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Fuzzy interpolation by convex completion of sparse rule bases

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Abstract— This paper proposes an approach to the interpolation between sparse fuzzy rules, founded on a unique principle which supplements the classical approximate reasoning machinery. The case of rules of the form $A \rightarrow B$, where A and B are intervals is first discussed, and then extended to fuzzy sets.

I. INTRODUCTION

Fuzzy rule-based systems have been either used as a convenient tool for synthesizing control laws from data, or in a knowledge representation and reasoning perspective in AI. Beside the natural way of representing imprecise and gradual pieces of knowledge that fuzzy rules provide, both applications take advantage of the partial overlapping of their condition parts in the inference mechanism. Indeed, when the input domain is covered by the premises of the rules, well-founded and established reasoning mechanisms associated to fuzzy rule-based systems have been proposed to generalize the Modus Ponens, as the so-called Compositional Rule of Inference (CRI) introduced by Zadeh in 1973 [1]. By contrast, in case of sparse rules, a special interpolative reasoning mechanism is needed.

Interpolation relies on the assumption of gradual and smooth variation of a variable with respect to another one. This is related to the idea of distance which underlies the natural way human beings fill the gap between extreme rules when they have to cope with an input “ X is A^* ” which does not fit any of the premises “ X is A_i ” of available sparse rules. Interpolative reasoning naturally subsumes some idea of similarity, where at least two rules of the form “if X is A_i then Y is B_i ” must be used simultaneously. Indeed the result of an interpolation depends on the reference values provided by the rules that are the closest to the input value “ X is A^* ”. Kóczy and Hirota [2], [3] have proposed some empirical interpolative methods for reasoning from sparse rules, i.e., rules where the A_i ’s do not achieve a coverage of the input universe U , but are rather disjoint and even distant from one another. The problem is then to compute some plausible conclusion “ Y is B^* ” for an input “ X is A^* ”, with B^* in between some B_i and B_j , just as A^* is in between the corresponding A_i and A_j , using distances between fuzzy sets. Working on the α -cuts $(A)_\alpha = [\underline{a}, \bar{a}]$ of fuzzy sets A , a natural idea is to compute the α -cut of B^* as $[\underline{b}^*, \bar{b}^*] = [\lambda \cdot \underline{b}_i + (1 - \lambda) \cdot \underline{b}_j, \lambda' \cdot \bar{b}_i + (1 - \lambda') \cdot \bar{b}_j]$ on the basis of the solutions λ and λ' of the equations $\underline{a}^* = \lambda \cdot \underline{a}_i + (1 - \lambda) \cdot \underline{a}_j$ and $\bar{a}^* = \lambda' \cdot \bar{a}_i + (1 - \lambda') \cdot \bar{a}_j$ where $[\underline{a}^*, \bar{a}^*]$ is the α -cut of A^* . Unfortunately, the α -cuts of such a B^* may not be nested and we may even have $\underline{b}^* > \bar{b}^*$. See Yan et al. [4], Qiao et al. [5], Hsiao et al. [6] for further developments along this line.

This interpolation problem can also be formalized as the computation of a convex sum of B_i and B_j based on a parameter λ imprecisely constrained by $u_0 = \lambda \cdot u_1 + (1 - \lambda) \cdot u_2$, with u_0 restricted by A^* , u_1 by A_i and u_2 by A_j [7], [8]. However, while the above-mentioned drawback is avoided, the result of the inference may be very imprecise.

A general method has been proposed by Bouchon-Meunier et al. [9], [10] on the basis of the concepts of location and shape of a fuzzy set and the evaluation of the distinguishability of two fuzzy sets with regard to these concepts and in agreement with the analogical scheme proposed in [11]. More precisely, if A^* is between A_i and A_{i+1} , with respect to some order on fuzzy sets, a conclusion B^* regarding variable Y is constructed between B_i and B_{i+1} , in such a way that the location and shape distinguishabilities of B^* and B_i , of B^* and B_{i+1} , are analogous to the location and shape distinguishabilities of A^* and A_i , of A^* and A_{i+1} . However, several degrees of freedom remain in this rather general scheme.

In this paper, the proposed approach relies on a unique linear interpolation principle which, once accepted, fully determines the result of the inference. The underlying idea of the approach, first briefly suggested at the end of [12], and closely related to some mechanism first partially described in [13] is the following. In classical reasoning, the two rules $(A \cap A') \rightarrow (B \cap B')$ and $(A \cup A') \rightarrow (B \cup B')$ are direct consequences of the set of rules $\{A \rightarrow B, A' \rightarrow B'\}$. The basic principle of the proposed approach is to augment this machinery on the real line, by assuming that it is also allowed to derive from the two rules any new rule $A_\lambda \rightarrow B_\lambda$, where $\lambda \in [0, 1]$, and where $A_\lambda = \lambda \cdot A + (1 - \lambda) \cdot A'$ and $B_\lambda = \lambda \cdot B + (1 - \lambda) \cdot B'$, obtained by interval arithmetics.

