^2dx+o(1) \qquad (5.32)$$

and

$$|v(b_1)|^2+|v(a_2)|^2\leq(\frac{\rho_1}{2k_1}\lambda^{1-\frac{\alpha}{2}}+2c_g')\int_{a_2}^{b_1}|v|^2dx+\frac{o(1)}{\lambda}. \qquad (5.33)$$

Proof. First, we will prove equation (5.32). Multiplying (5.5) by $-2\frac{\rho_1}{k_1}g\overline{S_{d_1}}$ and integrating over (a_2,b_1), we get

$$|S_{d_1}(b_1)|^2+|S_{d_1}(a_2)|^2=\int_{a_2}^{b_1}g'|S_{d_1}|^2dx+\Re\left\{2\frac{\rho_1 i\lambda}{k_1}\int_{a_2}^{b_1}g\,v\,\overline{S_{d_1}}dx\right\}-\Re\left\{\frac{2\rho_1}{k_1}\,\lambda^{\frac{\alpha}{2}-1}\int_{a_2}^{b_1}g\,f_2\,\overline{S_{d_1}}dx\right\}, \qquad (5.34)$$

consequently,

$$|S_{d_1}(b_1)|^2+|S_{d_1}(a_2)|^2\leq c_g'\int_{a_2}^{b_1}|S_{d_1}|^2dx+2\frac{\rho_1\lambda}{k_1}c_g\int_{a_2}^{b_1}|v|\,|S_{d_1}|\,dx-\frac{2\rho_1}{k_1}\,\lambda^{\frac{\alpha}{2}-1}c_g\int_{a_2}^{b_1}|f_2|\,|S_{d_1}|\,dx. \qquad (5.35)$$

Using Young's inequality and first estimation of (5.13), we obtain

$$\frac{2\rho_1\lambda c_g}{k_1}\left|v\right|\left|S_{d_1}\right| \leq \frac{\rho_1\lambda^{1+\frac{\alpha}{2}}}{2k_1}\left|v\right|^2+\frac{2\rho_1\lambda^{1-\frac{\alpha}{2}}c_g^2}{k_1}\left|S_{d_1}\right|^2 \leq \frac{\rho_1\lambda^{1+\frac{\alpha}{2}}}{2k_1}\left|v\right|^2+o(1). \tag{5.36}$$

Using Cauchy-Schwarz inequality, first estimation of (5.13) and using the fact that $||f_2||=o(1)$, we obtain

$$\left\{\begin{array}{l} \displaystyle\int_{a_2}^{b_1}\left|S_{d_1}\right|^2dx=\frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}},\\[2ex] \displaystyle\lambda^{\frac{\alpha}{2}-1}\int_{a_2}^{b_1}\left|f_2\right|\left|S_{d_1}\right|dx=\frac{o(1)}{\lambda^{\frac{3}{2}-\frac{3\alpha}{4}}}. \end{array}\right. \tag{5.37}$$

Inserting (5.36) and (5.37) in (5.35) and since $0<\alpha<1$, we get

$$\left|S_{d_1}(b_1)\right|^2+\left|S_{d_1}(a_2)\right|^2 \leq \frac{\rho_1\lambda^{1+\frac{\alpha}{2}}}{2k_1}\int_{a_2}^{b_1}\left|v\right|^2dx+o(1). \tag{5.38}$$

Now, we will prove (5.33). From equation (5.4), we have

$$v_x=i\lambda u_x-\lambda^{\frac{\alpha}{2}-1}(f_1)_x. \tag{5.39}$$

Multiplying equation (5.39) by $2g\overline{v}$ and integrating over (b_1,a_2), then taking the real part, we get

$$\left|v(b_1)\right|^2+\left|v(a_2)\right|^2=\int_{a_2}^{b_1}g'|v|^2dx+\Re\left\{2i\lambda\int_{a_2}^{b_1}gu_x\overline{v}dx\right\}-\Re\left\{2\lambda^{\frac{\alpha}{2}-1}\int_{a_2}^{b_1}g(f_1)_x\overline{v}dx\right\}.$$

