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Abstract. In this paper, we begin by reviewing the calculus induced by the framework of [10]. In there, we extended Polylogarithm functions over a subalgebra of noncommutative rational power series, recognizable by finite state (multiplicity) automata over the alphabet $X = \{x_0, x_1\}$. The stability of this calculus under shuffle products relies on the nuclearity of the target space [31]. We also concentrated on algebraic and analytic aspects of this extension allowing to index polylogarithms, at non positive multi-indices, by rational series and also allowing to regularize divergent polyzetas, at non positive multi-indices [10]. As a continuation of works in [10] and in order to understand the bridge between the extension of this "polylogarithmic calculus" and the world of harmonic sums, we propose a local theory, adapted to a full calculus on indices of Harmonic Sums based on the Taylor expansions, around zero, of polylogarithms with index x_1 on the rightmost end. This theory is not only compatible with Stuffle products but also with the Analytic Model. In this respect, it provides a stable and fully algorithmic model for Harmonic calculus. Examples by computer are also provided[6].

Keywords: theory of domains, harmonic sums, polylogarithms

1 Introduction

Riemann's zeta function is defined by the series

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s} \tag{1}$$

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where s is a complex number. It is absolutely for $\Re(s) > 1$ (for any $s \in \mathbb{C}$, $\Re(s)$ stands for the real part of s).

It can be extended to a meromorphic function on the complex plane $\mathbb{C}$ with a single pole at $s = 1$ [30][7].). In fact, the story began with Euler's works to find the solution of Basel's problem. In these works, Euler proved that [14]

$$\zeta(2) = \sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{2}$$

Moreover, for any $s_1, s_2 \in \mathbb{C}$ such that $\Re(s_1) > 1$ and $\Re(s_2) > 1$, Euler gave an important identity as follows[8]:

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_1 + s_2) + \zeta(s_2, s_1). \tag{3}$$

where, for any $s_1, s_2 \in \mathbb{C}$ such that $\Re(s_1) > 1$ and $\Re(s_2) > 1$,

$$\zeta(s_1, s_2) := \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}}. \tag{4}$$

The numbers $\zeta(s_1, s_2)$ were called "double zeta values" at (s_1, s_2). More generally, for any $r \in \mathbb{N}_+$ and $s_1, \ldots, s_r \in \mathbb{C}$, we denote

$$\zeta(s_1, \ldots, s_r) := \sum_{n_1 > \ldots > n_r \geq 1} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}. \tag{5}$$

Then the results of K. Matsumoto [16] showed that the series $\zeta(s_1, \ldots, s_r)$ converges absolutely for $s \in \mathscr{H}_r$ where

$$\mathscr{H}_r := \{\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r | \forall m = 1, \ldots, r;\ \Re(s_1) + \ldots + \Re(s_m) > m\}. \tag{6}$$

In the convergent cases, $\zeta(s_1, \ldots, s_r)$ were called "polyzeta values" at multi-index $\mathbf{s} = (s_1, \ldots, s_r)$. Indeed $\mathbf{s} \mapsto \zeta(\mathbf{s})$ is holomorphic on $\mathscr{H}_r$ and has been extended to $\mathbb{C}^r$ as a meromorphic function (see [17, 34]).

In fact, for any r-uplet $(s_1, \ldots, s_r) \in \mathbb{N}_+^r$, $r \in \mathbb{N}_+$, the polyzeta $\zeta(s_1, \ldots, s_r)$ is also the limit at $z = 1$ of the *polylogarithmic function*, defined by:

$$\mathrm{Li}_{s_1, \ldots, s_r}(z) := \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}} \tag{7}$$

for any $z \in \mathbb{C}$ such that $|z| < 1$. It is easily seen that, for any $s_i \in \mathbb{N}_+,\ r > 1$,

$$z\frac{d}{dz}\mathrm{Li}_{s_1, \ldots, s_r}(z) = \mathrm{Li}_{s_1 - 1, \ldots, s_r}(z) \text{ if } s_1 > 1 \tag{8}$$

$$(1 - z)\frac{d}{dz}\mathrm{Li}_{1, s_2, \ldots, s_r}(z) = \mathrm{Li}_{s_2, \ldots, s_r}(z) \text{ if } r > 1 \tag{9}$$

[7] Whence the famous sum $\zeta(-1) = 1 + 2 + 3 + \cdots = -\frac{1}{12}$ by which, among other "results", S. Ramanujan was noticed by G. H. H. Hardy (see [1]).

[8] In fact, in Euler's formula, $s_1, s_2 \in \mathbb{N}_+$. This identity appeared under the name "Prima Methodus ..." (see [15] pp 141-144).

and this formulas will be ended at the "seed" $\mathrm{Li}_1(z)=\log\left(\frac{1}{1-z}\right)$.

Moreover, if X^* is the free monoid of rank two (generators, or alphabet, $X=\{x_0,x_1\}$ and neutral 1_{X^*}) then the polylogarithms indexed by a list

$$(s_1,\ldots,s_r)\in\mathbb{N}_+^r \text{ can be reindexed by the word } x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1\in X^*x_1 \tag{10}$$

In order to reverse the recursion introduced in Eqns. 8, we introduce two differential forms

$$\omega_0(z)=z^{-1}dz \text{ and } \omega_1(z)=(1-z)^{-1}dz, \tag{11}$$

on Ω[9]. We then get an integral representation[10] of the functions (7) as follows[11] [20]

$$\mathrm{Li}_w(z)=\begin{cases}1_{\mathscr{H}(\Omega)} & \text{if } w=1_{X^*}\\ \displaystyle\int_0^z\omega_1(s)\,\mathrm{Li}_u(s) & \text{if } w=x_1u\\ \displaystyle\int_1^z\omega_0(s)\,\mathrm{Li}_u(s) & \text{if } w=x_0u \text{ and } |u|_{x_1}=0, i.e. w\in x_0^*\\ \displaystyle\int_0^z\omega_0(s)\,\mathrm{Li}_u(s) & \text{if } w=x_0u \text{ and } |u|_{x_1}>0, i.e. w\notin x_0^*,\end{cases} \tag{12}$$

the upper bound z belongs to Ω (we recall that $\Omega=\mathbb{C}\setminus(]-\infty,0]\cup[1,+\infty[)$ is simply connected domain so that the intergrals, which can be proved to be convergent in all cases, depend only on their bounds). The the neutral element of the algebra of analytic functions $\mathscr{H}(\Omega)$, a constant function will be here denoted $1_{\mathscr{H}(\Omega)}$.
This provides not only the analytic continuation of (7) to Ω but also extends the indexation to the whole alphabet X, allowing to study the complete generating series

$$\mathrm{L}(z)=\sum_{w\in X^*}\mathrm{Li}_w(z)w \tag{13}$$

and show that it is the solution of the following first order noncommutative differential equation

$$\begin{cases}\mathbf{d}(S)=(\omega_0(z)x_0+\omega_1(z)x_1)S, & (NCDE)\\ \displaystyle\lim_{z\in\Omega,z\to 0}S(z)e^{-x_0\log(z)}=1_{\mathscr{H}(\Omega)\langle\langle X\rangle\rangle}, & \text{asymptotic initinial condition,}\end{cases} \tag{14}$$

where, for any $S\in\mathscr{H}(\Omega)\langle\langle X\rangle\rangle$.
Through term by term derivation, one gets [13]

$$\mathbf{d}(S)=\sum_{w\in X^*}\frac{d}{dz}(\langle S\mid w\rangle)w. \tag{15}$$

[9] Ω is the simply connected domain $\mathbb{C}\setminus(]-\infty,0]\cup[1,+\infty[)$.

[10] In here, we code the moves $z\frac{d}{dz}$ (resp. $(1-z)\frac{d}{dz}$) - or more precisely sections $\int_0^z\frac{f(s)}{s}ds$ (resp. $\int_0^z\frac{f(s)}{1-s}ds$) - with x_0 (resp. x_1).

[11] Given a word $w\in X^*$, we note $|w|_{x_1}$ the number of occurrences of x_1 within w.

This differential system allows to show that L is a $\sqcup\!\sqcup$-character[12] [24], *i.e.*

$$\forall u,v\in X^*,\quad \langle \mathrm{L}\mid u\sqcup\!\sqcup v\rangle=\langle \mathrm{L}\mid u\rangle\langle \mathrm{L}\mid v\rangle \text{ and } \langle \mathrm{L}\mid 1_{X^*}\rangle=1_{\mathscr{H}(\Omega)}. \tag{16}$$

Note that, in what precedes, we used the pairing $\langle\bullet\mid\bullet\rangle$ between series and polynomials, classically defined by, for $T\in\mathbf{k}\langle\langle X\rangle\rangle$ and $P\in\mathbf{k}\langle X\rangle$

$$\langle T\mid P\rangle=\sum_{w\in X^*}\langle T\mid w\rangle\langle P\mid w\rangle, \tag{17}$$

where, when w is a word, $\langle S\mid w\rangle$ stands for the coefficient of w in S and $\mathbf{k}$ any commutative ring (as here $\mathscr{H}(\Omega)$). With this at hand, we extend at once the indexation of Li from X^* to $\mathbb{C}\langle X\rangle$ by

$$\mathrm{Li}_P:=\sum_{w\in X^*}\langle P\mid w\rangle\,\mathrm{Li}_w=\sum_{n\geq 0}\Big(\sum_{|w|=n}\langle P\mid w\rangle\,\mathrm{Li}_w\Big). \tag{18}$$

In [10], it has been established that the polylogarithm, well defined locally by (7), could be extended to some series (with conditions) by the last part of formula (18) where the polynomial P is replaced by some series. A complete theory of global domains was presented in [10], the present work concerns the whole project of extending $\mathrm{H}_\bullet$ [9, 19]. over stuffle subalgebras of rational power series on the alphabet Y, in particular the stars of letters and some explicit combinatorial consequences of this extension.

