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Piecewise derivatives versus short memory concept: Analysis and application

Abdon ATANGANA\(^1\), Seda İĞRET ARAZ\(^{1,2}\)
\(^1\)AtanganaA@ufs.ac.za, Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, South Africa
\(^2\)Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

Abdon ATANGANA\(^1\), Seda İĞRET ARAZ\(^{1,2}\)
\(^1\)AtanganaA@ufs.ac.za, Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, South Africa
\(^2\)Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

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Abstract
We have provided a detailed analysis to show the fundamental difference between the concept of short memory and piecewise differential and integral operators. While the concept of short memory leads to different long tails in different intervals of time or space as results of power law with different fractional orders. The concept of piecewise helps to depict crossover behaviours of different patterns. We presented some examples with different numerical simulations. In some cases, models with piecewise led to crossover behaviours from deterministic to stochastics which is indeed the reason this concept was introduced.

Keywords: Piecewise calculus, short memory concept, crossover behaviours, chaos and epidemiology.

1 Introduction
When searching the literature, we have notice that several propositions has been made by several researchers to find kernels that can be used to obtain fractional differential operators. The main reason for this is that real world problems show sign of processes resembling behaviors of some mathematical functions. Riemann, Liouville, Cauchy and Abel works lead a fractional calculus with power law kernel. Their work was latter modified by Caputo; this version has been used in many fields of science because of its ability to allow classical initial conditions [1]. Prabhakar suggested a different kernel as product of power law and the generalized Mittag-Leffler function with three parameters. This version has also attracted attention of many researchers, studies have been done on theory as well as on applications. Indeed, the two kernels have their specific values, for example power law helps only to replicate processes exhibiting power law behaviors. While the product of the power law and the generalized three-parameters Mittag-Leffler has also his field of application [2]. As nature is complex, a new kernel was suggested by Caputo and Fabrizio, a special exponential kernel with Delta Dirac properties. A differential operator that is in fashion nowadays due to its ability to replicate process following fading memory. Indeed,
this kernel brought a new shift into fractional calculus, as the concept of fractional derivative with non-
singular kernel was introduced [3]. A point made by some researcher about the non-fractionality of the
kernel led to a new kernel, the Generalized Mittag-Leffler function with one parameter. This version was
suggested by Atangana and Baleanu, another new step forward in the field of fractional calculus. The
operators have been used in several field of studies with great success [4]. While looking at nature and
its complexities, one can with no doubt conclude that these suggested kernels are not enough to predict
all complex behaviours of our universe. On this note, one will proceed to searching different kernel or
modified kernel, or class of functions that will be used to introduce new differential operators. Sabatier
recently presented some variant of kernels that will also open new doors for investigation [5]. Addition
to these outstanding contributions, several other ideas were suggested, for example the concept of short
memory was suggested, a fractional derivative in Caputo sense is defined for different values of fractional
orders. The idea is initiated to have a different type of variable order derivative unlike the well-known
version that considers a fractional order to be a function if time. This case was suggested by Wu et al.
[6] and applied in chaos. On the other hand, researchers have noticed that several real-world problems
exhibit processes with different behaviors as function of time and space. A particular case is a passage
from deterministic to stochastic, or from power law to exponential decay. It was noted that, existing dif-
ferential operators may not be able to account for these behaviors, thus piecewise differential and integral
operators were introduced to deal with problems exhibiting crossover behaviors [7]. The main aim of this
note is to provide a critical analysis, possible applications, advantages, and disadvantages of these two
concepts.

2 Motivation for piecewise derivatives

We will illustrate the motivation with same examples.

Death body decay in different temperature

Consider a corpse found in a snowish place, assume that such body has been found after 20 days. The
corpse is taken and brought to the house and kept in normal temperature for few days later put in
mortuary later the corpse is buried. The main aim here is to replicate the process of decay. The first
part will provide a very low decay. The second part will provide a fast decay, the third part will again
provide fast process and finally slow process. Indeed slow and fast processes can be characterized by
some mathematical functions. Power-law function is

$$ t^{-\alpha}. $$

Exponential decay function with Dirac-Delta property is

$$ \frac{1}{1-\alpha} \exp \left[ -\frac{\alpha}{1-\alpha} t \right]. $$

However slow and later fast flow this crossover can be modeled using

$$ \frac{1}{1-\alpha} E_\alpha \left[ -\frac{\alpha}{1-\alpha} t^\alpha \right]. $$

The process can be divided in several interval to capture each behavior. In the first part, one can have

$$ C_0 D_1^\alpha y(t) = -\lambda_1 y(t) \text{ if } 0 \leq t \leq T_1. $$
The second process can be
\[
\frac{CF}{T_1} D_t^\alpha y (t) = -\lambda_2 y (t) \quad \text{if} \quad T_1 \leq t \leq T_2. \tag{5}
\]
The two last parts will be characterized by
\[
\frac{ABC}{T_2} D_t^\alpha y (t) = -\lambda_3 y (t) \quad \text{if} \quad T_2 \leq t \leq T. \tag{6}
\]
Thus the whole process will be a system with crossover behaviors
\[
\begin{aligned}
\frac{C}{0} D_t^\alpha y (t) &= -\lambda_1 y (t) \quad \text{if} \quad 0 \leq t \leq T_1, \\
\frac{CF}{T_1} D_t^\alpha y (t) &= -\lambda_2 y (t) \quad \text{if} \quad T_1 \leq t \leq T_2, \\
\frac{ABC}{T_2} D_t^\alpha y (t) &= -\lambda_3 y (t) \quad \text{if} \quad T_2 \leq t \leq T.
\end{aligned}
\tag{7}
\]
Therefore in general the concept of piecewise derivative was introduced.