Then,

$$\left|v(b_1)\right|^2+\left|v(a_2)\right|^2\leq c_{g'}\int_{a_2}^{b_1}|v|^2dx+2c_g\lambda\int_{a_2}^{b_1}|u_x||v|dx+2c_g\lambda^{\frac{\alpha}{2}-1}\int_{a_2}^{b_1}|(f_1)_x||v|dx. \tag{5.40}$$

Using Young's inequality, we have

$$2c_g\lambda|u_x||v|\leq\frac{\rho_1}{2k_1}\lambda^{1-\frac{\alpha}{2}}|v|^2+\frac{2k_1c_g^2}{\rho_1}\lambda^{1+\frac{\alpha}{2}}|u_x|^2 \quad\text{and}\quad 2c_g\lambda^{-\ell}|(f_1)_x||v|\leq c_{g'}|v|^2+\frac{c_g^2}{c_{g'}}\lambda^{\alpha-2}|(f_1)_x|^2. \tag{5.41}$$

Using equation (5.41) and the fact that $||(f_1)_x||=o(1)$, then equation (5.40) becomes

$$\left|v(b_1)\right|^2+\left|v(a_2)\right|^2\leq\left(\frac{\rho_1}{2k_1}\lambda^{1-\frac{\alpha}{2}}+2c_{g'}\right)\int_{a_2}^{b_1}|v|^2dx+\frac{2k_1c_g^2}{\rho_1}\lambda^{1+\frac{\alpha}{2}}\int_{a_2}^{b_1}|u_x|^2dx+\frac{o(1)}{\lambda^{2-\alpha}}. \tag{5.42}$$

Using the first estimation of (5.12) and the second estimation of (5.22), we get

$$\begin{array}{rcl} \displaystyle\lambda^{1+\frac{\alpha}{2}}\int_{a_2}^{b_1}|u_x|^2dx & \leq & \displaystyle 2\lambda^{1+\frac{\alpha}{2}}\int_{a_2}^{b_1}|u_x+y|^2dx+2\lambda^{1+\frac{\alpha}{2}}\int_{a_2}^{b_1}|y|^2dx\\ & \leq & \displaystyle\frac{o(1)}{\lambda}+\frac{o(1)}{\lambda^2}\\ & \leq & \displaystyle\frac{o(1)}{\lambda}. \end{array} \tag{5.43}$$

Now, inserting (5.43) in (5.42) and $0<\alpha<1$, we get

$$\left|v(b_1)\right|^2+\left|v(a_2)\right|^2\leq\left(\frac{\rho_1}{2k_1}\lambda^{1-\frac{\alpha}{2}}+2c_{g'}\right)\int_{a_2}^{b_1}|v|^2dx+\frac{o(1)}{\lambda}. \tag{5.44}$$

□

Lemma 5.6. Assume that $\eta>0$ and assumption (A_1) holds. Then, for $j=1,2$, the solution $(u,v,y,z,\omega^1,\omega^2)\in D\left(\mathcal{A}_j\right)$ of system (5.4)-(5.9) satisfies the following asymptotic behavior estimations:

$$\int_{a_2}^{b_1}|v|^2dx=\frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}}\quad\text{and}\quad\int_{a_2}^{b_1}|u|^2dx=\frac{o(1)}{\lambda^{3+\frac{\alpha}{2}}}. \tag{5.45}$$