Considering two sparse rules $A_i \rightarrow B_i$ and $A_{i+1} \rightarrow B_{i+1}$, and an observation A^* in between A_i and A_{i+1} , the method consists in computing a union or an intersection of several A_λ which best fit A^* . Then, the union or intersection of the corresponding B_λ gives the corresponding output B^* , according to a fuzzy linear interpolation scheme between the two initial rules, as it will be shown.

In Section II, this paper first studies linear interpolative reasoning between two non fuzzy rules, i.e., rules of the form $A \rightarrow B$, where A and B are intervals. The case of one input and one output rules is first considered, followed by a brief discussion of interpolation with multiple input rules. Then, in Section III the results are extended to fuzzy rules, by approximating A^* with a union or an intersection of A_λ s (and then B^* with B_λ s).

II. CASE OF NON-FUZZY RULES

This section deals with interpolative reasoning between the two non-fuzzy sparse rules of the set $\mathcal{K} = \{A_0 \rightarrow B_0, A_1 \rightarrow B_1\}$. The operator $\rightarrow$ is then the classical logic implication. The rules in $\mathcal{K}$ being sparse (or disjoint) means that $A_0 \cap A_1 = \emptyset$. In this section, as only intervals (instead of fuzzy sets) are considered, an interval A is denoted $[\underline{a}, \bar{a}]$.

Now, consider a piece of information A^* (e.g., an observation) on the input variable X . The set A^* is also supposed to be an interval, which lies in between A_0 and A_1 , on the input universe U . It encompasses the case where A^* somewhat overlaps with A_0 and/or A_1 , but still being in between (i.e., $\underline{a}_0 \leq \underline{a}^* \leq \bar{a}^* \leq \bar{a}_1$), as the interpolation mechanism also has influence on A_0 and A_1 , as shown on Figure 5.c.

The problem is then to determine an interval B^* in between B_0 and B_1 (which restricts the possible values for the variable Y , on the output universe V), such that $A^* \rightarrow B^*$ is a valid rule w.r.t. $A_0 \rightarrow B_0$ and $A_1 \rightarrow B_1$ according to the principle proposed in the introduction.

For the sake of simplicity, the case of one input and one output rules is first detailed. Moreover, in all this section, only the increasing case is considered, i.e., the case where A_1 is "greater" than A_0 , and B_1 is "greater" than B_0 (see for instance Figure 1). All the results obtained in this case can be easily carried over to the decreasing case.

A. One input, one output rules

In this first case, the premises of the two rules in $\mathcal{K}$, namely A_0 and A_1 , as well as their conclusions B_0 and B_1 , are compact subsets of the real line $\mathbb{R}$.

Assuming a linear interpolation model between the two rules in $\mathcal{K}$ leads to the family of rules $A_\lambda \rightarrow B_\lambda$, for all $\lambda \in [0, 1]$, where $A_\lambda = \lambda \cdot A_1 + (1 - \lambda) \cdot A_0$ and $B_\lambda = \lambda \cdot B_1 + (1 - \lambda) \cdot B_0$. An example of such an A_λ and a B_λ , with $\lambda = 0.5$, is shown on Figure 1.

Fig. 1. Linear interpolation between two non fuzzy rules

Thus, given that $A_\lambda \rightarrow B_\lambda$ are the rules obtained from $\mathcal{K}$ by linear interpolation, the idea is to compute a union or an intersection of A_λ s that corresponds to (or best fits) A^* . Then, the corresponding union or intersection of B_λ s gives the output interval B^* . This idea comes from the fact that the two rules $(A \cap A') \rightarrow (B \cap B')$ and $(A \cup A') \rightarrow (B \cup B')$ are logical

consequences of a set of two rules $\{A \rightarrow B, A' \rightarrow B'\}$, and the generalization of this property to a set of more than 2 rules.

Then, given the set of two rules $\mathcal{K}$, and an observation A^* in between the premises of the rules, three different situations can occur:

1. $A^* = A_\lambda$ for some $\lambda \in [0, 1]$. Then, the rule $A_\lambda \rightarrow B_\lambda$ obtained by linear interpolation directly gives the conclusion $B^* = B_\lambda$.
2. $A^* \supset A_\lambda$ for some $\lambda \in [0, 1]$. Then, with $S_{\max} = \{\lambda, A^* \supset A_\lambda\} \subseteq [0, 1]$, the set A^* is obtained by:

$$\bigcup_{\lambda \in S_{\max}} A_\lambda = A^*, \quad \text{and then} \quad B^* = \bigcup_{\lambda \in S_{\max}} B_\lambda. \quad (1)$$

3. $A^* \subset A_\lambda$ for some $\lambda \in [0, 1]$. Then, with $S_{\max} = \{\lambda, A^* \subset A_\lambda\} \subseteq [0, 1]$, the set A^* is obtained by:

$$\bigcap_{\lambda \in S_{\max}} A_\lambda = A^*, \quad \text{and then} \quad B^* = \bigcap_{\lambda \in S_{\max}} B_\lambda. \quad (2)$$

Clearly, only the last two cases need to be detailed.

