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HYPERSURFACES OF CONSTANT GAUSS-KRONECKER CURVATURE WITH LI-NORMALIZATION IN AFFINE SPACE

XIN NIE AND ANDREA SEPPI

ABSTRACT. For convex hypersurfaces in the affine space \mathbb{A}^{n+1} ($n \geq 2$), A.-M. Li introduced the notion of α -normal field as a generalization of the affine normal field. By studying a Monge-Ampère equation with gradient blowup boundary condition, we show that regular domains in \mathbb{A}^{n+1} , defined with respect to a proper convex cone and satisfying some regularity assumption if $n \geq 3$, are foliated by complete convex hypersurfaces with constant Gauss-Kronecker curvature relative to the Li-normalization. When $n = 2$, a key feature is that no regularity assumption is required, and the result extends our recent work about the $\alpha = 1$ case.

1. INTRODUCTION

As shown by B. Chen, A.-M. Li, U. Simon and G. Zhao [17, 18], Euclidean-complete convex hypersurfaces of constant affine Gauss-Kronecker curvature in the $(n + 1)$ -dimensional real affine space \mathbb{A}^{n+1} ($n \geq 2$) are governed by the Monge-Ampère problem

$$(1.1) \quad \begin{cases} \det D^2 u = (-w_\Omega)^{-(n+2)} & \text{in } \Omega, \\ u|_{\partial\Omega} = \varphi, \\ \|Du(x)\| \rightarrow +\infty & \text{as } x \in \Omega \text{ tends to } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded convex domain, φ is a function on $\partial\Omega$, and $w_\Omega \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ is the unique convex solution, established by Cheng-Yau [8], of another Monge-Ampère equation

$$(1.2) \quad \begin{cases} \det D^2 w = (-w)^{-(n+2)} & \text{in } \Omega, \\ w|_{\partial\Omega} = 0. \end{cases}$$

More specifically, the domain Ω and the boundary function φ determine a convex cone $C \subset \mathbb{R}^{n+1}$ and a C -regular domain $D \subset \mathbb{A}^{n+1}$, respectively, whereas a solution u of Eq.(1.1) corresponds, via Legendre transformation, to a convex hypersurface $\Sigma \subset D$ asymptotic to ∂D with constant affine Gauss-Kronecker curvature, and the last gradient blowup condition in (1.1) is equivalent to the completeness of Σ . As a well studied particular case, if C is the light cone in the Minkowski space $\mathbb{R}^{n,1}$, so that Ω is the unit ball and $w_\Omega(x) = -\sqrt{1 - |x|^2}$, then Σ is a spacelike hypersurface in $\mathbb{R}^{n,1}$ whose Gauss-Kronecker curvature in the classical sense is constant (see [5, 6, 10, 26]). In [17, 18] and most of the follow-up works, in order to guarantee the solvability of (1.1), $\partial\Omega$ and φ are assumed to be C^2 and Ω to be strictly convex.

Recently, we investigated the subject again in [21, 22], with emphasis on the geometry of regular domains and group actions, through which the link with *higher Teichmüller theory* and the theory of *globally hyperbolic spacetimes* is made. Because of this perspective, a key feature of our works is that the regularity assumptions on $\partial\Omega$ and φ are relaxed, which is necessary for the applications. But this also forces us to restrict to the $n = 2$ case, as it is well-known that for $n \geq 3$, non-smooth boundary data can result in undesirable Pogorelov-type singular solutions. Bonsante-Fillastre [4] studied the geometry of such solutions in globally hyperbolic spacetimes.

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Three-dimensional affine space. The purpose of this paper is twofold. The first is to extend the results in [22] to convex surfaces in \mathbb{A}^3 relative to the *Li-normalization*. Namely, while the above notion of affine Gauss-Kronecker curvature for a hypersurface $\Sigma \subset \mathbb{A}^{n+1}$ is defined using the usual *affine normal field* $N_1 : \Sigma \rightarrow \mathbb{R}^{n+1}$, here we replace N_1 by a more general transversal vector field $N_\alpha : \Sigma \rightarrow \mathbb{R}^{n+1}$ defined by A.-M. Li (see [29, 30, 31, 32]), which depends on a parameter $\alpha \in \mathbb{R}$ and coincides with N_1 when $\alpha = 1$. For $\alpha \neq 0$, the condition that the Gauss-Kronecker curvature of Σ relative to N_α is constant turns out to be similar to equations (1.1) and (1.2), only with the exponent $-(n+2)$ in both equations replaced by $-\frac{n+2}{\alpha}$. The new equations and the problem of prescribing the curvature relative to N_α have been studied by Wu-Zhao [29] under the aforementioned C^2 and strict convexity assumptions. We show the following extension in the $n = 2$ case, with the weakest possible regularity assumption on φ . Here, for any extended-real-valued function f , we let $\text{dom}(f)$ denote the subset of the domain where f is real-valued.

Theorem A (Simplified version of Theorem 8.1). *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain satisfying the exterior circle condition, and $\varphi : \partial\Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function such that $\text{dom}(\varphi)$ has at least three points. Then for any $\alpha \in (0, 1]$, there exists a unique lower semicontinuous convex function $u : \bar{\Omega} \rightarrow \mathbb{R} \cup \{+\infty\}$ which is smooth in the interior U of $\text{dom}(u)$ and satisfies*

$$(1.3) \quad \begin{cases} \det D^2 u = (-w)^{-\frac{4}{\alpha}} & \text{in } U := \text{int dom}(u), \\ u|_{\partial\Omega} = \varphi, \\ \|Du(x)\| \rightarrow +\infty & \text{as } x \in U \text{ tends to } \partial U, \end{cases}$$

where $w \in C^0(\bar{\Omega}) \cap C^\infty(\Omega)$ is the unique convex solution to

$$\begin{cases} \det D^2 w = (-w)^{-\frac{4}{\alpha}} & \text{in } \Omega, \\ w|_{\partial\Omega} = 0. \end{cases}$$

Moreover, this u has the property that $\text{dom}(u)$ coincides with the convex hull of $\text{dom}(\varphi)$ in \mathbb{R}^2 , and u coincides with the convex envelope $\bar{\varphi}$ of φ on the boundary of $\text{dom}(u)$.

Remark (about the assumptions). It has been observed in [22, Prop. E] that if Ω is not strictly convex, then the gradient blowup condition in (1.3) might not be fulfillable for certain φ that takes finite values exactly at three points. The *exterior circle condition* on Ω (i.e. the condition that for every $p \in \partial\Omega$, there is a circle passing through p surrounding Ω) is intended as a sufficient condition to guarantee the solvability of (1.3). Meanwhile, we also show in Proposition 8.4 that for any $\alpha > 1$, (1.3) is not solvable when Ω is the unit disk and φ takes finite values exactly at three points, so the assumption $\alpha \leq 1$ is sharp.

In order to give an affine-differential-geometric interpretation of the theorem, we first pointed out that although the affine normal field N_1 of Σ is uniquely determined once a translation-invariant volume form on \mathbb{A}^{n+1} is chosen, the definition of Li's α -normal field N_α with $\alpha \neq 1$ depends furthermore on the choice of a *vertical unit vector* v in the underlying vector space \mathbb{R}^{n+1} (c.f. Remark 2.3 below). In fact, for $\alpha = 0$, N_0 is exactly the constant vector field given by v . With this in mind, we can state the geometric counterpart of Theorem A as follows:

Theorem A' (Simplified version of Theorem 4.3). *Let $\alpha \in (0, 1]$ be a constant, $v \in \mathbb{R}^3$ be a non-zero vector and $C \subset \mathbb{R}^3$ be a proper convex cone containing v such that a planar section Ω of the dual cone $C^* \subset \mathbb{R}^{3*}$ satisfies the exterior circle condition. Then every proper C -regular domain $D \subset \mathbb{A}^3$ is foliated by smooth, complete, locally strongly convex surfaces asymptotic to ∂D , whose Gaussian curvatures with respect to Li's α -normal fields (defined using v as the vertical unit vector) are constants ranging from 0 to $+\infty$. Moreover, the function on D which assigns to each leaf its curvature is log-convex.*

Theorems A and B generalize the results in [22] about the $\alpha = 1$ case.

Higher dimensions. The other purpose of this paper is to give a careful exposition of affine hyperspheres and constant Gauss-Kronecker curvature hypersurfaces relative to Li-normalization, including their link with Monge-Ampère equations. In the process, we improve and simplify the past works [17, 18, 29, 30] on the PDE side and also provide a clearer geometric picture. In particular, we prove statements similar to Theorems A and A' which hold for all dimensions $n \geq 2$ (with stronger assumptions required). We first state the analytic version:

Theorem B (Simplified version of Theorem 7.1). *For any bounded convex domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$), suppose $\varphi \in C^0(\partial\Omega)$ satisfies the following condition: for every $p \in \partial\Omega$, there exists an affine function $a : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $a(p) = \varphi(p)$ and $a \geq \varphi$ on $\partial\Omega$. Then for any $\gamma > n$ and $\lambda > 0$, there exists a unique convex generalized solution u to the problem*

$$(1.4) \quad \begin{cases} \det D^2 u = \lambda(-w)^{-\gamma} & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi \end{cases}$$

which is in $C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ and has the gradient blowup property on $\partial\Omega$, where w is the unique convex solution to (1.5) (see Theorem C below).

The affine-differential-geometric counterpart of Theorem B is contained the following statement (the parameters in the two statements are related by $\gamma = \frac{n+2}{\alpha}$):

Theorem B' (Simplified version of Theorem 4.1). *Let $\alpha \in (0, 1 + \frac{2}{n})$, $v \in \mathbb{R}^{n+1}$ be a non-zero vector, $C \subset \mathbb{R}^{n+1}$ be a proper convex cone containing v , and $D \subset \mathbb{A}^{n+1}$ be a C -regular domain satisfying the following condition: for every C -null subspace $L \subset \mathbb{R}^{n+1}$, there exists a translation $C' \subset \mathbb{A}^{n+1}$ of the cone C such that C' contains D and the supporting hyperplane L' of C' parallel to L is an asymptotic hyperplane of D . Then D is foliated by smooth, complete, locally strongly convex surfaces asymptotic to ∂D , whose Gaussian curvatures with respect to Li's α -normal fields (defined using v as the vertical unit vector) are constants ranging from 0 to $+\infty$. Moreover, the function on D which assigns to each leaf its curvature is log-convex.*

As a prerequisite for Theorems B and B', we establish a classification result for affine hyperspheres with Li-normalization, stated in PDE form and geometric form separately as follows (as before, the parameters are related by $\gamma = \frac{n+2}{\alpha}$):

Theorem C (Simplified version of Theorem 6.1). *For any bounded convex domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) and any $\gamma > n$, there exists a unique convex generalized solution w to the Dirichlet problem*

$$(1.5) \quad \begin{cases} \det D^2 w = (-w)^{-\gamma} & \text{in } \Omega, \\ w|_{\partial\Omega} = 0, \end{cases}$$

which is in $C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ and has the gradient blowup property on $\partial\Omega$.

Theorem C can be seen as a generalization of Theorem B, which corresponds to the case $\varphi \equiv 0$. Concerning the range of the exponent γ , we remark that:

- For $\gamma > 1$, the same conclusions in Theorem B and Theorem C hold if Ω satisfies both the exterior and the interior sphere conditions at every boundary point, see Theorem 6.1 and Remark 7.2.
- No convex solution to Eq.(1.5) with $0 < \gamma \leq 1$ can have the gradient blowup property throughout $\partial\Omega$, see Theorem 6.1.

On the geometric side, we summarize Theorem C and the following remarks in the following statement.

Theorem C' (Theorem 3.2). *Let $\alpha \in (0, 1 + \frac{2}{n})$ and $C \subset \mathbb{R}^{n+1}$ be a proper convex cone containing the vertical unit vector v . Then for every $c > 0$, C contains a unique complete hyperbolic affine hypersphere with α -normalization which is centered at the origin $0 \in \mathbb{R}^{n+1}$, asymptotic to ∂C , and has shape operator cI . More generally, for $\alpha \in (0, n+2)$, the same conclusion holds if C has the property that a hyperplane section of it is a convex domain in \mathbb{R}^n satisfying both the exterior and the interior sphere conditions at every boundary point. On the other hand, if $\alpha \geq n+2$, there does not exist such complete affine hyperspheres.*

The $\alpha = 1$ case is a well known result of Cheng-Yau [8, 9] (with clarifications due to Gigena [12], Li [15, 16] and Sasaki [24]) proving a conjecture of Calabi [7]. The generalization here to Li-normalization is first given by Xiong-Yang [30] for $\alpha \in (0, 1]$ under smoothness and strict convexity assumptions on $\partial\Omega$. While Xiong-Yang's work is based on Lazer-McKenna [14], which studies Eq.(1.5) (with a more general right-hand side) under those assumptions, our proof is more similar to [8, p.67], based on barrier functions on simplices and balls.

Organization of the paper. In Sections 2 and 3, we review the theory of Li-normalization and affine hyperspheres. In Section 4, we review the theory of C -regular domains and define *affine (C, k) -hypersurfaces*, which form the subclass of constant Gauss-Kronecker curvature hypersurfaces involved in this paper, then we use these notions to give precise statements of the mains results. In Section 5, we explain how Legendre transformation and other basic constructions in convex analysis permit us to reformulate the geometric statements into PDE ones. Finally, we prove the PDE statements in Sections 6, 7 and 8.