### 3 Short memory concept

In this section, we present definition of both short memory and piecewise differentiation. For short memory case, the idea was already discussed, for example a paper published by Deng in 2007 has already discussed the short memory principle, which was also called fixed memory principle, logarithmic memory principle. An alternative idea was already discussed in 2001 by Fordani called rested memory concept. We are more interested in the paper published by Wu and co-authors, we claimed to have introduced a new fractional variable order derivative and their definition follows
\[
\begin{aligned}
\frac{C}{0} D_t^\alpha y (t) &= f (x, t) \quad \text{for} \quad t \in [t_0, t_1], \\
\frac{CF}{T_1} D_t^\alpha y (t) &= f (x, t) \quad \text{for} \quad t \in [t_1, t_2], \\
\frac{ABC}{T_2} D_t^\alpha y (t) &= f (x, t) \quad \text{for} \quad t \in [t_2, t_3].
\end{aligned}
\tag{8}
\]
The fractional derivative used here with Caputo power law derivative which is known to have singularity at the origin for a class of functions. The space of the functions is not also define therefore, if we assume a class of the functions for which \( \frac{d}{dt} \) is continuous then at each \( t = t_i \), \( f (x, t) \) should be zero.

On the other hand, the piecewise concept is defined as follows.

1. A piecewise derivative of \( f \) with classical, power-law and stochastic processes is given by
\[
\begin{aligned}
D y (t) &= f (t, y (t)) \quad \text{if} \quad 0 < t < t_1, \\
\frac{C}{t_1} D_t^\alpha y (t) &= f (t, y (t)) \quad \text{if} \quad t_1 < t < t_2, \\
d y (t) &= f (t, y (t)) \frac{dt + \sigma y (t) dB (t)}{dt} \quad \text{if} \quad t_2 < t < t_3.
\end{aligned}
\tag{9}
\]

2. A piecewise derivative of \( f \) with power-law, exponential and Mittag-Leffler law is given as
\[
\begin{aligned}
\frac{C}{0} D_t^\alpha y (t) &= f (t, y (t)) \quad \text{if} \quad 0 < t < t_1, \\
\frac{CF}{T_1} D_t^\alpha y (t) &= f (t, y (t)) \quad \text{if} \quad t_1 < t < t_2, \\
\frac{ABC}{T_2} D_t^\alpha y (t) &= f (t, y (t)) \quad \text{if} \quad t_2 < t < t_3.
\end{aligned}
\tag{10}
\]
Several definitions can be found in [7].
In the next section, we will provide a deep analysis of these concepts. To point out their advantages, disadvantages theirs possibility applications. We note that

\[ C_0 D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{df(\tau)}{d\tau} (t-\tau)^{-\alpha} d\tau, \]  

(11)

\[ C_F D_t^\alpha f(t) = \frac{1}{1-\alpha} \int_0^t \frac{df(\tau)}{d\tau} \exp \left[ -\frac{\alpha}{1-\alpha} (t-\tau) \right] d\tau, \]  

(12)

\[ A_{BC} D_t^\alpha f(t) = \frac{1}{1-\alpha} \int_0^t \frac{df(\tau)}{d\tau} E_\alpha \left[ -\frac{\alpha}{1-\alpha} (t-\tau)^\alpha \right] d\tau. \]  

(13)

4 Analysis and difference

Here we present a deep analysis that will help readers to see the difference between both concepts. Let us start the short memory. A deep look inside short memory concept defined by we show that

1. Short memory principle considers a change in constant variable order, however use a single kernel. Indeed, this can be viewend as variable order where the order is changed within shorter intervals. Nonetheless, it describes same process, in the case of Caputo derivation, this process will only describe power-law process. There is no crossover behavior will be depicted.

2. The concept is unable to capture classical processes as the fractional order should change in each interval.

On a serious note, however, if we have \( f(t, y(t)) \neq 0, \forall t \in [0, T] \). So at each \( t_j \)

\[ \lim_{t \to t_j} C_{t_j} D_t^\alpha y(t) = f(t, y(t)) \bigg|_{t=t_j}. \]  

(14)

Since \( \frac{dy}{dt} \) is continuous,

\[ \lim_{t \to t_j} C_{t_j} D_t^\alpha y(t) = f(t_j, y(t_j)) \neq 0. \]  

(15)

However the right hand side produces , which is contradiction. It infers that \( f(t_j, y(t_j)) = 0 \). This may be used to explain process with renewal force that follows power-law process. For example, the trajectory of flea with a constant jump. There are fewer real world problems that present these behaviors. However, due to the power-law singularity the definition suggested by Wu may some problems at the boundaries. Nonetheless using non-singular kernels, the concept of short memory will be well-posed since the conditions at the boundaries will be well controlled and the renewal processes would be well-posed.