Proposition 1: These three cases are mutually exclusive.

Proof: As previously said, each interval A is denoted by its lower and upper bounds $[\underline{a}, \bar{a}]$. The values $\underline{\lambda}$ and $\bar{\lambda}$ given by:

- $\underline{a}^* = \underline{\lambda} \cdot \underline{a}_1 + (1 - \underline{\lambda}) \cdot \underline{a}_0$, and then: $\underline{\lambda} = (\underline{a}^* - \underline{a}_0) / (\underline{a}_1 - \underline{a}_0)$,
- $\bar{a}^* = \bar{\lambda} \cdot \bar{a}_1 + (1 - \bar{\lambda}) \cdot \bar{a}_0$, and then: $\bar{\lambda} = (\bar{a}^* - \bar{a}_0) / (\bar{a}_1 - \bar{a}_0)$,

are such that the left side of $A_{\underline{\lambda}}$ matches the left side of A^* , and the right side of $A_{\bar{\lambda}}$ matches the right side of A^* . Then, from the values $\underline{\lambda}$ and $\bar{\lambda}$, three cases can occur (see Figure 2):

- $\underline{\lambda} = \bar{\lambda}$ corresponds to the first case, as $A_{\underline{\lambda}} = A_{\bar{\lambda}} = A^*$,
 - $\underline{\lambda} < \bar{\lambda}$ means that $A_{\underline{\lambda}} \subset A^*$ and $A_{\bar{\lambda}} \subset A^*$ (case number 2),
 - $\underline{\lambda} > \bar{\lambda}$ means that $A_{\underline{\lambda}} \supset A^*$ and $A_{\bar{\lambda}} \supset A^*$ (case number 3),
- and these three cases are obviously exclusive. ■

Fig. 2. The 3 cases depending on $\underline{\lambda}$ and $\bar{\lambda}$

Now, since the method can be viewed as the replacement of the initial set of two rules $\mathcal{K} = \{A_0 \rightarrow B_0, A_1 \rightarrow B_1\}$, with the (infinite) set of rules $\mathcal{L} = \{A_\lambda \rightarrow B_\lambda, \lambda \in [0, 1]\}$ (obtained from the two rules in $\mathcal{K}$), the following proposition gives a fundamental result.

Proposition 2: The set of rules $\mathcal{L} = \{A_\lambda \rightarrow B_\lambda, \lambda \in [0, 1]\}$, obtained by linear interpolation from $\mathcal{K} = \{A_0 \rightarrow B_0, A_1 \rightarrow B_1\}$, may be potentially incoherent, in the sense of coherence given in [14]. It means that two rules in $\mathcal{L}$ can be simultaneously triggered (i.e., $A_\lambda \cap A_{\lambda'} \neq \emptyset$), while they give incompatible conclusions (i.e., $B_\lambda \cap B_{\lambda'} = \emptyset$).

Proof: See an example of coherence and incoherence cases on Figure 3 and a characterization of the incoherence cases given by Proposition 8. ■

It is clear from the definition that the incoherence can be especially detected in the third case, when $\bigcap_{\lambda \in S} A_\lambda = A^* (\neq \emptyset)$ and $\bigcap_{\lambda \in S} B_\lambda = \emptyset$ in (2).

As a consequence, when $\mathcal{L}$ is a potentially incoherent set of rules, intersections of A_{λ} s (i.e., case 3) shouldn't be used. A more cautious approach could even consider that the interpolative inference method proposed here may not be used when $\mathcal{L}$ is not coherent. Indeed, it will be shown in the following that this method is equivalent to the CRI applied to $\mathcal{L}$ only when $\mathcal{L}$ is coherent. Thus, for incoherent cases, a slightly different method will be proposed.

B. Case of A^* s bigger than A_{λ} s

First consider the case of a large A^* , i.e., an interval such that $A^* \supset A_{\lambda}$ for some $\lambda \in [0, 1]$.

Proposition 3: Given A^* such that $A^* \supset A_{\lambda}$, for one $\lambda \in [0, 1]$, the set $\mathcal{S}_{\max} = \{\lambda, A_{\lambda} \supset A^*\}$ is the maximal (for inclusion) subset of $[0, 1]$ which verifies (1). This set $\mathcal{S}_{\max}$ is the interval $[\underline{\lambda}, \bar{\lambda}]$, where $\underline{\lambda} = (\underline{a}^* - \underline{a}_0)/(\underline{a}_1 - \underline{a}_0)$ and $\bar{\lambda} = (\bar{a}^* - \bar{a}_0)/(\bar{a}_1 - \bar{a}_0)$.