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2. LI-NORMALIZATION

Let \mathbb{A}^{n+1} denote the $(n+1)$ -dimensional real affine space endowed with a translation-invariant volume form ω , and \mathbb{R}^{n+1} denote the underlying vector space. Given a hypersurface $\Sigma \subset \mathbb{A}^{n+1}$ and a transversal vector field $N : \Sigma \rightarrow \mathbb{R}^{n+1}$, we recall the following fundamental notions in affine differential geometry (see e.g. [19, 23, 25] for details):

- The *induced volume form* v , *induced affine connection* ∇ , *affine metric* h , *shape operator* S and *transversal connection form* τ on Σ relative to N are defined by the equalities

$$\begin{aligned} v(X_1, \dots, X_n) &= \omega(X_1, \dots, X_n, N), \\ D_X Y &= \nabla_X Y + h(X, Y)N, \\ D_X N &= S(X) + \tau(X)N, \end{aligned}$$

for any tangent vector fields X, Y and X_1, \dots, X_n on Σ , where D is the covariant derivative in the ambient affine space.

- If the affine metric h is non-degenerate (i.e. a pseudo-Riemannian metric), then Σ is said to be *non-degenerate* as well, and this property is independent of the choice of N . In particular, h is a Riemannian metric if and only if Σ is locally strongly convex¹ and N points towards the convex side of Σ .
- The transversal vector field N is said to be *equi-affine* if $\tau = 0$.
- If Σ is non-degenerate, N is said to be an *affine normal field* of Σ if it is equi-affine and the volume form v_h of the metric h coincides with the induced volume form v . Such an N exists and is unique up to sign.
- The *dual covector field* N^* of N is by definition the map $N^* : \Sigma \rightarrow \mathbb{R}^{(n+1)*}$ pointwise satisfying

$$(2.1) \quad \langle N^*, N \rangle = 1, \quad \langle N^*, X \rangle = 0$$

for any tangent vector field X on Σ , where $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathbb{R}^{(n+1)*}$ and \mathbb{R}^{n+1} . In general, N^* cannot determine N because adding a tangent vector field to N does not change its dual. Nevertheless, when Σ is non-degenerate, a map $N^* : \Sigma \rightarrow \mathbb{R}^{(n+1)*}$ satisfying the second equality in (2.1) does determine a unique equi-affine transversal vector field N which it is dual to.

We henceforth fix a nonzero vector $v \in \mathbb{R}^{n+1}$ as the vertical unit vector. A locally strongly convex hypersurface $\Sigma \subset \mathbb{A}^{n+1}$ is a *locally strongly convex graph* (with respect to v) if the constant vector field

$$Y : \Sigma \rightarrow \mathbb{R}^{n+1}, \quad Y \equiv v$$

¹For a C^2 function or hypersurface, by *locally strongly convexity*, we mean positive definiteness of the Hessian.

is transversal to Σ and points towards the convex side of Σ . In this case, we let ν_0 and h_0 denote the induced volume form and affine metric on Σ relative to Y , respectively, and let ν_{h_0} denote the volume form of the Riemannian metric h_0 .

Remark 2.1. In the literature, one often works with a fixed coordinate system on \mathbb{R}^{n+1} and \mathbb{A}^{n+1} , and take $v = (0, \dots, 0, 1)$. In this setting, a convex graph is exactly a hypersurface of the form

$$\Sigma = \{(x, F(x)) \mid x \in U\},$$

where U is a domain in \mathbb{R}^n and $F : U \rightarrow \mathbb{R}$ is a locally strongly convex function, and one readily checks that

$$\nu_0 = dx_1 \wedge \dots \wedge dx_n, \quad h_0 = \sum_{1 \leq i, j \leq n} \partial_{ij}^2 F(x) dx_i dx_j, \quad \nu_{h_0} = (\det D^2 F)^{\frac{1}{2}} \nu_0.$$

Proposition/Definition 2.2 (An-Min Li, see [31, 32, 30, 29]). Let $\Sigma \subset \mathbb{A}^{n+1}$ be a locally strongly convex graph, $N_1 : \Sigma \rightarrow \mathbb{R}^{n+1}$ be its affine normal field pointing towards the convex side, and $Y : \Sigma \rightarrow \mathbb{R}^{n+1}$ be the constant vertical unit vector field as above. Then N_1 is exactly the equi-affine transversal vector field on Σ with dual (c.f. the last bullet point above) given by

$$N_1^* = \left(\frac{\nu_{h_0}}{\nu_0} \right)^{-\frac{2}{n+2}} Y^*,$$

where $\frac{\nu_{h_0}}{\nu_0}$ is the density function of ν_{h_0} with respect to ν_0 . More generally, given $\alpha \in \mathbb{R}$, Li's α -normal field $N_\alpha : \Sigma \rightarrow \mathbb{R}^{n+1}$ is by definition the equi-affine transversal vector field with dual given by

$$N_\alpha^* = \left(\frac{\nu_{h_0}}{\nu_0} \right)^{-\frac{2\alpha}{n+2}} Y^*.$$

Proof. Take a coordinate system, let v and Σ be as in Remark 2.1 and parameterize Σ by the domain $U \subset \mathbb{R}^n$ through the function $F : U \rightarrow \mathbb{R}$. By computations, we may check the following facts:

- Letting $DF := (\partial_1 F, \dots, \partial_n F)$ denote the gradient of F , we have

$$Y^* = (-DF, 1).$$

- Any equi-affine transversal vector field $N = (N', N^{(n+1)}) : U \cong \Sigma \rightarrow \mathbb{R}^{n+1}$, where N' and $N^{(n+1)}$ denote the horizontal and vertical components, respectively, is determined by the component $N^{(n+1)}$ along with the function

$$\lambda := N^{(n+1)} - DF \cdot N'.$$

In fact, the equi-affine condition implies that $N' = -(D^2 F)^{-1} D\lambda$.

- The induced volume form ν and the affine metric h on Σ relative to an equi-affine transversal vector field N as above are related to ν_0 and h_0 (c.f. Remark 2.1) by

$$(2.2) \quad \nu = \lambda \nu_0, \quad h = \lambda^{-1} h_0.$$

Moreover, the dual covector field of N is

$$(2.3) \quad N^* = \lambda^{-1} (-DF, 1).$$

Now take $N = N_1$. By definition of affine normal fields and (2.2), we have $\nu_h = \lambda^{-\frac{n}{2}} \nu_{h_0} = \nu = \lambda \nu_0$, hence

$$\lambda = \left(\frac{\nu_{h_0}}{\nu_0} \right)^{\frac{2}{n+2}}.$$

Combining with (2.3), we arrive at the required statement. \square

Remark 2.3. While the affine normal field N_1 is covariant with respect to volume-preserving affine transformations of \mathbb{A}^{n+1} and hence independent of the choice of v , Li's α -normal field N_α with $\alpha \neq 1$ does not have this property. This can be seen from the unit hypersphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ or the hyperboloid $\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_n^2 - x_{n+1}^2 = -1\}$, for which N_1 equals the position vector field $P(x) := x$ up to a sign. When $\alpha \neq 1$, one checks that N_α^* (defined with respect to $v = (0, \dots, 0, 1)$) is not proportional to N_1^* ,

hence N_α is not proportional to P . Due to the affine symmetries of \mathbb{S}^n and \mathbb{H}^n , this implies that N_α is not affine-covariant.

We refer to Lemma 5.4 below for a more explicit expression of N_α under a particular parametrization of Σ , as well as expressions of the affine metric and shape operator of Σ relative to N_α .

When $\alpha = 0$, N_0 is just the vertical unit vector field Y , and we will only consider the $\alpha \neq 0$ case below. The following lemma characterizes N_α in a way similar to the definition of affine normal fields:

Lemma 2.4. *Let Σ be as above and suppose $\alpha \neq 0$. Then N_α is the unique equi-affine transversal vector field on Σ pointing towards the convex side with the following property: if v and h are the induced volume form and affine metric on Σ relative to N_α , respectively, and v_h is the volume form of the metric h , then the densities of v and v_h with respect to v_0 are related by*

$$(2.4) \quad \frac{v_h}{v_0} = \left(\frac{v}{v_0} \right)^\beta, \quad \text{where } \beta := \frac{1}{2} \left(\frac{n+2}{\alpha} - n \right).$$

Proof. Following the computations in the previous proof, the condition for an equi-affine transversal vector field $N = (N', N^{(n+1)})$ to be Li's α -normal field can be written as

$$(2.5) \quad \lambda = \left(\frac{v_{h_0}}{v_0} \right)^{\frac{2\alpha}{n+2}},$$

where $\lambda := N^{(n+1)} - DF \cdot N'$. On the other hand, by (2.2), we have

$$\frac{v_h}{v_0} = \frac{v_h}{v_{h_0}} \frac{v_{h_0}}{v_0} = \lambda^{-\frac{n}{2}} \frac{v_{h_0}}{v_0}, \quad \frac{v}{v_0} = \lambda,$$

so condition (2.4) can be rewritten as

$$(2.6) \quad \lambda^{-\frac{n}{2}} \frac{v_{h_0}}{v_0} = \lambda^\beta.$$

Conditions (2.5) and (2.6) are clearly equivalent to each other, which proves the required statement. \square

3. AFFINE HYPERSPHERES AND CONSTANT CURVATURE WITH LI-NORMALIZATION

In this section, we fix a constant $\alpha \neq 0$, a vertical unit vector $v \in \mathbb{R}^{n+1}$, let $\Sigma \subset \mathbb{A}^{n+1}$ be a smooth locally strongly convex graph (with respect to v), $N_\alpha : \Sigma \rightarrow \mathbb{R}^{n+1}$ be Li's α -normal field on Σ , and S_α be the shape operator of Σ relative to N_α . The following notions are straightforward generalizations of the classical ones:

- Σ is called a *hyperbolic affine hypersphere with α -normalization*² if

$$S_\alpha = cI$$

for a constant $c > 0$, where I is the identity (1,1)-tensor on Σ . This condition is equivalent to the existence of a point $o \in \mathbb{A}^{n+1}$, the *center* of Σ , such that $N_\alpha(p) = c\overrightarrow{op}$ for all $p \in \Sigma$. The same definition can be made for $c \leq 0$ and yields elliptic or parabolic affine hyperspheres, but they are not the concern of this paper.

- The *Gauss-Kronecker curvature* (which we simply call *Gaussian curvature* if $n = 2$) of Σ with α -normalization is by definition the function

$$\kappa_\alpha := \det(S_\alpha) : \Sigma \rightarrow \mathbb{R}.$$

As a simple fact for any equi-affine transversal vector field $N : \Sigma \rightarrow \mathbb{R}^{n+1}$, if the shape operator S of Σ relative to N is non-degenerate (i.e. $\det(S) \neq 0$ everywhere), then N defines an immersion of Σ into \mathbb{R}^{n+1} . With this in mind, we can state the following result, which links the above two notions together and generalizes [22, Prop. 3.5 (2)] about the $\alpha = 1$ case:

Proposition 3.1. *Suppose S_α is non-degenerate. Then the following conditions are equivalent:*

²In the literature [31, 32, 30, 29] and the introduction, " α -normalization" has been phrased alternatively as "Li-normalization". Here we choose to emphasize the dependence on the parameter α in order to facilitate the statement of some results below.

- (a) The eigenvalues of S_α are all positive, and κ_α is a constant.
 (b) $N_\alpha(\Sigma) \subset \mathbb{R}^{n+1}$ is a hyperbolic affine hypersphere with α -normalization, centered at the origin.

Moreover, when these conditions are fulfilled, the affine hypersphere in (b) has shape operator $\kappa_\alpha^{-\frac{\alpha}{n+2}} I$.

Proof. For any equi-affine transversal vector field $N : \Sigma \rightarrow \mathbb{R}^{n+1}$, if v , h and S are the induced volume form, affine metric and shape operator of Σ relative to N , and v' , h' and S' are the induced volume form, affine metric and shape operator of N viewed as a centro-affine immersion (i.e. relative to the position vector field), then it can be shown that

$$(3.1) \quad v' = \det(S)v, \quad h'(\cdot, \cdot) = h(S(\cdot), \cdot), \quad S' = I$$

(see [22, Lemma 2.6]). A similar computation also shows that if v_0 and v'_0 are the induced volume forms of Σ and the immersion N , respectively, relative to the vertical unit vector field, then

$$(3.2) \quad v'_0 = \det(S)v_0.$$

Now if condition (a) holds, applying (3.1) and (3.2) to $N = N_\alpha$ and using Lemma 2.4, we conclude that

$$(3.3) \quad \frac{v_{h'}}{v'_0} = \frac{\det(S)^{\frac{1}{2}} v_h}{\det(S)v_0} = \kappa_\alpha^{-\frac{1}{2}} \left(\frac{v}{v_0} \right)^\beta = \kappa_\alpha^{-\frac{1}{2}} \left(\frac{v'}{v'_0} \right)^\beta,$$

where $\beta := \frac{1}{2} \left(\frac{n+2}{\alpha} - n \right)$. In general, scaling a transversal vector field by a constant r amounts to scaling the induced volume form by r and the volume form of affine metric by $r^{-\frac{n}{2}}$. Therefore, (3.3) and Lemma 2.4 imply that for the hypersurface $N_\alpha(\Sigma)$, scaling the position vector field by

$$r := \kappa_\alpha^{-\frac{1}{2\beta+n}} = \kappa_\alpha^{-\frac{\alpha}{n+2}}$$

yields Li's α -normal field. It follows that $N_\alpha(\Sigma)$ is a hyperbolic affine hypersphere with α -normalization, centered at the origin, and its shape operator is rI . This proves the implication “(a) \Rightarrow (b)” and the last statement of the proposition.

