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► To cite this version:

Zhiqing Kui, Jean Baccou, Jacques Liandrat. On the construction of multiresolution analyses Associated to general subdivision schemes. *Mathematics of Computation*, 2021, 90, pp.2185-2208. 10.1090/mcom/3646 . hal-03377373

HAL Id: hal-03377373

<https://hal.science/hal-03377373>

Submitted on 14 Oct 2021

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ON THE CONSTRUCTION OF MULTIREOLUTION ANALYSES ASSOCIATED TO GENERAL SUBDIVISION SCHEMES

ZHIQING KUI, JEAN BACCOU, AND JACQUES LIANDRAT

ABSTRACT. Subdivision schemes are widely used in numerical mathematics such as signal/image approximation, analysis and control of data or numerical analysis. However, to develop their full power, subdivision schemes should be incorporated into a multiresolution analysis that, mimicking wavelet analyses, provides a multi-scale decomposition of a function, a curve, or a surface. The ingredients needed to define a multiresolution analysis associated to a subdivision scheme are a decimation scheme and detail operators. Their construction is not straightforward as soon as the subdivision scheme is non-interpolatory.

This paper is devoted to the construction of decimation schemes and detail operators compatible with general subdivision schemes, including non-linear ones. Analysis of the performances of the constructed analyses is carried out. Some numerical applications are presented in the framework of image approximation.

Multiresolution, Subdivision schemes, Detail operators, Decimation schemes, Non-linear schemes

1. INTRODUCTION

Subdivision schemes have been introduced and developed in many fields, leading to specific schemes targeting smoothness, stability, shape preserving, manifold preserving and many other properties.

The complete framework into which subdivision schemes can develop their powerful properties is the multiresolution framework that, mimicking the wavelet framework, involves two other ingredients : a decimation operator and the detail operators. When subdivision schemes (only binary subdivision scheme will be considered in this paper) are uniform, interpolatory, linear and stationary, the decimation operator $\tilde{h}$ and detail operators $(g, \tilde{g})$ are easily defined. This is not the case as soon as the subdivision scheme is more general (e.g. [13]).

This paper is devoted to a generic construction of multiresolution frameworks for general subdivision schemes. The originality of the constructions relies on the development of a systematic approach that can be used to generate new types of multiresolution analysis associated to non-standard subdivision schemes. In terms of applications, these developments should allow the design of new frameworks for the decomposition and reconstruction of functions, signals and images.

The paper is organized as follows : we first provide an overview of the multiresolution framework associated to subdivision schemes (Section 2). Two examples of schemes for which no associated multiresolution existed are also given to emphasize the main motivation of our work. Then starting from first results presented in [15], we focus on the construction of decimation operators consistent with general subdivision schemes (Section 3). Section 4 is devoted to the construction of detail

operators while the analysis of linear and non-linear multiresolutions is provided in section 5. Different examples and applications are detailed in the last two sections.

2. OVERVIEW ON SUBDIVISION SCHEMES AS BASIC ELEMENTS OF THE MULTIREOLUTION FRAMEWORK

2.1. Subdivision schemes. A binary subdivision operator h [11] is defined through a real-valued sequence $(h_k)_{k \in \mathbb{Z}}$ having a finite number of non-zero values such that $(f_k)_{k \in \mathbb{Z}} \in l^\infty(\mathbb{Z}) \mapsto ((hf)_k)_{k \in \mathbb{Z}} \in l^\infty(\mathbb{Z})$ with

$$(hf)_k = \sum_{l \in \mathbb{Z}} h_{k-2l} f_l .$$

Any set $\{h_k : k_0 \leq k \leq k_1, k \in \mathbb{Z}\}$ containing all the non-zero values of $(h_k)_{k \in \mathbb{Z}}$ is called a mask of the operator h of length $k_1 - k_0 + 1$ and is denoted M_h .

Subdivision is generally iterated starting from an initial sequence $(f_k^{j_0})_{k \in \mathbb{Z}}$ to generate $(f_k^j)_{k \in \mathbb{Z}}$, $j > j_0$ as

$$(1) \quad f^{j+1} = hf^j.$$

At each scale j , the polygon $(k2^{-j}, f_k^j), k \in \mathbb{Z}$ is considered. The advantage of using subdivision for data prediction relies on the flexibility in the choice of the mask. The simplest strategy that consists in considering the same mask for every position, scale and data f^j (leading to linear uniform and stationary operators) [10] can be improved by introducing a dependence on k (non-uniform scheme) [9] or j (non-stationary scheme) [6]. The linearity can be also relaxed by taking into account the values of the sequence $(f_l^j)_{l \in \mathbb{Z}}$ for l in the vicinity of k [2].

The analysis of a subdivision scheme relies on the study of two important properties related to its convergence and stability. These notions are recalled in the following definition where $\|\cdot\|$ denotes any norm. When required, $\|\cdot\|_\infty$ stands for the infinity norm of sequences (l^∞ norm) or functions (L^∞ norm).

Definition 1 (Convergence of a subdivision scheme). *A subdivision scheme is uniformly convergent if*

$$\forall f \in l^\infty(\mathbb{Z}), \exists f^\infty \in C^0(\mathbb{R}), \text{ such that } \lim_{j \rightarrow \infty} \|h^j f - f^\infty(2^{-j} \cdot)\|_\infty = 0.$$

Definition 2 (Stability of a subdivision scheme). *A convergent subdivision scheme h is stable in the norm $\|\cdot\|$ if there exists a constant $C \in \mathbb{R}$ such that for all $f, f_\epsilon \in l^\infty(\mathbb{Z})$,*

$$\forall i \in \mathbb{N}, \quad \|h^i f - h^i f_\epsilon\| \leq C \|f - f_\epsilon\|.$$

Note that if the subdivision h is linear, the stability ($\exists C \in \mathbb{R}, \forall i \in \mathbb{N}, \|h^i\| \leq C$) is a direct consequence of the convergence of the subdivision scheme.

2.2. Multiresolution transforms. In the multiresolution framework, if $(V^j)_{j \in \mathbb{Z}}$ denotes a family of separable spaces with $f^j \in V_j$, relation (1) can be exploited to define the prediction from V^j to V^{j+1} ([14], [3], [4]). The connection from fine to coarse levels is performed thanks to a decimation operator denoted $\tilde{h}$. If $f^{j+1} \in V^{j+1}$ is obtained after subdivision i.e. $f^{j+1} = hf^j$, a decimation must satisfy the following consistency condition in order to recover f^j from f^{j+1} :

Definition 3 (Consistency property). *A decimation operator $\tilde{h}$ is said to be consistent with the subdivision scheme h if*

$$\tilde{h}h = I$$

where I stands for the identity operator.

However, if $f^j \in V^j$ is obtained after decimation of $f^{j+1} \in V^{j+1}$, hf^j usually does not coincide with f^{j+1} . Thus, a prediction error is introduced as :

$$(2) \quad e^{j+1} = f^{j+1} - h\tilde{h}f^{j+1}.$$

The relation $f^j = \tilde{h}f^{j+1}$ associated to formula (2) defines a mapping from V^{j+1} to $V^j \times V^{j+1}$ sending f^{j+1} to $\{f^j, e^{j+1}\}$. A left inverse is given by $f^{j+1} = hf^j + e^{j+1}$. This mapping can be reduced to a bijective mapping from V^{j+1} to $V^j \times W^j$ with W^j a suitable subspace of V^{j+1} . This leads to introduce $\tilde{g}$, a detail decimation operator and its right inverse $g : W^j \mapsto V^{j+1}$. The quantity $d^j = \tilde{g}f^{j+1}$ is called a detail.

