# Functor homology over an additive category

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# FUNCTOR HOMOLOGY OVER AN ADDITIVE CATEGORY

AURÉLIEN DJAMENT AND ANTOINE TOUZÉ

Abstract. We prove a series of results which open the way to computations of functor homology over arbitary additive categories. In particular, we generalize the strong comparison theorem of Franjou Friedlander Scorichenko and Suslin to an arbitrary $\mathbb{F}_p$-linear additive category. This result allows us to compare the cohomology of the classical algebraic groups and the cohomology of their discrete groups of $k$-points when $k$ is an infinite perfect field of positive characteristic, in the spirit of the celebrated theorem of Cline Parshall Scott and van der Kallen over finite fields.

Résumé. Nous établissons une série de résultats rendant accessibles de nombreux calculs d'homologie des foncteurs sur une catégorie additive arbitraire. En particulier, nous généralisons le théorème de comparaison forte de Franjou, Friedlander, Scorichenko et Suslin à une catégorie source additive $\mathbb{F}_p$-linéaire quelconque. Cela nous permet de comparer la cohomologie des groupes algébriques classiques à celle des groupes discrets de leurs points sur un corps parfait infini de caractéristique non nulle, dans l'esprit d'un célèbre théorème de Cline, Parshall, Scott et van der Kallen pour les corps finis.

## 1. Introduction

Given a small category $\mathcal{C}$ and a commutative ring $k$, we denote by $k[\mathcal{C}]$-**Mod** (resp. **Mod**-$k[\mathcal{C}]$) the category of covariant (resp. contravariant) functors from $\mathcal{C}$ to $k$-modules, and natural transformations between such functors. As the notation suggests, $k[\mathcal{C}]$-**Mod** behaves very much like a category of modules over a ring $R$. In particular, one can compute Ext in these functor categories, and there is also a tensor product over $k[\mathcal{C}]$

$$- \otimes_{k[\mathcal{C}]} - : \mathbf{Mod}\text{-}k[\mathcal{C}] \times k[\mathcal{C}]\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$$

which can be derived to define Tor-modules. Such Ext and Tor-modules are sometimes referred to as 'functor (co)homology'. Functor homology is a classical subject of study, going back to Mitchell's influential article [31] in the early seventies. Close relations with $K$-theory were discovered later, namely stable $K$-theory [42, 17], topological Hochschild homology [36] and stable homology of groups with twisted coefficients [14, 8, 24] can all be expressed in terms of Ext or Tor groups over $k[\mathcal{C}]$ with $\mathcal{C} = \mathbf{P}_R$, the category of finitely generated projective $R$-modules.

The first success of the functor homology approach to these $K$-theoretic invariants was the quick computation of THH of finite fields in [15]. This showed the

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computational power of functor homology over finite fields, and prompted the development of computational techniques. The strong comparison theorem [14, Thm 3.10] is one of the most important results. It bridges, for a finite field $k$, Ext-computations in $k[\mathbf{P}_k]$-**Mod** with Ext-computations in the much nicer category $\mathcal{P}_k$ of strict polynomial functors in the sense of Friedlander and Suslin [18]. The latter is an avatar of modules over Schur algebras and it is much more amenable to computation. By using the strong comparison theorem jointly with the theorem describing the homological effect of Frobenius twists [5, 46, 52] one indeed gets an effective way of performing many homological computations in $k[\mathbf{P}_k]$-**Mod**, when $k$ is a finite field.

However, except a few insights [16, 33] for $R = \mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$, explicit homological computations in $k[\mathbf{P}_R]$-**Mod** have been limited to finite fields $R$ over the past twenty years, the case of more general rings staying out of reach. Quoting Suslin from [14, Appendix p.717] the Ext-modules in $k[\mathbf{P}_R]$-**Mod** 'do not seem to be computable unless we are dealing with finite fields'.

The purpose of the present article is to provide a series of fundamental results which allow to compute functor homology (much) beyond the case of finite fields – in fact, many of our results hold when $\mathbf{P}_R$ is replaced by a small additive $\mathbb{F}_p$-linear category $\mathcal{A}$ or even an arbitrary additive category $\mathcal{A}$. In order to understand the results, the reader should keep in mind that functor (co)homology is analogous to group (co)homology, where the group is replaced by the additive category $\mathcal{A}$. (As already mentioned, this is actually more than a mere analogy since functor (co)homology *is* group (co)homology in some cases [14, 8, 24].) Our results can then be classified into three types.

- **Tensor products:** theorem 4.4 allows to compute functor homology involving tensor products $F \otimes F'$ from the functor homology involving $F$ and $F'$ separately when $F$ and $F'$ have different structures (that is are respectively polynomial and antipolynomial). It is a homological companion of a Steinberg tensor product decomposition proved in [11].
- **Restriction:** the results of section 5 as well as theorem 6.9 tell us when restriction along an additive functor $\mathcal{A} \to \mathcal{A}'$ induces isomorphism in homology. We call these results 'excision results' because of the analogy with Suslin's excision results in K-theory [45, 43], see remark 5.9. They can be typically used to reduce the computation to an additive category with finite Hom-sets, or to discard the 'cross characteristic part' of $\mathcal{A}$ which is not seen by $k$.
- **Algebraic versus discrete:** the results of sections 8 and 10 are functor homology analogues of the natural comparison between the algebraic group cohomology (a.k.a. rational cohomology in [22]) of an algebraic group and the cohomology of the underlying discrete group of $k$-points. They can be typically used to compute functor homology over $\mathcal{A}$ from classical algebraic data, such as generic homology of Schur algebras and Hochschild homology, which are more amenable to computations.

Each of these results is interesting in itself, and can be used independently of the others to obtain explicit functor homology computations. For example, we use excision to prove a homological vanishing result in corollary 5.11. As another example, the comparison between algebraic and discrete cohomology given by corollary 10.11 allows us to quickly generalize in corollary 11.1 the cohomological computations of

[14] to an infinite perfect field, which was (as noted by Suslin) previously completely out of reach. However, these independent results are also intended to be used jointly. Indeed, we proved in [11] that reasonable functors over an additive category are all constructed (via tensor products and restrictions along additive functors) from simpler standard pieces. The results of the present paper explain how to compute the homology of a given functor from the homology of these simpler standard pieces. This perspective on our results is explained in details the last part of this introduction, entitled 'a short guide to functor homology'.

Recall that an important motivation for studying functor homology over additive categories such as $\mathbf{P}_R$ is the connection with K-theory and group homology. We give one notable application of our results to the cohomology of groups in section 11 (leaving further applications to future work). Namely, by exploiting the connections between functor homology and stable group homology, we compare the rational cohomology [22] and the discrete group cohomology [3] of the classical groups over infinite perfect fields. To be more specific, we show in theorem 11.13 that if $V$ and $W$ are two finite-dimensional polynomial representations of degree $d$ of $\mathrm{GL}_n(k)$, the natural morphism is an isomorphism in low degrees (an explicit degree range is given in our theorem, the range increases with $n$):

$$\underline{\mathrm{Ext}}^*_{\mathrm{GL}_n(k)}(V^{[r]}, W^{[r]}) \to \mathrm{Ext}^*_{\mathrm{GL}_n(k)}(V, W) \ .$$

Here $\underline{\mathrm{Ext}}$ denotes the extensions computed in the category of rational representations, while the other Ext is computed in the category of all representations of $\mathrm{GL}_n(k)$. The symbol $^{[r]}$ refers to the $r$-th Frobenius twist of a rational representation (one knows that the left hand side does not depend on $r$ in low degrees provided $r$ is big enough, and it is called *generic extensions of* $\mathrm{GL}_n(k)$, see remark 11.14). This gives an analogue of the celebrated result of Cline, Parshall, Scott and van der Kallen [6] in the context of infinite perfect fields $k$ of positive characteristic. Similar results for symplectic and orthogonal groups are given in theorem 11.15.

Note that contrarily to the finite field case [6], our comparison result requires a stabilization on the rank $n$ of the group. Our bounds on $n$ are maybe not optimal, but some stabilization is most probably needed, already when $V = W = k$ are trivial representations. Indeed, in this case, the left hand-side of the comparison map vanishes in positive degrees for all $r$ by Kempf theorem [22, II cor 4.11]. Thus an isomorphism for all $n$ would be equivalent to the vanishing of the mod $p$ cohomology of the discrete group $\mathrm{GL}_n(k)$ for all $n$ and an arbitrary perfect field $k$. Such a vanishing follows from classical $p$-divisibility results in K-theory if $n$ is large (see lemma 11.5) but it is not known (nor conjectured) for small values $n$.

Now let us say a brief word about the proofs. In order to establish our results on functor homology, we have to overcome two fundamental problems which cannot be addressed with the standard functor homology techniques found e.g. in [31, 14, 35]. Firstly, considerable problems with (co)limits arise when one tries to replace the source category $\mathbf{P}_k$ over a finite field $k$ by a small additive category $\mathcal{A}$ which does not enjoy similar nice finiteness property. Getting around these problems required a number of technical detours, the most visible ones being the necessity of considering both Tor and Ext (even if we are primarily interested in computing Ext, see e.g. remark 10.14), and the modification of $\mathcal{A}$ by the means of the $\aleph$-additivization procedure of section 2.6. The second fundamental problem concerns the comparison of the homological properties of $k[\mathcal{A}]$-**Mod** together with that of its full subcategory ${}_k\mathcal{A}$-**Mod** of additive functors. Our solution to this problem involves classical homotopy theory. Indeed, in several places we encode the properties of functor categories into the homology of certain simplicial objects while we encode the properties of the full subcategories of additive functors into the homotopy groups of the same simplicial objects. Variants of the Hurewicz theorem given in section 3 allow us to compare these quantities.

The remainder of this introduction is written as a self-contained short guide to functor homology, which describes in details the main results of the present article, and places them in the wider perspective of the structural results established in [11] and recalled below.

## A short guide to functor homology

We fix a small additive category $\mathcal{A}$ (that is a small $\mathbb{Z}$-linear category $\mathcal{A}$ having biproducts), and $k$ denotes a commutative ring. Unadorned tensor products are taken over $k$. As before, $k[\mathcal{A}]$-**Mod** stands for the category of (non necessarily additive) functors $\mathcal{A} \to k$-**Mod**, and natural transformations between them.

*Polynomial versus antipolynomial functors.* Functors $F : \mathcal{A} \to k$-**Mod** can be rather wild objects. However, their study is substantially simplified by the rigidity results established in [11]. For example, if $k$ is a big field (e.g. an algebraically closed field) and if $F$ is a simple functor with finite dimensional values there is a unique tensor decomposition [11, Cor 4.13]:

$$F(a) \simeq F^{\mathrm{anti}}(a) \otimes F^{\mathrm{pol}}(a) \;.$$

The functor $F^{\mathrm{pol}}$ is *polynomial* in the sense of Eilenberg and Mac Lane [12], hence quite closely related to additive functors (typical polynomial functors are tensor powers of additive functors). The functor $F^{\mathrm{anti}}$ is *antipolynomial*, which means that it factors through an additive functor $\pi : \mathcal{A} \to \mathcal{B}$ whose codomain is a *$k$-trivial category*, that is, an additive category $\mathcal{B}$ with finite Hom-sets and such that $\mathcal{B}(b, b') \otimes_{\mathbb{Z}} k = 0$ for all pairs of objects $(b, b')$. By their definition, antipolynomial functors are quite far from polynomial functors, actually the only functors which are both polynomial and antipolynomial are the constant functors. If $F$ satisfies finiteness hypotheses milder than simplicity, and the field $k$ may be not big enough, a slightly weaker result subsists. Namely, there is [11, Cor 4.11] a unique bifunctor $B : \mathcal{A} \times \mathcal{A} \to k$-**Mod**, which is antipolynomial with respect to its left variable and polynomial with respect to its right variable such that

$$F(a) \simeq B(a, a) \;.$$

We say in this article that such a bifunctor $B$ is of *AP-type*. If $k$ is only a commutative ring, results similar to the ones of [11] have not been investigated, but we nonetheless obtain a very large variety of functors $F : \mathcal{A} \to k$-**Mod** by restricting bifunctors of AP-type along the diagonal functor $\Delta : \mathcal{A} \to \mathcal{A} \times \mathcal{A}$.

Our first result deals with functor homology involving bifunctors of AP-type. We denote by $\Delta^* B$ the functor obtained by restricting $B$ along the diagonal: $\Delta^* B(a) = B(a, a)$.

**Theorem 1.1.** *Let $B$, $B'$ and $C$ be three bifunctors of AP-type, with $B$ contravariant in both variables. Restriction along the diagonal $\Delta$ yields isomorphisms:*

$$\mathrm{Ext}^*_{k[\mathcal{A}\times\mathcal{A}]}(B',C) \simeq \mathrm{Ext}^*_{k[\mathcal{A}]}(\Delta^*B',\Delta^*C)\;,$$
$$\mathrm{Tor}_*^{k[\mathcal{A}]}(\Delta^*B,\Delta^*C) \simeq \mathrm{Tor}_*^{k[\mathcal{A}\times\mathcal{A}]}(B,C)\;.$$

Theorem 1.1 should be interpreted as the fact that polynomial functors and antipolynomial functors cannot interact homologically, a fact which greatly simplifies the computations. For example, assume to simplify that $k$ a field, and that

$$B(a,b) = F^{\mathrm{anti}}(a)\otimes F^{\mathrm{pol}}(b)\;, \qquad\qquad C(a,b) = G^{\mathrm{anti}}(a)\otimes G^{\mathrm{pol}}(b)\;.$$

Then in view of the classical Künneth formula (recalled in proposition 2.21), theorem 1.1 actually expresses $\mathrm{Tor}_*^{k[\mathcal{A}]}(\Delta^*B,\Delta^*C)$ as the tensor product of two terms of a different nature, namely

$$\mathrm{Tor}_*^{k[\mathcal{A}]}(F^{\mathrm{anti}},G^{\mathrm{anti}}) \qquad \text{and} \qquad \mathrm{Tor}_*^{k[\mathcal{A}]}(F^{\mathrm{pol}},G^{\mathrm{pol}})\;,$$

which we respectively call *antipolynomial* homology and *polynomial* homology. In the remainder of the introduction, we examine how to compute these two terms.

*Remark* 1.2. In the remainder of the introduction, we privilege the computation of Tor in our exposition. Indeed, Ext computations follow the same pattern, but often require additional finiteness assumptions, which lengthen the exposition. As an example, the Künneth isomorphism for Ext holds only under suitable finiteness assumptions, see proposition 2.22 and remark 2.23.

*Antipolynomial homology.* For every pair of antipolynomial functors $(F^{\mathrm{anti}}, G^{\mathrm{anti}})$, we can always find such a full, additive, and essentially surjective $\pi:\mathcal{A}\to\mathcal{B}$ with $k$-trivial codomain such that $F^{\mathrm{anti}}$ and $G^{\mathrm{anti}}$ both factor through $\pi$. The next result allows to transport the computation of antipolynomial homology in the category $k[\mathcal{B}]$-**Mod**. Since $\mathcal{B}$ is $k$-cotrivial – in particular it has finite Hom-sets – the latter category has a much nicer structure than $k[\mathcal{A}]$-**Mod**, see e.g. [11, Prop 11.7], and it is potentially more accessible via combinatorial methods.

**Theorem 1.3.** *Let $k$ be a commutative ring and let $\pi:\mathcal{A}\to\mathcal{B}$ be a full, additive, and essentially surjective functor, with $k$-trivial codomain $\mathcal{B}$. Let $F$, $F'$ and $G$ be three functors from $\mathcal{B}$ to $k$-**Mod**, with $F$ contravariant. Let $\pi^*F$ denote the composition $F\circ\pi$. Restriction along $\pi$ yields isomorphisms:*

$$\mathrm{Ext}^*_{k[\mathcal{B}]}(F',G) \simeq \mathrm{Ext}^*_{k[\mathcal{A}]}(\pi^*F',\pi^*G)\;,$$
$$\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^*F,\pi^*G) \simeq \mathrm{Tor}_*^{k[\mathcal{B}]}(F,G)\;.$$

We actually obtain theorem 1.3 as a special case of the excision theorem 5.1, which is analogous to Suslin's excision theorems in algebraic K-theory [45, 43], see remark 5.9.

Functor homology over a $k$-trivial source category $\mathcal{B}$ has not received much attention until now, appart from Kuhn's structure results [24]. Building on the latter, we prove a new cancellation result in corollary 5.11. Namely, if $k$ is a field of characteristic zero and if $F^{\mathrm{anti}}$ and $G^{\mathrm{anti}}$ both factor through $\mathbf{P}_R$ for a finite semi-simple ring $R$, then $\mathrm{Tor}_*^{k[\mathcal{A}]}(F^{\mathrm{anti}},G^{\mathrm{anti}})$ and its Ext analogue are zero in positive degrees.

*Polynomial homology: preliminary reductions.* In our computations of polynomial homology, we assume that $k$ is a field. For all field extensions $k \to K$, the Künneth formula yields an isomorphism

$$K \otimes_k \mathrm{Tor}_*^{k[\mathcal{A}]}(F^{\mathrm{pol}}, G^{\mathrm{pol}}) \simeq \mathrm{Tor}_*^{K[\mathcal{A}]}(K \otimes_k F^{\mathrm{pol}}, K \otimes_k G^{\mathrm{pol}}) \;.$$

So we can reduce ourselves to the case where $k$ is an infinite perfect field, and we will make this assumption in the remainder of the introduction.

Many polynomial functors of interest are constructed from the so-called *strict polynomial functors* introduced by MacDonald in characteristic zero [28, I, App. A] and by Friedlander and Suslin over arbitrary fields [18]. If $k$ is an infinite field, a functor $F : \mathbf{P}_k \to k\text{-}\mathbf{Mod}$ with finite-dimensional values is $d$-homogeneous strict polynomial if for all pairs $(v, w)$ of finite-dimensional vector spaces, the coordinate functions of the map

$$M_{p,q}(k) \simeq \mathrm{Hom}_k(v, w) \xrightarrow{f \mapsto F(f)} \mathrm{Hom}_k(F(v), F(w)) \simeq M_{r,s}(k)$$

are $d$-homogeneous polynomials of the $pq$ entries of the matrix representing $f$. Frequent examples of $d$-homogeneous strict polynomial functors are the symmetric $d$-th powers $S^d$, the exterior $d$-th powers $\Lambda^d$ or the Schur functors associated with a partition of $d$ as in the work of Akin, Buchsbaum and Weymann [2]. Strict polynomial functors also arise in tight connection with classical Schur algebras as in [20, 30]. Indeed the full subcategory $\mathcal{P}_{d,k}$ of $k[\mathbf{P}_k]$-$\mathbf{Mod}$ on the $d$-homogeneous strict polynomial functors is equivalent [18, Thm 3.2] to modules over some classical Schur algebras.

In general, being strict polynomial is a more restrictive condition than being polynomial in the sense of Eilenberg and Mac Lane. That's why, unlike MacDonald, we insist on keeping the word 'strict'. Strict polynomial functors are building blocks for more general polynomial functors from $\mathcal{A}$ to $k$-$\mathbf{Mod}$, by considering tensor products of the form

$$(1) \qquad F^{\mathrm{pol}} = \pi_1^* F_1 \otimes \cdots \otimes \pi_m^* F_m \qquad\qquad G^{\mathrm{pol}} = \rho_1^* G_1 \otimes \cdots \otimes \rho_n^* G_n$$

where each $\pi_i^* F_i$ (resp. $\rho_i^* G_i$) is the composition of a homogeneous strict polynomial functor $F_i$ (resp. $G_i$) and an additive functor $\pi_i$ (resp. $\rho_i$) from $\mathcal{A}$ to $k$-vector spaces. It is known [11, Thm 5.5] that when the field is big enough, every simple functor with finite-dimensional values has this form.

*Remark* 1.4. Since the domain of a strict polynomial functor is the category of finite-dimensional vector spaces, the composition $\pi^* F = F \circ \pi$ makes sense only when $\pi$ has finite-dimensional values. However, we go further and we actually allow additive functors $\pi_i$ and $\rho_i$ with infinite-dimensional values in the tensor products (1). If so, we interpret an expression like $\pi^* F$ as the composition of $\overline{F} \circ \pi$ where $\overline{F}$ denotes the left Kan extension of $F$ to all vector spaces, see definition 6.1. This extension to additive functors with infinite-dimensional values allows to incorporate additional interesting examples, and it is also a crucial point for several proofs where additive functors with infinite-dimensional values naturally appear.

In the sequel, we concentrate on computing functor homology involving functors $F^{\mathrm{pol}}$ and $G^{\mathrm{pol}}$ of the form (1). Then, some standard techniques recalled in section 6 allow to reduce the computation of $\mathrm{Tor}_*^{k[\mathcal{A}]}(F^{\mathrm{pol}}, G^{\mathrm{pol}})$ to the case of only one factor in the tensor products.

In fact, there are *two* reduction procedures, depending on the characteristic of the field $k$. In the advantageous situation where the characteristic of $k$ is *large*, that is, each $F_i$ is $d_i$-homogeneous and each $G_j$ is $e_j$-homogeneous for some integers such that $d_i!$ and $e_j!$ are nonzero in $k$ (thus, characteristic zero is always large!), the computation of $\mathrm{Tor}_*^{k[\mathcal{A}]}(F^{\mathrm{pol}}, G^{\mathrm{pol}})$ reduces to the computation of

$$\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi_i, \rho_j) \tag{2}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. If the characteristic of $k$ is not large (in particular $k$ is a field of positive characteristic), the situation is more complicated and we cannot reduce ourselves to computing functor homology between additive functors. Instead, we can reduce ourselves to computing terms of the form

$$\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi_i^* F, \rho_j^* G) \tag{3}$$

for some strict polynomial functors $F$ and $G$ computed from $F^{\mathrm{pol}}$ and $G^{\mathrm{pol}}$.

The computation of the terms (2) and (3) is examined in the next two paragraphs. In these paragraphs, we assume further that $\mathcal{A}$ is $\mathbb{F}_p$-linear. This is a simplifying hypothesis, which is satisfied in many situations of interest. The case of a general additive source category $\mathcal{A}$, which is not linear over a field, remains somewhat mysterious, but in several such situations, one may also replace the category $\mathcal{A}$ by an $\mathbb{F}_p$-linear one, for example by using the polynomial analogue of the excision theorem that we prove in theorem 6.9.

*Polynomial homology: the case of additive functors.* Let us examine the computation of Ext and Tor between additive functors. There are two categories in which we can perform such homological computations. Firstly, we can consider Ext and Tor in the category $k[\mathcal{A}]$-**Mod** of all functors with domain $\mathcal{A}$ and codomain $k$-vector spaces – this is what we are really interested in, and which appears in (2). Secondly, we can consider Ext and Tor in the full abelian subcategory ${}_k\mathcal{A}$-**Mod** of all *additive* functors. The latter category has a simpler structure, so one expects easier homological computations there[1].

In the advantageous case of a field $k$ of characteristic zero, it is known [9, Thm 1.2] that we have graded isomorphisms:

$$\mathrm{Ext}^*_{k[\mathcal{A}]}(\pi', \rho) \simeq \mathrm{Ext}^*_{{}_k\mathcal{A}}(\pi', \rho)\ , \qquad \mathrm{Tor}_*^{k[\mathcal{A}]}(\pi, \rho) \simeq \mathrm{Tor}_*^{{}_k\mathcal{A}}(\pi, \rho)\ .$$

These providential isomorphisms allow us to perform homological computations in the much nicer category ${}_k\mathcal{A}$-**Mod**. Unfortunately, if $k$ has positive characteristic it is well-known [9] that there are no such graded isomorphisms. Nonetheless, we prove the following theorem, which still allows to reduce homological computations to the category ${}_k\mathcal{A}$-**Mod**.

**Theorem 1.5.** *Let $k$ be an infinite perfect field of positive characteristic $p$, and let $\pi, \pi', \rho$ be three additive functors from $\mathcal{A}$ to $k$-**Mod**, with $\pi$ contravariant. If $\mathcal{A}$ is $\mathbb{F}_p$-linear, there are graded isomorphisms, natural with respect to $\pi$, $\pi'$ and $\rho$:*

$$\mathrm{Ext}^*_{{}_k\mathcal{A}}(\pi', \rho) \otimes E^*_\infty \simeq \mathrm{Ext}^*_{k[\mathcal{A}]}(\pi', \rho)\ ,$$
$$\mathrm{Tor}_*^{{}_k\mathcal{A}}(\pi, \rho) \otimes T_*^\infty \simeq \mathrm{Tor}_*^{k[\mathcal{A}]}(\pi, \rho)\ ,$$

[1] In particular, when $\mathcal{A}$ is the category $\mathbf{P}_R$ of finitely generated projective right $R$-modules, of most importance for applications to K-theory and group homology of $GL_n(R)$ or other classical groups, ${}_k\mathcal{A}$-**Mod** is nothing but the category of $R \otimes_{\mathbb{Z}} k$-modules under disguise by the Eilenberg-Watts theorem, see e.g. [11, Prop 2.1].

*where $E^*_\infty$ and $T_*^\infty$ denote the graded vector spaces equal to $k$ if $*$ is even and non-negative, and to zero in the other degrees.*

*Remark* 1.6. The graded vector spaces $E^*_\infty$ and $T_*^\infty$ have an interpretation in terms of the generic homology that we recall below, namely $E^*_\infty = \mathrm{Ext}^*_{\mathrm{gen}}(I,I)$ and $T_*^\infty = \mathrm{Tor}_*^{\mathrm{gen}}(I^\vee, I)$, where $I$ and $I^\vee$ are defined by $I(v) = v$ and $I^\vee(v) = \mathrm{Hom}_k(v,k)$ for all finite-dimensional vector spaces $v$.

If $\mathcal{A}$ is the category of finitely generated projective $R$-modules, then $\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi,\rho)$ is related [36] to the topological Hochschild homology of $R$, while $\mathrm{Tor}_*^{k\mathcal{A}}(\pi,\rho)$ is related to the Hochschild homology of $R$. Thus, although technically very different, our theorem 1.5 is similar in spirit to the main theorems of [25, 26] (see also [34, Cor 4.2]) which compare Hochschild homology and topological Hochschild homology over smooth $\mathbb{F}_p$-algebras.

*Polynomial homology: the generalized comparison theorem.* The previous paragraph substantially simplifies the task of computing polynomial functor homology of the form (2). Now we are going to examine the much harder situation of functor homology of the form (3), that is:

$$\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^*F, \rho^*G)$$

where $F$ and $G$ are $d$-homogeneous strict polynomial functors, $\pi$ and $\rho$ are additive functors from $\mathcal{A}$ to the category of $k$-vector spaces, respectively contravariant and covariant, and $k$ is an infinite perfect field of positive characteristic $p$.

Contrarily to what happens with additive functors in theorem 1.5, it seems very difficult to produce a closed formula computing these graded vector spaces without any further simplifying assumption. We shall provide such a formula under a suitable vanishing of $\mathrm{Tor}_*^{k\mathcal{A}}(\pi,\rho)$ – this is a restrictive assumption, but still covers a large number of situations of interest.

Our formula expresses the result in terms of *generic homology of strict polynomial functors*. This generic homology is a classical object: it already appears (in the Ext form) in [14] and it plays a central role in making the bridge between the cohomology of reductive algebraic groups and that of the corresponding finite groups of Lie type [6] — see also [52] for a survey of these topics including a formula allowing explicit computations. We denote by $F^{(r)}$ the $r$-fold precomposition of a $d$-homogeneous strict polynomial functor $F$ by the Frobenius twist functor $I^{(1)}$

$$F^{(r)} = F \circ \underbrace{I^{(1)} \circ \cdots \circ I^{(1)}}_{r \text{ times}} .$$

This notation applies to strict polynomial functors which may be covariant or contravariant, and yields $dp^r$-homogeneous strict polynomial functors with the same variance. If $2p^r > i$, the vector space $\mathrm{Tor}_i(F^{(r)}, G^{(r)})$ computed within the category of $dp^r$-homogeneous strict polynomial functors does not depend on $r$, it is called the generic Tor between $F$ and $G$, and denoted by $\mathrm{Tor}_i^{\mathrm{gen}}(F,G)$.

The next result is a generalization of the strong comparison theorem [14, Thm 3.10] to the small additive categories which are $\mathbb{F}$-linear over a big enough field $\mathbb{F}$. It is a special case of the generalized comparison theorem 10.1 that we state and prove in section 10. In the latter, there is no assumption on the size of $\mathbb{F}$, in particular we can use it when $\mathbb{F} = \mathbb{F}_p$, but this assumption is removed at the price of a more complicated formula. We also refer the reader to section 10.4 for similar results for Ext, which hold under suitable finiteness hypotheses.

**Theorem 1.7** (The generalized comparison theorem - $\mathbb{F}$-linear case)**.** *Let $k$ be an infinite perfect field containing a (finite or infinite) subfield $\mathbb{F}$ and let $\mathcal{A}$ be an additive $\mathbb{F}$-linear category. Let $\pi$ and $\rho$ be two $\mathbb{F}$-linear functors from $\mathcal{A}$ to $k$-modules, respectively contravariant and covariant, and let $F$ and $G$ be two objects of $\mathcal{P}_{d,k}$ with $d$ less or equal to the cardinal of $\mathbb{F}$. Assume furthermore that*

$$\mathrm{Tor}_i^{{}_k\mathcal{A}}(\pi,\rho)=0 \quad \textit{for } 0<i<e.$$

*Then for $0\le i<e$ there are $k$-linear isomorphisms*

$$\mathrm{Tor}_i^{k[\mathcal{A}]}(\pi^*F,\rho^*G)\simeq \mathrm{Tor}_i^{\mathrm{gen}}(D_{\pi,\rho}^*F,G)$$

*where $D_{\pi,\rho}$ refers to the contravariant functor $D_{\pi,\rho}(v)=\mathrm{Hom}_k(v,\pi\otimes_{k[\mathcal{A}]}\rho)$.*

We point out that generic Tor can be expressed in the language of Schur algebras. Indeed, $F(k^n)$ and $G(k^n)$ are respectively right and left modules over the Schur algebra $S(n,d)$ in a natural way, and if $2p^r>i$ and $n\ge dp^r$ we have an isomorphism

$$\mathrm{Tor}_i^{\mathrm{gen}}(F,G)\simeq \mathrm{Tor}_i^{S(n,dp^r)}(F(k^n)^{[r]},G(k^n)^{[r]})$$

where the exponent '${}^{[r]}$' indicates restriction along the morphism of Schur algebras $S(n,dp^r)\to S(n,d)$ provided by the Frobenius morphism. Thus theorem 1.7 compares a rather mysterious functor homology $\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^*F,\rho^*G)$ with a familiar Tor over a Schur algebra. This drastically simplifies explicit computations, and this also implies some surprising qualitative properties of polynomial functor homology (such as finiteness of dimensions in each homological degree), since Schur algebras have good homological finiteness properties.

Theorem 1.7 is already nontrivial and interesting for $\mathcal{A}=\mathbf{P}_k$. In this case, the Tor-vanishing condition follows from the Hochschild-Kostant-Rosenberg theorem, and we obtain the following result in corollaries 10.10 and 10.11. It is the infinite field version of the main comparison theorem of [14].

**Corollary 1.8.** *Let $k$ be an infinite perfect field, and let $F$ and $G$ be two objects of $\mathcal{P}_{d,k}$. Let also $F^\vee$ denote the contravariant strict polynomial functor $F^\vee(v)=F(\mathrm{Hom}_k(v,k))$. There are graded isomorphisms*

$$\mathrm{Ext}^*_{\mathrm{gen}}(F,G)\simeq \mathrm{Ext}^*_{k[\mathbf{P}_k]}(F,G)\ , \qquad \mathrm{Tor}^{\mathrm{gen}}_*(F^\vee,G)\simeq \mathrm{Tor}_*^{k[\mathbf{P}_k]}(F^\vee,G)\ .$$

**Some general notations and terminology.** For the convenience of the reader, we gather here some notations and terminology which are used throughout the article.

- The letter $k$ denotes a commutative ring, unadorned tensor products are taken over $k$, $k[X]$ denotes the free $k$-module on a set $X$.
- Categories have a class of objects, and we require that the morphisms between any two objects form a set. We say that a category $\mathcal{C}$ is *small* if the isomorphism classes of objects of $\mathcal{C}$ form a set (those categories are often called 'essentially small' in the literature, we prefer to call them 'small' to avoid heavy terminology). We say that a category $\mathcal{C}$ is *additive* if it is a $\mathbb{Z}$-category with a zero object and finite biproducts [27, VIII.2].
- The letter $\mathcal{A}$ denotes a small additive category, $k[\mathcal{A}]$ is the free $k$-category on $\mathcal{A}$, and ${}_k\mathcal{A}=k\otimes_{\mathbb{Z}}\mathcal{A}$ is the $k$-category obtained by base change. If $k=\mathbb{Z}/n\mathbb{Z}$, the latter is also denoted by $\mathcal{A}/n$.

- The letter $R$ denotes a ring, and $\mathbf{P}_R$ denotes the category of finitely generated projective left $R^{\mathrm{op}}$-modules and $R^{\mathrm{op}}$-linear morphisms (or equivalently finitely generated right modules over $R$). When $R$ is a commutative ring, we identify $\mathbf{P}_R$ with a full subcategory of $R$-**Mod** in the obvious way.
- Given a functor $\phi : \mathcal{C} \to \mathcal{D}$, we use the same notation to denote the induced functor on the opposite categories $\phi : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.
- Finally, we use the standard terminology about connectedness: a morphism of cohomologically graded $k$-modules $f : H^* \to K^*$ is $e$-connected if it is bijective in degrees $* < e$, and injective in degree $* = e$. A morphism of cohomologically graded $k$-modules $f : L_* \to M_*$ is $e$-connected if it is bijective in degrees $* < e$, and surjective in degree $* = e$.

## 2. Prerequisites of functor homology

2.1. **Functor categories.** We recall some standard notations and terminology from [31]. In particular, the term *$k$-category* refers to a $k$-linear category, that is, a category whose homomorphism sets are equipped with a $k$-module structure and such that the composition is $k$-bilinear. The $k$-functors are the $k$-linear functors, that is the functors whose action on morphisms is given by a $k$-linear map.

Given a small $k$-category $\mathcal{K}$, we denote by $\mathcal{K}$-**Mod** the category whose objects are the $k$-functors $F : \mathcal{K} \to k$-**Mod**, whose morphisms are the natural transformations between such functors, the composition being the usual composition of natural transformations. We let **Mod**-$\mathcal{K} = \mathcal{K}^{\mathrm{op}}$-**Mod**. Throughout the article, we will use the following notation, which makes a typographical distinction between the level of the source category $\mathcal{K}$ and the level of the functor category $\mathcal{K}$-**Mod**.

**Notation 2.1.** Objects of $\mathcal{K}$ are denoted by lowercase letters $x, y, \dots$ and homomorphism groups in $\mathcal{K}$ are denoted by $\mathcal{K}(x, y)$. Objects of $\mathcal{K}$-**Mod** are denoted by uppercase letters $F, G, \dots$ and $k$-modules of homomorphism in $\mathcal{K}$-**Mod** are denoted by $\mathrm{Hom}_{\mathcal{K}}(F, G)$.

*Remark* 2.2 (smallness). Smallness of $\mathcal{K}$ ensures that the morphisms between two objects $\mathcal{K}$-**Mod** form a set.

The category $\mathcal{K}$-**Mod** has the structure of a $k$-category, and it is a Grothendieck category (i.e. an AB5 abelian category with a set of generators) with enough projectives and injectives. To be more specific, limits and colimits in $\mathcal{K}$-**Mod** are computed objectwise, in particular a sequence $0 \to F' \to F \to F'' \to 0$ is exact if and only if the sequence of $k$-modules $0 \to F'(x) \to F(x) \to F''(x) \to 0$ is exact for all objects $x$ of $\mathcal{K}$. The representable functors $h^x_{\mathcal{K}} = \mathcal{K}(x, -)$ yield a set of projective generators, which are called the *standard projectives*. Finally, if $M$ is an injective cogenerator of $k$-**Mod**, the dual functors $\mathrm{Hom}_k(h^x_{\mathcal{K}^{\mathrm{op}}}, M)$, which send an object $y$ to $\mathrm{Hom}_k(h^x_{\mathcal{K}^{\mathrm{op}}}(y), M) = \mathrm{Hom}_k(\mathcal{K}(y, x), M)$ yield a set of injective cogenerators, called the *standard injectives*.

We are chiefly interested in the following classical examples of functor categories. The third one, namely, the category of homogeneous strict polynomial functors, will not be used before section 6.

**Example 2.3. Ordinary functors.** Let $\mathcal{C}$ be a small category, and let $k[\mathcal{C}]$ be the free $k$-category on $\mathcal{C}$, that is, the category with the same objects as $\mathcal{C}$, and such that $k[\mathcal{C}](a, b) = k[\mathcal{C}(a, b)]$. The category $k[\mathcal{C}]$-**Mod** identifies with the category

$\mathcal{F}(\mathcal{C};k)$ of all functors $F:\mathcal{C}\to k$-**Mod** and natural transformations between such functors. The latter category is called the category of ordinary functors, the term 'ordinary' refering to the fact that there is no $k$-linearity condition on the functors of this category.

**Example 2.4. Additive functors.** Let $\mathcal{A}$ be a small additive category, and let ${}_k\mathcal{A}$ be the $k$-category obtained by base change. Thus ${}_k\mathcal{A}$ has the same objects as $\mathcal{A}$, and ${}_k\mathcal{A}(a,b)=k\otimes_{\mathbb{Z}}\mathcal{A}(a,b)$. The category ${}_k\mathcal{A}$-**Mod** identifies with the category $\mathbf{Add}(\mathcal{A};k)$ of all additive functors $F:\mathcal{A}\to k$-**Mod** and natural transformations.

**Example 2.5. Homogeneous strict polynomial functors.** Let $\Gamma^d\mathbf{P}_k$ denote the *Schur category*. This $k$-category is defined as follows. First, we denote by $\Gamma^d(v)=(v^{\otimes d})^{\mathfrak{S}_d}$ the $d$-th divided power of a finitely generated projective $k$-module $v$. Then $\Gamma^d\mathbf{P}_k$ has the same objects as $\mathbf{P}_k$ and

$$\Gamma^d\mathbf{P}_k(v,w)=\Gamma^d(\mathrm{Hom}_k(v,w))=\mathrm{Hom}_{k\mathfrak{S}_d}(v^{\otimes d},w^{\otimes d})$$

(i.e. the module of $k$-linear morphisms from $v^{\otimes d}$ to $w^{\otimes d}$ which are equivariant for the action of the symmetric group $\mathfrak{S}_d$ which permutes the factors of the tensor product). The composition is the usual composition of equivariant morphisms. The category $\Gamma^d\mathbf{P}_k$-**Mod** is an avatar of representations of Schur algebras. Indeed, let $S(n,d)=\mathrm{End}_{\Gamma^d\mathbf{P}_k}(k^n)$ denote the classical Schur algebra as in [20]. Then evaluation on $k^n$ yields an equivalence of categories $\Gamma^d\mathbf{P}_k$-**Mod** $\simeq S(n,d)$-**Mod**.

In [18], Friedlander and Suslin introduce the category of $d$-homogeneous strict polynomial functors $\mathcal{P}_{d,k}$ over a field $k$. As observed in [35], this category $\mathcal{P}_{d,k}$ identifies with the full subcategory category $\Gamma^d\mathbf{P}_k$-**mod** of $\Gamma^d\mathbf{P}_k$-**Mod** on the functors $F$ such that $F(v)$ is finite-dimensional for all $v$.

In the case of a field, the equivalence of categories between $\Gamma^d\mathbf{P}_k$-**Mod** and modules over Schur algebras restricts to an equivalence of categories $\Gamma^d\mathbf{P}_k$-**mod** $\simeq$ $S(n,d)$-**mod**. Thus, just as in the case of modules over Schur algebras, the Ext computed in the category $\Gamma^d\mathbf{P}_k$-**mod** are the same as the Ext computed in the bigger category $\Gamma^d\mathbf{P}_k$-**Mod**. Hence working with the former or the latter is rather a matter of taste. In this article, we choose to work in the bigger category $\Gamma^d\mathbf{P}_k$-**Mod**, and we call its objects the *$d$-homogeneous strict polynomial functors* over $k$.

2.2. **Tensor products and** Tor **over a $k$-category.** Given a small $k$-category $\mathcal{K}$ there is a tensor product over $\mathcal{K}$:

$$-\otimes_{\mathcal{K}}-:\mathbf{Mod}\text{-}\mathcal{K}\times\mathcal{K}\text{-}\mathbf{Mod}\to k\text{-}\mathbf{Mod}$$

which is defined by the coend formula [27, IX.6] (recall that unadorned tensor products are taken over $k$):

$$F\otimes_{\mathcal{K}}G=\int^x F(x)\otimes G(x)\;.$$

Thus, $F\otimes_{\mathcal{K}}G$ is the quotient module of $\bigoplus_{x\in\mathrm{Ob}(\mathcal{K})}F(x)\otimes G(x)$ by the relations $F(f)(t)\otimes s=t\otimes G(f)(s)$ for all $t\otimes s\in F(y)\otimes G(x)$ and all $f\in\mathcal{K}(x,y)$.

**Example 2.6.** If $R$ is a $k$-algebra, and $\mathcal{K}=*^R$ is the category with one object $*$ with endomorphism ring equal to $R$, then $\mathcal{K}$-**Mod** $=R$-**Mod** and $\otimes_{\mathcal{K}}$ is nothing but the usual tensor product of $R$-modules.

Usual properties of tensor products over a ring generalize to tensor products over a category. In particular, the tensor product over $\mathcal{K}$ is characterized by the fact that it is $k$-linear and preserves colimits with respect to each of its variables, together with the 'Yoneda isomorphisms', natural with respect to $F$, $G$, $x$, $y$:

$$F \otimes_{\mathcal{K}} h^x_{\mathcal{K}} \simeq F(x) \quad \text{and} \quad h^y_{\mathcal{K}^{\mathrm{op}}} \otimes_{\mathcal{K}} G \simeq G(y)\ . \tag{4}$$

To be more specific, the first Yoneda isomorphism sends the class $[\![s \otimes f]\!]$ of an element $s \otimes f \in F(y) \otimes \mathcal{K}(x, y)$ to $F(f)(s) \in F(x)$. The second Yoneda isomorphism is given by a similar formula.

Alternatively, the tensor product over $\mathcal{K}$ is characterized by the isomorphism natural with respect to the functors $F$ and $G$ and the $k$-module $M$:

$$\alpha : \mathrm{Hom}_k(F \otimes_{\mathcal{K}} G, M) \simeq \mathrm{Hom}_{\mathcal{K}}(G, \mathrm{Hom}_k(F, M))\ , \tag{5}$$

where the functor $\mathrm{Hom}_k(F, M)$ is defined by $\mathrm{Hom}_k(F, M)(x) = \mathrm{Hom}_k(F(x), M)$. To be more explicit, the natural isomorphism $\alpha$ is defined by sending a morphism $f : F \otimes_{\mathcal{K}} G \to M$ to the natural transformation $\alpha(f)$ such that the $k$-linear map $\alpha(f)_x : G(x) \to \mathrm{Hom}_k(F(x), M)$ sends an element $s$ to $s' \mapsto f([\![s \otimes s']\!])$ where the brackets refer to the class of $s \otimes s' \in F(x) \otimes G(x)$ in the quotient $F \otimes_{\mathcal{K}} G$.