Proof: Clearly, $\underline{\lambda}$ (resp. $\bar{\lambda}$) is the smallest (resp. the greatest) λ such that $A_{\lambda} \supset A^*$. Then, it can be easily shown that $\forall \lambda \in [\underline{\lambda}, \bar{\lambda}]$, $A_{\lambda} \supset A^*$. Thus, $\mathcal{S}_{\max} = [\underline{\lambda}, \bar{\lambda}]$, and this set is maximal for inclusion. ■

This proposition gives one possible set $\mathcal{S}$ which verifies (1), and thus such that $B^* = \bigcup_{\lambda \in \mathcal{S}} B_{\lambda}$. However, this set does not really enable the computation of B^* , since it is an infinite subset of $[0, 1]$.

A first way to solve this problem could consist in computing a *finite* subset $\mathcal{S}$ of $\mathcal{S}_{\max}$ such that $A^* = \bigcup_{\lambda \in \mathcal{S}} A_{\lambda}$. However, as one can get a set $\mathcal{S}$ such that $\bigcup_{\lambda \in \mathcal{S}} A_{\lambda} = A^*$ and $\bigcup_{\lambda \in \mathcal{S}} B_{\lambda} \neq B^*$ (with A^* and B^* defined by (1)) because of lack of convexity of $\bigcup_{\lambda \in \mathcal{S}} B_{\lambda}$, this approach is not developed here.

Another approach is based on the fact that B^* is an interval on the continuous universe V , namely $[\underline{b}^*, \bar{b}^*]$. Thus, in order to define B^* , it is sufficient to compute its two boundaries $\underline{b}^*$ and $\bar{b}^*$.

Proposition 4: Given the two sparse rules in $\mathcal{K}$, and an observation A^* in between A_0 and A_1 , such that $A^* \supset A_{\lambda}$, for one $\lambda \in [0, 1]$, the corresponding output B^* is given by $[\underline{b}^*, \bar{b}^*]$, with $\underline{b}^* = \underline{b}_{\underline{\lambda}}$ and $\bar{b}^* = \bar{b}_{\bar{\lambda}}$, or equivalently:

$$B^* = \left[\frac{\underline{b}_1 - \underline{b}_0}{\underline{a}_1 - \underline{a}_0} \underline{a}^* + \frac{\underline{a}_1 \underline{b}_0 - \underline{a}_0 \underline{b}_1}{\underline{a}_1 - \underline{a}_0}, \frac{\bar{b}_1 - \bar{b}_0}{\bar{a}_1 - \bar{a}_0} \bar{a}^* + \frac{\bar{a}_1 \bar{b}_0 - \bar{a}_0 \bar{b}_1}{\bar{a}_1 - \bar{a}_0} \right].$$

Proof: The set B^* is given by the interval $\bigcup_{\lambda \in \mathcal{S}_{\max}} B_{\lambda}$, and $\mathcal{S}_{\max}$ is the interval $[\underline{\lambda}, \bar{\lambda}]$. The lower bound of B^* is then the smallest value of the lower bounds of the B_{λ} s, for $\lambda \in \mathcal{S}_{\max}$. And it is clear that this value is the lower bound of $B_{\underline{\lambda}}$. Thus $\underline{b}^* = \underline{b}_{\underline{\lambda}}$. The proof is the same for the upper bound. ■

C. Case of A^* s smaller than A_{λ} s

Now consider the third case, where A^* is included in one A_{λ} . The next proposition is the counterpart of Proposition 3 for small A^* s (the same notations are used).

Proposition 5: Given A^* such that $A^* \subset A_{\lambda_0}$, for one $\lambda_0 \in [0, 1]$, the set $\mathcal{S}_{\max} = \{\lambda, A_{\lambda} \subset A^*\}$ is the maximal (for inclusion) subset of $[0, 1]$ which verifies (2). This set $\mathcal{S}_{\max}$ is

the interval $[\bar{\lambda}, \underline{\lambda}]$, where $\underline{\lambda} = (\underline{a}_1 - \underline{a}^*)/(\underline{a}_1 - \underline{a}_0)$ and $\bar{\lambda} = (\bar{a}_1 - \bar{a}^*)/(\bar{a}_1 - \bar{a}_0)$.

Proof: It is similar to the proof of Proposition 3. ■

As in section II-B, the set $\mathcal{S}_{\min}$ is infinite, and is not very useful for the computation of B^* . But in case of A^* s smaller than A_{λ} s, only two values in $\mathcal{S}$ are necessary to obtain A^* . Indeed, $A^* = A_{\underline{\lambda}} \cap A_{\bar{\lambda}}$, as shown on Figure 2.

Proposition 6: The set $\mathcal{S}_{\min} = \{\bar{\lambda}, \underline{\lambda}\}$ is the minimal (for inclusion) set $\mathcal{S}$ which verifies (2).