The quadruplet of operators $(h, \tilde{h}, g, \tilde{g})$ fully defines a multiresolution associated to the subdivision scheme h and satisfies the relation

$$h\tilde{h} + g\tilde{g} = I.$$

This quadruplet is said to be **compatible** if and only if, moreover,

$$\tilde{h}h = I, \quad \tilde{g}g = I, \quad \tilde{h}g = 0 \text{ and } \tilde{g}h = 0.$$

Iterating, for a fixed level $j_0 \leq j$, the multi-scale decomposition of a sequence f^j is the element $\{f^{j_0}, d^{j_0}, d^{j_0+1}, \dots, d^{j-1}\}$. Similarly a reconstruction transform can be introduced to recover f^j from $\{f^{j_0}, d^{j_0}, d^{j_0+1}, \dots, d^{j-1}\}$.

The performance of the multiresolution process is controlled by the behavior of the prediction error and more precisely by the amount of small error values at each scale. Generally, the norm of the prediction error exponentially decays with the scale j , and the decay rate plays a key role in the sparsity of the multi-scale decomposition. The definition of the decay rate is recalled as follows.

Definition 4 (Decay of the norm of the prediction error). *A prediction error e^j is said to decay with a decay rate p in the norm $\|\cdot\|$ if and only if for all $f^j \in l^\infty(\mathbb{Z})$, there exists a constant C independent on j such that*

$$\forall j' < j, \|e^{j'}\| \leq C2^{-pj'}.$$

Since truncation is a key ingredient in image approximation algorithms, the stability has to be considered. In addition to Definition 2, this leads to the following stability definitions.

Definition 5 (Stability of a decimation scheme). *A decimation scheme $\tilde{h}$ is stable in the norm $\|\cdot\|$ if there exists a constant $C \in \mathbb{R}$ such that for all $f, f_\epsilon \in l^\infty(\mathbb{Z})$,*

$$\forall i \in \mathbb{N}, \quad \|\tilde{h}^i f - \tilde{h}^i f_\epsilon\| \leq C\|f - f_\epsilon\|.$$

Definition 6 (Stability of the multiresolution). *Let $\{f^j, d^j, \dots, d^{j-1}\}$ (resp. $\{f_\epsilon^j, d_\epsilon^j, \dots, d_\epsilon^{j-1}\}$) stand for the multiresolution decompositions of f^j (resp. f_ϵ^j).*

The **reconstruction** transform is said to be stable in the norm $\|\cdot\|$ if there exists a constant C such that for all $\{f^{j_0}, d^{j_0}, \dots, d^{j-1}\}$ and $\{f_\epsilon^{j_0}, d_\epsilon^{j_0}, \dots, d_\epsilon^{j-1}\}$,

$$\|f^j - f_\epsilon^j\| \leq C(\|f^{j_0} - f_\epsilon^{j_0}\| + \sum_{i=j_0}^{j-1} \|d^i - d_\epsilon^i\|).$$

The **decomposition** transform is said to be stable in the norm $\|\cdot\|$ if there exists a constant C such that for all (f^j, f_ϵ^j) ,

$$\|f^{j_0} - f_\epsilon^{j_0}\| + \sum_{i=j_0}^{j-1} \|d^i - d_\epsilon^i\| \leq C\|f^j - f_\epsilon^j\|.$$

The multiresolution is said to be stable if the associated decomposition transform and reconstruction transform are stable.

Subdivision-based multiresolution is an appealing framework for data analysis since it inherits the flexibility of the construction of subdivision operators. It has been considered as an interesting alternative to classical orthogonal or biorthogonal wavelet approaches in several scientific fields such as image processing [5] or computational mathematics [7]. However, the construction of the full multiresolution process is not straightforward when the subdivision scheme is non-interpolatory¹ or non-linear. The contributions of this paper are developed to specifically tackle this problem. To illustrate the typical situations addressed by our work, we first recall two types of schemes that will be further studied in Section 6 and emphasize the open issues concerning the construction of the associated multiresolution.

2.3. Examples of subdivision schemes. Two specific schemes are described in this section. The first one is the 4-point shifted Lagrange subdivision scheme [12]. It is non-interpolatory and is defined by

$$\begin{cases} f_{2k}^{j+1} = -\frac{7}{128}f_{k-1}^j + \frac{105}{128}f_k^j + \frac{35}{128}f_{k+1}^j - \frac{5}{128}f_{k+2}^j, \\ f_{2k+1}^{j+1} = -\frac{5}{128}f_{k-1}^j + \frac{35}{128}f_k^j + \frac{105}{128}f_{k+1}^j - \frac{7}{128}f_{k+2}^j. \end{cases}$$

The mask associated to this subdivision scheme is $M_h = \{h_k\}$, $-4 \leq k \leq 3$ with

$$(3) \quad M_h = \left\{ -\frac{5}{128}, -\frac{7}{128}, \frac{35}{128}, \frac{105}{128}, \frac{105}{128}, \frac{35}{128}, -\frac{7}{128}, -\frac{5}{128} \right\}.$$

The second one is the non-interpolatory and non-linear 4-point shifted PPH scheme [2]. Introducing

$$\Delta^2 f_k = (f_{k+1}^j - f_k^j) - (f_k^j - f_{k-1}^j),$$

it is defined as :

if $|\Delta^2 f_k| \leq |\Delta^2 f_{k+1}|$,

$$(4) \quad \begin{cases} f_{2k}^{j+1} &= -\frac{7}{128}f_{k-1}^j + \frac{105}{128}f_k^j + \frac{35}{128}f_{k+1}^j - \frac{5}{128}f_{k+2}^j - \frac{5}{64}DHA_k, \\ f_{2k+1}^{j+1} &= -\frac{5}{128}f_{k-1}^j + \frac{35}{128}f_k^j + \frac{105}{128}f_{k+1}^j - \frac{7}{128}f_{k+2}^j - \frac{7}{64}DHA_k, \end{cases}$$

if $|\Delta^2 f_k| > |\Delta^2 f_{k+1}|$,

$$(5) \quad \begin{cases} f_{2k}^{j+1} &= -\frac{7}{128}f_{k-1}^j + \frac{105}{128}f_k^j + \frac{35}{128}f_{k+1}^j - \frac{5}{128}f_{k+2}^j - \frac{7}{64}DHA_k, \\ f_{2k+1}^{j+1} &= -\frac{5}{128}f_{k-1}^j + \frac{35}{128}f_k^j + \frac{105}{128}f_{k+1}^j - \frac{7}{128}f_{k+2}^j - \frac{5}{64}DHA_k, \end{cases},$$

¹A subdivision scheme h is said to be interpolatory if $\forall f, \forall k, (hf)_{2k} = f_k$

where

$$DHA_k = H(\Delta^2 f_k, \Delta^2 f_{k+1}) - A(\Delta^2 f_k, \Delta^2 f_{k+1}), A(x, y) = \frac{x+y}{2},$$

$$H(x, y) = \frac{xy}{x+y}(\text{sign}(xy) + 1), \quad \text{and } \text{sign}(x, y) = \begin{cases} -1, & \text{if } xy < 0 \\ 1, & \text{if } xy \geq 0 \end{cases},$$

Note that the 4-point shifted PPH scheme can be considered as the linear 4-point shifted Lagrange scheme with a non-linear perturbation.