The tensor product over $\mathcal{K}$ is a left balanced bifunctor, hence can be left derived in by taking a projective resolution of $F$ or a projective resolution of $G$. We denote by $\mathrm{Tor}^{\mathcal{K}}_*(F, G)$ these derived functors. By deriving the adjunction isomorphism (5) we obtain the following result.

**Lemma 2.7.** *For all $k$-functors $F$, $G$ and for all injective $k$-modules $M$, the adjunction morphism $\alpha$ derives to a graded isomorphism:*

$$\mathrm{Hom}_k(\mathrm{Tor}^{\mathcal{K}}_*(F, G), M) \xrightarrow[\simeq]{\alpha} \mathrm{Ext}^*_{\mathcal{K}}(G, \mathrm{Hom}_k(F, M))\ .$$

2.3. **Restriction functors.** If $\phi : \mathcal{K} \to \mathcal{L}$ is a $k$-functor, there is a $k$-functor called *restriction along* $\phi$ and denoted by $\phi^*$:

$$\begin{array}{cccc} \phi^* : & \mathcal{L}\text{-}\mathbf{Mod} & \to & \mathcal{K}\text{-}\mathbf{Mod} \\ & F & \mapsto & \phi^* F := F \circ \phi \end{array}\ .$$

In this article, we mainly deal with the following examples of restriction functors.

**Example 2.8. Ordinary functors.** If $\phi : \mathcal{C} \to \mathcal{D}$ is a functor between two small categories, we still denote by $\phi : k[\mathcal{C}] \to k[\mathcal{D}]$ the induced $k$-functor. If we identify $k[\mathcal{C}]$-$\mathbf{Mod}$ with the category of ordinary functors $\mathcal{F}(\mathcal{C}; k)$, then the restriction functor $\phi^* : k[\mathcal{D}]\text{-}\mathbf{Mod} \to k[\mathcal{C}]\text{-}\mathbf{Mod}$ identifies with the functor $\mathcal{F}(\mathcal{D}; k) \to \mathcal{F}(\mathcal{C}; k)$ given by precomposition by $\phi$.

**Example 2.9. Additive functors.** If $\phi : \mathcal{A} \to \mathcal{B}$ is an additive functor between two small additive categories, we also denote by $\phi : {}_k\mathcal{A} \to {}_k\mathcal{B}$ the induced $k$-functor. If we identify ${}_k\mathcal{A}$-$\mathbf{Mod}$ with the category of additive functors $\mathbf{Add}(\mathcal{A}; k)$, then the restriction functor $\phi^* : {}_k\mathcal{B}\text{-}\mathbf{Mod} \to {}_k\mathcal{A}\text{-}\mathbf{Mod}$ identifies with the functor $\mathbf{Add}(\mathcal{B}; k) \to \mathbf{Add}(\mathcal{A}; k)$ given by precomposition by $\phi$.

**Example 2.10. Ordinary versus additive functors.** Let $\pi : k[\mathcal{A}] \to {}_k\mathcal{A}$ be the $k$-functor which is the identity on objects and which is defined on morphisms by $\pi(\sum \lambda_f f) = \sum \lambda_f \otimes f$. Then the restriction functor $\pi^* : {}_k\mathcal{A}\text{-}\mathbf{Mod} \to k[\mathcal{A}]\text{-}\mathbf{Mod}$ identifies with the embedding $\mathbf{Add}(\mathcal{A}; k) \hookrightarrow \mathcal{F}(\mathcal{A}; k)$. In order to avoid overloaded notations, the restriction functor $\pi^*$ will often be omitted, i.e. an additive functor will be denoted by the same letter $F$ when viewed as an object of ${}_k\mathcal{A}$-$\mathbf{Mod}$ or as an object of $k[\mathcal{A}]$-$\mathbf{Mod}$.

**Example 2.11. Ordinary versus strict polynomial functors.** Let $\gamma^d : k[\mathbf{P}_k] \to \Gamma^d\mathbf{P}_k$ be the $k$-functor which is the identity on objects and which is defined on morphisms by $\gamma^d(\sum \lambda_f f) = \sum \lambda_f f^{\otimes d}$. Then the restriction functor $\gamma^{d\,*} : \Gamma^d\mathbf{P}_k\text{-}\mathbf{mod} \to k[\mathbf{P}_k]\text{-}\mathbf{Mod}$ identifies with the forgetful functor $\mathcal{P}_{d,k} \to \mathcal{F}(\mathbf{P}_k; k)$. It is fully faithful if $k$ is field with at least $d$ elements [14, Prop 1.4]. In order to avoid overloaded notations, the restriction functor $\gamma^{d\,*}$ will often be omitted, i.e. a strict polynomial functor will be denoted by the same letter $F$ when viewed as an object of $\Gamma^d\mathbf{P}_k$-**Mod** or as an object of $k[\mathbf{P}_k]$-**Mod**.

Restriction along $\phi$ yields $k$-linear morphisms

$$\phi^* : \mathrm{Hom}_{\mathcal{L}}(G, F) \to \mathrm{Hom}_{\mathcal{K}}(\phi^* G, \phi^* F)$$

that we also call restriction morphisms. Similarly there are $k$-linear restriction morphisms, natural with respect to $F$ and $G$:

$$\mathrm{res}^{\phi} : \phi^* F \otimes_{\mathcal{K}} \phi^* G \to F \otimes_{\mathcal{L}} G \;.$$

To be more specific, $\mathrm{res}^{\phi}$ sends the class of an element $s \otimes s' \in \phi^* F(x) \otimes \phi^* G(x)$ to the class of the same element $s \otimes s'$ now viewed as an element of $F(y) \otimes G(y)$ with $y = \phi(x)$. These two restriction maps derive to morphisms of graded $k$-modules, which are also denoted by the same letters:

$$\phi^* : \mathrm{Ext}^*_{\mathcal{L}}(G, F) \to \mathrm{Ext}^*_{\mathcal{K}}(\phi^* G, \phi^* F) \;, \tag{6}$$

$$\mathrm{res}^{\phi} : \mathrm{Tor}^{\mathcal{K}}_*(\phi^* F, \phi^* G) \to \mathrm{Tor}^{\mathcal{L}}_*(F, G) \;. \tag{7}$$

These two restriction maps are related by the following proposition.

**Proposition 2.12.** *For all objects $F$ and $G$ of* **Mod**-$\mathcal{K}$ *and* $\mathcal{K}$-**Mod***, for all $k$-functors $\phi : \mathcal{K} \to \mathcal{L}$, and for all injectives $k$-modules $M$, there is a commutative square:*

$$\begin{array}{ccc}
\mathrm{Ext}^*_{\mathcal{L}}(G, \mathrm{Hom}_k(F, M)) & \xrightarrow[\simeq]{\alpha} & \mathrm{Hom}_k(\mathrm{Tor}^{\mathcal{L}}_*(F, G), M) \\
\big\downarrow{\scriptstyle \phi^*} & & \big\downarrow{\scriptstyle \mathrm{Hom}_k(\mathrm{res}^{\phi}, M)} \\
\mathrm{Ext}^*_{\mathcal{K}}(\phi^* G, \phi^* \mathrm{Hom}_k(F, M)) & \xrightarrow[\simeq]{\alpha} & \mathrm{Hom}_k(\mathrm{Tor}^{\mathcal{K}}_*(\phi^* F, \phi^* G), M)
\end{array} .$$

*Proof.* Commutativity of the diagram in degree 0 is a straightforward verification from the definitions. The commutativity in higher degrees follows from the fact that the arrows are all obtained by deriving the arrows in degree zero. □

Several results in the article assert that under certain conditions, the two restriction maps (6) and (7) are isomorphisms. We will use the next corollaries to reduce the proofs of these results to checking that *one* of these maps is an isomorphism.

**Corollary 2.13.** *Given two functors $F$ and $G$ and an integer $i$, the restriction map* (9) *below is an isomorphism if the restriction map* (8) *is an isomorphism for all injective $k$-modules $M$.*

$$\phi^* : \mathrm{Ext}^i_{\mathcal{L}}(G, \mathrm{Hom}_k(F, M)) \to \mathrm{Ext}^i_{\mathcal{K}}(\phi^* G, \phi^* \mathrm{Hom}_k(F, M)) \tag{8}$$

$$\mathrm{res}^{\phi} : \mathrm{Tor}^{\mathcal{K}}_i(\phi^* F, \phi^* G) \to \mathrm{Tor}^{\mathcal{L}}_i(F, G) \tag{9}$$

The following standard terminology will be often used in the article.

**Definition 2.14.** Let $e$ be a positive integer. A morphism $f : H^* \to K^*$ between cohomologically graded $k$-modules is called *$e$-connected* if it is an isomorphism in degrees $i < e$ and if it is injective in degree $i = e$. Similarly, a morphism $g : L_* \to M_*$ between homologically graded $k$-modules is called *$e$-connected* if it is an isomorphism in degrees $i < e$ and if it is surjective in degree $i = e$.

A morphism of graded $k$-modules is $\infty$-connected if it is $e$-connected for all positive integers $e$.

The following global version of definition 2.14 will be used in section 5.

**Proposition-Definition 2.15.** Let $e$ be a positive integer or $\infty$. The restriction functor $\phi^* : \mathcal{L}$-**Mod** $\to \mathcal{K}$-**Mod** is called *$e$-excisive* if it satisfies one of the following equivalent assertions.

(1) For all $F, G$, the map (6) is an isomorphism in degrees $0 \le * < e$.
($1^+$) For all $F, G$, the map (6) is $e$-connected.
(2) For all $F, G$, the map (7) is an isomorphism in degrees $0 \le * < e$.
($2^+$) For all $F, G$, the map (7) is $e$-connected.
(3) The restriction functor $\phi^* : \mathcal{L}$-**Mod** $\to \mathcal{K}$-**Mod** is fully faithful and for all objects $x$, $y$ of $\mathcal{L}$:

$$\bigoplus_{0<i<e} \mathrm{Tor}_i^{\mathcal{K}}(\phi^* h^x_{\mathcal{L}^{\mathrm{op}}}, \phi^* h^y_{\mathcal{L}}) = 0 \ .$$

*Proof.* It suffices to prove that the five assertions are equivalent for all positive integers $e$. It is clear that ($1^+$)⇒(1) and ($2^+$)⇒(2), and the converses follow from inspecting the long exact sequences associated to a short exact sequence $0 \to K \to P \to G \to 0$ with $P$ projective.

Let us prove (1)⇔(2). We claim that (1) is equivalent to the following assertion.

(1') For all standard injectives $F$ and for all $G$, the map (6) is an isomorphism in degrees $0 \le * < e$.

Indeed, it is clear that (1)⇒(1'). Conversely, every $F$ has a coresolution $J$ by direct products of standard injectives. We have two spectral sequences:

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathcal{K}}(G, J^p)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{K}}(G, F) \ ,$$
$$'E_1^{p,q} = \mathrm{Ext}^q_{\mathcal{L}}(\phi^* G, \phi^* J^p) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{L}}(\phi^* G, \phi^* F) \ ,$$

and $\phi^*$ induces a morphism of spectral sequences. If (1') holds then the two spectral sequences have isomorphic first pages, hence isomorphic abutments, hence (1) holds. Now (1') is equivalent to (2) by corollary 2.13.

Finally, let us prove (2)⇔(3). If (2) holds, then $\phi^*$ is fully faithful by (1), and moreover $\mathrm{Tor}_i^{\mathcal{K}}(\phi^* h^x_{\mathcal{L}^{\mathrm{op}}}, \phi^* h^y_{\mathcal{L}})$ is isomorphic to $\mathrm{Tor}_i^{\mathcal{L}}(h^x_{\mathcal{L}^{\mathrm{op}}}, h^y_{\mathcal{L}})$ for all $0 < i < e$, hence it is zero by projectivity of $h^x_{\mathcal{L}^{\mathrm{op}}}$, which proves (3). Conversely, if $\phi^*$ is fully faithful then (2) holds for $e = 1$, hence $\mathrm{res}^\phi$ is an isomorphism in degree 0 by (1). If in addition the Tor-vanishing is satisfied, then the map (7) is an isomorphism in degrees $0 \le * < e$ for all projective objects $F$ and $G$, hence for all objects $F$ and $G$ by a décalage argument, which proves (2). □

**Example 2.16.** If $\phi : \mathcal{K} \to \mathcal{L}$ is full and essentially surjective, then $\phi^* : \mathcal{L}$-**Mod** $\to$ $\mathbb{K}$-**Mod** is 1-excisive. Indeed, $\phi^*$ is easily checked to be fully faithful.

2.4. **Adjoint functors in homology.** We now recall from [35, Lm 1.3 and Lm 1.5] the good homological properties that restriction along $\phi:\mathcal{K}\to\mathcal{L}$ enjoys when $\phi$ admits a right adjoint.

**Proposition 2.17.** *Let $\phi:\mathcal{K}\leftrightarrows\mathcal{L}:\psi$ be an adjoint pair and let $u:\mathrm{id}\to\psi\circ\phi$ and $e:\phi\circ\psi\to\mathrm{id}$ denote the unit and the counit of an adjunction. Then the following composition is an isomorphism:*

$$\mathrm{Ext}^*_{\mathcal{L}}(\psi^*F,G)\xrightarrow{\phi^*}\mathrm{Ext}^*_{\mathcal{K}}(\phi^*\psi^*F,\phi^*G)\xrightarrow{\mathrm{Ext}^*_{\mathcal{K}}(F(u),G)}\mathrm{Ext}^*_{\mathcal{K}}(F,\phi^*G)\;,$$

*whose inverse is given by the composition:*

$$\mathrm{Ext}^*_{\mathcal{K}}(F,\phi^*G)\xrightarrow{\psi^*}\mathrm{Ext}^*_{\mathcal{L}}(\psi^*F,\psi^*\phi^*G)\xrightarrow{\mathrm{Ext}^*_{\mathcal{L}}(F,G(e))}\mathrm{Ext}^*_{\mathcal{L}}(\psi^*F,G)\;.$$

There is a similar result for Tor.

**Proposition 2.18.** *Let $\phi:\mathcal{K}\leftrightarrows\mathcal{L}:\psi$ be an adjoint pair and let $u:\mathrm{id}\to\psi\circ\phi$ and $e:\phi\circ\psi\to\mathrm{id}$ denote the unit and the counit of an adjunction. Then the following composition is an isomorphism:*

$$\mathrm{Tor}^{\mathcal{K}}_*(\phi^*F,G)\xrightarrow{\mathrm{Tor}^{\mathcal{K}}_*(\phi^*F,G(u))}\mathrm{Tor}^{\mathcal{K}}_*(\phi^*F,\phi^*\psi^*G)\xrightarrow{\mathrm{res}^{\phi}}\mathrm{Tor}^{\mathcal{L}}_*(F,\psi^*G)\;,$$

*whose inverse is given by the composition:*

$$\mathrm{Tor}^{\mathcal{L}}_*(F,\psi^*G)\xrightarrow{\mathrm{Tor}^{\mathcal{L}}_*(F(e),\psi^*G)}\mathrm{Tor}^{\mathcal{L}}_*(\psi^*\phi^*F,\psi^*G)\xrightarrow{\mathrm{res}^{\psi}}\mathrm{Tor}^{\mathcal{K}}_*(\phi^*F,G)\;.$$

**Example 2.19.** If $n\ge 2$, the $k$-functor $\Delta:k[\mathcal{A}]\to k[\mathcal{A}^{\times n}]$ such that $\Delta(x)=(x,\dots,x)$ is adjoint on both sides to the $k$-functor $\Sigma:k[\mathcal{A}^{\times n}]\to k[\mathcal{A}]$ such that $\Sigma(x_1,\dots,x_n)=x_1\oplus\cdots\oplus x_n$. The unit and counit for these two adjunctions are given by the morphisms

$$\mathrm{diag}:a\to a^{\oplus n}\;,\qquad \mathrm{proj}:(\bigoplus_{1\le i\le n}a_i,\dots,\bigoplus_{1\le i\le n}a_i)\to(a_1,\dots,a_n)\;,$$
$$\mathrm{sum}:a^{\oplus n}\to a\;,\qquad \mathrm{incl}:(a_1,\dots,a_n)\to(\bigoplus_{1\le i\le n}a_i,\dots,\bigoplus_{1\le i\le n}a_i)\;,$$

such that the coordinate morphisms of diag and $\rho$ are equal to $\mathrm{id}_a$ and the coordinate morphisms of proj and incl are given by the canonical projections and inclusions. The Ext-isomorphisms

$$\mathrm{Ext}^*_{k[\mathcal{A}]}(\Delta^*G,F)\simeq\mathrm{Ext}^*_{k[\mathcal{A}^{\times n}]}(G,\Sigma^*F)\;,$$
$$\mathrm{Ext}^*_{k[\mathcal{A}]}(F,\Delta^*G)\simeq\mathrm{Ext}^*_{k[\mathcal{A}^{\times n}]}(\Sigma^*F,G)\;,$$

and the analogue Tor-isomorphisms will be often refered to as *sum-diagonal adjunction isomorphisms.*

If the functor $\phi:\mathcal{K}\to\mathcal{L}$ does not have adjoints, the isomorphisms of propositions 2.17 and 2.18 are replaced by spectral sequences. To be more explicit, for all integers $i$, let $L^{\phi}_i,R^i_{\phi}:\mathcal{K}\text{-}\mathbf{Mod}\to\mathcal{L}\text{-}\mathbf{Mod}$ denote the functors such that

$$(L^{\phi}_iF)(y)=\mathrm{Tor}^{\mathcal{K}}_i(\phi^*h^y_{\mathcal{L}^{\mathrm{op}}},F)\;,\qquad (R^i_{\phi}F)(y)=\mathrm{Ext}^i_{\mathcal{K}}(\phi^*h^y_{\mathcal{L}},F)\;.$$

Then $L^{\phi}_0$ and $R^0_{\phi}$ are respectively left ajoint and right adjoint to the restriction functor $\phi^*:\mathcal{L}\text{-}\mathbf{Mod}\to\mathcal{K}\text{-}\mathbf{Mod}$, and the $L^{\phi}_i$ and $R^i_{\phi}$ are the derived functors of $L^{\phi}_0$ and $R^0_{\phi}$ respectively. The following proposition is proved in the same way as the usual base change spectral sequences [54, Chap 5, Thm 5.6.6].

**Proposition 2.20.** *There are base change spectral sequences:*

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{L}}(F, R^t_\phi G) \Rightarrow \mathrm{Ext}^{s+t}_{\mathcal{K}}(F, \phi^* G)\ ,$$
$$'E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{L}}(L^\phi_t F, G) \Rightarrow \mathrm{Ext}^{s+t}_{\mathcal{K}}(F, \phi^* G)\ ,$$
$$''E^2_{s,t} = \mathrm{Tor}^{\mathcal{L}}_s(F', L^\phi_t G) \Rightarrow \mathrm{Tor}^{\mathcal{K}}_{s+t}(\phi^* F', G)\ .$$

2.5. **External tensor products.** We recall from [31] the tensor product $\mathcal{K}\otimes\mathcal{L}$ of two $k$-categories $\mathcal{K}$ and $\mathcal{L}$. The objects of this tensor category are the pairs $(x, y)$ where $x$ is an object of $\mathcal{K}$ and $y$ is an object of $\mathcal{L}$, and its $k$-module morphisms are the tensor products $\mathcal{K}(x, x') \otimes \mathcal{L}(y, y')$. There is an external tensor product operation

$$\boxtimes : \mathcal{K}\text{-}\mathbf{Mod} \times \mathcal{L}\text{-}\mathbf{Mod} \to \mathcal{K}\otimes\mathcal{L}\text{-}\mathbf{Mod}$$

which sends a pair $(F, G)$ to the $k$-functor $(F \boxtimes G)(x, y) = F(x) \otimes G(y)$.

**Proposition 2.21** (Künneth)**.** *Assume that $k$ is a field. There is an isomorphism of graded $k$-vector spaces, natural with respect to $F$, $G$, $H$, $K$:*

$$\mathrm{Tor}^{\mathcal{K}}_*(F, H) \otimes \mathrm{Tor}^{\mathcal{L}}_*(G, K) \simeq \mathrm{Tor}^{\mathcal{K}\otimes\mathcal{L}}_*(F \boxtimes G, H \boxtimes K)\ .$$

*Proof.* Since $h^{(x,y)}_{\mathcal{K}\otimes\mathcal{L}}$ is equal to $h^x_{\mathcal{K}} \boxtimes h^y_{\mathcal{L}}$, there is an isomorphism

$$(F \boxtimes G) \otimes_{\mathcal{K}\otimes\mathcal{L}} (h^x_{\mathcal{K}} \boxtimes h^y_{\mathcal{L}}) \simeq F(x) \otimes G(y) \simeq (F \otimes_{\mathcal{K}} h^x_{\mathcal{K}}) \otimes (G \otimes_{\mathcal{L}} h^y_{\mathcal{L}})\ .$$

Tensor products preserve arbitrary direct sums with respect to each variable. So if $P$, resp. $Q$, is a projective resolution of $H$, resp. $K$, the complex $(F \boxtimes G) \otimes_{\mathcal{K}\otimes\mathcal{L}} (P \boxtimes Q)$ is isomorphic to the complex $(F \otimes_{\mathcal{K}} P) \otimes (G \otimes_{\mathcal{L}} Q)$. Now $P \boxtimes Q$ is a projective resolution of $H \boxtimes K$, hence the result follows from the usual Künneth isomorphism for complexes. □

There is a similar result on the level of extension groups. Namely, the tensor product induces a graded morphism:

$$\mathrm{Ext}^*_{\mathcal{K}}(F, H) \otimes \mathrm{Ext}^*_{\mathcal{L}}(G, K) \to \mathrm{Ext}^*_{\mathcal{K}\otimes\mathcal{L}}(F \boxtimes G, H \boxtimes K)\ . \tag{10}$$

However Hom only preserve finite direct sums, so one needs an additional assumption to adapt the proof of proposition 2.21 for Ext. One says [35, Section 2.3] that a functor is *of type $fp_\infty$* if it has a projective resolution by finite direct sums of standard projectives (or equivalently if it has a resolution by finitely generated projectives). With this additional hypothesis, one easily proves:

**Proposition 2.22** (Künneth)**.** *Let $k$ be a field. Assume that $F$ and $G$ are of type $fp_\infty$, or assume that $F$ is of type $fp_\infty$ and that $H$ has only finite-dimensional values. Then morphism* (10) *is an isomorphism.*

*Remark* 2.23. Being of type $fp_\infty$ is a rather strong property, which is usually very hard to check in an elementary way on a given functor. When $\mathcal{K} = \mathbf{P}_{\mathbb{F}_q}$, one may prove such properties by using Schwartz's $fp_\infty$-lemma [15, Prop 10.1], or by using that $k[\mathbf{P}_{\mathbb{F}_q}]$-**Mod** is locally noetherian [39]. When $\mathcal{K} = k[\mathcal{A}]$ over more general additive categories $\mathcal{A}$, local noetherianity often fails but one can use generalizations of Schwartz's $fp_\infty$-lemma given in [10].

2.6. **The $\aleph$-additivization of a $k$-category.** Let $\aleph$ denote a cardinal. A $k$-category is called $\aleph$-*additive* if it has all $\aleph$-*direct sums*, that is, if all direct sums indexed by sets of cardinality less or equal to $\aleph$ exist. A $k$-functor between two such categories is called $\aleph$-*additive* if it preserves $\aleph$-direct sums. The following construction will be used in sections 8 and 9.

**Definition 2.24.** The $\aleph$-*additivization* of a small $k$-category $\mathcal{K}$ is the category $\mathcal{K}^\aleph$ whose objects are the families of objects of $\mathcal{K}$ indexed by sets of cardinality less or equal to $\aleph$. Such an object is denoted by a formal direct sum $\bigoplus_{i\in\mathcal{I}} x_i$. The morphisms $f:\bigoplus_{j\in\mathcal{J}} x_j \to \bigoplus_{i\in\mathcal{I}} y_i$ are the 'matrices' $[f_{ij}]_{(i,j)\in\mathcal{I}\times\mathcal{J}}$ such that each $f_{ij}\in\mathcal{K}(x_j,y_i)$, and such that for all $j_0$ only a finite number of morphisms $f_{ij_0}$ are nonzero. The composition of morphisms is given by matrix multiplication.

We identify $\mathcal{K}$ with the full $k$-subcategory of $\mathcal{K}^\aleph$ on the formal direct sums with only one object.

The definition of morphisms in $\mathcal{K}^\aleph$ shows that the formal direct sum $\bigoplus_{i\in\mathcal{I}} x_i$ is the categorical coproduct of the $x_i$ in $\mathcal{K}^\aleph$, and also the categorical product if $\mathcal{I}$ is finite, which justifies the direct sum notation. The next elementary proposition gathers the basic properties of $\aleph$-additivization.

**Proposition 2.25.** *The $k$-category $\mathcal{K}^\aleph$ is small and $\aleph$-additive. Moreover:*

1. *An object $x$ of $\mathcal{K}^\aleph$ is isomorphic to a finite direct sum of objects of $\mathcal{K}$ if and only if the functor $\mathcal{K}^\aleph(x,-):\mathcal{K}^\aleph\to k\text{-}\mathbf{Mod}$ is $\aleph$-additive.*
2. *Every $k$-functor $F:\mathcal{K}\to\mathcal{L}$ whose codomain is a $\aleph$-additive extends to a unique (up to isomorphism) $\aleph$-additive functor $F^\aleph:\mathcal{K}^\aleph\to\mathcal{L}$.*

*Proof.* Let us prove (1). The objects of $\mathcal{K}$ (hence their finite direct sums) are $\aleph$-additive by the definition of morphisms in $\mathcal{K}^\aleph$. Conversely, if $x=\bigoplus x_i$ is an object such that $\mathcal{K}^\aleph(x,-)$ is $\aleph$-additive, the isomorphism $\mathcal{K}^\aleph(x,\bigoplus x_i)\simeq\bigoplus\mathcal{K}^\aleph(x,x_i)$ shows that $\mathrm{id}_x$ factors through a finite direct sum of the $x_i$, hence $x$ is isomorphic to a finite direct sum of objects of $\mathcal{K}$. Now we prove (2). For all objects $x=\bigoplus x_i$ we choose a direct sum $\bigoplus F(x_i)$ in $\mathcal{L}$. The assignment $F^\aleph(x)=\bigoplus F(x_i)$ defines an $\aleph$-additive functor such that $F^\aleph\circ\iota=F$. Uniqueness follows from the fact that given any pair of $\aleph$-additive functors $F',G':\mathcal{K}^\aleph\to\mathcal{L}$, every natural transformation $\theta$ between their restrictions to $\mathcal{K}$ extends uniquely into a natural transformation $\theta':F'\to G'$. Indeed it suffices to define $\theta'_{\bigoplus x_i}$ as the unique morphism fitting in the commutative square in which the vertical arrows are the canonical isomorphisms:

$$\begin{array}{ccc} F'(\bigoplus x_i) & \xrightarrow{\theta'_{\bigoplus x_i}} & G'(\bigoplus x_i) \\ \simeq\uparrow & & \simeq\uparrow \\ \bigoplus F(x_i) & \xrightarrow{\bigoplus \theta_{x_i}} & \bigoplus G(x_i) \end{array}.$$

□

The following examples are easily checked by using the universal properties of the categories in play.

**Examples 2.26.** 1. Let $\mathbf{F}_k$ denote the full subcategory of $k$-**Mod** on the free $k$-modules. Its $\aleph$-additivization $\mathbf{F}_k^\aleph$ is equivalent to $\mathbf{F}_k$ if $\aleph$ is finite, and to the full subcategory of $k$-**Mod** on free modules of rank less or equal to $\aleph$ if $\aleph$ is infinite.
2. We have canonical equivalences of categories $k[\mathcal{A}^\aleph]\simeq k[\mathcal{A}]^\aleph$ and ${}_k(\mathcal{A}^\aleph)\simeq({}_k\mathcal{A})^\aleph$.

**Proposition 2.27.** *Assume that for all pairs $(x,y)$ of objects of $\mathcal{K}$, the $k$-module $\mathcal{K}(x,y)$ belongs to $\mathbf{F}_k^{\aleph}$. Then for all objects $x$ of $\mathcal{K}$, the functor $\mathcal{K}^{\aleph}(x,-):\mathcal{K}^{\aleph}\to\mathbf{F}_k^{\aleph}$ has a left adjoint $-\otimes x:\mathbf{F}_k^{\aleph}\to\mathcal{K}^{\aleph}$.*

*Proof.* Let $v$ be a free $k$-module with basis $(b_i)_{i\in I}$. We let $v\otimes x=\bigoplus_{i\in I}x_i$ where each $x_i$ denotes a copy of $x$. The $k$-linear map $v\mapsto\mathcal{K}^{\aleph}(x,v\otimes x)$, sending $b_i$ to the canonical inclusion of $x$ as the factor $x_i$ of $v\otimes x$ is initial in $v\downarrow\mathcal{K}^{\aleph}(x,-)$, and the result follows from [38, Lm 4.6.1]. □

**Proposition 2.28.** *Let $x$ be an object of $\mathcal{K}^{\aleph}$ decomposed as a direct sum $x=\bigoplus_{i\in I}x_i$ of objects of $\mathcal{K}$. Let $\mathcal{F}$ denote the poset of finite direct summands $\bigoplus_{i\in J}x_i$ ordered by the canonical inclusions. The left Kan extension of a $k$-functor $F:\mathcal{K}\to k$-$\mathbf{Mod}$ along the inclusion $\iota:\mathcal{K}\hookrightarrow\mathcal{K}^{\aleph}$ is given by*

$$\mathrm{Lan}_\iota F(x)=\operatorname*{colim}_{y\in\mathcal{F}}F(y)\;.$$

*Thus the functor $\mathrm{Lan}_\iota:\mathcal{K}$-$\mathbf{Mod}\to\mathcal{K}^{\aleph}$-$\mathbf{Mod}$ is exact and restriction along $\iota$ yields isomorphisms*

$$\mathrm{Ext}^*_{\mathcal{K}}(F,\iota^*G)\simeq\mathrm{Ext}^*_{\mathcal{K}^{\aleph}}(\mathrm{Lan}_\iota F,G)\;,\qquad\mathrm{Tor}_*^{\mathcal{K}}(\iota^*G,F)\simeq\mathrm{Tor}_*^{\mathcal{K}^{\aleph}}(G,\mathrm{Lan}_\iota F)\;.$$

*Proof.* Since $\mathcal{F}$ is a filtered category which is cofinal in the comma category $\mathcal{K}\downarrow x$, we have $\mathrm{Lan}_\iota F(x)=\mathrm{colim}_{y\in\mathcal{F}}F(y)$. Since filtered colimits of $k$-modules are exact, and since exactness of a short exact sequence of functors is tested objectwise, this implies that $\mathrm{Lan}_\iota$ is exact. Moreover, $\mathrm{Lan}_\iota$ preserves projectives (because it has an exact right adjoint). This implies the Ext and Tor isomorphisms. □

## 3. Simplicial techniques

3.1. **Simplicial objects in abelian categories.** We denote by $s\mathcal{M}$ the category of simplicial objects in an abelian category $\mathcal{M}$. The Dold-Kan correspondence [19, Cor 2.3] or [54, Section 8.4] yields two mutually inverse equivalences of categories

$$\mathbf{N}:s\mathcal{M}\rightleftarrows\mathrm{Ch}_{\geq 0}(\mathcal{M}):\mathbf{K}$$

between the category $s\mathcal{M}$ of simplicial objects in $\mathcal{M}$ and the category of non-negatively graded chain complexes in $\mathcal{M}$. The functor $\mathbf{N}$ is the functor of normalized chains, and the functor $\mathbf{K}$ is the Kan functor. Two simplicial morphisms $f,g:X\to Y$ are *homotopic* if the morphisms of chain complexes $\mathbf{N}f$ and $\mathbf{N}g$ are homotopic. The Kan functor preserves homotopies.

The *homotopy groups* $\pi_*X$ of an object $X$ of $s\mathcal{M}$ are defined as the homology groups of the normalized chains $\mathbf{N}X$.

The category $\mathcal{M}$ is called *concrete* if it is equipped with a faithful functor to the category of sets. If $\mathcal{M}$ is concrete, every object $X$ of $s\mathcal{M}$ has an underlying simplicial set and we let $H_*^{\mathrm{ss}}(X;k)$ denote the *homology with coefficients in $k$* of the underlying simplicial set of $X$, that is, $H_*^{\mathrm{ss}}(X;k)=\pi_*k[X]$.

*Remark* 3.1. If the faithful functor $\mathcal{M}\to\mathbf{Set}$ is the composition of an exact functor to the category of abelian groups $\mathcal{M}\to\mathbf{Ab}$ and the usual forgetful functor $\mathbf{Ab}\to\mathbf{Set}$, then the homotopy groups of $X$ coincide with the homotopy groups of its underlying simplicial set by [19, Cor 2.7] or [54, Thm 8.3.8].

Let $e$ be a non-negative integer. An object $X$ of $s\mathcal{M}$ is *$e$-connected* if $\pi_iX=0$ for all $i\leq e$. A morphism $f:X\to Y$ is *$e$-connected* if $\pi_if:\pi_iX\to\pi_iY$ is bijective for all $i < e$ and surjective for $i = e$. (Thus $f$ is $e$-connected if and only if its homotopy cofiber is $e$-connected). A morphism $f : X \to Y$ is a *weak equivalence* if it is $e$-connected for all $e \geq 0$.

A *simplicial projective resolution* of an object $X$ of $s\mathcal{M}$ is a weak equivalence $f : P \to X$ where $P$ is degreewise projective in $\mathcal{M}$. As usual, we identify $\mathcal{M}$ with the full subcategory of $s\mathcal{M}$ on the constant simplicial objects. Hence a simplicial projective resolution of an object $X$ of $\mathcal{M}$ is a degreewise projective simplicial object $P$ such that $\pi_i P = 0$ for $i > 0$, equipped with an isomorphism $\pi_0 P \simeq X$. If $\mathcal{M}$ has enough projectives then by the Dold-Kan equivalence every object of $s\mathcal{M}$ has a simplicial resolution, and every morphism between simplicial objects can be lifted to a morphism of simplicial resolutions, unique up to homotopy.

3.2. **Eilenberg MacLane spaces and Hurewicz theorems.** For all abelian groups $A$ and all $n \geq 0$, we denote by $K(A, n)$ any simplicial free abelian group such that $\pi_i K(A, n) = 0$ for $i \neq n$ and $\pi_n K(A, n) \simeq A$. Such a simplicial free abelian group is called an *Eilenberg-MacLane space* and is unique up to homotopy equivalence. The study of simplicial abelian groups often reduces to that of Eilenberg-MacLane spaces by the following classical lemma, see e.g. [19, Prop 2.20]. We impose that Eilenberg-MacLane spaces are degreewise free abelian groups by definition in order to have genuine maps rather than zig-zags in this lemma.

**Lemma 3.2.** *For all simplicial abelian groups $A$, there is a weak equivalence (unique up to homotopy)*

$$\prod_{i \geq 0} K(\pi_i A, i) \to A\ .$$

*Moreover for all morphisms of simplicial abelian groups $f : A \to B$, let $K(\pi_i f, i) : K(\pi_i A, i) \to K(\pi_i B, i)$ denote a lift of $\pi_i f : \pi_i A \to \pi_i B$ to the level of the simplicial resolutions. Then the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} \prod_{i\geq 0} K(\pi_i A, i) & \xrightarrow{\prod K(\pi_i f, i)} & \prod_{i\geq 0} K(\pi_i B, i) \\ \downarrow & & \downarrow \\ A & \xrightarrow{\quad f \quad} & B \end{array} .$$

If $A$ is a simplicial abelian group, the morphism of simplicial sets $A \to \mathbb{Z}[A]$ induces a natural morphism of graded abelian groups

$$h_* : \pi_* A \to H_*^{\mathrm{ss}}(A; \mathbb{Z})$$

called the *Hurewicz morphism*. The following well-known proposition recalls the classical Hurewicz theorems in the context of simplicial abelian groups (injectivity of $h_*$ as well as the fact that no hypothesis on fundamental groups is needed for the relative version are specific to the abelian group setting).

**Proposition 3.3** (Classical Hurewicz Theorems)**.** *Let $e$ be a nonnnegative integer.*

1. *(Absolute theorem) The Hurewicz map $h_*$ is split injective. Moreover, if $A$ is $e$-connected then $h_i$ is an isomorphism for $i \leq e + 1$.*
2. *(Relative theorem) Every $e$-connected morphism of simplicial abelian groups $f : A \to B$ induces an $e$-connected map $H_*^{\mathrm{ss}}(f; \mathbb{Z}) : H_*^{\mathrm{ss}}(A; \mathbb{Z}) \to H_*^{\mathrm{ss}}(B; \mathbb{Z})$.*

*Proof.* (1) The canonical morphism of abelian groups $\mathbb{Z}[A] \to A$ yields a retract of $h_*$. The isomorphism is given by [19, III Thm 3.7]. (2) Since simplicial groups are fibrant simplicial sets [19, I Lm 3.4], any weak equivalence between simplicial groups yields a homotopy equivalence of simplicial sets [19, II Thm 1.10], hence it induces an isomorphism in homology. Therefore, lemma 3.2 and the Künneth theorem reduce the proof of the isomorphism to the case where $A$ and $B$ are Eilenberg-MacLane spaces, with nonzero homotopy groups placed in the same degree $i$. If $i < e$, $f$ is $e$-connected if and only if it is a weak equivalence, hence if and only if it induces an isomorphism in homology. If $i \geq e$, then $A$ and $B$ are $e$-connected hence the result follows from (1). □

For our purposes, we need a $k$-local version of the absolute Hurewicz theorem. We shall derive it from the following presumably well-known property of Eilenberg-MacLane spaces, which we have not found in the literature – though the case of a prime field $k$ is of course given by the classical calculations of Cartan [1].

**Lemma 3.4.** *Let $k$ be a commutative ring, let $A$ be an abelian group. If $k \otimes_{\mathbb{Z}} A = 0$ and $\mathrm{Tor}_1^{\mathbb{Z}}(k, A) = 0$, then $H_i^{\mathrm{ss}}(K(A,n), k) = 0$ for all positive integers $n$ and $i$.*

*Proof.* We say that an abelian group $A$ is $k$-negligible if $\mathrm{Tor}_1^{\mathbb{Z}}(k, A) = 0 = k \otimes_{\mathbb{Z}} A$.

We first take $n = 1$. The lemma holds if $A$ is a $k$-negligible torsion-free group because $H_i^{\mathrm{ss}}(K(A,1); k) \simeq k \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}^i(A) = \Lambda_k^i(k \otimes_{\mathbb{Z}} A) = 0$. The lemma also holds if $A$ is a $k$-negligible torsion group. Indeed, if $A$ is finite, the lemma holds by a direct computation. If $A$ is infinite, then $A$ is the filtered union of all its finite subgroups $A_\alpha$. And since any subgroup of a $k$-negligible torsion group is $k$-negligible, we have: $H_i^{\mathrm{ss}}(K(A,1); k) = \mathrm{colim}_\alpha H_i^{\mathrm{ss}}(K(A_\alpha,1); k) = 0$. Now let $A$ be an arbitrary abelian group with torsion subgroup $A_{\mathrm{tors}}$. If $A$ is $k$-negligible, then so are $A_{\mathrm{tors}}$ and $A/A_{\mathrm{tors}}$. So the lemma holds for $A$ as a consequence of the Hochschild-Serre spectral sequence of the fibration $K(A_{\mathrm{tors}},1) \to K(A,1) \to K(A/A_{\mathrm{tors}},1)$.

Assume now that $n > 1$. Then $K(A,1) \otimes_{\mathbb{Z}} K(\mathbb{Z}, n-1)$ is an Eilenberg Mac Lane space $K(A,n)$. Thus $H_*^{\mathrm{ss}}(K(A,n))$ is the abutment of the spectral sequence of the bisimplicial $k$-module $M_{pq} = k[K(\mathbb{Z},n-1)_q \otimes_{\mathbb{Z}} K(\pi,1)_p]$. Let us choose $K(\mathbb{Z}, n-1)$ such that it is free of finite rank $r(q)$ in each degree $q$ (e.g. take the image of the complex $\mathbb{Z}[-n]$ by the Kan functor). Then for $q$ fixed, the simplicial $k$-module $M_{\bullet q}$ is isomorphic to $k[K(A^{\times r(q)}, 1)]$. Thus the simplicial spectral sequence of $M_{pq}$ can be rewriten as:

$$E^1_{pq} = H_p^{\mathrm{ss}}\big(K(A^{\times r(q)},1)\big) \Longrightarrow H_{p+q}^{\mathrm{ss}}(K(A,n)) \ .$$

The first page is zero by the case $n = 1$, whence the result. □

**Proposition 3.5** ($k$-local absolute Hurewicz theorem)**.** *Let $A$ be a simplicial abelian group, let $k$ be a commutative ring and let $e$ be a non-negative integer. Assume that for $0 < i \leq e$ and for $0 < j < e$ we have $k \otimes_{\mathbb{Z}} \pi_i A = 0 = \mathrm{Tor}^{\mathbb{Z}}(k, \pi_j A)$. Then*

*(1) $H_0^{\mathrm{ss}}(A; k) = k[\pi_0 A]$;*
*(2) $H_i^{\mathrm{ss}}(A; k) = 0$ for $0 < i \leq e$;*
*(3) $H_{e+1}^{\mathrm{ss}}(A; k)$ contains the following $k$-module as a direct summand:*

$$k[\pi_0 A] \otimes_k \Big(k \otimes_{\mathbb{Z}} \pi_{e+1} A \,\oplus\, \mathrm{Tor}^{\mathbb{Z}}(k, \pi_e A)\Big) \ .$$

*Proof.* Lemma 3.2 and the Künneth theorem reduce the proof to the case of an Eilenberg-MacLane space $A$. Assume that the nonzero homotopy group of $A$ is placed in degree $i$. If $i = 0$, the result holds by a direct computation. If $0 < i < e$ then the result follows from lemma 3.4. If $i \geq e$, the result follows from the classical absolute Hurewicz theorem of proposition 3.3 together with the universal coefficient theorem which says that the graded $k$-module $H_*^{\mathrm{ss}}(A;k)$ is (non canonically) isomorphic to $k\otimes_{\mathbb{Z}} H_*^{\mathrm{ss}}(A;\mathbb{Z})\oplus \mathrm{Tor}^{\mathbb{Z}}(k,H_{*-1}^{\mathrm{ss}}(A;\mathbb{Z}))$. □

**Corollary 3.6.** *The $k$-modules $H_i^{\mathrm{ss}}(A;k)$ vanish for $0<i\le e$ if and only if $k\otimes_{\mathbb{Z}}\pi_i A$ and $\mathrm{Tor}_1^{\mathbb{Z}}(k,\pi_j A)$ vanish for $0<i\le e$ and $0<j<e$.*

3.3. **Functors of simplicial objects.** Assume that $\mathcal{M}$ is an abelian category. Evaluating a functor $F:\mathcal{M}\to s(k\text{-}\mathbf{Mod})$ on a simplicial object $M$ in $\mathcal{M}$ yields a bisimplicial object $F(M_p)_q$, and we denote by $F(M)$ the associated diagonal simplicial $k$-module. This construction is natural with respect to $F$ and $M$.

We shall say that a natural transformation $f:F\to F'$ is *$e$-connected* if for all $M$ in $\mathcal{M}$, the morphism of simplicial $k$-modules $F(M)\to F'(M)$ is $e$-connected.