On the other hand, however, the concept of piecewise was introduced for different purposes. The following example will give light to situation. We consider evaluating the velocity of water within a geological formation with fractures. We record the velocity as function of time. The velocity will obviously be low in the matrix rock, however as the water reaches the fracture, there will be a crossover behavior as the velocity will suddenly increase. The first part of this process follow behaviors resembling declining process and later an almost constant high velocity. Now if one was to withdraw water from such geological formation and record the water level. One will observe a fading level of groundwater level in earlier time then when water is being taken from the fracture, we observe a steady level of groundwater
level. Thus, there is a crossover behavior from fading process to long range, which can be captured with exponential function then later power-law. Therefore, the differential operator able to replicate this process is

\[
\begin{aligned}
C_0^0 h (r, t) &= f (t, r, h (r, t)) \text{ if } 0 \leq t \leq T_1, \\
C_{T_1}^1 h (r, t) &= f_1 (t, r, h (r, t)) \text{ if } T_1 \leq t \leq T.
\end{aligned}
\]

(16)

Therefore, one can see a clear difference and objectives of both concepts. Piecewise differential operators are thus conceived to capture processes exhibiting different patterns, crossover behaviors. Here the order does not change within interval as changing order does not change the process but only the memory with same pattern, rather, the kernel change to bring into mathematical formulation crossover behaviors by each kernel.

For example, using piecewise derivative the decay equation

\[
\begin{aligned}
y (t) &= -\lambda y (t) \text{ if } 0 < t < T_1, \\
C_{T_1}^1 y (t) &= -\lambda y (t) \text{ if } T_1 < t < T_2, \\
dy (t) &= -\lambda y (t) dt + \sigma y (t) dB (t) \text{ if } T_2 < t < T.
\end{aligned}
\]

(17)

will lead to decay will crossover behaviors. From 0 < t < T_1, we shall observe normal decay process. From T_1 < t < T_2, we shall observe power-law decay or the Mittag-Leffler decay. From T_2 < t < T, we shall observe decay with randomness. On the other hand, with short memory

\[
\begin{aligned}
C_0^1 y (t) &= -\lambda y (t) \text{ if } 0 < t < T_1, \\
C_{T_1}^1 y (t) &= -\lambda y (t) \text{ if } T_1 < t < T_2, \\
C_{T_2}^2 y (t) &= -\lambda y (t) \text{ if } T_2 < t < T.
\end{aligned}
\]

(18)

which leads only to power-law process or Mittag-Leffler process.

5 Application to real world problems

In this section, we shall present clearly the difference between both concepts by applying on some problems. We shall start with simple problem about the decay problem with two intervals.

Example 1. We consider a simple decay model within [0, T_1] and [T_1, T_2]. In the case of short memory, we have

\[
\begin{aligned}
C_0^1 y (t) &= -\lambda y (t) \text{ if } t \in [0, T_1], \\
C_{T_1}^1 y (t) &= -\lambda y (t) \text{ if } t \in [T_1, T_2].
\end{aligned}
\]

(19)

In the case of piecewise, we can consider the following

\[
\begin{aligned}
\frac{dy (t)}{dt} &= -\lambda y (t) \text{ if } t \in [0, T_1], \\
C_{T_1}^1 y (t) &= -\lambda y (t) \text{ if } t \in [T_1, T_2].
\end{aligned}
\]

(20)

or

\[
\begin{aligned}
A_{BC} C_{T_1}^1 y (t) &= -\lambda y (t) \text{ if } t \in [0, T_1], \\
A_{BC} \frac{dy (t)}{dt} &= -\lambda y (t) \text{ if } t \in [T_1, T_2].
\end{aligned}
\]

(21)

or

\[
\begin{aligned}
C_{T_1}^1 y (t) &= -\lambda y (t) \text{ if } t \in [0, T_1], \\
C_{T_1}^1 y (t) &= -\lambda y (t) \text{ if } t \in [T_1, T_2].
\end{aligned}
\]

(22)

Using the Laplace transform on the above system yields in the case of short memory

\[
\begin{aligned}
y (t) &= y (0) E_{\alpha_1} \left[-\lambda t^{\alpha_1}\right] \text{ if } t \in [0, T_1], \\
y (t) &= y (T_1) E_{\alpha_2} \left[-\lambda (t - T_1)^{\alpha_2}\right] \text{ if } t \in [T_1, T_2].
\end{aligned}
\]

(23)
In the case of piecewise, we have the following solution

\[
\begin{align*}
  y(t) &= y(0) \exp[-\lambda t] \text{ if } t \in [0, T_1], \\
  y(t) &= y(T_1) E_{\alpha}[-\lambda (t - T_1)\Gamma] \text{ if } t \in [T_1, T_2]
\end{align*}
\]  

(24)

or

\[
\begin{align*}
  y(t) &= y(0) \text{ if } t = 0, \\
  y(t) &= \frac{y(0)}{1 + \lambda T_1} E_{\alpha} \left[ -\frac{\lambda}{1 + \lambda T_1} t^{\alpha} \right] \text{ if } t \in [0, T_1], \\
  y(t) &= y(T_1) \exp[-\lambda (t - T_1)] \text{ if } t \in [T_1, T_2]
\end{align*}
\]  

(25)

or

\[
\begin{align*}
  y(t) &= y(0) \text{ if } t = 0, \\
  y(t) &= \frac{y(0)}{1 + \lambda T_1} \exp \left[ -\frac{\lambda}{1 + \lambda T_1} t^{\alpha} \right] \text{ if } t \in [0, T_1], \\
  y(t) &= y(T_1) E_{\alpha}[-\lambda (t - T_1)\Gamma] \text{ if } t \in [T_1, T_2]
\end{align*}
\]  

(26)

The numerical simulations are presented in Figures below.