Both schemes are non-interpolatory. They are therefore well suited to take into account noise in the data. Moreover, the PPH approach is data-dependent. It leads to accurate prediction of data exhibiting non-regular behavior. However, up to now, the efficiency of these schemes within the multiresolution framework has been ignored since no multiresolution framework (i.e. the operators $\tilde{h}$, g and $\tilde{g}$) was available.

The two following sections are therefore devoted to the introduction of generic methods to derive $\tilde{h}$ and $(g, \tilde{g})$ for such general subdivision operators.

Before ending this section, we introduce some notations that will be used in the sequel :

- A linear decimation operator $\tilde{h}$ is defined through a real-valued sequence $(\tilde{h}_k)_{k \in \mathbb{Z}}$ having a finite number of non-zero values such that $(f_k)_{k \in \mathbb{Z}} \in l^\infty(\mathbb{Z}) \mapsto ((\tilde{h}f)_k)_{k \in \mathbb{Z}} \in l^\infty(\mathbb{Z})$ with

$$(\tilde{h}f)_k = \sum_{l \in \mathbb{Z}} \tilde{h}_{l-2k} f_l.$$

Any set of the form $\{\tilde{h}_k : k_0 \leq k \leq k_1, k \in \mathbb{Z}\}$ containing all non-zero values of $(\tilde{h}_k)_{k \in \mathbb{Z}}$ is called a mask of the operator $\tilde{h}$ of length $k_1 - k_0 + 1$ and is denoted $M_{\tilde{h}}$.

- For any operator h (resp. $\tilde{h}$) defined through the real value sequence $(h_k)_{k \in \mathbb{Z}}$ (resp. $(\tilde{h}_k)_{k \in \mathbb{Z}}$) and any integer $t \in \mathbb{N}$, the **translated operator** $T_t(h)$ (resp. $T_t(\tilde{h})$) is defined through the sequence $(h_{k-t})_{k \in \mathbb{Z}}$ (resp. $(\tilde{h}_{k-t})_{k \in \mathbb{Z}}$).
- We call (σ, σ') the pair of **subsampling operators** defined as

$$\forall e \in l^\infty(\mathbb{Z}), \quad \begin{cases} (\sigma e)_k = e_{2k+1}, \\ (\sigma' e)_k = e_{2k}, \end{cases} \quad k \in \mathbb{Z}.$$

and τ the **interlacing operator**,

$$\forall u, v \in l^\infty(\mathbb{Z}), \quad (\tau(u, v))_k = \begin{cases} u_i, & k = 2i + 1, \\ v_i, & k = 2i, \end{cases} \quad i \in \mathbb{Z}.$$

- We denote $\tilde{h}^e$ and $\tilde{h}^o$ the two operators associated to even and odd terms of a linear decimation operator $\tilde{h}^L$,

$$\forall k \in \mathbb{Z}, f \in l^\infty(\mathbb{Z}), \quad \begin{cases} (\tilde{h}^o f)_k = \sum_l \tilde{h}_{2l+1-2k}^L f_{2l+1} \\ (\tilde{h}^e f)_k = \sum_l \tilde{h}_{2l-2k}^L f_{2l} \end{cases}.$$

3. CONSTRUCTION OF CONSISTENT DECIMATION OPERATORS

In the sequel, the subdivision h is fixed but general.

3.1. Linear case. A generic method to construct the mask of any decimation consistent with a given linear uniform subdivision has been proposed in [15]. The main results are recalled in the two following propositions. The first one is devoted to the construction of a finite family of elementary operators while the second one describes how all consistent decimation operators can be recovered using linear combinations of translated versions of elementary operators.

Proposition 1. *Let h be a linear subdivision operator whose mask is constructed from the sequence*

$$\{h_{n-2\alpha}, h_{n-2\alpha+1}, \dots, h_n, h_{n+1}\}.$$

Introducing, $H_{M_h} =$

$$\begin{bmatrix} h_n & h_{n-2} & \cdots & h_{n-2\alpha} & 0 & \cdots & 0 \\ h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & 0 & \cdots & 0 \\ 0 & h_n & h_{n-2} & \cdots & h_{n-2\alpha} & \cdots & 0 \\ 0 & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & \cdots & 0 \\ \vdots & & & \vdots & & & \\ 0 & 0 & \cdots & h_n & h_{n-2} & \cdots & h_{n-2\alpha} \\ 0 & 0 & \cdots & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} \end{bmatrix},$$

if $\det(H_{M_h}) \neq 0$, there exists 2α consistent elementary decimation operators whose masks are of length 2α . These masks are given by each row of $H_{M_h}^{-1}$.

Proposition 2. *The subdivision operator h being fixed and satisfying the hypotheses of Proposition 1, let $\{\tilde{h}^{(i)}\}_{1 \leq i \leq 2\alpha}$ be the set of elementary consistent decimation operators. Then, any consistent decimation operator can be constructed as*

$$(6) \quad \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{I}} c_{i,t} T_{2t}(\tilde{h}^{(i)})$$

with

$$\forall t \in \mathcal{T} \subset \mathbb{Z}, \sum_{i \in \mathcal{I}} c_{i,t} = \delta_{t,0}, \text{ and } 0 \in \mathcal{T}.$$

The large choice of decimation masks generated by this approach is of prime importance in practice since it allows to tune some specific characteristics of the scheme according to given objectives.

The key ingredient in this construction of linear consistent decimation operators is the translation invariance of the subdivision mask. In the case of non-linear schemes, since this property is lost, the method introduced above is not tractable. This situation is studied in the next section.

3.2. Non-linear case. Introducing $(h^L, \tilde{h}^L)$ a couple of consistent linear subdivision and decimation operators, the following proposition provides a generic method to derive a non-linear decimation operator $\tilde{h}$ consistent with a non-linear subdivision scheme h .

Proposition 3. *Let h be a subdivision operator. If there exists a linear decimation $\tilde{h}^L$ so that $\tilde{h}^L h - I$ is contractive in the l^∞ norm², then for any $f^{j+1} \in l^\infty(\mathbb{Z})$, the fixed-point equation*

$$(7) \quad f^j = \tilde{h}^L f^{j+1} - (\tilde{h}^L h - I)f^j$$

has a unique solution.

Moreover, $\tilde{h} : f^{j+1} \mapsto f^j$ is a decimation operator consistent with h .

Proof. The existence and uniqueness of the solution of the fixed-point equation is a consequence of the Banach fixed point theorem since $\tilde{h}^L h - I$ is contractive.

Starting from f^{j+1} such that $f^{j+1} = h\hat{f}^j$, it is clear that $\hat{f}^j$ is a solution of equation (7). Since the solution is unique, $\hat{f}^j$ is the decimated sequence from f^{j+1} and (7) defines a consistent decimation. $\square$

Remark 1.

- (1) *Given f^{j+1} , the different choices of linear decimation operators $\tilde{h}^L$ lead to different f^j and associated prediction errors. This flexibility is exploited in the numerical tests of Section 7.*
- (2) *If $\tilde{h}^L$ is consistent with h , then $\tilde{h}^L h = I$ and the fixed-point equation reduces to $f^j = \tilde{h}^L f^{j+1}$.*

4. CONSTRUCTION OF DETAIL OPERATORS

In this section the construction of the multiresolution transforms is completed exhibiting a couple of operators $(g, \tilde{g})$ compatible with $(h, \tilde{h})$.