In fact, we focus on what happens in (well choosen) neighbourhoods of zero (see section 3), therefore, the aim of this work is manyfold.

a) Use the extension to local Taylor expansions[13] as in (7) and the coefficients of their quotients by $1-z$, namely the harmonic sums, denoted $\mathrm{H}_\bullet$ and defined, for any $w\in X^*x_1$, as follows[14] ([22] see also related literature [4, 19])

$$\frac{\mathrm{Li}_w(z)}{1-z}=\sum_{N\geq 0}\mathrm{H}_{\pi_X(w)}(N)z^N, \tag{19}$$

by a suitable theory of local domains which assures to carry over the computation of these Taylor coefficients and preserves the stuffle indentity, again true for polynomials over the alphabet $Y=\{y_n\}_{n\geq 1}$, *i.e.*[15]

$$\forall S,T\in\mathbb{C}\langle Y\rangle,\ \ \mathrm{H}_{S\,\uplus\, T}=\mathrm{H}_S\mathrm{H}_T \text{ and } \mathrm{H}_{1_{\mathbb{C}\langle Y\rangle}}=1_{\mathbb{C}^{\mathbb{N}}}, \tag{20}$$

note that $1_{\mathbb{C}\langle Y\rangle}$ is identified with 1_{Y^*} and $1_{\mathbb{C}^{\mathbb{N}}}$ is the constant (to one) function[16] $\mathbb{N}\to\mathbb{C}$. This means that $\mathrm{H}_\bullet:(\mathbb{C}\langle Y\rangle,\uplus,1_{Y^*})\longrightarrow(\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*},\times,1)$ mapping any word

[12] Here, the shuffle product is denoted by $\sqcup\!\sqcup$. It will be redefined in the section 5 .

[13] Around zero.

[14] Here, the conc-morphism $\pi_X:(\mathbb{C}\langle Y\rangle,\mathtt{conc},1_{Y^*})\to(\mathbb{C}\langle X\rangle,\mathtt{conc},1_{X^*})$ is defined by $\pi_X(y_n)=x_0^{n-1}x_1$ and π_Y is its inverse on $\mathrm{Im}(\pi_X)$. See [8, 10] for more details and a full Definition of π_Y.

[15] Here, $\uplus$ stands for the stuffle product which will be recalled as in the section 5.

[16] In fact, it could be $\mathbb{Q}$ but we will use afterwards $\mathbb{C}$-linear combinations.

$w = y_{s_1} \dots y_{s_r} \in Y^*$ to

$$\mathrm{H}_w = \mathrm{H}_{s_1,\dots,s_r} = \sum_{N \geq n_1 > \dots > n_r > 0} \frac{1}{n_1^{s_1} \dots n_r^{s_r}}, \tag{21}$$

is a $\uplus$ (unital) morphism[17].

b) Extend these correspondences (*i.e.* $\mathrm{Li}_\bullet, \mathrm{H}_\bullet$) to some series (over X and Y, respectively) in order to preserve the identity[18] [22]

$$\frac{\mathrm{Li}_{\pi_X(S)}(z)}{1-z} \odot \frac{\mathrm{Li}_{\pi_X(T)}(z)}{1-z} = \frac{\mathrm{Li}_{\pi_X(S \uplus T)}(z)}{1-z}. \tag{22}$$

true for polynomials $S, T \in \mathbb{C}\langle Y \rangle$.

To this end, we use the explicit parametrization of the `conc`-characters obtained in [8, 10] and the fact that, under stuffle products, they form a group.

2 Polylogarithms: from global to local domains

Now we are facing the following constraint:

In order that the results given by symbolic computation reflect the reality with complex numbers (and analytic functions), we have to introduce some topology [19].

Let $\mathscr{H}(\Omega) = C^\omega(\Omega; \mathbb{C})$ be the algebra (for the pointwise product) of complex -valued functions which are holomorphic on Ω. Endowed with the topology of compact convergence [20], it is a nuclear space[21].

Definition 1. *i) Let $S \in \mathbb{C}\langle\langle X \rangle\rangle$ be a series decomposed in its homogeneous (w.r.t. the length) components $S_n = \sum_{|w|=n} \langle S \mid w \rangle w$ (so that $S = \sum_{n \geq 0} S_n$) is in the domain of* Li *iff the family $(\mathrm{Li}_{S_n})_{n \geq 0}$ is summable in $\mathscr{H}(\Omega)$ in other words, due to the fact that the space is complete (see [31]), if one has*

$$(\forall W \in \mathscr{B}_{\mathscr{H}(\Omega)})(\exists N)(\forall n \geq N)(\forall k)\Big(\sum_{n \leq j \leq n+k} \mathrm{Li}_{S_j} \in W\Big) . \tag{23}$$

where $\mathscr{B}_{\mathscr{H}(\Omega)}$ is the set of neighbourhoods of 0 in $\mathscr{H}(\Omega)$.

*ii) The set of these series will be noted Dom(*Li*) and, for $S \in$ Dom(*Li*), the sum $\sum_{n \geq 0} \mathrm{Li}_{S_n}$ will be noted Li_S.*

Of course criterium 23 is only a theoretical tool to establish properties of the Domain of Li. In further calculations (i.e. in practice), we will not use it but the stability of the domain under certain operations.

[17] It can be proved that this morphism is into [22].

[18] Here $\odot$ stands for the Hadamard product [18].

[19] Readers who are not keen on topology or functional analysis may skip the details of this section and hold its conclusions.

[20] This topology is defined by the seminorms

$$p_K(f) = \sup_{s \in K} |f(s)| \ (K \subset \Omega \text{ is compact}).$$

[21] Space where commutatively convergent and absolutely convergent series are the same. This will allow the domain of the polylogarithm to be closed by shuffle products (i.e. the possiblity to compute legal polylogarithms through shuffle products).

Example 1 ([20]). For example, the classical polylogarithms: dilogarithm Li_2, trilogarithm Li_3, etc... are defined and obtained through the coding (10) by

$$\mathrm{Li}_k(z) = \sum_{n\geq 1} \frac{z^n}{n^k} = \mathrm{Li}_{x_0^{k-1}x_1}(z) = \langle \mathrm{L}(z) \mid x_0^{k-1}x_1 \rangle$$

(where $\mathrm{L}(z)$ is as in Eq. 13) but, one can check that, for $t \geq 0$ (real), the series $(tx_0)^*x_1$ belongs to $\mathrm{Dom}(\mathrm{Li}_\bullet)$ (see Def. 1.ii) iff $0 \leq t < 1$. In fact, in this case,

$$\mathrm{Li}_{(tx_0)^*x_1}(z) = \sum_{n\geq 1} \frac{z^n}{n-t}.$$

This opens the door of Hurwitz polyzetas [25].

The map $\mathrm{Li}_\bullet$ is now extended to a subdomain of $\mathbb{C}\langle\langle X\rangle\rangle$, called $\mathrm{Dom}(\mathrm{Li}_\bullet)$ (see also [8, 10]).

Example 2. For any $\alpha, \beta \in \mathbb{C}$, $(\alpha x_0)^*, (\beta x_1)^*$, and $(\alpha x_0 + \beta x_1)^* = (\alpha x_0)^* ⧢ (\beta x_1)^*$. We have

$$\mathrm{Li}_{\alpha x_0^*}(z) = z^\alpha \ ; \ \mathrm{Li}_{\beta x_1^*}(z) = (1-z)^{-\beta} \ ; \ \mathrm{Li}_{(\alpha x_0 + \beta x_1)^*}(z) = z^\alpha (1-z)^{-\beta}$$

where $z \in \Omega$.