Proof. Multiplying Equation (5.5) by $-i\lambda^{-1}\overline{v}$ and integrating over (a_2, b_1), then taking the real part, we get

$$\int_{a_2}^{b_1} |v|^2 dx = -\frac{k_1}{\rho_1}\lambda^{-1}\Re\left\{i\int_{a_2}^{b_1} (S_{d_1})_x\, \overline{v} dx\right\} - \lambda^{-2+\frac{\alpha}{2}}\Re\left\{i\int_{a_2}^{b_1} f_2\, \overline{v} dx\right\},$$

consequently,

$$\begin{aligned}\int_{a_2}^{b_1} |v|^2 dx \le{}& \frac{k_1}{\rho_1}\lambda^{-1}\left|\int_{a_2}^{b_1} (S_{d_1})\, \overline{v}_x dx\right| + \frac{k_1}{\rho_1}\lambda^{-1}\left|S_{d_1}(b_1)\right|\left|v(b_1)\right| + \frac{k_1}{\rho_1}\lambda^{-1}\left|S_{d_1}(a_2)\right|\left|v(a_2)\right| \\ &+\lambda^{-2+\frac{\alpha}{2}}\int_{a_2}^{b_1} |f_2||v| dx.\end{aligned} \tag{5.46}$$

From the fact that v is uniformly bounded in $L^2(0, L)$ and $||f_2|| = o(1)$, we get

$$\lambda^{-2+\frac{\alpha}{2}}\int_{a_2}^{b_1} |f_2||v| dx = o(\lambda^{-2+\frac{\alpha}{2}}). \tag{5.47}$$

Using the second estimation in (5.12), the first estimation in (5.22) and the first one in (5.13), we get

$$\begin{aligned}\frac{k_1}{\rho_1}\lambda^{-1}\left|\int_{a_2}^{b_1} (S_{d_1})\, \overline{v}_x dx\right| &\le \frac{k_1}{\rho_1}\lambda^{-1}\left|\int_{a_2}^{b_1} (S_{d_1})\, (\overline{v_x + z}) dx\right| + \frac{k_1}{\rho_1}\lambda^{-1}\left|\int_{a_2}^{b_1} (S_{d_1})\, \overline{z} dx\right| \\ &\le \frac{o(1)}{\lambda^{\frac{3}{2}}} + \frac{o(1)}{\lambda^2} \\ &\le \frac{o(1)}{\lambda^{\frac{3}{2}}}.\end{aligned} \tag{5.48}$$

Inserting (5.47) and (5.48) in (5.46), we get

$$\begin{aligned}\int_{a_2}^{b_1} |v|^2 dx &\le \frac{k_1}{2\rho_1}\lambda^{-1+\frac{\alpha}{2}}\left(|v(b_1)|^2 + |v(a_2)|^2\right) + \frac{k_1}{2\rho_1}\lambda^{-1-\frac{\alpha}{2}}\left(|S_{d_1}(b_1)|^2 + |S_{d_1}(a_2)|^2\right) \\ &+ \frac{o(1)}{\lambda^{\frac{3}{2}}} + \frac{o(1)}{\lambda^{2-\frac{\alpha}{2}}}.\end{aligned} \tag{5.49}$$

Now, inserting (5.32) and (5.33) in the above estimation, we get

$$\left(\frac{1}{2} - \frac{k_1}{\rho_1}c_g'\lambda^{-1+\frac{\alpha}{2}}\right)\int_{a_2}^{b_1} |v|^2 dx \le \frac{o(1)}{\lambda^{2-\frac{\alpha}{2}}} + \frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}} + \frac{o(1)}{\lambda^{\frac{3}{2}}}. \tag{5.50}$$

Since $0 < \alpha < 1$ and $|\lambda| \longrightarrow +\infty$, for λ large enough, we get

$$0 < \left(\frac{1}{2} - \frac{k_1}{\rho_1}c_g'\lambda^{-1+\frac{\alpha}{2}}\right)\int_{a_2}^{b_1} |v|^2 dx \le \frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}}. \tag{5.51}$$

Hence, we get the first asymptotic estimate of (5.45). Inserting the first asymptotic estimate of (5.45) in (5.4), we get the second estimate of (5.45). Thus, the proof is complete. □