Figure 1. Decay model with short memory for $\lambda = 3, \alpha_1 = 0.9, \alpha_2 = 0.6$. 
Figure 2. Decay model with piecewise derivative for $\lambda = 3$.

Figure 3. Decay model with piecewise derivative for $\lambda = 3$. 
Figure 4. Decay model with piecewise derivative for $\lambda = 3$.

**Example 2.** We consider a spread of an infectious disease. We consider a SEIR model. It was proven that by Atangana and Seda that such model may not be able to predict waves [8]. Thus to introduce wave, in particular two waves, we consider the spread to be in period $[0, T_1]$ and $[T_1, T_2]$. In the case of short memory, we have

\[
\begin{aligned}
&\left\{
\begin{array}{l}
\frac{C}{\alpha} D_t^\alpha S = \mu N - \mu S - \frac{\beta SI}{N}, \\
\frac{C}{\alpha} D_t^\alpha E = \frac{\beta SI}{N} - (\mu + a) E, \\
\frac{C}{\alpha} D_t^\alpha I = aE - (\gamma + \mu) I, \\
\frac{C}{\alpha} D_t^\alpha R = \gamma I - \mu R
\end{array}
\right. \\
\text{if } t \in [0, T_1],
\end{aligned}
\]

\[
\begin{aligned}
&\left\{
\begin{array}{l}
\frac{C}{\beta} D_t^\beta S = \mu N - \mu S - \frac{\beta SI}{N}, \\
\frac{C}{\beta} D_t^\beta E = \frac{\beta SI}{N} - (\mu + a) E, \\
\frac{C}{\beta} D_t^\beta I = aE - (\gamma + \mu) I, \\
\frac{C}{\alpha} D_t^\alpha R = \gamma I - \mu R
\end{array}
\right. \\
\text{if } t \in [T_1, T_2].
\end{aligned}
\]
In the case of piecewise several scenarios can be obtained. So we can have

\[
\begin{align*}
0 \leq t \leq T_1, \quad S &= \mu N - \mu S - \frac{\beta S I}{N} \\
0 \leq t \leq T_1, \quad \dot{E} &= \frac{\beta S I}{N} - (\mu + a) E \\
0 \leq t \leq T_1, \quad \dot{I} &= aE - (\gamma + \mu) I \\
0 \leq t \leq T_1, \quad \dot{R} &= \gamma I - \mu R \\
\end{align*}
\]

or

\[
\begin{align*}
0 \leq t \leq T_1, \quad dS &= \left(\mu N - \mu S - \frac{\beta S I}{N}\right) dt + \sigma_1 S dB_1 (t) \\
0 \leq t \leq T_1, \quad dE &= \left(\frac{\beta S I}{N} - (\mu + a) E\right) dt + \sigma_2 E dB_2 (t) \\
0 \leq t \leq T_1, \quad dI &= (aE - (\gamma + \mu) I) dt + \sigma_3 I dB_3 (t) \\
0 \leq t \leq T_1, \quad dR &= (\gamma I - \mu R) dt + \sigma_4 R dB_4 (t) \\
\end{align*}
\]

\[
\begin{align*}
\text{if } t \in [0, T_1], \\
\text{if } t \in [T_1, T_2],
\end{align*}
\]

Noting that several more scenarios can be considered, however, we will only consider these two in the case of piecewise. Before proceeding with analysis of these models, we shall first provide an interpolation of each case in terms of wave behaviors. In the case of short memory, the first and second waves show the non-Gaussian distribution associated to power-law $t^{a_1}$ and $t^{a_2}$. Therefore, one will observe different long tail accordingly to $t^{a_1}$ and $t^{a_2}$. On the other hand, however, the first considered model shows that the first wave has a normal distribution, while the second wave has random behavior. The second model shows that, the first wave has a long tail spread while the second has random behavior.

We now present the numerical solutions of the considered models. We shall consider the numerical scheme suggested by Ghanbari et. al. [9] for fractional cases. The short memory case can be simplified to

\[
\begin{align*}
\text{if } t \in [0, T_1], \\
\text{if } t \in [T_1, T_2],
\end{align*}
\]

\[
\begin{align*}
\frac{C}{0} D_{t}^\alpha y_1 (t) &= F_i (t, y_i (t)) \\
\frac{C}{T_1} D_{t}^\beta y_1 (t) &= F_i (t, y_i (t)) \quad \text{if } t \in [T_1, T_2]
\end{align*}
\]

Applying the Ghanbari method yields

\[
\begin{align*}
y_i^{n_1} &= y_i (0) + h^\alpha \left( \frac{\alpha_{n_1}}{\alpha_{n_1}} F_i (t_0, y_i (t_0)) + \sum_{k=2}^{n_1-2} \frac{\alpha_{n_1-k}}{\alpha_{n_1-k}} F_i (t_k, y_i (t_k)) \right) \quad \text{if } t \in [0, T_1], \\
y_i^{n_2} &= y_i (T_1) + h^\beta \left( \frac{\alpha_{n_2}}{\alpha_{n_2}} F_i (T_1, y_i (T_1)) + \sum_{j=2}^{n_2} \frac{\alpha_{n_2-j}}{\alpha_{n_2-j}} F_i (t_j, y_i (t_j)) \right) \quad \text{if } t \in [T_1, T_2]
\end{align*}
\]
where

\[ a_n^{(\alpha)} = \frac{(n_1 - 1)^{\alpha+1} - n_1^n (n_1 - \alpha - 1)}{\Gamma(\alpha + 2)}, \quad (32) \]

\[ a_n^{(\beta)} = \begin{cases} \frac{1}{1(\beta+2)} & \text{if } n = 0, \\ \frac{(n_2-1)^{\beta+1} - 2n_2^n + (n_2+1)^{\beta+1}}{1(\beta+2)} & \text{if } n \geq 1. \end{cases} \]