Proposition 1. *i) The domain $Dom(\mathrm{Li})$ is a shuffle subalgebra of $(\mathbb{C}\langle\langle X\rangle\rangle, ⧢, 1_{X^*})$.*
ii) The extended polylogarithm

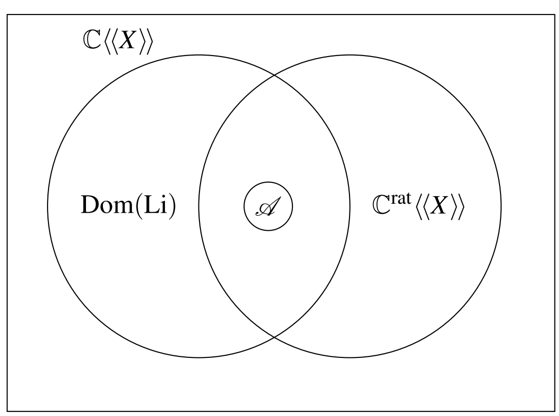
$$\mathrm{Li}: \ Dom(\mathrm{Li}) \to \mathscr{H}(\Omega)$$

is a shuffle morphism,i.e. $S, T \in Dom(\mathrm{Li})$, we still have

$$\mathrm{Li}_{S ⧢ T} = \mathrm{Li}_S \mathrm{Li}_T \ \textit{and} \ \mathrm{Li}_{1_{X^*}} = 1_{\mathscr{H}(\Omega)} \tag{24}$$

Proof. This proof has been done in [10].

The picture about $Dom(\mathrm{Li})$ within the algebra $(\mathbb{C}\langle\langle X\rangle\rangle, ⧢, 1_{X^*})$, the positioning of $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ (rational series, see [2, 8, 10]) and shuffle subalegbras as, for example, $\mathscr{A} = \mathbb{C}\langle X\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$ read as follows:



3 From Polylogarithms to Harmonic sums

Definition of $Dom(\mathrm{Li})$ has many merits[22] and can easily be adapted to arbitrary (open and connected) domains. However this definition, based on a global condition of a fixed domain Ω, does not provide a sufficiently clear interpretation of the stable symbolic computations around a point, in particular at $z = 0$. One needs to consider a sort of "symbolic local germ" worked out explicitely. Indeed, as the harmonic sums (or MZV[23]) are the coefficients of the Taylor expansion at zero of the convergent polylogarithms divided by $1-z$, we only need to know locally these functions. In order to gain more indexing series and to describe the local situation at zero, we reshape and define a new domain of Li around zero to $\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet)$.
The first step will be to characterize the polylogarithms having a removable singularity at zero The following Proposition helps us characterize their indices.

Proposition 2. *Let $P \in \mathbb{C}\langle X\rangle$ and $f(z) = \langle \mathrm{L} \mid P\rangle = \sum_{w\in X^*}\langle P \mid w\rangle \mathrm{Li}_w$.*
1) The following conditions are equivalent

i) f can be analytically extended around zero.
ii) $P \in \mathbb{C}\langle X\rangle x_1 \oplus \mathbb{C}.1_{X^*}$.

2) In this case Ω itself[24] can be extended to $\Omega_1 = \mathbb{C}\setminus(]-\infty,-1]\cup[1,+\infty[)$.

Proof (Sketch). (ii) $\Longrightarrow$ (i) being straightforward, it remains to prove that (ii) $\Longrightarrow$ (i).
Let then $P \in \mathbb{C}\langle X\rangle$ such that $f(z) = \langle \mathrm{L} \mid P\rangle$ has a removable singularity at zero. As a consequence of Radford's results [29], one can write down a basis of any free shuffle algebra in terms of Lyndon words. This implies that our polynomial reads

$$P = \sum_{k\geq 0} \alpha_k (P_k ⧢ x_0^{⧢\, k}) \text{ with } \alpha_k \in \mathbb{C},\ P_k \in \mathbb{C}\langle X\rangle x_1 \oplus \mathbb{C}.1_{X^*} \tag{25}$$

the family $(P_k)_{k\geq 0}$ being unique and finitely supported.
Using (25) and (16), we get

$$\mathrm{Li}_P(z) = \sum_{k\geq 0} \alpha_k \mathrm{Li}_{P_k}(z)\log(z)^k$$

the result now follows easily using asymptotic scale $x^n\log(x)^m$ along the axis $]0,+\infty[$ (and for $x \to 0_+$).

The second step will be provided by the following Proposition which says that, for appropriate series, the Taylor coefficients behave nicely.

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[22] As the fact that, due to special properties of $\mathscr{H}(\Omega)$ (it is a nuclear space [31], see details in [8]), one can show that Dom(Li) is closed by shuffle products.
[23] Multiple Zeta Values.
[24] The domain, for z of Li_P.

Proposition 3. *Let $S \in \mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{X^*}$ such that $S = \sum_{n\geq 0}[S]_n$ where $[S]_n = \sum_{w\in X^*,|w|=n}\langle S \mid w\rangle w$, ($[S]_n$ are the homogeneous components of S), we suppose that $0 < R \leq 1$ and that $\sum_{n\geq 0}\mathrm{Li}_{[S]_n}$ is unconditionally convergent (for the standard topology) within the open disk $|z|< R$*[25]. *Remarking that $\frac{1}{1-z}\sum_{n\geq 0}\mathrm{Li}_{[S]_n}(z)$ is unconditionally convergent in the same disk, we set*

$$\frac{1}{1-z}\sum_{n\geq 0}\mathrm{Li}_{[S]_n}(z) = \sum_{N\geq 0} a_N z^N \ .$$

Then, for all $N \geq 0$, $\sum_{n\geq 0}\mathrm{H}_{\pi_Y([S]_n)}(N) = a_N$.

Proof. Let us recall that, for any $w \in X^*$, the function $(1-z)^{-1}\,\mathrm{Li}_w(z)$ is analytic in the open disk $|z| < R$. Moreover, one has

$$\frac{1}{1-z}\mathrm{Li}_w(z) = \sum_{N\geq 0}\mathrm{H}_{\pi_Y(w)}(N)z^N.$$

Since $[S]_n = \sum_{w\in X^*,|w|=n}\langle S \mid w\rangle w$ and $(1-z)^{-1}\sum_{n\geq 0}\mathrm{Li}_{[S]_n}$ absolutely converges (for the standard topology[26].) within the open disk $D_{<R}$, one obtains, for all $|z| < R$

$$\begin{aligned}\frac{1}{1-z}\sum_{n\geq 0}\mathrm{Li}_{[S]_n}(z) &= \frac{1}{1-z}\sum_{n\geq 0}\sum_{w\in X^*,|w|=n}\langle S \mid w\rangle w\,\mathrm{Li}_w(z) = \sum_{n\geq 0}\sum_{w\in X^*,|w|=n}\langle S \mid w\rangle w\frac{\mathrm{Li}_w(z)}{1-z}\\ &= \sum_{n\geq 0}\sum_{w\in X^*,|w|=n}\langle S \mid w\rangle w\sum_{N\geq 0}\mathrm{H}_{\pi_Y(w)}(N)z^N\\ &\overset{(*)}{=} \sum_{N\geq 0}\sum_{n\geq 0}\sum_{w\in X^*,|w|=n}\langle S \mid w\rangle w\mathrm{H}_{\pi_Y(w)}(N)z^N = \sum_{N\geq 0}\mathrm{H}_{\pi_Y([S]_n)}(N)z^N.\end{aligned}$$

$(*)$ being possible because $\sum_{w\in X^*,|w|=n}$ is finite.
This implies that, for any $N \geq 0$, $a_N = \sum_{n\geq 0}\mathrm{H}_{\pi_Y([S]_n)}(N)$.

To prepare the construction of the "symbolic local germ" around zero, let us set, in the same manner as in [8, 10],

$$\begin{aligned}\mathrm{Dom}_R(\mathrm{Li}) := \{&S \in \mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{X^*} \mid\\ &\sum_{n\geq 0}\mathrm{Li}_{[S]_n} \text{ is unconditionally convergent in } \mathscr{H}(D_{<R})\}\end{aligned} \tag{26}$$

and prove the following:

[25] With the definition given later (26) this amounts to say that $S \in \mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{X^*} \cap Dom_R(\mathrm{Li})$.

[26] For this topology, unconditional and absolute convergence coincide [31].

Proposition 4. *With the notations as above, we have:*

1. *The map given by* $R \mapsto \mathrm{Dom}_R(\mathrm{Li})$ *from* $]0,1]$ *to* $2^{\mathbb{C}\langle\langle X\rangle\rangle}$ *(the target is the set of subsets*[27] *of* $\mathbb{C}\langle\langle X\rangle\rangle$ *ordered by inclusion) is strictly decreasing*
2. *Each* $\mathrm{Dom}_R(\mathrm{Li})$ *is a shuffle (unital) subalgebra of* $\mathbb{C}\langle\langle X\rangle\rangle$.