For linear subdivision schemes, the prediction error (2) belongs, by construction, to the kernel of the associated linear consistent decimation operator. This statement guarantees the existence of a couple of detail subdivision and detail decimation operators $(g, \tilde{g})$ which are compatible with the couple of subdivision and decimation operators $(h, \tilde{h})$.

A similar result can be derived for general subdivision and decimation constructed following Proposition 3. It is given by the next proposition,

Proposition 4. *Let h be a general subdivision operator and $\tilde{h}$ be a consistent decimation operator given by Proposition 3 with $\tilde{h}^L$ the involved linear decimation operator. The associated prediction error e^{j+1} satisfies*

$$(8) \quad \tilde{h}^L e^{j+1} = 0,$$

and

$$(9) \quad \tilde{h} e^{j+1} = 0.$$

Proof. Under the contraction condition, the unique solution for equation (7) is denoted $\hat{f}^j = \tilde{h} f^{j+1}$.

Then, the prediction error can be written as

$$\begin{aligned} e^{j+1} &= f^{j+1} - h\tilde{h}f^{j+1} \\ &= f^{j+1} - h^L \hat{f}^j - (h - h^L) \hat{f}^j \\ &= (I - h^L \tilde{h}^L) f^{j+1} - (I - h^L \tilde{h}^L)(h - h^L) \hat{f}^j. \end{aligned}$$

²The operator U is said to be contractive in the norm $\|\cdot\|$ if there exists $c \in \mathbb{R}, 0 < c < 1$, such that for all (u, v) , $\|Uu - Uv\| \leq c\|u - v\|$.

Applying $\tilde{h}^L$ and using the consistency relation leads to (8).
Let $w^j = \tilde{h}e^{j+1}$. Then

$$\begin{aligned} w^j &= \tilde{h}^L e^{j+1} - \tilde{h}^L (h - h^L) w^j \\ &= -(\tilde{h}^L h - I) w^j. \end{aligned}$$

According to the fixed-point theorem, $w^j = 0$ which leads to (9). $\square$

Then the detail operators can be constructed as follows,

Proposition 5. *Let h be a subdivision operator and $\tilde{h}$ a consistent decimation constructed following Proposition 3. If there exists a linear left inverse operator of $\tilde{h}^e$, denoted $(\tilde{h}^e)^{-1}$, then $(g, \tilde{g})$ defined as*

$$(10) \quad \begin{cases} \tilde{g} = \sigma(I - h\tilde{h}) \\ g = \tau(\cdot, -(\tilde{h}^e)^{-1}\tilde{h}^o \cdot) \end{cases}$$

are detail operators compatible with $(h, \tilde{h})$.

Proof. Thanks to Proposition 4, we have $\tilde{h}^L e^{j+1} = 0$ that can be written as $\tilde{h}^o \sigma e^{j+1} + \tilde{h}^e \sigma' e^{j+1} = 0$. Taking $d^j = \sigma e^{j+1}$, we get $\sigma' e^{j+1} = -(\tilde{h}^e)^{-1} \tilde{h}^o d^j$. Therefore, $e^{j+1} = \tau(\sigma e^{j+1}, \sigma' e^{j+1}) = \tau(d^j, -(\tilde{h}^e)^{-1} \tilde{h}^o d^j)$. Detail operators then read

$$\begin{cases} d^j = \sigma e^{j+1} = \tilde{g} f^{j+1} \\ e^{j+1} = g d^j \end{cases}$$

where $(g, \tilde{g})$ are defined by (10).

Finally, we prove the compatibility of $(h, \tilde{h}, g, \tilde{g})$:

According to Proposition 4, we have $\tilde{h}^L g = 0$ and $\tilde{h} g = 0$. Then $\tilde{g} g = \sigma(I - h\tilde{h})g = \sigma g - \sigma h\tilde{h}g = \sigma g = I$.

Under the consistency condition $\tilde{h}h = I$, we have $\tilde{g}h = \sigma(I - h\tilde{h})h = 0$, which concludes the compatibility proof. $\square$

Remark 2. *If h is an interpolatory subdivision, the subsampling operator $\tilde{h}$ defined by $\forall f, \forall k, (\tilde{h}f)_k = f_{2k}$ is consistent with h and $\tilde{h}^o$ stands for the identity operator.*

5. ANALYSIS

Here, we investigate the stability and the prediction error decay of linear and non-linear multiresolutions. In the first case, the stability is ensured as soon as the subdivision and the decimation are stable. It is not true in the non-linear framework where extra conditions are required (see [1] for example in the case of non-linear interpolatory schemes). However, we restrict in the next section the stability analysis to the subdivision and decimation operators. We refer to Section 7.1 for a numerical evidence of the stability of the non-linear shifted PPH multiresolution transforms.

5.1. Stability of the subdivision and of the decimation.

5.1.1. *Linear case.* In the linear case, the stability of the subdivision is a consequence of its convergence. We focus on the stability study of the decimation operator.

The following proposition provides a necessary and sufficient condition for stability.

Proposition 6. *A linear decimation operator $\tilde{h}$ is stable if and only if the **subdivision** of mask constructed from the sequence $2(\tilde{h}_l)_{l \in \mathbb{Z}}$ is stable.*

Proof. If $f^j = h^j f^0$, then

$$\begin{aligned} f_l^j &= \sum_{l_{j-1}} h_{l-2l_{j-1}} \sum_{l_{j-2}} h_{l_{j-1}-2l_{j-2}} \cdots \sum_{l_1} h_{l_2-2l_1} \sum_{l_0} h_{l_1-2l_0} f_{l_0}^0 \\ &= \sum_{l_0} h_{l-2^j l_0}^j f_{l_0}^0. \end{aligned}$$

The operator h is stable if and only if there exists $C \in \mathbb{R}$, such that $\forall l, \sum_{l_0} |h_{l-2^j l_0}^j| \leq C$.

Since $\sum_l |h_{l-2^j l_0}^j|$ is independent of l_0 , then

$$(11) \quad \forall l_0, \quad \sum_l |h_{l-2^j l_0}^j| \leq 2^j C.$$

If $f^0 = \tilde{h}^j f^j$, then

$$\begin{aligned} f_{l_0}^0 &= \sum_{l_1} \tilde{h}_{l_1-2l_0} \sum_{l_2} \tilde{h}_{l_2-2l_1} \cdots \sum_{l_{j-1}} \tilde{h}_{l_{j-1}-2l_{j-2}} \sum_{l_j} \tilde{h}_{l_j-2l_{j-1}} f_l^j \\ (12) \quad &= \sum_l \tilde{h}_{l-2^j l_0}^j f_l^j. \end{aligned}$$

If the subdivision is constructed from the sequence $2(\tilde{h}_l)_{l \in \mathbb{Z}}$, (11) is equivalent to

$$\sum_l |\tilde{h}_{l-2^j l_0}^j| = 2^{-j} \sum_l |h_{l-2^j l_0}^j| \leq C,$$

which is exactly the stability condition for the decimation $\tilde{h}$. □

The assumption on the stability of the subdivision associated to $2(\tilde{h}_l)_{l \in \mathbb{Z}}$ can be replaced by a general condition that can be easier to verify in practice. Let us first recall without proof the following useful lemma,

Lemma 1. *The decimation operator $\tilde{h}$ is stable if and only if, for some $i \in \mathbb{N}^*$, $\tilde{h}^i$ is stable.*

Proposition 7. *The decimation operator $\tilde{h}$ is stable if and only if there exists $i \in \mathbb{N}^*$, such that the subdivision h constructed from the sequence $2(\tilde{h}_l^i)_{l \in \mathbb{Z}}$ is stable.*