Conversely, if (b) holds, in view of the expression of h' in (3.1) and the fact that h' and h are both Riemannian metrics (as Σ and $N_\alpha(\Sigma)$ are locally strongly convex; c.f. §2), we first conclude that the eigenvalues of S_α are positive. Then, by Lemma 2.4 and the above scaling argument again, we have

$$\frac{v_{h'}}{v'_0} = c \left(\frac{v'}{v'_0} \right)^\beta$$

for a constant $c > 0$. Using this and equations (3.1) and (3.2), we get

$$\frac{v_h}{v_0} = \kappa_\alpha^{\frac{1}{2}} \frac{v_{h'}}{v'_0} = c \kappa_\alpha^{\frac{1}{2}} \left(\frac{v'}{v'_0} \right)^\beta = c \kappa_\alpha^{\frac{1}{2}} \left(\frac{v}{v_0} \right)^\beta.$$

Therefore, again by Lemma 2.4, the assumption that N_α is Li's α -normal field implies $c \kappa_\alpha^{\frac{1}{2}} \equiv 1$, hence κ_α is a constant. This shows “(b) \Rightarrow (a)” and completes the proof. \square

We henceforth restrict ourselves to *complete*, i.e. properly embedded³, affine hyperspheres with α -normalization. Recall that a *convex cone* in \mathbb{R}^{n+1} is by definition a convex domain invariant under positive scalings, and is said to be *proper* if it does not contain any entire affine line. The classification theorem of usual hyperbolic affine hyperspheres, conjectured by Calabi [7] and proved by Cheng-Yau [8, 9] (with clarifications due to Gigena [12], Li [15, 16] and Sasaki [24]), roughly claims a 1-to-1 correspondence between proper convex cones in \mathbb{R}^{n+1} and complete affine hyperspheres centered at the origin. Xiong-Yang [30] generalized the classification to the setting of α -normalization under some extra assumptions. Theorem C' in the introduction, which we restate below, improves on Xiong-Yang's result:

³This notion is usually referred to as *Euclidean completeness* in the literature on affine differential geometry, in order to make the distinction with *affine completeness*, i.e. completeness of the affine metric. We do not study the latter notion in this paper.

Theorem 3.2 (Theorem C). *Let $\alpha \in (0, 1 + \frac{2}{n})$ and $C \subset \mathbb{R}^{n+1}$ be a proper convex cone containing the vertical unit vector v . Then for every $c > 0$, C contains a unique complete hyperbolic affine hypersphere with α -normalization which is centered at the origin $0 \in \mathbb{R}^{n+1}$, asymptotic to ∂C , and has shape operator cI . More generally, for $\alpha \in (0, n+2)$, the same conclusion holds if C has the property that a hyperplane section of it is a convex domain in \mathbb{R}^n satisfying both the exterior and the interior sphere conditions at every boundary point. On the other hand, if $\alpha \geq n+2$, there does not exist such complete affine hypersphere.*

Here, a convex domain $\Omega \subset \mathbb{R}^n$ is said to “satisfy both the exterior and the interior sphere conditions at every boundary point” if for every $p \in \partial\Omega$ there exist round balls $B_1, B_2 \subset \mathbb{R}^n$ such that $B_1 \subset \Omega \subset B_2$ and $\partial B_1 \cap \partial B_2 = \{p\}$.

Remark 3.3. Using the definitions, it can be shown that if Σ_0 is an affine hypersphere with α -normalization, centered at the origin, whose shape operator is the identity I , then the scaled hypersurface $t\Sigma_0$ ($t > 0$) is also such an affine hypersphere, with shape operator $t^{-(1+\frac{\alpha n}{n+2})}I$. Therefore, the affine hyperspheres given by Theorem 3.2 form a homothetic family.

We will outline in Section 5 the standard process of transforming Theorem 3.2 into an equivalent PDE statement via Legendre transformation, then prove the equivalent statement (*i.e.* Theorem C in the introduction) in Section 6.

In view of Proposition 3.1 and Theorem 3.2, we now define the following particular class of convex graphs with constant Gauss-Kronecker curvature with α -normalization, which we study in the sequel:

Definition 3.4. Given $\alpha, k > 0$ and a proper convex cone $C \subset \mathbb{R}^{n+1}$ containing the vertical unit vector v , a (C, k) -hypersurface with α -normalization is a smooth locally strongly convex graph $\Sigma \subset \mathbb{A}^{n+1}$ such that

- the Gauss-Kronecker curvature κ_α of Σ is constantly k ;
- Li’s α -normal field N_α of Σ , viewed as an immersion of Σ into \mathbb{R}^{n+1} , has image in a complete hyperbolic affine hypersphere with α -normalization asymptotic to ∂C .

Remark 3.5. We only consider the $\alpha < 1 + \frac{2}{n}$ case of this definition, where the hyperspheres described in the last bullet point uniquely exist up to scaling by Theorem 3.2 and Remark 3.3. The exactly hypersphere in this scaling family that contains the image of N_α is actually determined by k : If we let Σ_C denote the one in this family with identity shape operator, then by Proposition 3.1 and Remark 3.3, N_α is contained in $k^{\frac{\alpha}{(\alpha+1)n+2}}\Sigma_C$, whose shape operator is $k^{-\frac{\alpha}{n+2}}I$. Therefore, the defining conditions are equivalent to the condition “ S_α is non-degenerate and N_α has image in the scaled affine hypersphere $k^{\frac{\alpha}{(\alpha+1)n+2}}\Sigma_C$ ”.

4. C -REGULAR DOMAINS AND FOLIATION BY (C, k) -HYPERSURFACES WITH α -NORMALIZATION

Given a proper convex cone $C \subset \mathbb{R}^{n+1}$, the following definitions introduced in [21, 22] generalize classical notions from Minkowski geometry:

- A subspace $L \subset \mathbb{R}^{n+1}$ of dimension n is said to be C -spacelike if L meets the closure \overline{C} of C only at the origin 0 , and is said to be C -null if $L \cap \overline{C}$ is contained in ∂C and is not the single point 0 .
- An affine hyperplane in \mathbb{A}^{n+1} is said to be C -spacelike/ C -null if the underlying vector subspace of \mathbb{R}^{n+1} is. Given such a hyperplane $H \subset \mathbb{A}^{n+1}$, we let \widehat{H} denote the closed half-space of \mathbb{A}^{n+1} bounded by H which contains a translation of C (*i.e.* the “upper half” of \mathbb{A}^{n+1} cut out by H).
- A C -regular domain in \mathbb{A}^{n+1} is by definition an unbounded convex domain $D \subset \mathbb{A}^{n+1}$ of the form

$$D = \text{int} \bigcap_{H \in \mathcal{S}} \widehat{H}$$

where int denotes the interior and \mathcal{S} is a collection of C -null hyperplanes. If \mathcal{S} consists of all C -null hyperplanes H such that \widehat{H} contains some set $E \subset \mathbb{A}^{n+1}$, then D is called the C -regular domain generated by E .

- A C -convex hypersurface is by definition an open subset Σ of the boundary of some convex domain $U \subset \mathbb{A}^{n+1}$ such that any supporting hyperplane H of U at any point of Σ is C -spacelike with $U \subset \widehat{H}$. So Σ is complete if it is the entire ∂U . The simplest examples are just C -spacelike hyperplanes.

The main class of examples that we study are affine (C, k) -hypersurfaces from Definition 3.4, which can be shown to be C -convex.

For the future light cone C_0 in the Minkowski space $\mathbb{R}^{n,1}$, C_0 -regular domains are classically known as *regular domains* or *domains of dependence*, and play an important role in the study of *globally hyperbolic flat spacetimes* from mathematical relativity, where one considers a discontinuous isometric action on such a domain with quotient a Lorentzian manifold of the form $M \times \mathbb{R}$, where M is a compact n -manifold and \mathbb{R} is the time direction (see e.g. [1, 2, 3, 20]). Mess [20] discovered a profound link between the $n = 2$ case and Teichmüller Theory. Some of Mess' results are generalized from C_0 to general C in [21].

The main results of this paper can be viewed as generalizations of Theorem 3.2 above, with the cone C replaced by C -regular domains, and the affine hyperspheres replaced by (C, k) -hypersurfaces. We first give a statement which holds for any $n \geq 2$:

Theorem 4.1 (Extended version of Theorem B'). *Let $0 < \alpha < 1 + \frac{2}{n}$, $C \subset \mathbb{R}^{n+1}$ be a proper convex cone containing the vertical unit vector v , and $D \subset \mathbb{A}^{n+1}$ be a C -regular domain satisfying the following condition: for every C -null subspace $L \subset \mathbb{R}^{n+1}$, there exists a translation $C' \subset \mathbb{A}^{n+1}$ of the cone C such that C' contains D and the supporting hyperplane L' of C' parallel to L is an asymptotic hyperplane⁴ of D . Then D contains, for every $k > 0$, a unique complete (C, k) -hypersurface with α -normalization Σ_k which generates D . Moreover, the following statements hold:*

- For each k , the distance from $x \in \Sigma_k$ to ∂D tends to 0 as x tends to infinity in Σ_k .
- $(\Sigma_k)_{k>0}$ is a foliation of D (i.e. the Σ_k 's are disjoint and their union is D).
- The function $K : D \rightarrow \mathbb{R}$ given by $K|_{\Sigma_k} = \log k$ is convex.

Remark 4.2. (1) The assumption on D is not satisfied, for example, when $D = C \cap \text{int } \widehat{L}'_0$ for some C -null subspace L_0 and a C -null hyperplane L'_0 obtained by moving L_0 towards C .
 (2) The condition that “the distance from $x \in \Sigma$ to ∂D tends to 0 as x goes to infinity in Σ ” is phrased as “ Σ is asymptotic to $\partial \Sigma$ ” in [22, §5.3], and it is explained therein that for a general complete C -convex hypersurface Σ , this condition is strictly stronger than “ Σ generates D ”.

Like Theorem 3.2, we will transform Theorem 4.1 and the other two results below into equivalent PDE statements via the construction in the next section, and then give proofs in Sections 7 and 8. The reader may check the PDE version Theorem 7.1 of Theorem 4.1 (see also Remark 7.2) for a more transparent interpretation of the assumption on D . All these results improve on the work of Wu-Zhao [29].

When $n = 2$, under some mild restrictions on α and C , we may extend Theorem 4.1 to any C -regular domain D which is proper (i.e. not containing any entire affine line), without any extra assumption:

Theorem 4.3 (Extended version of Theorem A'). *Suppose $0 < \alpha \leq 1$ and let $C \subset \mathbb{R}^3$ be a proper convex cone containing the vertical unit vector such that a plane section $\Omega \subset \mathbb{R}^2$ of the dual cone $C^* \subset \mathbb{R}^{3*}$ satisfies the exterior circle condition. Then the conclusions of Theorem 4.1 hold for any proper C -regular domain $D \subset \mathbb{A}^3$.*

This is a generalization of our recent work [22] about the $\alpha = 1$ case.

In Prop. E and Cor. 8.6 of [22], we showed in the $\alpha = 1$ case that if Ω is not strictly convex, then the conclusion does not hold for certain examples of D . We believe there also exist such examples where Ω is strictly convex but does not satisfy the exterior circle condition. On the other hand, the necessity of the condition $\alpha \leq 1$ is shown by the following result:

Proposition 4.4. *Suppose $n = 2$ and $\alpha > 1$. Let C_0 be the future light cone in the Minkowski space $\mathbb{R}^{2,1}$ and D be a triangular cone bounded by three null planes, which is a C_0 -regular domain. Then D does not contain any complete (C_0, k) -surface with α -normalization which generates D .*

⁴An *asymptotic hyperplane* of an unbounded convex domain $U \subset \mathbb{A}^{n+1}$ is by definition a hyperplane H disjoint from U such that any hyperplane H' obtained by moving H parallelly towards U intersects U along an unbounded set.

5. LEGENDRE TRANSFORMATION

We henceforth identify \mathbb{A}^{n+1} with \mathbb{R}^{n+1} by choosing a base point, and fix a coordinate system on \mathbb{R}^{n+1} under which the vertical unit vector is $v = (0, \dots, 0, 1)$. We usually write a point of \mathbb{R}^{n+1} as $X = (x, \xi)$, where $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$ are the horizontal and vertical components, respectively.

Given a proper convex cone $C \subset \mathbb{R}^{n+1}$ containing v , an important bounded convex domain $\Omega \subset \mathbb{R}^n$ associated with C is the section of the dual cone $C^* := \{X \in \mathbb{R}^{n+1} \mid X \cdot Y < 0, \forall Y \in \overline{C} \setminus \{0\}\}$ ⁵ by the hyperplane $\mathbb{R}^n \times \{-1\}$. That is, Ω is defined by

$$C^* = \{t(x, -1) \mid x \in \Omega, t > 0\}.$$

As an alternative definition, if we suppose $C = \{t(y, 1) \mid y \in C_1, t > 0\}$, namely C_1 is the section of C by the hyperplane $\mathbb{R}^n \times \{1\}$, then we have

$$\Omega = \{x \in \mathbb{R}^n \mid x \cdot y < 1, \forall y \in \overline{C}_1\}.$$

That is, Ω is the dual convex domain of C_1 in the sense of Sasaki [24].

The significance of Ω lies in the fact that all the geometric objects in \mathbb{A}^{n+1} introduced above with respect to C , such as C -null/ C -spacelike hyperplanes, C -regular domains, (C, k) -hypersurfaces, *etc.*, can be interpreted in terms of convex functions on Ω through Legendre transformation, which we now explain.

Let us first recall some facts about convex function. Let $\text{LC}(\mathbb{R}^n)$ denote the space of $\mathbb{R} \cup \{+\infty\}$ -valued, lower semicontinuous, convex functions on \mathbb{R}^n that are not constantly $+\infty$. Given $u \in \text{LC}(\mathbb{R}^n)$, if the effective domain $\text{dom}(u) := \{x \mid u(x) < +\infty\}$ has nonempty interior $U := \text{int dom}(u)$, then the values of u on ∂U (hence the values on the whole \mathbb{R}^n) are determined by the restriction $u|_U$, because we have

$$(5.1) \quad u(p) = \liminf_{U \ni x \rightarrow p} u(x) = \lim_{s \rightarrow 0^+} u((1-s)p + sx) \in (-\infty, +\infty]$$

for any $p \in \partial U$ and $x \in U$ (see [22, §4.1]).