Proof. 1. For $0 < R_1 < R_2 \leq 1$ it is straightforward that $\mathrm{Dom}_{R_2}(\mathrm{Li}) \subset \mathrm{Dom}_{R_1}(\mathrm{Li})$. Let us prove that the inclusion is strict. Take $|z| < 1$ and let us, be it finite or infinite, evaluate the sum

$$M(z) = \sum_{n\geq 0} |\mathrm{Li}_{[S]_n(t)}(z)| = \sum_{n\geq 0} \langle S(t) \mid x_1^n \rangle \, |\mathrm{Li}_{x_1^n}(z)|$$

then, by means of Lemma 1, with $x_1^+ = x_1 x_1^* = x_1^* - 1$ and $S(t) = \sum_{m\geq 0} t^m (x_1^+)^{\sqcup\!\sqcup\, m}$, we have

$$\begin{aligned} M(z) &= \sum_{n\geq 0} |S(t)| \, x_1^n \, |\mathrm{Li}_{x_1^n}(z)| = \sum_{n\geq 0}\sum_{m\geq 0} |t^m (x_1^+)^{\sqcup\!\sqcup\, m}| \, x_1^n \, |\mathrm{Li}_{x_1^n}(z)| \\ &= \sum_{m\geq 0} m! t^m \sum_{n\geq 0} S_2(n,m) \frac{|\mathrm{Li}_{x_1}(z)|^n}{n!} \leq \sum_{m\geq 0} m! t^m \sum_{n\geq 0} S_2(n,m) \frac{\mathrm{Li}_{x_1}^n(|z|)}{n!}, \end{aligned}$$

due to the fact that $|\mathrm{Li}_{x_1}(z)| \leq \mathrm{Li}_{x_1}(|z|)$ (Taylor series with positive coefficients). Finally, in view of equation (29), we get, on the one hand, for $|z| < (t+1)^{-1}$,

$$M(z) \leq \sum_{m\geq 0} t^m (e^{\mathrm{Li}_{x_1}(|z|)} - 1)^m = \sum_{m\geq 0} t^m \Big(\frac{|z|}{1-|z|}\Big)^m = \frac{1-|z|}{1-(t+1)\,|z|}.$$

This proves that, for all $r \in]0, \frac{1}{t+1}[$, $\sum_{n\geq 0} \| \mathrm{Li}_{[S]_n(t)}(z) \|_r < +\infty$.

On the other hand, if $(t+1)^{-1} \leq |z| < 1$, one has $M(|z|) = +\infty$, and the preceding calculation shows that, with t choosen such that

$$0 \leq \frac{1}{R_2} - 1 < t < \frac{1}{R_1} - 1,$$

we have $S(t) \in \mathrm{Dom}_{R_1}(\mathrm{Li})$ but $S(t) \notin \mathrm{Dom}_{R_2}(\mathrm{Li})$ whence, for $0 < R_1 < R_2 \leq 1$, $\mathrm{Dom}_{R_2}(\mathrm{Li}) \subsetneq \mathrm{Dom}_{R_1}(\mathrm{Li})$.

2. One has (proofs as in [10])
   (a) $1_{X^*} \in \mathrm{Dom}_R(\mathrm{Li})$ (because $1_{X^*} \in \mathbb{C}\langle X\rangle$) and $\mathrm{Li}_{1_{X^*}} = 1_{\mathscr{H}(\Omega)}$.
   (b) Taking $S, T \in \mathrm{Dom}_R(\mathrm{Li})$ we have, by absolute convergence, $S \sqcup\!\sqcup\, T \in \mathrm{Dom}_R(\mathrm{Li})$. It is easily seen that $S \sqcup\!\sqcup\, T \in \mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}1_{X^*}$ and, moreover, that $\mathrm{Li}_S \mathrm{Li}_T = \mathrm{Li}_{S \sqcup\!\sqcup\, T}$[28].

The combinatorial Lemma needed in the Theorem 1 is as follows

[27] For any set E, the set of its subsets is noted 2^E.
[28] Proof by absolute convergence as in [10].

Lemma 1. *For a letter "a", one has*

$$|\,(a^+)^{\sqcup\!\sqcup\, m}\,|\,a^n = m!S_2(n,m) \tag{27}$$

($S_2(n,m)$ being the Stirling numbers of the second kind). The exponential generating series of R.H.S. in equation (27) *(w.r.t. n) is given by*

$$\sum_{n\geq 0} m!S_2(n,m)\frac{x^n}{n!} = (e^x-1)^m. \tag{28}$$

Proof. The expression $(a^+)^{\sqcup\!\sqcup\, m}$ is the specialization of

$$L_m = a_1^+ \sqcup\!\sqcup\, a_2^+ \sqcup\!\sqcup \ldots \sqcup\!\sqcup\, a_m^+$$

to $a_j \to a$ (for all $j = 1,2\ldots m$). The words of L_m are in bijection with the surjections $[1\ldots n] \to [1\ldots m]$, therefore the coefficient $\langle (a^+)^{\sqcup\!\sqcup\, m} | a^n\rangle$ is exactly the number of such surjections namely $m!S_2(n,m)$. A classical formula[29] says that

$$\sum_{n\geq 0} m!S_2(n,m)\frac{x^n}{n!} = (e^x-1)^m. \tag{29}$$

In Theorem 1 below, we study, for series taken in $\mathbb{C}\langle\langle X\rangle\rangle x_1 \oplus \mathbb{C}.1_{X^*}$, the correspondence $\mathrm{Li}_\bullet$ to some $\mathscr{H}(D_{<R})$, first (point 1) establishes its surjectivity (in a certain sense) and then (points 2 and 3) examine the relation between summability of the functions and that of their Taylor coefficients. For that, let us begin with a very general Lemma on sequences of Taylor series which adapts, for our needs, the notion of *normal families* as in [27].

Lemma 2. *Let $\tau = (a_{n,N})_{n,N\geq 0}$ be a double sequence of complex numbers. Setting*

$$I(\tau) := \{r \in]0,+\infty[\,|\, \sum_{n,N\geq 0} |a_{n,N} r^N| < +\infty\},$$

one has

1. *$I(\tau)$ is an interval of $]0,+\infty[$, it is not empty iff there exists $z_0 \in \mathbb{C}\setminus\{0\}$ such that*

$$\sum_{n,N\geq 0} |a_{n,N} z_0^N| < +\infty \tag{30}$$

   In this case, we set $R(\tau) := \sup(I(\tau))$, one has
   (a) *For all N, the series $\sum_{n\geq 0} a_{n,N}$ converges absolutely (in $\mathbb{C}$). Let us note a_N - with one subscript - its limit*
   (b) *For all n, the convergence radius of the Taylor series $T_n(z) = \sum_{N\geq 0} a_{n,N} z^N$ is at least $R(\tau)$ and $\sum_{n\in\mathbb{N}} T_n$ is summable for the standard topology of $\mathscr{H}(D_{<R(\tau)})$ with sum $T(z) = \sum_{n,N\geq 0} a_N z^N$.*

[29] See [28], the twelvefold way, formula (1.94b)(pp. 74) for instance.

2. *Conversely, we suppose that it exists $R > 0$ such that*
   (a) *Each Taylor series $T_n(z) = \sum_{N\geq 0} a_{n,N} z^N$ converges in $\mathscr{H}(D_{<R})$.*
   (b) *The series $\sum_{n\in\mathbb{N}} T_n$ converges unconditionnally in $\mathscr{H}(D_{<R})$.*
   Then $I(\tau) \neq \emptyset$ and $R(\tau) \geq R$.

Proof. 1. The fact that $I(\tau) \subset]0,+\infty[$ is straightforward from the Definition. If it exists $z_0 \in \mathbb{C}$ such that $\sum_{n,N\geq 0} \mid a_{n,N} z_0^N \mid < +\infty$ then, for all $r \in]0, |z_0|[$, we have

$$\sum_{n,N\geq 0} \mid a_{n,N} r^N \mid = \sum_{n,N\geq 0} \mid a_{n,N} z_0^N \mid \left(\frac{r}{\mid z_0 \mid}\right)^N \leq \sum_{n,N\geq 0} |a_{n,N} z_0^N| < +\infty$$

in particular $I(\tau) \neq \emptyset$ and it is an interval of $]0,+\infty[$ with lower bound zero.
(a) Take $r \in I(\tau)$ (hence $r \neq 0$) and $N \in \mathbb{N}$, then we get the expected result as

$$r^N \sum_{n\geq 0} \mid a_{n,N} \mid = \sum_{n\geq 0} \mid a_{n,N} r^N \mid \leq \sum_{n,N\geq 0} \mid a_{n,N} r^N \mid < +\infty.$$

(b) Again, take any $r \in I(\tau)$ and $n \in \mathbb{N}$, then $\sum_{N\geq 0} \mid a_{n,N} r^N \mid < +\infty$ which proves that $R(T_n) \geq r$, hence the result[30]. We also have

$$\mid \sum_{N\geq 0} a_N r^N \mid \leq \sum_{N\geq 0} r^N \mid \sum_{n\geq 0} a_{n,N} \mid \leq \sum_{n,N\geq 0} \mid a_{n,N} r^N \mid < +\infty$$

and this proves that $R(T) \geq r$, hence $R(T) \geq R(\tau)$.