*Remark* 3.7. If $\mathcal{M}$ is small, the category $k[\mathcal{M}]$-$\mathbf{Mod}$ is well-defined. In that case the functors $F:\mathcal{M}\to s(k\text{-}\mathbf{Mod})$ (resp. the natural transformations between such functors) identify with the simplicial objects in $k[\mathcal{M}]$-$\mathbf{Mod}$ (resp. the morphisms between such simplicial objects), and the definition of $e$-connectedness given here coincides with the one given in section 3.1.

**Proposition 3.8.** *Let $\mathcal{M}$ be an abelian category of global dimension 0. For all $e$-connected natural transformations $f:F\to F'$ and for all $e$-connected simplicial morphisms $g:M\to M'$, the induced morphism $F(M)\to F'(M')$ is $e$-connected.*

*Proof.* We prove that $f(M):F(M)\to F'(M)$ and $F'(g):F'(M)\to F'(M')$ are $e$-connected. The $e$-connectedness of $f(M)$ follows from the spectral sequence [19, IV, section 2.2], which is natural with respect to $F$:

$$E^1_{m,n}(F)=\pi_m(F(M_n))\Rightarrow \pi_{m+n}(F(M))\ .$$

Let us prove the $e$-connectedness of $F'(g)$. We choose a small additive subcategory $\mathcal{M}'\subset\mathcal{M}$ which contains the objects $M_n$ and $M'_n$ for $n\ge 0$. Let $\pi:P\to F'$ be a simplicial projective resolution in $s(\mathcal{M}'\text{-}\mathbf{Mod})$. We have a commutative square of simplicial $k$-modules

$$\begin{array}{ccc} P(M) & \xrightarrow{P(g)} & P(M') \\ \downarrow{\scriptstyle \pi(M)} & & \downarrow{\scriptstyle \pi(M')} \\ F(M) & \xrightarrow{F(g)} & F(M') \end{array}$$

whose vertical arrows are weak equivalences by the preceding paragraph, so it suffices to prove that $P(g)$ is $e$-connected. By using the spectral sequence natural with respect to $M$:

$$F^1_{m,n}(M)=\pi_n(\overline{P}_m(M))\Rightarrow \pi_{m+n}(P(M))\ ,$$

the proof reduces further to showing that for all projective objects $Q$ in $\mathcal{M}'$-$\mathbf{Mod}$, the morphism of simplicial $k$-modules $Q(g)$ is $e$-connected. Since $Q$ is a direct summand of a direct sum of standard projectives, we can assume that $Q=k[\mathcal{M}(x,-)]$. Since $\mathcal{M}$ has global dimension zero, the functor $\mathcal{M}(x,-)$ is exact, in particular $\mathcal{M}(x,g)$ is $e$-connected. Hence, the result follows from the relative Hurewicz theorem of proposition 3.3. □

## 4. Homology of bifunctors of AP-type

In this section we prove theorem 1.1 from the introduction. So we fix a commutative ring $k$ and a small additive category $\mathcal{A}$.

We first come back to the definition of bifunctors of AP-type. Recall from [31, p.18] that an *ideal* of $\mathcal{A}$ is a subfunctor of $\mathcal{A}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Ab}$. Given such an ideal $\mathcal{I}$, we can form the additive quotient $\mathcal{A}/\mathcal{I}$ of $\mathcal{A}$, with the same objects as $\mathcal{A}$ and with morphisms $(\mathcal{A}/\mathcal{I})(x,y) = \mathcal{A}(x,y)/\mathcal{I}(x,y)$. We let $\pi_{\mathcal{I}} : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ denote the additive quotient functor. The following definitions are introduced in [11].

**Definition 4.1.** An additive category $\mathcal{B}$ is *$k$-trivial* if for all objects $x$ and $y$ the abelian group $\mathcal{B}(x,y)$ is finite and such that $k \otimes_{\mathbb{Z}} \mathcal{B}(x,y) = 0$. An ideal $\mathcal{I}$ of $\mathcal{A}$ is *$k$-cotrivial* if $\mathcal{A}/\mathcal{I}$ is $k$-trivial. A functor $F : \mathcal{A} \to k\text{-}\mathbf{Mod}$ is *antipolynomial* if there is a $k$-cotrivial ideal $\mathcal{I}$ of $\mathcal{A}$ such that $F$ factors though $\pi_{\mathcal{I}} : \mathcal{A} \to \mathcal{A}/\mathcal{I}$.

We also need the polynomial functors introduced by Eilenberg and Mac Lane in [12]. Given a positive integer $d$, an object $F$ of $k[\mathcal{A}]\text{-}\mathbf{Mod}$ is *polynomial of degree less than $d$* if its $d$-th cross-effect $\mathrm{cr}_d F$ vanishes. This $d$-th cross-effect is an object of $k[\mathcal{A}^{\times d}]\text{-}\mathbf{Mod}$, defined as the image of a certain idempotent endomorphism of the functor $F(x_1 \oplus \cdots \oplus x_d)$. We refer the reader to [11, Section 1] for a detailed description of polynomial functors and further references. We will use the following classical properties. (The Tor vanishing can be obtained from the Ext-vanishing and the isomorphism of lemma 2.7.)

**Proposition 4.2.** *The full subcategory $k[\mathcal{A}]\text{-}\mathbf{Mod}_{<d}$ of $k[\mathcal{A}]\text{-}\mathbf{Mod}$ on the polynomial functors of degree less than $d$ is stable under subobjects, extensions, arbitrary direct sums and products. Moreover, if $F$ and $F'$ are polynomial of degree less than $d$, then for all objects $F_i$ of $k[\mathcal{A}]\text{-}\mathbf{Mod}$ and $F'_i$ of $\mathbf{Mod}\text{-}k[\mathcal{A}]$ satisfying $F_i(0) = 0 = F'_i(0)$, we have*

$$\mathrm{Ext}^*_{k[\mathcal{A}]}(F_1 \otimes \cdots \otimes F_d, F) = 0 = \mathrm{Ext}^*_{k[\mathcal{A}]}(F, F_1 \otimes \cdots \otimes F_d)\ , \tag{11}$$

$$\mathrm{Tor}^{k[\mathcal{A}]}_*(F'_1 \otimes \cdots \otimes F'_d, F) = 0 = \mathrm{Tor}^{k[\mathcal{A}]}_*(F', F_1 \otimes \cdots \otimes F_d)\ . \tag{12}$$

**Definition 4.3.** A bifunctor $B : \mathcal{A} \times \mathcal{A} \to k\text{-}\mathbf{Mod}$ is of *antipolynomial-polynomial type* (AP-type) if for all objects $x$ of $\mathcal{A}$ the functor $y \mapsto B(y,x)$ is antipolynomial and the functor $y \mapsto B(x,y)$ is polynomial.

The remainder of the section is devoted to the proof of theorem 1.1, which we restate here for the convenience of the reader.

**Theorem 4.4.** *Let $B$, $C$, $B'$ be three bifunctors of AP-type, with $B'$ contravariant in both variables. Restriction along the diagonal $\Delta : \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ yields isomorphisms:*

$$\mathrm{Ext}^*_{k[\mathcal{A}\times\mathcal{A}]}(B, C) \simeq \mathrm{Ext}^*_{k[\mathcal{A}]}(\Delta^* B, \Delta^* C)\ ,$$
$$\mathrm{Tor}^{k[\mathcal{A}]}_*(\Delta^* B', \Delta^* C) \simeq \mathrm{Tor}^{k[\mathcal{A}\times\mathcal{A}]}_*(B', C)\ .$$

(Notice that the Hom-isomorphism is included in [11, Prop 4.9], which is proved by another method.)

If $B' : \mathcal{A}^{\mathrm{op}} \times \mathcal{A}^{\mathrm{op}} \to k\text{-}\mathbf{Mod}$ is of AP-type, then for all injective $k$-modules $M$, the bifunctor $\mathrm{Hom}_k(B', M) = \mathrm{Hom}_k(-, M) \circ B'$ is also of AP-type. Therefore, by proposition 2.12, it suffices to prove the Ext-isomorphism of theorem 4.4. We shall prove this Ext-isomorphism in two steps. We first reduce the proof to bifunctors $B$

'of special-AP-type', that is, bifunctors of the form $B(x,y) = A(x) \otimes P(y)$ for some particular antipolynomial functor $A$. Then we establish the isomorphism for these bifunctors of special AP-type.

*Step 1: Reduction to bifunctors of special-AP-type.* Given an ideal $\mathcal{I}$ of $\mathcal{A}$ and a positive integer $d$, we denote by $\mathcal{C}_{\mathcal{I},d}$ the full subcategory of $k[\mathcal{A}\times\mathcal{A}]$-**Mod** whose objects are the bifunctors $B$ such that:

i) $B$ factors through $\pi_{\mathcal{I}} \times \mathrm{id} : \mathcal{A}\times\mathcal{A} \to (\mathcal{A}/\mathcal{I})\times\mathcal{A}$, and
ii) for all $x$, the functor $y \mapsto B(x,y)$ is polynomial of degree less than $d$.

The subcategory $\mathcal{C}_{\mathcal{I},d}$ of $k[\mathcal{A}\times\mathcal{A}]$-**Mod** is stable under limits and colimits. Stability under colimits ensures that any object $B$ of $k[\mathcal{A}\times\mathcal{A}]$-**Mod** has a largest subobject $B_{\mathcal{I},d}$ belonging to $\mathcal{C}_{\mathcal{I},d}$.

**Lemma 4.5.** *If $B$ is a bifunctor of AP-type then $B = \bigcup_{\mathcal{I},d} B_{\mathcal{I},d}$, where $\mathcal{I}$ runs over the set of $k$-cotrivial ideals of $\mathcal{A}$ and $d$ runs over the set of positive integers.*

*Proof.* We fix two objects $x, y$ of $\mathcal{A}$. Let $d$ be the degree of $t \mapsto B(x,t)$ and let $\mathcal{I}$ be a $k$-cotrivial ideal such that $s \mapsto B(s,y)$ factors through $\mathcal{A}/\mathcal{I}$. To prove the lemma, it suffices to show that the inclusion $B_{\mathcal{I},d} \hookrightarrow B$ induces an equality $B_{\mathcal{I},d}(x,y) = B(x,y)$.

Let $B_d(a,-)$ be the largest subfunctor of $B(a,-)$ of degree less than $d$. Any map $f : a \to b$ induces a map $B_d(a,-) \to B_d(b,-)$, so that the functors $B_d(a,-)$ assemble into a bifunctor $B_d : \mathcal{A}\times\mathcal{A} \to k$-**Mod** which is a subfunctor of $B$, polynomial of degree less of equal to $d$ with respect to its first variable. By construction $B_d(x,y) = B(x,y)$. Similarly, let $B_{\mathcal{I}}(-,b)$ be the largest subfunctor of $B(-,b)$ factorizing through $\mathcal{A}/\mathcal{I}$. These functors assemble into a bifunctor $B_{\mathcal{I}} : \mathcal{A}\times\mathcal{A} \to k$-**Mod** factorizing through $\mathcal{A}/\mathcal{I}\times\mathcal{A}$. By construction $B_{\mathcal{I}}(x,y) = B(x,y)$. Since $B_{\mathcal{I}} \cap B_d \subset B_{\mathcal{I},d}$ we finally obtain that $B_{\mathcal{I},d}(x,y) = B(x,y)$. □

An object $B$ of $k[\mathcal{A}\times\mathcal{A}]$-**Mod** is *of special-AP-type* if there is an object $z$ of $\mathcal{A}$, a $k$-cotrivial ideal $\mathcal{I}$ and a polynomial functor $F$ in $k[\mathcal{A}]$-**Mod** such that

$$B(x,y) = k[\mathcal{A}/\mathcal{I}(z,x)] \otimes F(y)\ .$$

**Lemma 4.6.** *Let $B$ be an object of $k[\mathcal{A}\times\mathcal{A}]$-**Mod** such that $B = \bigcup_{\mathcal{I},d} B_{\mathcal{I},d}$, where $\mathcal{I}$ runs over the the set of $k$-cotrivial ideals of $\mathcal{A}$ and $d$ over the set of positive integers. Then $B$ has a resolution $Q$ whose terms are direct sums of bifunctors of special-AP-type.*

*Proof.* It suffices to prove that every $B_{\mathcal{I},d}$, is a quotient of a direct sum of bifunctors of special-AP-type. By definition $B_{\mathcal{I},d} = (\pi_{\mathcal{I}} \times \mathrm{id}_{\mathcal{A}})^* B'$ for some bifunctor $B' : \mathcal{A}/\mathcal{I} \times \mathcal{A} \to k$-**Mod** such that each $B'_z(-) := B'(z,-)$ is polynomial of degree less than $d$. The standard resolution of $B'$ [31, section 17] yields an epimorphism $\bigoplus_z h^z_{k[\mathcal{A}/\mathcal{I}]} \boxtimes B'_z \to B'$, where the sum is indexed by a set of representatives $z$ of isomorphism classes of objects of $\mathcal{A}/\mathcal{I}$. The result follows by restricting this epimorphism along $\pi_{\mathcal{I}} \times \mathrm{id}_{\mathcal{A}}$. □

**Proposition 4.7.** *Let $C$ be an arbitrary object of $k[\mathcal{A}\times\mathcal{A}]$-**Mod**. If the map:*

$$\Delta^* : \mathrm{Ext}^*_{k[\mathcal{A}\times\mathcal{A}]}(B,C) \to \mathrm{Ext}^*_{k[\mathcal{A}]}(\Delta^*B, \Delta^*C)$$

*is an isomorphism for all bifunctors $B$ of special-AP-type then it is an isomorphism for all bifunctors $B$ of AP-type.*

*Proof.* By lemmas 4.5 and 4.6, $B$ has a resolution $Q$ by direct sums of bifunctors of special AP-type. We have two spectral sequences:

$$E_1^{p,q} = \mathrm{Ext}^q_{k[\mathcal{A}\times\mathcal{A}]}(Q_p, C) \Rightarrow \mathrm{Ext}^{p+q}_{k[\mathcal{A}\times\mathcal{A}]}(B, C)\ ,$$
$$'E_1^{p,q} = \mathrm{Ext}^q_{k[\mathcal{A}]}(\Delta^* Q_p, \Delta^* C)) \Rightarrow \mathrm{Ext}^{p+q}_{k[\mathcal{A}]}(\Delta^* B, \Delta^* C)\ ,$$

and $\Delta^*$ induces a morphism of spectral sequences. So it suffices to prove that $\Delta^*$ is an isomorphism when $B = Q_p$, hence when $B$ is a functor of special-AP-type. □

*Step 2: Proof for bifunctors of special-AP-type.* The proof relies on three vanishing lemmas. The first two lemmas are quite general, and we will also use them later in the article, in the proof of proposition 5.10 and theorem 6.9.

**Lemma 4.8.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two small categories and let $B$ and $C$ be two objects of $k[\mathcal{C}\times\mathcal{D}]$-**Mod**. There is a first quadrant spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}^p_{k[\mathcal{D}^{\mathrm{op}}\times\mathcal{D}]}(k[\mathcal{D}]; E^q_{\mathcal{C}}) \Rightarrow \mathrm{Ext}^{p+q}_{k[\mathcal{C}\times\mathcal{D}]}(B, C)$$

*where $k[\mathcal{D}]$ and $E^q_{\mathcal{C}}$ are the objects of $k[\mathcal{D}^{\mathrm{op}}\times\mathcal{D}]$-**Mod** respectively defined by*

$$k[\mathcal{D}](x,y) = k[\mathcal{D}(x,y)]\ , \qquad E^q_{\mathcal{C}}(x,y) = \mathrm{Ext}^q_{k[\mathcal{C}]}(B(-,x), C(-,y))\ .$$

*In particular, if $E^*_{\mathcal{C}} = 0$ then $\mathrm{Ext}^*_{k[\mathcal{C}\times\mathcal{D}]}(B, C) = 0$.*

*Proof.* We use the notation $\mathrm{Hom}_{k[C]}(B, C) := E^0_{\mathcal{C}}$. There is an isomorphism, natural with respect to $B$, $C$, and the object $D$ of $k[\mathcal{D}^{\mathrm{op}}\times\mathcal{D}]$-**Mod**:

$$\mathrm{Hom}_{k[\mathcal{D}^{\mathrm{op}}\times\mathcal{D}]}(D, \mathrm{Hom}_{k[\mathcal{C}]}(B, C)) \simeq \mathrm{Hom}_{k[\mathcal{C}\times\mathcal{D}]}(B \otimes_{k[\mathcal{D}^{\mathrm{op}}]} D, C)\ .$$

This isomorphism is the functor analogue of [4, IX.2 Prop 2.2], and we may construct it as follows. Firstly, there is a natural isomorphism when $D$ is a standard projective. Indeed $D(x,y) = k[\mathcal{D}(x,c)] \otimes k[\mathcal{D}(d,y)]$ and the the two sides are naturally isomorphic to $E^0_{\mathcal{C}}(c,d)$ by the Yoneda lemma. Now the isomorphism extends to every functor $D$ by taking a projective presentation of $D$.

Thus we have two spectral sequences converging to the same abutment (the construction of these spectral sequences is exactly the same as the one given for categories of modules in [4, XVI.4]):

$$\mathrm{I}^{p,q} = \mathrm{Ext}^p_{k[\mathcal{D}^{\mathrm{op}}\times\mathcal{D}]}(D, E^q_{\mathcal{C}}) \Rightarrow H^{p+q}$$
$$\mathrm{II}^{p,q} = \mathrm{Ext}^p_{k[\mathcal{C}\times\mathcal{D}]}(\mathrm{Tor}_q^{k[\mathcal{D}^{\mathrm{op}}]}(B, D), C) \Rightarrow H^{p+q}\ .$$

If $D = k[\mathcal{D}]$ then for all $x$, $k[\mathcal{D}](-,x)$ is a projective object of $k[\mathcal{D}^{\mathrm{op}}]$-**Mod**, hence the functor $\mathrm{Tor}_q^{k[\mathcal{D}^{\mathrm{op}}]}(B, k[\mathcal{D}])$ is zero for positive $q$, and the Yoneda isomorphism (4) shows that for $q = 0$ this functor is isomorphic to $B$. Thus the second spectral sequence collapses at the second page and $H^* = \mathrm{Ext}^*_{k[\mathcal{C}\times\mathcal{D}]}(B, C)$. Hence the first spectral sequence gives the result. □

The next two lemmas use the notion of a reduced functor. A functor of $k[\mathcal{A}]$-**Mod** is *reduced* if it satisfies $F(0) = 0$. In general, we denote by $F^{\mathrm{red}}$ the *reduced part* of a functor $F$ in $k[\mathcal{A}]$-**Mod**. Thus $F^{\mathrm{red}}$ is a reduced functor such that $F \simeq F(0) \oplus F^{\mathrm{red}}$ in $k[\mathcal{A}]$-**Mod**. This decomposition is natural with respect to $F$, in particular we have a decomposition of Ext (since the full subcategory of constant functors is equivalent to $k$-**Mod**, the Ext between $F(0)$ and $G(0)$ can be indifferently computed in $k[\mathcal{A}]$-**Mod** or $k$-**Mod**):

$$\mathrm{Ext}^*_{k[\mathcal{A}]}(F, G) \simeq \mathrm{Ext}^*_k(F(0), G(0)) \oplus \mathrm{Ext}^*_{k[\mathcal{A}]}(F^{\mathrm{red}}, G^{\mathrm{red}})$$

and a similar decomposition for Tor.

**Lemma 4.9.** *Let $G : \mathcal{A}^{\mathrm{op}} \to \mathbb{Z}$-**Mod** be such that $\mathrm{Tor}_1^{\mathbb{Z}}(k, G(x)) = 0$ and $k \otimes_{\mathbb{Z}} G(x) = 0$ for all objects $x$ of $\mathcal{A}$, and let $k[G]$ denote its composition with the $k$-linearization functor $k[-]$. For all functors $H$ and for all reduced polynomial functors $F$ and $F'$ we have:*

$$\mathrm{Tor}_*^{k[\mathcal{A}]}(k[G]^{\mathrm{red}} \otimes H, F) = 0 = \mathrm{Ext}^*_{k[\mathcal{A}^{\mathrm{op}}]}(k[G]^{\mathrm{red}} \otimes H, F') \;.$$

*Proof.* We prove the Tor-vanishing, the proof of the Ext-vanishing is similar. The functor $k[G]$ is isomorphic to $k[G(0)] \otimes k[G^{\mathrm{red}}]$, hence to a direct sum of copies of $k[G^{\mathrm{red}}]$. Therefore, to prove the vanishing, we may assume that $G$ is reduced.

Let $K$ be the kernel of the augmentation $\epsilon : k[G] \to k$, such that $\epsilon(\sum \lambda_f [f]) = \sum \lambda_f$. Then $k[G]^{\mathrm{red}} = K$. By lemma 3.4, the hypotheses on $G(x)$ imply that this abelian group has trivial homology with coefficients in $k$. Thus the reduced normalized bar construction of $k[G(x)]$ yields an exact sequence

$$\cdots \to K^{\otimes i+1} \to K^{\otimes i} \to \cdots \to K \to 0$$

in **Mod**-$k[\mathcal{A}]$. Since $K$ has $k$-free values, this complex becomes a *split* exact complex of $k$-modules after evaluation on every object $x$ or $\mathcal{A}$. Therefore, if we tensor this sequence by $K^{\otimes r-1} \otimes H$ for a positive integer $r$, we obtain an exact complex with associated hypercohomology spectral sequence:

$$E^1_{s,t}(r) = \mathrm{Tor}_t(K^{\otimes s+r+1} \otimes H, F) \Rightarrow \mathrm{Tor}_t^{k[\mathcal{A}]}(K^{\otimes r} \otimes H, F) \;.$$

Now we can prove that $\mathrm{Tor}_*^{k[\mathcal{A}]}(K^{\otimes r} \otimes H, F) = 0$ for all positive integers $r$. Since $K^{\otimes r} \otimes H$ is the direct sum of $K^{\otimes r} \otimes H^{\mathrm{red}}$ and $K^{\otimes r} \otimes H(0)$, this is true if $r > d$ by proposition 3.4. Now if this is true for a given $r$, then $E^1_{*,*}(r-1) = 0$, hence $\mathrm{Tor}_*^{k[\mathcal{A}]}(K^{\otimes r-1} \otimes H, F) = 0$. The result follows. □

**Lemma 4.10.** *Let $A$, $P$, and $F$ be three objects of $k[\mathcal{A}]$-**Mod**. Assume that $A$ is antipolynomial and that $P$ is polynomial. Then $\mathrm{Ext}^*_{k[\mathcal{A}]}(A \otimes F, P) = 0$ if $A$ is reduced, and $\mathrm{Ext}^*_{k[\mathcal{A}]}(P \otimes F, A) = 0$ if $P$ is reduced.*

*Proof.* We prove the first Ext-vanishing, the proof of second one is similar. By sum-diagonal adjunction $\mathrm{Ext}^*_{k[\mathcal{A}]}(A \otimes F, P)$ is isomorphic to $\mathrm{Ext}^*_{k[\mathcal{A}\times\mathcal{A}]}(A \boxtimes F, P)$, where $B(x,y) = P(x \oplus y)$ (see example 2.19). For all $x$, the functor $x \mapsto P(x \oplus y)$ is polynomial, hence by lemma 4.8, it suffices to prove that $\mathrm{Ext}^*_{k[\mathcal{A}]}(A, P) = 0$ for all reduced antipolynomial functors $A$ and for all polynomial functors $P$. Since $A$ is reduced, we may as well assume that $P$ is reduced. Since $A = \pi_{\mathcal{I}}^* A'$ for some $k$-cotrivial ideal $\mathcal{I}$ and some functor $A'$ in $k[\mathcal{A}/\mathcal{I}]$-**Mod**, the first base change spectral sequence of proposition 2.20 shows that is suffices to prove that $\mathrm{Ext}^*_{k[\mathcal{A}]}(k[\mathcal{A}/\mathcal{I}(-,x)], P) = 0$ for all $x$ in $\mathcal{A}$. The latter follows from lemma 4.9 (with $G = \mathcal{A}/\mathcal{I}(-,x)$, $H = k$ and $F = P$). □

The next proposition finishes the proof of theorem 4.4.

**Proposition 4.11.** *Let $B$ be a bifunctor of special-AP-type, and let $C$ be a bifunctor of AP-type. Restriction along the diagonal yields an isomorphism:*

$$\Delta^* : \mathrm{Ext}^*_{k[\mathcal{A}\times\mathcal{A}]}(B, C) \xrightarrow{\simeq} \mathrm{Ext}^*_{k[\mathcal{A}]}(\Delta^* B, \Delta^* C) \;.$$

*Proof.* We have $B = A \boxtimes F$ and $\Delta^* B = A \otimes F$, with $F$ polynomial of degree less than $d$, and $A = k[\mathcal{A}/\mathcal{I}(z, -)]$ for a $k$-cotrivial ideal $\mathcal{I}$ and an object $z$ of $\mathcal{A}$.

We have $A \simeq k \oplus A^{\mathrm{red}}$. We also have an isomorphism $F(x \oplus y) \simeq F(x) \oplus G(x, y)$ for some bifunctor $G$. Since $A(x \oplus y) \simeq A(x) \otimes A(y)$, we have a decomposition

$$(A \otimes F)(x \oplus y) \simeq \big(A(x) \otimes F(y)\big) \;\oplus\; X(x, y) \;\oplus Y(x, y)$$

where the bifunctors $X$ and $Y$ are defined by

$$X(x, y) := A(y) \otimes G(x, y)\;, \qquad Y(x, y) := A(x) \otimes A^{\mathrm{red}}(y) \otimes F(x \oplus y)\;.$$

Moreover, let $q : A \boxtimes F \oplus X \oplus Y \twoheadrightarrow A \boxtimes F$ denote the canonical projection. Then we readily check from the expression of the adjunction isomorphism $\alpha$ given in example 2.19 that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Ext}^*_{k[\mathcal{A}\times\mathcal{A}]}(A \boxtimes F, C) & \xrightarrow{\;\Delta^*\;} & \mathrm{Ext}^*_{k[\mathcal{A}]}(A \otimes F, \Delta^* C) \\ {\scriptstyle \mathrm{Ext}^*_{k[\mathcal{A}\times\mathcal{A}]}(q, C)}\big\downarrow & \swarrow{\scriptstyle \alpha}\,{\scriptstyle\simeq} & \\ \mathrm{Ext}^*_{k[\mathcal{A}\times\mathcal{A}]}(A \boxtimes F \oplus X \oplus Y, C) & & \end{array} \;.$$

Therefore, it suffices to prove that there is no nonzero Ext between $X \oplus Y$ and $C$.

But for all objects $y$ and $y'$, there is no nonzero Ext between $X(x, -)$ and $C(x, -)$ by lemma 4.10 since $C(-, y')$ is antipolynomial and $X(-, y)$ is polynomial and such that $X(0, y) = 0$. Hence there is no nonzero Ext between $X$ and $C$ by lemma 4.8. Similarly, there is no nonzero Ext between $Y$ and $C$, whence the result. □

## 5. Excision in functor homology

In this section, we prove theorem 1.3 from the introduction. In fact, we prove theorem 1.3 in corollary 5.6, by deriving it from the following more general excision theorem, which is a variant of the result sketched in [7, Proposition 2.19.4]. This excision theorem is a functor homology analogue of the excision theorem of Suslin and Wodzicki in $K$-theory [45], see remark 5.9.

**Theorem 5.1** (excision)**.** *Let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive functor between two small additive categories, such that $\phi^* : k[\mathcal{B}]\text{-}\mathbf{Mod} \to k[\mathcal{A}]\text{-}\mathbf{Mod}$ is fully faithful. For all positive integers $e$, the following assertions are equivalent.*

*(1) The restriction functor $\phi^*$ is $e$-excisive.*

*(2) For all objects $x$ and $y$ of $\mathcal{B}$ we have:*

$$\bigoplus_{0 \le i < e} k \otimes_{\mathbb{Z}} \pi_i A(x, y) = 0 \text{ and } \bigoplus_{0 \le i < e-1} \mathrm{Tor}^{\mathbb{Z}}(k, \pi_i A(x, y)) = 0\;,$$

*where $A(x, y)$ is the simplicial abelian group $A(x, y) := Q^x \otimes_{\mathcal{A}} \phi^* h^y_{\mathcal{B}}$, where $Q^x$ is a projective simplicial resolution of the functor $\phi^* h^x_{\mathcal{B}^{\mathrm{op}}}$ in $\mathbf{Mod}\text{-}\mathcal{A}$.*

*Proof.* By lemma 5.2 below, the simplicial $k$-module $k[A(x, y)]$ is isomorphic to $k[Q^x] \otimes_{k[\mathcal{A}]} k[\phi^* h^y_{\mathcal{B}}]$. But we have:

$$k[\phi^* h^y_{\mathcal{B}}] = k[\mathcal{B}(y, \phi(-))] = \phi^* h^y_{k[\mathcal{B}]}\;, \qquad k[\phi^* h^x_{\mathcal{B}^{\mathrm{op}}}] = k[\mathcal{B}(\phi(-), x)] = \phi^* h^x_{k[\mathcal{B}^{\mathrm{op}}]}\;,$$

and $k[Q^x]$ is a simplicial projective resolution of $\phi^* h^x_{k[\mathcal{B}^{\mathrm{op}}]}$ by the relative Hurewicz theorem of proposition 3.3 (with $e = \infty$). Whence an isomorphism:

$$\pi_* k[A(x, y)] \simeq \mathrm{Tor}^{k[\mathcal{A}]}_*(\phi^* h^x_{k[\mathcal{B}^{\mathrm{op}}]}, \phi^* h^y_{k[\mathcal{B}]})\;. \tag{13}$$

The second assertion of the theorem is equivalent to the vanishing of $\pi_i k[\mathcal{A}(x,y)]$ for $0 < i < e$ by the $k$-local Hurewicz theorem of corollary 3.6. Thus, the result follows from the isomorphism (13) and proposition 2.15. □

The following result is well-known to experts, but we do not know any written reference for it.

**Lemma 5.2.** *Let $A : \mathcal{A}^{\mathrm{op}} \to \mathbb{Z}$-**Mod** and $B : \mathcal{A} \to \mathbb{Z}$-**Mod** be two additive functors, and let $k[A]$ and $k[B]$ denote the composition of these functors with the $k$-linearization functor $k[-]$. There is an isomorphism of $k$-modules, natural with respect to $A$ and $B$:*

$$k[A] \otimes_{k[\mathcal{A}]} k[B] \simeq k[A \otimes_{\mathcal{A}} B] \; .$$

*Proof.* For all objects $x$ of $\mathcal{A}$, we let $\theta_{A,B,x} : k[A(x)] \otimes k[B(x)] \to k[A \otimes_{\mathcal{A}} B]$ be the $k$-linear map such that $\theta_{A,B,x}(s \otimes t) = [\![ s \otimes t ]\!]$ for all $s$ in $A(x)$ and all $t \in B(x)$. (The brackets refer to the class of an element of $A(x) \otimes B(x)$ in the quotient $A \otimes_{\mathcal{A}} B$.) These maps $\theta_{A,B,x}$ induce a $k$-linear map, natural in $A$ and $B$:

$$\theta_{A,B} : k[A] \otimes_{k[\mathcal{A}]} k[B] \to k[A \otimes_{\mathcal{A}} B] \; .$$

Assume that $B = \mathcal{A}(x,-)$, hence $k[B]$ is a standard projective in $k[\mathcal{A}]$-**Mod**. One checks on the explicit formulas that the composition of $\theta_{A,B}$ with the $k$-linearization of the Yoneda isomorphism $A \otimes_{{}_k\mathcal{A}} \mathcal{A}(x,-) \simeq A(x)$ is equal to the Yoneda isomorphism $k[A] \otimes_{k[\mathcal{A}]} k[\mathcal{A}(x,-)] \simeq k[A(x)]$. Thus $\theta_{A,\mathcal{A}(x,-)}$ is an isomorphism. Since every finitely generated projective object of $\mathcal{A}$-**Mod** is a direct summand of a standard projective, it follows that $\theta_{A,B}$ is an isomorphism for all finitely projective functor $B$ in $\mathcal{A}$-**Mod**.

Every projective functor $\mathcal{A}$-**Mod** is the filtered colimit of its finitely generated projective subfunctors. Since both the source and the target of $\theta_{A,B}$, viewed as functors of $B$, preserve filtered colimits of monomorphisms, we obtain that $\theta_{A,B}$ is an isomorphism for all projectives $B$.

Now let $B$ be arbitrary and let $P \to B$ be a projective simplicial resolution of $B$ in $\mathcal{A}$-**Mod**. Then we have a commutative square of simplicial $k$-modules in which the top row is an isomorphism, the bottom row features constant simplicial $k$-modules and the vertical arrows are induced by the simplicial maps $P \to B$:

$$\begin{array}{ccc} k[A] \otimes_{k[\mathcal{A}]} k[P] & \xrightarrow[\simeq]{\Theta_{A,P}} & k[A \otimes_{\mathcal{A}} P] \\ \downarrow & & \downarrow \\ k[A] \otimes_{k[\mathcal{A}]} k[B] & \xrightarrow{\Theta_{A,B}} & k[A \otimes_{\mathcal{A}} B] \end{array} \; .$$

By right exactness of tensor products and by the relative Hurewicz theorem of proposition 3.3, the vertical morphisms are isomorphisms in $\pi_0$. Hence $\Theta_{A,B}$ is an isomorphism. □

*Some special cases of the excision theorem.* We now investigate concrete conditions which imply the second assertion of theorem 5.1.

**Theorem 5.3.** *Let $\mathcal{A}$ and $\mathcal{B}$ be small additive categories, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a full and essentially surjective additive functor. Assume that for all $x$, $\mathcal{B}(x,x) \otimes_{\mathbb{Z}} k = 0$. Then the restriction functor $\phi^* : k[\mathcal{B}]$-**Mod** $\to k[\mathcal{A}]$-**Mod** is $\infty$-excisive.*

*Proof.* Let $\mathcal{C}$ denote the following full subcategory of abelian groups:

- if $\operatorname{char} k \neq 0$, the objects of $\mathcal{C}$ are the groups on which multiplication by $\operatorname{char} k$ is invertible;
- if $\operatorname{char} k = 0$, the objects of $\mathcal{C}$ are the torsion groups whose elements have orders invertible in $k$.

This subcategory is stable under kernels, cokernels and direct sums. Moreover, for all $A \in \mathcal{C}$ we have $\operatorname{Tor}_1^{\mathbb{Z}}(A, k) = 0 = A \otimes_{\mathbb{Z}} k$.

Now, the hypothesis on $\mathcal{B}$ implies that $\mathcal{B}(x, y) \in \mathcal{C}$ for all $(x, y)$, thanks to lemma 5.4 below (applied to the rings $k$ and $\mathcal{B}(y, y)$, using that $\mathcal{B}(x, y)$ is a $\mathcal{B}(y, y)$-module). Thus, for all standard projectives $h^a_{\mathcal{A}^{\mathrm{op}}}$ in **Mod**-$\mathcal{A}$, the abelian group $h^a_{\mathcal{A}^{\mathrm{op}}} \otimes_{\mathcal{A}} \phi^* h^y_{\mathcal{B}} = \mathcal{B}(y, \phi(a))$ belongs to $\mathcal{C}$. Therefore, $A(x, y)$ is a simplicial group in $\mathcal{C}$. In particular, its homotopy groups belong to $\mathcal{C}$. Thus the second assertion of theorem 5.1 is satisfied for all $e$, whence the result. □

**Lemma 5.4.** *Let $R$ and $S$ be rings such that $R \otimes_{\mathbb{Z}} S = 0$. Let us denote $r := \operatorname{char} R$ and $s := \operatorname{char} S$. Then $(r, s) \neq (0, 0)$. Moreover, if $r \neq 0$, then $r$ belongs to $S^{\times}$.*

*Proof.* If a tensor product of abelian groups is zero, at least one of them is torsion, whence $(r, s) \neq (0, 0)$. If $r \neq 0$, then $\mathbb{Z}/r$ is a direct summand of the additive group of $R$, whence $\mathbb{Z}/r \otimes_{\mathbb{Z}} S = 0$, what implies $r \in S^{\times}$. □

The next two corollaries are direct consequences of theorem 5.3. Corollary 5.6 provides a proof of theorem 1.3 from the introduction.

**Corollary 5.5.** *Let $\mathcal{A}$ be a small additive category and let $n$ be an integer invertible in $k$. The restriction functor $\phi^* : k[\mathcal{A}/n]$-**Mod** $\to k[\mathcal{A}]$-**Mod** induced by the quotient functor $\phi : \mathcal{A} \to \mathcal{A}/n$ is $\infty$-excisive.*

**Corollary 5.6.** *Let $\mathcal{I}$ be a $k$-cotrivial ideal of a small additive category $\mathcal{A}$. The restriction functor $\pi^* : k[\mathcal{A}/\mathcal{I}]$-**Mod** $\to k[\mathcal{A}]$-**Mod** induced by the quotient functor $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is $\infty$-excisive.*

Under some favorable assumptions on $\mathcal{A}$ and $\mathcal{B}$, theorem 5.1 can also be reformulated in terms of categories of additive functors.

**Definition 5.7.** Let $k$ be a commutative ring. We say that an additive category $\mathcal{C}$ is *$k$-torsion-free* if $\operatorname{Tor}^{\mathbb{Z}}(k, \mathcal{C}(x, y)) = 0$ for all objects $x$, $y$ of $\mathcal{C}$.

**Theorem 5.8.** *Let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive functor between two small additive categories, such that $\phi^* : k[\mathcal{B}]$-**Mod** $\to k[\mathcal{A}]$-**Mod** is fully faithful. Assume that that $\mathcal{A}$ and $\mathcal{B}$ are both $k$-torsion free. Then the following assertions are equivalent.*

*(1) The functor $\phi^* : k[\mathcal{B}]$-**Mod** $\to k[\mathcal{A}]$-**Mod** is $e$-excisive.*
*(2) The functor $\phi^* : {}_k\mathcal{B}$-**Mod** $\to {}_k\mathcal{A}$-**Mod** is $e$-excisive.*

*Proof.* We first claim that for all objects $x, y$ and all integers $i$ we have a short exact sequence, where $A(x, y)$ is the simplicial group defined in theorem 5.1:

$$0 \to k \otimes_{\mathbb{Z}} \pi_i A(x, y) \to \pi_i(k \otimes_k A(x, y)) \to \operatorname{Tor}^{\mathbb{Z}}(k, \pi_{i-1} A(x, y)) \to 0\ .$$

If $\mathcal{B}$ is $k$-torsion-free, then for all objects $a$ of $\mathcal{A}$, $\operatorname{Tor}_1^{\mathbb{Z}}(k, -)$ vanishes on the abelian group $h^a_{\mathcal{A}^{\mathrm{op}}} \otimes_{\mathcal{A}} \phi^* h^y_{\mathcal{B}^{\mathrm{op}}} \simeq \mathcal{B}(y, \phi(a))$, hence on the abelian groups $A(x, y)_q$ for all $q \geq 0$, and the short exact sequence is given by the universal coefficient theorem [29, XII Thm 12.1]. Thus, the second assertion of theorem 5.1 is equivalent to the vanishing of the homotopy groups of $k \otimes_{\mathbb{Z}} A(x, y) \simeq (k \otimes_{\mathbb{Z}} Q^x) \otimes_{{}_k\mathcal{A}} (\phi^* h^y_{{}_k\mathcal{B}})$ in degrees $0 < i < e$.

Next, we claim that $k\otimes_{\mathbb{Z}} Q^x$ is a simplicial resolution of $k\otimes_{\mathbb{Z}}\phi^* h^x_{\mathcal{B}^{\mathrm{op}}} = k\otimes_{\mathbb{Z}} \mathcal{B}(\phi(-),x)$ in $\mathbf{Mod}\text{-}_k\mathcal{A}$. If $\mathcal{A}$ is $k$-torsion-free, then $\mathrm{Tor}_1^{\mathbb{Z}}(k,-)$ vanishes on the objects of $Q^x$, and also on $\pi_0 Q^x$ because $\mathcal{B}$ is $k$-torsion-free. Thus the claim follows from the universal coefficient theorem [29, XII Thm 12.1]. As a consequence, proposition 2.15 tells us that the vanishing of the homotopy groups of $k\otimes_{\mathbb{Z}} A(x,y)$ is equivalent to $\phi^* : {}_k\mathcal{B}\text{-}\mathbf{Mod}\to {}_k\mathcal{A}\text{-}\mathbf{Mod}$ being $e$-excisive. □

*Remark* 5.9. The above theorem is a functor homology analogue of Suslin-Wodzicki's excision theorem in rational algebraic $K$-theory [45] (see also [43] for the non-rational case). Indeed, the second assertion in theorem 5.8 is a natural generalization of the '$H$-unital' condition which governs excision in $K$-theory.

To be more specific, if $I$ is a two-sided ideal of a ring $R$, and if we consider $\mathcal{A}=\mathbf{P}_R$, $\mathcal{B}=\mathbf{P}_{R/I}$ and $\phi = -\otimes_R R/I$, then the second assertion of theorem 5.8 is easily seen to be equivalent to

$$(3)\quad \mathrm{Tor}_i^{R\otimes_{\mathbb{Z}}k}((R/I)\otimes_{\mathbb{Z}} k,(R/I)\otimes_{\mathbb{Z}} k)=0 \text{ for } 0<i<e.$$

(To prove the equivalence, use proposition 2.15 and the fact that ${}_k\mathcal{A}\text{-}\mathbf{Mod}$ is equivalent to $R\otimes_{\mathbb{Z}} k$ by the Eilenberg-Watts theorem.) In the situation considered in [45], that is, if $R=\mathbb{Z}\oplus I$ where $I$ is a ring without unit (seen as an ideal in the unital ring $R$ constructed by adding formally a unit to $I$) and $k=\mathbb{Q}$, the Tor appearing in assertion (3) can be computed with a bar complex, hence assertion (3) is equivalent to $R$ being $H$-unital.

*An application of excision.* The next proposition is a consequence of Kuhn's structure results [24], and corollary 5.11 is the consequence for antipolynomial homology that one immediately deduces from corollary 5.6.

**Proposition 5.10.** *If $R$ is a finite semi-simple ring and if $k$ is a field of characteristic zero, the $k$-vector spaces $\mathrm{Ext}^i_{k[\mathbf{P}_R]}(F,G)$ and $\mathrm{Tor}_i^{k[\mathbf{P}_R]}(F,G)$ vanish in positive degrees $i$ for all functors $F$, $G$.*

*Proof.* The main result of [24] says that $k[\mathbf{P}_{\mathbb{F}_q}]\text{-}\mathbf{Mod}$ is equivalent to the infinite product $\prod_{n\geq 0} k[\mathrm{GL}_n(\mathbb{F}_q)]\text{-}\mathbf{Mod}$, which implies the vanishing result when $R$ is a finite field. Assume now that $R$ is a finite simple ring. Then $R$ is isomorphic to a matrix ring $M_n(\mathbb{F}_q)$ and $\mathbf{P}_R$ is therefore equivalent to $\mathbf{P}_{\mathbb{F}_q}$ by Morita theory, which implies that the vanishing result holds for finite simple rings. Finally, assume that $R$ is a finite semi-simple ring. Then $R$ is isomorphic to a product $R_1\times\cdots\times R_n$ of simple rings, hence $\mathbf{P}_R$ is equivalent to $\mathbf{P}_{R_1}\times\cdots\times\mathbf{P}_{R_n}$. The vanishing result can then be retrieved from the vanishing result for simple rings by iterated uses of the spectral sequence of lemma 4.8. □

**Corollary 5.11.** *Let $k$ be a field of characteristic zero, and let $F,F',G$ be three functors from $\mathcal{A}$ to $k$-$\mathbf{Mod}$, with $F$ contravariant. If there is a finite semi-simple ring $R$ such that these three functors factor through $\mathbf{P}_R$, then for all $i>0$ we have:*

$$\mathrm{Ext}^i_{k[\mathcal{A}]}(F',G)=0\ ,\qquad\qquad \mathrm{Tor}_i^{k[\mathcal{A}]}(F,G)=0\ .$$

## 6. Preliminaries on polynomial homology

We use the term 'polynomial homology' as a shorthand for the computation of Tor and Ext over $k[\mathcal{A}]$ between polynomial functors. Sections 6 to 10 deal with the computation of polynomial homology. We will assume that $k$ is a field and we will focus on the polynomial functors of the form

$$T_F := \pi_1^* F_1 \otimes \cdots \otimes \pi_n^* F_n \tag{14}$$

where the $F_i$ are strict polynomial functors over $k$ and the $\pi_i$ are additive functors from $\mathcal{A}$ to $k$-vector spaces. The purpose of this short section is to make preliminary remarks on polynomial homology, which justify and explain the assumptions of our theorems in sections 8 and 10. All the material presented in this section is more or less standard, the new result is the polynomial analogue of excision given in theorem 6.9.