Therefore, given a convex domain $U \subset \mathbb{R}^n$ and a convex function $u : U \rightarrow \mathbb{R}$, we define the *boundary value* $u|_{\partial U}$ of u as the function on ∂U whose value at any $p \in \partial U$ is the liminf or the limit in (5.1). The extension of u to \mathbb{R}^n given by $u|_{\partial U}$ and by setting $u = +\infty$ outside of \overline{U} is an element of $\text{LC}(\mathbb{R}^n)$. This gives a canonical way of viewing every convex function on a convex domain as an element of $\text{LC}(\mathbb{R}^n)$.

Given $p \in \partial U$, we say that u has *infinite slope* (or *infinite inner derivatives*) at p if either $u(p) = +\infty$ or $u(p)$ is finite but

$$(5.2) \quad \lim_{s \rightarrow 0^+} \frac{u((1-s)p + sx) - u(p)}{s} = -\infty$$

for some $x \in U$ (by convexity of u , the fraction decreases as s decreases to 0, hence the limit exists in $[-\infty, +\infty)$). We refer to [22, §4] for the following fundamental facts about this notion:

- It is independent of the choice of $x \in U$. That is, if (5.2) holds for one x , then it holds for all.
- The following conditions are equivalent:
 - u has infinite slope at $p \in \partial U$;
 - u does not have any *subgradient* at p (see *e.g.* [22, §4.4] for the definition), or in other words, the graph of u does not have any non-vertical supporting hyperplane at p ;
 - for any sequence $(x_i)_{i=1,2,\dots}$ in U tending to p such that u is differentiable at every x_i , we have $\|Du(x_i)\| \rightarrow +\infty$.

In particular, when $u \in C^1(U)$, the condition that u has infinite slope at every point of ∂U is equivalent to the gradient blowup condition “ $\|Du(x)\| \rightarrow +\infty$ as $x \in U$ tends to ∂U ”.

We will consider some particular classes of functions $u \in \text{LC}(\mathbb{R}^n)$ with $\text{dom}(u)$ contained in the closure $\overline{\Omega}$ of the above convex domain Ω . The first consists of *convex envelopes* of functions on $\partial\Omega$, defined as

⁵Conceptually, C^* is a convex cone in the dual vector space $\mathbb{R}^{(n+1)*}$, but here we use the standard inner product “ \cdot ” to identify $\mathbb{R}^{(n+1)*}$ with \mathbb{R}^{n+1} . Also, the definition of C^* in the literature (*e.g.* [22]) sometimes differs from the one here by a sign.

follows: given a function $\psi : \partial\Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ which is bounded from below and not constantly $+\infty$, the convex envelope $\bar{\psi} \in \text{LC}(\mathbb{R}^n)$ of ψ is

$$\bar{\psi}(x) := \sup \{a(x) \mid a : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is an affine function with } a|_{\partial\Omega} \leq \psi\}.$$

As shown in [22, §4.3], the assignment $\varphi \mapsto \bar{\varphi}$ is a bijection from the set of functions

$$\text{LC}(\partial\Omega) := \left\{ u : \partial\Omega \rightarrow \mathbb{R} \cup \{+\infty\} \mid \begin{array}{l} u \text{ is lower semicontinuous and restricts to a} \\ \text{convex function on any line segment in } \partial\Omega. \end{array} \right\}$$

to the subset of $\text{LC}(\mathbb{R}^n)$ consisting of all convex envelopes, with inverse the restriction map $u \mapsto u|_{\partial\Omega}$. We will make use of the following properties of the convex envelope $\bar{\varphi}$ of $\varphi \in \text{LC}(\partial\Omega)$:

- $\text{dom}(\bar{\varphi})$ is the convex hull of $\text{dom}(\varphi) \subset \partial\Omega$ (see [22, Prop. 4.8]).
- For any $u \in \text{LC}(\mathbb{R}^n)$ with $u|_{\partial\Omega} \leq \varphi$, we have $u \leq \bar{\varphi}$ throughout \mathbb{R}^n (see [22, Cor. 4.5]).
- Suppose $u : \Omega \rightarrow \mathbb{R}$ is a convex function with $u|_{\partial\Omega} \leq \varphi$. Then either of the following conditions is sufficient for the strict inequality $u < \bar{\varphi}$ to hold in Ω :
 - u is strictly convex;
 - u has infinite slope at every point of $\partial\Omega$.
 This can be proved by using [22, Lemma 4.9].
- If φ is \mathbb{R} -valued, so that $\bar{\varphi}$ is also \mathbb{R} -valued in Ω , then the Monge-Ampère measure of $\bar{\varphi}$ is the zero measure (see [13, Thm. 1.5.2]).

Two other classes of $u \in \text{LC}(\mathbb{R}^n)$ that we will consider are

$$\text{S}(\Omega) := \left\{ u \in \text{LC}(\mathbb{R}^n) \mid \text{dom}(u) \subset \bar{\Omega}, u \text{ does not admit any subgradient at any point of } \partial\Omega \right\}$$

and the subset of $\text{S}(\Omega)$ given by

$$\text{S}_0(\Omega) := \left\{ u \in \text{LC}(\mathbb{R}^n) \mid \begin{array}{l} \text{the interior } U \text{ of } \text{dom}(u) \text{ is nonempty and contained} \\ \text{in } \Omega; u \text{ is smooth and locally strongly convex in } U, \\ \text{and has infinite slope at every point of } \partial U \end{array} \right\}.$$

The *Legendre transform* of $u \in \text{LC}(\mathbb{R}^n)$ is by definition the function $u^* \in \text{LC}(\mathbb{R}^n)$ given by

$$u^*(y) := \sup_{x \in \mathbb{R}^n} (x \cdot y - u(x)).$$

It is a fundamental fact that the Legendre transformation $u \mapsto u^*$ is an involution on $\text{LC}(\mathbb{R}^n)$ (see e.g. [22, §4.5]). If u is an \mathbb{R} -valued convex function only defined on a convex domain $U \subset \mathbb{R}^n$, we can still define u^* , either by changing the range of x in the supremum into U , or by viewing u as an element of $\text{LC}(\mathbb{R}^n)$ through the canonical extension mentioned above. We will need the following properties of u^* :

- $u_1 \leq u_2$ on \mathbb{R}^n if and only if $u_1^* \geq u_2^*$ on \mathbb{R}^n ;
- if u is differentiable at a point x , then the value of u^* at the gradient $Du(x)$ is given by

$$u^*(Du(x)) = x \cdot Du(x) - u(x).$$

Concerning the last property, we further note that if u is strictly convex and C^1 in a convex domain $U \subset \mathbb{R}^n$, then the gradient map $x \mapsto Du(x)$ is a homeomorphism from U to the image $Du(U)$. The property implies that the graph of u^* over $Du(U)$ is $\{(Du(x), x \cdot Du(x) - u(x)) \in \mathbb{R}^{n+1} \mid x \in U\}$. Also note that $Du(U)$ is the whole \mathbb{R}^n if and only if $U = \text{int dom}(u)$ and u has infinite slope at every point of ∂U .

With the above notions and facts in mind, we can now formulate the relationship between convex functions on Ω and the geometric objects in \mathbb{A}^{n+1} as the following theorem. Here and below, given a function $f : \mathbb{R}^n \supset E \rightarrow \mathbb{R} \cup \{+\infty\}$, we let

$$\text{gr}(f) := \{(x, \xi) \in \mathbb{R}^{n+1} \mid x \in E, \xi = f(x)\}, \quad \text{ep}^\circ(f) := \{(x, \xi) \in \mathbb{R}^{n+1} \mid x \in E, \xi > f(x)\}$$

denote the graph and strict epigraph of f , respectively.

Theorem 5.1. *Suppose $u \in \text{LC}(\mathbb{R}^n)$ and $\text{dom}(u) \subset \bar{\Omega}$. Then for each row of Table 1, u has the given property on the left if and only if the graph of the Legendre transform u^* fulfills the description on the right. Moreover, the following statements about the rows ③ and ④ hold true:*

	<i>property of u</i>	<i>nature of the graph of u^*</i>
①	$u = +\infty$ except at a single point of Ω (resp. $\partial\Omega$)	a C -null (resp. C -spacelike) hyperplane
②	u is an affine function on $\overline{\Omega}$	the boundary of a translation of C
③	$u = \overline{\varphi}$ for some $\varphi \in \text{LC}(\partial\Omega)$	the boundary of a C -regular domain D (i.e. $D = \text{ep}^\circ(u^*)$ is a C -regular domain)
④	$u \in S(\Omega)$	complete C -convex hypersurface
⑤	$u \in S_0(\Omega)$	complete, smooth, locally strongly convex, C -convex hypersurface
⑥	$u \in S_0(\Omega)$, $u _{\partial\Omega} = 0$, $\det D^2u = (-u)^{-\frac{n+2}{\alpha}}$ in Ω	complete hyperbolic affine hypersphere with α -normalization, centered at 0 and asymptotic to ∂C
⑦	$u \in S_0(\Omega)$, $\det D^2u = k^{-\frac{n+2}{(\alpha+1)n+2}}(-w)^{-\frac{n+2}{\alpha}}$ in the interior of $\text{dom}(u)$ for some $w \in S_0(\Omega)$ satisfying the condition in ⑥	complete (C, k) -hypersurface with α -normalization

TABLE 1. Correspondence between convex functions on Ω and objects in \mathbb{A}^{n+1}

- (1) Given $\varphi \in \text{LC}(\partial\Omega)$, the C -regular domain $D = \text{ep}^\circ(\overline{\varphi}^*)$ is proper if and only if the convex set $\text{dom}(\overline{\varphi})$ (which is the convex hull of $\text{dom}(\varphi)$) has nonempty interior.
- (2) Suppose $u \in S(\Omega)$ and denote $\varphi := u|_{\partial\Omega} \in \text{LC}(\partial\Omega)$. Then the hypersurface $\Sigma := \text{gr}(u^*)$ generates the domain $D := \text{ep}^\circ(\overline{\varphi}^*)$ and is contained in D . In this case, the distance from $x \in \Sigma$ to ∂D tends to 0 as x goes to infinity in Σ (c.f. Remark 4.2) if and only if the following conditions are satisfied:
- $\text{dom}(u)$ coincides with $\text{dom}(\overline{\varphi})$;
 - $\overline{\varphi}(x) - u(x) \rightarrow 0$ as $x \in U := \text{int dom}(u)$ tends to ∂U .
- (3) Let $(u_t)_{t \in \mathbb{R}}$ be a one-parameter family in $S(\Omega)$ with the same boundary value $\varphi = u_t|_{\partial\Omega} \in \text{LC}(\partial\Omega)$, satisfying the following conditions:
- for any $x \in \mathbb{R}^n$ and $t_1 < t_2$, we have $u_{t_1}(x) \leq u_{t_2}(x)$, and the inequality is strict if u_{t_1} admits a subgradient at x ;
 - as $t \rightarrow +\infty$, u_t pointwise converges to $\overline{\varphi}$;
 - u_t is not bounded from below uniformly in t ;
 - for every fix $x \in \mathbb{R}^n$, the function $\mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$, $t \rightarrow u_t(x)$ is concave.

Then the hypersurfaces $\Sigma_t := \text{gr}(u_t^*)$, $t \in \mathbb{R}$ form a foliation of the domain $D := \text{ep}^\circ(\overline{\varphi}^*)$ and has the property that the function $K : D \rightarrow \mathbb{R}$ given by $K|_{\Sigma_t} = t$ is convex. Meanwhile, the above conditions are also necessary for the Σ_t 's to form a foliation of D with that property.

Remark 5.2. A special instance of row ③ is the whole affine space \mathbb{A}^{n+1} as a C -regular domain, whose corresponding u is the constant function $+\infty$ (although u is not in $\text{LC}(\mathbb{R}^n)$, the Legendre transform u^* is well defined and equals $-\infty$). Consequently, as the simplest example of statement (3), given a point $x_0 \in \Omega$, the family $(u_t)_{t \in \mathbb{R}}$ given by $u_t(x_0) = t$ and $u_t = +\infty$ on $\mathbb{R}^n \setminus \{x_0\}$ corresponds to the foliation of \mathbb{A}^{n+1} by the translations of a C -spacelike hyperplane.

The equivalences in the rows ①~⑤ and statements (1), (2) and (3) of the theorem are essentially the content of [22, §5], so we omit the proof of these parts. The affine differential geometry computations leading to the rows ⑥ and ⑦ have been done in [30, 29] and can be summarized as follows:

Proposition 5.3. *Let $U \subset \mathbb{R}^n$ be a convex domain and $u \in C^\infty(U)$ be a locally strongly convex function. Let Σ denote the graph of u^* over $Du(U)$, i.e.*

$$\Sigma := \text{gr}(u^*|_{Du(U)}) = \{(Du(x), x \cdot Du(x) - u(x)) \mid x \in U\} \subset \mathbb{R}^{n+1}.$$

Then the following statements hold:

- (1) Σ is a hyperbolic affine hypersphere with α -normalization, centered at the origin $0 \in \mathbb{R}^{n+1}$, and has shape operator cI , if and only if u satisfies

$$\det D^2u = (-cu)^{-\frac{n+2}{\alpha}}.$$

- (2) Let $N_\alpha : \Sigma \rightarrow \mathbb{R}^{n+1}$ be Li's α -normal field of Σ , $\tilde{u} \in C^\infty(U)$ be another locally strongly convex function and $\tilde{\Sigma}$ be the graph of \tilde{u}^* over $D\tilde{u}(U)$. Then N_α is an immersion with image contained in $\tilde{\Sigma}$ if and only if u satisfies

$$\det D^2u = (-\tilde{u})^{-\frac{n+2}{\alpha}}.$$

Moreover, in this case, N_α is actually a diffeomorphism from Σ to $\tilde{\Sigma}$.

The equivalence between the properties of u and $\text{gr}(u^*)$ in row ⑥ of Theorem 5.1 follows immediately from part (1) of the proposition and row ⑤, whereas the the equivalence in row ⑦ follows from part (2) in combination with Remark 3.5.