2. Let $0 < r < r_1 < R$ and consider the path $\gamma(t) = r_1 e^{2i\pi t}$, we have

$$\mid a_{n,N} \mid = \mid \frac{1}{2i\pi} \int_\gamma \frac{T_n(z)}{z^{N+1}} dz \mid \leq \frac{2\pi}{2\pi} \frac{r_1 \|T_n\|_K}{r_1^{N+1}} \leq \frac{\|T_n\|_K}{r_1^N}$$

with $K = \gamma([0,2\pi])$, hence

$$\sum_{n,N\geq 0} \mid a_{n,N} r^N \mid \leq \sum_{n,N\geq 0} \mid T_n \mid_K (\frac{r}{r_1})^N \leq \frac{r_1}{r_1 - r} \sum_{n\geq 0} \|T_n\|_K < +\infty.$$

Remark 1. (i) First point says that every function analytic at zero can be represented around zero as $\mathrm{Li}_S(z)$ for some $S \in \mathbb{C}\langle\langle x_1 \rangle\rangle$.
(ii) In point 2, the arithmetic functions $\mathrm{H}_{\pi_Y(S)} \in \mathbb{Q}^{\mathbb{N}}$, for $S \in Dom(\mathrm{Li})$ are quickly defined (and in a way extending the old definition) and we draw a very important bound saying that, in this condition, for some $r > 0$ the array $\left(\mathrm{H}_{\pi_Y([S]_n)}(N) r^N\right)_{n,N}$ converges (then, in particular, horizontally and vertically).
(iii) Point 3 establishes the converse.

[30] For a Taylor series T, we note $R(T)$ the radius of convergence of T.

Theorem 1. *1. Let $T(z)=\sum_{N\ge 0} a_N z^N$ be a Taylor series converging on some non-empty disk centered at zero i.e. such that $\limsup_{N\to+\infty} |a_N|^{\frac{1}{n}} = B < +\infty$, then the series*

$$S=\sum_{N\ge 0} a_N(-(-x_1)^+)^{\sqcup\!\sqcup N} \tag{31}$$

is summable in $\mathbb{C}\langle\langle X\rangle\rangle$ (with sum in $\mathbb{C}\langle\langle x_1\rangle\rangle$), $S\in \mathrm{Dom}_R(\mathrm{Li})$ with $R=(B+1)^{-1}$ and $\mathrm{Li}_S=T$.

2. Let $S\in\mathrm{Dom}_R(\mathrm{Li})$ and $S=\sum_{n\ge 0}[S]_n$ (homogeneous decomposition), we define $N\longmapsto \mathrm{H}_{\pi_Y(S)}(N)$ by[31]

$$\frac{\mathrm{Li}_S(z)}{1-z}=\sum_{N\ge 0}\mathrm{H}_{\pi_Y(S)}(N)z^N. \tag{32}$$

Then,

$$\forall r\in]0,R[,\ \sum_{n,N\ge 0}|\mathrm{H}_{\pi_Y([S]_n)}(N)r^N|<+\infty. \tag{33}$$

In particular, for all $N\in\mathbb{N}$, the series (of complex numbers), $\sum_{n\ge 0}\mathrm{H}_{\pi_Y([S]_n)}(N)$ converges absolutely to $\mathrm{H}_{\pi_Y(S)}(N)$.

3. Conversely, let $Q\in\mathbb{C}\langle\langle Y\rangle\rangle$ with $Q=\sum_{n\ge 0}Q_n$ (decomposition by weights), we suppose that it exists $r\in]0,1]$ such that

$$\sum_{n,N\ge 0}|\mathrm{H}_{Q_n}(N)r^N|<+\infty, \tag{34}$$

in particular, for all $N\in\mathbb{N}$, $\sum_{n\ge 0}\mathrm{H}_{Q_n}(N)=\ell(N)\in\mathbb{C}$ unconditionally. Under such circumstances, $\pi_X(Q)\in\mathrm{Dom}_r(\mathrm{Li})$ and, for all $z\in\mathbb{C},|z|\le r$,

$$\frac{\mathrm{Li}_S(z)}{1-z}=\sum_{N\ge 0}\ell(N)z^N, \tag{35}$$

Proof. 1. The fact that the series (31) is summable comes from the fact that $\omega(a_N(-(-x_1)^+)^{\sqcup\!\sqcup N})\ge N$. Now from the Lemma 1, we get

$$(S)_n=\sum_{N\ge 0}(a_N(-(-x_1)^+)^{\sqcup\!\sqcup N})_n=(-1)^{N+n}a_N N!S_2(n,N)x_1^n.$$

Then, with $r=\sup_{z\in K}|z|$ (we have indeed $r=||Id||_K$) and taking into account that $\|Li_{x_1}\|_K\le\log(1/(1-r))$, we have

$$\sum_{n\ge 0}\|\mathrm{Li}_{(S)_n}\|_K\le\sum_{n\ge 0}\sum_{N\ge 0}|a_N|N!S_2(n,N)\|\mathrm{Li}_{x_1^n}\|_K\le\sum_{n\ge 0}\sum_{N\ge 0}|a_N|N!S_2(n,N)\frac{\|\mathrm{Li}_{x_1}\|_K^n}{n!}$$

[31] This Definition is compatible with the old one when S is a polynomial.

$$\leq \sum_{N\geq 0} |a_N| \sum_{n\geq 0} N!S_2(n,N)\frac{|\mathrm{Li}_{x_1}|_K^n}{n!} \leq \sum_{N\geq 0} |a_N| \,(e^{\log(\frac{1}{1-r})}-1)^N$$
$$= \sum_{N\geq 0} |a_N| \left(\frac{r}{1-r}\right)^N.$$

Now if we suppose that $r \leq (B+1)^{-1}$, we have $r(1-r)^{-1} \leq \dfrac{1}{B}$ and this shows that the last sum is finite.

2. This point and next point are consequences of Lemma 2. Now, considering the homogeneous decomposition $S = \sum_{n\geq 0}[S]_n \in \mathrm{Dom}_R(\mathrm{Li})$. We first establish inequation (33). Let $0 < r < r_1 < R$ and consider the path $\gamma(t) = r_1 e^{2i\pi t}$, we have

$$|\mathrm{H}_{\pi_Y([S]_n)}(N)| = |\frac{1}{2i\pi}\int_\gamma \frac{\mathrm{Li}_{[S]_n}(z)}{(1-z)z^{N+1}}dz| \leq \frac{2\pi}{2\pi}\frac{\|\mathrm{Li}_{[S]_n}\|_K}{(1-r_1)r_1^{N+1}},$$

$K = \gamma([0,1])$ being the circle of center 0 and radius r_1. Taking into account that, for $K \subset_{compact} \mathscr{D}_{<R}$, we have a decomposition $\sum_{n\in\mathbb{N}} |\mathrm{Li}_{[S]_n}|_K = M < +\infty$, we get

$$\sum_{n,N\geq 0} |\mathrm{H}_{\pi_Y([S]_n)}(N)r^N| = \sum_{n,N\geq 0} |\mathrm{H}_{\pi_Y([S]_n)}(N)r_1^N| \,(\frac{r}{r_1})^N = \sum_{N\geq 0}(\frac{r}{r_1})^N \sum_{n\geq 0}|\mathrm{H}_{\pi_Y([S]_n)}(N)r_1^N|$$
$$\leq \sum_{N\geq 0}(\frac{r}{r_1})^N\frac{M}{(1-r_1)r_1} \leq \frac{M}{(1-r_1)(r_1-r)} < +\infty.$$

The series $\sum_{n\geq 0}\mathrm{Li}_{[S]_n}(z)$ converges to $\mathrm{Li}_S(z)$ in $\mathscr{H}(D_{<R})$ ($D_{<R}$ is the open disk defined by $|z| < R$). For any $N \geq 0$, by Cauchy's formula, one has,

$$\mathrm{H}_{\pi_Y(S)}(N) = \frac{1}{2i\pi}\int_\gamma \frac{\mathrm{Li}_S(z)}{(1-z)z^{N+1}}dz = \frac{1}{2i\pi}\int_\gamma \frac{\sum_{n\geq 0}\mathrm{Li}_{[S]_n}(z)}{(1-z)z^{N+1}}dz$$
$$= \frac{1}{2i\pi}\sum_{n\geq 0}\int_\gamma \frac{\mathrm{Li}_{[S]_n}(z)}{(1-z)z^{N+1}}dz = \sum_{n\geq 0}\mathrm{H}_{\pi_Y([S]_n)}(N)$$

the exchange of sum and integral being due to the compact convergence. The absolute convergence comes from the fact that the convergence of $\sum_{n\geq}\mathrm{Li}_{[S]_n}(z)$ is unconditional [31].

3. Fixing $N \in \mathbb{N}$, from inequation (34), we get $\sum_{n\geq 0}|\mathrm{H}_{Q_n}(N)| < +\infty$ which proves the absolute convergence. Remark now that $(\pi_X(Q))_n = \pi_X(Q_n)$ and $\pi_Y(\pi_X(Q_n)) = Q_n$, one has, for all $|z| \leq r$, $|\mathrm{Li}_{\pi_X(Q_n)}(z)| = |\sum_{N\in\mathbb{N}}\mathrm{H}_{Q_n}(N)z^N| \leq |\sum_{N\in\mathbb{N}}\mathrm{H}_{Q_n}(N)r^N|$, in other words $\|\mathrm{Li}_{\pi_X(Q_n)}\|_{D\leq r} \leq |\sum_{N\in\mathbb{N}}\mathrm{H}_{Q_n}(N)r^N|$ and

$$\sum_{n\in\mathbb{N}}\|\mathrm{Li}_{\pi_X(Q_n)}\|_{D\leq r} \leq |\sum_{n,N\in\mathbb{N}}\mathrm{H}_{Q_n}(N)r^N| < +\infty$$

which shows that $\pi_X(Q) \in \mathrm{Dom}_r(\mathrm{Li})$. The equation (35) is a consequence of point 2, taking $S = \pi_X(Q)$.