From what precedes and from Lemma 5.2-5.6, we deduce that

$$\|U\|_{\mathcal{H}_j} = o(1), \quad \text{over} \quad (a_2, b_1).$$

Lemma 5.7. Assume that $\eta > 0$ and assumption (A_1) holds . Let $h \in C^1([0, L])$ and $h(0) = h(L) = 0$ be a given function. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$ of system (5.4)-(5.9) satisfies the following asymptotic behavior estimation:

$$\int_0^L h'\left(\rho_1|v|^2 + k_2\left|y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2(x)}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2 + \rho_2|z|^2 + k_1\left|u_x + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\right|^2\right) dx = o(1).$$

Proof. Let $S := u_x + \frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi$, from Lemma 5.2, the definition of $D_1(x)$ and the fact that u_x is uniformly bounded in $L^2(0, L)$, we get S is uniformly bounded in $L^2(0, L)$. First, multiplying (5.5) by

$2\rho_1 h\overline{S}$, integrating over $(0,L)$, taking the real part, and using the fact that $\|f_2\| = o(1)$, we obtain

$$\Re\left\{\rho_1\int_0^L 2hi\lambda v\overline{S}dx\right\} - \Re\left\{2k_1\int_0^L hy_x\overline{S}dx\right\} - k_1\int_0^L h\left(|S|^2\right)_x dx = \underbrace{\Re\left\{2\rho_1\int_0^L h\lambda^{\frac{\alpha}{2}-1}f_2\overline{S}dx\right\}}_{o(\lambda^{\frac{\alpha}{2}-1})}. \tag{5.52}$$

From equation (5.4), we have

$$i\lambda\overline{u_x} = -\overline{v_x} - \lambda^{\frac{\alpha}{2}-1}(\overline{f_1})_x.$$

Then,

$$i\lambda\overline{S} = -\overline{v_x} - \lambda^{\frac{\alpha}{2}-1}(\overline{f_1})_x + i\lambda\frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^1(x,\xi)}d\xi. \tag{5.53}$$

Moreover, from the definition of S and $D_1(x)$, Cauchy-Schwarz inequality, the fact that $0<\alpha<1$ and $\eta>0$, Lemma 5.2, Lemma 5.6, equation (5.53), the fact that y_x and v are uniformly bounded in $L^2(0,L)$ and $\|(f_1)_x\| = o(1)$, we get

$$\begin{cases} \Re\left\{2k_1\int_0^L hy_x\overline{S}dx\right\} = \Re\left\{2k_1\int_0^L hy_x\overline{u_x}dx\right\} + \underbrace{\Re\left\{2\kappa(\alpha)d_1\int_{a_2}^{b_1} hy_x\int_{\mathbb{R}}\mu(\xi)\overline{\omega^1(x,\xi)}d\xi dx\right\}}_{=\frac{o(1)}{\lambda^{\frac{3}{2}}}}, \\ \Re\left\{\rho_1\int_0^L 2hi\lambda v\overline{S}dx\right\} = -\rho_1\int_0^L h\left(|v|^2\right)_x dx - \underbrace{\lambda^{\frac{\alpha}{2}-1}2\rho_1\int_0^L hv\overline{(f_1)_x}dx}_{=\frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}} + \underbrace{\Re\left\{2\rho_1\frac{\kappa(\alpha)}{k_1}d_1\int_{a_2}^{b_1} hi\lambda v\int_{\mathbb{R}}\mu(\xi)\overline{\omega^1(x,\xi)}d\xi dx\right\}}_{=o(1)}. \end{cases}$$

Inserting the above estimations in equation (5.52), we obtain

$$-\rho_1\int_0^L h\left(|v|^2\right)_x dx - k_1\int_0^L h\left(|S|^2\right)_x dx - \Re\left\{2k_1\int_0^L hy_x\overline{u_x}dx\right\} = o(1). \tag{5.54}$$