For the sake of exposition, we give a proof of Prop. 5.3 using the lemma below, which provides an explicit expressions of Li's α -normal field N_α , as well as the affine metric and shape operator relative to N_α , although the latter two are not needed in Prop. 5.3. Here, for any $u \in C^1(U)$ (not necessarily convex) we call the map $U \rightarrow \mathbb{R}^{n+1}$, $x \mapsto (Du(x), x \cdot Du(x) - u(x))$ the *Legendre map* of u .

Lemma 5.4. *Let U and u be as in Proposition 5.3 and $f : U \rightarrow \mathbb{R}^{n+1}$ be the Legendre map of u , viewed as a convex hypersurface embedding. Put $w_\alpha := -(\det D^2u)^{-\frac{\alpha}{n+2}} \in C^\infty(U)$. Then the Legendre map*

$$N_\alpha(x) := (Dw_\alpha(x), x \cdot Dw_\alpha(x) - w_\alpha(x))$$

of w_α is Li's α -normal field for the embedding f (with respect to the vertical unit vector $v = (0, \dots, 0, 1)$). Moreover, the affine metric h_α and the shape operator S_α of f relative to N_α have the following matrix expressions (with respect to the frame $(\partial_1, \dots, \partial_n)$):

$$h_\alpha = -\frac{1}{w} D^2u, \quad S_\alpha = (D^2u)^{-1} D^2w_\alpha.$$

Proof. By standard computations, one finds that for a general vector field $N = (N', N^{(n+1)}) : U \rightarrow \mathbb{R}^{n+1}$ transversal to f (where $N' : U \rightarrow \mathbb{R}^n$ and $N^{(n+1)} : U \rightarrow \mathbb{R}$ are respectively the horizontal and vertical components), if we define

$$w : U \rightarrow \mathbb{R}, \quad w(x) := x \cdot N'(x) - N^{(n+1)}(x),$$

then the induced volume form v , affine metric h , shape operator S and transversal connection form τ of f relative to N are given by

$$v = -w \det D^2u dx_1 \wedge \dots \wedge dx_n, \quad \tau = -\frac{1}{w} (dN^{(n+1)} - x \cdot dN'),$$

$$h = -\frac{1}{w} D^2u, \quad S = (D^2u)^{-1} (\partial_1 N', \dots, \partial_n N') - N'(\tau_1, \dots, \tau_n),$$

where in the last matrix expression, N' is understood as a row vector of functions, and τ_1, \dots, τ_n are the functions such that $\tau = \tau_1 dx_1 + \dots + \tau_n dx_n$.

The expression of τ implies that if N is equi-affine (i.e. $\tau = 0$), then we have

$$Dw(x) = D(x \cdot N'(x) - N^{(n+1)}(x)) = N'(x),$$

which implies that N is the Legendre map of w . Also, in this case, we have

$$S = (D^2u)^{-1} (\partial_1 N', \dots, \partial_n N') = (D^2u)^{-1} D^2w.$$

Thus, in order to prove the required statements, it now remains to be shown that if $w = -(\det D^2 u)^{-\frac{\alpha}{n+2}}$ then N is the α -normal field of f .

To this end, we use the characterization of the α -normal field in Lemma 2.4. By a simple computation, the induced volume form v_0 relative to the constant vertical unit vector field is given by

$$v_0 = \det D^2 u dx_1 \wedge \cdots \wedge dx_n.$$

So the density functions of v_h and v with respect to v_0 are given by

$$\frac{v_h}{v_0} = \frac{(-w)^{-\frac{\alpha}{2}} (\det D^2 u)^{\frac{1}{2}}}{\det D^2 u} = (-w)^{-\frac{\alpha}{2}} (\det D^2 u)^{-\frac{1}{2}}, \quad \frac{v}{v_0} = \frac{-w \det D^2 u}{\det D^2 u} = -w.$$

By Lemma 2.4, the condition for N to be the α -normal field is

$$(-w)^{-\frac{\alpha}{2}} (\det D^2 u)^{-\frac{1}{2}} = (-w)^\beta, \quad \text{where } \beta := \frac{1}{2} \left(\frac{n+2}{\alpha} - n \right).$$

This is equivalent to the required equality $w = -(\det D^2 u)^{-\frac{\alpha}{n+2}}$, so the proof is completed. \square

For the proof of Prop. 5.3, we will also need the following facts about the Legendre maps f_1 and f_2 of two functions $u_1, u_2 \in C^1(U)$ (not necessarily convex):

- We have $f_1 = cf_2$ for a constant $c \neq 0$ if and only if $u_1 = cu_2$.
- Suppose u_1 is strictly convex and f_2 is an immersion of U into \mathbb{R}^{n+1} with image contained in the image of f_1 . Then $u_1 = u_2$.

The first can be checked directly using the definition. For the second statement, first note that the assumption $f_2(U) \subset f_1(U)$ means every supporting hyperplane of the graph $\text{gr}(u_2)$ is also a supporting hyperplane of $\text{gr}(u_1)$ (for example, u_2 can be the affine function whose graph is a supporting hyperplane of $\text{gr}(u_1)$). On the other hand, it can be shown that f_2 is an immersion with locally strictly convex image if and only if u_2 is a locally strictly convex function. Thus, we can only have $u_1 = u_2$.

Proof of Proposition 5.3. (1) Let f be the Legendre map of u , so that Σ is the image of f . By definitions, Σ is as described in the statement if and only if Li's α -normal field $N_\alpha : U \rightarrow \mathbb{R}^{n+1}$ of f satisfies

$$(5.3) \quad N_\alpha = cf.$$

Lemma 5.4 says that N_α is the Legendre map of $w_\alpha := -(\det D^2 u)^{-\frac{\alpha}{n+2}}$. Therefore, by the first statement before the proof, Eq.(5.3) is equivalent to $w_\alpha = cu$, which is in turn equivalent to the required equation.

(2) Similarly as above, the condition “ N_α is an immersion with image contained in $\tilde{\Sigma}$ ” means that the Legendre map of w_α is an immersion with image contained in the image of the Legendre map of \tilde{u} . By the second statement before the proof, this is equivalent to $w_\alpha = \tilde{u}$, which is in turn equivalent to the required equation and implies the “Moreover” statement. \square

6. MONGE-AMPÈRE PROBLEM FOR AFFINE HYPERSPHERES

In the rest of the paper, we translate the geometric statements above, namely Theorems 3.2, 4.1, 4.3 and Proposition 4.4, into equivalent PDE statements via Theorem 5.1, and then give proofs. We will make use of basic notions in Monge-Ampère theory, such as *Monge-Ampère measure* and *convex generalized solutions* (see e.g. [11, 13, 27]), as well as the notions and facts about convex functions reviewed in Section 5. In particular, recall from Section 5 that the gradient blowup property “ $\|Du(x)\| \rightarrow +\infty$ as $x \in \Omega$ tends to $\partial\Omega$ ” for a convex function $u : \Omega \rightarrow \mathbb{R}$ is equivalent to the *infinite slope property*, which we use in all the statements below.

In this section, we treat Theorem 3.2, for which it suffices to consider the $c = 1$ case by Remark 3.3. By Theorem 5.1, this case is equivalent to the following result, which is stated in the introduction as Theorem C and the subsequent discussion:

Theorem 6.1 (Extended version of Theorem C, PDE version of Theorem 3.2). *Let Ω be a bounded convex domain in \mathbb{R}^n ($n \geq 2$) and consider the Dirichlet problem*

$$(6.1) \quad \begin{cases} \det D^2 w = (-w)^{-\gamma} & \text{in } \Omega, \\ w|_{\partial\Omega} = 0. \end{cases}$$

- (1) *If $\gamma > n$, then Eq.(6.1) has a unique convex generalized solution, which is smooth in Ω and has infinite slope at every point of $\partial\Omega$.*
- (2) *If $\gamma > 1$ and $\Omega \subset \mathbb{R}^n$ satisfies both the exterior and the interior sphere conditions at every boundary point, then the same conclusions as (1) holds.*
- (3) *If $0 < \gamma \leq 1$, the Eq.(6.1) does not have any convex generalized solution with infinite slope at every boundary point.*

Remark 6.2. The proof of Part (3) shows more specifically that if Ω satisfies the exterior sphere condition at some $p \in \partial\Omega$, then any convex function w satisfying $\det D^2 w \leq (-w)^{-\gamma}$ in the generalized sense with $0 < \gamma \leq 1$ must have finite slope at p .

The exponent $-\gamma$ in Eq.(6.1) corresponds to the exponent $-\frac{n+2}{\alpha}$ in the equations displayed in Theorem 5.1, so the conditions $\gamma > n$ and $\gamma > 1$ here are equivalent to the conditions $\alpha \in (0, 1 + \frac{2}{n})$ and $\alpha \in (0, n+2)$ in Theorem 3.2, respectively. We will discuss more about the range of γ at the end of this section. Also note that although the domain Ω here is supposed to be a hyperplane section of the dual cone C^* rather than C itself (see Section 5 for details), it is actually easy to see that Ω satisfies the exterior and the interior sphere conditions if and only if a hyperplane section of C does, so the assumption in Part (2) coincides with that in the second statement of Theorem 3.2.

We give the proof of Theorem 6.1 after establishing some lemmata. Although the proof is a simple application of the Monge-Ampère theory, we do not know any reference from which the theorem can be deduced immediately. In fact, most past works on equations of the form (6.1) impose strictly convexity and C^2 assumptions on $\partial\Omega$. The improvement to general Ω in Part (1) is achieved by using barrier functions from the following lemma, similarly as in [8, p.67]:

Lemma 6.3. *Let $\Delta \subset \mathbb{R}^n$ be a simplex and p_0, \dots, p_n be its vertices, so that every $x \in \overline{\Delta}$ can be written uniquely as*

$$x = t_0(x)p_0 + \dots + t_n(x)p_n, \quad \text{where } t_0(x), \dots, t_n(x) \in [0, 1], \quad t_0(x) + \dots + t_n(x) = 1.$$

Given $\gamma > -n$, consider the function $v \in C^0(\overline{\Delta}) \cap C^\infty(\Delta)$ defined by

$$v(x) := -\left(t_0(x) \cdots t_n(x)\right)^{\frac{2}{n+\gamma}},$$

which vanishes on $\partial\Delta$. Then v is convex if and only if $\gamma \geq n$. If $\gamma > n$, then there are constants $C_1, C_2 > 0$ only depending on n, γ and the volume of Δ , such that

$$C_1(-v)^{-\gamma} \leq \det D^2 v \leq C_2(-v)^{-\gamma}$$

in Δ , and v has infinite slope at every boundary point of Δ .

The proof consists of elementary but cumbersome calculations, so we postpone it to the appendix.

Remark 6.4. From the expression of $\det D^2 v$ in the proof of Lemma 6.3, we see that the equality $\det D^2 v = C(-v)^{-\gamma}$ holds only when $\gamma = n+2$. This gives a solution of Eq.(6.1) for $\Omega = \Delta$, $\gamma = n+2$, which corresponds to a hyperbolic affine hypersphere in \mathbb{R}^{n+1} of the form $\{x_1 \cdots x_{n+1} = c\}$ (c.f. [7]) via Thm. 5.1.

The following version of Comparison Principle for Monge-Ampère equations is a special case of [8, Prop. 2]. The proof here is adapted from the classical version (see [13, Thm. 1.4.6]).

Lemma 6.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain and $u_+, u_- \in C^0(\partial\Omega)$ be convex functions with boundary values $\varphi_{\pm} := u_{\pm}|_{\partial\Omega}$ such that $\varphi_- \leq \varphi_+$ and $u_{\pm} < \overline{\varphi}_+$ in Ω . Suppose $F(x, t)$ is a positive continuous function on $\{(x, t) \in \Omega \times \mathbb{R} \mid t < \overline{\varphi}_+(x)\}$ and is non-decreasing in t , and u_+ (resp. u_-) is a generalized supersolution (resp. subsolution) of the Monge-Ampère equation*

$$(6.2) \quad \det D^2 u = F(x, u).$$

Then we have $u_- \leq u_+$ on $\overline{\Omega}$.

Proof. Suppose by contradiction that $\epsilon := u_-(x_0) - u_+(x_0) > 0$ for some $x_0 \in \Omega$. We may take a sufficiently small $\delta > 0$ such that $\epsilon - \delta|x - x_0|^2 > 0$ on Ω . Then

$$E := \{x \in \Omega \mid u_-(x) - u_+(x) \geq \epsilon - \delta|x - x_0|^2\}$$

is a compact set with nonempty interior. Since the convex function $u'_- := u_- - \epsilon + \delta|x - x_0|^2$ equals u_+ on ∂E and is bounded from below by u_+ on E , the mass of the Monge-Ampère measure of u'_- on E is no larger than that of u_+ (see e.g. [13, Lemma 1.4.1]). That is,

$$(6.3) \quad \mathcal{M}[u_+](E) \geq \mathcal{M}[u'_-](E) = \mathcal{M}[u_- + \delta|x - x_0|^2] \geq \mathcal{M}[u_-](E) + (2\delta)^n \mu(E),$$

where $\mathcal{M}[u]$ denotes the Monge-Ampère measure of u , μ is the Lebesgue measure, and the last inequality follows from the super-additivity of Monge-Ampère measures (see e.g. [11, Lemma 2.9]). On the other hand, u_{\pm} being a generalized super-/sub-solution means $\det D^2 u_+ \leq F(x, u_+)$ and $\det D^2 u_- \geq F(x, u_-)$ in the generalized sense, which implies the inequality of measures

$$F(x, u_+(x))^{-1} \mathcal{M}[u_+] \leq F(x, u_-(x))^{-1} \mathcal{M}[u_-]$$

(where $F(x, u_{\pm}(x))^{-1}$ are understood as density functions). Since $u_+ < u_-$ on E and $F(x, t)$ is non-decreasing in t , this implies $\mathcal{M}[u_+](E) \leq \mathcal{M}[u_-](E)$, contradicting (6.3). \square

The smoothness assertion in Theorem 6.1 relies on the following regularity result, whose proof is suggested to us by Connor Mooney:

Lemma 6.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain and let $\varphi \in C^0(\partial\Omega)$. Suppose $F(x, t)$ is a smooth function on $\{(x, t) \in \Omega \times \mathbb{R} \mid t < \overline{\varphi}(x)\}$ and is non-decreasing in t , and $u \in C^0(\overline{\Omega})$ is a convex generalized solution to Eq.(6.2) with $u|_{\partial\Omega} = \varphi$, such that u has infinite slope at every point of $\partial\Omega$ (this implies $u(x) < \overline{\varphi}(x)$ for all $x \in \Omega$, see Section 5). Then $u \in C^\infty(\Omega)$.*

Proof. In order to prove that u is smooth around a point $x \in \Omega$, we may assume without loss of generality that $u(x) = 0$ and $u \geq 0$ on $\overline{\Omega}$. The infinite slope assumption implies that $u > 0$ on $\partial\Omega$, hence for sufficiently small $h > 0$, the section $S_h(x) := \{y \in \Omega \mid u(y) \leq h\}$ is a convex domain with closure contained in Ω .