6.1. **The size of the field $k$.** We can most often assume that the field $k$ is as big as we want, in particular infinite and perfect. Indeed, for all field extensions $k \to K$ proposition 2.21 yields a base change isomorphism:

$$\mathrm{Tor}_*^{k[\mathcal{A}]}(F,G)\otimes K \simeq \mathrm{Tor}_*^{K[\mathcal{A}]}(F\otimes K, G\otimes K)\;. \tag{15}$$

We have similar situation for Ext, however one needs suitable finiteness assumptions (namely $F$ is $fp_\infty$, or $k \to K$ is a finite extension of fields, see proposition 2.22) to ensure that the following map is an isomorphism

$$\mathrm{Ext}^*_{k[\mathcal{A}]}(F,G)\otimes K \to \mathrm{Ext}^*_{K[\mathcal{A}]}(F\otimes K, G\otimes K)\;. \tag{16}$$

6.2. **Additive functors $\pi_i$ with infinite dimensional values.** Recall that over an infinite field, the category of $d$-homogeneous polynomial functors is a full subcategory of the category $k[\mathbf{P}_k]$-**Mod**. In other words, strict polynomial functors are functors from *finite-dimensional* vector spaces to all vector spaces, hence the meaning of $\pi_i^* F_i = F_i \circ \pi_i$ is clear only when $\pi_i(x)$ is finite-dimensional for all $x$. In general we use the following definition.

**Definition 6.1.** Let $\overline{F} : k\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$ denote the left Kan extension to all vector spaces of a functor $F : \mathbf{P}_k \to k\text{-}\mathbf{Mod}$. That is, $\overline{F}(v)$ is the colimit of the vector spaces $F(u)$ taken over the poset of finite-dimensional subspaces $u \subset v$ ordered by inclusion. For all additive functors $\pi : \mathcal{A} \to k\text{-}\mathbf{Mod}$ we define $\pi^* F$ as the composition

$$\pi^* F := \overline{F} \circ \pi\;.$$

The following lemma follows from the fact that $\overline{F}$ is defined by taking filtered colimits and that limits and colimits in functor categories are computed objectwise.

**Lemma 6.2.** *Sending a strict polynomial functor $F$ to the composition $\pi^* F$ yields an exact and colimit preserving functor*

$$\pi^* : \Gamma^d \mathbf{P}_k\text{-}\mathbf{Mod} \to k[\mathcal{A}]\text{-}\mathbf{Mod}\;.$$

*As a consequence, we have induced maps on the level of functor homology:*

$$\mathrm{Ext}^*_{\Gamma^d \mathbf{P}_k}(F,G) \to \mathrm{Ext}^*_{k[\mathcal{A}]}(\pi^* F, \pi^* G)\;,$$

$$\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^* F, \pi^* G) \to \mathrm{Tor}_*^{\Gamma^d \mathbf{P}_k}(F,G)\;.$$

*Remark* 6.3. The notation $\pi^* F$ used in definition 6.1 is compact and it extends the classical notation for composition. However, we shall be careful about the following phenomenon. If $\phi : k\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$ is an additive functor, we often denote by the same letter $\phi : \mathbf{P}_k \to k\text{-}\mathbf{Mod}$ its restriction to finite-dimensional vector spaces. Now the functor $\overline{F} \circ \phi \circ \pi$ need not be be isomorphic to $\overline{F \circ \phi} \circ \pi$ – though these two functors do coincide if $\phi$ preserves filtered colimits of monomorphisms of vector spaces or if $\pi$ has finite-dimensional values. As a consequence, $\pi^*(\phi^* F)$ might have two different meanings, depending on the fact that we consider $\phi : k\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$ or its restriction to $\mathbf{P}_k$. For this reason, we shall cautiously avoid iterating the notation of definition 6.1 and we turn back to notations with compositions whenever there is a risk of ambiguity.

6.3. **Reducing the number of factors in tensor products.** Computation of Ext and Tor between tensor products of the form (14) can always be reduced to the case where there is only one factor in the tensor products. The reduction procedure is well-known (at least to the experts) and we briefly explain it here. We consider two cases, according to the characteristic of the field $k$.

**Definition 6.4.** We say that the characteristic of the field $k$ is *large with respect to the tensor product* (14) if each $F_i$ is a $d_i$-homogeneous strict polynomial functor such that $d_i!$ is invertible in $k$.

Notice that according to definition 6.4, characteristic zero is large. The following well-known fact is a consequence of classical Schur-Weyl duality for Schur algebras, together with the fact [18, Thm 3.2] that the category of $d$-homogeneous strict polynomial functors is equivalent to the category of modules over the Schur algebra $S(n, d)$, if $n \geq d$.

**Lemma 6.5.** *Assume that $d!$ is invertible in the field $k$. Then for all $d$-homogeneous strict polynomial functor $F$ there is a $k[\mathfrak{S}_d]$-module $M$ and an isomorphism, natural with respect to the vector space $v$*

$$F(v) \simeq v^{\otimes d} \otimes_{k[\mathfrak{S}_d]} M \ .$$

Assume that the characteristic of $k$ is large with respect to the tensor product (14) and let $\mathfrak{S}_{\mathbf{d}} = \mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_n}$. Lemma 6.5 yields a $k[\mathfrak{S}_{\mathbf{d}}]$-module $N$ such that

$$T_F \simeq (\pi_1^{\otimes d_1} \otimes \cdots \otimes \pi_n^{\otimes d_n}) \otimes_{k[\mathfrak{S}_{\mathbf{d}}]} N$$

where the action of $\mathfrak{S}_{\mathbf{d}}$ on $\pi_1^{\otimes d_1} \otimes \cdots \otimes \pi_n^{\otimes d_n}$ is given by permuting the factors of the tensor product. If the characteristic of $k$ is also large with respect to another tensor product $T_G := \rho_1^* G_1 \otimes \cdots \otimes \rho_m^* G_m$ where each $G_i$ is a $e_i$-homogeneous strict polynomial functor then we have a similar isomorphism:

$$T_G \simeq (\rho_1^{\otimes e_1} \otimes \cdots \otimes \rho_m^{\otimes e_m}) \otimes_{k[\mathfrak{S}_{\mathbf{e}}]} M \ .$$

The assumption on the characteristic also ensures that $k[\mathfrak{S}_{\mathbf{d}}]$ and $k[\mathfrak{S}_{\mathbf{e}}]$ are semisimple, hence we have an isomorphism:

$$\text{(17)} \qquad \mathrm{Tor}_*^{k[\mathcal{A}]}(T_F, T_G) \simeq \mathbb{T}_* \otimes_{k[\mathfrak{S}_{\mathbf{e}}] \otimes k[\mathfrak{S}_{\mathbf{e}}]} (N \otimes M)$$

where $\mathbb{T}_*$ denotes the right $k[\mathfrak{S}_{\mathbf{d}}] \otimes k[\mathfrak{S}_{\mathbf{e}}]$-module (with action of $\mathfrak{S}_{\mathbf{d}}$ and $\mathfrak{S}_{\mathbf{e}}$ induced by permuting the factors of the tensor product in the first argument of Tor and in the second argument of Tor respectively):

$$\mathbb{T}_* := \mathrm{Tor}_*^{k[\mathcal{A}]}(\pi_1^{\otimes d_1} \otimes \cdots \otimes \pi_n^{\otimes d_n}, \rho_1^{\otimes e_1} \otimes \cdots \otimes \rho_n^{\otimes e_m}) \ .$$

Similarly $\mathrm{Ext}^*_{k[\mathcal{A}]}(T_F, T_G)$ can be computed from

$$\mathbb{E}^* := \mathrm{Ext}^*_{k[\mathcal{A}]}(\pi_1^{\otimes d_1} \otimes \cdots \otimes \pi_n^{\otimes d_n}, \rho_1^{\otimes e_1} \otimes \cdots \otimes \rho_n^{\otimes e_m}) \ .$$

Thus it remains to compute $\mathbb{T}_*$ and $\mathbb{E}^*$. This can be achieved by the standard technique using sum-diagonal adjunction isomorphisms (see example 2.19) and Künneth formulas (see section 2.5). Some special instances of this computation can be found in the literature, see e.g. [13, Thm 1.8] or [49, Prop 5.4]. The general formula is not harder to prove but it is combinatorially slightly more involved. We give the result for $\mathbb{T}_*$ (and leave its proof as an exercise to the reader).

**Proposition 6.6.** *Let $d = d_1 + \cdots + d_n$ and $e = e_1 + \cdots + e_m$. If $d \neq e$ then $\mathbb{T}_*$ is zero in all degrees. If $d = e$, let*

$$\alpha : \{1, \ldots, d\} \twoheadrightarrow \{1, \ldots, n\} \text{ and } \beta : \{1, \ldots, d\} \twoheadrightarrow \{1, \ldots, m\}$$

*be the nondecreasing surjective maps such that $\alpha^{-1}(i)$ has cardinal $d_i$ and $\beta^{-1}(i)$ has cardinal $e_i$ for all $i$. There is an isomorphism of graded vector spaces:*

$$\mathbb{T}_* \simeq \bigoplus_{\sigma \in \mathfrak{S}_d} \mathbf{T}^\sigma_* \ , \text{ with } \mathbf{T}^\sigma_* = \bigotimes_{1 \le i \le d} \mathrm{Tor}^{k[\mathcal{A}]}_*(\pi_{\alpha\sigma(i)}, \rho_{\beta(i)}) \ .$$

*The action of $(\tau, \mu) \in \mathfrak{S}_{\mathbf{d}} \times \mathfrak{S}_{\mathbf{e}}$ on the right hand side of this isomorphism can be described as follows. If $t = t_1 \otimes \cdots \otimes t_n \in \mathbf{T}^\sigma_*$ then $(\tau, \mu) \cdot t$ equals*

$$\epsilon t_{\mu(1)} \otimes \cdots \otimes t_{\mu(n)} \in \mathbb{T}^{\tau^{-1}\sigma\mu}_*$$

*where $\epsilon \in \{\pm 1\}$ is the Koszul sign such that $t_1 \cdots t_n = \epsilon t_{\mu(1)} \cdots t_{\mu(n)}$ in the free graded commutative algebra generated by $t_1, \ldots, t_n$.*

There is a similar result for Ext, provided the $\pi_i$ are of type $fp_\infty$ – this assumption is needed for the Künneth formula, cf. proposition 2.22 and remark 2.23. Isomorphism (17) and proposition 6.6 lead us to the following conclusion.

**Conclusion 6.7.** If the characteristic of the field $k$ is large with respect to the tensor products $T_F$ and $T_G$, then the computation of $\mathrm{Tor}^{k[\mathcal{A}]}_*(T_F, T_G)$ reduces to the computation of $\mathrm{Tor}^{k[\mathcal{A}]}_*(\pi, \rho)$ where $\pi$ and $\rho$ are some of the additive functors appearing in the definition of $T_F$ and $T_G$. A similar reduction to additive functors holds for the computation of Ext-groups, provided the additive functors $\pi_i$ appearing in the definition of $T_F$ are of type $fp_\infty$. (See remark 2.23 and the article [10] for more details on the $\mathrm{fp}_\infty$ condition).

If the characteristic of the field $k$ is not large with respect to $T_F$ and $T_G$, we may not be able to reduce ourselves to the computation of Ext an Tor between additive functors. But in principle, we can still reduce the computations to something simpler. Namely sum-diagonal adjunction yields an isomorphism

$$\mathrm{Tor}^{k[\mathcal{A}]}_*(T_F, T_G) \simeq \mathrm{Tor}^{k[\mathcal{A}^{\times m}]}_*(T_F^{\boxplus^m}, \rho_1^* G_1 \boxtimes \cdots \boxtimes \rho_m^* G_m) \ ,$$

where $T_F^{\boxplus^m}$ is the functor such that

$$T_F^{\boxplus^m}(x_1, \ldots, x_m) = T_F(x_1 \oplus \cdots \oplus x_m) = \bigotimes_{1 \le i \le n} F_i(\pi_i(x_1) \oplus \cdots \oplus \pi_i(x_m))$$

Each $F_i(v_1 \oplus \cdots \oplus v_m)$ has a finite filtration (e.g. its Loewy filtration) whose layers are direct sums of tensor products of the form $L_1(v_1) \otimes \cdots \otimes L_m(v_m)$, in which the $L_i$ are strict polynomial functors. Thus $T_F^{\boxplus^m}$ has a finite filtration whose layers are direct sums of functors of the form $T_{H_1} \boxtimes \cdots \boxtimes T_{H_m}$, where each $T_{H_j}$ is a tensor product of the form (14):

$$T_{H_j} = \pi_1^* H_{1,j} \otimes \cdots \otimes \pi_n^* H_{n,j} \ .$$

Therefore, the Künneth formula of proposition 2.21 reduces the computation of $\mathrm{Tor}_*^{k[\mathcal{A}]}(T_F, T_G)$ to the computation of the graded $k$-modules

$$\mathrm{Tor}_*^{k[\mathcal{A}]}(T_{H_j}, \rho_j^* G_j)$$

In other words, we have reduced the computation of $\mathrm{Tor}_*^{k[\mathcal{A}]}(T_F, T_G)$ to a similar computation, where the tensor product in the right argument of Tor is now a tensor product with only one factor. (Admittedly, this reduction may be hard to work out in practice since it involves a computation of the filtration of $T_F^{\boxplus^m}$ and the study of the associated long exact sequences in Tor.)

A similar reasoning then allows to reduce the number of factors of the tensor product in the left hand side argument of Tor, and we obtain the following conclusion.

**Conclusion 6.8.** In principle, the computation of $\mathrm{Tor}_*^{k[\mathcal{A}]}(T_F, T_G)$ can be reduced to the computation of Tor-groups of the form $\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^* H, \rho^* K)$, where $H$, $K$ are strict polynomial functors and $\pi$ and $\rho$ are some of the additive functors used in the definition of $T_F$ and $T_G$. A similar reduction holds for the computation of Ext-groups under some suitable $fp_\infty$ assumptions.

6.4. **Simplifying the source category $\mathcal{A}$.** If $k$ is a field of positive characteristic $p$, let $\mathcal{I}$ be a ideal of $\mathcal{A}$ contained in $p\mathcal{A}$. Then every additive functor $\mathcal{A} \to k$-**Mod** factors through $\mathcal{A}/\mathcal{I}$, hence every tensor product of the form (14) factors through $\mathcal{A}/\mathcal{I}$. The next theorem is a polynomial analogue of the excision theorem. Under good hypotheses on $\mathcal{I}$, it reduces the computation of polynomial homology over $\mathcal{A}$ to the computation of polynomial homology over $\mathcal{A}/\mathcal{I}$. A typical use of this theorem is when $\mathcal{I}(x, y)$ is the abelian subgroup of all the elements of $\mathcal{A}(x, y)$ whose orders are finite and invertible in $k$.

**Theorem 6.9.** *Let $k$ be an arbitrary commutative ring, and let $\mathcal{I}$ be an ideal of $\mathcal{A}$ such that $k \otimes_{\mathbb{Z}} \mathcal{I}(x, y) = 0 = \mathrm{Tor}_1^{\mathbb{Z}}(k, \mathcal{I}(x, y))$ for all $x$ and $y$ in $\mathcal{A}$. Then for all polynomial functors $F$ in $k[\mathcal{A}/\mathcal{I}]$-**Mod** and for all functors $G$ in $k[\mathcal{A}/\mathcal{I}]$-**Mod** and $G'$ in **Mod**-$k[\mathcal{A}/\mathcal{I}]$, restriction along $\pi : \mathcal{A} \twoheadrightarrow \mathcal{A}/\mathcal{I}$ yields isomorphisms*

$$\mathrm{Ext}^*_{k[\mathcal{A}/\mathcal{I}]}(G, F) \simeq \mathrm{Ext}^*_{k[\mathcal{A}]}(\pi^* G, \pi^* F) , \qquad \mathrm{Tor}_*^{k[\mathcal{A}/\mathcal{I}]}(G', F) \simeq \mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^* G', \pi^* F) .$$

*Proof.* We prove the Ext-isomorphism, the proof of the Tor-isomorphism is similar. Since $\pi^* : k[\mathcal{A}/\mathcal{I}]$-**Mod** $\to k[\mathcal{A}]$-**Mod** is full and faithful, we only have to check that for all $x$ in $\mathcal{A}$ the functor $\pi^* h^x_{\mathcal{A}/\mathcal{I}} = k[\mathcal{A}/\mathcal{I}(x, -)]$ is $\mathrm{Hom}_{k[\mathcal{A}]}(-, F)$-acyclic.

For all objects $x$ and $y$ of $\mathcal{A}$, $\mathrm{Tor}_*^{k[\mathcal{I}(x,y)]}(k, k[\mathcal{A}(x, y)])$ equals zero in positive degrees and $k[\mathcal{A}/\mathcal{I}(x, y)]$ in degree zero, hence the normalized bar construction yields an exact complex in $k[\mathcal{A}]$-**Mod**:

$$\cdots \to K^{\otimes i} \otimes h^x_{\mathcal{A}} \to K^{\otimes i-1} \otimes h^x_{\mathcal{A}} \to \cdots \to h^x_{\mathcal{A}} \to \pi^* h^x_{\mathcal{A}/\mathcal{I}} \to 0$$

where $K$ denotes the reduced part of the functor $k[\mathcal{I}(x, -)]$. The functor $h^x_{\mathcal{A}}$ is projective, and the functors $K^{\otimes i} \otimes h^x_{\mathcal{A}}$ are $\mathrm{Hom}_{k[\mathcal{A}]}(-, F)$-acyclic by lemma 4.9. Hence $\pi^* h^x_{\mathcal{A}/\mathcal{I}}$ is $\mathrm{Hom}_{k[\mathcal{A}]}(-, F)$-acyclic by a dimension shifting argument. □

In some cases, $\mathcal{A}/\mathcal{I}$ is $\mathbb{F}_p$-linear. If not, one can at least hope to obtain information on the polynomial homology over $k[\mathcal{A}/\mathcal{I}]$ from the polynomial homology over $k[\mathcal{A}/p]$ via the base change spectral sequences of proposition 2.20. In the sequel of the article, we bound ourselves to the study of polynomial homology over an $\mathbb{F}_p$-linear source category.

## 7. Polynomial homology over $\mathbf{P}_{\mathbb{F}_q}$

Let $k$ be a perfect field of positive characteristic $p$ containing a finite subfield $\mathbb{F}_q$ with $q$ elements and let $t : \mathbf{P}_{\mathbb{F}_q} \to \mathbf{P}_k$ be the additive functor given by extensions of scalars $t(v) = k \otimes_{\mathbb{F}_q} v$. In this section we compute the graded vector spaces $\mathrm{Ext}^*_{k[\mathbf{P}_{\mathbb{F}_q}]}(t^*F, t^*G)$, where $F$ and $G$ are strict polynomial functors over $k$, in terms of their so-called generic cohomology. For this purpose, we heavily rely on the results and the proofs of the fundamental work of Franjou, Friedlander, Scorichenko and Suslin [14]. The results of this section are one of the key ingredients for our results on polynomial homology over arbitrary categories in sections 8 and 10.

7.1. **Frobenius twists and generic homology.** Let $k$ be a perfect field of positive characteristic $p$. For all integers $r$ and all $k$-vector spaces $v$ we denote by ${}^{(r)}v$ the $k$-vector space which equals $v$ as an abelian group, with action of $k$ given by

$$\lambda \cdot x := \lambda^{p^{-r}} x \;.$$

Thus ${}^{(0)}v = v$ and ${}^{(s)}({}^{(r)}v) = {}^{(s+r)}v$. We note that ${}^{(r)}-$ is an additive endofunctor of $k$-vector spaces which preserves dimension.

**Notation 7.1.** If $L$ is a perfect field and $F$ is an object of $k[\mathbf{P}_L]$-**Mod** we denote by $F^{(r)}$ the composition of $F$ and ${}^{(r)}- : \mathbf{P}_L \to \mathbf{P}_L$.

When $r > 0$, the functor ${}^{(r)}- : \mathbf{P}_k \to k$-**Mod** is the underlying ordinary functor of a certain strict polynomial functor. Indeed, let sym : $S^{p^r} \to \otimes^{p^r}$ be the symmetrization morphism. This is a morphism of $p^r$-homogeneous strict polynomial functors such that for all finite-dimensional $k$-vector spaces $v$:

$$\begin{array}{rccc} \mathrm{sym}: & S^{p^r}(v) & \to & v^{\otimes p^r} \\ & x_1 \cdots x_{p^r} & \mapsto & \sum_{\sigma \in \mathfrak{S}_{p^r}} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p^r)} \end{array} .$$

The natural morphism ${}^{(r)}v \to S^{p^r}(v)$ which maps $x$ to $x^{p^r}$ identifies ${}^{(r)}v$ with the kernel of sym. Hence ${}^{(r)}v$ is actually the underlying functor of a $p^r$-homogeneous strict polynomial functor, namely the kernel of sym, which is called the *$r$-th Frobenius twist functor* and which is denoted by $I^{(r)}$. The following notation was introduced in [18], it is the strict polynomial functor analogue of notation 7.1.

**Notation 7.2.** For all $d$-homogeneous strict polynomial functors $F$, we denote by $F^{(r)}$ the $dp^r$-homogeneous strict polynomial functor $F^{(r)} := F \circ I^{(r)}$.

*Remark* 7.3. The definition of composition of strict polynomial functors is the obvious one if we think of strict polynomial functors in the way they are defined in [18]. If we use the description of strict polynomial functors as $k$-linear functors, then $F^{(r)}$ is the restriction of $F$ along the $k$-linear functor $\Gamma^{dp^r}\mathbf{P}_k \to \Gamma^d\mathbf{P}_k$ which sends a vector space $v$ to its Frobenius twist $v^{(r)}$, and whose action on morphisms is induced by the verschiebung map

$$\mathbf{v} : \Gamma^{dp^r}\mathrm{Hom}_k(u,v) \to \Gamma^d({}^{(r)}\mathrm{Hom}_k(u,v)) \simeq \Gamma^d(\mathrm{Hom}_k({}^{(r)}u, {}^{(r)}v)) \;.$$

One checks that $(I^{(r)})^{(s)} = I^{(r+s)}$ for all positive integers $r, s$. This formula extends to all non-negative integers if we define $I^{(0)}$ as the 1-homogeneous functor such that $I^{(0)}(v) = v$.

*Remark* 7.4. The strict polynomial functor $I^{(0)}$ is known under many different names, namely we have isomorphisms of 1-homogeneous strict polynomial functors $I^{(0)} \simeq S^1 \simeq \Lambda^1 \simeq \Gamma^1 \simeq \otimes^1$. The functor is also commonly denoted by the letter $I$, this simpler notation being consistent with notation 7.2.

The next result is established in [14] when $k$ is a finite field, but the case of an infinite field $k$ follows easily.

**Proposition-Definition 7.5.** Let $F$ and $G$ be two $d$-homogeneous strict polynomial functors. The maps given by precomposition by $I^{(1)}$:

$$\mathrm{Ext}^i_{\Gamma^{dp^r}\mathbf{P}_k}(F^{(r)}, G^{(r)}) \to \mathrm{Ext}^i_{\Gamma^{dp^{r+1}}\mathbf{P}_k}(F^{(r+1)}, G^{(r+1)})$$

are always injective, and they are isomorphisms if $i < 2p^r$. The stable value is called the vector space of *generic extensions of degree* $i$ and denoted by $\mathrm{Ext}^i_{\mathrm{gen}}(F, G)$:

$$\mathrm{Ext}^i_{\mathrm{gen}}(F, G) := \operatorname*{colim}_r \mathrm{Ext}^i_{\Gamma^{dp^r}\mathbf{P}_k}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^i_{\Gamma^{dp^r}\mathbf{P}_k}(F^{(r)}, G^{(r)}) \text{ if } r \gg 0.$$

*Proof.* Injectivity on Ext is proved in [14, Cor 1.3] when $k$ is a finite field, but the proof also applies without change over when $k$ is infinite. Isomorphism is proved in [14, Cor 4.6] over a finite field. Thus, the isomorphism holds if $k$ is infinite and if $F$ is a standard projective and $G$ is a standard injective by base change as in [44, 2.7]. The isomorphism for arbitrary $F$ and $G$ follows by considering a projective resolution of $F$, an injective resolution of $G$, and by using a standard spectral sequence argument. □

We refer the reader to [52] for a survey of these generic extensions and some formulas to compute them (which simplify and generalize [14]). See also section 11.1. We can define generic Tor in the same fashion as generic Ext. In order to compute Tor, we need objects of $\mathbf{Mod}\text{-}\Gamma^d\mathbf{P}_k$ that we call *contravariant d-homogeneous strict polynomial functors*. The following proposition follows from proposition 2.12.

**Proposition-Definition 7.6.** Let $F$ and $G$ be two $d$-homogeneous strict polynomial functors, with $F$ contravariant. The maps given by precomposition by $I^{(1)}$

$$\mathrm{Tor}^i_{\Gamma^{dp^{r+1}}\mathbf{P}_k}(F^{(r+1)}, G^{(r+1)}) \to \mathrm{Tor}^i_{\Gamma^{dp^r}\mathbf{P}_k}(F^{(r)}, G^{(r)})$$

are always surjective, and they are isomorphisms if $i < 2p^r$. The stable value is called the vector space of *generic torsion of degree* $i$ and denoted by $\mathrm{Tor}_i^{\mathrm{gen}}(F, G)$:

$$\mathrm{Tor}_i^{\mathrm{gen}}(F, G) := \lim_r \mathrm{Tor}_i^{\Gamma^{dp^r}\mathbf{P}_k}(F^{(r)}, G^{(r)}) \simeq \mathrm{Tor}_i^{\Gamma^{dp^r}\mathbf{P}_k}(F^{(r)}, G^{(r)}) \text{ for } r \gg 0.$$

## 7.2. The strong comparison theorem.

Recall from lemma 6.2 the exact functor:

$$t^* : \Gamma^d\mathbf{P}_k\text{-}\mathbf{Mod} \to k[\mathbf{P}_{\mathbb{F}_q}]\text{-}\mathbf{Mod}\ .$$

induced by forgetful functor from strict polynomial functors to ordinary functors together and restriction along the base change functor $t : \mathbf{P}_{\mathbb{F}_q} \to \mathbf{P}_k$. If $q = p^r$, there are canonical isomorphisms $t^*I^{(nr)} \simeq t$ in $k[\mathbf{P}_{\mathbb{F}_q}]$-$\mathbf{Mod}$, which sends an element $\lambda \otimes x$ in ${}^{(nr)}(k \otimes_{\mathbb{F}_q} v)$ to the element $\lambda^{p^{nr}} \otimes x$ in $k \otimes_{\mathbb{F}_q} v$. So if $F$ and $G$ are $d$-homogeneous strict polynomial functors, we have canonical isomorphisms:

$$t^*F \simeq t^*(F^{(nr)})\ , \qquad\qquad t^*(G^{(nr)}) \simeq t^*G\ .$$

So if $n$ is big enough, by combining these isomorphisms with the morphism of Ext induced by $t^*$ we obtain a graded $k$-linear map:

$$\mathrm{Ext}^i_{\mathrm{gen}}(F,G) \simeq \mathrm{Ext}^i_{\Gamma^{dp^{nr}}\mathbf{P}_k}(F^{(nr)},G^{(nr)}) \to \mathrm{Ext}^i_{k[\mathbf{P}_{\mathbb{F}_q}]}(t^*F,t^*G)\ . \tag{18}$$

The next result follows from the strong comparison theorem [14, Thm 3.10].

**Theorem 7.7.** *Let $k$ be an infinite perfect field containing a finite subfield with $q$ elements, and let $F$ and $G$ be two $d$-homogeneous strict polynomial functors. If $q \geq d$, the map* (18) *is an isomorphism in all degrees $i$.*

*Proof.* Theorem 7.7 generalizes the strong comparison theorem of [14] in two ways. Firstly, contrarily to [14], we do not assume that $k = \mathbb{F}_q$. Secondly, we work with $\Gamma^d\mathbf{P}_k$-**Mod** rather than with the category $\Gamma^d\mathbf{P}_k$-**mod**, i.e. we allow our strict polynomial functors to have infinite-dimensional values.

We overcome these two technical points as follows. The standard projective objects of $\Gamma^d\mathbf{P}_k$-**Mod** are the divided power functors $\Gamma^{d,s} = \Gamma^d(\mathrm{Hom}_k(k^s,-))$ and the standard injectives are the symmetric power functors $S^{d,s} = S^d(k^s \otimes -)$. These two kinds of functors commute with base change: there are canonical isomorphisms

$$t^*\Gamma^{d,s}(v) \simeq \Gamma^{d,s}_{\mathbb{F}_q}(v) \otimes_{\mathbb{F}_q} k \text{ and } t^*S^{d,s}(v) \simeq S^{d,s}_{\mathbb{F}_q}(v) \otimes_{\mathbb{F}_q} k$$

where the indices $\mathbb{F}_q$ indicate their counterparts in the category $\Gamma^d\mathbf{P}_{\mathbb{F}_q}$-**Mod** of strict polynomial functors over $\mathbb{F}_q$. Thus the morphism (18) fits into a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^i_{\Gamma^d\mathbf{P}_{\mathbb{F}_q}}(\Gamma^{d,s\,(nr)}_{\mathbb{F}_q}, S^{d,s\,(nr)}_{\mathbb{F}_q}) \otimes_{\mathbb{F}_q} k & \xrightarrow{\simeq} & \mathrm{Ext}^i_{\Gamma^d\mathbf{P}_k}(\Gamma^{d,s\,(nr)}, S^{d,s\,(nr)}) \\
\downarrow & & \downarrow{\scriptstyle (18)} \\
\mathrm{Ext}^i_{\mathbb{F}_q[\mathbf{P}_{\mathbb{F}_q}]}(\Gamma^{d,s}_{\mathbb{F}_q}, S^{d,s}_{\mathbb{F}_q}) \otimes_{\mathbb{F}_q} k & \xrightarrow{\simeq} & \mathrm{Ext}^i_{k[\mathbf{P}_{\mathbb{F}_q}]}(t^*\Gamma^{d,s}, t^*S^{d,s})
\end{array}$$

where the upper horizontal isomorphism is the base change functor for strict polynomial functors [44, 2.7], and the lower horizontal isomorphism is induced by tensoring by $k$ over $\mathbb{F}_q$ (that tensoring by $k$ yields an isomorphism follows from the Künneth formula of proposition 2.22 and the fact that $\Gamma^{d,s}_{\mathbb{F}_q}$ is $\mathrm{fp}_\infty$ by Schwartz's $\mathrm{fp}_\infty$ lemma [15, Prop 10.1]), and the vertical morphism on the left hand side is induced by the forgetful functor from strict polynomial functors to ordinary functors. This latter morphism is an isomorphism if $n \gg 0$ by the strong comparison theorem [14, Thm 3.10], so that our morphism (18) is an isomorphism when $F$ is a standard projective and $G$ is a standard injective. Next, we observe that the source and the target of morphism (18) both turn sums into products when viewed as functors of $F$, and they both turn products into products when viewed as functors of $G$, which implies that morphism (18) is an isomorphism when $F$ is an arbitrary projective strict polynomial functor and $G$ is an arbitrary injective strict polynomial functors (possibly with infinite-dimensional values). This implies that (18) is an isomorphism when $F$ and $G$ are arbitrary objects of $\Gamma^d\mathbf{P}_k$-**Mod** by a standard spectral sequence argument. □

We have a similar situation with generic Tor. Namely, if $F$ is a $d$-homogeneous contravariant strict polynomial functors, and if $G$ is a $d$-homogeneous strict polynomial functor, restriction along $t$ (as in lemma 6.2) together with the isomorphisms

$t^*F \simeq t^*F^{(nr)}$ and $t^*G \simeq t^*G^{(nr)}$ induce a morphism:

$$\mathrm{Tor}_i^{k[\mathbf{P}_{\mathbb{F}_q}]}(t^*F, t^*G) \to \mathrm{Tor}_i^{\Gamma^{dp^{nr}}\mathbf{P}_k}(F^{(nr)}, G^{(nr)}) \simeq \mathrm{Tor}_i^{\mathrm{gen}}(F,G)\;. \tag{19}$$

The next corollary follows from theorem 7.7 and proposition 2.12.

**Corollary 7.8.** *If $q \geq d$, the map (19) is an isomorphism in all degrees $i$.*

7.3. **Recollections of non-homogeneous strict polynomial functors.** Non-homogeneous strict polynomial functors are used in the generalizations of the strong comparison theorem 7.7 that we give in sections 7.4 and 7.5. We abuse notations (see remark 7.11 below) and we denote by $\Gamma\mathbf{P}_k$-**Mod** the category of strict polynomial functors of bounded degree over a field $k$. This category is defined by

$$\Gamma\mathbf{P}_k\text{-}\mathbf{Mod} = \bigoplus_{d\geq 0} \Gamma^d\mathbf{P}_k\text{-}\mathbf{Mod}\;.$$

Thus a strict polynomial functor of bounded degree $F$ is simply defined as a family of $d$-homogeneous strict polynomial functors $F_d$, which are called the $d$-homogeneous components of $F$, and all the $F_d$ are zero but a finite number of them. The highest $d$ such that $F_d \neq 0$ is called the *weight of* $F$, and denoted by $\mathrm{w}(F)$. We have $F = \bigoplus_{d\geq 0} F_d$. Morphisms of strict polynomial functors preserve these decompositions into homogeneous components. More generally we have (only finitely many terms of the sum are nonzero):

$$\mathrm{Ext}^*_{\Gamma\mathbf{P}_k}(F,G) = \bigoplus_{d\geq 0} \mathrm{Ext}^*_{\Gamma^d\mathbf{P}_k}(F_d, G_d)\;.$$

We define generic extensions by:

$$\mathrm{Ext}^*_{\mathrm{gen}}(F,G) = \bigoplus_{d\geq 0} \mathrm{Ext}^*_{\mathrm{gen}}(F_d, G_d)\;.$$

*Remark* 7.9. Let $\Gamma\mathbf{P}_k$-**mod** denote the full subcategory of $\Gamma\mathbf{P}_k$-**Mod** on the functors $F$ such that $F(v) = \bigoplus_{d\geq 0} F(v)$ has finite dimension for all $v$. Then $\Gamma\mathbf{P}_k$-**mod** identifies with the category $\mathcal{P}_k$ introduced in [18]. The inclusion $\Gamma\mathbf{P}_k$-**mod** $\hookrightarrow$ $\Gamma\mathbf{P}_k$-**Mod** induces an isomorphism on Ext, so that working with the former category or the latter is largely a matter of taste.

The forgetful functor from homogeneous strict polynomial functors to ordinary functors described in section 2.3 extends to the non-homogeneous case. Namely, we have a forgetful functor:

$$\gamma^* : \Gamma\mathbf{P}_k\text{-}\mathbf{Mod} = \bigoplus_{d\geq 0} \Gamma^d\mathbf{P}_k\text{-}\mathbf{Mod} \xrightarrow{\sum \gamma^{d\,*}} k[\mathbf{P}_k]\text{-}\mathbf{Mod}\;.$$

If $k$ is an infinite field, this forgetful functor is fully faithful, and an ordinary functor $F$ with finite-dimensional values is the image of a strict polynomial functor of weight $d$ if and only if the coordinate functions of the maps $\mathrm{Hom}_k(v,w) \to \mathrm{Hom}_k(F(v), F(w))$, $f \mapsto F(f)$, are polynomials of degree $d$.

Most often we will omit $\gamma^*$ from the notations, and simply denote by $F$ the underlying ordinary functor of a strict polynomial functor $F$.

*Remark* 7.10. The underlying ordinary functor of a strict polynomial functor of bounded degree $F$ is always polynomial in the sense of Eilenberg and Mac Lane, used in section 4. Thus $F$ has a weight $\mathrm{w}(F)$ and a degree $\deg F$. We have $\mathrm{w}(F) \leq$

$\deg F$, but the inequality may be strict. For example $\mathrm{w}(I^{(r)}) = p^r$ and $\deg I^{(r)} = 1$. More detailed relations between these two notions of degree can be found in [51]. The discrepancy between these two notions is our reason for using the term 'weight' instead of the term 'degree' used in [18].

Similarly there is a category $\mathbf{Mod}\text{-}\Gamma\mathbf{P}_k$ of contravariant strict polynomial functors of bounded degree and we have a similar decompositions (with finitely many nonzero terms in the direct sum)

$$\mathrm{Tor}_*^{\Gamma\mathbf{P}_k}(F,G) = \bigoplus_{d\geq 0}\mathrm{Tor}_*^{\Gamma^d\mathbf{P}_k}(F_d,G_d)\;,\quad \mathrm{Tor}_*^{\mathrm{gen}}(F,G) = \bigoplus_{d\geq 0}\mathrm{Tor}_*^{\mathrm{gen}}(F_d,G_d)\;.$$

*Remark* 7.11. Despite its notation, the category $\Gamma\mathbf{P}_k$-$\mathbf{Mod}$ is not a category of $k$-linear functors from some category $\Gamma\mathbf{P}_k$ to $k$-modules. However this abuse of notation emphasizes the fact the properties of the category $\Gamma\mathbf{P}_k$-$\mathbf{Mod}$ are very close to those of the categories $\Gamma^d\mathbf{P}_k$-$\mathbf{Mod}$. It also allows compact notations for Ext and Tor, with the fictious category $\Gamma\mathbf{P}_k$ as a decoration.

7.4. **Strong comparison without homogeneity.** If $F$ and $G$ are two strict polynomial functors of bounded degrees, we define a comparison map

$$\begin{aligned}&\mathrm{Ext}^*_{\mathrm{gen}}(F,G) = \\ (20)\qquad &\bigoplus_{d\geq 0}\mathrm{Ext}^*_{\mathrm{gen}}(F_d,G_d) \;\to \bigoplus_{d\geq 0}\mathrm{Ext}^*_{k[\mathbf{P}_{\mathbb{F}_q}]}(t^*F_d,t^*G_d) \to \mathrm{Ext}^*_{k[\mathbf{P}_{\mathbb{F}_q}]}(t^*F,t^*G)\end{aligned}$$

where the map on the left hand side is the direct sum of the comparison maps (18) used in theorem 7.7 while the map on the right hand side is the canonical inclusion into

$$\mathrm{Ext}^*_{k[\mathbf{P}_k]}(t^*F,t^*G) = \bigoplus_{d,e\geq 0}\mathrm{Ext}^*_{k[\mathbf{P}_k]}(t^*F_d,t^*G_e)\;.$$

We will often refer to morphism (20) as the *strong comparison map*. The next result extends the strong comparison theorem 7.7 to the non-homogeneous case.

**Theorem 7.12.** *Let $k$ be an infinite perfect field containing a finite subfield with $q$ elements, and let $F$ and $G$ be two strict polynomial functors, with weights less or equal to $q$. Then the map* (20) *is an isomorphism.*

*Proof.* The first map of the composition (20) is an isomorphism by the strong comparison theorem 7.7. Thus it suffices to prove that $\mathrm{Ext}^*_{k[\mathbf{P}_k]}(t^*F_d,t^*G_e)=0$ as soon as $d\neq e$, which follows from the vanishing lemma 7.13 below. □

We shall explain the elementary vanishing result used in theorem 7.12 in a general context, in order to use it again later. Let $\mathcal{A}$ be a small additive category. We assume that $\mathcal{A}$ is $\mathbb{F}$-linear, over a subfield $\mathbb{F}\subset k$. In the next lemma, we say that a functor $F$ of $k[\mathcal{A}]$-$\mathbf{Mod}$ is *$d$-homogeneous* (with respect to the field $\mathbb{F}$) if $F(\lambda f) = \lambda^d F(f)$ for all morphisms $f$ in $\mathcal{A}$ and all $\lambda\in\mathbb{F}$.

**Lemma 7.13.** *Let $\mathbb{F}$ be a subfield of $k$, and let $d\neq e$ be two non-negative integers such that* $\mathrm{card}\,\mathbb{F}\geq d,e$. *If $F$ and $G$ are two objects of* $k[\mathcal{A}]$-$\mathbf{Mod}$ *which are respectively $d$-homogeneous and $e$-homogeneous, then* $\mathrm{Ext}^*_{k[\mathcal{A}]}(F,G)=0$.

*Proof.* Let $F$ and $G$ be two arbitrary objects of $k[\mathcal{A}]$-$\mathbf{Mod}$. Since $\mathcal{A}$ is $\mathbb{F}$-linear, every element of $\mathbb{F}$ yields a natural transformation $\lambda_F\in\mathrm{End}_{k[\mathcal{A}]}(F)$ whose component at $x$ equals $F(\lambda\mathrm{id}_x)$. Thus $\mathrm{Ext}^*_{k[\mathcal{A}]}(F,G)$ has an $\mathbb{F}$-$\mathbb{F}$-bimodule structure given by $\lambda\cdot[\xi]\cdot\mu=[\mu_G\circ\xi\circ\lambda_F]$, where $-\circ\lambda_F$ is the pullback of an extension along $\lambda_F$ and $\lambda_G\circ -$ is the pushout of an extension along $\mu_G$. Moreover, for all morphisms $f:H\to K$ in $k[\mathcal{A}]$-**Mod** we have $f\circ\lambda_H=\mu_G\circ f$, which implies that the two $\mathbb{F}$-module structures coincide: $\lambda_F\cdot[\xi]=[\xi]\cdot\lambda_G$. Assume now that $F$ is $d$ homogeneous and $G$ is $e$-homogeneous. Then $\lambda_F=\lambda^d\mathrm{id}_F$ and $\lambda_G=\lambda^e\mathrm{id}_G$. Thus for all extensions $[\xi]$ we have $\lambda^d[\xi]=\lambda_F\cdot[\xi]=[\xi]\cdot\lambda_G=\lambda^e[\xi]$. Since the cardinal of $\mathbb{F}$ is greater or equal to $d$ and $e$, the maps $\lambda\mapsto\lambda^d$ and $\lambda\mapsto\lambda^e$, seen as maps from $\mathbb{F}$ to $k$, are not equal. Hence the equality $\lambda^d[\xi]=\lambda^e[\xi]$ implies that $[\xi]=0$. □

As before, this result can be dualized. Namely, if $F$ and $G$ are two strict polynomial functors of bounded degrees, respectively contravariant and covariant, we have a comparison map (where the first map is the canonical projection and the second map is given by the direct sum of the comparison maps 19)

$$\mathrm{Tor}_*^{k[\mathbf{P}_k]}(t^*F,t^*G)\to\bigoplus_{d\geq 0}\mathrm{Tor}_*^{k[\mathbf{P}_k]}(t^*F_d,t^*G_d)\to \begin{array}{c}\mathrm{Tor}_*^{\mathrm{gen}}(F,G)=\\ \bigoplus_{d\geq 0}\mathrm{Tor}_*^{\mathrm{gen}}(F_d,G_d)\end{array}\;. \tag{21}$$

We will often refer to morphism (21) as the *strong comparison map (for* Tor*)*. By using proposition 2.12 we deduce the following result from theorem 7.12.