Now, we have have a better understanding of what can (and will) be the domain, $\mathrm{Dom}(\mathrm{H}_\bullet)$, of harmonic sums.

Definition 2. *We set* $\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}) = \bigcup_{0<R\le 1} \mathrm{Dom}_R(\mathrm{Li}); \mathrm{Dom}(\mathrm{H}_\bullet) = \pi_Y(\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}))$ *and, for* $S \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li})$, $\mathrm{Li}_S(z) = \sum_{n\ge 0} \mathrm{Li}_{[S]_n}(z)$ *and* $\frac{\mathrm{Li}_S(z)}{1-z} = \sum_{N\ge 0} \mathrm{H}_{\pi_Y(S)}(N) z^N$.

4 Applications

We remark that formula (7), *i.e.*, $\mathrm{Li}_{s_1,\ldots,s_r}(z) := \sum_{n_1>\ldots>n_r>0} \frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}}$, still makes sense for $|z|<1$ and $(s_1,\ldots,s_r) \in \mathbb{C}^r$ so that we will freely use the list indexing to get index lists with $s_i \in \mathbb{Z}$ for any $i = 1,\ldots,r$ and $r \in \mathbb{N}^+$.

Recall that for any $s_1,\ldots,s_r \in \mathbb{N}$, we can present $\mathrm{Li}_{-s_1,\ldots,-s_r}(z)$ as a polynomial of $\frac{1}{1-z}$ with integer coefficients. Then, using (24) and $(kx_1)^* = [(x_1)^*]^{\sqcup\!\sqcup\, k}$, we get $\frac{1}{(1-z)^k} = \mathrm{Li}_{(kx_1)^*}(z)$, $\forall k \in \mathbb{N}^+$, we obtain a polynomial $P \in \mathrm{Dom}(\mathrm{Li}) \cap \mathbb{C}[x_1^*] = \mathbb{C}[x_1^*]$ such that $\mathrm{Li}_{-s_1,\ldots,-s_r} = \mathrm{Li}_P$ (see [10]). Using Theorem 1, we have $\frac{\mathrm{Li}_P(z)}{1-z} = \sum_{N\ge 0} \mathrm{H}_{\pi_Y(P)}(N) z^N$. This means that we can provide a class of elements of $\mathrm{Dom}(\mathrm{H}_\bullet)$ (as in Definition 2) relative to the set of indices of harmonic sums at negative integer multiindices. Here are some examples.

Example 3. For any $|z|<1$, we have

$$\mathrm{Li}_{x_1^*}(z) = \frac{1}{1-z}\ ;\ \mathrm{Li}_{x_1^*-1_{X^*}}(z) = \frac{z}{1-z} = \mathrm{Li}_0(z)\ ;\ \mathrm{Li}_{(2x_1)^*-x_1^*}(z) = \frac{z}{(1-z)^2} = \mathrm{Li}_{-1}(z);$$

$$\mathrm{Li}_{(2x_1)^*-2x_1^*+1_{X^*}}(z) = \frac{z^2}{(1-z)^2} = \mathrm{Li}_{0,0}(z);$$

$$\mathrm{Li}_{12(5x_1)^*-33(4x_1)^*+31(3x_1)^*-11(2x_1)^*+x_1^*}(z) = \frac{z^4+7z^3+4z^2}{(1-z)^5} = \mathrm{Li}_{-2,-1}(z);$$

$$\mathrm{Li}_{40(6x_1)^*-132(5x_1)^*+161(4x_1)^*-87(3x_1)^*+19(2x_1)^*-x_1^*}(z) = \frac{z^5+14z^4+21z^3+4z^2}{(1-z)^6} = \mathrm{Li}_{-2,-2}(z);$$

$$\begin{aligned}&\mathrm{Li}_{1260(8x_1)^*-5400(7x_1)^*+9270(6x_1)^*-8070(5x_1)^*+3699(4x_1)^*-829(3x_1)^*+71(2x_1)^*-x_1^*}(z)\\ &= \frac{z^7+64z^6+424z^5+584z^4+179z^3+8z^2}{(1-z)^8} = \mathrm{Li}_{-3,-3}(z);\end{aligned}$$

$$\mathrm{Li}_{10(6x_1)^*-38(5x_1)^*+55(4x_1)^*-37(3x_1)^*+11(2x_1)^*-x_1^*}(z) = \frac{z^5+6z^4+3z^3}{(1-z)^6} = \mathrm{Li}_{-1,0,-2}(z);$$

$$\begin{aligned}&\mathrm{Li}_{280(8x_1)^*-1312(7x_1)^*+2497(6x_1)^*-2457(5x_1)^*+1310(4x_1)^*-358(3x_1)^*+41(2x_1)^*-x_1^*}(z)\\ &= \frac{z^7+34z^6+133z^5+100z^4+12z^3}{(1-z)^8} = \mathrm{Li}_{-1,-2,-2}(z).\end{aligned}$$

Thus, for any $N \in \mathbb{N}$, for readability, below 1 stands for 1_{X^*}

$$\mathrm{H}_{\pi_Y(x_1^*)}(N) = N+1,\ \mathrm{H}_{\pi_Y(x_1^*-1)}(N) = N = \sum_{n=1}^{N} n^0,\ \mathrm{H}_{\pi_Y((2x_1)^*-x_1^*)}(N) = \frac{1}{2}N^2 + \frac{1}{2}N = \sum_{n=1}^{N} n^1;$$

$$\mathrm{H}_{\pi_Y((2x_1)^*-2x_1^*+1)}(N) = \frac{1}{2}N^2 - \frac{1}{2}N = \sum_{n_1=1}^{N} n_1^0 \sum_{n_2=1}^{n_1-1} n_2^0;$$

$$\mathrm{H}_{\pi_Y(12(5x_1)^*-33(4x_1)^*+31(3x_1)^*-11(2x_1)^*+x_1^*)}(N) = \frac{1}{10}N^5 + \frac{1}{8}N^4 - \frac{1}{12}N^3 - \frac{1}{60}N - \frac{1}{8}N^2 = \sum_{n_1=1}^{N} n_1^2 \sum_{n_2=1}^{n_1-1} n_2^1;$$

$$\mathrm{H}_{\pi_Y(40(6x_1)^*-132(5x_1)^*+161(4x_1)^*-87(3x_1)^*+19(2x_1)^*-x_1^*)}(N) = \frac{1}{15}N^5 + \frac{1}{18}N^6 - \frac{5}{72}N^4 + \frac{1}{72}N^2$$
$$+\frac{1}{60}N - \frac{1}{12}N^3 = \sum_{n_1=1}^{N} n_1^2 \sum_{n_2=1}^{n_1-1} n_2^2;$$

$$\mathrm{H}_{\pi_Y(1260(8x_1)^*-5400(7x_1)^*+9270(6x_1)^*-8070(5x_1)^*+3699(4x_1)^*-829(3x_1)^*+71(2x_1)^*-x_1^*)}(N) = \sum_{n_1=1}^{N} n_1^3 \sum_{n_2=1}^{n_1-1} n_2^3;$$

$$\mathrm{H}_{\pi_Y(10(6x_1)^*-38(5x_1)^*+55(4x_1)^*-37(3x_1)^*+11(2x_1)^*-x_1^*)}(N) = -\frac{1}{40}N^5 + \frac{1}{72}N^6 - \frac{1}{36}N^4 + \frac{1}{72}N^2$$
$$+\frac{1}{24}N^3 - \frac{1}{60}N = \sum_{n_1=1}^{N} n_1^1 \sum_{n_2=1}^{n_1-1} \sum_{n_3=1}^{n_2-1} n_3^2;$$

$$\mathrm{H}_{\pi_Y(280(8x_1)^*-1312(7x_1)^*+2497(6x_1)^*-2457(5x_1)^*+1310(4x_1)^*-358(3x_1)^*+41(2x_1)^*-x_1^*)}(N) = -\frac{13}{1260}N^7$$
$$+\frac{1}{144}N^8 - \frac{7}{240}N^6 + \frac{1}{24}N^4 - \frac{7}{360}N^2 + \frac{23}{720}N^5 + \frac{1}{210}N - \frac{19}{720}N^3 = \sum_{n_1=1}^{N} n_1^1 \sum_{n_2=1}^{n_1-1} n_2^2 \sum_{n_3=1}^{n_2-1} n_3^2.$$

Observe that, from this Definition, Theorem 2 will show us that $\mathrm{Dom}(\mathrm{H}_\bullet)$ is a stuffle subalgebra of $\mathbb{C}\langle\langle Y\rangle\rangle$. Let us however remark that some series are not in this domain as shown below

(i) The series $T = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{y_n}{n} \in \mathbb{C}\langle\langle Y\rangle\rangle$ is not in $\mathrm{Dom}(\mathrm{H}_\bullet)$ because we see that its decomposition by weights ($T = \sum_{n=1}^{\infty} T_n$ as in (34)) provides $T_n = \frac{(-1)^{n-1}}{n} y_n$ for $n \geq 1$ and $T_0 = 0$. Direct calculation, gives, for $n \geq 1$ $\mathrm{H}_{y_n}(N) = \sum_{k=1}^{N} \frac{1}{k^n}$, so that we have $\mathrm{H}_{y_n}(N) \geq 1, \forall n \in \mathbb{N}^+; N \in \mathbb{N}^+$, because $\mathrm{H}_{y_n}(0) = 0$, for all $0 < r < 1$, one has

$$\sum_{n,N} |\,\mathrm{H}_{T_n}(N) r^N| = \sum_{N\geq 0}\sum_{n\geq 1} |\,\frac{1}{n}\mathrm{H}_{y_n}(N) r^N| \geq \left(\sum_{n\geq 0}\frac{1}{n}\right)\frac{r}{1-r} = +\infty. \tag{36}$$

However one can get unconditional convergence using a sommation by pairs (odd + even).