Proof. The proof is straightforward by taking $j = ik$ in the proof of Proposition 6, since (12) can be rewritten as

$$\begin{aligned} f_{l_0}^0 &= \sum_{l_1} \tilde{h}_{l_1-2l_0} \sum_{l_2} \tilde{h}_{l_2-2l_1} \cdots \sum_{l_{ik-1}} \tilde{h}_{l_{ik-1}-2l_{ik-2}} \sum_{l_{ik}} \tilde{h}_{l_{ik}-2l_{ik-1}} f_{l_{ik}}^{ik} \\ &= \sum_{l_i} \left(\sum_{l_{i-1}} \cdots \sum_{l_1} \tilde{h}_{l_{i-1}-2l_{i-2}} \cdots \tilde{h}_{l_1-2l_0} \right) \cdots \sum_{l_{ik}} \left(\sum_{l_{ik-1}} \cdots \sum_{l_{ik-i+1}} \tilde{h}_{l_{ik}-2l_{ik-1}} \cdots \tilde{h}_{l_{ik-i+1}-2l_{ik-i}} \right) f_{l_{ik}}^{ik} \\ &= \sum_{l_i} \tilde{h}_{l_i-2^i l_0}^i \cdots \sum_{l_{ik}} \tilde{h}_{l_{ik}-2^i l_{ik-i}}^i f_{l_{ik}}^{ik}. \end{aligned}$$

Lemma 1 provides the conclusion. $\square$

5.1.2. Non-linear case. In the non-linear case, there already exist some results for the stability of the subdivision [1]. We focus again on the decimation.

Proposition 8. *As above, we suppose that $h = h^L + h^N$ and note $\tilde{h}^L$ a stable decimation consistent with h^L such that $\tilde{h}^L h^N$ is $\mu_1 < 1$ Lipschitz and $(I + \tilde{h}^L h^N)^{-1}$ is β_0 Lipschitz.*

Let μ_k , $k > 1$, denote the Lipschitz constant of $(\tilde{h}^L)^k h^N$. If there exists $L > 1$ and $\alpha < 1$ such that for all $k \geq L$, $\mu_k \leq \alpha^k$ then, the non-linear decimation defined through the fixed point equation (7) is stable as soon as the constant $C_{\mu_2, \dots, \mu_L, \alpha, \beta_0} < 1$ where $C_{\mu_2, \dots, \mu_L, \alpha, \beta_0} = \beta_0 \left(\sum_{k=1}^{L-1} \mu_{k+1} + \frac{\alpha^{L+1}}{1-\alpha} \right)$.

Proof. Iterating equation (7) one gets that for all $p > 0$, $f^{j-p} = (\tilde{h}^L)^{p+1} f^{j+1} - \sum_{k=0}^p (\tilde{h}^L)^{k+1} h^N f^{j-p+k}$. It follows that for any couple of sequences (f^{j+1}, g^{j+1}) , if we note $\delta^l = \|f^l - g^l\|$, we get, using the stability of $\tilde{h}^L$ and the Lipschitz hypotheses,

$$\begin{aligned} \delta^j &\leq \beta_0 C \delta^{j+1}, \\ \forall p \geq 1, \delta^{j-p} &\leq \beta_0 C \delta^{j+1} + \beta_0 \sum_{k=1}^p \mu_{k+1} \delta^{j-p+k}, \\ &\leq \beta_0 C \delta^{j+1} + \beta_0 \left(\sum_{k=1}^{L-1} \mu_{k+1} \delta^{j-p+k} + \sum_{k=L}^p \alpha^{k+1} \delta^{j-p+k} \right). \end{aligned}$$

If we call $s_p = \max\{\delta^{j-k}, -1 \leq k \leq p\}$, we get that for all $p \geq 0$,

$$s_p \leq \beta_0 C s_{-1} + \beta_0 \left(\sum_{k=1}^{L-1} \mu_{k+1} + \frac{\alpha^{L+1} - \alpha^{p+2}}{1-\alpha} \right) s_{p-1}.$$

Since $\alpha < 1$, as soon as $C_{\mu_2, \dots, \mu_L, \alpha, \beta_0} < 1$, then, s_p is bounded by $C' s_{-1}$. That concludes the proof. $\square$

5.2. Decay of the prediction error. In the linear case, the decay of the prediction error for consistent scheme $(h, \tilde{h})$ was studied in [15] and is recalled in the following proposition,

Proposition 9. *Let h be a linear subdivision scheme such that*

$$\forall n = 0, 1, 2, \dots, p-1, \quad \sum_{l \in \mathbb{Z}} h_l l^n (-1)^l = 0,$$

then for any consistent linear decimation $\tilde{h}$, the decay rate (defined in Definition 4) in the l^∞ norm of the associated prediction error is at least p .

Remark 3. Since the derivation of Proposition 9 relies on the Taylor expansion, this theoretical decay rate is significant only when the data are regular.

This result can be generalized to the non-linear case as follows :

Proposition 10. Let h be a non-linear subdivision scheme with $h = h^L + h^N$ where h^L verifies Proposition 9 and such that for all $f^j \in l^\infty(\mathbb{Z})$, there exists a constant C independent on j such that

$$(13) \quad \|h^N f^j\|_\infty \leq C 2^{-q(j+1)}.$$

If $\tilde{h}$ is a stable (in the l^∞ norm) and consistent decimation operator constructed according to Proposition 3, then the decay rate of the associated prediction error is at least $\min(p, q)$.

Proof. For a consistent couple $(h, \tilde{h})$ deduced by Proposition 3,

$$\hat{f}^j = \tilde{h} f^{j+1} = \tilde{h}^L f^{j+1} - \tilde{h}^L h^N \hat{f}^j.$$

The associated prediction error can be written as

$$e^{j+1} = (I - h^L \tilde{h}^L) f^{j+1} - (I - h^L \tilde{h}^L) h^N \hat{f}^j.$$

Since $(I - h^L \tilde{h}^L)$ is a linear operator and $\tilde{h}$ is stable, there exists constants C_1 and C_2 such that

$$\|e^{j+1}\| \leq C_1 2^{-p(j+1)} + C_2 2^{-q(j+1)} \leq (C_1 + C_2) 2^{-\min(p, q)(j+1)},$$

and the decay rate is at least $\min(p, q)$. □

6. EXAMPLES

We focus in this section on the linear and non-linear multiresolutions associated to the subdivision schemes recalled in Section 2.3. For each of them, the construction of a consistent decimation is addressed. Exploiting the results established in the previous section, a stability analysis is performed and the prediction error decay of the multiresolutions is studied as well.

6.1. 4-point Shifted Lagrange Scheme. Construction of consistent linear decimations.

