On the semi-positive definition of the product of a positive diagonal matrix and a symmetric semi-positive definite matrix

Noelie Ramuzat

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Noëlie Ramuzat - LAAS CNRS

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Problem statement: We have Λ a real symmetric semi-positive definite matrix and D a real positive diagonal matrix (thus also symmetric). The product ΛD is a square matrix but non-symmetric. We want to prove that ΛD quadratic form, $x^T \Lambda D x$, is semi-positive definite, i.e. $x^T \Lambda D x \geq 0 \, \forall x \in \mathbb{R}^n \backslash \{0\}$.

If the elements of the diagonal D are equal, i.e. $D = aI$ with with $a \in \mathbb{R}^+ \backslash \{0\}$, then the product $\Lambda D = a\Lambda$ is a real symmetric semi-positive definite matrix. Thus, for a diagonal D with equal elements we have directly $x^T \Lambda D x \geq 0 \, \forall x \in \mathbb{R}^n \backslash \{0\}$.

For different elements on the diagonal, we have the following results:

- The eigenvalues of ΛD are real and non-negatives
- If $\frac{\Lambda D + (\Lambda D)^T}{2}$ has non-negative eigenvalues, then the quadratic form of ΛD is semi-positive definite

Proof. Let us recall some useful matrix definitions and properties:

Definition 1 (Real symmetric semi-positive definite matrix)**.** *A is a real symmetric matrix (its eigenvalues are thus real): A is semi-positive definite $\iff$ all its eigenvalues are non-negatives.*

Definition 2 (Quadratic form semi-positive definition)**.** *A matrix A is semi-positive definite $\iff$ $x^T A x \geq 0, \, \forall x \in \mathbb{R}^n \backslash \{0\}$*

Property 1. *If A is a real semi-positive definite matrix, then $B^T A B$ is semi-positive definite for any matrix B.*

Property 2 (Matrix congruent to a symmetric matrix)**.** *Any matrix congruent to a symmetric matrix is again symmetric: If A is a symmetric matrix then so is $B^T A B$ for any matrix B.*

Proof that the eigenvalues of ΛD are real and non-negatives

Because D is diagonal and positive we can write $D = D^{\frac{1}{2}}D^{\frac{1}{2}}$, $D^{\frac{1}{2}}$ is also real positive and symmetric thus invertible. Let us reformulate the matrix ΛD:

$$\begin{aligned} \Lambda D &= \Lambda D^{\frac{1}{2}}D^{\frac{1}{2}} \\ &= D^{-\frac{1}{2}}(D^{\frac{1}{2}}\Lambda D^{\frac{1}{2}})D^{\frac{1}{2}} \end{aligned} \tag{1}$$

It corresponds to a change of basis of $D^{\frac{1}{2}}$. Because the eigenvalues (denoted λ) are invariant to change of basis we have:

$$\lambda(\Lambda D) = \lambda(D^{\frac{1}{2}}\Lambda D^{\frac{1}{2}}) \tag{2}$$

Using the Properties. 1 and 2, because Λ is a real symmetric semi-positive definite matrix, $(D^{\frac{1}{2}})^T\Lambda D^{\frac{1}{2}} = D^{\frac{1}{2}}\Lambda D^{\frac{1}{2}}$ is a real symmetric semi-positive definite matrix. Thus its eigenvalues are real and non-negatives (Definition.1) and so are the ones of ΛD because of Eq.2. Then we have proven our first result: the eigenvalues of ΛD are real and non-negatives.

Semi-positive definition of ΛD

In this part we look at a way to prove the semi-positive definition of ΛD by studying its symmetric part in the Toeplitz decomposition:

Definition 3 (Toeplitz decomposition)**.** *Every square matrix A can be decomposed uniquely as the sum of two matrices U and V, where U is symmetric and V is skew-symmetric.*

$$A = U + V = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \tag{3}$$

In our case $A = \Lambda D$ and $U = \frac{1}{2}(\Lambda D + (\Lambda D)^T)$ is symmetric.

We recall the fact that the quadratic form of a skew-symmetric matrix equals to zero. Indeed, by definition $V^T = -V$ and thus $x^TVx = (x^TV^Tx)^T = -x^TVx$ which holds only if it equals to zero. Thus, the quadratic form of $x^T\Lambda Dx$ is the same one of x^TUx, i.e:

– If $\frac{\Lambda D+(\Lambda D)^T}{2}$ has non-negative eigenvalues, then the quadratic form of ΛD is semi-positive definite

$$x^T\Lambda Dx = x^T(\frac{\Lambda D + (\Lambda D)^T}{2})x \tag{4}$$

One can thus prove the semi-positive definition of the symmetric matrix U to prove the semi-positive definition of ΛD. Indeed, if $x^TUx \geq 0$, $\forall x \in \mathbb{R}^n\backslash\{0\}$, using Eq.4, we obtain $x^T\Lambda Dx \geq 0$, $\forall x \in \mathbb{R}^n\backslash\{0\}$: proving of the semi-positive definition of ΛD.

One way to prove the semi-positive definition of U is to look at its eigenvalues. Because U is symmetric, if its eigenvalues are non-negatives then U is semi-positive definite (see Definition 1). This gives our second result: if U has non-negative eigenvalues, then ΛD is semi-positive definite.

On the eigenvalues of U

One may notice that we have further information on the eigenvalues of U with respect to the ones of ΛD. Using the following theorem of Fan on matrices [1] (Chapter 10, Theorem 10.28):

Theorem 1. *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1(A), ..., \lambda_n(A)$ and $\mathcal{R}_e\lambda_i(A)$ their real parts. Then:*

$$\sum_{i=1}^{n} \mathcal{R}_e\lambda_i(A) \leq \sum_{i=1}^{n} \lambda_i(\frac{A + A^T}{2}) \tag{5}$$

Then, because the eigenvalues of ΛD are real and non-negatives (as proven in the first part), we have using the Theorem 1:

$$0 \leq \sum_{i=1}^{n} \lambda_i(\Lambda D) \leq \sum_{i=1}^{n} \lambda_i(U) \tag{6}$$

Thus, we know that the sum of the eigenvalues of U is non-negative, however to prove the semi-positive definition of U it is needed to prove that all its eigenvalues are non-negatives.

□

References

[1] Fuzhen Zhang. *Matrix theory: basic results and techniques.* Springer, 2011.