**Corollary 7.14.** *Let $k$ be an infinite perfect field containing a finite subfield with $q$ elements, and let $F$ and $G$ be two strict polynomial functors, with weights less or equal to $q$. Then the map* (21) *is an isomorphism.*

7.5. **Strong comparison over small fields.** We are now going to generalize the strong comparison theorem 7.12 to the case when $q=p^r$ is not big enough with respect to the degrees of $F$ and $G$. Our approach, in particular lemma 7.16 and proposition 7.19, is inspired by the proof of [14, Thm 6.1].

**Notation 7.15.** Let $L$ be a perfect field. For all positive integers $a$ and $s$ and for all $L$-vector spaces $v$, we let ${}^{(a,s)}v={}^{(0)}v\oplus{}^{(a)}v\oplus\cdots\oplus{}^{((s-1)a)}v$. For all functors $F$ in $k[\mathbf{P}_L]$-**Mod**, we denote by $F^{(a,b)}$ the composition of $F$ with the functor ${}^{(a,s)}-:\mathbf{P}_L\to\mathbf{P}_L$.

Assume that $\mathbb{F}_q\subset L$ is an extension of perfect fields, and let $\tau:\mathbf{P}_{\mathbb{F}_q}\to\mathbf{P}_L$ be the associated extension of scalars. Then if $a$ is divisible by $r$ then for all $\mathbb{F}_q$-vector spaces $v$ there is a canonical isomorphism of $L$-vector spaces ${}^{(a,s)}\tau(v)\simeq\tau(v)^{\oplus s}$, which sends an element $\lambda_i\otimes x_i$ of the $i$-th summand ${}^{(ia)}\tau(v)$ of ${}^{(a,s)}\tau(v)$ to the element $\lambda_i^{p^{ai}}\otimes x_i$ of the $i$-th summand of $\tau(v)^{\oplus i}$. If we denote by $\mathrm{diag}:\tau\to\tau^{\oplus s}$ and $\mathrm{sum}:\tau^{\oplus s}\to\tau$ the morphisms whose restrictions to the components of $\tau^{\oplus s}$ are all equal to the identity of $\tau$, we can define two morphisms in $k[\mathbf{P}_{\mathbb{F}_q}]$-**Mod**:

$$\tau^*F\xrightarrow{F(\mathrm{diag})}(\tau^{\oplus s})^*F\simeq\tau^*(F^{(r,s)})\;,\qquad \tau^*(G^{(rs,s)})\simeq(\tau^{\oplus s})^*G\xrightarrow{G(\mathrm{sum})}\tau^*G\;.$$

These two morphisms, together with restriction along $\tau$, yield a morphism of graded $k$-vector spaces:

$$\mathrm{Ext}^*_{k[\mathbf{P}_L]}(F^{(r,s)},G^{(rs,s)})\to\mathrm{Ext}^*_{k[\mathbf{P}_{\mathbb{F}_q}]}(\tau^*F,\tau^*G)\;. \tag{22}$$

**Lemma 7.16.** *If $\mathbb{F}_q\subset L$ is an extension of fields of degree $s^2$ and $q=p^r$, then for all $F$ and $G$ in $k[\mathbf{P}_L]$-**Mod**, the map* (22) *is an isomorphism.*

*Proof.* We have a sequence of extensions of fields $\mathbb{F}_q \subset K \subset L$ of degree $s$. We are going to convert Ext over $k[\mathbf{P}_L]$ into Ext over $k[\mathbf{P}_{\mathbb{F}_q}]$ in two steps, by using the effect on Ext of the restriction of scalars $\mathbf{P}_L \to \mathbf{P}_K$ and $\mathbf{P}_K \to \mathbf{P}_{\mathbb{F}_q}$, and their adjoints (with the help of proposition 2.17), and we are going to check that the isomorphism obtained coincides with the map (22).

By [14, Prop 3.1] we have an adjoint pair

$$\rho' : \mathbf{P}_L \leftrightarrows \mathbf{P}_K : \tau'$$

where $\tau'$ is the extension of scalars and $\rho'$ the restriction of scalars associated to the extension $K \subset L$. (Note that $\tau'(v) = L \otimes_K v$ is the *right* adjoint.) First, for all $L$-vector spaces $v$, we have a natural isomorphism of $(L, L)$-bimodules $\phi_v : L \otimes_K v \simeq \bigoplus_{0 \le i < s} {}^{(irs)}v$ given by sending $\lambda \otimes x$ to $\sum_{0 \le i < s} \lambda^{p^{-rsi}} x$. Whence an isomorphism in $k[\mathbf{P}_L]$-**Mod**:

$$G(\phi^{-1}) : G^{(rs,s)} \simeq {\rho'}^* {\tau'}^* G \; . \tag{23}$$

Next, since $\rho'$ is left adjoint to $\tau'$, we have an isomorphism, for all $H$ in $k[\mathbf{P}_L]$-**Mod**:

$$\mathrm{Ext}^*_{k[\mathbf{P}_L]}(H, {\rho'}^* {\tau'}^* G) \simeq \mathrm{Ext}^*_{k[\mathbf{P}_K]}({\tau'}^* H, {\tau'}^* G) \; . \tag{24}$$

To be more specific, isomorphism (24) is given by restriction along $\tau'$ and by the map $({\tau'}^* G)(\epsilon)$ where $\epsilon$ is the counit of the adjunction $\rho' \dashv \tau'$. By [14, Prop 3.1], this counit of adjunction $\epsilon_u : L \otimes_K u \to u$ is given by $\epsilon_u(\lambda \otimes x) = T(\lambda)x$, where $T(\lambda) = \sum_{0 \le i < s} \lambda^{p^{rsi}}$ is the trace of $\lambda$. Thus for all $K$-vector spaces $u$ we have a commutative square of $L$-vector spaces, in which the upper horizontal arrow is the canonical isomorphism:

$$\begin{array}{ccc}
\bigoplus_{0 \le i < s} {}^{(-rsi)}\tau'(u) & \xrightarrow{\simeq} & \bigoplus_{0 \le i < s} \tau'(u) \\
{\scriptstyle \phi_{\tau'(u)}} \uparrow & & \downarrow {\scriptstyle \mathrm{sum}} \\
\tau'(\rho'(\tau'(u))) & \xrightarrow{\tau'(\epsilon_u)} & \tau'(u)
\end{array} \; .$$

It follows that the isomorphism

$$\mathrm{Ext}^*_{k[\mathbf{P}_L]}(H, G^{(rs,s)}) \xrightarrow{\simeq} \mathrm{Ext}^*_{k[\mathbf{P}_K]}({\tau'}^* H, {\tau'}^* G) \tag{25}$$

induced by isomorphisms (23) and (24) admits another description: it equals the composition of ${\tau'}^*$ with the map induced by ${\tau'}^*(G^{(rs,s)}) \simeq ({\tau'}^{\oplus s})^* G \xrightarrow{G(\mathrm{sum})} G$.

Similarly, we have a pair of adjoints, in which $\tau''$ and $\rho''$ are the extension of scalars and the restriction of scalars associated to the extension $\mathbb{F}_q \subset K$ (and $\tau''$ is this time seen as a left adjoint):

$$\tau'' : \mathbf{P}_{\mathbb{F}_q} \leftrightarrows \mathbf{P}_K : \rho'' \; .$$

For all $K$-vector spaces $v$, the isomorphism $\psi_v : L \otimes_{\mathbb{F}_q} v \simeq \bigoplus_{0 \le i < s} {}^{(ri)}(L \otimes_K v)$ given by $\psi_v(\lambda \otimes x) = \sum_{0 \le i < s} \lambda^{p^{-ri}} x$ induces an isomorphism in $k[\mathbf{P}_K]$-**Mod**:

$$F(\psi) : {\rho''}^* \tau^* F \simeq {\tau'}^* (F^{(r,s)}) \; . \tag{26}$$

Since $\rho''$ is right adjoint to $\tau''$ we have an isomorphism for all $K$ in $k[\mathbf{P}_K]$-**Mod**:

$$\mathrm{Ext}^*_{k[\mathbf{P}_K]}({\rho''}^* \tau^* F, K) \simeq \mathrm{Ext}^*_{k[\mathbf{P}_{\mathbb{F}_q}]}(\tau^* F, {\tau''}^* K) \; . \tag{27}$$

To be more specific, isomorphism (27) is induced by restriction along $\tau''$ and by $(\tau^* F)(\eta)$, where $\eta$ is the unit of the adjunction $\tau'' \dashv \rho''$. This unit of adjunction

$\eta_v : v \to K \otimes_{\mathbb{F}_q} v$ is given by $\eta_v(x) = 1 \otimes x$, whence a commutative diagram of $L$-vector spaces, in which the lower horizontal arrow is the canonical isomorphism:

$$\begin{array}{ccc} \tau(\rho''(\tau''(u))) & \xleftarrow{\tau(\eta_u)} & \tau(u) \\ \downarrow{\scriptstyle \psi_{\tau''(u)}} & & \downarrow{\scriptstyle \mathrm{diag}} \\ \bigoplus_{0\le i<s} {}^{(ri)}\tau'(\tau''(u)) & \xleftarrow{\simeq} & \bigoplus_{0\le i<s} \tau(u) \end{array} .$$

This shows that the isomorphism

$$\mathrm{Ext}^*_{k[\mathbf{P}_K]}({\tau'}^*(F^{(r,s)}), K) \xrightarrow{\simeq} \mathrm{Ext}^*_{k[\mathbf{P}_{\mathbb{F}_q}]}(\tau^* F, {\tau''}^* K) \tag{28}$$

induced by isomorphisms (26) and (27) admits another description: it is the composition of ${\tau''}^*$ with the map induced by $\tau^* F \xrightarrow{F(\mathrm{diag})} (\tau^{\oplus s})^* F \simeq {\tau''}^* {\tau'}^* (F^{(r,s)})$.

Thus, the graded morphism (22) is the composition of the maps (25) (with $H = F^{(r,s)}$) and (28) (with $K = {\tau'}^* G$), hence it is an isomorphism. □

**Notation 7.17.** For all positive integers $r$ and $s$, we denote by $I^{(r,s)}$ the (non-homogeneous) strict polynomial functor of weight $p^{(s-1)r}$ defined by:

$$I^{(r,s)} := I^{(0)} \oplus I^{(r)} \oplus \cdots \oplus I^{(\,(s-1)r\,)}.$$

For all strict polynomial functors $F$ we let $F^{(r,s)} = F \circ I^{(r,s)}$. If $\mathrm{w}(F) = d$ then $\mathrm{w}(F^{(r,s)}) = p^{(s-1)r} d$.

*Remark* 7.18. The definition of composition of strict polynomial functors is the obvious one if we think of strict polynomial functors in the way they are defined in [18]. If we use the description of strict polynomial functors as families of $k$-linear functors as we pretend to do it, then composition can be defined as follows. First we can consider $F(v_0 \oplus \cdots \oplus v_{s-1})$ as a strict polynomial functor of $s$ variables as in [51, Section 3.2]. Then we precompose each variable $v_i$ by the Frobenius twist $I^{(ir)}$. The strict polynomial functor $F^{(r,s)}$ is then defined as the evaluation of the resulting strict polynomial functor with $s$ variables on the $s$-tuple $(v, \dots, v)$.

For all strict polynomial functors $F$ and $G$ we define a morphism $\Xi_k$ of graded $k$-vector spaces as the composition

$$\Xi_k : \mathrm{Ext}^*_{\mathrm{gen}}(F^{(r,s)}, G^{(rs,s)}) \to \mathrm{Ext}^*_{k[\mathbf{P}_{\mathbb{F}_q}]}(F^{(r,s)}, G^{(rs,s)}) \to \mathrm{Ext}^*_{k[\mathbf{P}_{\mathbb{F}_q}]}(t^* F, t^* G) \tag{29}$$

where the first map is the strong comparison map (20), and the second map is morphism (22) for $L = \mathbb{F}_q$.

**Proposition 7.19.** *Assume that $k$ contains a finite field $\mathbb{F}_q$ of cardinal $q = p^r$, and let $s$ be a positive integer. Then for all strict polynomial functors $F$ and $G$ of weights less or equal to $q^s$, the map* (29) *is an isomorphism in all degrees.*

*Proof.* It suffices to prove the result when $k$ contains a subfield $L$ with $q^{s^2}$ elements. Indeed, let $k \to K$ be a finite extension of fields and let $\tau : k\text{-}\mathbf{Mod} \to K\text{-}\mathbf{Mod}$ be the extension of scalars. By [44, Section 2] there is an exact $k$-linear base change functor

$$-_K : \Gamma\mathbf{P}_k\text{-}\mathbf{Mod} \to \Gamma\mathbf{P}_K\text{-}\mathbf{Mod}$$

such that for all strict polynomial functors $F'$ over $k$ there are canonical isomorphisms of functors $\tau^* F'_K \simeq \tau \circ F'$. Moreover this base change functor induces an isomorphism on the level of Ext (See [44, cor 2.7] for the case of functors with finite-dimensional values. The proof extends to arbitrary functors when $k \to K$ is a finite extension of fields). There is a commutative square

$$\begin{array}{ccc} \mathrm{Ext}^*_{\mathrm{gen}}(F_K^{(r,s)}, G_K^{(rs,s)}) & \xrightarrow{\Xi_K} & \mathrm{Ext}^*_K[\mathbf{P}_{\mathbb{F}_q}](t^*\tau^*F_K, t^*\tau^*G_K) \\ \simeq\uparrow & & \simeq\uparrow \\ K\otimes \mathrm{Ext}^*_{\mathrm{gen}}(F^{(r,s)}, G^{(rs,s)}) & \xrightarrow{K\otimes\Xi_k} & K\otimes \mathrm{Ext}^*_K[\mathbf{P}_{\mathbb{F}_q}](t^*F, t^*G) \end{array}$$

in which the vertical isomorphism on the left hand side is induced by the base change functor $-_K$ and the vertical isomorphism on the right hand side is induced by $\tau$ (see the map (16) in section 6) and by the isomorphisms $H_K\circ\tau\circ t \simeq \tau\circ H\circ t$, for $H = F$ or $G$. Therefore $\Xi_k$ is an isomorphism if an only if $\Xi_K$ is an isomorphism.

So we assume that $k$ contains a subfield $L$ with $q^{s^2}$ elements. We denote by $t' : \mathbf{P}_L \to \mathbf{P}_k$ the extension of scalars associated to the extension of fields $L \subset k$. In this case $\Xi_k$ is an isomorphism because we can rewrite it as the composition of three isomorphisms:

$$\begin{array}{ccc} \mathrm{Ext}^*_{\mathrm{gen}}(F^{(r,s)}, G^{(rs,s)}) & \xrightarrow{\Xi_k} & \mathrm{Ext}^*_{k[\mathbf{P}_{\mathbb{F}_q}]}(t^*F, t^*G) \\ \downarrow\simeq & & \simeq\uparrow \\ \mathrm{Ext}^*_{k[\mathbf{P}_L]}(t'^*(F^{(r,s)}), t'^*(G^{(rs,s)})) & \xrightarrow{\simeq} & \mathrm{Ext}^*_{k[\mathbf{P}_L]}((t'^*F)^{(r,s)}, (t'^*G)^{(rs,s)}) \end{array} .$$

To be more specific, the vertical map on the left hand side is the strong comparison map (20) relative to the finite field $L$. By our assumptions on $s$, the weights of $F^{(r,s)}$ and $G^{(rs,s)}$ are less or equal to the cardinal of $L$, hence this map is an isomorphism by theorem 7.12. The lower horizontal map is induced by the canonical isomorphisms $t'^*(F^{(r,s)}) \simeq (t'^*F)^{(r,s)}$ and $t'^*(G^{(rs,s)}) \simeq (t'^*G)^{(rs,s)}$. And the vertical map on the right hand side is the isomorphism provided by lemma 7.16. □

Now we introduce a variant of the map $\Xi_k$ which will be better adapted to our later purposes. With this new map $\Xi'_k$, direct sums of Frobenius twists only appear inside $F$ in the generic extensions. To be more specific, we let

$$\Xi'_k : \mathrm{Ext}^*_{\mathrm{gen}}(F^{(r,s^2)}, G^{(rs^2-rs)}) \to \mathrm{Ext}^*_{k[\mathbf{P}_{\mathbb{F}_q}]}(t^*F, t^*G) \; . \tag{30}$$

be the graded $k$-linear map induced by the strong comparison map (20), with $F$ and $G$ respectively replaced by $F^{(r,s^2)}$ and $G^{(rs^2-rs)}$, and the following morphisms:

$$t^*F \xrightarrow{F(\mathrm{diag})} (t^{\oplus s^2})^*F \simeq F^{(r,s^2)} \; , \qquad\qquad t^*(G^{(rs^2-rs)}) \simeq t^*G \; .$$

Next theorem subsumes the strong comparison theorem 7.12. To be more specific, on recovers theorem 7.12 by taking $s = 1$ in the statement.

**Theorem 7.20.** *Let $k$ be a perfect field containing a finite subfield with $q = p^r$ elements, and let $s$ be a positive integer. Assume that $F$ and $G$ are two strict polynomial functors with weights less or equal to $q^s$. Then the map* (30) *is a graded isomorphism.*

*Proof.* We shall use strict polynomial multifunctors, as in [44, Section 3], [48, Section 2] or [51, Section 3.2]. To be more specific, we consider the category of strict polynomial multifunctors of $n$ variables:

$$\Gamma(\mathbf{P}_k^{\times n})\text{-}\mathbf{Mod} = \bigoplus_{d\geq 0} \Gamma^d(\mathbf{P}_k^{\times n})\text{-}\mathbf{Mod}\;.$$

The operation of precomposition by Frobenius twist extends to the multivariable setting, namely given a strict polynomial multifunctors $F$ and an $n$-tuple of non-negative integers $\underline{r} = (r_1,\dots,r_n)$ we let $F^{(\underline{r})}$ denote the strict polynomial multifunctor such that

$$F^{(\underline{r})}(v_1,\dots,v_n) = F({}^{(r_1)}v_1,\dots,{}^{(r_n)}v_n)\;.$$

Precomposition by Frobenius twists yield a morphism on Ext:

$$-\circ I^{(\underline{m})} : \mathrm{Ext}^i_{\Gamma(\mathbf{P}_k^{\times n})}(F^{(\underline{r})},G^{(\underline{r})}) \to \mathrm{Ext}^i_{\Gamma(\mathbf{P}_k^{\times n})}(F^{(\underline{r}+\underline{m})},G^{(\underline{r}+\underline{m})})$$

which is an isomorphism provided that all the integers $r_i$ are big enough (with respect to $i$, $F$ and $G$). Indeed, by a spectral sequence argument, it suffices to check the result when $F$ is a standard projective and $G$ is a standard injective. In this case, $F(v_1,\dots,v_n) = F_1(v_1)\otimes\cdots\otimes F_n(v_n)$ for some standard projective strict polynomial functors $F_i$, and $G(v_1,\dots,v_n) = G_1(v_1)\otimes\cdots\otimes G_n(v_n)$ for some standard injective strict polynomial functors $G_i$, hence the isomorphism follows from the Ext-isomorphism for functors with one variable and the Künneth formula.

There is a forgetful functor $\gamma^* : \Gamma(\mathbf{P}_k^{\times n})\text{-}\mathbf{Mod} \to k[\mathbf{P}_k^{\times n}]\text{-}\mathbf{Mod}$ and the sum-diagonal adjunction lifts to the setting of strict polynomial functors.

Now in order to prove theorem 7.20, we observe that we may choose strict polynomial multifunctors $F'$, $G'$, $F''$ and $G''$ such that there is a commutative diagram, with $n \gg 0$:

$$\begin{array}{ccc}
\mathrm{Ext}^i_{\Gamma\mathbf{P}_k^{s^2}}(F',G') & \xrightarrow[(*)]{\simeq} & \mathrm{Ext}^i_{\Gamma\mathbf{P}_k}\Big((F^{(r,s^2)})^{(nr)},(G^{(rs-r)})^{(nr)}\Big) \\
{\scriptstyle -\circ I^{(\underline{m})}}\big\uparrow{\scriptstyle\simeq} & & \\
\mathrm{Ext}^i_{\Gamma\mathbf{P}_k^{s^2}}(F'',G'') & & \big\downarrow{\scriptstyle \Xi'_k} \\
{\scriptstyle (**)}\big\downarrow{\scriptstyle\simeq} & & \\
\mathrm{Ext}^i_{\Gamma\mathbf{P}_k}\Big((F^{(r,s)})^{(nr)},(G^{(rs,s)})^{(nr)}\Big) & \xrightarrow{\Xi_k} & \mathrm{Ext}^i_{k[\mathbf{P}_{\mathbb{F}_q}]}(t^*F,t^*G)
\end{array}$$

To be more specific, the strict polynomial multifunctors $F'$, $G'$, $F''$ and $G''$ of the $s^2$ variables $v_{ij}$, $0\leq i,j<s$, are respectively given by

$$\begin{aligned}
F'(\dots,v_{ij},\dots) &= F\Big(\bigoplus_{0\leq i,j<s} {}^{(nr+ri+rsj)}v_{ij}\Big),\\
G'(\dots,v_{ij},\dots) &= G\Big(\bigoplus_{0\leq i,j<s} {}^{(nr+rs^2-rs)}v_{ij}\Big),\\
F''(\dots,v_{ij},\dots) &= F\Big(\bigoplus_{0\leq i,j<s} {}^{(nr+ri)}v_{ij}\Big),\\
G''(\dots,v_{ij},\dots) &= G\Big(\bigoplus_{0\leq i,j<s} {}^{(nr+rsj)}v_{ij}\Big).
\end{aligned}$$

The $s^2$-tuple $\underline{m}$ is given by $m_{ij} = rs^2 - rs - rsj$ and $- \circ I^{(\underline{m})}$ is an isomorphism because $n$ is big enough. The maps $(*)$ and $(**)$ are given by sum-diagonal adjunction. To be more explicit, the map $(*)$ is given by setting $v_{ij} = v$ for all $i$ and $j$, and by composing the resulting extensions of strict polynomial functors of the variable $v$ by the morphism $G(\mathrm{sum}')$, where

$$\mathrm{sum}' : {}^{(rs^2-rs+nr)}v^{\oplus s^2} \to {}^{(rs^2-rs+nr)}v$$

is the morphism which restricts to the identity of ${}^{(rs^2-rs+nr)}v$ on each summand of ${}^{(rs^2-rs+nr)}v^{\oplus s^2}$. Similarly, the map $(**)$ is given by setting $v_{ij} = v$ for all $i$ and $j$, and by composing the resulting extensions of strict polynomial functors of the variable $v$ by the morphisms $F(\mathrm{diag}'')$ and $G(\mathrm{sum}'')$ where each of the morphisms

$$\mathrm{diag}'' : \bigoplus_{0\le i<s} {}^{(ri+nr)}v \to \bigoplus_{0\le i<s} {}^{(ri+nr)}v^{\oplus s}$$
$$\mathrm{sum}'' : \bigoplus_{0\le j<s} {}^{(rsj+nr)}v^{\oplus s} \to \bigoplus_{0\le j<s} {}^{(rsj+nr)}v$$

restricts to identity morphisms between any two summands with the same number of Frobenius twists.

Under the hypotheses of theorem 7.20 the map $\Xi_k$ is an isomorphism by proposition 7.19, hence $\Xi'_k$ is an isomorphism by commutativity of the above diagram. □

Let us give the analogue of theorem 7.20 for Tor. Let $F$ and $G$ be two strict polynomial functors, with $F$ contravariant. The strong comparison map (21) for Tor and the morphisms

$$t^*F \xrightarrow{F(\mathrm{sum})} (t^{\oplus s^2})^*F \simeq F^{(r,s^2)}\;, \qquad\qquad t^*G \simeq t^*(G^{(rs^2-rs)})\;,$$

(where $\mathrm{sum} : t^{\oplus s^2} \to t$ is the morphism whose restriction to each direct summand $t$ of $t^{\oplus s^2}$ equals the identity of $t$) induce a graded $k$-linear map:

$$\mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}_q}]}(t^*F, t^*G) \to \mathrm{Tor}_*^{\mathrm{gen}}(F^{(r,s^2)}, G^{(rs^2-rs)})\;. \tag{31}$$

Proposition 2.12 allows to dualize theorem 7.20, and we obtain the following result.

**Corollary 7.21.** *Let $k$ be a perfect field containing a finite subfield with $q = p^r$ elements, and let $s$ be a positive integer. Assume that $F$ ans $G$ are two strict polynomial functors (respectively contravariant and covariant) with weights less or equal to $q^s$. Then the map* (31) *is a graded isomorphism.*

## 8. The homology of additive functors

In this section, we prove theorem 1.5 from the introduction, which compares functor homology over $\mathcal{A}$ with functor homology over $k[\mathcal{A}]$. As we are going to explain it now, we actually prove a more precise statement for the Ext-isomorphism.

Let $k$ be an infinite perfect field of positive characteristic $p$. The self-extensions of the strict polynomial functor $I^{(r)}$ were first computed in [18, Thm 4.5]. We know that they are $k$-vector spaces of dimension 1 in all degrees $2i$ such that $i < p^r$, and zero in the other degrees. By [18, Cor 4.9] or by [14, Thm 2.6], we also know that these non-zero $k$-vector spaces of self-extensions are equal to the $k$-vector spaces of generic self-extensions of $I$. Thus, for all even integers $i \geq 0$, we may choose an integer $r \geq \log_p(i/2)$ and a nonzero class

$$0 \neq e_i \in \mathrm{Ext}^i_{\Gamma^{p^r}\mathbf{P}_k}(I^{(r)}, I^{(r)}) = \mathrm{Ext}^i_{\mathrm{gen}}(I, I) ,$$

which is a basis vector of this one-dimensional vector space.

**Notation 8.1.** For all additive functors $\rho : \mathcal{A} \to k$-**Mod** and all integers $j$, we denote by ${}^{(j)}\rho$ the additive functor such that $({}^{(j)}\rho)(a) := {}^{(j)}\rho(a)$.

*Remark* 8.2. The exponent ${}^{(j)}$ is written on the left to suggest that ${}^{(j)}\rho$ is *postcomposition* by the $(j)$-th Frobenius twist. Compare with notation 7.2, in which $F^{(j)}$ which indicates precomposition by the $j$-th Frobenius twist.

Recall from section 7.1 that $I^{(r)}({}^{(-r)}v) = v$. Thus, by lemma 6.2, evaluation of $e_i$ on ${}^{(-r)}\rho$ yields an extension

$$e_i' \in \mathrm{Ext}^i_{k[\mathcal{A}]}(\rho, \rho) .$$

Now let $E^*_\infty$ be the $k$-vector space which equals $k$ in even non-negative degrees and which equals zero in the other degrees. For all even integers $i$ and all integers $j$, we denote by $\Upsilon_{ij}$ the composition

$$\Upsilon_{ij} : \mathrm{Ext}^j_{{}_k\mathcal{A}}(\pi, \rho) \otimes E^i_\infty \to \mathrm{Ext}^j_{k[\mathcal{A}]}(\pi, \rho) \otimes E^i_\infty \to \mathrm{Ext}^{i+j}_{k[\mathcal{A}]}(\pi, \rho)$$

where the first map is induced by the forgetful functor ${}_k\mathcal{A}$-**Mod** $\to k[\mathcal{A}]$-**Mod** and the second one sends $e \otimes 1$ to the Yoneda splice $e_i' \circ e$. The maps $\Upsilon_{ij}$ assemble into a graded $k$-linear map

$$\Upsilon : \mathrm{Ext}^*_{{}_k\mathcal{A}}(\pi, \rho) \otimes E^*_\infty \to \mathrm{Ext}^*_{k[\mathcal{A}]}(\pi, \rho) . \tag{32}$$

The following result is the main result of this section, and our improved form of the Ext-isomorphism of theorem 1.5.

**Theorem 8.3.** *Let $k$ be an infinite perfect field of positive characteristic $p$, and let $\mathcal{A}$ be a small additive category, which we assume to be $\mathbb{F}_p$-linear. For all additive functors $\rho, \pi : \mathcal{A} \to k$-**Mod**, the map $\Upsilon$ defined in equation (32) is an isomorphism of graded $k$-vector spaces.*

*Remark* 8.4. Since $\Upsilon$ is constructed in a very natural way, one can easily check that it is compatible with a variety of operations that we may consider on Ext. For example, assume that $\rho = \pi$, and interpret $E^*_\infty$ as generic extensions. Then Yoneda splices naturally endow the source and the target of $\Upsilon$ with $k$-algebra structures, and $\Upsilon$ is a morphism of algebras.

Before proving theorem 8.3, we observe that theorem 8.3 implies not only the Ext-isomorphism in theorem 1.5 but also the Tor-isomorphism therein. Hence the whole of theorem 1.5 is actually a direct consequence of theorem 8.3.

**Corollary 8.5.** *If $T^\infty_*$ denote the graded vector space which is $k$ in even non-negative degrees and $0$ in the other degrees. There is a graded isomorphism, natural with respect to $\pi$ and $\rho$:*

$$\mathrm{Tor}^{{}_k\mathcal{A}}_*(\pi, \rho) \otimes T^\infty_* \simeq \mathrm{Tor}^{k[\mathcal{A}]}_*(\pi, \rho) .$$

*Proof of corollary 8.5.* The graded vector spaces $\mathrm{Hom}_k(\mathrm{Tor}_*^{\mathcal{K}}(\pi,\rho),M)$ are naturally isomorphic to $\mathrm{Ext}^*_{\mathcal{K}}(\rho,\mathrm{Hom}_k(\pi,M))$ for $\mathcal{K} = {}_k\mathcal{A}$ or $\mathcal{K} = k[\mathcal{A}]$ and for all vector spaces $M$. So theorem 8.3 yields an isomorphism, natural in $\pi$, $\rho$ and $M$:

$$\mathrm{Hom}_k(\mathrm{Tor}_*^{{}_k\mathcal{A}}(\pi,\rho),M)\otimes E^*_\infty \simeq \mathrm{Hom}_k(\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi,\rho),M) \tag{33}$$

Note that $E^*_\infty$ is the $k$-linear graded dual of $T_*^\infty$. Since $E_*^\infty$ is degreewise finite-dimensional, there is a canonical isomorphism:

$$\mathrm{Hom}_k(\mathrm{Tor}_*^{{}_k\mathcal{A}}(\pi,\rho),M)\otimes E^*_\infty \simeq \mathrm{Hom}_k(\mathrm{Tor}_*^{{}_k\mathcal{A}}(\pi,\rho)\otimes T_*^\infty,M)\;. \tag{34}$$

Thus the right hand sides of (34) and (33) are naturally isomorphic, and we conclude the proof by applying the Yoneda lemma. □

The remainder of section 8 is devoted to the proof of theorem 8.3. We first establish two reduction lemmas.

**Lemma 8.6.** *Fix $\rho$ in ${}_k\mathcal{A}$-**Mod**. If the morphism* (32) *is an isomorphism for all standard projectives $\pi$ in ${}_k\mathcal{A}$-**Mod** then it is an isomorphism for all $\pi$ in ${}_k\mathcal{A}$-**Mod**.*

*Proof.* The source and the target of $\Upsilon$, regarded as functors of the variable $\pi$, turn direct sums into products. Since every projective of ${}_k\mathcal{A}$-**Mod** is a direct summand of a direct sum of standard projectives, this implies that the map (32) is an isomorphism whenever $\pi$ is projective in ${}_k\mathcal{A}$-**Mod**.

Now let $\pi$ be an arbitrary object of ${}_k\mathcal{A}$-**Mod**, let $P$ be a projective resolution of $\pi$ in ${}_k\mathcal{A}$-**Mod**, and let $Q$ be an injective resolution of $\rho$ in $k[\mathcal{A}]$-**Mod**. We consider the bicomplexes:

$$C^{pq} = \mathrm{Hom}_{{}_k\mathcal{A}}(P_p,\rho)\otimes E^q_\infty\;,\qquad D^{pq} = \mathrm{Hom}_{k[\mathcal{A}]}(P_p,Q^q)\;.$$

Here we consider $E^*_\infty$ as a complex with zero differential, hence the second differential of $C$ is zero. We have two associated spectral sequences:

$$\begin{aligned}E_1^{pq}(C) &= \mathrm{Hom}_{{}_k\mathcal{A}}(P_p,\rho)\otimes E^q_\infty \Rightarrow (\mathrm{Ext}_{{}_k\mathcal{A}}(\pi,\rho)\otimes E_\infty)^{p+q}\;,\\ E_1^{pq}(D) &= \mathrm{Ext}^q_{k[\mathcal{A}]}(P_p,\rho) \Rightarrow \mathrm{Ext}^{p+q}_{k[\mathcal{A}]}(\pi,\rho)\;.\end{aligned}$$

For all even integers $q$, we choose a cycle $z'_q$ representing $e'_q$ in the complex $\mathrm{Hom}_{k[\mathcal{A}]}(\rho,Q)$. Then the morphism of bicomplexes $\Phi^{pq}: C^{pq}\to D^{pq}$ such that $\Phi^{pq}(f\otimes e_q) = z'_q\circ f$ induces a morphism of spectral sequences $E(\Phi)$. By construction, the morphisms

$$\begin{aligned}E_1^{p,*}(\Phi) &: \mathrm{Hom}_{{}_k\mathcal{A}}(P_p,\rho)\otimes E^*_\infty \to \mathrm{Ext}^*_{k[\mathcal{A}]}(P_p,\rho)\\ \mathrm{Tot}\,(\Phi) &: \mathrm{Ext}^*_{{}_k\mathcal{A}}(\pi,\rho)\otimes E^*_\infty \to \mathrm{Ext}^*_{k[\mathcal{A}]}(\pi,\rho)\end{aligned}$$

are equal to $\Upsilon$. Thus $E_1(\Phi)$ is an isomorphism (since the morphism (32) is an isomorphism on projective objects of ${}_k\mathcal{A}$-**Mod**), which implies that $\mathrm{Tot}\,(\Phi)$ is an isomorphism. □

**Lemma 8.7.** *If theorem 8.3 holds for $\mathcal{A} = \mathbf{P}_{\mathbb{F}_p}$ then it holds for all small additive categories $\mathcal{A}$ which are $\mathbb{F}_p$-linear.*

*Proof.* By lemma 8.6, it suffices to prove that the map (32) is an isomorphism when $\pi = k\otimes_{\mathbb{Z}}\mathcal{A}(a,-)$.

Let $\aleph$ be a cardinal larger than the cardinal of $\mathcal{A}(x,y)$ for all $x$ and $y$, and let $\mathcal{A}^{\aleph}$ be the $\aleph$-additivization of the $\mathbb{F}_p$-category $\mathcal{A}$, as in definition 2.24. Let $\pi' = k\otimes_{\mathbb{Z}}\mathcal{A}^{\aleph}(a,-)$ and let $\rho' : \mathcal{A}^{\aleph}\to k$-**Mod** be an arbitrary extension of $\rho$. We have a commutative diagram in which the vertical arrows are induced by restriction along the inclusions $\mathcal{A} \hookrightarrow \mathcal{A}^{\aleph}$:

$$\begin{array}{ccc} \mathrm{Ext}^*_{{}_k\mathcal{A}^{\aleph}}(\pi',\rho') \otimes E^*_\infty & \xrightarrow{\Upsilon} & \mathrm{Ext}^*_{k[\mathcal{A}^{\aleph}]}(\pi',\rho') \\ \downarrow\simeq & & \downarrow\simeq \\ \mathrm{Ext}^*_{{}_k\mathcal{A}}(\pi,\rho) \otimes E^*_\infty & \xrightarrow{\Upsilon} & \mathrm{Ext}^*_{k[\mathcal{A}]}(\pi,\rho) \end{array} \quad . \tag{35}$$

The explicit formula for Kan extensions given in proposition 2.28 shows that $\pi'$ is the left Kan extension of $\pi$ (regarded as an object of ${}_k\mathcal{A}$-**Mod** or as an object of $k[\mathcal{A}]$-**Mod**), hence the vertical arrows are isomorphisms. Thus it suffices to check that the upper $\Upsilon$ is an isomorphism.

Proposition 2.27 gives an adjoint pair $\mathcal{A}^{\aleph}(a,-) : \mathcal{A}^{\aleph} \leftrightarrows \mathbf{P}^{\aleph}_{\mathbb{F}_p} : a \otimes -$. One can write $\pi'$ as the composition of the functor $\mathcal{A}^{\aleph}(a,-)$ with the functor $I' : \mathbf{P}^{\aleph}_{\mathbb{F}_p} \to \mathbf{P}^{\aleph}_k$ such that $I'(v) = k \otimes_{\mathbb{F}_p} v$. Hence we have adjunction isomorphisms

$$\mathrm{Ext}^*_{{}_k\mathcal{A}^{\aleph}}(\pi',\rho') \simeq \mathrm{Ext}^*_{\mathbf{P}^{\aleph}_{\mathbb{F}_p}}(I',\rho'(a\otimes -)) \;,$$
$$\mathrm{Ext}^*_{k[\mathcal{A}^{\aleph}]}(\pi',\rho') \simeq \mathrm{Ext}^*_{k[\mathbf{P}^{\aleph}_{\mathbb{F}_p}]}(I',\rho'(a\otimes -)) \;.$$

These adjunction isomorphisms are given by evaluation on $a \otimes -$ and restriction along the unit of adjunction $v \to \mathcal{A}^{\aleph}(a, a\otimes v)$ (see the beginning of section 2.4). Thus they fit into a commutative square:

$$\begin{array}{ccc} \mathrm{Ext}^*_{{}_k\mathcal{A}^{\aleph}}(\pi',\rho') \otimes E^*_\infty & \xrightarrow{\Upsilon} & \mathrm{Ext}^*_{k[\mathcal{A}^{\aleph}]}(\pi',\rho') \\ \downarrow\simeq & & \downarrow\simeq \\ \mathrm{Ext}^*_{{}_k\mathbf{P}^{\aleph}_{\mathbb{F}_p}}(I',\rho'(a\otimes -)) \otimes E^*_\infty & \xrightarrow{\Upsilon} & \mathrm{Ext}^*_{k[\mathbf{P}^{\aleph}_{\mathbb{F}_p}]}(I',\rho'(a\otimes -)) \end{array} \quad . \tag{36}$$

Thus in order to prove lemma 8.7, it suffices to prove that the lower $\Upsilon$ in diagram (36) is an isomorphism.

Finally, we observe that $I' = k \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p,-) = {}_k(\mathbf{P}^{\aleph}_{\mathbb{F}_p})(\mathbb{F}_p,-)$, hence diagram (35) with $\mathcal{A}$, $\pi'$, $\rho'$ respectively taken as $\mathbf{P}_{\mathbb{F}_p}$, $I'$, $\rho'(a\otimes -)$, shows that the lower $\Upsilon$ in diagram (36) is an isomorphism as soon as we know that theorem 8.3 holds for the additive category $\mathcal{A} = \mathbf{P}_{\mathbb{F}_p}$, whence the result. □

*Proof of theorem 8.3.* By lemma 8.7 it suffices to prove theorem 8.3 when $\mathcal{A} = \mathbf{P}_{\mathbb{F}_p}$. The Eilenberg-Watts theorem yields and equivalence of categories

$$k\text{-}\mathbf{Mod} \simeq {}_k(\mathbf{P}_{\mathbb{F}_p})\text{-}\mathbf{Mod}$$

which sends a $k$-vector space $u$ to the functor $v \mapsto v \otimes_{\mathbb{F}_p} u$. In particular, the category ${}_k(\mathbf{P}_{\mathbb{F}_p})$-**Mod** is semi-simple, with only simple object the functor $t(v) = v \otimes_{\mathbb{F}_p} k$. Since both the source and the target of morphism (32) preserve finite direct sum when they are viewed as a functor of $\pi$, we reduce ourselves further to the case $\pi = t$. By using the fact that the source and the target of the morphism (32) turn direct sums into products when they are viewed as functors of $\pi$, and that they preserve products when they are viewed as functors of $\rho$, we reduce ourselves to proving that (32) is an isomorphism when $\pi = \rho = t$.

Since the functor $t$ is representable by $\mathbb{F}_p$, the Yoneda lemma shows that the $k$-vector space $\mathrm{Hom}_{{}_k(\mathbf{P}_{\mathbb{F}_p})}(t,t)$ has dimension 1, with basis the identity morphism of $t$. Thus, if we come back to the definition of $\Upsilon$, we see that when $\pi = \rho = t$, it coincides with the morphism induced by precomposition by $t$:

$$k \otimes \mathrm{Ext}^*_{\mathrm{gen}}(I, I) \simeq \mathrm{Ext}^*_{\mathrm{gen}}(I, I) \xrightarrow{t^*} \mathrm{Ext}^*_{k[\mathbf{P}_{\mathbb{F}_p}]}(t^*I, t^*I) = \mathrm{Ext}^*_{k[\mathbf{P}_{\mathbb{F}_p}]}(t, t) .$$

But the latter is an isomorphism by the strong comparison theorem 7.7. This concludes the proof of theorem 8.3. □

## 9. An auxiliary comparison map

Throughout this section $k$ is a commutative ring, $\mathbb{F}$ is a field, $\mathcal{A}$ is a small additive category, and we consider functors

$$F, G : \mathbf{P}_{\mathbb{F}} \to k\text{-}\mathbf{Mod} , \quad \pi : \mathcal{A}^{\mathrm{op}} \to \mathbb{F}\text{-}\mathbf{Mod} , \quad \rho : \mathcal{A} \to \mathbb{F}\text{-}\mathbf{Mod} ,$$

with $\rho$ and $\pi$ additive. In particular $\rho$ and $\pi$ may be considered as objects of the $\mathbb{F}$-categories $\mathbf{Mod}\text{-}_{\mathbb{F}}\mathcal{A}$ and $_{\mathbb{F}}\mathcal{A}\text{-}\mathbf{Mod}$ respectively, and for all vector spaces $v$ over $\mathbb{F}$, we define a dual vector space $D_{\pi,\rho}(v)$ by

$$D_{\pi,\rho}(v) = \mathrm{Hom}_{\mathbb{F}}(v, \pi \otimes_{_{\mathbb{F}}\mathcal{A}} \rho) .$$

Recall from definition 6.1 that a notation such as $\pi^*F$ refers to the composition $\overline{F} \circ \pi$, where $\overline{F}$ is the left Kan extension of $F$ to all $\mathbb{F}$-vector spaces. The purpose of this section is to introduce a comparison map:

$$\Theta_{\mathbb{F}} : \mathrm{Tor}^{k[\mathcal{A}]}_*(\pi^*F, \rho^*G) \to \mathrm{Tor}^{k[\mathbf{P}_{\mathbb{F}}]}_*(D^*_{\pi,\rho}F, G) .$$

and to establish its main properties. We will use these properties (in the special case where $\mathbb{F} = \mathbb{F}_q$ and $k$ is an overfield of $\mathbb{F}_q$) in the proof of the generalized comparison theorem in section 10.