Now, multiplying (5.7) by $2\rho_2 h\overline{S_{d_2}} = 2\rho_2 h\left(\overline{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2(x)}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)$, integrating over $(0,L)$, taking the real part, then using the fact that $\left(y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2(x)}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right)$ is uniformly bounded in $L^2(0,L)$, equation (5.10), equation (5.13), $\|y\| = O(|\lambda|^{-1})$ and $\|f_4\| = o(1)$, we obtain

$$\begin{aligned} &\Re\left\{2\rho_2 i\lambda\int_0^L hz\overline{S_{d_2}}dx\right\} - k_2\int_0^L h\frac{d}{dx}\left|y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2 dx \\ &+\Re\left\{2k_1\int_0^L hu_x\overline{S_{d_2}}dx\right\} + \underbrace{\Re\left\{2k_1\int_0^L hy\overline{S_{d_2}}dx\right\}}_{=\frac{O(1)}{\lambda}} + \underbrace{\Re\left\{2\kappa(\alpha)\int_0^L h\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\omega^1(x,\xi)d\xi\overline{S_{d_2}}dx\right\}}_{=\frac{o(1)}{1-\frac{\alpha}{2}}} \\ &= \underbrace{\Re\left\{2\rho_2\int_0^L h\lambda^{\frac{\alpha}{2}-1}f_4\overline{S_{d_2}}dx\right\}}_{=\frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}}. \end{aligned} \tag{5.55}$$

From equation (5.6), we have

$$i\lambda\overline{y_x} = -\overline{z_x} - \lambda^{\frac{\alpha}{2}-1}(\overline{f_3})_x.$$

Then,

$$i\lambda\overline{S_{d_2}} = -\overline{z_x} - \lambda^{\frac{\alpha}{2}-1}(\overline{f_3})_x + i\lambda\frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi. \tag{5.56}$$

Moreover, from the definition of S_{d_2} and $D_2(x)$, Cauchy-Schwarz inequality, the fact that $0 < \alpha < 1$ and $\eta > 0$, Lemma 5.2, Lemma 5.4, equation (5.56), the fact that u_x and z are uniformly bounded in $L^2(0,L)$ and $\|(f_3)_x\| = o(1)$, we have

$$\begin{cases} \Re\left\{2k_1\int_0^L hu_x\overline{S_{d_2}}dx\right\} = \Re\left\{2k_1\int_0^L hu_x\overline{y_x}dx\right\} + \underbrace{\Re\left\{2\frac{k_1}{k_2}\kappa(\alpha)d_2\int_{a_2}^{b_1} hu_x\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi dx\right\}}_{=\frac{o(1)}{\lambda^{\frac{1}{2}-\frac{\alpha}{4}}}}, \\ \Re\left\{\rho_2\int_0^L 2hi\lambda z\overline{S_{d_2}}dx\right\} = -\rho_2\int_0^L h\left(|z|^2\right)_x dx - \underbrace{\lambda^{\frac{\alpha}{2}-1}2\rho_1\int_0^L hz\overline{(f_3)_x}dx}_{=\frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}} \\ \qquad + \underbrace{\Re\left\{\frac{2\rho_2\kappa(\alpha)}{k_2}d_2\int_{a_2}^{b_1} hi\lambda z\int_{\mathbb{R}}\mu(\xi)\overline{\omega^1(x,\xi)}d\xi dx\right\}}_{=o(1)}. \end{cases}$$

Inserting the above estimations in (5.55), we obtain
(5.57)

$$-\rho_2\int_0^L h\frac{d}{dx}|z|^2dx - k_2\int_0^L h\frac{d}{dx}\left|y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2 dx + \Re\left\{2k_1\int_0^L hu_x\overline{y_x}dx\right\} = o(1).$$

Adding (5.54) and (5.57), we get our desired result. □

Lemma 5.8. Assume that $\eta > 0$ and assumption (A_1) holds. Then, for $j = 1,2$, the solution $(u,v,y,z,\omega^1,\omega^2) \in D(\mathcal{A}_j)$ of system (5.4)-(5.9) satisfies the following asymptotic behavior estimation:

$$\|U\|_{\mathcal{H}_j} = o(1).$$

Proof. Proceeding in a similar way as in Lemma 3.10, we get our desired result. □