Proof: It is clear that $\mathcal{S}_{\min}$ verifies (2). The set $\mathcal{S}_{\min}$ is minimal since the suppression of one of its elements leads to a set which clearly do not verify (2). And it is the minimal set since the replacement of one element in $\mathcal{S}_{\min}$ also leads to a set which do not verify (2), as clear from Figure 2. ■

Here, the set B^* can be computed as

$$B^* = \bigcap_{\lambda \in \mathcal{S}_{\min}} B_{\lambda} = B_{\bar{\lambda}} \cap B_{\underline{\lambda}}. \quad (3)$$

An equivalent way to compute B^* is given by the counterpart of Proposition 4 for A^* s smaller than A_{λ} s.

Proposition 7: Given the two sparse rules in $\mathcal{K}$, and an observation A^* in between A_0 and A_1 , such that $A^* \subset A_{\lambda}$, for one $\lambda \in [0, 1]$, the corresponding output B^* is given, when it exists, by $[\underline{b}^*, \bar{b}^*]$, with $\underline{b}^* = \underline{b}_{\underline{\lambda}}$ and $\bar{b}^* = \bar{b}_{\bar{\lambda}}$, or equivalently:

$$B^* = \left[\frac{\underline{b}_1 - \underline{b}_0}{\underline{a}_1 - \underline{a}_0} \underline{a}^* + \frac{\underline{a}_1 \underline{b}_0 - \underline{a}_0 \underline{b}_1}{\underline{a}_1 - \underline{a}_0}, \frac{\bar{b}_1 - \bar{b}_0}{\bar{a}_1 - \bar{a}_0} \bar{a}^* + \frac{\bar{a}_1 \bar{b}_0 - \bar{a}_0 \bar{b}_1}{\bar{a}_1 - \bar{a}_0} \right].$$

Proof: It comes straightforwardly from the computation of the intersection $B_{\bar{\lambda}} \cap B_{\underline{\lambda}} = [\underline{b}_{\bar{\lambda}}, \bar{b}_{\underline{\lambda}}]$, since $\underline{\lambda} > \bar{\lambda}$. ■

Clearly, the expression of B^* is the same in Propositions 4 and 7. However, in this case, the interval B^* does not always exist.

D. Detection of an incoherent situation

The two rules $A_{\underline{\lambda}} \rightarrow B_{\underline{\lambda}}$ and $A_{\bar{\lambda}} \rightarrow B_{\bar{\lambda}}$ are said to be incoherent when $A^* = A_{\underline{\lambda}} \cap A_{\bar{\lambda}} \neq \emptyset$ and $B^* = B_{\underline{\lambda}} \cap B_{\bar{\lambda}} = \emptyset$.

These cases of incoherence depend on properties on the antecedent (A_0 and A_1) and conclusion parts (B_0 and B_1) of the two rules in $\mathcal{K}$. Thus, it is possible to determine a necessary and sufficient coherence condition for the set $\mathcal{L}$.

Proposition 8: Let $\underline{\Gamma}$ be the line defined by the two points $(\underline{a}_0, \underline{b}_0)$ and $(\underline{a}_1, \underline{b}_1)$, and $\bar{\Gamma}$ the line defined by the two points $(\bar{a}_0, \bar{b}_0)$ and $(\bar{a}_1, \bar{b}_1)$, as shown on Figure 3. The set $\mathcal{L}$ is *potentially incoherent* if the line $\underline{\Gamma}$ is *above* the line $\bar{\Gamma}$ for at least one point in $[\bar{a}_0, \underline{a}_1]$, i.e., if $\exists u \in [\bar{a}_0, \underline{a}_1], \underline{\Gamma}(u) > \bar{\Gamma}(u)$.

Proof: Consider a value $u \in U$ such that $\underline{\Gamma}(u) > \bar{\Gamma}(u)$, and $A^* = \{u\}$. The computation of $B_{\underline{\lambda}}$ and $B_{\bar{\lambda}}$ (as defined in Section II-C) shows that $B_{\underline{\lambda}} = [\underline{\Gamma}(u), \underline{b}_{\underline{\lambda}}]$ and that $B_{\bar{\lambda}} = [\bar{b}_{\bar{\lambda}}, \bar{\Gamma}(u)]$. And thus $B^* = B_{\underline{\lambda}} \cap B_{\bar{\lambda}} = \emptyset$ (since $\underline{\Gamma}(u) > \bar{\Gamma}(u)$), which means incoherence. ■

Note that when the two lines $\underline{\Gamma}$ and $\bar{\Gamma}$ intersect, $\mathcal{L}$ is potentially incoherent on a subset of $[\bar{a}_0, \underline{a}_1]$ only. As a corollary of Proposition 8, one can compute the interval of potential incoherence in this case, which corresponds to the subinterval of $[\bar{a}_0, \underline{a}_1]$ for which $\underline{\Gamma}$ is above $\bar{\Gamma}$.

