The physics of higher-order interactions in complex systems

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Perspective article:
Higher-order interactions in complex networks: an opportunity for new physics

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ABSTRACT

For many years, complex networks have represented the main paradigm to model the dynamics of interacting systems. However, networks are intrinsically limited to describing pairwise interactions, while higher-order interactions (HOIs), i.e. interactions in groups of three or more units, are ubiquitous. Higher-order structures, such as hypergraphs and simplicial complexes are therefore a better tool to map the real organization of many social, biological and man-made systems. Here, we highlight recent evidence of novel collective behaviours induced by higher-order interactions, and we outline three key challenges for the new physics of higher-order systems.

Network science allows us to understand how the highly interconnected world we live in evolves [1]. How do rumours spread in a social network? How do large ecosystems stabilise despite competing interactions among species? What is the origin of abrupt changes in biological oscillators? A key shared feature of all such systems is that they are characterised by a complex relational structure, which largely governs their emergent dynamics [2, 3, 4]. In the last decades, the architecture of social networks, ecosystems and the human brain have all been modelled as graphs, collections of nodes describing the units of the systems – humans, animals, neurons – and links, encoding their pairwise interactions. This approach has brought new

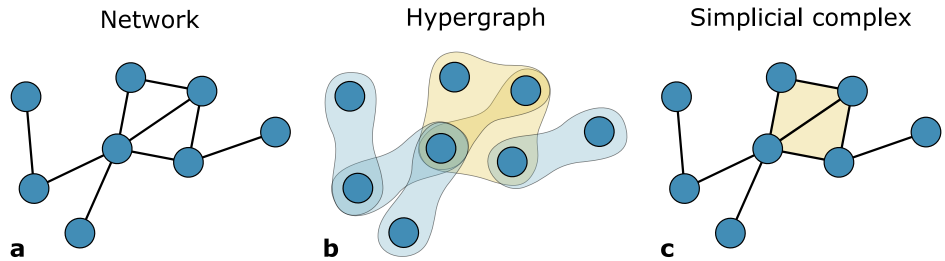

Figure 1. Pairwise and higher-order representations. Systems composed by many interacting units have long been represented as networks (**a**), with interactions restricted to pairs of nodes and represented as the network edges. However, it is not always possible to describe group interactions as sums of pairwise interactions only. Representations allowing for genuine group interactions include hypergraphs (**b**), which can encode interactions among an arbitrary number of units without further constraints, and simplicial complexes (**c**), which –although more constrained than hypergraphs– provide access to powerful mathematical formalisms [9]. In (**b**) shaded groups of nodes represent hyperedges, while in (**c**) edges (1-simplices) are shown in black, full triangles (2-simplices) in yellow. Note that in simplicial complexes all subfaces of a simplex (e.g. edges of a triangle) need to be included. This constraint does not hold for hypergraphs.

insights in different realms. For instance, we have discovered that a heavy-tailed distribution in the number of contacts within a population makes us at risk of pandemics because of the vanishing of the epidemic threshold [5, 6], that small-world networks and clustering promote synchronization [7], and that efficient communication structures are suited to reach rapid and diffused consensus but also prone to the unwanted spreading of misinformation [8].

Graphs, however, while very convenient representations of a wide variety of systems, can only provide a limited description of reality. Indeed, they are inherently constrained to represent systems with pairwise interactions only. Yet, in many biological, physical and social systems, units may interact in larger groups, and such interactions cannot always be decomposed as a linear combination of dyadic couplings [10] (Fig. 1). For example, evidence from neural systems shows, on the one hand, that higher-order effects are present and important both statistically [11, 12, 13] and topologically [14, 15], and, on the other hand, that such higher-order signatures might in some cases be an epiphenomenon of low-order ones [16, 17]. In ecological systems, evidence shows clearly the existence of complex many-body interactions between multiple species [18, 19, 20], although the effects induced by their interaction patterns have only recently started being investigated formally [21]. Other examples include metabolic and genetic systems [22], social coordination [23], and group formation [24].