Proof of Theorem 5.1. From Lemma 5.8 we get that $\|U\|_{\mathcal{H}_j} = o(1)$, which contradicts (5.2). This implies that

$$\sup_{\lambda\in\mathbb{R}}\left\|(i\lambda I - \mathcal{A}_j)^{-1}\right\|_{\mathcal{L}(\mathcal{H}_j)} = O\left(\lambda^{1-\frac{\alpha}{2}}\right).$$

The result follows from Theorem A.3. □

6. Conclusion

We have studied the stabilization of a one-dimensional Timoshenko system with localized internal fractional kelvin-Voigt damping via non-smooth coefficients. We proved the strong stability of the system using Arendt-Batty criteria. Polynomial stability results has been proved in three cases: The case of fractional kelvin-Voigt damping acting on the bending moment equation. We showed a polynomial energy decay rate of type t^{-1}. The case of fractional kelvin-Voigt damping acting on the shear force equation. We proved a polynomial energy decay rate of type t^{-1}. In the last case, the fractional kelvin-Voigt damping acting on the shear force and bending moment equations. We established a polynomial energy decay rate of type $t^{\frac{-4}{2-\alpha}}$. Thereby, we highlight the following important open problems: the *optimality* of the obtained decay rates and the *generalization* of our results to a Bresse system.

Appendix A. Some notions and stability theorems

In order to make this paper more self-contained, we recall in this short appendix some notions and stability results used in this work.

Definition A.1. Let $A : D(A) \subset H \to H$ generates a C_0-semigroup of contractions $\left(e^{tA}\right)_{t\geq 0}$ on H. The C_0-semigroup $\left(e^{tA}\right)_{t\geq 0}$ is said to be

1. Strongly stable if
$$\lim_{t\to+\infty} \|e^{tA}x_0\|_H = 0, \quad \forall\ x_0 \in H.$$
2. Exponentially (or uniformly) stable if there exist two positive constants M and ϵ such that
$$\|e^{tA}x_0\|_H \leq Me^{-\epsilon t}\|x_0\|_H, \quad \forall\ t > 0,\ \forall\ x_0 \in H.$$
3. Polynomially stable if there exists two positive constants C and α such that
$$\|e^{tA}x_0\|_H \leq Ct^{-\delta}\|Ax_0\|_H, \quad \forall\ t > 0,\ \forall\ x_0 \in D\left(A\right).$$

□

Now, we look for sufficient conditions to show the strong stability of the C_0-semigroup $\left(e^{tA}\right)_{t\geq 0}$. We will rely on the following result obtained by Arendt and Batty [1].

Theorem A.2. (**Arendt and Batty** [1]) Let $A : D(A) \subset H \to H$ generates a C_0-semigroup of contractions $\left(e^{tA}\right)_{t\geq 0}$ on H. If

1. A has no pure imaginary eigenvalues,
2. $\sigma\left(A\right) \cap i\mathbb{R}$ is countable,

where $\sigma\left(A\right)$ denotes the spectrum of A, then the C_0-semigroup $\left(e^{tA}\right)_{t\geq 0}$ is strongly stable. □

Concerning the characterization of polynomial stability stability of a C_0-semigroup of contraction $\left(e^{tA}\right)_{t\geq 0}$ we rely on the following result due to Borichev and Tomilov [3] (see also [2] and [6])

Theorem A.3. (Batty in [2], Borichev and Tomilov in [3]). Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ on $\mathcal{H}$. If $\sigma\left(\mathcal{A}\right) \cap\ i\mathbb{R} = \emptyset$, then for a fixed $\ell > 0$ the following conditions are equivalent:

1. $\sup_{\lambda\in\mathbb{R}} \left\|\left(i\lambda I - \mathcal{A}\right)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = O\left(|\lambda|^{\ell}\right).$
2. $\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} \leq \dfrac{C}{t^{\frac{1}{\ell}}}\ \|U_0\|_{D(\mathcal{A})},\ \forall\ t > 0,\ U_0 \in D\left(\mathcal{A}\right),$ for some $C > 0$.

□

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