Beside the calculations, it is important to understand the meaning of *potential incoherence* for sparse rules. Indeed,

Fig. 3. Coherence and potential incoherence cases

considering the CRI (as a generalization of the Modus Ponens) for the reasoning mechanism, it has been shown that sparse rules cannot lead to incoherence (see [14]). Strictly speaking, $\mathcal{K}$ is never incoherent. Only $\mathcal{L}$ may be incoherent. However, when $\mathcal{L}$ is incoherent, $\mathcal{K}$ can be said incoherent *w.r.t. the interpolative inference mechanism* proposed here, according to a generalization of the notion of coherence for rule-based systems: “for any allowed input, the system must give a nonempty output”.

From the equations, it can be seen that potential incoherence may occur either when the B s are sparser than the A s, or when the B s are more precise than the A s (or a combination of these two effects since they can compensate). The B s being sparser than the A s means that they are divergent, as the conclusion parts are less similar than the condition parts of the rules. It seems a natural extension of the notion of incoherence from a similarity point of view (see Figure 4). Thus, $\mathcal{K}$ can be said potentially incoherent if two rules in $\mathcal{L}$ are potentially incoherent in the sense of [14], as shown on Figure 4.b for $A_{0.4} \rightarrow B_{0.4}$ and $A_{0.6} \rightarrow B_{0.6}$.

Fig. 4. Generalized notion of incoherence

Moreover, in this special case of incoherence, applying (2) with a well-chosen interval A^* could lead to a singleton for B^* . In this particular case, obtaining a singleton, i.e., a precise value for B^* , from an imprecise observation A^* and two (even more) imprecise rules seems hardly acceptable. In such a case, if A^* would be made still more precise, B^* becomes empty. By contrast, it can be shown that B^* s obtained in the coherent case are always more imprecise than the corresponding A^* . This is why this method, and especially the intersection of rules shouldn't be used in the incoherent case.

E. Comparisons and an inference algorithm based on the CRI

It is interesting to compare this interpolative reasoning method with previously proposed ones, and especially the two

ones in [15], shown on Figure 5.a–b. These two methods are similar to the one proposed in this paper (for coherence cases) since they are all equivalent to the definition of a relation on the Cartesian product of the input output universes $U \times V$ in between the rules, and the application of the CRI as the reasoning mechanism.

Fig. 5. Three different interpolative mechanisms

From a semantic point of view, the first method considers the pair (A, B) (from the rule $A \rightarrow B$), as a fuzzy point and applies the extension principle (see [15], [8]). It means that the relation is the set of all segments of the form $[m_0, m_1]$, where m_0 is a point in $A_0 \times B_0$, and m_1 a point in $A_1 \times B_1$. The obtained relation is the convex hull of $(A_0 \times B_0) \cup (A_1 \times B_1)$.

The second method considers the pair (A, B) as a part of a fuzzy function from U to V . Thus, for a point $(u, v) \in U \times V$, if $u \in A$, then $v \in B$. The interpolation between these two rules only considers the right side of A_0 and the left side of A_1 , and takes the corresponding convex hull. It leads to the relation depicted on Figure 5.b. Although this relation is more restrictive than the previous one, it is more in accordance with the notion of rule. Indeed, if $u \in A$, then v cannot be outside B , while it is the case in Figure 5.a (see the grey region above the rectangle representing $A_0 \times B_0$ on $U \times V$).

The relations defined by these interpolation processes can be represented by two bounding lines. It is also the case with the third figure (5.c), which corresponds to the method proposed here in a *coherent case*. It can be represented by the two lines $\underline{L}$ and $\overline{F}$ defined above, and corresponds to the relation $\bigcap_{\lambda \in [0,1]} A_\lambda \rightarrow B_\lambda$, defined by the set of rules $\mathcal{L}$. This relation is more restrictive than the two others.

In these three cases, the reasoning mechanism can be the CRI applied to the relation defined on $U \times V$ by two bounding lines, as shown on Figure 5. Thus, it comes down to the evaluation of a function in two points. For the proposed method, it is $B^* = [\underline{L}(a^*), \overline{F}(a^*)]$ for the case of an increasing interpolative function (as on Figure 5.c). This result is in accordance with $B^* = A^* \circ \bigcap_{\lambda \in [0,1]} A_\lambda \rightarrow B_\lambda$, i.e., with the CRI.

In the incoherent case, the CRI cannot be applied, since the relation represented by the set of rules $\mathcal{L}$ on $U \times V$ is empty. However, if the entire set $\mathcal{L}$ is incoherent in this case, considering only one rule from $\mathcal{L}$ still makes sense, as well as considering one rule, obtained by union of rules in $\mathcal{L}$. Indeed, a rule $A_\lambda \rightarrow B_\lambda$, for one $\lambda \in [0, 1]$ such that A_λ overlap neither A_0 , nor A_1 is always coherent w.r.t. $\mathcal{K}$. And it is clear that the case 1 of Section II-A (for which $\exists \lambda \in [0, 1]$ such that $A^* = A_\lambda$) corresponds to the CRI with A^* and such a rule. Similarly, a rule $\bigcup_{\lambda \in S} A_\lambda \rightarrow \bigcup_{\lambda \in S} B_\lambda$ such that $\bigcup_{\lambda \in S} A_\lambda$ overlap neither A_0 , nor A_1 is always coherent w.r.t. $\mathcal{K}$. And it can be easily shown that the case 2 of Section II-A (for which $\exists \lambda \in [0, 1]$ such that $A^* \supseteq A_\lambda$) corresponds to the CRI with A^* and such a rule. Thus it is sound to use the proposed method for these two cases, while it is not for the third one.