Applying Proposition 1, we get 6 elementary decimations whose masks are given by the rows of $\tilde{H}_{M_h}$ with

$$\tilde{H}_{M_h} = \begin{bmatrix} \frac{24367}{1152} & -\frac{63605}{1152} & \frac{31115}{576} & -\frac{10325}{576} & -\frac{4165}{1152} & \frac{2975}{1152} \\ \frac{2975}{175} & -\frac{4165}{245} & \frac{1771}{875} & -\frac{565}{245} & -\frac{1152}{133} & \frac{1152}{95} \\ \frac{1152}{95} & -\frac{1152}{133} & -\frac{576}{245} & -\frac{576}{875} & -\frac{1152}{245} & \frac{1152}{175} \\ \frac{1152}{175} & -\frac{1152}{245} & -\frac{576}{565} & \frac{576}{1771} & -\frac{1152}{4165} & \frac{1152}{2975} \\ \frac{1152}{2975} & -\frac{1152}{4165} & -\frac{576}{10325} & \frac{576}{31115} & -\frac{1152}{63605} & \frac{1152}{24367} \\ 1152 & -1152 & -576 & 576 & -1152 & 1152 \end{bmatrix}.$$

With equation (6), a symmetrical consistent decimation operator $\tilde{h}$ of length 8 can be formed by combining the 3rd and 4th rows of $\tilde{H}_{M_h}$ as

$$(14) \quad M_{\tilde{h}} = \{\tilde{h}_{-4}, \tilde{h}_{-3}, \tilde{h}_{-2}, \tilde{h}_{-1}, \tilde{h}_0, \tilde{h}_1, \tilde{h}_2, \tilde{h}_3\} \\ = \left\{ \frac{95}{2304}, -\frac{133}{2304}, -\frac{35}{256}, \frac{1505}{2304}, \frac{1505}{2304}, -\frac{35}{256}, -\frac{133}{2304}, \frac{95}{2304} \right\}.$$

This operator is used in the next section to construct a consistent decimation associated to the non-linear scheme PPH.

Stability. The shifted Lagrange scheme has been proven to be convergent (and therefore stable) in [12]. We have moreover

Proposition 11. *The decimation operator given by (14) is stable.*

Proof. Considering the subdivision operator constructed from $2(\tilde{h}_l^3)_{l \in \mathbb{Z}}$ where $\tilde{h}$ is given by (14), a direct calculation gives

$$\forall i_1, i_2, \dots, i_9 \in \{0, 1\}, \quad \max \left(\left(\frac{1}{2} \right)^9 \|A_{i_1}^{(1)} A_{i_2}^{(1)} \cdots A_{i_9}^{(1)}\| \right) = 0.98334 < 1,$$

where $A_0^{(1)}, A_1^{(1)}$ are associated refinement matrices for differences. According to [11], this proves that the subdivision $2(\tilde{h}_l^3)_{l \in \mathbb{Z}}$ is convergent. Therefore, $\tilde{h}$ is stable thanks to Proposition 7. $\square$

Remark 4. *The subdivision related to the sequence $2(\tilde{h}_l)_{l \in \mathbb{Z}}$ with $\tilde{h}$ given by (14) is not convergent since there exists eigenvalue of the associated refinement matrices larger than 1. The proposed pair of consistent and stable subdivision/decimation operators is therefore out of the range covered by biorthogonal multiresolutions described in [8].*

Decay of the prediction error. A direct calculation provides that the mask (3) satisfies Proposition 9 with $p = 5$. Therefore, the decay rate of the prediction error associated to the 4-point shifted Lagrange subdivision and any consistent decimation scheme is 5.

6.2. 4-point Shifted PPH Scheme. To derive a consistent decimation operator for this scheme, following Proposition 4, a linear decimation operator $\tilde{h}^L$ consistent with the linear part of the subdivision scheme h^L and such that $\tilde{h}^L h^N$ is contractive should be constructed. Before introducing the main result, two lemmas are required:

Lemma 2. $\forall x, y, a, b \in \mathbb{R}$,

$$|(H(x, y) - A(x, y)) - (H(a, b) - A(a, b))| \leq 2 \cdot \max(|x - a|, |y - b|).$$

Proof. we distinguish different cases:

(1) $x, a > 0, y, b < 0$, then $H(x, y) = H(a, b) = 0$,

$$\begin{aligned} & |(H(x, y) - A(x, y)) - (H(a, b) - A(a, b))| \\ & \leq |A(x, y) - A(a, b)| \\ & \leq \max(|x - a|, |y - b|). \end{aligned}$$

(2) $x, y, a > 0, b < 0$, then $H(a, b) = 0$,

$$\begin{aligned}
& |(H(x, y) - A(x, y)) - (H(a, b) - A(a, b))| \\
&= \left| \frac{a-x}{2} + \frac{b-y}{2} + \frac{2xy}{x+y} \right| \\
&\leq \left| \frac{a-x}{2} \right| + \left| \frac{b}{2} + \frac{y}{2} \frac{x-y}{x+y} \right| + \left| \frac{1}{2} \frac{2xy}{x+y} \right| \\
&\leq \frac{1}{2}|x-a| + \frac{1}{2}|y-b| + \max(|x|, |y|) \\
&\leq 2 \cdot \max(|x-a|, |y-b|).
\end{aligned}$$

(3) $x, y, a, b > 0$ or $x, y > 0, a, b < 0$,

$$\begin{aligned}
& |(H(x, y) - A(x, y)) - (H(a, b) - A(a, b))| \\
&= \left| \left(\frac{2xa}{(x+y)(a+b)} - \frac{1}{2} \right) (y-b) + \left(\frac{2yb}{(x+y)(a+b)} - \frac{1}{2} \right) (x-a) \right| \\
&\leq \max \left(\left| \frac{2xa+2yb}{(x+y)(a+b)} - 1 \right|, \left| \frac{2xa-2yb}{(x+y)(a+b)} \right| \right) \cdot \max(|x-a|, |y-b|) \\
&\leq 2 \cdot \max(|x-a|, |y-b|).
\end{aligned}$$

and we can conclude the other cases by symmetry. $\square$

Lemma 3. Given $x, y, a, b \in \mathbb{R}$, if $(|x| - |y|)(|a| - |b|) < 0$, then $\forall p, q \in \mathbb{R}$,

$$|p(H(x, y) - A(x, y)) - q(H(a, b) - A(a, b))| \leq \max(|p|, |q|) \cdot \max(|x-a|, |y-b|).$$

Proof. We first consider $p = q = 1$,

$$|(H(x, y) - A(x, y)) - (H(a, b) - A(a, b))| \leq \max(|x-a|, |y-b|),$$

under the condition $(|x| - |y|)(|a| - |b|) < 0$ by distinguishing different cases:

(1) $xy < 0, ab < 0$, then $H(x, y) = H(a, b) = 0$,

$$\begin{aligned}
& |(H(x, y) - A(x, y)) - (H(a, b) - A(a, b))| \\
&\leq \left| \frac{x-a}{2} + \frac{y-b}{2} \right| \\
&\leq \max(|x-a|, |y-b|).
\end{aligned}$$

(2) $xy > 0, ab < 0$ then $H(a, b) = 0$,

$$\begin{aligned}
& |(H(x, y) - A(x, y)) - (H(a, b) - A(a, b))| \\
&\leq \left| \frac{x-y}{2} \right| + \left| \frac{a+b}{2} \right| \\
&\leq \max(|x-a|, |y-b|).
\end{aligned}$$

(3) $xy > 0, ab > 0$,

$$\begin{aligned}
& |(H(x, y) - A(x, y)) - (H(a, b) - A(a, b))| \\
&= \left| \frac{xa+xb+ya-yb}{2(x+y)(a+b)} (x-a) \right| + \left| \frac{-xa+xb+ya+yb}{2(x+y)(a+b)} (y-b) \right| \\
&\leq \frac{1}{2}|x-a| + \frac{1}{2}|y-b| \\
&\leq \max(|x-a|, |y-b|).
\end{aligned}$$

and we can conclude the other cases by symmetry.