By the interior regularity result of Cheng-Yau [8], if $v \in C^0(\overline{U})$ is a generalized convex solution to $\det D^2 v = G(x, v)$ in bounded convex domain U with $v|_{\partial U} = 0$, where the function $G \in C^\infty(U \times (-\infty, 0])$ is positive, bounded, and non-decreasing in the last variable, then $v \in C^\infty(U)$ ⁶. We obtain the required smoothness of u by applying this to $v = u - h$ on $U = S_h(x)$. \square

Remark 6.7. (1) In the applications below of Lemmas 6.5 and 6.6, the function $F(x, t)$ is either $(-t)^{-\gamma}$ or only a function of x . In the former case, a consequence of Lemma 6.5 that we will often invoke is that if $\Omega_1 \subset \Omega_2$ and $w_i \in C^0(\overline{\Omega}_i)$ is a convex generalized solution to Eq.(6.1) for $\Omega = \Omega_i$ ($i = 1, 2$), then $0 \geq w_1 \geq w_2$ on Ω_1 . In the latter case, Lemma 6.5 is a special case of the classical Comparison Principle (see [13, Thm. 1.4.6]) and the assumption “ $u_{\pm} < \overline{\varphi}_+$ in Ω ” is unnecessary.

⁶The original statement [8, Thm. 5] assumes that U satisfies the exterior sphere condition at every boundary point, but this assumption can be removed since it is only used to get paraboloid barriers, while for general U we can use the barriers from Lemma 6.3 instead. Alternatively, we can use a standard approximation argument (approximate U from the inside by nice subdomains, then apply Pogorelov C^2 estimate and Schauder theory to the solutions on these domains) to deduce the result for general U .

- (2) The assumption that u is continuous up to $\partial\Omega$ is inessential in Lemmas 6.5 and 6.6. Namely, both lemmas are valid for any convex $u \in C^0(\Omega)$ if we interpret the boundary value $u|_{\partial\Omega} : \partial\Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ in the way explained in Section 5. Also, the non-decreasing assumption on F is inessential in Lemma 6.6, as we can use the interior regularity result of Urbas [28] instead of Cheng-Yau's.

Proof of Theorem 6.1 (1). First suppose that a convex generalized solution w of Eq.(6.1) exists. It must be strictly negative in Ω because otherwise the convexity and the condition $w|_{\partial\Omega} = 0$ would imply that $w \equiv 0$ in Ω . Therefore, may apply Lemma 6.5 to infer that w is unique. Then we can show that w has infinite slope at every $p \in \partial\Omega$ as follows: Let $\Delta \subset \Omega$ be a simplex with a vertex at p , $v \in C^0(\bar{\Delta}) \cap C^\infty(\Delta)$ be the function given by Lemma 6.3, and v_1 be a constant multiple of v such that $\det D^2 v_1 \leq (-v_1)^{-\gamma}$ in Δ . Applying Lemma 6.5 to $u_+ := v_1$ and $u_- := w|_{\bar{\Delta}}$, we get $v_1 \geq w$ in $\bar{\Delta}$. Since v_1 has infinite slope at p and has the same value as w at p , it follows that w also has infinite slope at p , as required. By Lemma 6.6, it follows that $w \in C^\infty(\Omega)$.

Thus, we now only need to show the existence of w . If Ω satisfies the exterior sphere condition, this is a special case of [8, Thm. 1]. For general Ω , we let $\Omega_1 \subset \Omega_2 \subset \dots$ be an exhausting sequence of convex subdomains of Ω satisfying the exterior sphere condition, and consider the generalized solution w_i on Ω_i . Given $p \in \partial\Omega$, for any simplex $\Delta \subset \mathbb{R}^n$ containing Ω with $p \in \partial\Delta$, Lemma 6.3 provides a convex function $w_\Delta \in C^0(\bar{\Delta})$ vanishing on $\partial\Delta$ and satisfying $\det D^2 w_\Delta \geq (-w_\Delta)^{-\gamma}$, which bounds every w_i from below by Lemma 6.5 and Remark 6.7. Therefore, by Arzelà-Ascoli Lemma and the fact that the sequence (w_i) is decreasing, the sequence converges uniformly on any compact subset of Ω to a convex function w on Ω whose boundary value at p (in the sense of Section 5) is zero. Since p is arbitrary, we actually have $w|_{\partial\Omega} = 0$, which also implies that $w \in C^0(\partial\Omega)$. Now that w_i is a generalized solution and its Monge-Ampère measure weakly converges to that of w , we conclude that w is a generalized solution of (6.1), as required. \square

Part (2) of Theorem 6.1 can be proved by a nearly identical argument, only with the barriers on simplices from Lemma 6.3 replaced by those on balls from the following lemma, so we omit the details.

Lemma 6.8. *Let $B := B(0, 1) \subset \mathbb{R}^n$ be the unit ball. Then for any $\gamma \geq 1$, the radially symmetric function $v \in C^0(\bar{B}) \cap C^\infty(B)$ given by*

$$v(x) := -(1 - |x|^2)^{\frac{n+1}{n+\gamma}}$$

is convex, and there are constants $C_1, C_2 > 0$ only depending on γ , such that

$$C_1(-v)^{-\gamma} \leq \det D^2 v \leq C_2(-v)^{-\gamma}$$

in B . Moreover, v has infinite slope at the boundary points if and only if $\gamma > 1$.

Proof. For a radially symmetric function $v(x) = f(|x|)$, we have

$$\partial_i v(x) = \frac{f'}{|x|} x_i, \quad \partial_{ij}^2 v(x) = \frac{f'}{|x|} \left(\delta_{ij} - \frac{x_i x_j}{|x|^2} + \frac{x_i x_j f''}{|x| f'} \right)$$

(f' and f'' are evaluated at $|x|$). If $f'(t) > 0, f''(t) > 0$ for all $0 < t < 1$, then the matrix

$$A := I + \left(\frac{f''}{|x| f'} - \frac{1}{|x|^2} \right) x^t x$$

(where x is viewed as a column vector and x^t is its transpose) is positive definite for $0 < |x| < 1$ because

$${}^t y A y = \begin{cases} |y|^2 & \text{if } y \text{ is orthogonal to } x \\ |x|^2 + \left(\frac{f''}{|x| f'} - \frac{1}{|x|^2} \right) |x|^4 = \frac{|x|^3 f''}{f'} & \text{if } y = x. \end{cases}$$

The expression of $\partial_{ij}^2 v$ can be written as $D^2 v = \frac{f'}{|x|} A$, so v is convex in B this case, and we have

$$\det D^2 v(x) = \frac{f'(|x|)^{n-1} f''(|x|)}{|x|^{n-1}}.$$

Now apply these calculations to $f(t) = -(1-t^2)^\eta$ with $\eta = \frac{n+1}{n+\gamma}$. By a little more calculations, we find that $f'(t), f''(t) > 0$ holds for $0 < t < 1$ if and only if $\eta \leq 1$, or equivalently, $\gamma \geq 1$, and in this case we have

$$\det D^2 v(x) = (2\eta)^n [1 + (1-2\eta)|x|^2] (-v)^{-\gamma}.$$

The required inequality follows. The ‘‘Moreover’’ statement is elementary to check. \square

Remark 6.9. Similarly as Remark 6.4, the equality $\det D^2 v = C(-v)^{-\gamma}$ holds for the function v in Lemma 6.8 also only when $\gamma = n+2$, and this gives a solution of Eq.(6.1) for $\Omega = B$, $\gamma = n+2$, whose corresponding affine hypersphere is the hyperboloid $\{x_1^2 + \dots + x_n^2 + 1 = x_{n+1}^2\}$.

We also deduce Part (3) of Theorem 6.1 from this lemma.

Proof of Theorem 6.1 (3). Fix $0 < \gamma \leq 1$ and take a ball $B \subset \mathbb{R}^n$ containing Ω with $\partial B \cap \partial\Omega \neq \emptyset$. By Lemma 6.8, there is a convex function $v_1 \in C^0(\overline{B})$ which satisfies $\det D^2 v_1 \geq (-v_1)^{-\gamma}$ and has finite slopes at boundary points. Any convex generalized solution u to Eq.(6.1) is bounded from below by v_1 by Lemma 6.5 while having the same value as v_1 at any $p \in \partial B \cap \partial\Omega$, hence has finite slope at p . \square

By Parts (2) and (3), if Ω satisfies the exterior and interior sphere conditions, then the condition $\gamma > 1$ is necessary and sufficient for the convex solution of Eq.(6.1) to have the infinite slope property. However, we do not know whether the condition $\gamma > n$ in Part (1) is the optimal sufficient condition for general Ω . An essential problem here is:

Question. When $\Omega \subset \mathbb{R}^n$ is a simplex and $\gamma = n$, does the convex solution w of Eq.(6.1) have finite or infinite slope at the vertices of Ω ?

In fact, as γ decreases from $> n$ to 0, we believe that w should start to have finite slope first at the vertices, then at the edges, next at the 2-facets, and so on (compare Lemma 8.3 below about the $n = 2$ case). So a more general question is: Given $k = 0, \dots, n-1$, what is the exact range of γ for w to have infinite slope at every point of the k -skeleton of Ω ? We leave the study of these questions to future works.

7. MONGE-AMPÈRE PROBLEM FOR AFFINE (C, k) -HYPERSURFACE

In this section, we prove the following PDE result which implies Theorem 4.1 via Theorem 5.1. Note that the equivalence between the assumption on the C -regular domain $D = \text{ep}^\circ(\overline{\varphi}^*)$ in Theorem 4.1 and the assumption on φ below can be seen by using the rows ① and ② in Theorem 5.1.

Theorem 7.1 (Extended version of Theorem B, PDE version of Theorem 4.1). *Let $\gamma > n$, $\Omega \subset \mathbb{R}^n$ be a bounded convex domain and $w \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ be the convex solution of Eq.(6.1) given by Theorem 6.1. Suppose $\varphi \in C^0(\partial\Omega)$ satisfies the following condition: for every $p \in \partial\Omega$, there exists an affine function $a : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $a(p) = \varphi(p)$ and $a \geq \varphi$ on $\partial\Omega$. Then for each $\lambda > 0$, a unique convex generalized solution u to*

$$(7.1) \quad \begin{cases} \det D^2 u = \lambda(-w)^{-\gamma} & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi \end{cases}$$

exists and has the following properties:

- u is smooth in Ω ;
- u has infinite slope at every point of $\partial\Omega$;
- if u_t denotes the solution with parameter $\lambda = e^{-t}$ in Eq.(7.1), then for every fixed $x \in \Omega$, $u_t(x)$ is a strictly increasing concave function in t , with value tending to $-\infty$ and $\overline{\varphi}(x)$ as t tends to $-\infty$ and $+\infty$, respectively.

Proof. The scheme of the proof is the same as Theorem 6.1. If a generalized solution u of Eq.(7.1) exists, then it is unique by Lemma 6.5 (see also Remark 6.7), and can be shown to have infinite slope at any $p \in \partial\Omega$ as follows: Let a be an affine function as in the hypothesis and put $w_1 := a + w$. We have $w_1 = a \geq \varphi = u$ on $\partial\Omega$ and $\det D^2 w_1 = \det D^2 u$ in Ω , hence $w_1 \geq u$ holds in Ω by Lemma 6.5. Since w_1 has infinite slope

at p by Theorem 6.1 and has the same value as u at p , it follows that u has infinite slope at p as well. Lemma 6.6 then implies $u \in C^\infty(\Omega)$.

To show the existence of a generalized solution, we let $\Omega_1 \subset \Omega_2 \subset \dots$ be an increasing sequence of strictly convex subdomains of Ω with $\bigcup_i \Omega_i = \Omega$, consider the generalized solution to

$$\begin{cases} \det D^2 u = \lambda(-w)^{-\gamma} & \text{in } \Omega_i, \\ u|_{\partial\Omega_i} = \bar{\varphi}|_{\partial\Omega_i}, \end{cases}$$

which exists because $\int_{\Omega_i} \lambda(-w)^{-\gamma} dx < +\infty$ (see e.g. [13, Theorem 1.6.2]), and use the convex function $\bar{\varphi} + \lambda^{\frac{1}{n}} w \in C^0(\bar{\Omega})$ as a lower barrier. By the super-additive property of Monge-Ampère measure (see e.g. [11, Lemma 2.9]), we have

$$\det D^2(\bar{\varphi} + \lambda^{\frac{1}{n}} w) \geq \det D^2 \bar{\varphi} + \det D^2(\lambda^{\frac{1}{n}} w) = \lambda(-w)^{-\gamma}$$

in the generalized sense. This allows us to apply Lemma 6.5 and conclude that $u_i \geq \bar{\varphi} + \lambda^{\frac{1}{n}} w$ in Ω_i . As in the proof of Theorem 6.1, using Arzelà-Ascoli, we infer that a subsequence of (u_i) converges uniformly on compact subsets to a convex generalized solution u of Eq.(7.1) on Ω , as required. An immediate estimate of u , which we will use later, is

$$(7.2) \quad \bar{\varphi} + \lambda^{\frac{1}{n}} w \leq u \leq \bar{\varphi}.$$

In particular, this implies that $u|_{\partial\Omega} = \varphi$ in the sense of Section 5. Since $\varphi \in C^0(\partial\Omega)$, we have $u \in C^0(\bar{\Omega})$.