In the case of piecewise first model, we have

\[
\begin{cases}
y_i^{n_1} = y_i(0) + \sum_{k=2}^{n_1} \left[ \frac{23}{12} F_i(t_k, y_i(t_k)) \Delta t ight. \\
- \frac{4}{3} F_i(t_{k-1}, y_i(t_{k-1})) \Delta t \\
+ \frac{5}{3} F_i(t_{k-2}, y_i(t_{k-2})) \Delta t \\
\left. + \sigma_i y_i (B_{i}^{n_1+1} - B_{i}^{n_2}) \right] & \text{if } t \in [0, T_1], \\
y_i^{n_2} = y_i(T_1) + \sum_{j=n_1+3}^{n_2} \left[ \frac{23}{12} F_i(t_j, y_i(t_j)) \Delta t \\
- \frac{4}{3} F_i(t_{j-1}, y_i(t_{j-1})) \Delta t \\
+ \frac{5}{3} F_i(t_{j-2}, y_i(t_{j-2})) \Delta t \\
\right. \\
+ \sigma_i y_i (B_{i}^{n_2} - B_{i}^{j}) & \text{if } t \in [T_1, T_2].
\end{cases} \quad (33)
\]

For Case 2, we obtain

\[
\begin{cases}
y_i^{n_1} = y_i(0) + \sum_{j=2}^{n_1} \left[ \frac{23}{12} F_i(t_j, y_i(t_j)) \Delta t \\
- \frac{4}{3} F_i(t_{j-1}, y_i(t_{j-1})) \Delta t \\
+ \frac{5}{3} F_i(t_{j-2}, y_i(t_{j-2})) \Delta t \\
\right. \\
+ \sigma_i \sum_{j=1}^{n_1} y_j (B_{i}^{j} - B_{i}^{j-1}) & \text{if } t \in [0, T_1], \\
y_i^{n_2} = y_i(T_1) + h^{\alpha} \left[ a_n^{(\alpha)} F_i(t_{n_1}, y_i(t_{n_1})) \\
+ \sum_{k=0}^{n_1} a_n^{(\alpha)} F_i(t_k, y_i(t_k)) \right] & \text{if } t \in [T_1, T_2].
\end{cases} \quad (34)
\]

The numerical simulation for short memory are depicted in Figure 5 with the following parameters

\[ N = 1000, \mu = 0.01, \beta = 0.6, a = 0.2, \gamma = 0.03. \quad (35) \]

and initial conditions

\[
\begin{align*}
S(0) &= 1000, E(0) = 25, I(50) = 22, R(50) = 5, S(50) = 39, \\
E(50) &= 67, I(50) = 505, R(50) = 390.
\end{align*} \quad (36)
\]
Figure 5. The numerical solutions of Covid-19 model with short memory for $\alpha_1 = 0.95, \alpha_2 = 0.75$.

The numerical simulations for Case 1 and 2 are depicted in Figures 6 and 7 with the following parameters

$$N = 1000, \mu = 0.01, \beta = 0.6, \alpha = 0.2, \gamma = 0.03.$$  \hspace{1cm} (37)

and initial conditions

$$S(0) = 1000, E(0) = 25, I(50) = 22, R(50) = 5, S(50) = 35$$ \hspace{1cm} (38)
$$E(50) = 44, I(50) = 424, R(50) = 527, \alpha = 0.7$$
Figure 6. The numerical solutions of Covid-19 model with Case 1 $\sigma_1 = 0.000005$, $\sigma_2 = 0.008, \sigma_3 = 0.0021, \sigma_4 = 0.009, \alpha = 0.7$.

Figure 7. The numerical solutions of Covid-19 model with Case 2 $\sigma_1 = 0.17$, $\sigma_2 = 0.25, \sigma_3 = 0.21, \sigma_4 = 0.19, \alpha = 0.7$. 
Figure 5 show case of short memory effect, where the first and the second parts present different long tails behaviours characterized by $\alpha_1$ and $\alpha_2$, however, both depict clearly the same process which is the power law process. On the other hand, Figures 6 and 7 are clear effects of the piecewise differential operators, the first part in Figure 6 shows normal distribution, which is characteristic of the classical differential operator, while the second part clearly shows the effect of randomness. The complete model therefore shows crossover behaviour from deterministic to stochastic with no steady state. In Figure 6, the process is different. We start with randomness then end up with long tail behaviours as a result of the power law kernel.

**Example 3.** We now consider a chaotic problem in particular the 8-Wing attractor [10,11]. This model has been studied by several researchers, the model is given as

\begin{align}
\dot{x} &= a(y - x) + f(t)yz \\
\dot{y} &= cx + dy - xz \\
\dot{z} &= -bz + xy
\end{align}

where the function $f(t) = M \text{sgn}(\sin(ut)) + K$. The constant $w$ is known to be a switch frequency, $M$ and $K$ are constant parameters.