In contrast with the other sections of the article, many constructions of this section are performed over the ground field $\mathbb{F}$ (that is, we use $\mathbb{F}$-linear categories, tensor products over $\mathbb{F}$…) rather than over $k$.

9.1. **Construction of $\Theta_{\mathbb{F}}$.** Taking $\mathcal{K} = {}_{\mathbb{F}}\mathcal{A}^{\mathrm{op}}$ in the isomorphism (5) of section 2.2 yields a pair of adjoint $\mathbb{F}$-functors $- \otimes_{_{\mathbb{F}}\mathcal{A}} \rho : \mathbf{Mod}\text{-}_{\mathbb{F}}\mathcal{A} \leftrightarrows \mathbb{F}\text{-}\mathbf{Mod} : \mathrm{Hom}_{\mathbb{F}}(\rho, -)$. We denote by $\theta_{\mathbb{F}}$ the unit of adjunction:

$$\theta_{\mathbb{F}} : \pi \to \mathrm{Hom}_{\mathbb{F}}(\rho, \rho \otimes_{_{\mathbb{F}}\mathcal{A}} \pi) = D_{\pi,\rho} \circ \rho . \tag{37}$$

Thus $(\theta_{\mathbb{F}})_a$ sends an element $x \in \pi(a)$ to the $\mathbb{F}$-linear map $y \mapsto [\![x \otimes y]\!]$ where $y \in \rho(a)$ and the brackets denote the image of the tensor in $\rho \otimes_{_{\mathbb{F}}\mathcal{A}} \pi$. We choose a cardinal $\aleph$ such that the images of $\rho$ and $\pi$ are contained in the category $\mathbf{P}^{\aleph}_{\mathbb{F}}$ of vector spaces of dimension less or equal to $\aleph$, and we let $\iota^{\aleph} : \mathbf{P}^{\aleph}_{\mathbb{F}} \hookrightarrow \mathbb{F}\text{-}\mathbf{Mod}$ be the inclusion of categories. We define $\Theta_{\mathbb{F}}$ as the unique graded $k$-linear map fitting into the commutative square (note that $\mathrm{res}^{\iota^{\aleph}}$ is an isomorphism by proposition 2.28):

$$\begin{array}{ccc} \mathrm{Tor}^{k[\mathcal{A}]}_*(\overline{F} \circ \pi, \overline{G} \circ \rho) & \xrightarrow{\Theta_{\mathbb{F}}} & \mathrm{Tor}^{k[\mathbf{P}_{\mathbb{F}}]}_*(\overline{F} \circ D_{\pi,\rho}, G) \\ {\scriptstyle \mathrm{Tor}^{k[\mathcal{A}]}_*(F(\theta_{\mathbb{F}}), \overline{G}\circ\rho)} \downarrow & & \simeq \downarrow {\scriptstyle \mathrm{res}^{\iota^{\aleph}}} \\ \mathrm{Tor}^{k[\mathcal{A}]}_*(\overline{F} \circ D_{\pi,\rho} \circ \rho, \overline{G} \circ \rho) & \xrightarrow{\mathrm{res}^{\rho}} & \mathrm{Tor}^{k[\mathbf{P}^{\aleph}_{\mathbb{F}}]}_*(\overline{F} \circ D_{\pi,\rho}, \overline{G}) \end{array} . \tag{38}$$

**Lemma 9.1.** *The map $\Theta_{\mathbb{F}}$ does not depend on the choice of $\aleph$.*

*Proof.* This is a consequence of the fact that for a cardinal $\beth$ greater than $\aleph$ we have a commutative diagram (where $\iota^{\aleph,\beth}$ is the inclusion of $\mathbf{P}_{\mathbb{F}}^{\aleph}$ into $\mathbf{P}_{\mathbb{F}}^{\beth}$):

$$\begin{array}{ccc}
 & & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}]}(\overline{F}\circ D_{\pi,\rho}F, G) \\
 & & \simeq\big\downarrow \mathrm{res}^{\iota^{\aleph}} \quad \searrow \mathrm{res}^{\iota^{\beth}}\ \simeq \\
\mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}\circ D_{\pi,\rho}\circ\pi, \overline{G}\circ\rho) & \xrightarrow{\mathrm{res}^{\rho}} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}^{\aleph}]}(\overline{F}\circ D_{\pi,\rho}, \overline{G}) \\
 & \searrow^{\mathrm{res}^{\rho}} & \big\downarrow \mathrm{res}^{\iota^{\aleph,\beth}} \\
 & & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}^{\beth}]}(\overline{F}\circ D_{\pi,\rho}, \overline{G})
\end{array}\;.$$

□

**Lemma 9.2.** *The map $\Theta_{\mathbb{F}}$ is natural with respect to $F$, $G$, $\pi$ and $\rho$.*

*Proof.* It is equivalent to prove the naturality of $\mathrm{res}^{\iota^{\aleph}}\circ\Theta_k$ with respect to $F$, $G$, $\pi$ and $\rho$. Naturality with respect to $F$, $G$ and $\pi$ is a straightforward verification. We check naturality with respect to $\rho$, which is less straightforward since $\theta_{\mathbb{F}}$ is not natural with respect to $\rho$. Let $f:\rho\to\rho'$ be a natural transformation, and let $D_f : D := D_{\pi,\rho}\to D' := D_{\pi,\rho'}$ be the natural transformation induced by $f$. We consider the following diagram of graded $k$-modules, in which the composition operator for functors is omitted, e.g. '$\overline{F}\pi$' means $\overline{F}\circ\pi$, and the arrows are labelled by the natural transformations which induce them.

$$\begin{array}{ccccc}
\mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}\pi,\overline{G}\rho) & & \xrightarrow{\quad\overline{G}f\quad} & & \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}\pi, G\rho') \\
\big\downarrow F\theta_{\mathbb{F}} & \searrow^{\overline{F}\theta_{\mathbb{F}}} & & & \big\downarrow \overline{F}\theta_{\mathbb{F}} \\
\mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}D\rho,\overline{G}\rho) & & \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}D'\rho',\overline{G}\rho) & \searrow^{\overline{G}f} & \\
 & \searrow_{\overline{F}D_f\rho} & \big\downarrow \overline{F}D'f & & \\
\big\downarrow \mathrm{res}^{\rho} & & \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}D'\rho,\overline{G}\rho) & & \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}D'\rho',\overline{G}\rho') \\
 & & & \searrow_{\mathrm{res}^{\rho}} & \big\downarrow \mathrm{res}^{\rho'} \\
\mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}^{\aleph}]}(\overline{F}D,\overline{G}) & & \xrightarrow{\quad\overline{F}D_f\quad} & & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}^{\aleph}]}(\overline{F}D',\overline{G})
\end{array}$$

The upper right triangle and the lower left triangle of the diagram are obviously commutative. The upper left parallelogram is commutative because of the dinaturality of $\theta_{\mathbb{F}}$, i.e. because the following square commutes:

$$\begin{array}{ccc}
\pi & \xrightarrow{\quad\theta_{\mathbb{F}}\quad} & D'\rho' = \mathrm{Hom}_{\mathbb{F}}(\rho',\pi\otimes_{\mathbb{F}\mathcal{A}}\rho') \\
\big\downarrow \theta_{\mathbb{F}} & & \big\downarrow \mathrm{Hom}_{\mathbb{F}}(f,\pi\otimes_{\mathbb{F}\mathcal{A}}\rho') \\
D\rho = \mathrm{Hom}_{\mathbb{F}}(\rho,\pi\otimes_{\mathbb{F}\mathcal{A}}\rho) & \xrightarrow{\mathrm{Hom}_{\mathbb{F}}(\rho,\pi\otimes_{\mathbb{F}\mathcal{A}}f)} & D'\rho = \mathrm{Hom}_{\mathbb{F}}(\rho,\pi\otimes_{\mathbb{F}\mathcal{A}}\rho')
\end{array}\;.$$

Finally, the lower right parallelogram commutes by dinaturality of restriction maps between Tor-modules (which comes from the fact that tensor products are defined by a coend formula). Thus the outer square is commutative, which shows that $\mathrm{res}^{\iota^{\aleph}}\circ\Theta_{\mathbb{F}}$, hence $\Theta_{\mathbb{F}}$, is natural with respect to $\pi$. □

9.2. **Base change.** We fix a field morphism $\mathbb{F} \to \mathbb{K}$, and we let $t : \mathbb{F}\text{-}\mathbf{Mod} \to \mathbb{K}\text{-}\mathbf{Mod}$ denote the extension of scalars $t(v) = \mathbb{K} \otimes_{\mathbb{F}} v$. The canonical isomorphisms $t(\pi(a)) \otimes_{\mathbb{K}} t(\rho(a)) \simeq t(\pi(a) \otimes_{\mathbb{F}} \rho(a))$ induce a canonical isomorphism:

$$t(\pi \otimes_{\mathbb{F}\mathcal{A}} \rho) \simeq (t \circ \pi) \otimes_{\mathbb{K}\mathcal{A}} (t \circ \rho)\ . \tag{39}$$

Thus, extension of scalars induces a $\mathbb{K}$-linear morphism:

$$\begin{aligned} \mathbb{K} \otimes_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}}(v, \pi \otimes_{\mathbb{F}\mathcal{A}} \rho) \xrightarrow{(f\otimes\lambda\mapsto f\otimes\lambda)} & \mathrm{Hom}_{\mathbb{K}}(t(v), t(\pi \otimes_{\mathbb{F}\mathcal{A}} \rho)) \\ & \simeq \mathrm{Hom}_{\mathbb{K}}(t(v), (t \circ \pi) \otimes_{\mathbb{K}\mathcal{A}} (t \circ \rho)) \end{aligned} \tag{40}$$

which is an isomorphism when $v$ has finite dimension. If we let $D_{\pi,\rho}$ and $D_{t\circ\pi,t\circ\rho}$ be the duality functors respectively defined by:

$$\begin{aligned} D_{\pi,\rho}(v) &= \mathrm{Hom}_{\mathbb{F}}(v, \pi \otimes_{\mathbb{F}\mathcal{A}} \rho) \\ D_{t\circ\pi,t\circ\rho}(w) &= \mathrm{Hom}_{\mathbb{K}}(w, (t \circ \pi) \otimes_{\mathbb{K}\mathcal{A}} (t \circ \rho)) \end{aligned}$$

then the morphism (40) can be written as a canonical morphism of functors

$$t \circ D_{\pi,\rho} \xrightarrow{\mathrm{can}} D_{t\circ\pi,t\circ\rho} \circ t \tag{41}$$

whose component at every finite-dimensional $\mathbb{F}$-vector space $v$ is an isomorphism.

**Proposition 9.3.** *Let $\mathbb{F} \to \mathbb{K}$ be a field morphism. For all additive functors $\pi : \mathcal{A}^{\mathrm{op}} \to \mathbb{F}\text{-}\mathbf{Mod}$ and $\rho : \mathcal{A} \to \mathbb{F}\text{-}\mathbf{Mod}$, and for all objects $F$ and $G$ in $k[\mathbf{P}_{\mathbb{K}}]\text{-}\mathbf{Mod}$, we have a commutative diagram in which the lower horizontal isomorphism is induced by the isomorphism $F(\mathrm{can})$:*

$$\begin{array}{ccc} \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F} \circ t \circ \pi, \overline{G} \circ t \circ \rho) & \xrightarrow{\Theta_{\mathbb{K}}} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{K}}]}(\overline{F} \circ D_{t\circ\pi,t\circ\rho}, G) \\ \downarrow \Theta_{\mathbb{F}} & & \mathrm{res}^t \uparrow \\ \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}]}(\overline{F} \circ t \circ D_{\pi,\rho}, G \circ t) & \xrightarrow[\simeq]{} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}]}(\overline{F} \circ D_{t\circ\pi,t\circ\rho} \circ t, G \circ t) \end{array} .$$

*Proof.* Let us denote $D = D_{t\circ\pi,t\circ\rho}$ and $D' = D_{\pi,\rho}$ for short and let $\aleph$ be a big enough cardinal. We have a diagram of graded $k$-modules, in which the composition symbol for functors has been omitted and the arrows are labelled by the name of the morphisms which induce them.

$$\begin{array}{ccccc} \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}t\pi, \overline{G}t\rho) & \xrightarrow{\overline{F}\theta_{\mathbb{K}}} & \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}Dt\rho, \overline{G}t\rho) & \xrightarrow{\mathrm{res}^{t\rho}} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{K}}^{\aleph}]}(\overline{F}D, \overline{G}) \\ \downarrow \overline{F}t\theta_{\mathbb{F}} & \nearrow^{\overline{F}\mathrm{can}\rho} & \downarrow \mathrm{res}^{\rho} & \nearrow^{\mathrm{res}^t} & \simeq \uparrow \mathrm{res} \\ \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}tD'\rho, \overline{G}t\rho) & & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}^{\aleph}]}(\overline{F}Dt, \overline{G}t) & & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{K}}]}(\overline{F}D, \overline{G}) \\ \downarrow \mathrm{res}^{\rho} & \nearrow^{\overline{F}\mathrm{can}} & & \nwarrow^{\mathrm{res}}_{\simeq} & \mathrm{res}^t \uparrow \\ \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}^{\aleph}]}(\overline{F}tD', \overline{G}t) & \xleftarrow[\mathrm{res}]{\simeq} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}]}(\overline{F}tD', \overline{G}t) & \xrightarrow{\overline{F}\mathrm{can}} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}^{\aleph}]}(\overline{F}Dt, \overline{G}t) \end{array}$$

One readily checks from the explicit expressions of $\theta_{\mathbb{K}}$, $\theta_{\mathbb{F}}$ and of the canonical morphism $\mathrm{can} : tD' \to Dt$ that $\theta_{\mathbb{K}} = (\mathrm{can}\,\rho) \circ (t\theta_{\mathbb{F}})$, hence the upper left triangle of the diagram commutes. The other cells of the diagram obviously commute. The commutativity of the outer square proves proposition 9.3. □

9.3. **Isomorphism conditions.** We now investigate some conditions which ensure that our comparison map $\Theta_{\mathbb{F}}$ is an isomorphism. The next proposition provides the base case.

**Proposition 9.4.** *If $\mathcal{A}$ is $\mathbb{F}$-linear and if $\pi = \mathcal{A}(-,a)$ and $\rho = \mathcal{A}(b,-)$, then $\Theta_{\mathbb{F}}$ is an isomorphism.*

*Proof.* By lemma 9.1, we may assume $\aleph$ as big as we want in the definition of $\Theta_{\mathbb{F}}$, so that the functor $\rho^{\aleph} := \mathcal{A}^{\aleph}(b,-) : \mathcal{A}^{\aleph} \to \mathbf{P}_{\mathbb{F}}^{\aleph}$ has a left adjoint $\tau := b \otimes_{\mathbb{F}} -$ by proposition 2.27. We also let $\pi^{\aleph} := \mathcal{A}^{\aleph}(-,a) : \mathcal{A}^{\aleph} \to \mathbb{F}\text{-}\mathbf{Mod}$.

Let us first reinterpret $\theta_{\mathbb{F}}$ in the situation of proposition 9.4. We have an isomorphism

$$\phi : \pi \otimes_{\mathbb{F}\mathcal{A}} \rho \xrightarrow{\simeq} \mathcal{A}(b,a)$$

which sends the class of $f \otimes g \in \mathcal{A}(x,a) \otimes_{\mathbb{F}} \mathcal{A}(b,x)$ to $f \circ g \in \mathcal{A}(b,a)$. (The inverse of $\phi$ sends an element $f \in \mathcal{A}(a,b)$ to the class of $\mathrm{id}_a \otimes f \in \mathcal{A}(a,a) \otimes_{\mathbb{F}} \mathcal{A}(b,a)$). From the explicit expressions of $\theta_{\mathbb{F}}$ and $\phi$, one sees that the lower left triangle of the following diagram commutes.

$$\begin{array}{ccc} \mathcal{A}(x,a) & \xrightarrow{\mathcal{A}(\epsilon_x,a)} & \mathcal{A}^{\aleph}(b \otimes_{\mathbb{F}} \mathcal{A}(b,x), a) \\ {\scriptstyle (\theta_{\mathbb{F}})_x}\downarrow & \searrow^{\mathcal{A}(b,-)} & \simeq\downarrow{\scriptstyle \alpha} \\ \mathrm{Hom}_{\mathbb{F}}(\mathcal{A}(b,x), \pi \otimes_{\mathbb{F}\mathcal{A}} \rho) & \xrightarrow[\mathrm{Hom}_{\mathbb{F}}(\mathcal{A}(b,x),\phi)]{\simeq} & \mathrm{Hom}_{\mathbb{F}}(\mathcal{A}(b,x), \mathcal{A}(b,a)) \end{array} \tag{42}$$

The upper right triangle of diagram (42) also commutes: here $\alpha$ is an adjunction isomorphism for the adjunction between $\tau$ and $\rho^{\aleph}$, and $\epsilon_x$ is the associated counit of adjunction. Diagram (42) is our new interpretation of $\theta_{\mathbb{F}}$.

Next we prove that $\mathrm{res}^{\iota^{\aleph}} \circ \Theta_{\mathbb{F}}$ is an isomorphism. We let $D := D_{\pi,\rho}$ for short, and we let $\chi : D \simeq \pi^{\aleph} \circ \tau$ be the isomorphism whose component at $v$ is given by the composition:

$$\mathrm{Hom}_{\mathbb{F}}(v, \pi \otimes_{\mathbb{F}\mathcal{A}} \rho) \xrightarrow[\simeq]{\mathrm{Hom}_{\mathbb{F}}(v,\phi)} \mathrm{Hom}_{\mathbb{F}}(v, \mathcal{A}(a,b)) \xrightarrow[\simeq]{\alpha^{-1}} \mathcal{A}^{\aleph}(b \otimes_{\mathbb{F}} v, a)\ .$$

We consider the following diagram of graded $k$-modules, in which the composition operator for functors is omitted and the arrows are labelled by the natural transformations which induce them. The vertical maps $\mathrm{res}^j$ are induced by restriction along the canonical inclusion $j : \mathcal{A} \hookrightarrow \mathcal{A}^{\aleph}$.

$$\begin{array}{ccccc} \mathrm{Tor}_*^{k[\mathcal{A}^{\aleph}]}(\overline{F}\pi^{\aleph}, \overline{G}\rho^{\aleph}) & \xrightarrow{\overline{F}\pi^{\aleph}\epsilon} & \mathrm{Tor}_*^{k[\mathcal{A}^{\aleph}]}(\overline{F}\pi^{\aleph}\tau\rho^{\aleph}, \overline{G}\rho^{\aleph}) & \xrightarrow{\mathrm{res}^{\rho}} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}^{\aleph}]}(\overline{F}\pi^{\aleph}\tau, \overline{G}) \\ \simeq\uparrow{\scriptstyle \mathrm{res}^j} & & \simeq\uparrow{\scriptstyle \mathrm{res}^j} & & \| \\ \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}\pi^{\aleph}, \overline{G}\rho^{\aleph}) & \xrightarrow{\overline{F}\pi^{\aleph}\epsilon} & \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}\pi^{\aleph}\tau\rho, \overline{G}\rho) & \xrightarrow{\mathrm{res}^{\rho}} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}^{\aleph}]}(\overline{F}\pi^{\aleph}\tau, \overline{G}) \\ \| & & \simeq\uparrow{\scriptstyle \overline{F}\chi\rho} & & \simeq\uparrow{\scriptstyle \overline{F}\chi} \\ \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}\pi, \overline{G}\rho) & \xrightarrow{\overline{F}\theta_{\mathbb{F}}} & \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}D\rho, \overline{G}\rho) & \xrightarrow{\mathrm{res}^{\rho}} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}^{\aleph}]}(\overline{F}D, \overline{G}) \end{array}$$

All the squares of the diagram are obviously commutative, but the lower left square which commutes by commutativity of diagram (42). The maps $\mathrm{res}^j$ are isomorphisms by proposition 2.28 because $\overline{G}\rho^{\aleph}$ is the left Kan extension of $\overline{G}\rho$ along $j$. (To see this, use that $\rho^{\aleph} = \mathcal{A}^{\aleph}(b,-) : \mathcal{A}^{\aleph} \to \mathbb{F}\text{-}\mathbf{Mod}$ is $\aleph$-additive by proposition 2.25, hence it is the left Kan extension of $\rho$ and $\overline{G}$ is already a left Kan extension.)

The composite in the top row is the Tor-map induced by the adjunction between $\tau$ and $\rho^{\aleph}$, hence it is an isomorphism by proposition 2.18. We deduce that the composite in the bottom row, which is nothing but $\mathrm{res}^{\iota^{\aleph}} \circ \Theta_{\mathbb{F}}$, is an isomorphism. Hence $\Theta_{\mathbb{F}}$ is an isomorphism. □

**Corollary 9.5.** *If $\mathcal{A}$ is $\mathbb{F}$-linear, and if*

$$\pi = \bigoplus_{i\in I} \mathcal{A}(-,a_i)\ , \qquad \rho = \bigoplus_{j\in J} \mathcal{A}(b_j,-)$$

*for some possibly infinite indexing sets $I$ and $J$, then $\Theta_{\mathbb{F}}$ is an isomorphism.*

*Proof.* If $I$ and $J$ are finite, then $\pi \simeq \mathcal{A}(-,a)$ and $\rho \simeq \mathcal{A}(b,-)$ for $a = \bigoplus a_i$ and $b = \bigoplus b_j$, hence $\Theta_{\mathbb{F}}$ is an isomorphism by proposition 9.4. For arbitrary $I$ and $J$, the functors $\pi$ and $\rho$ are filtered colimits of monomorphisms of functors of the form $\mathcal{A}(-,a)$ and $\mathcal{A}(b,-)$. So the result follows from the fact that the target and the source of $\Theta_{\mathbb{F}}$ both preserve filtered colimits of monomorphisms of functors, when viewed as functors of the variables $\pi$ and $\rho$. (Indeed $\mathrm{Tor}_*$, $\overline{F}\circ\pi$, $\overline{G}\circ\rho$ and $\overline{F}\circ D_{\pi,\rho}$ preserve filtered colimits of monomorphisms – for $\overline{F}\circ\pi$, $\overline{G}\circ\rho$, this follows from the fact that $\overline{F}$ and $\overline{G}$ are left Kan extensions of $F$ and $G$, and for $\overline{F}\circ D_{\pi,\rho}$, one uses in addition the isomorphism $D_{\pi,\rho}(v) \simeq \mathrm{Hom}_{\mathbb{F}}(v,\mathbb{F}) \otimes_{\mathbb{F}} (\pi \otimes_{\mathbb{F}\mathcal{A}} \rho)$, which holds because we view $D_{\pi,\rho}$ as a functor from $\mathbf{P}_{\mathbb{F}}$ to $\mathbb{F}$-**Mod**.) □

We are going to extend corollary 9.5 to more general $\mathbb{F}$-linear functors $\pi$ and $\rho$ by taking simplicial resolutions. We refer the reader to section 3 for recollections of simplicial techniques.

If $X$ is a simplicial object in $k[\mathbf{P}_{\mathbb{F}}]$-**Mod**, and $\mu$ is a simplicial object in the category of additive functors $\mathcal{A} \to \mathbb{F}$-**Mod**, we let $\mu^*X$ be the diagonal simplicial object $\overline{X}_n \circ \mu_n$. Thus $\mu^*X$ is a simplicial object in $k[\mathcal{A}]$-**Mod** natural with respect to $\mu$ and $X$. The next two lemmas are our main tools to contruct convenient simplicial resolutions.

**Lemma 9.6.** *If $X_1 \to X_2$ and $\mu_1 \to \mu_2$ are e-connected morphisms, the induced morphism $\mu_1^*X_1 \to \mu_2^*X_2$ is e-connected.*

*Proof.* We have to show that for all $a$ in $\mathcal{A}$ the morphism of simplicial $k$-modules $\overline{X_1}(\mu_1(a)) \to \overline{X_2}(\mu_2(a))$ is $e$-connected. The morphism $\overline{X_1} \to \overline{X_2}$ is $e$-connected because it is defined as a filtered colimit and homotopy groups commute with filtered colimits. Hence the result follows from proposition 3.8. □

**Lemma 9.7** (linearization of projective additive functors)**.** *Let $\rho : \mathcal{A} \to \mathbb{F}$-**Mod** be an additive functor, and let $P$ be a projective object in $k[\mathbf{P}_{\mathbb{F}}]$-**Mod**. If $\rho$ is projective as an* additive *functor from $\mathcal{A}$ to* abelian groups*, then $\rho^*P$ is a projective over $k[\mathcal{A}]$.*

*Proof.* It suffices to prove the result when $P = k[\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^n,-)]$. As $\rho$, seen as an object of $\mathcal{A}$-**Mod**, is a direct summand of a direct sum of representable functors, it is enough to see that $k[\bigoplus_{i\in E}\mathcal{A}(a_i,-)]$ is projective over $k[\mathcal{A}]$ for every family $(a_i)_{i\in E}$ of objects of $\mathcal{A}$. The result follows from the cross-effect type decomposition

$$k[\bigoplus_{i\in E}\mathcal{A}(a_i,-)] \simeq \bigoplus_{I\in\mathcal{P}_f(E)} \bigotimes_{i\in I} k[\mathcal{A}(a_i,-)]^{\mathrm{red}}$$

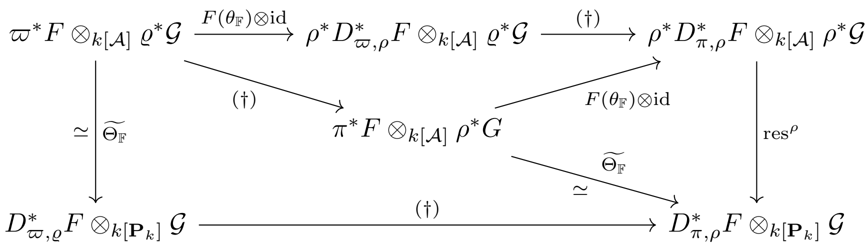
where $^{\mathrm{red}}$ refers to the reduced part of a functor (see lemma 4.9 in section 4) and $\mathcal{P}_f(E)$ denotes the set of finite subsets of $E$, as each functor $\bigotimes_{i\in I} k[\mathcal{A}(a_i,-)]^{\mathrm{red}}$ is a direct summand of the projective functor $k[\mathcal{A}(\bigoplus_{i\in I} a_i,-)]$. □

**Theorem 9.8.** *Let $k$ be a commutative ring and let $\mathcal{A}$ be a small additive $\mathbb{F}$-category for a field $\mathbb{F}$. Assume that $\pi$ and $\rho$ are $\mathbb{F}$-linear, consider them as objects of the $\mathbb{F}$-categories* **Mod**-$\mathcal{A}$ *and* $\mathcal{A}$-**Mod**, *and let $e$ be a positive integer such that* $\mathrm{Tor}_i^{\mathcal{A}}(\pi,\rho) = 0$ *for* $0 < i < e$. *Then for all objects $F$, $G$ of* $k[\mathbf{P}_{\mathbb{F}}]$-**Mod**, *the map*

$$\Theta_{\mathbb{F}} : \mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^*F, \rho^*G) \to \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}]}(D^*_{\pi,\rho}F, G)$$

*defined by diagram* (38) *is $e$-connected.*

*Proof.* Let $\mathcal{G} \to G$, $\varpi \to \pi$ and $\varrho \to \rho$ be simplicial resolutions by direct sums of standard projectives in the categories $k[\mathbf{P}_{\mathbb{F}}]$-**Mod**, **Mod**-$\mathcal{A}$ and $\mathcal{A}$-**Mod** respectively. We have a commutative diagram of simplicial $k$-modules, in which the maps (†) are induced by the morphisms $\varpi \to \pi$ and $\varrho \to \rho$, and the maps $\widetilde{\Theta_{\mathbb{F}}}$ are degree-wise equal to the degree zero component of $\Theta_{\mathbb{F}}$, hence they are isomorphisms by corollary 9.5:




The homotopy groups of $D^*_{\pi,\rho}F \otimes_{k[\mathbf{P}_{\mathbb{F}}]} \mathcal{G}$ compute $\mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}}]}(D^*_{\pi,\rho}F, G)$ because $\mathcal{G} \to G$ is a projective simplicial resolution. The homotopy groups of $\varpi^*F \otimes_{k[\mathcal{A}]} \varrho^*\mathcal{G}$ compute $\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^*F, \rho^*G)$ because $\varrho^*\mathcal{G} \to \rho^*G$ is a simplicial projective resolution by lemmas 9.6 and 9.7. And the map induced on the level of homotopy groups by the top right corner of the diagram is $\Theta_{\mathbb{F}}$. Thus, to prove the theorem, it remains to prove that the bottom horizontal map (†) is $e$-connected.

In order to do this, we first observe that the $\mathbb{F}$-category $\mathcal{A}$-**Mod** of $\mathbb{F}$-functors from $\mathcal{A}$ to $\mathbb{F}$-**Mod** is a full subcategory of the $\mathbb{F}$-category ${}_{\mathbb{F}}\mathcal{A}$-**Mod** of additive functors from $\mathcal{A}$ to $\mathbb{F}$-**Mod**. Hence for all $\mathbb{F}$-functors $\pi$ and $\rho$, restriction along the functor ${}_{\mathbb{F}}\mathcal{A} \to \mathcal{A}$, $f \otimes \lambda \mapsto \lambda f$ induces an isomorphism $\pi \otimes_{{}_{\mathbb{F}}\mathcal{A}} \rho \simeq \pi \otimes_{\mathcal{A}} \rho$ by corollary 2.15. Thus the Tor-condition in the theorem ensures that $\varpi \otimes_{\mathcal{A}} \varrho \to \pi \otimes_{\mathcal{A}} \rho$ is $e$-connected. Thus $D_{\varpi,\varrho} \to D_{\pi,\rho}$ is $e$-connected, hence $D^*_{\varpi,\varrho}F \to D^*_{\pi,\rho}F$ is $e$-connected by lemma 9.6. This implies that the bottom horizontal map is $e$-connected by a standard spectral sequence argument (use the spectral sequence of a bisimplicial $k$-module as in [19, IV section 2.2]). □

## 10. The generalized comparison theorem

Throughout this section, $k$ is an infinite perfect field of positive characteristic $p$, $\pi : \mathcal{A}^{\mathrm{op}} \to k$-**Mod** and $\rho : \mathcal{A} \to k$-**Mod** are two additive functors and $F$ and $G$ are two strict polynomial functors over $k$ (possibly non-homogeneous, cf. section 7.3). We also fix two positive integers $r$ and $s$, and for $0 \le i < s^2$ we let $T_i$ denote the $k$-vector space:

$$T_i = ({}^{(-ri)}\pi) \otimes_{{}_k\mathcal{A}} ({}^{(rs-rs^2)}\rho) \tag{43}$$

where a notation such as ${}^{(-ri)}\pi$ refers to the additive functor obtained as the composition of the functor $\pi$ and the Frobenius twist ${}^{(-ri)}-$ as in notation 8.1. We define a functor $F'$ in $\mathbf{Mod}\text{-}k[\mathbf{P}_k]$ by

$$F'(v) = \overline{F}\Big(\bigoplus_{0\le i<s^2} {}^{(ri)}\mathrm{Hom}_k(v,T_i)\Big) \tag{44}$$

where as in definition 6.1, the notation $\overline{F}$ refers to the left Kan extension of $F$ to all vector spaces. Then $F'$ is actually a (non-homogeneous) strict polynomial functor of the variable $v$. In this section, we will first construct a certain morphism of graded vector spaces, natural with respect to $F$, $G$, $\pi$ and $\rho$:

$$\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^*F,\rho^*G) \to \mathrm{Tor}_*^{\mathrm{gen}}(F', G^{(rs^2-rs)})\;. \tag{45}$$

We will refer to morphism (45) as the *generalized comparison map* in the remainder of the article. Then we will prove the following theorem, and spell out some special cases and consequences.

**Theorem 10.1** (generalized comparison)**.** *Let $k$ be an infinite perfect field of characteristic $p$, containing a finite field $\mathbb{F}_q$ of cardinal $q=p^r$. Let $\mathcal{A}$ be a small additive category, let $\pi:\mathcal{A}^{\mathrm{op}}\to k\text{-}\mathbf{Mod}$ and $\rho:\mathcal{A}\to k\text{-}\mathbf{Mod}$ be two additive functors. Assume that there are positive integers $s$ and $e$ such that*

$$\mathrm{Tor}_j^{k\mathcal{A}}\left({}^{(-ri)}\pi, {}^{(rs-rs^2)}\rho\right) = 0$$

*for $0<j<e$ and $0\le i<s^2$. Assume further that $\mathcal{A}$ is $\mathbb{F}_q$-linear and that $\rho$ and $\pi$ are $\mathbb{F}_q$-linear. Then for all strict polynomial functors $F$ and $G$ of weights less or equal to $q^s$, the generalized comparison map (45) is $e$-connected.*

*Remark* 10.2. If $\pi$ and $\rho$ are $\mathbb{F}_q$-linear, so are ${}^{(ri)}\pi$ and ${}^{(rs-rs^2)}\rho$, and we can consider them as objects of $k\otimes_{\mathbb{F}_q}\mathcal{A}\text{-}\mathbf{Mod}$ and $\mathbf{Mod}\text{-}k\otimes_{\mathbb{F}_q}\mathcal{A}$. Hence it is natural to ask about the relation between the Tor hypothesis in theorem 10.1 and the vanishing of $\mathrm{Tor}_j^{k\otimes_{\mathbb{F}_q}\mathcal{A}}({}^{(ri)}\pi, {}^{(rs-rs^2)}\rho)$. These two conditions are actually equivalent as we shall see it in lemma 10.8.

*Remark* 10.3. In theorem 10.1 the target category of $\pi$ and $\rho$ is the category $k\text{-}\mathbf{Mod}$ though these functors are $\mathbb{F}_q$-linear. This is in contrast with theorem 9.8, and this explains why the Tor hypotheses of these two theorems are different.

10.1. **Construction of the generalized comparison map.** We define the generalized comparison map (45) as the composition of three maps $\Lambda_k$, $\Theta'_k$ and $\Phi_k$:

$$\begin{array}{ccc}
\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^*F,\rho^*G) & \xrightarrow{\Lambda_k} & \mathrm{Tor}_*^{k[\mathcal{A}]}((\pi^{\oplus s^2})^*F,\rho^*G) \\
\downarrow{\scriptstyle (45)} & & \downarrow{\scriptstyle \Theta'_k} \\
\mathrm{Tor}_*^{\mathrm{gen}}(F', G^{(rs^2-rs)}) & \xleftarrow{\Phi_k} & \mathrm{Tor}_*^{k[\mathbf{P}_k]}(F', G^{(rs^2-rs)})
\end{array}\;.$$

The map $\Lambda_k$ is induced by the morphism of additive functors $\mathrm{diag}:\pi\to\pi^{\oplus s^2}$ whose components all equal the identity morphism of $\pi$.

The map $\Phi_k$ is an immediate generalization of the strong comparison map (21) over an infinite perfect field $k$. Namely, the additive functor ${}^{(n)}-:\mathbf{P}_k\to\mathbf{P}_k$ has a quasi-inverse ${}^{(-n)}-$ for all positive integers $n$. Restriction along this quasi-inverse yields an isomorphism of graded vector spaces, natural in the functors $H$ and $K$:

$$\mathrm{Tor}_*^{k[\mathbf{P}_k]}(H,K) \xrightarrow{\simeq} \mathrm{Tor}_*^{k[\mathbf{P}_k]}(H^{(n)},K^{(n)})\;,$$

where a notation such as $H^{(n)}$ denotes the precomposition of $H$ by ${}^{(n)}-$. Since $k$ is infinite, the category $\Gamma\mathbf{P}_k$-**Mod** of strict polynomial functors identifies with a full subcategory of $k[\mathbf{P}_k]$-**Mod** (see section 7.3). If $H$ and $K$ are strict polynomial functors, we let $\Phi_k$ be the unique morphism of graded $k$-vector spaces fitting in the commutative diagrams for all $i \geq 0$ and all $r \gg 0$:

$$\begin{array}{ccc} \mathrm{Tor}_i^{k[\mathbf{P}_k]}(H,K) & \overset{\Phi_k}{\dashrightarrow} & \mathrm{Tor}_i^{\mathrm{gen}}(H,K) \\ \simeq\downarrow \mathrm{res}^{(-n)}- & & \downarrow\simeq \\ \mathrm{Tor}_i^{k[\mathbf{P}_k]}(H^{(n)},K^{(n)}) & \xrightarrow{\mathrm{res}} & \mathrm{Tor}_i^{\Gamma\mathbf{P}_k}(H^{(n)},K^{(n)}) \end{array} \quad . \tag{46}$$

In order to apply $\Phi_k$ to $H = F'$ and $K = G^{(rs^2-rs)}$, we have to prove that these functors are actually strict polynomial functors. This is obvious for $H$, and for $K$ this is proved in the following lemma.

**Lemma 10.4.** *The functor $F'_{\pi,\rho}$ defined by* (44) *is a strict polynomial functor.*

*Proof.* The vector spaces $T_i$ can be written as a filtered colimit of finite-dimensional subspaces $T_{i,\alpha}$. Hence for all finite-dimensional vector spaces $v$, $F'(v)$ is the filtered colimit of $F(\bigoplus_{0\leq i<s^2} {}^{(ri)}\mathrm{Hom}_k(v, T_{i,\alpha}))$. The latter are strict polynomial functors of the variable $v$, all having the same weight as $F$. But $\bigoplus_{d\leq \mathrm{w}(F)} \Gamma^d\mathbf{P}_k$-**Mod** is a full subcategory of $k[\mathbf{P}_k]$-**Mod** stable under colimits, hence $F'$ is a strict polynomial functor. □

It remains to define $\Theta'_k$. The latter is a generalization of the comparison map $\Theta_k$ constructed in section 9. In order to avoid heavy notations, we set:

$$\pi_i := {}^{(-ri)}\pi\ , \qquad \sigma := {}^{(rs-rs^2)}\rho\ .$$

We use the morphisms $\theta_k$ defined by equation (37) in section 9 to construct a morphism $\theta'_k$ of additive functors:

$$\theta'_k : \pi^{\oplus s^2} = \bigoplus_{0\leq i<s^2} {}^{(ri)}\pi_i \xrightarrow{\bigoplus {}^{(ri)}\theta_k} \bigoplus_{0\leq i<s^2} {}^{(ri)}D_{\pi_i,\sigma}\circ\sigma\ . \tag{47}$$

Observe that $F' = \overline{F}\circ\big(\bigoplus_{0\leq i<s^2} {}^{(ri)}D_{\pi_i,\sigma}\big)$, whence a morphism in **Mod**-$k[\mathcal{A}]$:

$$\overline{F}(\theta'_k) : F\circ(\pi^{\oplus s^2}) \to F'\circ\sigma\ .$$

Furthermore, we have $\rho^*G = \overline{G}\circ\rho = \overline{G}^{(rs^2-rs)}\circ\sigma$. Thus, if we choose a cardinal $\aleph$ greater than the dimension of $\sigma(a)$ for all objects $a$ of $\mathcal{A}$, then we can define our map $\Theta'_k$ as the unique map making the following square commute:

$$\begin{array}{ccc} \mathrm{Tor}_*^{k[\mathcal{A}]}(F\circ(\pi^{\oplus s^2}),\overline{G}\circ\rho) & \overset{\Theta'_k}{\dashrightarrow} & \mathrm{Tor}_*^{k[\mathbf{P}_k]}(F',G^{(rs^2-rs)}) \\ \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}(\theta'_k),\overline{G}\circ\rho)\downarrow & & \simeq\downarrow \mathrm{res}^{\iota^\aleph} \\ \mathrm{Tor}_*^{k[\mathcal{A}]}(F'\circ\sigma,\overline{G}^{(rs^2-rs)}\circ\sigma) & \xrightarrow{\mathrm{res}^\sigma} & \mathrm{Tor}_*^{k[\mathbf{P}_k^\aleph]}(F',\overline{G}^{(rs^2-rs)}) \end{array} \quad . \tag{48}$$

Note that the map $\mathrm{res}^{\iota^\aleph}$ in this diagram is an isomorphism as a consequence of proposition 2.28 and of the fact that $\overline{G}^{(rs^2-rs)}$ is the left Kan extension of $G^{(rs^2-rs)}$ to all vector spaces (because $-^{(rs^2-rs)}$ is an autoequivalence of the category of $k$-vector spaces, hence it preserves filtered colimits).

The following lemma is proved exactly in the same way as lemmas 9.1 and 9.2.

**Lemma 10.5.** *The map $\Theta'_k$ does not depend on the cardinal $\aleph$. Moreover, it is natural with respect to $F$, $G$, $\pi$ and $\rho$.*

Next, we clarify the relation between $\Theta'_k$ and the comparison map $\Theta_k$ from section 9. Assume that we are given isomorphisms of additive functors $\pi \simeq {}^{(-ri)}\pi$ and $\rho \simeq {}^{(rs-rs^2)}\rho$. Then these isomorphisms induce isomorphisms (where $F^{(r,s^2)}$ stands for the strict polynomial functor $v \mapsto F(\bigoplus_{0\le i<s^2} {}^{(ri)}v)$ as in notation 7.17):

$$(\pi^{\oplus s^2})^*F \simeq \pi^*(F^{(r,s^2)})\;, \qquad \rho^*G \simeq \rho^*(G^{(rs^2-rs)})\;, \qquad F' \simeq D^*_{\pi,\rho}(F^{(r,s^2)})\;,$$

and the next lemma is a straightforward verification.

**Lemma 10.6.** *There is a commutative square, whose vertical isomorphisms are induced by the above isomorphisms of functors:*

$$\begin{array}{ccc}
\mathrm{Tor}_*^{k[\mathcal{A}]}((\pi^{\oplus s^2})^*F,\rho^*G) & \xrightarrow{\Theta'_k} & \mathrm{Tor}_*^{k[\mathbf{P}_k]}(F',G^{(rs^2-rs)}) \\
\downarrow\simeq & & \downarrow\simeq \\
\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^*(F^{(r,s^2)}),\rho^*G^{(rs^2-rs)}) & \xrightarrow{\Theta_k} & \mathrm{Tor}_*^{k[\mathbf{P}_k]}(D^*_{\pi,\rho}(F^{(r,s^2)}),G^{(rs^2-rs)})
\end{array}.$$

10.2. **Proof of the generalized comparison theorem 10.1.** The proof of the generalized comparison theorem follows the same strategy as the proof of theorem 9.8. Namely we first give a proof of a base case relying on adjunctions (the homological algebra part of the proof), and then we deduce the general case by taking simplicial resolutions (the homotopical algebra part of the proof).

Throughout this section, we assume that the hypotheses of theorem 10.1 are satisfied, in particular $\mathcal{A}$ is $\mathbb{F}_q$-linear over some finite subfield $\mathbb{F}_q$ of $k$, and $\pi$ and $\rho$ are $\mathbb{F}_q$-linear. Next proposition is the base case of the proof.