It remains to show the assertion in the last bullet point. The proof is the same as [22, §8.5]. First, using the fact that $\log \det$ is a concave function on the space of positive definite matrices, we get

$$\begin{aligned} \log \det D^2 \left(\frac{u_{t_1} + u_{t_2}}{2} \right) &= \log \det \left(\frac{D^2 u_{t_1} + D^2 u_{t_2}}{2} \right) \\ &\geq \frac{\log \det D^2 u_{t_1} + \log \det D^2 u_{t_2}}{2} = -\frac{t_1 + t_2}{2} + \log(-w)^{-\gamma} = \log \det D^2 u_{t_3} \end{aligned}$$

in Ω for any $t_1, t_2 \in \mathbb{R}$ and $t_3 := \frac{t_1 + t_2}{2}$. This implies $\frac{u_{t_1} + u_{t_2}}{2} \leq u_{t_3}$ by Lemma 6.5, which means $t \mapsto u_t(x)$ is a concave function for any fixed x . Next, the required limit $\lim_{t \rightarrow +\infty} u_t(x) = \bar{\varphi}(x)$ follows from Inequality (7.2), whereas the other limit $\lim_{t \rightarrow -\infty} u_t(x) = -\infty$ can be shown by taking a ball $B = B(x, \varepsilon)$ with $\bar{B} \subset \Omega$ and considering the upper barrier

$$v_t(y) = a e^{-\frac{t}{n}} (|y - x|^2 - \varepsilon^2) + b$$

of u_t in B , where the constants $a, b > 0$ are chosen to ensure that

$$\det D^2 v_t = (2a)^n e^{-t} \leq e^{-t} (-w)^{-\gamma} = \det D^2 u_t$$

in B and $v_t \geq \bar{\varphi} \geq u_t$ on ∂B for all $t \in \mathbb{R}$. By Lemma 6.5 again, the inequality $v_t \geq u_t$ holds in B , and in particular at x , so the required limit follows. Finally, it follows from Lemma 6.5 yet again that for any fixed $x \in \Omega$, the function $t \mapsto u_t(x)$ is non-decreasing. If it is not strictly increasing, then the concavity and limiting properties just established would imply that $u_t(x) = \bar{\varphi}(x)$ for all t larger than some t_0 . But since u_t is strictly convex in Ω (as it is smooth, convex, with $\det D^2 u_t > 0$), this is impossible (see Section 5). The proof is completed. \square

Remark 7.2. (1) Since the assumption $\gamma > n$ is inherited from Part (1) of Theorem 6.1, by using Part (2) instead, we may relax it to $\gamma > 1$ if Ω satisfies both the exterior and interior sphere conditions.

(2) If Ω is a bounded convex domain with C^2 boundary and satisfying the exterior sphere condition, then a sufficient condition for φ to fulfill the assumption in Theorem 7.1 is $\varphi \in C^2(\partial\Omega)$. Indeed, fix a point p and write $\partial\Omega$ locally as a graph of the form $\{x + f(x)v \mid x \in U\}$, where $U \subset H$ is an open subset, H is the tangent hyperplane of $\partial\Omega$ through p , v is its inward unit normal vector, and $f \in C^2(U)$ is a convex function. Then the exterior sphere condition at p is equivalent to the condition that the Hessian of f is positive definite at p . Up to adding an affine function, we can

suppose that φ has a critical point at p ; now one easily sees that the affine function mapping $x + tv$ to $x + ctv$ is larger than φ on $\partial\Omega$ if $c > 0$ is sufficiently large.

- (3) As one can see from the proof, the existence and uniqueness of the convex generalized solution u to Eq.(7.1) hold independently of the assumption on φ . The assumption is only used to ensure the regularity and the infinite slope property.
- (4) When Ω is the unit ball in \mathbb{R}^n with $n \geq 3$, the examples constructed by Bonsante-Fillastre [4, §3.7] correspond to certain $\varphi \in C^0(\partial\Omega)$ such that the convex generalized solution u of Eq.(7.1) is not smooth or strictly convex. In fact, u restricts to affine functions on many line segments in Ω joining boundary points, and coincides with $\bar{\varphi}$ on these segments.

8. MONGE-AMPÈRE PROBLEM FOR AFFINE (C, k) -SURFACE ($n = 2$ CASE).

In this final section, we generalize Theorem 7.1 along the line of our previous work [22] and obtain Theorem 8.1 below, which implies Theorem 4.3 (= Theorem B in the introduction) via Theorem 5.1.

Recall that the main novelty of [22] on the PDE aspect is the study of Eq.(7.1) when $n = 2$ and φ is merely an $\mathbb{R} \cup \{+\infty\}$ -valued lower semicontinuous function. The simplest nontrivial example is

$$(8.1) \quad \varphi(x) = \begin{cases} 0 & \text{if } x \in \{p_1, p_2, p_3\}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $p_1, p_2, p_3 \in \partial\Omega$. Another typical example is $\varphi = 0$ throughout $\partial\Omega$ except at a single point p , with $\varphi(p) = -1$. When Ω is the unit disk, the solution of Eq.(7.1) for the latter example is computed in [5].

When $\varphi = +\infty$ on a part of $\partial\Omega$, we interpret Eq.(7.1) as the problem of seeking a lower semicontinuous convex function u on \mathbb{R}^2 with $u|_{\partial\Omega} = \varphi$, such that $\text{dom}(u)$ is contained in $\bar{\Omega}$ and has nonempty interior U , with the Monge-Ampère equation satisfied in U . We shall add to this problem the extra constraint that u has infinite slope at every boundary point of U within Ω , otherwise the solution would not be unique for trivial reason: for example, if φ is as in (8.1), then for any closed convex set $E \subset \bar{\Omega}$ with $E \cap \partial\Omega = \{p_1, p_2, p_3\}$, modifying the values of the function w outside of E into $+\infty$ would give a solution.

The main result below asserts that Eq.(7.1), interpreted in this way, has a unique solution. Here the spaces of functions $\text{LC}(\mathbb{R}^2)$ and $\text{LC}(\partial\Omega)$ were introduced in Section 5. Also recall that the essential domain $\text{dom}(\bar{\varphi})$ of the convex envelope $\bar{\varphi}$ of any $\varphi \in \text{LC}(\partial\Omega)$ coincides with the convex hull of $\text{dom}(\varphi)$.

Theorem 8.1 (Extended version of Theorem A, PDE version of Theorem 4.3). *Let $\gamma > 2$, $\Omega \subset \mathbb{R}^2$ be a bounded convex domain, and $w \in C^0(\bar{\Omega}) \cap C^\infty(\Omega)$ be the convex solution of Eq.(6.1) given by Theorem 6.1. Let $\varphi \in \text{LC}(\partial\Omega)$ be such that $\text{dom}(\bar{\varphi})$ has nonempty interior. Then for any $\lambda > 0$, there exists a unique $u \in \text{LC}(\mathbb{R}^2)$ satisfying*

$$(8.2) \quad \begin{cases} U := \text{int dom}(u) \neq \emptyset, U \subset \Omega, \\ \det D^2 u = \lambda(-w)^{-\gamma} \text{ in } U, \\ u|_{\partial\Omega} = \varphi, \\ u \text{ has infinite slope at every point of } \partial U \cap \Omega, \end{cases}$$

in the generalized sense, and it has the following properties:

- (a) u is smooth in U ;
- (b) there exists a convex function $f \in C^0(\bar{U})$ with $f|_{\partial U} = 0$ such that $\bar{\varphi} + f \leq u \leq \bar{\varphi}$ on \bar{U} ;
- (c) if u_t denotes the solution with parameter $\lambda = e^{-t}$ in Eq.(8.2), then for every fixed $x \in U$, $u_t(x)$ is a strictly increasing concave function in t , with value tending to $-\infty$ and $\bar{\varphi}(x)$ as t tends to $-\infty$ and $+\infty$, respectively.

Moreover, if $\gamma \geq 4$ and Ω satisfies the exterior circle condition at $p \in \partial U \cap \partial\Omega$, then u has infinite slope at p .

In particular, if Ω satisfies the exterior circle condition at every boundary point, then the resulting solutions belong to the space $S_0(\Omega)$ defined in Section 5. Therefore, using Theorem 5.1, one readily checks that this theorem implies the geometric result, Theorem 4.3.

Before giving the proof of Theorem 8.1, we first note that for a general $u \in \text{LC}(\mathbb{R}^2)$ with $u|_{\partial\Omega} = \varphi$, since $u \leq \bar{\varphi}$ on \mathbb{R}^2 , the convex set $\text{dom}(u)$ contains $\text{dom}(\bar{\varphi})$, but the two sets might not coincide. Even if they coincide, the values of u and $\bar{\varphi}$ on the boundary of the set might not be the same. However, we do have $\text{dom}(u) = \text{dom}(\bar{\varphi})$ and $u|_{\partial U} = \bar{\varphi}|_{\partial U}$ for u satisfying (8.2). The proof of this relies on and the following Generalized Comparison Principle from [22]:

Lemma 8.2 ([22, Lemma 6.4]). *Let $U \subset \mathbb{R}^n$ be a bounded convex domain, $u_+ : \bar{U} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function taking finite values in U and $u_- : \bar{U} \rightarrow \mathbb{R}$ be a continuous convex function such that*

- $\det D^2 u_+ \leq \det D^2 u_-$ in U in the generalized sense;
- $u_+(x) \geq u_-(x)$ for every $x \in \partial U$ where u_+ has finite slope, as well as every $x \in \partial U$ where u_- has infinite slope.

Then we have $u_+ \geq u_-$ throughout \bar{U} .

The infinite slope property in Theorem 8.1 will be treated by using the following results:

Lemma 8.3 ([22, Lemmas 8.3 and 8.4]). *Let $\Delta \subset \mathbb{R}^2$ be a triangle, $p \in \partial\Delta$ be a vertex, $I \subset \partial\Delta$ be an edge, and $u \in C^0(\bar{\Delta})$ be a convex function. Then the following statements hold.*

- (1) *If there are constants $c > 0$ and $\beta > -2$ such that*

$$\begin{cases} \det D^2 u(x) \leq c|x-p|^\beta \text{ for } x \in \Delta \\ u|_{\partial\Delta} = 0 \end{cases}$$

in the generalized sense, then u has finite slope at p .

- (2) *If there is a constant $c > 0$ such that*

$$\det D^2 u(x) \geq c|x-p|^{-2} \text{ for } x \in \Delta$$

in the generalized sense, then u has infinite slope at p .

- (3) *If there is a constant $c > 0$ such that*

$$\begin{cases} \det D^2 u \geq c \text{ in } \Delta \\ u|_I = 0 \end{cases}$$

in the generalized sense, then u has infinite slope at every interior point of I .

We now present the proof of Theorem 8.1. Since it is very similar to the $\gamma = 4$ case treated in [22, Theorem A'] except a few modifications, we do not give all the details.

Proof of Theorem 8.1. First, we fix a solution $u \in \text{LC}(\mathbb{R}^2)$ of Eq.(8.2) with parameter $\lambda = 1$ and prove properties (a) and (b). The proof easily adapts to general $\lambda > 0$.

Since $(-w)^\gamma$ is a positive smooth function in Ω , the regularity theory of Monge-Ampère equations in two variables (see e.g. [22, Thm. 6.7]) immediately implies property (a).

We proceed to show property (b) when $f = w_0 \in C^0(\bar{U}) \cap C^\infty(U)$ is the convex solution to

$$(8.3) \quad \begin{cases} \det D^2 w_0 = (-w_0)^{-\gamma} \text{ in } U, \\ w_0|_{\partial U} = 0, \end{cases}$$

(whose unique-existence is given by Theorem 6.1). Namely, we shall show

$$(8.4) \quad u \geq \bar{\varphi} + w_0 \text{ on } \bar{U}.$$

For the sake of clearness, let us first show the weaker inequality $u \geq \bar{\varphi} + w$ on \bar{U} . Since $\bar{\varphi}$ is the pointwise supremum of all affine functions $a : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying $a|_{\partial\Omega} \leq \varphi$, it suffices to show $u \geq a + w$ for any such a . But this follows from Lemma 8.2 because on one hand, the Monge-Ampère measures of u and $a + w$ coincide; on the other hand, a point $x \in \partial U$ where either u has finite slope or $a + w$ has infinite slope can only lie on $\partial\Omega$, where we have $u(x) \geq a(x) = a(x) + w(x)$ by assumptions.

This argument does not work immediately if we replace w by w_0 : although we still have the correct comparison of Monge-Ampère measures required by Lemma 8.2, namely

$$\det D^2 u = (-w)^{-\gamma} \geq (-w_0)^{-\gamma} = \det D^2(a + w_0)$$

(this follows from Lemma 6.5), we lack the comparison of boundary values, as $a + w_0$ now has infinite slope on the whole ∂U rather than just on $\partial U \cap \partial \Omega$. Nevertheless, it can be adjusted as follows. Let $U_\delta := \{x \in \Omega \mid d(x, U) < \delta\}$ be the intersection of the δ -neighborhood of U with Ω , and $w_\delta \in C^0(\overline{U}_\delta) \cap C^\infty(U_\delta)$ be the solution of Eq.(8.3) with U replaced by U_δ . Then the argument is valid for w_δ and yields

$$(8.5) \quad u \geq a + w_\delta \text{ on } \overline{U}.$$

As δ tends to 0, the restriction of w_δ to U increases and converges uniformly to w_0 (the proof of this is given in [22, Prop. 7.8] for the $\gamma = n + 2$ case, which can be generalized to any $\gamma > n$ by using the barrier functions from Lemma 6.3). Therefore, by taking the pointwise supremum of (8.5) for all affine function a with $a|_{\partial \Omega} \leq \varphi$ and all $\delta > 0$, we obtain the required inequality (8.4) and finish the proof of (b).