The idea of higher-order interactions (HOIs) is not new in physics, in particular in many-body physics , e.g. in strong interactions [25, 26] or van der Waals interactions [27], as well as in statistical mechanics [28]. However, in all these cases, the representations of HOIs are simple in the sense that they do not contribute to the emerging complexity of the problem, which is instead what typically happens in complex systems. To take these HOIs explicitly into account, different mathematical structures, such as hypergraphs and simplicial complexes, are better suited [10]. Several early investigations have already shown that the presence of higher-order interactions may significantly impact the dynamics on networked systems, from diffusion [29, 30] and synchronization [31, 32] to social [33, 34, 35] and evolutionary processes [36], possibly leading to the emergence of abrupt (explosive) transitions between states. In addition, while most research in complex systems focuses on the dynamical evolution of the states of the nodes, it becomes natural to consider that higher-order structures (hyperedges) can themselves possess a dynamical state, leading to a whole new panorama of dynamical processes. Finally, while many datasets directly come under the form of networks, very few are directly under a hypergraph representation. How to go from the dynamics of units, and possibly information about their pairwise interactions, to a meaningful pattern of higher-order interactions between these units, remains a largely open issue. In this Perspective, we outline the main signatures of new physics arising in higher-order systems, and we propose three key directions for future research.

A general pathway to explosive transitions. Most processes on networks, from the dynamical evolution of coupled oscillators to the spreading of diseases, display emerging collective behaviours. Typically, such phenomena are described by continuous phase transitions: the order parameter describing, e.g. the emergence of synchronisation among the oscillators, increases continuously as the control parameter crosses a critical threshold. Similar transitions are also well-known for percolation on networks, where initially separated small clusters start merging together to span a non-vanishing fraction of the system size at a critical point. In contrast, an explosive transition was first found a dozen years ago for particular link selection rules [37]: the size of the largest cluster seemed to jump abruptly to a finite value at the transition. While this specific transition was in fact later on classified as continuous, but with an anomalous scaling [38, 39], an intense activity followed in a

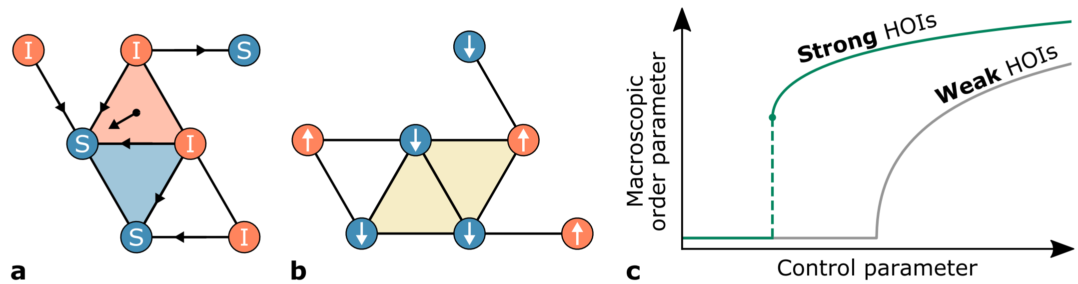

Figure 2. Higher-order interactions lead to explosive phenomena. Edges and hyperedges encode pairwise and group-level couplings among the nodes of a complex system. The latter modulate group infection and many-body feedback in higher-order processes of contagion (**a**) and synchronization (**b**). In (**a**) susceptible nodes (S, blue) can be infected by infectious ones (I, orange) in the usual way along edges, but also by groups containing a large fraction of infected nodes (e.g. orange 2-simplices). Similarly, in (**b**) oscillators on nodes can be coupled along edges, or in groups via higher-order interactions. Abrupt transitions emerge when increasing the strength of such interactions, suggesting a general pathway to explosive phenomena (**c**).

quest for explosive phenomena [40]. Several discontinuous phase transitions were confirmed for different processes, such as synchronisation. However, explosive phenomena are rather difficult to obtain for systems represented as networks, i.e., with only pairwise interactions. They are typically achieved by adding artificial elements or rules to the most natural dynamical setups in an attempt to prevent the transition. Eventually though, these produce abrupt jumps in the order parameter once the transition becomes inevitable. For instance, synchronisation can become explosive in heterogeneous networks by correlating the natural frequency of oscillators to their degree [41]. Explosive phenomena however are known to exist in nature, e.g. tipping points that lead to strong changes when crossed, and are of extreme interest as they are more difficult to handle, predict, and potentially control than continuous ones. Reaching a better understanding of the conditions under which they can occur, and of the mechanisms that can produce them,

constitutes thus a major challenge in complexity science.