**Proposition 10.7.** *Assume that*

$$\pi = \bigoplus_{i\in I} k\otimes_{\mathbb{F}_q}\mathcal{A}(-,a_i)\;, \qquad \rho = \bigoplus_{j\in J} k\otimes_{\mathbb{F}_q}\mathcal{A}(b_j,-)$$

*for some possibly infinite indexing sets $I$ and $J$. Then the generalized comparison map* (45) *is a graded isomorphism.*

*Proof.* If $t(v) = k\otimes_{\mathbb{F}_q} v$ denotes the extension of scalars from $\mathbb{F}_q$ to $k$, then we have $\pi \simeq t\circ\mu$ and $\rho\simeq t\circ\nu$ for some additive functors $\mu:\mathcal{A}^{\mathrm{op}}\to\mathbb{F}_q$-**Mod** and $\nu:\mathcal{A}\to\mathbb{F}_q$-**Mod**. The canonical isomorphisms of functors $t\simeq {}^{(ri)}t$ induce isomorphisms $\pi\simeq {}^{(ri)}\pi$ and $\rho\simeq {}^{(rs-rs^2)}\rho$, so by lemma 10.6 and by naturality of $\Phi_k$ with respect to the isomorphism $F'\xrightarrow{\simeq}\overline{F}^{(r,s^2)}\circ D_{\pi,\rho}$, we have a commutative square (in which the composition operator of functors is omitted):

$$\begin{array}{ccc}
\mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}\pi,\overline{G}\rho) & \xrightarrow{\Lambda'_k} & \mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}^{(r,s^2)}\pi,\overline{G}^{(rs^2-rs)}\rho) \\
 & & \downarrow\Theta_k \\
\downarrow(45) & & \mathrm{Tor}_*^{k[\mathbf{P}_k]}(\overline{F}^{(r,s^2)}D_{\pi,\rho},G^{(rs^2-rs)}) \\
 & & \downarrow\Phi_k \\
\mathrm{Tor}_*^{\mathrm{gen}}(F',G^{(rs^2-rs)}) & \xrightarrow{\simeq} & \mathrm{Tor}_*^{\mathrm{gen}}(\overline{F}^{(r,s^2)}D_{\pi,\rho},G^{(rs^2-rs)})
\end{array}$$

in which $\Lambda'_k$ is induced by the canonical isomorphism $t \simeq {}^{(rs^2-rs)}t$ and by the morphism of additive functors:

$$\mathfrak{d} : t \xrightarrow{\text{diag}} t^{\oplus s^2} \simeq \bigoplus_{0\le i<s^2} {}^{(ri)}t \;.$$

Therefore in order to prove proposition 10.7 it suffices to prove that the composition $\Phi_k \circ \Theta_k \circ \Lambda'_k$ in top right corner of the diagram is an isomorphism.

Next, since $\pi = t\circ\mu$ and $\rho = t\circ\nu$, the base change property of proposition 9.3 gives a commutative square:

$$\begin{array}{ccc}
\mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}^{(r,s^2)}\pi, \overline{G}^{(rs^2-rs)}\rho) & \xrightarrow{\Theta_{\mathbb{F}_q}} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}_q}]}(\overline{F}^{(r,s^2)} tD_{\mu,\nu}, \overline{G}^{(rs^2-rs)}t) \\
\downarrow{\scriptstyle \Theta_k} & & \downarrow{\scriptstyle \simeq} \\
\mathrm{Tor}_*^{k[\mathbf{P}_k]}(\overline{F}^{(r,s^2)} D_{\pi,\rho}, G^{(rs^2-rs)}) & \xleftarrow[\mathrm{res}^t]{} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}_q}]}(\overline{F}^{(r,s^2)} D_{\pi,\rho}t, \overline{G}^{(rs^2-rs)}t)
\end{array} .$$

Hence, by naturality of $\Theta_{\mathbb{F}_q}$ with respect to the isomorphism $\overline{G}t \simeq \overline{G}^{(rs^2-rs)}t$ and the morphism $F(\mathfrak{d}) : \overline{F}t \to \overline{F}^{(r,s^2)}$, we obtain a commutative square:

$$\begin{array}{ccc}
\mathrm{Tor}_*^{k[\mathcal{A}]}(\overline{F}\pi, \overline{G}\rho) & \xrightarrow{\Theta_{\mathbb{F}_q}} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}_q}]}(\overline{F}tD_{\mu,\nu}, \overline{G}t) \\
\downarrow{\scriptstyle \Theta_k\circ\Lambda'_k} & & \downarrow{\scriptstyle \Lambda''_k} \\
\mathrm{Tor}_*^{k[\mathbf{P}_k]}(\overline{F}^{(r,s^2)} D_{\pi,\rho}, G^{(rs^2-rs)}) & \xleftarrow[\mathrm{res}^t]{} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}_q}]}(\overline{F}^{(r,s^2)} D_{\pi,\rho}t, \overline{G}^{(rs^2-rs)}t)
\end{array}$$

where $\Lambda''_{\mathbb{F}_q}$ is induced by $\mathfrak{d}$, by the isomorphism $t \simeq {}^{(rs^2-rs)}t$ and by the canonical isomorphism $\mathrm{can} : t\circ D_{\mu,\nu} \simeq D_{\pi,\rho}\circ t$. The map $\Theta_{\mathbb{F}_q}$ on the top row of this square is an isomorphism by corollary 9.5. Hence, to prove proposition 10.7 it remains to prove that the composition $\Phi_k \circ \mathrm{res}^t \circ \Lambda''_k$ is an isomorphism.

For this purpose, we are going to rewrite the composition $\Phi_k \circ \mathrm{res}^t \circ \Lambda''_k$ into yet another form. We claim that there is a $k$-linear isomorphism, natural with respect to $v$, $\mu$ and $\nu$:

$$\psi_v : {}^{(r,s^2)}D_{t\mu,t\nu}(v) \to D_{t\mu,t\nu}({}^{(r,s^2)}v) \;,$$

Indeed, we have isomorphisms of vector spaces, natural with respect to $\mu$ and $\nu$:

$$\phi_i : {}^{(ri)}\,(t\mu \otimes_{{}_k\mathcal{A}} t\nu) \xrightarrow{\simeq} t\mu \otimes_{{}_k\mathcal{A}} t\nu$$

which send the class $[\![(\alpha\otimes x)\otimes(\beta\otimes y)]\!]$ where $\alpha,\beta\in k$, $x\in\mu(a)$ and $y\in\nu(a)$ to the class $[\![(\alpha^{p^{ri}}\otimes x)\otimes(\beta^{p^{ri}}\otimes y)]\!]$. We define $\psi_v$ as the following composition, where $T$ stands for $t\mu\otimes_{{}_k\mathcal{A}} t\nu$, the first and last isomorphisms are the canonical ones and the second morphism is induced by the $\phi_i$:

$$\begin{aligned}
{}^{(r,s^2)}D_{t\mu,t\nu}(v) = \bigoplus_{0\le i<s^2} {}^{(ri)}\mathrm{Hom}_k(v,T) &\xrightarrow{\simeq} \bigoplus_{0\le i<s^2} \mathrm{Hom}_k({}^{(ri)}v, {}^{(ri)}T) \\
&\xrightarrow{\simeq} \bigoplus_{0\le i<s^2} \mathrm{Hom}_k({}^{(ri)}v, T) \\
&\xrightarrow{\simeq} \mathrm{Hom}_k(\bigoplus_{0\le i<s^2} {}^{(ri)}v, T) = D_{t\mu,t\nu}({}^{(r,s^2)}v) \;.
\end{aligned}$$

Moreover, one readily checks that $\psi$ fits into a commutative diagram in the category of functors from $\mathbf{P}_{\mathbb{F}_q}$ to $k$-**Mod**:

$$\begin{array}{ccccc} tD_{\mu,\nu} & \xrightarrow{\mathfrak{d}(D_{\mu,\nu})} & {}^{(r,s^2)}tD_{\mu,\nu} & \xrightarrow[\simeq]{{}^{(r,s^2)}\mathrm{can}} & {}^{(r,s^2)}D_{\pi,\rho}t \\ \simeq\downarrow\mathrm{can} & & & & \simeq\downarrow\psi(t) \\ D_{\pi,\rho}t & & \xrightarrow{D_{\pi,\rho}(\mathfrak{s})} & & D_{\pi,\rho}{}^{(r,s^2)}t \end{array} \tag{49}$$

where $\mathfrak{s}$ denotes the composition ${}^{(r,s^2)}t \simeq t^{\oplus s^2} \xrightarrow{\mathrm{sum}} t$.

Finally, diagram (49) and naturality of $\Phi_k$ and $\mathrm{res}^t$ with respect to the map $\overline{F}(\psi)$ yield a commutative square, in which the horizontal isomorphisms are induced by the isomorphisms $\overline{F}(\mathrm{can})$ and $\overline{F}(\psi)$ and the map $\Lambda_k'''$ is induced by $\overline{F}(D_{\pi,\rho}(\mathfrak{s}))$:

$$\begin{array}{ccc} \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}_q}]}(\overline{F}tD_{\mu,\nu},\overline{G}t) & \xrightarrow{\simeq} & \mathrm{Tor}_*^{k[\mathbf{P}_{\mathbb{F}_q}]}(\overline{F}D_{\pi,\rho}t,\overline{G}t) \\ \downarrow\Phi_k\circ\mathrm{res}^t\circ\Lambda_k'' & & \downarrow\Phi_k\circ\mathrm{res}^t\circ\Lambda_k''' \\ \mathrm{Tor}_*^{\mathrm{gen}}(\overline{F}^{(r,s^2)}D_{\pi,\rho},G^{(rs^2-rs)}) & \xrightarrow{\simeq} & \mathrm{Tor}_*^{\mathrm{gen}}(\overline{F}D_{\pi,\rho}{}^{(r,s^2)},G^{(rs^2-rs)}) \end{array} .$$

We observe that $\Phi_k\circ\mathrm{res}^t\circ\Lambda_k'''$ equals the comparison map of equation (31), hence it is an isomorphism by corollary 7.21. This concludes the proof. □

We are going to extend the isomorphism of proposition 10.7 to more general $\mathbb{F}_q$-linear functors $\pi:\mathcal{A}^{\mathrm{op}}\to k$-**Mod** and $\rho:\mathcal{A}\to k$-**Mod** by taking simplicial resolutions. We first need an elementary lemma regarding the computation of Tor between $\mathbb{F}_q$-linear functors. The functor $\rho$ is an object of the $k$-category of all additive functors from $\mathcal{A}$ to $k$-**Mod**, which identifies with ${}_k\mathcal{A}$-**Mod**. But $\rho$ is also an object an object of the category of all $\mathbb{F}_q$-linear functors from $\mathcal{A}$ to $k$-**Mod**, which identifies with $(k\otimes_{\mathbb{F}_q}\mathcal{A})$-**Mod**. Similarly, $\pi$ can be viewed as an object of **Mod**-${}_k\mathcal{A}$ or **Mod**-$(k\otimes_{\mathbb{F}_q}\mathcal{A})$.

**Lemma 10.8.** *For all $\mathbb{F}_q$-linear functors $\pi$ and $\rho$, there is an isomorphism, natural in $\pi$ and $\rho$:*

$$\mathrm{Tor}_*^{k\otimes_{\mathbb{F}_q}\mathcal{A}}(\pi,\rho)\simeq\mathrm{Tor}_*^{{}_k\mathcal{A}}(\pi,\rho)\ .$$

*Proof.* Let $\phi: k\otimes_{\mathbb{F}_p}\mathbb{F}_q\to k$ denote the surjective morphism of $(k,\mathbb{F}_q)$-bimodules such that $\phi(x\otimes y)=xy$. Restriction along the functor $\phi\otimes_{\mathbb{F}_q}\mathcal{A}: {}_k\mathcal{A}\to k\otimes_{\mathbb{F}_q}\mathcal{A}$ yields a fully faithful functor $(k\otimes_{\mathbb{F}_q}\mathcal{A})$-**Mod** $\to {}_k\mathcal{A}$-**Mod** hence an isomorphism $\pi\otimes_{k\otimes_{\mathbb{F}_q}\mathcal{A}}\rho\simeq\pi\otimes_{{}_k\mathcal{A}}\rho$ for all $\mathbb{F}_q$-linear functors $\pi$ and $\rho$ by corollary 2.15. Moreover, $\phi$ admits a section (as a morphism of $(k,\mathbb{F}_q)$-bimodules) because $\mathbb{F}_q$ is a finite separable extension of $\mathbb{F}_p$. Hence for all objects $a$ in $\mathcal{A}$, the additive functor $k\otimes_{\mathbb{F}_q}\mathcal{A}(a,-)$ is a direct summand of the additive functor $k\otimes_{\mathbb{F}_p}\mathcal{A}(a,-)$. Thus, every projective resolution $P$ of $\rho$ in the category of $\mathbb{F}_q$-linear functors may be regarded as a projective resolution of $\rho$ in the category of additive functors. As a result we have

$$\mathrm{Tor}_*^{k\otimes_{\mathbb{F}_q}\mathcal{A}}(\pi,\rho)=H_*(\pi\otimes_{k\otimes_{\mathbb{F}_q}\mathcal{A}}P)\simeq H_*(\pi\otimes_{{}_k\mathcal{A}}P)=\mathrm{Tor}_*^{{}_k\mathcal{A}}(\pi,\rho)\ .$$

where the middle isomorphism is given by restriction along $\phi\otimes_{\mathbb{F}_q}\mathcal{A}$. □

We are now ready to prove the generalized comparison theorem.

*Proof of theorem 10.1.* We set $\sigma := {}^{(rs-rs^2)}\rho$ and $H = G^{(rs^2-rs)}$ for the sake of concision. Thus, $\rho^*G = \sigma^*H$. We also emphasize the dependence of $F'$ on $\pi$ and $\sigma$ by setting:

$$F'_{\pi,\sigma}(v) := \overline{F}\Big(\bigoplus_{0\le i<s^2} {}^{(ri)}\mathrm{Hom}_k\big(v, ({}^{(-ri)}\pi)\otimes_{k\mathcal{A}}\sigma\big)\Big)\;.$$

We fix an integer $n \gg 0$ ($n > \log_p(e/2)$ suffices) such that the canonical map

$$\mathrm{Tor}_*^{\mathrm{gen}}(F'_{\pi,\sigma}, H) \to \mathrm{Tor}_*^{\Gamma\mathbf{P}_k}((F'_{\pi,\sigma})^{(n)}, H^{(n)}) \tag{50}$$

is $e$-connected. Let

$$\Psi_k : \mathrm{Tor}_*^{k[\mathbf{P}_k]}(F'_{\pi,\sigma}, H) \to \mathrm{Tor}_*^{\Gamma\mathbf{P}_k}((F'_{\pi,\sigma})^{(n)}, H^{(n)})$$

be the morphism given by restriction along ${}^{(-n)}-$ and by the restriction from ordinary functors to strict polynomial functors. Then $\Psi_k$ is the composition of the canonical map (50) with $\Phi_k$, so that it suffices to prove that the composition $\Psi_k\circ\Theta'_k\circ\Lambda_k$ is $e$-connected.

We consider simplicial resolutions by direct sums of standard projectives

$$\mathcal{H}\to H\;,\qquad \mathcal{H}'\to H^{(n)}\;,\qquad \varpi\to\pi\;,\qquad \varrho\to\rho\;,$$

respectively in the categories

$$k[\mathbf{P}_k]\text{-}\mathbf{Mod}\;,\qquad \Gamma\mathbf{P}_k\text{-}\mathbf{Mod}\;,\qquad \mathbf{Mod}\text{-}(k\otimes_{\mathbb{F}_q}\mathcal{A})\;,\qquad (k\otimes_{\mathbb{F}_q}\mathcal{A})\text{-}\mathbf{Mod}\;.$$

Standard projectives in $(k\otimes_{\mathbb{F}_q}\mathcal{A})$-**Mod** are of the form $t\circ\mathcal{A}(a,-)$, and we have ${}^{(rs-rs^2)}(t\circ\mathcal{A}(a,-))\simeq t\circ\mathcal{A}(a,-)$, hence

$$\varsigma := {}^{(rs-rs^2)}\varrho \to {}^{(rs-rs^2)}\rho = \sigma$$

is also a simplicial resolution in $(k\otimes_{\mathbb{F}_q}\mathcal{A})$-**Mod**. Then it follows from lemmas 9.6 and 9.7 that $\varsigma^*\mathcal{H}$ is a simplicial projective resolution of $\sigma^*H$ in $k[\mathcal{A}]$-**Mod**, and that $(\varpi^{\oplus i})^*F$ is a simplicial (not projective) resolution of $(\pi^{\oplus i})^*F$ in **Mod**-$k[\mathcal{A}]$ for all positive integers $i$. Therefore we have identifications of homotopy groups:

$$\begin{aligned}
&\pi_*\big(\varpi^*F\otimes_{k[\mathcal{A}]}\varsigma^*\mathcal{H}\big) = \mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^*F,\sigma^*H)\;,\\
&\pi_*\Big((\varpi^{\oplus s^2})^*F\otimes_{k[\mathcal{A}]}\varsigma^*\mathcal{H}\Big) = \mathrm{Tor}_*^{k[\mathcal{A}]}((\pi^{\oplus s^2})^*F,\sigma^*H)\;,\\
&\pi_*\big(F'_{\pi,\sigma}\otimes_{k[\mathcal{A}]}\mathcal{H}\big) = \mathrm{Tor}_*^{k[\mathbf{P}_k]}(F'_{\pi,\sigma},H)\;,\\
&\pi_*\Big((F'_{\pi,\sigma})^{(n)}\otimes_{\Gamma\mathbf{P}_k}\mathcal{H}'\Big) = \mathrm{Tor}_*^{k[\mathbf{P}_k]}((F'_{\pi,\sigma})^{(n)},H^{(n)})\;.
\end{aligned}$$

Moreover, let $f:\mathcal{H}\to\mathcal{H}'^{(-n)}$ be a simplicial morphism in $k[\mathbf{P}_k]$-**Mod** lifting the identity morphism of $H$. Then the maps $\Lambda_k$, $\Theta'_k$ and $\Psi_k$ are respectively induced by the morphisms of simplicial $k$-vector spaces:

$$\begin{aligned}
&\varpi^*F\otimes_{k[\mathcal{A}]}\varsigma^*\mathcal{H}\xrightarrow{\widetilde{\Lambda_k}}(\varpi^{\oplus s^2})^*F\otimes_{k[\mathcal{A}]}\varsigma^*\mathcal{H}\;,\\
&(\varpi^{\oplus s^2})^*F\otimes_{k[\mathcal{A}]}\varsigma^*\mathcal{H}\xrightarrow{\widetilde{\Theta'_k}}F'_{\varpi,\varsigma}\otimes_{k[\mathbf{P}_k]}\mathcal{H}\to F'_{\pi,\sigma}\otimes_{k[\mathbf{P}_k]}\mathcal{H}\;,\\
&F'_{\pi,\sigma}\otimes_{k[\mathbf{P}_k]}\mathcal{H}\xrightarrow{\mathrm{id}\otimes f}(F'_{\pi,\sigma})^{(n)(-n)}\otimes_{k[\mathbf{P}_k]}\mathcal{H}'^{(-n)}\xrightarrow{\widetilde{\Psi_k}}(F'_{\pi,\sigma})^{(n)}\otimes_{\Gamma\mathbf{P}_k}\mathcal{H}'\;.
\end{aligned}$$

Here, the unadorned simplicial morphism is induced by the simplicial morphisms $\varpi\to\pi$ and $\varsigma\to\sigma$. The simplicial morphisms $\widetilde{\Lambda_k}$, $\widetilde{\Theta'_k}$ and $\widetilde{\Psi_k}$ are degreewise equal to the degree zero component of $\Lambda_k$, $\Theta'_k$ and $\Psi_k$, for example the component of $\widetilde{\Lambda_k}$ in simplicial degree $i$ is equal to

$$\mathrm{Tor}_0^{k[\mathcal{A}]}(\varpi_i^* F, \varsigma_i^*\mathcal{H}_i) \xrightarrow{\Lambda_k} \mathrm{Tor}_0^{k[\mathcal{A}]}((\varpi_i^{\oplus s^2})^* F, \varsigma_i^*\mathcal{H}_i)\ .$$

(That these simplicial morphisms induce our maps $\Lambda_k$, $\Theta'_k$ and $\Psi_k$ is obvious for the first one and the last one. For $\Theta'_k$, this follows by the same reasoning as in the proof of theorem 9.8). We deduce that the comparison map $\Psi_k \circ \Theta'_k \circ \Lambda_k$ equals the map induced on homotopy groups by the composition of the simplicial morphism:

$$\left(\widetilde{\Psi_k} \circ (\mathrm{id} \otimes f) \circ \widetilde{\Theta'_k} \circ \widetilde{\Lambda'_k}\right)\ :\ \varpi^* F \otimes_{k[\mathcal{A}]} \varsigma^*\mathcal{H} \to (F'_{\varpi,\varsigma})^{(n)} \otimes_{\Gamma \mathbf{P}_k} \mathcal{H}' \tag{51}$$

together with the simplicial morphism induced by $\varpi \to \pi$ and $\varsigma \to \sigma$:

$$(F'_{\varpi,\varsigma})^{(n)} \otimes_{\Gamma \mathbf{P}_k} \mathcal{H}' \to (F'_{\pi,\sigma})^{(n)} \otimes_{\Gamma \mathbf{P}_k} \mathcal{H}' \tag{52}$$

Let us regard the source and the target of (51) as the diagonal of bisimplicial objects $\varpi_i^* F \otimes_{k[\mathcal{A}]} \varsigma_i^*\mathcal{H}_j$ and $(F'_{\varpi_i,\varsigma_i})^{(n)} \otimes_{\Gamma \mathbf{P}_k} \mathcal{H}'_j$ with bisimplicial degrees $(i,j)$. Then the simplicial morphism (51) actually comes from a bisimplicial morphism. Spectral sequences of bisimplicial $k$-modules as in [19, IV section 2.2] yield two spectral sequences:

$$\mathrm{I}^1_{ij} = \mathrm{Tor}_j^{k[\mathcal{A}]}(\varpi_i^* F, \varsigma_i^* H) \Longrightarrow \pi_{i+j}\left(\varpi^* F \otimes_{k[\mathcal{A}]} \varsigma^*\mathcal{H}\right)\ ,$$
$$\mathrm{II}^1_{ij} = \mathrm{Tor}_j^{\Gamma \mathbf{P}_k}(F'_{\varpi_i,\varsigma_i}, H) \Longrightarrow \pi_{i+j}\left((F'_{\varpi,\varsigma})^{(n)} \otimes_{\Gamma \mathbf{P}_k} \mathcal{H}'\right)\ ,$$

And there is a morphism of spectral sequences I $\to$ II which coincides with the morphism (51) on the abutment, and with the map $\Psi_k \circ \Theta'_k \circ \Lambda_k$ on the first page. By proposition 10.7, the map $\Phi_k \circ \Theta'_k \circ \Lambda_k$ is an isomorphism when $\pi$ and $\rho$ are direct sums of standard projectives, hence by our choice of $n$, the morphism of spectral sequences is $e$-connected on the first page. Hence the simplicial morphism (51) is $e$-connected.

Thus, it remains to prove that the simplicial morphism (52) is $e$-connected. The Tor-vanishing hypothesis of theorem 10.1 and lemma 7.16 imply that the maps $({}^{(-ri)}\varpi) \otimes_{{}_k\mathcal{A}} \varsigma \to ({}^{(-ri)}\pi) \otimes_{{}_k\mathcal{A}} \sigma$ are $e$-connected. Hence the map $F'_{\varpi,\varsigma} \to F'_{\pi,\sigma}$ is e-connected by lemma 9.6, hence the simplicial morphism (52) is $e$-connected by the usual bisimplicial spectral sequence argument. □

10.3. **Consequences of the generalized comparison theorem.** We first prove theorem 1.7 from the introduction. We consider a simplification of the generalized comparison map (45), namely for all strict polynomial functors $F$ and $G$ and all additive functors $\pi : \mathcal{A}^{\mathrm{op}} \to k$-**Mod** and $\rho : \mathcal{A} \to k$-**Mod** we consider the composition:

$$\mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^* F, \rho^* G) \xrightarrow{\Theta_k} \mathrm{Tor}_*^{k[\mathbf{P}_k]}(D^*_{\pi,\rho} F, G) \xrightarrow{\Phi_k} \mathrm{Tor}_*^{\mathrm{gen}}(D^*_{\pi,\rho} F, G)\ , \tag{53}$$

where $\Theta_k$ is the auxiliary comparison map of section 9. Theorem 1.7 is a direct consequence of the following result.

**Theorem 10.9.** *Let $k$ be an infinite perfect field of positive characteristic, containing a subfield $\mathbb{F}$ and let $\mathcal{A}$ be an additive $\mathbb{F}$-linear category. Let $\pi$ and $\rho$ be two $\mathbb{F}$-linear functors from $\mathcal{A}$ to $k$-modules, respectively contravariant and covariant, and let $F$ and $G$ be two strict polynomial functors with weights less or equal to the cardinal of $\mathbb{F}$. Assume furthermore that*

$$\mathrm{Tor}_i^{{}_k\mathcal{A}}(\pi, \rho) = 0 \quad \text{for } 0 < i < e.$$

*Then the comparison map* (53) *is* $e$*-connected.*

*Proof.* There are two cases. Assume first that $\mathbb{F}$ is a finite field. Then we can apply theorem 10.1 with $s = 1$. If $s = 1$ the generalized comparison map (45) is equal to the simplified comparison map (53) and the result follows.

Now assume that $\mathbb{F}$ is infinite. We claim that for all $\mathbb{F}$-linear functors $\alpha, \beta : \mathcal{A} \to k$-$\mathbf{Mod}$, we have $\mathrm{Ext}^*_{{}_k\mathcal{A}}({}^{(i)}\alpha, {}^{(j)}\beta) = 0$ for $i \neq j$. Indeed, this can be proved by repeating the argument of the vanishing lemma 7.13 in the case of the $k$-category ${}_k\mathcal{A}$, or alternatively by combining this vanishing lemma 7.13 with theorem 8.3. By lemma 2.7, this implies that

$$\mathrm{Tor}_*^{{}_k\mathcal{A}}({}^{(i)}\pi, {}^{(j)}\rho) = 0 \text{ for } i \neq j. \tag{54}$$

Thus we may apply the generalized comparison theorem 10.1 with $q = p$ (i.e. $\mathbb{F}_q$ is the prime field), and with an integer $s$ such that $p^s$ is greater or equal to the weights of $F$ and $G$. In this situation, the expression of $F'$ simplifies because of equation (54). Namely if we let $\alpha = {}^{(s-s^2)}\pi$ and $\beta = {}^{(s-s^2)}\rho$ then we have:

$$F' = \overline{F}^{(s^2-s)} \circ D_{\alpha,\beta}\ .$$

Moreover one readily checks that the composite map

$$\Theta'_k \circ \Lambda_k \ : \ \mathrm{Tor}_*^{k[\mathcal{A}]}(\pi^* F, \rho^* G) \to \mathrm{Tor}_*^{k[\mathbf{P}_k]}(D^*_{\alpha,\beta}(F^{(s^2-s)}), G^{(s^2-s)})$$

is equal to the map $\Theta_k$ relative to $\alpha$ and $\beta$.

Now we consider the following diagram, in which the composition of functors is omitted, $T_*$ stands for Tor, and we use the following notations $x := s^2 - s$, $D = D_{\pi,\rho}$, $D' = D_{\alpha,\beta}$. The Frobenius twist functor ${}^{(x)}- : k$-$\mathbf{Mod} \to k$-$\mathbf{Mod}$ is isomorphic to the extension of scalars along the morphism of fields $k \to k$, $\lambda \mapsto \lambda^{p^x}$, hence we have a canonical isomorphism can : ${}^{(x)}D' \simeq D^{(x)}$, and the isomorphisms $(*)$ in this diagram are induced by this canonical isomorphism.

$$\begin{array}{ccccc}
T_*^{k[\mathcal{A}]}(\overline{F}\pi, \overline{G}\rho) & \xrightarrow{\Theta_k} & T_*^{k[\mathbf{P}_k]}(\overline{F}^{(x)}D', G^{(x)}) & \xrightarrow{\Phi_k} & T_*^{\mathrm{gen}}(\overline{F}^{(x)}D', G^{(x)}) \\
\downarrow{\scriptstyle \Theta_k} & & \simeq\downarrow{\scriptstyle \mathrm{res}^{(x)}-} & & \simeq\downarrow{\scriptstyle \mathrm{res}^{(x)}-} \\
T_*^{k[\mathbf{P}_k]}(\overline{F}D, G) & \xrightarrow[(*)]{\simeq} & T_*^{k[\mathbf{P}_k]}(\overline{F}^{(x)}D'^{(-x)}, G) & \xrightarrow{\Phi_k} & T_*^{\mathrm{gen}}(\overline{F}^{(x)}D'^{(-x)}, G) \\
\downarrow{\scriptstyle \Phi_k} & & & \nearrow{\scriptstyle \simeq\,(*)} & \\
T_*^{\mathrm{gen}}(\overline{F}D, G) & & & &
\end{array}$$

The diagram is commutative. To be more specific, the upper left square of the diagram commutes by the base change property of proposition 9.3, the upper right square and the triangle commute by naturality of $\Phi_k$. As explained above, the composite map corresponding to the upper row is $e$-connected by theorem 10.1. Therefore, the composite given by the first column is also $e$-connected. But this composite is nothing but the simplified comparison map (53). This finishes the proof of the theorem. □

**Corollary 10.10.** *Let* $k$ *be an infinite perfect field of positive characteristic, let* ${}^\vee- : \mathbf{P}_k^{\mathrm{op}} \to \mathbf{P}_k$ *denote the* $k$*-linear duality functor* ${}^\vee- = \mathrm{Hom}_k(-, k)$*, and let* $F^\vee$ *denote the composition* $F \circ {}^\vee-$*. The map*

$$\Phi_k : \mathrm{Tor}_*^{k[\mathbf{P}_k]}(F^\vee, G) \to \mathrm{Tor}_*^{\mathrm{gen}}(F^\vee, G)$$

*is an isomorphism for all strict polynomial functors $F$ and $G$.*

*Proof.* The Eilenberg-Watts theorem gives an equivalence of categories between the category of additive functors $\mathbf{P}_R \to k$-**Mod** and the category of $(R,k)$-bimodules. Under this equivalence, an $(R,k)$-bimodule $M$ corresponds to the functor $-\otimes_R M$. Therefore, if we let $R = k$, $\pi(v) = v^\vee$ and $\rho(v) = v$, we obtain:

$$\mathrm{Tor}_*^{k(\mathbf{P}_k)}(\pi,\rho) \simeq \mathrm{Tor}_*^{k\otimes_{\mathbb{Z}} k}(k,k) = \mathrm{HH}_*(k)\ .$$

But $k \otimes_{\mathbb{Z}} k = k \otimes_{\mathbb{F}_p} k$ and every field extension of $\mathbb{F}_p$ is a filtered colimit of $\mathbb{F}_p$-subalgebras which are smooth and essentially of finite type. Thus $\mathrm{HH}_*(k)$ is an exterior algebra over the $k$-vector space of Kähler forms $\Omega^1(k/\mathbb{F}_p)$ by the Hochschild-Kostant-Rosenberg theorem [21, Cor 2.13]. Since $k$ is perfect, $\Omega^1(k/\mathbb{F}_p)$ is zero, so that $\mathrm{HH}_i(k) = k$ for $i = 0$, and zero otherwise. Hence $D_{\pi,\rho} \simeq {}^\vee-$, and $\Phi_k \circ \Theta_k$ is an isomorphism by theorem 10.9.

Furthermore, the categories of $k$-linear functors **Mod**-$\mathbf{P}_k$ and $\mathbf{P}_k$-**Mod** are respectively equivalent to **Mod**-$k$ and $k$-**Mod**. Under these equivalences of categories, the functors $\pi$ and $\rho$ correspond to $k$. Hence $\mathrm{Tor}_i^{\mathbf{P}_k}(\pi,\rho)$ equals $k$ if $i = 0$ and zero otherwise. Thus $\Theta_k$ is an isomorphism by theorem 9.8.

Since $\Phi_k \circ \Theta_k$ and $\Theta_k$ are both isomorphisms, so is $\Phi_k$. □

10.4. **Comparison of** Ext**.** We now indicate how the results of the previous section can be dualized to compare Ext. Let $G$ and $K$ be two strict polynomial functors. We let $\Phi_k$ be the unique map making the following diagrams commute for all $i$ and all $n \gg 0$, where the vertical isomorphism on the left hand side is the canonical isomorphism, and the one on the right hand side is given by restriction along the Frobenius twist ${}^{(-n)}-$:

$$\begin{array}{ccc} \mathrm{Ext}^i_{\Gamma\mathbf{P}_k}(G^{(n)},K^{(n)}) & \longrightarrow & \mathrm{Ext}^i_{k[\mathbf{P}_k]}(G^{(n)},K^{(n)}) \\ \downarrow\simeq & & \simeq\downarrow \\ \mathrm{Ext}^i_{\mathrm{gen}}(G,K) & \xrightarrow{\Phi_k} & \mathrm{Ext}^i_{k[\mathbf{P}_k]}(G,K) \end{array} .$$

**Corollary 10.11.** *Let $k$ be an infinite perfect field of positive characteristic. For all strict polynomial functors $G$ and $K$ the comparison map*

$$\Phi_k : \mathrm{Ext}^i_{\mathrm{gen}}(G,K) \to \mathrm{Ext}^i_{k[\mathbf{P}_k]}(G,K)$$

*is a graded isomorphism.*

*Proof.* By a standard spectral sequence argument, the proof reduces to the case where $K$ is a standard injective, hence when $K = \mathrm{Hom}_k(F^\vee,k)$, where $F^\vee$ is the precomposition of a standard projective $F$ in $\Gamma\mathbf{P}_k$-**Mod** by the duality functor ${}^\vee- = \mathrm{Hom}_k(-,k)$. In this latter case, $\Phi_k$ is an isomorphism because proposition 2.12 shows that it is dual to the isomorphism $\Phi_k$ of corollary 10.10. □

Similarly, one can dualize theorem 10.9. To be more specific, given two additive functors $\rho,\sigma : \mathcal{A} \to k$-**Mod** and a $k$-vector space $v$, we let

$$T_{\rho,\sigma}(v) = \mathrm{Hom}_{\,k\mathcal{A}}(\rho,\sigma) \otimes v\ .$$

Then for all strict polynomial functors $G$ and $K$ we have a map

$$\Theta_k : \mathrm{Ext}^*_{k[\mathbf{P}_k]}(G, T^*_{\rho,\sigma}K) \to \mathrm{Ext}^*_{k[\mathcal{A}]}(\rho^* G, \sigma^* K)$$

induced by restriction along $\rho$ and by the canonical evaluation morphism $\mathrm{ev}$ : $\mathrm{Hom}_{{}_k\mathcal{A}}(\rho,\sigma)\otimes\rho\to\sigma$.

**Corollary 10.12.** *Let $k$ be an infinite perfect field of positive characteristic, containing a subfield $\mathbb{F}$ and let $\mathcal{A}$ be an additive $\mathbb{F}$-linear category. Let $\rho,\sigma:\mathcal{A}\to\mathbf{P}_k$ be two $\mathbb{F}$-linear functors such that $\mathrm{Hom}_{{}_k\mathcal{A}}(\rho,\sigma)$ is finite-dimensional. Assume that*

$$\mathrm{Ext}^i_{{}_k\mathcal{A}}(\rho,\sigma)=0\quad \text{for } 0<i<e.$$

*Then for all strict polynomial functors $G$ and $K$ with weights less or equal to the cardinal of $\mathbb{F}$, the graded map*

$$\Theta_k\circ\Phi_k:\mathrm{Ext}^*_{\mathrm{gen}}(G,T^*_{\rho,\sigma}K)\to\mathrm{Ext}^*_{k[\mathcal{A}]}(\rho^*G,\sigma^*K)$$

*is $e$-connected.*

*Proof.* In this proof, we let ${}^\vee-=\mathrm{Hom}_k(-,k)$ and we omit the composition operator for functors, e.g. if $F$ is a strict polynomial functor, ${}^\vee F^\vee$ stands for $({}^\vee-)\circ F\circ({}^\vee-)$.

We first prove the result when $K={}^\vee F^\vee$ for some $F$ in $\Gamma\mathbf{P}_k$-**Mod** and $\sigma={}^\vee\pi$ for some additive functor $\pi:\mathcal{A}^{\mathrm{op}}\to k$-**Mod**. Let $\xi:T_{\sigma,\rho}\to{}^\vee D_{\pi,\rho}$ be the morphism of functors whose component $\xi_v$ at a vector space $v$ is given by the composition

$$\xi_v\ :\ v\otimes\mathrm{Hom}_{{}_k\mathcal{A}}(\rho,{}^\vee\pi)\xrightarrow{\simeq}v\otimes{}^\vee(\pi\otimes_{{}_k\mathcal{A}}\rho)\to{}^\vee\mathrm{Hom}_k(v,\pi\otimes_{{}_k\mathcal{A}}\rho)$$

where the first map is provided by lemma 2.7 and the second map is the canonical map $\mathrm{can}:v\otimes{}^\vee w\to{}^\vee\mathrm{Hom}_k(v,w)$ such that $\mathrm{can}(x\otimes f)(\phi)=f(\phi(x))$, and which is an isomorphism if $v$ is finite dimensional. One readily checks that the composition

$$T_{\rho,\sigma}\rho\xrightarrow{\xi\rho}{}^\vee D_{\pi,\rho}\rho\xrightarrow{{}^\vee\theta_{\mathbb{F}}}{}^\vee\pi=\sigma$$

equals the canonical evaluation map ev. The finite dimensionality hypotheses on the values of $\rho$ and $\sigma$ and on $\mathrm{Hom}_{{}_k\mathcal{A}}(\rho,\sigma)$ respectively imply that:

i) $\xi:T_{\sigma,\rho}\to{}^\vee D_{\pi,\rho}$ is an isomorphism,
ii) $\sigma^*(F^\vee)=F^\vee\sigma=F\pi=\pi^*F$,
iii) ${}^{\vee\vee}D_{\pi,\rho}$ identifies with $D_{\pi,\rho}$.

Hence we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^*_{k[\mathbf{P}_k]}(G,{}^\vee F^\vee T_{\rho,\sigma}) & \xrightarrow{\rho^*} & \mathrm{Ext}^*_{k[\mathcal{A}]}(G\rho,{}^\vee F^\vee T_{\rho,\sigma}\rho) & \xrightarrow{{}^\vee F^\vee(\mathrm{ev})} & \mathrm{Ext}^*_{k[\mathcal{A}]}(G\rho,{}^\vee F^\vee\sigma)\\
\simeq\uparrow\alpha & & \simeq\uparrow\alpha & & \simeq\uparrow\alpha\\
{}^\vee\mathrm{Tor}^{k[\mathbf{P}_k]}_*(F^\vee T_{\rho,\sigma},G) & \xrightarrow{{}^\vee\mathrm{res}_\rho} & {}^\vee\mathrm{Tor}^{k[\mathcal{A}]}_*(F^\vee T_{\rho,\sigma}\rho,G\rho) & \xrightarrow{F^\vee(\mathrm{ev})} & {}^\vee\mathrm{Tor}^{k[\mathcal{A}]}_*(F^\vee\sigma,G\rho)\\
\simeq\downarrow F^\vee(\xi) & & \simeq\downarrow F^\vee(\xi\rho) & & \|\\
{}^\vee\mathrm{Tor}^{k[\mathbf{P}_k]}_*(FD_{\pi,\rho},G) & \xrightarrow{{}^\vee\mathrm{res}_\rho} & {}^\vee\mathrm{Tor}^{k[\mathcal{A}]}_*(FD_{\pi,\rho}\rho,G\rho) & \xrightarrow{F(\theta_{\mathbb{F}})} & {}^\vee\mathrm{Tor}^{k[\mathcal{A}]}_*(F\pi,G\rho)
\end{array}$$

from which we deduce that the graded map $\Theta_k\circ\Phi_k$ fits into a commutative square

$$\begin{array}{ccc}
\mathrm{Ext}^*_{\mathrm{gen}}(G,T^*_{\rho,\sigma}({}^\vee F^\vee)) & \xrightarrow{\Theta_k\circ\Phi_k} & \mathrm{Ext}^*_{k[\mathcal{A}]}(\rho^*G,\sigma^*({}^\vee F^\vee))\\
\downarrow\simeq & & \downarrow\simeq\\
{}^\vee\mathrm{Tor}^{\mathrm{gen}}_*(D^*_{\pi,\rho}F,G) & \longrightarrow & {}^\vee\mathrm{Tor}^{k[\mathcal{A}]}_*(\pi^*F,\rho^*G)
\end{array}\tag{55}$$

where the bottom arrow is dual to the comparison map (53) of theorem 10.9. Since $\mathrm{Ext}^*_{{}_k\mathcal{A}}(\rho,\sigma)\simeq{}^\vee\mathrm{Tor}^{{}_k\mathcal{A}}_*(\pi,\rho)$, we deduce from the latter theorem that this bottom map is $e$-connected, hence $\Theta_k\circ\Phi_k$ is $e$-connected.

The case $K = {}^\vee F^\vee$ proves corollary 10.12 for all strict polynomial functors $K$ with finite dimensional values, in particular for the standard injectives. For an arbitrary $K$ one can consider an injective resolution and the result follows by a standard spectral sequence argument. $\square$

With the same strategy, one can also dualize theorem 10.1. Given a strict polynomial functor $K$, we denote by $K'$ the strict polynomial functor such that

$$K''(v) = \overline{K}\left(\bigoplus_{0\le i<s^2} {}^{(ri)}\big(v\otimes \mathrm{Hom}_{{}_k\mathcal{A}}({}^{(rs-rs^2)}\rho, {}^{(-ri)}\sigma)\big)\right) .$$

One defines a comparison map in the same fashion as the map of corollary 10.12:

$$\mathrm{Ext}^*_{\mathrm{gen}}(G^{(rs^2-rs)}, K'') \to \mathrm{Ext}^*_{k[\mathcal{A}]}(\rho^*G, \sigma^*K) . \tag{56}$$

The proof of the following corollary is similar to the proof of corollary 10.12 and is left to the reader.

**Corollary 10.13.** *Let $k$ be an infinite perfect field of characteristic $p$, containing a finite field $\mathbb{F}_q$ of cardinal $q = p^r$. Let $\mathcal{A}$ be a small additive category, let $\rho, \sigma : \mathcal{A} \to \mathbf{P}_k$ be two additive functors such that $\mathrm{Hom}_{{}_k\mathcal{A}}(\rho,\sigma)$ is finite dimensional. Assume that there are positive integers $s$ and $e$ such that*

$$\mathrm{Ext}^j_{{}_k\mathcal{A}}\left({}^{(rs-rs^2)}\rho, {}^{(ri)}\sigma\right) = 0$$

*for $0 < j < e$ and $0 \le i < s^2$. Assume further that $\mathcal{A}$ is $\mathbb{F}_q$-linear, that $\rho$ and $\sigma$ are $\mathbb{F}_q$-linear. Then for all strict polynomial functors $G$ and $K$ of weights less or equal to $q^s$, the map (56) is $e$-connected.*

*Remark* 10.14. The finite dimensionality hypotheses on the values of $\rho$, $\sigma$ and on $\mathrm{Hom}_{{}_k\mathcal{A}}(\rho,\sigma)$ are necessary in the proof of corollary 10.12 in order that i), ii) and iii) are satisfied. Without them, we would not obtain a commutative square (55) with vertical *isomorphisms*. Similarly, the finite dimensionality hypotheses are needed for the proof of corollary 10.13. Instead of dualizing, one could try to prove corollaries 10.12 and 10.13 by a direct approach, following the same strategy as the proofs of theorems 10.1 and 10.9. However, such a direct approach seems to raise inextricable problems with (co)limits.

## 11. Applications of corollary 10.11

The goal of this section is to prove applications of the generalized comparison theorem, or to be more specific, of the corollaries 10.10 and 10.11 which deal with the very specific case $\mathcal{A} = \mathbf{P}_k$. We first generalize the computations of [14] over an infinite perfect field. We also generalize some results of [13] to infinite perfect fields. Finally, the most important applications are probably theorems 11.13 and 11.15. These theorems are the analogues for classical groups over infinite perfect fields of the main result of Cline Parshall Scott and van der Kallen [6] which compares the cohomology of and algebraic group with the cohomology of its underlying discrete group. Throughout the section, $k$ is a field of positive characteristic $p$.