In the case of short memory, we have

\begin{align}
\begin{cases}
\frac{C}{0} D^{\alpha_1}_t x = a(y - x) + f(t)yz \\
\frac{C}{0} D^{\alpha_1}_t y = cx + dy - xz \\
\frac{C}{0} D^{\alpha_1}_t z = -bz + xy \\
\frac{C}{0} D^{\alpha_1}_t x = a(y - x) + f(t)yz \\
\frac{C}{0} D^{\alpha_1}_t y = cx + dy - xz \\
\frac{C}{0} D^{\alpha_1}_t z = -bz + xy \\
\end{cases}
& \text{if } t \in [0, T_1], \\
\frac{C}{T_1} D^{\alpha_2}_t x = a(y - x) + f(t)yz \\
\frac{C}{T_1} D^{\alpha_2}_t y = cx + dy - xz \\
\frac{C}{T_1} D^{\alpha_2}_t z = -bz + xy \\
& \text{if } t \in [T_1, T_2].
\end{align}

In the case of piecewise, we can consider three cases. For Case 1, we can write

\begin{align}
\begin{cases}
\frac{A}{0} B C D^{\alpha}_t x = a(y - x) + f(t)yz \\
\frac{A}{0} B C D^{\alpha}_t y = cx + dy - xz \\
\frac{A}{0} B C D^{\alpha}_t z = -bz + xy \\
\frac{A}{0} B C D^{\alpha}_t x = a(y - x) + f(t)yz \\
\frac{A}{0} B C D^{\alpha}_t y = cx + dy - xz \\
\frac{A}{0} B C D^{\alpha}_t z = -bz + xy \\
\end{cases}
& \text{if } t \in [0, T_1], \\
dx = (a(y - x) + f(t)yz) dt + \sigma_1 x dB_1(t) \\
dy = (cx + dy - xz) dt + \sigma_2 y dB_2(t) \\
dz = (-bz + xy) dt + \sigma_3 z dB_3(t) \\
& \text{if } t \in [T_1, T_2].
\end{align}

For Case 2, we have

\begin{align}
\begin{cases}
\frac{C}{0} D^{\alpha}_t x = a(y - x) + f(t)yz \\
\frac{C}{0} D^{\alpha}_t y = cx + dy - xz \\
\frac{C}{0} D^{\alpha}_t z = -bz + xy \\
\frac{C}{0} D^{\alpha}_t x = a(y - x) + f(t)yz \\
\frac{C}{0} D^{\alpha}_t y = cx + dy - xz \\
\frac{C}{0} D^{\alpha}_t z = -bz + xy \\
\end{cases}
& \text{if } t \in [0, T_1], \\
dx = (a(y - x) + f(t)yz) dt + \sigma_1 x dB_1(t) \\
dy = (cx + dy - xz) dt + \sigma_2 y dB_2(t) \\
dz = (-bz + xy) dt + \sigma_3 z dB_3(t) \\
& \text{if } t \in [T_1, T_2].
\end{align}
For Case 3, we can get

\[
\begin{aligned}
   \begin{cases}
      x = a(y - x) + f(t)yz \\
      y = cx + dy - xz \\
      z = -bz + xy
   \end{cases}
\end{aligned}
\]
if \( t \in [0, T_1] \),

\[
\begin{aligned}
   \begin{cases}
      dx = (a(y - x) + f(t)yz) \, dt + \sigma_3 x dB_1(t) \\
      dy = (cx + dy - xz) \, dt + \sigma_2 y dB_2(t) \\
      dz = (-bz + xy) \, dt + \sigma_3 z dB_3(t)
   \end{cases}
\end{aligned}
\]
if \( t \in [T_1, T_2] \).

We shall now present the numerical solution of each case. In the case of short memory, one can use the numerical method based on Lagrange interpolation \[12\]. We can first simplify the model as follows.

\[
\begin{aligned}
   \begin{cases}
      \frac{\partial}{\partial t} D_{i}^{p_1} M_i(t) = \Pi(t, M_i(t)) \, & \text{if } t \in [0, T_1], \\
      \frac{\partial}{\partial t} D_{i}^{p_2} M_i(t) = \Pi(t, M_i(t)) \, & \text{if } t \in [T_1, T_2]
   \end{cases}
\end{aligned}
\]

Thus applying such method on the above system yields

\[
\begin{aligned}
   M_i^{p_1} = M_i(0) + \left[ \begin{array}{l}
      \left( \frac{\Delta t}{(\alpha_1 + 2)} \right)^{p_1} \sum_{q=0}^{n_1} \Pi(t_q, M_i^q) \\
      - (n_1 - q + 1) \alpha_1 (n_1 - q + 2 + 2\alpha_1)
   \end{array} \right] \\
   \times \left[ \begin{array}{l}
      \sum_{q=1}^{n_1} \Pi(t_{q-1}, M_i^{q-1}) \\
      (n_1 - q + 1) \alpha_1 (n_1 - q + 1 + \alpha_1)
   \end{array} \right] \\
   \times \left[ \begin{array}{l}
      \sum_{k=n_1+1}^{n_2} \Pi(t_k, M_i^k) \\
      (n_2 - q + 1) \alpha_2 (n_2 - q + 2 + 2\alpha_2)
   \end{array} \right] \\
   \times \left[ \begin{array}{l}
      \sum_{k=n_1+1}^{n_2} \Pi(t_{k-1}, M_i^{k-1}) \\
      (n_2 - q + 1) \alpha_2 (n_2 - q + 1 + \alpha_2)
   \end{array} \right]
\end{aligned}
\]
if \( t \in [0, T_1] \),