Next, we show the unique-existence of solution $u \in \text{LC}(\mathbb{R}^2)$ to Eq.(8.2). By property (b), if we let U denote the interior of $\text{dom}(\overline{\varphi})$, then a solution $u \in \text{LC}(\mathbb{R}^2)$ of (8.2) is equivalent to a convex generalized solution $\tilde{u} \in C^0(U)$ of

$$(8.6) \quad \begin{cases} \det D^2 \tilde{u} = (-w)^{-\gamma} \text{ in } U, \\ \tilde{u}|_{\partial U} = \overline{\varphi}|_{\partial U}, \\ \tilde{u} \text{ has infinite slope at every point of } \partial U \cap \Omega, \end{cases}$$

(we obtain u by setting the value of \tilde{u} outside \overline{U} to be $+\infty$). If the last slope condition in Eq.(8.6) is removed, one easily shows the unique-existence of \tilde{u} by a standard approximation argument, as in the proofs of Theorems 6.1 and 7.1, using the function $\overline{\varphi} + w_0$ as a lower barrier. But such a \tilde{u} automatically has infinite slope at any $q \in \partial U \cap \Omega$ because of Lemma 8.3 (3) and the fact that q lies on a line segment in ∂U (since $\text{dom}(\overline{\varphi})$ is the convex envelope of points on $\partial \Omega$). This shows the unique-solvability of (8.6), and hence that of (8.2).

Now it only remains to show (c) and the last ‘‘Moreover’’ statement. Property (c) has already been proven in Theorem 7.1 in a slightly different setting. The proof still works in the current setting, so we omit the details. The idea of proof for the last statement is to estimate w near p using the function from Lemma 6.8 as lower barrier, then apply Lemma 8.3 (2). More precisely, let $B \subset \mathbb{R}^2$ be a disk containing Ω such that ∂B passes through p . Assuming without loss of generality that $B = B(0, 1)$, we use Lemma 6.8 to get a convex function $v \in C^0(\overline{B})$ of the form $v = -C(1 - |x|^2)^{\frac{3}{2+\gamma}}$ satisfying $\det D^2 v \geq (-v)^{-\gamma}$, which bounds w from below by Lemma 6.5. Given a triangle $\Delta \subset U$ with a vertex at p , we then have the following estimate for the right-hand side of the Monge-Ampère equation for u on Δ :

$$(-w)^{-\gamma} \geq (-v)^{-\gamma} = C^{-\gamma}(1 - |x|^2)^{-\frac{3\gamma}{2+\gamma}} \geq C_1|x - p|^{-\frac{3\gamma}{2+\gamma}} \geq C_2|x - p|^{-2} \quad (\text{for all } x \in \Delta),$$

where $C_1, C_2 > 0$ only depends on γ and Δ , the last inequality uses the assumption $\gamma \geq 4$, and the second-to-last inequality is because we have $c|x - p| \leq 1 - |x|^2 \leq c^{-1}|x - p|$ for all $x \in \Delta$ and some $0 < c < 1$ only depending on Δ . Therefore, we obtain the required statement by applying Lemma 8.3 (2) to $w|_{\overline{\Delta}}$. \square

Finally, the following result corresponds to Proposition 4.4 and explains the necessity of the condition $\gamma \geq 4$ in the last statement of Theorem 8.1:

Proposition 8.4 (PDE version of Prop. 4.4). *Under the hypotheses of Theorem 8.1, further assume that $\gamma < 4$, $\Omega = B := B(0, 1)$ is the unit disk, and φ is the function given by (8.1), namely $\varphi = 0$ at $p_1, p_2, p_3 \in \partial B$ and $\varphi = +\infty$ everywhere else. Then the solution $u \in \text{LC}(\mathbb{R}^2)$ to Eq.(8.2) has finite slope at p_1, p_2 and p_3 .*

Proof. Let us show that u has finite slope at p_1 . Let Δ denote the triangle with vertices p_1, p_2 and p_3 , so that $\text{dom}(u) = \text{dom}(\overline{\varphi}) = \overline{\Delta}$ by property (b) in Theorem 8.1. Lemma 6.8 gives a convex function $v \in C^0(\overline{B})$ of the form $v(x) = -C(1 - |x|^2)^{\frac{3}{2+\gamma}}$ satisfying $\det D^2 v \leq (-v)^{-\gamma}$, which bounds w from above by Lemma 6.5.

Similarly as in the last paragraph of the previous proof, we get the following estimate for the right-hand side of the Monge-Ampère equation for u :

$$(-w)^{-\gamma} \leq (-v)^{-\gamma} = C^{-\gamma}(1 - |x|^2)^{-\frac{3\gamma}{2+\gamma}} \leq C_1|x - p_1|^{-\frac{3\gamma}{2+\gamma}} \quad (\text{for all } x \in \Delta).$$

Since we have $-\frac{3\gamma}{2+\gamma} > -2$ by the assumption $\gamma < 4$ and $u|_{\partial\Delta} = 0$ by Theorem 8.1, we can apply Lemma 8.3 (1) and conclude that u has finite slope at p_1 . \square

APPENDIX A. PROOF OF LEMMA 6.3

By applying an affine transformation to the simplex Δ , which changes the Monge-Ampère measure of any convex function by a constant factor only depending on the Jacobian of the affine transformation, we may assume that the vertices of Δ are

$$p_0 = 0, p_1 = (1, 0, \dots, 0), \dots, p_n = (0, \dots, 0, 1),$$

so that we have $t_0(x) = 1 - x_1 - \dots - x_n$ and $t_i(x) = x_i$. This reduces Lemma 6.3 to:

Lemma A.1. *Let $\Delta := \{x \in \mathbb{R}^n \mid x_1, \dots, x_n > 0, x_1 + \dots + x_n < 1\}$. Given $\gamma > -n$, let $v \in C^0(\overline{\Delta})$ be defined by*

$$v(x) := -(x_0 x_1 \dots x_n)^{\frac{2}{n+\gamma}},$$

where given any $x \in \overline{\Delta}$, we denote $x_0 := 1 - x_1 - \dots - x_n$. Then v is convex if and only if $\gamma \geq n$. If $\gamma > n$, then there are constants $C_1, C_2 > 0$ only depending on n and γ , such that

$$(A.1) \quad C_1(-v)^{-\gamma} \leq \det D^2 v \leq C_2(-v)^{-\gamma}$$

in Δ , and v has infinite slope at every boundary point of Δ .

Proof. The last infinite slope property when $\gamma > n$ is elementary to check using the expression of v . To prove the other assertions, let us first compute $\det D^2 v$. Fix $\gamma > -n$ and put

$$\eta := \frac{2}{n+\gamma}, \quad V(x) := \eta(\log x_0 + \log x_1 + \dots + \log x_n).$$

Then we have $v = -e^V$ and

$$\partial_{ij}^2 v = v(\partial_{ij}^2 V + \partial_i V \partial_j V) =: (-v)(A_{ij} - B_i B_j),$$

where we set

$$B_i := \partial_i V = \eta \left(\frac{1}{x_i} - \frac{1}{x_0} \right), \quad A_{ij} := -\partial_{ij}^2 V = \eta \left(\frac{\delta_{ij}}{x_i^2} + \frac{1}{x_0^2} \right).$$

We have the following equalities for the determinant of a symmetric block matrix:

$$\det \begin{pmatrix} A & B \\ {}^t B & C \end{pmatrix} = \det(A) \det(C - {}^t B A^{-1} B) = \det(C) \det(A - B C^{-1} {}^t B)$$

(the first equality can be shown by eliminating the upper-right block, and the second by eliminating the lower-left one). As a consequence, when C is the 1×1 identity, we get

$$(A.2) \quad \det(A - B {}^t B) = (1 - {}^t B A^{-1} B) \det(A).$$

In order to apply this formula to the above-defined A and B , we need to compute ${}^t B A^{-1} B$ and $\det(A)$. To this end, note that A can be written as

$$A = \frac{\eta}{x_0^2} \left[\begin{pmatrix} (x_0/x_1)^2 & & \\ & \ddots & \\ & & (x_0/x_n)^2 \end{pmatrix} + \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (1, \dots, 1) \right] =: \frac{\eta}{x_0^2} (\Lambda + E),$$

It follows that A^{-1} can be calculated through series expansion as

$$A^{-1} = \frac{x_0^2}{\eta} (\Lambda^{-1} - \Lambda^{-1} E \Lambda^{-1} + \Lambda^{-1} E \Lambda^{-1} E \Lambda^{-1} - \dots) = \frac{x_0^2}{\eta} \sum_{k \geq 0} (-1)^k (\Lambda^{-1} E)^k \Lambda^{-1}.$$

When $k \geq 1$, we have

$$(\Lambda^{-1}E)^k \Lambda^{-1} = \Lambda^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \left[(1, \dots, 1) \Lambda^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right]^{k-1} (1, \dots, 1) \Lambda^{-1} = \left(\frac{|x|^2}{x_0^2} \right)^{k-1} \Lambda^{-1} E \Lambda^{-1},$$

where $|x|^2 := x_1^2 + \dots + x_n^2$. Therefore, we obtain

$$A^{-1} = \frac{x_0^2}{\eta} \left(\Lambda^{-1} - \frac{x_0^2}{x_0^2 + |x|^2} \Lambda^{-1} E \Lambda^{-1} \right) = \frac{1}{\eta} \begin{pmatrix} x_1^2 & & \\ & \ddots & \\ & & x_n^2 \end{pmatrix} - \frac{1}{\eta(x_0^2 + |x|^2)} \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix} (x_1^2, \dots, x_n^2),$$

$${}^t B A^{-1} B = \eta \left\{ \sum_{i=1}^n \left(1 - \frac{x_i}{x_0} \right)^2 - \frac{1}{x_0^2 + |x|^2} \left[\sum_{i=1}^n \left(1 - \frac{x_i}{x_0} \right) x_i \right]^2 \right\}.$$

The two sums in the expression of ${}^t B A^{-1} B$ can be simplified as follows:

$$\sum_{i=1}^n \left(1 - \frac{x_i}{x_0} \right)^2 = \sum_{i=1}^n \left(1 + \frac{x_i^2}{x_0^2} - \frac{2x_i}{x_0} \right) = n + \frac{|x|^2}{x_0^2} - \frac{2(1-x_0)}{x_0} = n + 2 + \frac{|x|^2}{x_0^2} - \frac{2}{x_0},$$

$$\sum_{i=1}^n \left(1 - \frac{x_i}{x_0} \right) x_i = (1-x_0) - \frac{|x|^2}{x_0} = 1 - \frac{x_0^2 + |x|^2}{x_0}.$$

It follows that

$${}^t B A^{-1} B = \eta \left\{ n + 2 + \frac{|x|^2}{x_0^2} - \frac{2}{x_0} - \frac{1}{x_0^2 + |x|^2} \left(1 - \frac{x_0^2 + |x|^2}{x_0} \right)^2 \right\} = \eta \left(n + 1 - \frac{1}{x_0^2 + |x|^2} \right)$$

On the other hand, noting that (A.2) still holds if both “-” signs are replaced by “+”, we get

$$\det(A) = \left(\frac{\eta}{x_0^2} \right)^n \left[1 + \left(\frac{x_1}{x_0} \right)^2 + \dots + \left(\frac{x_n}{x_0} \right)^2 \right] \left(\frac{x_0^n}{x_1 \cdots x_n} \right)^2 = \frac{\eta^n (x_0^2 + |x|^2)}{(x_0 x_1 \cdots x_n)^2}.$$

Putting everything together, we obtained the expression of $\det D^2 v$ as

$$\begin{aligned} \det D^2 v &= (-v)^n \det(A - B {}^t B) = (-v)^n (1 - {}^t B A^{-1} B) \det(A) \\ &= (x_0 x_1 \cdots x_n)^{\eta n} \left(1 - \eta(n+1) + \frac{\eta}{x_0^2 + |x|^2} \right) \frac{\eta^n (x_0^2 + |x|^2)}{(x_0 x_1 \cdots x_n)^2} \\ &= \eta^{n+1} [1 - (n+1 - \eta^{-1})(x_0^2 + |x|^2)] (-v)^{-\gamma}. \end{aligned}$$

The condition $\gamma > n$ is equivalent to $n+1 - \eta^{-1} < 1$. On the other hand, it is elementary to check that $x_0^2 + |x|^2 \leq 1$ for all $x \in \bar{\Delta}$, with equality exactly when x is a vertex. Thus, we obtain the required inequality (A.1) when $\gamma > n$. We may also conclude that if $\gamma \geq n$, then $\det D^2 v > 0$ throughout Δ , otherwise we have $\det D^2 v < 0$ near a vertex. In particular, v is not convex if $-n < \gamma < n$.

Now it only remains to be shown that v is indeed convex when $\gamma \geq n$. We shall fix such a γ and show that $D^2 v > 0$ in Δ . Since we already know $\det D^2 v > 0$ in Δ , if $D^2 v(x)$ is not positive definite for $x \in \Delta$, then it has at least two negative eigenvalues. But by the above computations, $D^2 v(x)$ can be written as $\tilde{A} - \tilde{B} {}^t \tilde{B}$ for the positive definite symmetric matrix $\tilde{A} = -v(x)A(x)$ and column vector $\tilde{B} = -v(x)B(x)$ (note that $v(x) < 0$). Hence it admits at most one negative eigenvalue, because ${}^t y(\tilde{A} - \tilde{B} {}^t \tilde{B})y = {}^t y \tilde{A} y > 0$ for any y orthogonal to \tilde{B} . This contradiction shows $D^2 v > 0$ and completes the proof. \square

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