(ii) For all $s \in]1,+\infty[$, the series $T(s) = \sum_{n=1}^{\infty} (-1)^{n-1} y_n n^{-s} \in \mathbb{C}\langle\langle Y\rangle\rangle$ is in $\mathrm{Dom}(\mathrm{H}_\bullet)$.

We can now state the

Theorem 2. *Let* $S,T \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li})$*, then* $S \sqcup\!\sqcup T \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}), \pi_X(\pi_Y(S) \uplus \pi_Y(T)) \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li})$ *and for all* $N \geq 0$*,*

$$\mathrm{Li}_{S \sqcup\!\sqcup T} = \mathrm{Li}_S \mathrm{Li}_T; \quad \mathrm{Li}_{1_{X^*}} = 1_{\mathscr{H}(\Omega)}, \tag{37}$$

$$\mathrm{H}_{\pi_Y(S) \uplus \pi_Y(T)}(N) = \mathrm{H}_{\pi_Y(S)}(N)\mathrm{H}_{\pi_Y(T)}(N). \tag{38}$$

$$\frac{\mathrm{Li}_S(z)}{1-z} \odot \frac{\mathrm{Li}_T(z)}{1-z} = \frac{\mathrm{Li}_{\pi_X(\pi_Y(S) \uplus \pi_Y(T))}(z)}{1-z}. \tag{39}$$

Proof. For equation (37), we get, from Lemma 4 that $Dom^{loc}(\mathrm{Li})$ is the union of an increasing set of shuffle subalgebras of $\mathbb{C}\langle\langle X \rangle\rangle$. It is therefore a shuffle subalgebra of the latter.

For equation (38), suppose $S \in \mathrm{Dom}_{R_1}(\mathrm{Li})$ (resp. $T \in \mathrm{Dom}_{R_2}(\mathrm{Li})$). By [18] and Theorem 1, one has $\dfrac{\mathrm{Li}_S(z)}{1-z} \odot \dfrac{\mathrm{Li}_T(z)}{1-z} \in \mathrm{Dom}_{R_1R_2}(\mathrm{Li})$ where $\odot$ stands for the Hadamard product [18]. Hence, for $|z| < R_1R_2$, one has

$$f(z) = \frac{\mathrm{Li}_S(z)}{1-z} \odot \frac{\mathrm{Li}_T(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}_{\pi_Y(S)}(N)\mathrm{H}_{\pi_Y(T)}(N)z^N \tag{40}$$

and, due to Theorem 1 point (32), for all N, $\sum\limits_{p \geq 0} \mathrm{H}_{\pi_Y(S_p)}(N) = \mathrm{H}_{\pi_Y(S)}(N)$ and $\sum\limits_{q \geq 0} \mathrm{H}_{\pi_Y(T_q)}(N) = \mathrm{H}_{\pi_Y(T)}(N)$ (absolute convergence) then, as the product of two absolutely convergent series is absolutely convergent (w.r.t. the Cauchy product), one has, for all N,

$$\begin{aligned}
\mathrm{H}_{\pi_Y(S)}(N)\mathrm{H}_{\pi_Y(T)}(N) &= \Big(\sum_{p \geq 0} \mathrm{H}_{\pi_Y(S_p)}(N)\Big)\Big(\sum_{q \geq 0} \mathrm{H}_{\pi_Y(T_q)}(N)\Big) \\
&= \sum_{p,q \geq 0} \mathrm{H}_{\pi_Y(S_p)}(N)\mathrm{H}_{\pi_Y(T_q)}(N) = \sum_{n \geq 0} \sum_{p+q=n} \mathrm{H}_{\pi_Y(S_p) \uplus \pi_Y(T_q)}(N) \\
&= \sum_{n \geq 0} \mathrm{H}_{(\pi_Y(S) \uplus \pi_Y(T))_n}(N).
\end{aligned} \tag{41}$$

Remains to prove that condition of Theorem 1, *i.e.* inequation (34) is fulfilled. To this end, we use the well-known fact that if $\sum_{m \geq 0} c_m z^m$ has radius of convergence $R > 0$, then $\sum\limits_{m \geq 0} |\, c_m \,|\, z^m$ has the same radius of convergence (use $1/R = \limsup_{m \geq 1} |\, c_m \,|^{-m}$), then from the fact that $S \in \mathrm{Dom}_{R_1}(\mathrm{Li})$ (resp. $T \in Dom_{R_2}(\mathrm{Li})$), we have (33) for each of them and, using the Hadamard product of these expressions, we get

$$\forall r \in]0, R_1.R_2[, \sum_{p,q,N \geq 0} |\mathrm{H}_{\pi_Y(S_p)}(N)\mathrm{H}_{\pi_Y(T_q)}(N)\, r^N| < +\infty,$$

and this assures, for $|z| < R_1R_2$, the convergence of

$$f(z) = \sum_{n,N \geq 0} \mathrm{H}_{(\pi_Y(S) \uplus \pi_Y(T))_n}(N)z^N \tag{42}$$

applying Theorem 1 point (3) to $Q = \pi_Y(S) \uplus \pi_Y(T)$ (with any $r < R_1R_2$), we get $\pi_X(Q) = \pi_X(\pi_Y(S) \uplus \pi_Y(T)) \in \mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li})$ and

$$f(z) = \sum_{N \geq 0} \Big(\sum_{n \geq 0} \mathrm{H}_{(\pi_Y(S) \uplus \pi_Y(T))_n}(N)\Big)z^N = \frac{\mathrm{Li}_{\pi_X(\pi_Y(S) \uplus \pi_Y(T))}(z)}{1-z}.$$

hence we obtain (38).

Recall that, as in Example 3, for any $s_1,\ldots,s_r \in \mathbb{N}$, we can find an elements $P \in \mathrm{Dom}(\mathrm{Li})$ such that $\dfrac{\mathrm{Li}_P(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}_{\pi_Y(P)(N)} z^N$. Theorem 2 proves that H is a stuffle character on Dom(H). Then for any mixed multiindices s, we can find the elements $P \in \mathrm{Dom}(\mathrm{Li})$ satisfying $\dfrac{\mathrm{Li}_{\mathrm{s}}(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}_{\pi_Y(P)}(N) z^N$.

Example 4.

$$\mathrm{H}_{\pi_Y(\frac{1}{2}(2x_1)^*-x_1^*+\frac{1}{2})}(N) = \frac{1}{4}N^2 - \frac{1}{4}N = \sum_{n_1=1}^{N} \frac{1}{n_1} \sum_{n_2=1}^{n_1-1} n_2^1,$$

$$\text{hence}\quad \mathrm{H}_{\pi_Y(x_1) \uplus \pi_Y((2x_1)^*-x_1^*)-\frac{1}{2}\pi_Y((2x_1)^*-1)}(N) = \sum_{n_1=1}^{N} n_1 \sum_{n_2=1}^{n_1-1} \frac{1}{n_2},$$

$$\mathrm{H}_{\pi_Y(\frac{2}{3}(3x_1)^*-\frac{3}{2}(2x_1)^*+x_1^*-\frac{1}{6})}(N) = \frac{1}{9}N^3 - \frac{1}{12}N^2 - \frac{1}{36}N = \sum_{n_1=1}^{N} \frac{1}{n_1} \sum_{n_2=1}^{n_1-1} n_2^2,$$

$$\text{then}\quad \mathrm{H}_{\pi_Y(x_1) \uplus \pi_Y(2(3x_1)^*-3(2x_1)^*+x_1^*)-\pi_Y(\frac{2}{3}(3x_1)^*-\frac{1}{2}(2x_1)^*-\frac{1}{6})}(N) = \sum_{n_1=1}^{N} n_1^2 \sum_{n_2=1}^{n_1-1} \frac{1}{n_2},$$

$$\mathrm{H}_{\pi_Y(\frac{1}{3}(2x_1)^*-\frac{5}{6}x_1^*+\frac{1}{2}+\frac{1}{6}x_1)}(N) = \sum_{n_1=1}^{N} \frac{1}{n_1^2} \sum_{n_2=1}^{n_1-1} n_2^2,$$