To summarize, the proposed method is equivalent to the CRI applied to $\mathcal{L}$ in the coherent case (as previously said). In the incoherent case, the method is sound (and still corresponds to the CRI, but only with one well-chosen rule instead of the set $\mathcal{L}$) only with input sets A^* greater or equal to one A_λ . Smaller A^* s should then be approximated by a well-chosen A_λ . A good choice can be the A_λ centered on A^* .

F. Rules with several inputs

This section briefly deals with the case of rules with several inputs, i.e., the case of a multi-dimensional input universe $U = U_1 \times \dots \times U_p$. Two main approaches can be considered, depending on the number of rules available for the interpolation process.

First consider that the set of rules $\mathcal{K}$ still contains two rules. The generalization of the method proposed in the previous sections consists in representing A^* by means of A_λ s, and then computing B^* using the λ s. It comes down to the computation, for each dimension i , of the conclusion B_i^* obtained from the two rules with a mono-dimensional input $A_{0,i} \rightarrow B_0$ and $A_{1,i} \rightarrow B_1$. And then to the computation of the convex union of the B_i^* s. In fact, it is acceptable when the (imprecise) point A^* is on the (imprecise) line defined by the two (imprecise) points A_0 and A_1 in the universe U , but in the general case it gives very imprecise results.

This is why a better method consists in considering more rules (if available) to define an imprecise hyper-plane of $U \times V$ (as well as the imprecise line defined by $[\underline{\Gamma}(u), \bar{\Gamma}(u)]$ depicted on Figure 5 for one input). If U has p dimensions (and V one) it requires $p + 1$ rules whose condition parts A_i are “around” A^* . This method resembles the one described at the end of [7] and in [8]. These two methods are not developed here for the sake of brevity. Note however that they involve some non-trivial problems of classical interpolation in multidimensional spaces.

III. FUZZY CASE

In this section, the sets involved in the considered rules may be fuzzy instead of just intervals. The considered fuzzy sets are trapezoidal-shaped, not only because they are simple to

deal with, but rather because they are the most commonly used in practical applications. The extension to more general fuzzy sets is left for further research.

In order to generalize the proposed interpolation method to fuzzy sets, let us first study some properties of the union and intersection of the corresponding A_λ s and the B_λ s. In this case, A_λ is obtained from A_0 and A_1 , applying the linear interpolation mechanism of the previous section on each of its α -cuts (which are supposed to be intervals): $(A_\lambda)_\alpha = \lambda \cdot (A_1)_\alpha + (1 - \lambda) \cdot (A_0)_\alpha$, where $(A)_\alpha$ is the α -cut of A . The following properties are obvious.

- If the fuzzy sets A_0 and A_1 are trapezoidal-shaped, then the fuzzy sets A_λ , for $\lambda \in [0, 1]$ are also trapezoidal-shaped.

Then, it can be shown that this property is also true for the union or intersections of A_λ s considered by the interpolation mechanism.

- If the fuzzy sets A_0 and A_1 are trapezoidal-shaped, then the union $\bigcup_{\lambda \in I} A_\lambda$, where $I \subseteq [0, 1]$ is an interval, is also trapezoidal-shaped.

The requirement for I to be an interval is not always necessary. The union of the cores of the A_λ , for $\lambda \in I$ being an interval is sufficient for the union to be trapezoidal-shaped. However, as in II-B, the set $I = S_{\max}$ which is used in the following is an interval.

- If the fuzzy sets A_0 and A_1 are trapezoidal-shaped, then the intersection $\bigcap_{\lambda \in I} A_\lambda$, for any $I \subseteq [0, 1]$, is also trapezoidal-shaped, which may be degenerated into a triangle, a sub-normalized triangle, or the empty set.

Although these results give interesting properties on the shapes of unions or intersections of A_λ s, it is rather clear that, in the general case, a trapezoidal-shaped input A^* cannot be recovered exactly by such union or intersections. It would require very particular properties on the slopes of A^* .

To generalize the interpolation method to the fuzzy case, the next section proposes an approximation of A^* by a union or an intersection of A_λ s. Then, Section III-B shows that applying the method proposed for intervals to an α -cuts decomposition of the fuzzy sets in order to obtain α -cuts of B^* correspond to the method proposed by Kóczy and Hirota [2] and thus raises the same problems.