\[
\begin{aligned}
   M_i^{p_2} = M_i(T_1) + \left[ \begin{array}{l}
      \sum_{q=0}^{n_1} \Pi(t_q, M_i^q) \\
      - (n_1 - q + 1) \alpha_1 (n_1 - q + 2 + 2\alpha_1)
   \end{array} \right] \\
   \times \left[ \begin{array}{l}
      \sum_{q=1}^{n_1} \Pi(t_{q-1}, M_i^{q-1}) \\
      (n_1 - q + 1) \alpha_1 (n_1 - q + 1 + \alpha_1)
   \end{array} \right] \\
   \times \left[ \begin{array}{l}
      \sum_{k=n_1+1}^{n_2} \Pi(t_k, M_i^k) \\
      (n_2 - q + 1) \alpha_2 (n_2 - q + 2 + 2\alpha_2)
   \end{array} \right] \\
   \times \left[ \begin{array}{l}
      \sum_{k=n_1+1}^{n_2} \Pi(t_{k-1}, M_i^{k-1}) \\
      (n_2 - q + 1) \alpha_2 (n_2 - q + 1 + \alpha_2)
   \end{array} \right]
\end{aligned}
\]
if \( t \in [T_1, T_2] \).

With piecewise case, we have the following numerical solution using a scheme based on the Newton polynomial interpolation \[13\]. Also, we simplify the model as follows.

\[
\begin{aligned}
   \begin{cases}
      \frac{\partial}{\partial t} D_{i}^{p_1} M_i(t) = \Pi(t, M_i(t)) \, & \text{if } t \in [0, T_1], \\
      dM_i(t) = \Pi(t, M_i(t)) \, dt + \sigma_i M_i dB_i(t) \, & \text{if } t \in [T_1, T_2]
   \end{cases}
\end{aligned}
\]
Then applying this method yields

\[
M_i^{n+1} = M_i(0) + \frac{1}{\Delta t} \sum_{k=2}^{n+1} \Pi(t_{n+k}, M_i^{n+1}) - \Pi(t_{n+k-2}, M_i^{n+1}) \times \left\{ \begin{array}{l}
(n_1 - k + 1)^\alpha -(n_1 - k)^\alpha \\
(n_1 - k + 1)^\alpha (n_1 - k + 2 + 2\alpha)
\end{array} \right\} + \frac{\alpha(\Delta t)^\alpha}{\Gamma(\alpha+2)} \sum_{k=2}^{n+1} \Pi(t_{n+k}, M_i^{n+1}) \times \left\{ \begin{array}{l}
(n_1 - k + 1)^\alpha (n_1 - k + 3 + 2\alpha)
\end{array} \right\}
\]

For Case 2, we apply again the scheme with Newton polynomial to obtain

\[
M_i^{n+4} = M_i^{n+3} + \sum_{j=1}^{n_3} \left[ \frac{23}{12} \Pi(t_{n+3}, M_i^{n+3}) - \frac{3}{4} \Pi(t_{n+2}, M_i^{n+2}) \right] \Delta t
\]

For Case 3, we use simple Adams-Bashforth as follows

\[
M_i^{n+3} = M_i^{n+2} + \left[ \frac{23}{12} \Pi(t_{n+2}, M_i^{n+2}) - \frac{3}{4} \Pi(t_{n+1}, M_i^{n+1}) \right] \Delta t, 0 \leq t \leq T_1
\]

\[
M_i^{n+4} = M_i^{n+3} + \left[ \frac{23}{12} \Pi(t_{n+3}, M_i^{n+3}) - \frac{3}{4} \Pi(t_{n+2}, M_i^{n+2}) \right] \Delta t, 0 \leq t \leq T_1
\]

Numerical simulations are obtained for the following parameters

\[
a = 14, b = 43, c = -1, d = 16, M = 9, K = 10, w = \frac{2\pi}{50}
\]
The numerical simulation for short memory are depicted in Figure 8 with the initial data \( x(0) = 22.3, y(0) = 4.9, z(0) = 2.6 \).

Figure 8. The numerical simulation for 8-Wing attractor with short memory for \( \alpha_1 = 0.96, \alpha_2 = 0.98 \).