In the other hand, $(|x| - |y|)(|a| - |b|) < 0$ leads to

$$|(H(x, y) - A(x, y)) + (H(a, b) - A(a, b))| \leq \max(|x - a|, |y - b|),$$

and

$$|H(x, y) - A(x, y)| \leq \max(|x - a|, |y - b|).$$

Then for $p, q \in \mathbb{R}$ satisfying,

(1) $p > q > 0$,

$$\begin{aligned} & |p(H(x, y) - A(x, y)) - q(H(a, b) - A(a, b))| \\ &= |q(H(x, y) - A(x, y)) - q(H(a, b) - A(a, b)) + (p - q)(H(x, y) - A(x, y))| \\ &\leq (|q| + |p - q|) \cdot \max(|x - a|, |y - b|) \\ &\leq p \cdot \max(|x - a|, |y - b|). \end{aligned}$$

(2) $p > 0 > q$, $|p| > |q|$,

$$\begin{aligned} & |p(H(x, y) - A(x, y)) - q(H(a, b) - A(a, b))| \\ &= |-q(H(x, y) - A(x, y)) - q(H(a, b) - A(a, b)) + (p + q)(H(x, y) - A(x, y))| \\ &\leq (|q| + |p + q|) \cdot \max(|x - a|, |y - b|) \\ &\leq p \cdot \max(|x - a|, |y - b|). \end{aligned}$$

and we can conclude the other cases by symmetry. $\square$

Construction of consistent non-linear decimations. We have:

Proposition 12. *Writing the 4-point shifted PPH subdivision scheme as $h = h^L + h^N$ with h^L given by (3) and taking $\tilde{h}^L$ given by (14), then $\tilde{h}^L h^N$ is contractive. Therefore, equation (7) defines a consistent decimation.*

Proof. In order to prove the contractivity, we focus on $\|\tilde{h}^L h^N u - \tilde{h}^L h^N v\|$. Let's denote $p = 2L_2(\frac{1}{4}) = 2L_{-1}(\frac{3}{4}) = -\frac{5}{64}$ and $q = 2L_2(\frac{3}{4}) = 2L_{-1}(\frac{1}{4}) = -\frac{7}{64}$.

(1) if $|\Delta^2 u_k| \leq |\Delta^2 u_{k+1}|$ and $|\Delta^2 v_k| \leq |\Delta^2 v_{k+1}|$,

$$\begin{aligned} & |(\tilde{h}^L h^N u)_l - (\tilde{h}^L h^N v)_l| \\ &\leq \sum_k \left| p \cdot \tilde{h}_{2k-2l} + q \cdot \tilde{h}_{2k+1-2l} \right| |DHA_k^u - DHA_k^v|. \end{aligned}$$

(2) if $|\Delta^2 u_k| > |\Delta^2 u_{k+1}|$ and $|\Delta^2 v_k| > |\Delta^2 v_{k+1}|$,

$$\begin{aligned} & |(\tilde{h}^L h^N u)_l - (\tilde{h}^L h^N v)_l| \\ &\leq \sum_k \left| q \cdot \tilde{h}_{2k-2l} + p \cdot \tilde{h}_{2k+1-2l} \right| |DHA_k^u - DHA_k^v|. \end{aligned}$$

(3) if $(|\Delta^2 u_k| - |\Delta^2 u_{k+1}|)(|\Delta^2 v_k| - |\Delta^2 v_{k+1}|) < 0$, according to Lemma 3,

$$\begin{aligned} & |(\tilde{h}^L h^N u)_l - (\tilde{h}^L h^N v)_l| \\ &\leq \sum_k \max \left(|p \cdot \tilde{h}_{2k-2l} + q \cdot \tilde{h}_{2k+1-2l}|, |q \cdot \tilde{h}_{2k-2l} + p \cdot \tilde{h}_{2k+1-2l}| \right) \\ &\quad \cdot \max(|\Delta^2 u_{k+1} - \Delta^2 v_{k+1}|, |\Delta^2 u_k - \Delta^2 v_k|). \end{aligned}$$

Since

$$|DHA_k^u - DHA_k^v| \leq 8 \cdot \|u - v\|,$$

$$\max(|\Delta^2 u_{k+1} - \Delta^2 v_{k+1}|, |\Delta^2 u_k - \Delta^2 v_k|) \leq 4 \cdot \|u - v\|,$$

combining the previous cases with (14) leads to

$$\|\tilde{h}^L h^N u - \tilde{h}^L h^N v\| \leq \frac{307}{384} \|u - v\|.$$

□

Stability. The shifted PPH scheme has been proven to be stable in [2]. Concerning the decimation, Proposition 8 requires evaluating the Lipschitz constants of several operators involved in the non-linear decimation. These constants are numerically estimated from a set of sequences constructed by random perturbations (10 realizations of a gaussian noise, $\mathcal{N}(0, 10)$) of the 260th column of the *man* image depicted on Figure 2. They are provided on Table 1.

Operators	Lipschitz constant
$(I + \tilde{h}h^N)^{-1}$	$\beta_0 = 1.12$
$\tilde{h}^L h^N$	$\mu_1 = 0.34$
$(\tilde{h}^L)^2 h^N$	$\mu_2 = 0.22$
$(\tilde{h}^L)^3 h^N$	$\mu_3 = 0.16$
$(\tilde{h}^L)^4 h^N$	$\mu_4 = 0.1$
$(\tilde{h}^L)^5 h^N$	$\mu_5 = 0.07$
$\{(\tilde{h}^L)^k h^N\}_{k \geq 6}$	$\alpha = 0.67$

TABLE 1. Numerical estimation of the Lipschitz constants associated to the operators involved in the non-linear decimation.

It follows that $C_{\mu_2, \dots, \mu_5, \alpha, \beta_0} = 0.92$ that leads to the stability of the non-linear decimation.

Decay of the prediction error. According to Proposition 10, and assuming that the decimation is stable, a direct calculation provides that (13) is satisfied with $q = 4$ for PPH scheme. Therefore, the decay rate of the prediction error associated to the 4-point shifted PPH subdivision and to the consistent decimation given by Proposition 3 with linear decimation (14) is 4.

A numerical estimate of the decay rate of the prediction error is carried out for the shifted Lagrange and the shifted PPH schemes (Figure 1). Two sequences of data are considered : one is a sampling of the function $\sin(x)$ in the interval $[1, 10]$ which is considered as a regular sequence, the other one is a line from image *man* presented in the left of Figure 2, which is considered as a non-regular sequence.

FIGURE 1. Logarithm of the prediction error norm $\|e^j\|$ versus the scale j associated to a regular sequence (Top) and a non-regular sequence (Bottom). For comparison, the slopes of the solid red line, the dash-dot black line and the solid black line are respectively 4, 5 and 1

The numerical decay of the prediction error is in full agreement with the theoretical one. For the regular data, the numerical evaluation of the slope is 4.420 for PPH and 5.173 for Lagrange. For the non-regular one, it is close to 1 (0.960 for PPH and 0.917 for Lagrange).

7. APPLICATION TO IMAGE APPROXIMATION

In this section, the capability of the two new linear and non-linear multiresolutions for the approximation of real and geometric images (Figures 3 and 4) is investigated.

Starting from the image of size 512×512 ($j = 9$), several decompositions are performed until $j_0 = 5$. Then, after truncation of the detail coefficients with different thresholds, the reconstruction transform is applied. Section 7.1 is first devoted to the numerical stability study of the shifted PPH multiresolution transforms. The performance of the two multiresolutions is studied in Section 7.2 and a comparison with existing approaches is given.