A. Approximation of A^*

Since A^* cannot be exactly retrieved in general, a first solution consists in computing an approximation of A^* . However, this approximation should be meaningful. Considering the interpretation of fuzzy rules based on possibility theory (see for instance [16]), the conclusion of a rule modeled with a genuine implication is a restriction of the set of possible values for the output variable. Thus, for this restriction imposed by B^* not being arbitrary, its approximation has to be a superset of B^* , which means that the approximation of A^* has also to be a superset of A^* . Moreover in order not to lose information, this superset has to be the smallest possible one. For a given A^* in between A_0 and A_1 , such an approximation obtained either by union, or by intersection of A_λ s is unique.

For trapezoidal-shaped fuzzy sets, the inclusion $A^* \subseteq A_\lambda$, defined by $A^* \subseteq A_\lambda$ if $\forall u \in U, \mu_{A^*}(u) \leq \mu_{A_\lambda}(u)$, is equivalent to the classical inclusion of the core and of the support ($C(A^*) \subseteq C(A_\lambda)$ and $S(A^*) \subseteq S(A_\lambda)$).

As well as in Section II, there are 3 different cases:

1. $A^* = A_\lambda$, for one $\lambda \in [0, 1]$,
2. $A^* \subset A_\lambda$, for one $\lambda \in [0, 1]$. In this case A^* is approximated by $\bigcap_{\lambda \in S_{\max}} A_\lambda$ with $S_{\max} = \{\lambda \in [0, 1], A^* \subset A_\lambda\}$.
3. $A^* \not\subseteq A_\lambda, \forall \lambda \in [0, 1]$ (i.e., all the other cases). Then, A^* is approximated by $\bigcup_{\lambda \in S_{\max}} A_\lambda$, where $S_{\max} = [\underline{\lambda}, \bar{\lambda}]$ is given by $\underline{\lambda} = \sup_{A^* \subseteq [A_\lambda, A_1]} \lambda$, (denoting $[A_\lambda, A_1]$ the convex hull of A_λ and A_1) and $\bar{\lambda} = \inf_{A^* \subseteq [A_0, A_\lambda]} \lambda$.

Then, considering the interval $S_{\max} = [\underline{\lambda}, \bar{\lambda}]$, the computation of the trapezoidal-shaped fuzzy set B^* can be done as in Section II, computing the left and right sides of the trapezoid. There are also incoherent cases, in particular (but not only) when the obtained B^* is sub-normalized.

B. Computation on the α -cut decomposition of A^*

This approach consists in i) decomposing the fuzzy sets into their α -cuts, ii) using the method proposed in Section II on the α -cuts, iii) and then computing B^* , also from its α -cuts. This method seems natural since it provides an exact representation of A^* . Moreover, computation with level cuts are often used in fuzzy logic, and especially for fuzzy interpolation.

However, it can be shown that it presents the same drawbacks as the method proposed by Kóczy and Hirota [2]. In particular, the computed α -cuts of B^* may not fulfill the nestedness requirement. Figure 6 shows an example. Although it corresponds to a coherent case, it shows the relations induced by this method for the support (in light grey) and core (in a darker grey). For the proposed A^* , it is immediate that the core and support of the B^* computed this way are disjoint !

Fig. 6. Interpolation by α -cut decomposition

Now, considering that all the fuzzy rules in $\mathcal{K}$ or $\mathcal{L}$ are gradual rules, modeled with Rescher-Gaines implication, the family of rules $(A_\lambda)_\alpha \rightarrow (B_\lambda)_\alpha, \alpha \in [0, 1]$, obtained from one rule by α -cutting A_λ and B_λ , gives an infinity of classical rules whose input and output parts are intervals. Computation with this (double) infinity of rules (or only some of them) and the CRI could lead to an acceptable result, while also processing by α -cut decomposition. However, the probability for inconsistency between rules is increased. This is a topic for further research.

IV. CONCLUDING REMARKS

This paper has outlined a method to interpolate between sparse fuzzy data, which has been also compared to approaches based on the Compositional Rule of Inference. Although this method corresponds to the CRI applied to one rule, defined by union or intersection of linearly interpolated rules, it is different from the CRI applied to the whole set of interpolated rules, which may be incoherent. This leads to a notion of incoherence for sparse rules which ought to be detailed in further developments. Besides, when three rules are considered, $A_i \rightarrow B_i, i = 1, \dots, 3$, such that A_1 is in between A_0 and A_2 , taking $A^* = A_1$ is a possible way to check whether or not this set of 3 rules is compatible with the linear interpolation principle in the sense that $B^* \supseteq B_1$. If $B^* \subseteq B_1$, removing the rule $A_1 \rightarrow B_1$ could be thought of, since the interpolation provides a more precise result. Otherwise, and in particular if $B^* \cap B_1 = \emptyset$, the interpolation should only be done between A_0 and A_1 , or A_1 and A_2 , and not between A_0 and A_2 .

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