For piecewise, we consider the following stochastic-deterministic problem with power-law and Mittag-Leffler kernel

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} D_t^\alpha x = a(y - x) + f(t)yz + \sigma_1 x B(t) \\
\frac{\partial}{\partial t} D_t^\alpha y = cx + dy - xz + \sigma_2 y B(t) \\
\frac{\partial}{\partial t} D_t^\alpha z = -bz + xy + \sigma_3 z B(t)
\end{array} \right. \quad \text{if } t \in [0, T_1], \tag{51}
\]

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} D_t^\alpha x = a(y - x) + f(t)yz \\
\frac{\partial}{\partial t} D_t^\alpha y = cx + dy - xz \\
\frac{\partial}{\partial t} D_t^\alpha z = -bz + xy
\end{array} \right. \quad \text{if } t \in [T_1, T_2]
\]

with the stochastic constants

\[
\sigma_1 = 0.031; \sigma_2 = 0.035; \sigma_3 = 0.032. \tag{52}
\]

The numerical simulation for such problem can be performed in Figure 9 using same initial data and
For piecewise, we consider the following stochastic-deterministic problem with power-law and Mittag-Leffler kernel

\[
\begin{align*}
\begin{cases}
0_{ABC} D^\alpha_t x &= a (y - x) + f (t) y z + \sigma_1 x B_{1_t} (t) \\
0_{ABC} D^\alpha_t y &= c x + d y - x z + \sigma_2 y B_{2_t} (t) \\
0_{ABC} D^\alpha_t z &= -b z + x y + \sigma_3 z B_{3_t} (t)
\end{cases}
& \quad \text{if } t \in [0, T_1], \\
\begin{cases}
c D^\alpha_t x &= a (y - x) + f (t) y z \\
c D^\alpha_t y &= c x + d y - x z \\
c D^\alpha_t z &= -b z + x y
\end{cases}
& \quad \text{if } t \in [T_1, T_2]
\end{align*}
\]

with the stochastic constants

\[
\sigma_1 = 0.031; \sigma_2 = 0.035; \sigma_3 = 0.032.
\]

The numerical simulation for such problem can be performed in Figure 10 using same initial data and
parameters.

Figure 10. Stochastic-deterministic 8-Wing attractor with piecewise derivative for $\alpha = 0.94$.

Also, we consider deterministic-stochastic problem with power-law and Mittag-Leffler kernel

\[
\begin{align*}
\frac{C}{0} D^\alpha_t x &= a(y - x) + f(t)yz \\
\frac{C}{0} D^\alpha_t y &= cx + dy - xz \\
\frac{C}{0} D^\alpha_t z &= -bz + xy
\end{align*}
\]

if $t \in [0, T_1], \quad (55)$

\[
\begin{align*}
0 ABC D^\alpha_t x &= a(y - x) + f(t)yz + \sigma_1 x B(t_1) \\
0 ABC D^\alpha_t y &= cx + dy - xz + \sigma_2 y B(t_2) \\
0 ABC D^\alpha_t z &= -bz + xy + \sigma_3 z B(t_3)
\end{align*}
\]

if $t \in [T_1, T_2]$ (56)

with the stochastic constant

\[\sigma_1 = 0.02; \sigma_2 = 0.012; \sigma_3 = 0.021.\]

The numerical simulation for above problem can be depicted in Figure 11 using same initial data and
parameters.

Figure 7 shows the results obtained by applying the power law short memory concept. To show the difference within the interval, we have opted to consider the first part of the interval to be in blue and second to be in red. However, a quick look at the results shows clearly the effect of power law, no great change from one interval to another is observed since both depict long tail behaviours as results of the power law kernel. In this case, there is no crossover behaviour, just a repetition of different long tails behaviours. On the other hand, in Figures 8 and 9, there is clear change in patterns where the first patterns show deterministic behaviours in particular, long-tails due to power law in Figure 8 and the second part shows randomness, there is therefore a clear crossover from power law to randomness. While, in Figure 9, we have three two crossovers, the first is due to the Mittag-Leffler kernel that shows a change from stretched exponential to power law and the second is stochastics.

Also, we consider deterministic-stochastic problem with power-law and Mittag-Leffler kernel

\[
\begin{align*}
0^{ABC} D_t^\alpha x &= a(y-x) + f(t)yz \\
0^{ABC} D_t^\alpha y &= cx + dy - xz \\
0^{ABC} D_t^\alpha z &= -bz + xy
\end{align*}
\]

if \( t \in [0,T_1] \),

\[
\begin{align*}
\int_0^\alpha D_t^\alpha x &= a(y-x) + f(t)yz + \sigma_1 x B_{t_1}(t) \\
\int_0^\alpha D_t^\alpha y &= cx + dy - xz + \sigma_2 y B_{t_2}(t) \\
\int_0^\alpha D_t^\alpha z &= -bz + xy + \sigma_3 z B_{t_3}(t)
\end{align*}
\]

if \( t \in [T_1,T_2] \)
with the stochastic constant

\[ \sigma_1 = 0.02; \sigma_2 = 0.012; \sigma_3 = 0.021. \]  

(58)

The numerical simulation for above problem can be depicted in Figure 12 using same initial data and parameters.

![Figure 12. Deterministic-stochastic 8-Wing attractor with piecewise derivative for \( \alpha = 0.9 \).](image)

6 Conclusion

In the last decades, researchers have devoted their attention to better understand complex real-world problems even on a small scale. They have therefore developed several methods, different differential, and integral operators. To understand the process by which different long tails occur in different intervals, the concept of short memory was introduced. This concept defines power law derivatives in different intervals and each interval has its own order. This order accounts for the long tail associated with that interval. However, the process is scale invariant in terms of patterns. On the other hand, because there are many real-world problems that exhibit passages from one process to another, for example a passage from power law to randomness, the concept of piecewise was introduced. In this work, a detailed analysis was given to show their different fields of applications and show also when short memory and piecewise
concepts can be applied. This paper thus helps researchers to identify what problem is suitable for short memory and piecewise.

References