7.1. Stability of the 4-point shifted PPH multiresolution. We first provide a numerical evidence of the stability of the decomposition transform. Given a sequence $f^j = (f_k^j)_{k \in \mathbb{Z}}$, we denote $f_\epsilon^j = ((f_\epsilon^j)_k)_{k \in \mathbb{Z}}$ a perturbed sequence and its associated decomposition $\{f_\epsilon^{j_0}, d_\epsilon^{j_0}, d_\epsilon^{j_0+1}, \dots, d_\epsilon^{j-1}\}$. The decomposition stability constant for the l_1 -norm is defined as

$$C_s = \frac{\|f^j - f_\epsilon^j\|_1}{\|f^{j_0} - f_\epsilon^{j_0}\|_1 + \sum_{i=j_0}^{j-1} \|d^i - d_\epsilon^i\|_1}.$$

This constant is evaluated for different columns of the images *man* (Figure 3) and *geometric* (Figure 4) with a perturbation generated by a white gaussian noise ($\mathcal{N}(0, 10)$). Figure 2 shows an example of f^j constructed from the 260-th column of each image. The numerical estimates of the decomposition stability constant for $j = 9$ and $j_0 = 5$ are shown in Table 2 and Table 3 for each image considering shifted PPH and shifted Lagrange schemes.

FIGURE 2. Sequence constructed from the 260-th column of image *man* (left) and image *geometric* (right)

column index	50	120	190	260	330	400	470
C_s - Lagrange	1.3256	1.3646	1.3420	1.3456	1.2602	1.3248	1.3647
C_s - PPH	1.2337	1.2668	1.2307	1.1794	1.2353	1.2920	1.2679

TABLE 2. Estimation of the stability constant for the decomposition associated to the shifted PPH and the shifted Lagrange schemes based on image *man*

column index	50	120	190	260	330	400	470
C_s - Lagrange	1.3011	1.2060	1.3170	1.2617	1.2334	1.3235	1.2929
C_s - PPH	1.2049	1.2296	1.2053	1.1848	1.2748	1.2904	1.2381

TABLE 3. Estimation of the stability constant for the decomposition associated to the shifted PPH and the shifted Lagrange schemes based on image *geometric*

Since the linear shifted Lagrange decomposition is known to be stable, the similarity between the evaluations of the stability constants associated to the two schemes is a good indicator for the stability of the non-linear shifted PPH decomposition.

The same type of evaluation can be performed for the shifted PPH reconstruction and leads to the same conclusion concerning the stability.

7.2. Performance of the multiresolutions. For sake of comparison, multiresolutions constructed from interpolatory schemes are also considered. The different multiresolutions are :

- **interpolatory Lagrange** : The 4-point interpolatory Lagrange subdivision scheme associated to the mask $\{-\frac{1}{16}, 0, \frac{9}{16}, 1, \frac{9}{16}, 0, -\frac{1}{16}, 0\}$ and the consistent subsampling decimation σ' .
- **shifted Lagrange** : The 4-point shifted Lagrange subdivision scheme given by (3) and the consistent decimation given by (14).
- **interpolatory PPH** : The 4-point interpolatory PPH subdivision scheme [2] and the consistent subsampling decimation σ' .
- **shifted PPH** : The 4-point shifted PPH subdivision scheme given by (4)-(5) and the consistent decimation constructed using Proposition 3 involving the linear decimation (14).

With the subdivisions and decimations defined above and the associated details subdivisions and decimations given by (10), the construction of the four corresponding compatible multiresolutions is completed.

Reconstructed and original images are compared by computing the so-called PSNR (Peak Signal Noise Ratio) defined as

$$PSNR = 10 \log_{10} \left(\frac{(\text{Maximum Signal Value})^2}{\text{Mean Squared Error}} \right)$$

versus the compression ratio defined as the ratio between the size of the original image and the size of the compressed image.

The results on the first image (Figure 3) show that the shifted schemes (that are non-interpolatory) exhibit a better performance than that of interpolatory ones. Moreover, the shifted PPH scheme outperforms the linear shifted one.

FIGURE 3. Top : Test image *man*, Bottom : PSNR versus compression ratio for interpolatory Lagrange, shifted Lagrange, interpolatory PPH and shifted PPH multiresolutions.

This difference between shifted Lagrange and shifted PPH schemes is more significant considering a sketchy image such as the geometric one of Figure 4.

FIGURE 4. Top : Test image *geometric*, Bottom : PSNR versus compression ratio for shifted Lagrange and shifted PPH multiresolutions.

The result stated by Proposition 3 allows different choices of linear decimation operators and therefore different prediction errors and details. To evaluate the effect of this choice on the performance of the 4-point shifted PPH multiresolution, three linear decimations leading to consistent non-linear decimations are considered in the sequel :

- **L8** : decimation $\tilde{h}^L$ of length 8 given by (14).
- **L12** : decimation $\tilde{h}^L$ of length 12 given by

$$\left\{ \frac{19}{16128}, -\frac{19}{11520}, \frac{19}{576}, -\frac{19}{576}, -\frac{2623}{16128}, \frac{7639}{11520}, \frac{7639}{11520}, -\frac{2623}{16128}, -\frac{19}{576}, \frac{19}{576}, -\frac{19}{11520}, \frac{19}{16128} \right\}$$
 which is consistent with the 4-point shifted Lagrange and satisfies the contraction of Proposition 3 with constant $\frac{15481}{20160}$.
- **Quarter** : decimation $\tilde{h}^L$ given by $\left\{ -\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, -\frac{1}{4} \right\}$ which satisfies the contraction of Proposition 3 with constant $\frac{48}{64}$.

Figure 5 displays the evolution of the PSNR with respect to the compression ratio for those multiresolutions.

FIGURE 5. PSNR versus compression ratio for the 4-point shifted PPH subdivision scheme coupled with three different consistent decimation operators, *man* image

It appears that the choice of the linear decimation has a non negligible effect on the capability of the multiresolution. In this test, the decimation **L8**, given by (14), leads to the best results. This clearly indicates that in practice, the choice of the linear decimation has to be carefully performed before applying Proposition 3 to optimize the performance of the non-linear multiresolution. This point will be studied in future investigations.

8. CONCLUSION

In this paper, we proposed a construction of a complete multiresolution framework associated to a general subdivision scheme. A method was first introduced to generate consistent decimation operators, starting from linear subdivision. Then we proposed a generalization to consistent decimations associated to any non-linear subdivision scheme h . It was based on the existence of a linear decimation $\tilde{h}^L$ such that $\tilde{h}^L h - I$ is contractive. In order to fully characterize the new multiresolutions, compatible detail operators were also constructed. The resulting framework leads to original multi-scale decomposition and reconstruction of data that can benefit from the flexibility of subdivision schemes. Various results have been provided concerning the stability of these operators and the decay of the details. Finally, examples of constructions were developed. They are associated to non-interpolatory schemes for which no multiresolution framework was available before. Their application to image approximation problem illustrated their promising potential compared to classical multiresolution approach.

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E-mail address: zhiqing.kui@centrale-marseille.fr

E-mail address: jean.baccou@irsn.fr

E-mail address: jacques.liandrat@centrale-marseille.fr

(ECM) AIX MARSEILLE UNIV., CNRS, CENTRALE MARSEILLE, I2M, UMR 7353, 13451 MARSEILLE, FRANCE

(IRSN) INSTITUT DE RADIOPROTECTION ET DE SÛRETÉ NUCLÉAIRE(IRSN), PSN-RES/SEMIA/LIMAR, CE CADARACHE, 13115 SAINT PAUL LES DURANCE, FRANCE