11.1. **A sample of functor homology computations.** Many computations of generic Ext can be found in the literature. Thus, the isomorphism of corollary 10.11 provides many concrete Ext-computations in $k[\mathbf{P}_k]$-**Mod** over an infinite field. We briefly illustrate this fact here.

We first point out that computations of generic Ext between strict polynomial functors are insensitive to field extensions. To be more specific, if $k \to L$ is any field extension, the base change formula [44, Section 2.7] yields an isomorphism

$$\mathrm{Ext}^*_{\mathrm{gen},k}(F,G)\otimes L \simeq \mathrm{Ext}^*_{\mathrm{gen},L}(F_L,G_L)$$

where the generic extensions on the left hand side are computed in the $k$-category $\Gamma\mathbf{P}_k$-**Mod**, while the generic extensions on the right hand side are computed in the $L$-category $\Gamma_L(\mathbf{P}_L)$-**Mod**. The functors $F_L$ and $G_L$ obtained by base change from $F$ and $G$ are usually easy to compute, e.g. if $F$ is the $d$-th symmetric power over $k$ then $F_L$ is the $d$-th symmetric power over $L$. In particular, all the computations of generic Ext over finite fields actually hold over arbitrary fields $k$ of positive characteristic, and can therefore be converted into computations in $k[\mathbf{P}_k]$-**Mod** by corollary 10.11 when $k$ is infinite and perfect. This is the case of the computations of generic Ext given in [14, Thm 5.8] (which are established in [50] by different methods, without spectral sequences). To be more specific, let $C^*$ be a graded coalgebra in $k[\mathbf{P}_k]$-**Mod** and let $A^*$ be a graded algebra in $k[\mathbf{P}_k]$-**Mod**. We consider the trigraded vector space

$$\mathrm{E}^*(C^*,A^*) := \bigoplus_{i,d,e\geq 0} \mathrm{Ext}^i_{k[\mathbf{P}_k]}(C^d,A^e)$$

equipped with the algebra structure given by convolution:

$$\mathrm{E}^i(C^d,A^e)\otimes \mathrm{E}^j(C^f,A^g) \xrightarrow{\cup} \mathrm{E}^{i+j}(C^d\otimes C^f,A^e\otimes A^g)\to \mathrm{E}^{i+j}(C^{d+f},A^{e+g})\;.$$

By letting $s\to\infty$ in [50, Thm 15.1] and by applying corollary 10.11 one obtains the following infinite field version of the computations of [14, Thm 6.3].

**Corollary 11.1.** *Let $k$ be an infinite perfect field of positive characteristic $p$, and let $r$ be a nonnegative integer. Let $V_{s,r}$ denote the trigraded vector space with homogeneous basis $(e_i)_{i\geq 0}$ where each $e_i$ is placed in tridegree $(2ip^r+sp^r-s,1,p^r)$. Then we have isomorphisms of trigraded algebras:*

$$\begin{aligned}
&\mathrm{E}^*(\Gamma^{*(r)},S^*)\simeq S(V_{0,r})\;, && \mathrm{E}^*(\Gamma^{*(r)},\Lambda^*)\simeq \Lambda(V_{1,r})\;,\\
&\mathrm{E}^*(\Lambda^{*(r)},S^*)\simeq \Lambda(V_{0,r})\;, && \mathrm{E}^*(\Lambda^{*(r)},\Lambda^*)\simeq \Gamma(V_{1,r})\;,\\
&\mathrm{E}^*(S^{*(r)},S^*)\simeq \Gamma(V_{0,r})\;, && \mathrm{E}^*(\Gamma^{*(r)},\Gamma^*)\simeq \Gamma(V_{2,r})\;.
\end{aligned}$$

The approach of [50] relies on a formula computing extensions between twisted strict polynomial functors, see [5], [46] and [52]. Namely, if $v$ is a finite-dimensional vector space and $G$ is a strict polynomial functor, we let $G_v$ be the strict polynomial functor 'with parameter $v$' defined by $G_v(-) := G(v\otimes -)$. If $v$ is graded, then $G_v$ inherits a grading. It is the unique grading natural with respect to $G$ and $v$, which coincides with the usual grading on symmetric powers of a graded vector space see [49, Section 2.5] and [52, Section 4.2]. Let $E_r$ denote the finite-dimensional graded vector space $E_r=\mathrm{Ext}^*_{\Gamma^{p^r}\mathbf{P}_k}(I^{(r)},I^{(r)})$ which equals $k$ in degrees $2i$ for $0\leq i<p^r$ and which is zero in the other degrees. Then we have a graded isomorphism, where the degree on the right hand side is obtained by totalizing the Ext-degree with the

degree of the functor $G_{E_r}$ (that is, if $G^j_{E_r}$ is the component of degree $j$ then the summand $\mathrm{Ext}^i_{\Gamma\mathbf{P}_k}(F, G^j_{E_r})$ is placed in degree $i+j$):

$$\mathrm{Ext}^*_{\Gamma\mathbf{P}_k}(F^{(r)}, G^{(r)}) \simeq \mathrm{Ext}^*_{\Gamma\mathbf{P}_k}(F, G_{E_r})\ .$$

We can extend the parametrization of a strict polynomial functor $G$ to infinite-dimensional graded vector spaces $v$ by letting $G_v := \mathrm{colim}\, G_u$, where the colimit is taken over the poset of all finite-dimensional graded vector spaces $u \subset v$. By taking the colimit over $r$ in the previous isomorphism, and by using corollary 10.11 we obtain the following result (in which no Frobenius twist appear in the Ext of the right hand side).

**Corollary 11.2.** *Let $k$ be an infinite perfect field of positive characteristic. Let $E_\infty$ be the graded vector space equal to $k$ in even degrees and to $0$ in odd degrees. There is a graded isomorphism, natural with respect to the strict polynomial functors $F$ and $G$, and where the degree on the right hand side is computed by totalizing the* Ext*-degree with the degree of the functor $G_{E_\infty}$:*

$$\mathrm{Ext}^*_{k[\mathbf{P}_k]}(F, G) \simeq \mathrm{Ext}^*_{\Gamma\mathbf{P}_k}(F, G_{E_\infty})\ .$$

11.2. **Bifunctor cohomology.** The words 'bifunctor cohomology' are sometimes used [13, 53] to denote the Hochschild cohomology of $k[\mathbf{P}_k]$ or $\Gamma^d\mathbf{P}_k$. The study of bifunctor cohomology was initiated in [13] for a finite field $k$. Here we extend two of the main results of [13] to infinite perfect fields of positive characteristic.

Let $\mathcal{K}$ denote either $k[\mathbf{P}_k]$ or $\Gamma^d\mathbf{P}_k$. The bifunctor cohomology of $B \in \mathcal{K}^{\mathrm{op}} \otimes \mathcal{K}$-**Mod** is defined as the extensions

$$\mathrm{HH}^*(\mathcal{K}, B) := \mathrm{Ext}^*_{\mathcal{K}^{\mathrm{op}}\otimes\mathcal{K}}(\mathcal{K}, B)$$

where the first argument in the Ext is the bifunctor given by homomorphisms in $\mathcal{K}$. Thus, if $\mathrm{gl}(v,w) := \mathrm{Hom}_k(v,w)$, then $\mathcal{K}(v,w) = k[\mathrm{gl}(v,w)]$ in the case of ordinary functors and $\mathcal{K}(v,w) = \Gamma^d(\mathrm{gl}(v,w))$ in the case of strict polynomial functors. If $B$ has *separable type*, that is, if $B(v,w) = \mathrm{Hom}_k(F(v), G(w))$ for some functors $F$ and $G$, we have isomorphisms natural with respect to $F$ and $G$ [13, Prop 2.2]:

$$\mathrm{HH}^*(\mathcal{K}, B) \simeq \mathrm{Ext}^*_{\mathcal{K}}(F, G)\ . \tag{57}$$

These isomorphisms can often be used to reduce questions regarding bifunctor cohomology to questions regarding functor cohomology, especially for strict polynomial bifunctors since the standard injectives of the category have separable type.

Just like for functors of one variable, we have a forgetful functor

$$k[\mathbf{P}_k^{\mathrm{op}}] \otimes k[\mathbf{P}_k] \to \Gamma^d\mathbf{P}_k^{\mathrm{op}} \otimes \Gamma^d\mathbf{P}_k$$

induced by restriction along the functor $\gamma^d\otimes\gamma^d$, where $\gamma^d$ is defined in example 2.11. If $k$ is infinite, this forgetful functor is fully faithful. By restricting extensions along $\gamma^d\otimes\gamma^d$ and by pulling back along the morphism of functors $\gamma^d(\mathrm{gl}) : k[\mathrm{gl}] \to \Gamma^d\mathrm{gl}$ we obtain a restriction map:

$$\mathrm{HH}^*(\Gamma^d\mathbf{P}_k, B) \to \mathrm{HH}^*(k[\mathbf{P}_k], B)\ .$$

For all $r \geq 0$, we let $\Phi'_k$ denote the composition of this restriction map together with the isomorphism induced by restriction along the $(-r)$-th Frobenius twist and by the canonical isomorphism $k[\mathrm{gl}]^{(-r)} = k[\mathrm{gl}^{(-r)}] \simeq k[^{(-r)}\mathrm{gl}] = k[\mathrm{gl}]$:

$$\Phi'_k : \mathrm{HH}^*(\Gamma^{dp^r}\mathbf{P}_k, B^{(r)}) \to \mathrm{HH}^*(k[\mathbf{P}_k], B^{(r)}) \xrightarrow{\simeq} \mathrm{HH}^*(k[\mathbf{P}_k], B). \tag{58}$$

The next proposition is the analogue of [13, Thm 7.6] for infinite perfect fields.

**Proposition 11.3.** *Let $k$ be an infinite perfect field of positive characteristic $p$ and let $B$ be a strict polynomial bifunctor in $\Gamma^d\mathbf{P}_k^{\mathrm{op}}\otimes\Gamma^d\mathbf{P}_k$-$\mathbf{Mod}$. Then the map (58) is $2p^r$-connected.*

*Proof.* By considering a coresolution of $B$ by products of standard injectives, we reduce the proof to the case where $B$ is a standard injective, hence when $B=\mathrm{Hom}_k(F,G)$. In this case, the map $\Phi_k'$ identifies with the composition

$$\mathrm{Ext}^*_{\Gamma^d p^r\mathbf{P}_k}(F^{(r)},G^{(r)})\to\mathrm{Ext}^*_{\mathrm{gen}}(F,G)\xrightarrow{\Phi_k}\mathrm{Ext}^*_{k[\mathbf{P}_k]}(F,G)$$

where the first map is the canonical inclusion (which is $2p^r$-connected by proposition-definition 7.5) and the isomorphism $\Phi_k$ of corollary 10.11. Whence the result. □

Proposition 11.3 should be seen as a way to obtain explicit bifunctor cohomology over $k[\mathbf{P}_k]$, in the spirit of section 11.1. For example, we obtain the following computation by letting $r\to\infty$ in [47, Thm 1] and applying proposition 11.3.

**Corollary 11.4.** *Let $E_\infty$ denote the graded vector space which equals $k$ in even degrees and $0$ in odd degrees and consider $k[\mathfrak{S}_d]$ as a vector space placed in degree zero. Let the symmetric group $\mathfrak{S}_d$ act on $E_\infty^{\otimes d}$ by permuting the factors of the tensor product, and by conjugation on $k[\mathfrak{S}_d]$. There is an isomorphism of graded vector spaces*

$$\mathrm{HH}^*(k[\mathbf{P}_k],S^d\mathrm{gl})\simeq(E_\infty^{\otimes d})\otimes_{\mathfrak{S}_d}k[\mathfrak{S}_d]\;.$$

We finish our section on bifunctor cohomology by describing its relation with the cohomology of $\mathrm{GL}_n(k)$. This relation provides a motivation for computing bifunctor cohomology over $k[\mathbf{P}_k]$, and we will also need it in the prooof of theorem 11.13 in section 11.4. We first need the following generalization of Quillen's vanishing of the mod $p$ homology of $\mathrm{GL}_\infty(\mathbb{F}_q)$.

**Lemma 11.5.** *Let $k$ be a perfect field of positive characteristic $p$. The mod $p$ homology of $\mathrm{GL}_\infty(k)$ is zero in positive degrees.*

*Proof.* By the $p$-local Hurewicz theorem [32, Thm 1.8.1], it is equivalent to prove that the mod $p$ homotopy groups of $B\,\mathrm{GL}(k)^+$ are trivial, or equivalently that the Quillen $K$-theory $K_n(k)$ is uniquely $p$-divisible for all positive $n$. As a consequence of the Geisser-Levine theorem [55, VI Thm 4.7], $K_n(k)$ is uniquely $p$-divisible if and only if the Milnor $K$-theory $K_n^M(k)$ is uniquely $p$-divisible. That the latter is $p$-divisible follows from the fact that $k^\times$ is $p$-divisible (because $k$ is perfect), hence that the abelian group $(k^\times)^{\otimes_{\mathbb{Z}}n}$ is $p$-divisible. It is *uniquely* $p$-divisible as a consequence of Izhboldin's theorem [55, III Thm 7.8]. □

If $B$ is an object of $k[\mathbf{P}_k^{\mathrm{op}}]\otimes k[\mathbf{P}_k]$-$\mathbf{Mod}$, then $B(k^n,k^n)$ is endowed with an action of $\mathrm{GL}_n(k)$. Namely, an element $g\in\mathrm{GL}_n(k)$ acts as $B(g^{-1},g)$ on $B(k^n,k^n)$. For example $\mathrm{GL}_n(k)$ acts by conjugation on $\mathrm{gl}(k^n,k^n)=\mathrm{End}_k(k^n)$. The identity of $k^n$ is invariant under conjugation, hence we have a morphism of representations

$$f_n:k\to k[\mathrm{gl}](k^n,k^n)=k[\mathrm{End}_k(k^n)]$$

defined by $f(\lambda)=\lambda f(\mathrm{id}_{k^n})$. Then evaluation on $k^n$ and pullback along $f_n$ yields a graded map

$$\mathrm{HH}^*(k[\mathbf{P}_k],B)\to\mathrm{H}^*(\mathrm{GL}_n(k),B(k^n,k^n))\;. \tag{59}$$

The next proposition 11.6 and its corollary 11.7 generalize Suslin's comparison result [14, Thm A.1] and its extension to bifunctors [13, Thm 7.4] from finite fields to arbitrary perfect fields of positive characteristic. It is a consequence of the stable $K$-theory computations of Scorichenko [42], which are reformulated in terms of stable homological calculations in [8], the homological stabilization result [37, Thm 5.11], together with the vanishing lemma 11.5.

**Proposition 11.6.** *Let $k$ be a perfect field of characteristic $p$. Assume that $B$ is polynomial of degree $d$ with finite-dimensional values. Then the comparison map (59) is $\frac{1}{2}(n-1-d)$-connected.*

*Proof.* Let $B^\sharp$ denote the Kuhn dual of $B$, that is the bifunctor defined by $B^\sharp(v,w) = {}^\vee B({}^\vee v, {}^\vee w)$ where ${}^\vee v$ refers to the dual of a $k$-vector space $v$. By using proposition 2.12, the symmetry of Tor (that is $\mathrm{Tor}_*^{\mathcal{K}}(F,G) \simeq \mathrm{Tor}_*^{\mathcal{K}^{\mathrm{op}}}(G,F)$) and the fact that ${}^\vee - : \mathbf{P}_k^{\mathrm{op}} \to \mathbf{P}_k$ is an equivalence of categories, the proof reduces to show that evaluation on $k^n$ and restriction along $f_n$ yields a $\frac{1}{2}(n-1-d)$-connected map:

$$\mathrm{H}_*(\mathrm{GL}_n(k), B^\sharp(k^n,k^n)) \to \mathrm{Tor}_*^{k[\mathbf{P}_k]}(k[\mathrm{gl}], B^\sharp)\ .$$

In order to achieve this, we compare the homology of $\mathrm{GL}_n(k)$ with the homology of $\mathrm{GL}_\infty(k)$. Namely we let $B^\sharp(k^\infty,k^\infty)$ denote representation of $\mathrm{GL}_\infty(k)$ obtained by taking the colimit of the $B^\sharp(k^n,k^n)$. Let $\rho_n$ denote the composition

$$\mathrm{H}_*(\mathrm{GL}_n(k), B^\sharp(k^n,k^n)) \to \mathrm{H}_*(\mathrm{GL}_n(k), B^\sharp(k^\infty,k^\infty)) \to \mathrm{H}_*(\mathrm{GL}_\infty(k), B^\sharp(k^\infty,k^\infty))$$

where the first map is induced by the canonical inclusion $B^\sharp(k^n,k^n) \to B^\sharp(k^\infty,k^\infty)$ and the second one is given by restriction along $\mathrm{GL}_n(k) \hookrightarrow \mathrm{GL}_\infty(k)$. We have a commutative square, in which the vertical isomorphism on the right hand side is the base change isomorphism of [31, Thm 14.2] and the bottom arrow (†) is induced by the map $f_\infty : \mathbb{Z} \to \mathbb{Z}[\mathbf{P}_k](k^\infty,k^\infty)$ and by evaluation on $k^\infty$.

$$\begin{array}{ccc}
\mathrm{H}_*(\mathrm{GL}_n(k), B^\sharp(k^n,k^n)) & \longrightarrow & \mathrm{Tor}_*^{k[\mathbf{P}_k^{\mathrm{op}}\times\mathbf{P}_k]}(k[\mathbf{P}_k], B^\sharp) \\
{\scriptstyle\rho_n}\downarrow & & \downarrow\simeq \\
\mathrm{H}_*(\mathrm{GL}_\infty(k), B^\sharp(k^\infty,k^\infty)) & \xrightarrow{(\dagger)} & \mathrm{Tor}_*^{\mathbb{Z}[\mathbf{P}_k^{\mathrm{op}}\times\mathbf{P}_k]}(\mathbb{Z}[\mathbf{P}_k], B^\sharp)
\end{array}\ .$$

The map (†) is an isomorphism by [8, Thm 5.6] and the vanishing lemma 11.5, and $\rho_n$ is $\frac{1}{2}(n-1-d)$-connected by [37, Thm 5.11]. Whence the result. □

**Corollary 11.7.** *Let $k$ be a perfect field of characteristic $p$. Assume either that (i) both $F$ and $G$ are polynomial functors of degree less or equal to $d$ with finite-dimensional values, or that (ii) both $F$ and $G$ are strict polynomial functors of weight less or equal to $d$. Evaluation on $k^n$ yields a $\frac{1}{2}(n-1-2d)$-connected map*

$$\mathrm{ev_n} : \mathrm{Ext}^*_{k[\mathbf{P}_k]}(F,G) \to \mathrm{Ext}^*_{\mathrm{GL}_n(k)}(F(k^n), G(k^n))\ .$$

*Proof.* Assume (ii). Take a resolution or $F$ by direct sums of standard projectives in $\Gamma\mathbf{P}_k$-**Mod** and a coresolution of $G$ by products of standard injectives in $\Gamma\mathbf{P}_k$-**Mod**. Then by a standard spectral sequence argument we can restrict ourselves to the case where $F$ is a standard projective and $G$ is a standard injective, in particular to the case where $F$ and $G$ have finite-dimensional values. Morover strict polynomial functors of weight less or equal to $d$ are polynomial of degree less or equal to $d$, see remark 7.10. Hence it suffices to prove the corollary under hypothesis (i).

Assume (i) and let $B$ denote the bifunctor $B(v,w) = \mathrm{Hom}_k(F(v), G(w))$. There is a commutative diagram whose horizontal maps are the canonical isomorphisms

$$\begin{array}{ccc} \mathrm{HH}^*(k[\mathbf{P}_k], B) & \xrightarrow{\simeq} & \mathrm{Ext}^*_{k[\mathbf{P}_k]}(F,G) \\ \downarrow{\scriptstyle (59)} & & \downarrow{\scriptstyle \mathrm{ev}_n} \\ \mathrm{H}^*(\mathrm{GL}_n(k), B(k^n,k^n)) & \xrightarrow{\simeq} & \mathrm{Ext}^*_{\mathrm{GL}_n(k)}(F(k^n), G(k^n)) \end{array} .$$

Hence the result follows from proposition 11.6. (Note that $B$ has degree less or equal to $2d$). ☐

11.3. **Orthogonal and symplectic cohomology.** Bifunctor cohomology and its relation to the cohomology of general linear groups has an analogue for symplectic and orthogonal groups that we now describe. Here we assume that $k$ has odd characteristic $p$.

Assume that $G = \mathrm{O}_{n,n}(k)$ or $G = \mathrm{Sp}_{2n}(k)$. We associate to $G$ a 'characteristic functor' $X : \mathbf{P}_k \to \mathbf{P}_k$, namely $X = S^2$ in the orthogonal case $X = \Lambda^2$ in the symplectic case. We define an analogue of bifunctor cohomology as follows. Let $F$ be an object of $k[\mathbf{P}_k]$-**Mod** or of $\Gamma^d\mathbf{P}_k$-**Mod**, we set:

$$\mathrm{H}^*_X(k[\mathbf{P}_k], F) = \mathrm{Ext}^*_{k[\mathbf{P}_k]}(k[X], F),$$
$$\mathrm{H}^*_X(\Gamma^d\mathbf{P}_k, F) = \begin{cases} \mathrm{Ext}^*_{\Gamma^d\mathbf{P}_k}(\Gamma^{d/2}\circ X, F) & \text{if } d \text{ is even,} \\ 0 & \text{if } d \text{ is odd.} \end{cases}$$

By restricting extensions along the functor $\gamma^d : k[\mathbf{P}_k] \to \Gamma^d\mathbf{P}_k$ and by pulling back along the morphism of functors $\gamma^{d/2}(X) : k[X] \to \Gamma^{d/2}X$ we obtain a restriction map (which is the zero map if $d$ is odd):

$$\mathrm{H}^*_X(\Gamma^d\mathbf{P}_k, F) \to \mathrm{H}^*_X(k[\mathbf{P}_k], F) .$$

For all $r \geq 0$, we let $\Phi'_{k,X}$ denote the composition of this restriction map together with the isomorphism induced by restriction along the $(-r)$-th Frobenius twist and by the canonical isomorphism $k[X]^{(-r)} = k[X^{(-r)}] \simeq k[^{(-r)}X] = k[X]$:

$$\Phi'_{k,X} : \mathrm{H}^*_X(\Gamma^{dp^r}\mathbf{P}_k, F^{(r)}) \to \mathrm{H}^*_X(k[\mathbf{P}_k], F^{(r)}) \xrightarrow{\simeq} \mathrm{H}^*_X(k[\mathbf{P}_k], F). \tag{60}$$

**Proposition 11.8.** *Let $k$ be an infinite perfect field of odd positive characteristic $p$ and let $F$ be a $d$-homogeneous strict polynomial functor. The map* (60) *is $2p^r$-connected.*

*Proof.* Since $p$ is odd, $X$ is a direct summand of the second tensor power functor $\otimes^2$, hence $\Phi'_{k,X}$ is a retract of $\Phi'_{k,\otimes^2}$. Thus we have to show that $\Phi'_{k,\otimes^2}$ is $2p^r$-connected. We achieve this by reformulating the problem in terms of bifunctor cohomology. Let $B$ be the object of $\Gamma^d(\mathbf{P}_k^{\mathrm{op}}\times\mathbf{P}_k)$-**Mod** such that $B(v,w) = F({}^\vee v\oplus w)$ where ${}^\vee v$ denotes the dual of the $k$-vector space $v$. We have a finite direct sum decomposition

$$\Gamma^d(\mathbf{P}_k^{\mathrm{op}}\times\mathbf{P}_k)\text{-}\mathbf{Mod} = \bigoplus_{i+j=d} \Gamma^i\mathbf{P}_k^{\mathrm{op}}\otimes\Gamma^j\mathbf{P}_k\text{-}\mathbf{Mod}$$

hence $B$ decomposes into a direct sum of homogeneous summands $B = \bigoplus_{i+j=d} B^{i,j}$.

We claim that for any bifunctor $B^{i,j} \in \Gamma^i\mathbf{P}_k^{\mathrm{op}}\otimes\Gamma^j\mathbf{P}_k$-**Mod**, the bifunctor cohomology $\mathrm{HH}^*(k[\mathbf{P}_k], B^{i,j})$ is zero if $i \neq j$. Indeed, by considering an injective coresolution of $B$ in $\Gamma^i\mathbf{P}_k^{\mathrm{op}}\otimes\Gamma^j\mathbf{P}_k$-**Mod** it suffices to prove the result when $B$ is a standard injective, hence when $B$ has separable type. In that case, the question reduces to the vanishing of Ext in $k[\mathbf{P}_k]$-**Mod** between a $i$-homogeneous strict polynomial functor and a $j$-homogeneous strict polynomial functor hence the claim follows from the vanishing lemma 7.13.

Sum-diagonal adjunction (as in example 2.19) and restriction along the equivalence of categories $k[\mathbf{P}_k^{\mathrm{op}}] \otimes k[\mathbf{P}_k] \simeq k[\mathbf{P}_k] \otimes k[\mathbf{P}_k]$ given by the duality functor ${}^\vee - : \mathbf{P}_k^{\mathrm{op}} \to \mathbf{P}_k$ yields an isomorphism

$$\mathrm{H}^*_{\otimes^2}(k[\mathbf{P}_k], F) \simeq \mathrm{HH}^*(k[\mathbf{P}_k], B)$$

Our claim implies that for $d$ odd, the right hand side is zero, hence that the comparison map (60) is an isomorphism.

Assume now that $d$ is even. We have a commutative diagram where the bottom isomorphism is described above, and the top isomorphism is its analogue for strict polynomial functors (induced by sum-diagonal adjunction, restriction along the equivalence of categories $\Gamma^{dp^r}(\mathbf{P}_k^{\mathrm{op}} \times \mathbf{P}_k) \simeq \Gamma^{dp^r}(\mathbf{P}_k \times \mathbf{P}_k)$ provided by the duality functor and projection onto the summand $B^{d,d}$ of $B$):

$$\begin{array}{ccc} \mathrm{H}^*_{\otimes^2}(\Gamma^{dp^r}\mathbf{P}_k, F^{(r)}) & \xrightarrow{\simeq} & \mathrm{HH}^*(k[\mathbf{P}_k], B^{d,d}) \\ \downarrow{\scriptstyle \Phi'_{k,\otimes^2}} & & \downarrow{\scriptstyle \Phi'_k} \\ \mathrm{H}^*_{\otimes^2}(k[\mathbf{P}_k], F) \xrightarrow{\simeq} \mathrm{HH}^*(k[\mathbf{P}_k], B) & = & \mathrm{HH}^*(k[\mathbf{P}_k], B^{d,d}) \end{array} .$$

Hence $\Phi'_{k,\otimes^2}$ is $2p^r$-connected by proposition 11.3. □

Now we explain the relation between the cohomology groups $\mathrm{H}^*_X(k[\mathbf{P}_k], F)$ and the cohomology of the symplectic and orthogonal group. We first need the following vanishing result. We gratefully thank Baptiste Calmès for helping us with the literature relative to hermitian $K$-theory.

**Lemma 11.9.** *Let $k$ be a perfect field of odd characteristic $p$. Then the mod $p$ homology of the groups $\mathrm{Sp}_\infty(k)$ and $\mathrm{O}_{\infty,\infty}(k)$ is zero in positive degrees.*

*Proof.* Let $G = \mathrm{Sp}_\infty(k)$ or $\mathrm{O}_{\infty,\infty}(k)$. By the universal coefficient theorem, it is equivalent to prove that $H_i(G, \mathbb{Z})$ is uniquely $p$-divisible for $i > 0$. If $A$ is an abelian group, we let $A[1/2]$ denote the tensor product $A \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$. Since $p$ is odd, $A$ is uniquely $p$-divisible if and only if $A[1/2]$ is uniquely $p$-divisible. And since $\mathbb{Z}[1/2]$ is flat we have $H_*(G, \mathbb{Z})[1/2] = H_*(G, \mathbb{Z}[1/2])$. Thus the statement of the lemma is equivalent to $H_i(G, \mathbb{Z}[1/2])$ being uniquely $p$-divisible for $i > 0$.

Since $k$ is a field of odd characteristic, the Witt groups $W(k)$ are an $\mathbb{F}_2$-vector space [40, Chap. 2, Thm 6.4]. Thus by [23, Thm 3.18] $H_*(G, \mathbb{Z}[1/2])$ is equal to $T_*(k)$, that is, to the homology of a space $\mathcal{C}(k)$ which is a retract of the localized classifying space $(B\mathcal{P}'^{+}_k)_{(2)}$, see [23, p. 253] for the latter point. Since the integral homology of $(B\mathcal{P}'^{+}_k)_{(2)}$ is equal to $H_*(B\mathcal{P}'^{+}_k, \mathbb{Z})[1/2]$, the lemma will be proved if we can prove that $B\mathcal{P}'^{+}_k$ has uniquely $p$-divisible positive integral homology groups.

But $B\mathcal{P}'^{+}_k$ has the weak homotopy type of $K_0(k) \times B\,\mathrm{GL}_\infty(k)^+$, hence its integral homology is direct sum of copies of the integral homology of $B\,\mathrm{GL}_\infty(k)^+$. Since $k$ is perfect, these integral homology groups are uniquely $p$-divisible in positive degrees by lemma 11.5 and by the universal coefficient theorem. □

*Remark* 11.10. Instead of relying on the results of [23], one could prove the lemma by using the formula of [41, Rk 7.8], which says that after tensoring by $\mathbb{Z}[1/2]$, the hermitian $K$-theory (hence [41, App A] the homotopy groups of $BG^+$ for $G = \mathrm{Sp}_\infty(k)$ or $\mathrm{O}_{\infty,\infty}(k)$) is the direct sum of a term computed from the $K$-theory of $k$ and a term given by Balmer's Witt groups of $k$ tensored with $\mathbb{Z}[1/2]$.

For all functors $F$ in $k[\mathbf{P}_k]$-**Mod**, the vector space $F(k^{2n})$ has a natural action of $G$, where $g \in G$ acts as $F(g)$. The quadratic form on $k^{2n}$ used to define $G$ yields an invariant $\omega \in X(k^{2n})$ under the action of $G$, hence we have a $G$-equivariant map

$$f_{2n,X} : k \to k[X](k^{2n}) = k[X(k^{2n})]$$

such that $f_{2n,X}(\lambda) = \lambda[\omega]$. Evaluation on $k^{2n}$ and pullback along $f_{2n,X}$ yields a graded map

$$\mathrm{H}^*_X(k[\mathbf{P}_k], F) \to \mathrm{H}^*(G, F(k^{2n})). \tag{61}$$

The following proposition and its corollary are proved exactly in the same fashion as proposition 11.6 and corollary 11.7, relying on the stable homology computations of [8, Cor 5.4], the homological stabilization result [37, Thm 5.15] and the vanishing lemma 11.9.

**Proposition 11.11.** *Let $k$ be a perfect field of odd characteristic $p$. Assume that $F$ is polynomial of degree $d$ with finite-dimensional values. Then the comparison map* (61) *is $\frac{1}{2}(n-2-d)$-connected.*

**Corollary 11.12.** *Let $k$ be a perfect field of odd characteristic $p$. Assume that $F$ is a strict polynomial functor of weight less or equal to $d$. Then the comparison map* (61) *is $\frac{1}{2}(n-2-d)$-connected.*

11.4. **Rational and discrete cohomology of classical groups.** Let $G$ be an algebraic group over $k$, let $\mathbf{Mod}_G$ denote the category of all representations of the discrete group $G$, and let $\mathbf{Rat}_G$ denote the full subcategory of $\mathbf{Mod}_G$ on the rational representations as in [22]. Extensions between two rational representations $V$ and $W$ can be computed in $\mathbf{Rat}_G$ or $\mathbf{Mod}_G$. In the sequel, we let

$$\underline{\mathrm{Ext}}^*_G(V,W) := \mathrm{Ext}^*_{\mathbf{Rat}_G}(V,W)\ , \qquad \underline{\mathrm{H}}^*(G,k) := \underline{\mathrm{Ext}}^*_G(k,k)\ ,$$
$$\mathrm{Ext}_G(V,W) := \mathrm{Ext}^*_{\mathbf{Mod}_G}(V,W)\ , \qquad \mathrm{H}^*(G,k) := \mathrm{Ext}^*_G(k,k)\ .$$

There is a canonical morphism:

$$\underline{\mathrm{Ext}}^*_G(V,W) \to \mathrm{Ext}^*_G(V,W)$$

which is far from being an isomorphism in general. An important difference between the source and the target of the canonical morphism is the behaviour of Frobenius morphisms. Namely assume that $G$ is one of the classical groups $\mathrm{GL}_n(k)$, $\mathrm{Sp}_{2n}(k)$ or $\mathrm{O}_{n,n}(k)$ (and if $k$ has characteristic $\neq 2$ in the latter case), and let let $\phi : G \to G$, $[a_{ij}] \mapsto [a_{ij}^p]$, denote the morphism of algebraic groups induced by the Frobenius endomorphism of $k$, and let $V^{[r]}$ denote the restriction of $V$ along $\phi^r$. We have a commutative ladder whose horizontal arrows are induced by restriction along $\phi$:

$$\begin{array}{ccccccc}
\underline{\mathrm{Ext}}^i_G(V,W) & \longrightarrow \cdots \longrightarrow & \underline{\mathrm{Ext}}^i_G(V^{[r]},W^{[r]}) & \longrightarrow & \underline{\mathrm{Ext}}^i_G(V^{[r+1]},W^{[r+1]}) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Ext}^i_G(V,W) & \longrightarrow \cdots \longrightarrow & \mathrm{Ext}^i_G(V^{[r]},W^{[r]}) & \longrightarrow & \mathrm{Ext}^i_G(V^{[r+1]},W^{[r+1]}) & \longrightarrow & \cdots .
\end{array}$$

Assume that $k$ is perfect. Then $\phi$ has an inverse $\phi^{-1}([a_{ij}]) = [a_{ij}^{-p}]$ so that the morphisms in the bottom row are all isomorphisms. However $\phi$ has no inverse in the sense of algebraic groups, so this argument does not apply to the top row. Instead, it is known [22, II 10.14] that the morphisms in the top row are all injective (and that they are isomorphisms for $r \gg 0$ only). Hence, if $k$ is perfect the ladder yields canonical maps:

$$\Phi_{k,G} : \underline{\mathrm{Ext}}^*_G(V^{[r]}, W^{[r]}) \to \mathrm{Ext}^*_G(V, W) \ . \tag{62}$$

Embed $G = \mathrm{GL}_n(k)$, $\mathrm{Sp}_{2n}(k)$ or $\mathrm{O}_{n,n}(k)$ in the multiplicative monoïd of matrices $M_m(k)$ in the usual way (here $m = n$ for $\mathrm{GL}_n(k)$ and $m = 2n$ for in the case of the symplectic or orthogonal groups). Then a finite-dimensional representation $V$ of is called *polynomial of degree less or equal to $d$* if it is the restriction to $G$ of a representation of $M_m(k)$, such that the coordinate maps of the action morphism

$$\rho_V : M_m(k) \to \mathrm{End}_k(V) \simeq M_{\dim V}(k)$$

are polynomials of degree less or equal to $d$ of the $m^2$ entries of $[a_{ij}] \in M_m(k)$. An infinite-dimensional representation $V$ is *polynomial of degree less or equal to $d$* if every element of $V$ is contained in a finite-dimensional subrepresentation which is polynomial of degree less or equal to $d$.

**Theorem 11.13.** *Let $k$ be an infinite perfect field of positive characteristic $p$, let $G = \mathrm{GL}_n(k)$, let $V$ and $W$ be two polynomial representations of degree less or equal to $d$, and let $r$ be a nonnegative integer. Assume that $n \ge \max\{dp^r, 4p^r + 2d + 1\}$. Then the comparison map in equation (62) is $2p^r$-connected.*

*Proof.* Evaluation on $k^n$ yields an exact functor $\mathrm{ev}_n : k[\mathbf{P}_k]\text{-}\mathbf{Mod} \to \mathbf{Mod}_{\mathrm{GL}_n(k)}$, which restricts to a exact functor $\mathrm{ev}_n : \Gamma\mathbf{P}_k\text{-}\mathbf{Mod} \to \mathbf{Rat}_{\mathrm{GL}_n(k)}$. Since $n \ge d$ we know from [18, Lm 3.4] that there exists two strict polynomial functors $F$ and $G$ of weight less or equal to $d$ such that $V \simeq F(k^n)$ and $W = G(k^n)$, hence such that $V^{[r]} \simeq F^{(r)}(k^n)$ and $W^{[r]} \simeq G^{(r)}(k^n)$ for all $r \ge 0$. We consider the following commutative diagram:

$$\begin{array}{ccc} \underline{\mathrm{Ext}}^*_{\mathrm{GL}_n(k)}(V^{[r]}, W^{[r]}) & \xleftarrow{\mathrm{ev}_n} & \mathrm{Ext}^*_{\Gamma\mathbf{P}_k}(F^{(r)}, G^{(r)}) \\ {\scriptstyle \Phi_{k,\mathrm{GL}_n(k)}}\downarrow & & \downarrow{\scriptstyle \Phi'_k} \\ \mathrm{Ext}^*_{\mathrm{GL}_n(k)}(V, W) & \xleftarrow{\mathrm{ev}_n} & \mathrm{Ext}^*_{k[\mathbf{P}_k]}(F, G) \end{array} .$$

in which $\Phi'_k$ is the composition

$$\mathrm{Ext}_{\Gamma\mathbf{P}_k}(F^{(r)}, G^{(r)}) \to \mathrm{Ext}^*_{\mathrm{gen}}(F, G) \xrightarrow[\simeq]{\Phi_k} \mathrm{Ext}^*_{k[\mathbf{P}_k]}(F, G)$$

where the second map is the isomorphism of corollary 10.11 and the first one is the canonical inclusion. This canonical inclusion is $2p^r$-connected by proposition-definition 7.5, hence $\Phi'_k$ is $2p^r$-connected. Moreover, since $n \ge dp^r$ the top horizontal map is an isomorphism by [18, Cor 3.13]. Finally, the bottom horizontal map is $2p^r$-connected by corollary 11.7. Thus $\Phi_{k,\mathrm{GL}_n(k)}$ is $2p^r$-connected. □

*Remark* 11.14 (Generic extensions of $\mathrm{GL}_n(k)$). Assume that $V$ and $W$ are finite dimensional polynomial representations of degree $d$ of $\mathrm{GL}_n(k)$. It is known [22, II.10.16] that the maps $\underline{\mathrm{Ext}}^*_{\mathrm{GL}_n(k)}(V^{[r]}, W^{[r]}) \to \underline{\mathrm{Ext}}^*_{\mathrm{GL}_n(k)}(V^{[r+1]}, W^{[r+1]})$ are injective, their colimit is called the *generic extensions* between $V$ and $W$. We denote it by $\underline{\mathrm{Ext}}^*_{\mathrm{gen}}(V,W)$. The comparison map $\Phi_{k,\mathrm{GL}_n(k)}$ factors through generic extensions:

$$\begin{array}{ccc} \underline{\mathrm{Ext}}^*_{\mathrm{gen}}(V,W) & & \\ \uparrow & \overset{\Phi_{\mathrm{gen}}}{\searrow} & \\ \underline{\mathrm{Ext}}^*_{\mathrm{GL}_n(k)}(V^{[r]},W^{[r]}) & \xrightarrow{\Phi_{k,\mathrm{GL}_n(k)}} & \mathrm{Ext}^*_{\mathrm{GL}_n(k)}(V,W) \end{array} .$$

The main theorem of [6] or rather the general linear group version given in [18, Thm 7.3] imply that the vertical arrow is $((p-1)r+2)$-connected provided that (i) $V$ and $W$ are defined over $\mathbb{F}_p$ and (ii) $k$ is a big enough finite field (with respect to $r$, $V$ and $W$). By base change [22, I.4.13], the vertical map is also an isomorphism when condition (ii) is replaced by: (ii') $k$ is an infinite field. Moreover, every finite dimensional polynomial representation has a filtration whose associated graded object is defined over $\mathbb{F}_p$, namely its Jordan-Hölder filtration. Hence, condition (i) can be removed by inspecting the long exact sequences associated to the Jordan-Hölder filtration. Thus we can state a version of theorem 11.13 in terms of generic cohomology, at the price of a worse connectivity bound. Namely, the comparison map $\Phi_{\mathrm{gen}}$ above is $((p-1)r+2)$-connected provided that $n \geq \max\{dp^r, 4p^r+2d+1\}$.

Theorem 11.13 has an analogue for orthogonal and symplectic groups. Here we take $V = k$ hence $V^{[r]} = k$ and the comparison map (62) can be rewritten as a map

$$\Phi_{k,G} : \mathrm{H}^*(G, W^{[r]}) \to \underline{\mathrm{H}}^*(G,W) . \tag{63}$$

**Theorem 11.15.** *Let $k$ be an infinite perfect field of odd characteristic $p$, let $G = \mathrm{Sp}_{2n}(k)$ or $\mathrm{O}_{n,n}(k)$ and let $W$ be a polynomial representation of degree less or equal to $d$. Assume that $2n \geq \max\{dp^r, 8p^r+4+2d\}$. Then the comparison map (63) is $2p^r$-connected.*

*Proof.* We proceed in the same way as in the proof of theorem 11.15. We know that $W = F(k^{2n})$ for some strict polynomial functor of weight less or equal to $d$. Furthermore, if $F_i$ is the $i$-homogeneous component of $F$ then $W = \bigoplus_{0\leq i\leq d} F_i(k^{2n})$, and since the source and the target of $\Phi_{k,G}$ are additive with respect to $W$, it suffices to prove the isomorphism when $F$ is homogeneous of weight (less or equal to) $d$.

We have a commutative diagram:

$$\begin{array}{ccc} \underline{\mathrm{H}}^*(G,W^{[r]}) & \longleftarrow & \mathrm{H}^*_X(\Gamma^{dp^r}\mathbf{P}_k, F^{(r)}) \\ {\scriptstyle \Phi_{k,G}}\downarrow & & \downarrow{\scriptstyle \Phi'_{k,X}} \\ \mathrm{H}^*(G,W) & \longleftarrow & \mathrm{H}^*_X(k[\mathbf{P}_k], F) \end{array} .$$

To be more specific, the bottom horizontal map of the diagram is the comparison map of corollary 11.12 hence it is $2p^r$-connected. The top horizontal map has a similar definition, namely it is zero if $d$ is odd, and if $d$ is even it is induced by evaluation on $k^{2n}$ and pullback along the $G$-equivariant morphism $f'_{2n,X} : k \to \Gamma^{d/2}(X(k^{2n}))$ such that $f_{2n,X'}(\lambda) = \lambda\omega^{\otimes d/2}$, where $\omega \in X(k^{2n})$ is the invariant element associated to the quadratic form defining $G$. This top horizontal arrow is an isomorphism by [48, Thm 3.17] or [48, Thm 3.24]. Moreover $\Phi'_k$ is $2p^r$-connected by proposition 11.8. The connectivity of $\Phi_{k,G}$ follows. □

*Remark* 11.16. Theorem 11.15 can be reformulated in terms of generic cohomology in the same fashion as we explained it for $\mathrm{GL}_n(k)$ in remark 11.14.

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