$$\text{which entails}\quad \mathrm{H}_{\pi_Y(x_0x_1) \uplus \pi_Y(2(3x_1)^*-3(2x_1)^*+x_1^*)-\pi_Y(\frac{1}{3}(2x_1)^*+\frac{1}{6}x_1^*-\frac{1}{2}+\frac{1}{6}x_1)} = \sum_{n_1=1}^{N} n_1^2 \sum_{n_2=1}^{n_1-1} \frac{1}{n_2^2},$$

$$\mathrm{H}_{\pi_Y(\frac{20}{3}(6x_1)^*-\frac{128}{5}(5x_1)^*+\frac{153}{4}(4x_1)^*-\frac{82}{3}(3x_1)^*+\frac{653}{72}(2x_1)^*-\frac{373}{360}x_1^*-\frac{1}{60})}(N) = \sum_{n_1=1}^{N} \frac{1}{n_1} \sum_{n_2=1}^{n_1-1} n_2^2 \sum_{n_3=1}^{n_2-1} n_3^2$$

$$\mathrm{H}_{\pi_Y(\frac{40}{3}(6x_1)^*-50(5x_1)^*+\frac{427}{6}(4x_1)^*-\frac{281}{6}(3x_1)^*+\frac{27}{2}(2x_1)^*-\frac{7}{6}x_1^*)}(N) = \sum_{n_1=1}^{N} n_1^2 \sum_{n_2=1}^{n_1-1} \frac{1}{n_2} \sum_{n_3=1}^{n_2-1} n_3^2$$

$$\text{thus}\quad \mathrm{H}_{\pi_Y(x_1) \uplus \pi_Y(40(6x_1)^*-132(5x_1)^*+161(4x_1)^*-87(3x_1)^*+19(2x_1)^*-x_1^*)}(N) - \sum_{n_1=1}^{N} n_1^2 \sum_{n_2=1}^{n_1-1} \frac{1}{n_2} \sum_{n_3=1}^{n_2-1} n_3^2$$
$$- \sum_{n_1=1}^{N} \frac{1}{n_1} \sum_{n_2=1}^{n_1-1} n_2^2 \sum_{n_3=1}^{n_2-1} n_3^2 - \sum_{n_1=1}^{N} n_1^2 \sum_{n_2=1}^{n_1-1} n_2 - \sum_{n_1=1}^{N} n_1 \sum_{n_2=1}^{n_1-1} n_2^2 = \sum_{n_1=1}^{N} n_1^2 \sum_{n_2=1}^{n_1-1} n_2^2 \sum_{n_3=1}^{n_2-1} \frac{1}{n_3}.$$

5 Some remarks about stuffle product and stuffle characters and their symbolic computations.

For the some reader's convenience, we recall here the Definitions of shuffle and stuffle products. As regards shuffle, the alphabet $\mathscr{X}$ is arbitrary and $\sqcup\!\sqcup$ is defined by the following recursion (for $a,b \in \mathscr{X}$ and $u,v \in \mathscr{X}^*$)

$$u \sqcup\!\sqcup 1_{\mathscr{X}^*} = 1_{\mathscr{X}^*} \sqcup\!\sqcup u = u;\ au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v). \tag{43}$$

As regards stuffle, the alphabet is $Y = Y_{\mathbb{N}_+} = \{y_s\}_{s\in\mathbb{N}_+}$ and $\uplus$ is defined by the following recursion

$$u \uplus 1_{Y^*} = 1_{Y^*} \uplus u = u, \tag{44}$$

$$y_s u \uplus y_t v = y_s(u \uplus y_t v) + y_t(y_s u \uplus v) + y_{s+t}(u \uplus v). \tag{45}$$

Be it for stuffle or shuffle, the noncommutative[32] polynomials equipped with this product form an associative commutative and unital algebra namely $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$ (resp. $(\mathbb{C}\langle Y\rangle, \uplus, 1_{Y^*})$).

Example 5. As examples of characters, we have already seen

- $\mathrm{Li}_\bullet$ from $(\mathrm{Dom}^{\mathrm{loc}}(\mathrm{Li}_\bullet), ш, 1_{X^*})$ to $\mathscr{H}(\Omega)$
- $\mathrm{H}_\bullet$ from $(\mathrm{Dom}(\mathrm{H}_\bullet), \uplus, 1_{Y^*})$ to $\mathbb{C}^{\mathbb{N}}$ (arithmetic functions $\mathbb{N} \to \mathbb{C}$)

In general, a character from a k-algebra[33] $(\mathscr{A}, *_1, 1_{\mathscr{A}})$ with values in $(\mathscr{B}, *_2, 1_{\mathscr{B}})$ is none other than a morphism between the k-algebras $\mathscr{A}$ and a commutative algebra[34] $\mathscr{B}$. The algebra $(\mathscr{A}, *_1, 1_{\mathscr{A}})$ does not have to be commutative for example characters of $(\mathbb{C}\langle\mathscr{X}\rangle, conc, 1_{\mathscr{X}^*})$ - *i.e.* `conc`-characters - where all proved to be of the form

$$\left(\sum_{x\in\mathscr{X}} \alpha_x x\right)^* \tag{46}$$

i.e. Kleene stars of the plane [8, 10]. They are closed under shuffle and stuffle and endowed with these laws, they form a group. Expressions like the infinite sum within brackets in (46) (*i.e.* homogeneous series of degree 1) form a vector space noted $\widehat{\mathbb{C}.Y}$.

As a consequence, given $P = \sum_{i\geq 1} \alpha_i y_i$ and $Q = \sum_{j\geq 1} \beta_j y_j$, we know in advance that their stuffle is a `conc`-character *i.e.* of the form $(\sum_{n\geq 1} c_n y_n)^*$. Examining the effect of this stuffle on each letter (which suffices), we get the identity

$$\left(\sum_{i\geq 1} \alpha_i y_i\right)^* \uplus \left(\sum_{j\geq 1} \beta_j y_j\right)^* = \left(\sum_{i\geq 1} \alpha_i y_i + \sum_{j\geq 1} \beta_j y_j + \sum_{i,j\geq 1} \alpha_i \beta_j y_{i+j}\right)^* \tag{47}$$

This suggests to take an auxiliary variable, say q, and code "the plane" $\widehat{\mathbb{C}.Y}$, *i.e.* expressions like (46), in the style of Umbral calculus by $\pi_Y^{\mathrm{Umbra}} : \sum_{n\geq 1} \alpha_n q^n \longmapsto \sum_{n\geq 1} \alpha_n y_n$ which is linear and bijective[35] from $\mathbb{C}_+[[q]]$ to $\widehat{\mathbb{C}.Y}$. With this coding at hand and for $S, T \in \mathbb{C}_+[[q]]$, identity (47) reads

$$(\pi_Y^{\mathrm{Umbra}}(S))^* \uplus (\pi_Y^{\mathrm{Umbra}}(T))^* = (\pi_Y^{\mathrm{Umbra}}((1+S)(1+T)-1))^*. \tag{48}$$

[32] For concatenation.
[33] Here we will use $k = \mathbb{Q}$ or $\mathbb{C}$.
[34] In this context all algebras are associative and unital.
[35] Its inverse will be naturally noted π_q^{Umbra}.

This shows that if one sets, for $z \in \mathbb{C}$ and $T \in \mathbb{C}_+[[x]]$, $G(z) = (\pi_Y^{\mathrm{Umbra}}(e^{zT} - 1))^*$, we get a one-parameter stuffle group[36] such that every coefficient is polynomial in z. Differentiating it we get

$$\frac{d}{dz}(G(z)) = (\pi_Y^{\mathrm{Umbra}}(T))G(z) \tag{49}$$

and (49) with the initial condition $G(0) = 1_{Y^*}$ integrates as

$$G(z) = \exp_{\uplus}(z\pi_Y^{\mathrm{Umbra}}(T)) \tag{50}$$

where the exponential map for the stuffle product is defined, for any $P \in \mathbb{C}\langle\langle Y \rangle\rangle$ such that $\langle P \mid 1_{Y^*}\rangle = 0$, is defined by $\exp_{\uplus}(P) := 1_{Y^*} + \frac{P}{1!} + \frac{P \uplus P}{2!} + \ldots + \frac{P^{\uplus n}}{n!} + \ldots$. In particular, from (50), one gets, for $k \geq 1$, $(zy_k)^* = \exp_{\uplus}\left(-\sum_{n\geq 1} y_{nk}\frac{(-z)^n}{n}\right)$.

6 Conclusion

Noncommutative symbolic calculus allows to get identities easy to check and to implement. With some amount of complex and functional analysis, it is possible to bridge the gap between symbolic, functional and number theoretic worlds. This was the case already for polylogarithms, harmonic sums and polyzetas. This is the project of this paper and will be pursued in forthcoming works.

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[36] *i.e.* $G(z_1 + z_2) = G(z_1) \uplus G(z_2); G(0) = 1_{Y^*}$.

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