In fact, it turns out that going beyond networks in modeling interactions and taking into account higher-order interactions provides a framework in which explosive phenomena emerge and can be studied more easily. An abrupt transition was recently observed in a model of social contagion on simplicial complexes [33], where individuals can assume either an infected of susceptible state. In contrast with previous proposals, here pairwise transmission does not operate alone, but can be reinforced by simplicial interactions associated to group pressure (Fig. 2a). The model can be solved analytically in the mean-field, showing that a discontinuous transition from a healthy to an endemic phase (in which a significant fraction of the population is infected) emerges when the relative weight of higher-order interactions crosses a threshold. Interestingly, the inclusion of three-body interactions is sufficient to obtain a bistable region where endemic and non-endemic states can co-exist, even for k-regular simplicial complexes. This result has been found to be robust and general. Explosive transitions have in fact been observed in heterogeneous [42] and time-varying [43, 44] structures, and in the more general setup of hypergraphs [34, 45, 46], where they can also be related to higher-order discontinuous percolation processes [47].

Explosiveness is not limited to spreading processes. Of paramount importance for biology and neuroscience are systems of coupled oscillators, where the states of the nodes are d-dimensional continuous variables that evolve over time under mutual influence (Fig. 2b). The most well-known setup is probably the one introduced by Kuramoto [48], where unidimensional phase oscillators are endowed with natural frequencies, and interactions occur through sinusoidal couplings. When generalised to account for structured higher-order interactions among oscillators, the additional non-linearity generates abrupt switches between synchronized and incoherent states [49]. The emergence of bistability and the appearance of hysteresis cycles are driven by the presence of higher-order interactions alone, without the need of ad-hoc coupling mechanisms between the dynamical evolution and the local connectivity of the nodes.

In both examples, the introduction of HOIs corresponds to having the state variable of a node influenced by a non-linear combination of the states of several other nodes. Tuning the relative importance of the strength of the higher-order and pairwise interactions provides a way in both cases to change the nature of the transition from continuous to discontinuous (Fig. 2c). The similarity of the mechanisms yielding a first-order transition in these two very different dynamical processes leads to conjecture that the introduction of non-linear HOIs and the tuning of their intensity is a general ingredient sufficient to provide abrupt transitions in a dynamical process. Despite this preliminary evidence however, a rigorous and general proof of this conjecture is still lacking. Approximate approaches based on linearization around a fixed point of ordinary differential equations link the stability of hypergraphs dynamics to their graph projections [50, 51, 52], suggesting general conditions for stability associated to the different orders of the interactions. Mean-field treatment allows for an analytical solution for diffusion and spreading

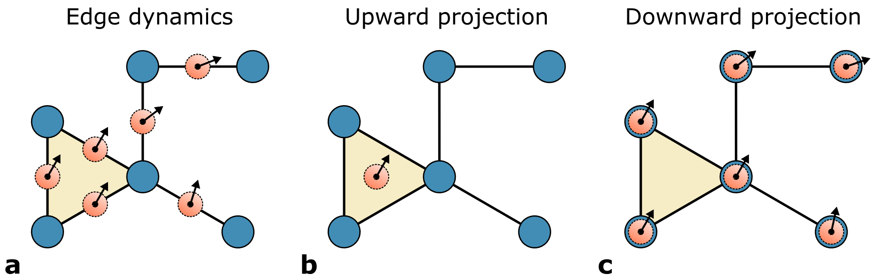

Figure 3. Higher-order systems are fully dynamical. As opposed to traditional descriptions focused on node dynamics, it is possible to define state variables for hyperedges or simplices of arbitrary order, for example by associating oscillators to edges (**a**) [57] and coupling them to each other using their higher-order adjacency. In so doing, the distinction between dynamical units and interactions dissolves, and dependencies and feedback loops between orders become possible. For example, it is possible to project the dynamics of hyperedges of order k (e.g. $k = 1$, edges or 1-simplices) onto their analogues of larger order (e.g. $k = 2$, 3-hyperedges or 2-simplices) (**b**) or smaller order ($k = 0$, nodes) (**c**).

processes on arbitrary structures, separating stability conditions into structural and dynamical terms [53, 54]. A general argument based on bifurcation theory shows that variations on pairwise models, such as adding higher-order interactions, can lead to a change of critical behaviour from a continuous to a discontinuous transition for a wide class of models, including epidemic, synchronization and percolation transitions [55]. Under some conditions, mathematicians were able to formally prove that higher-order interactions are sufficient to induce bi-stable behaviour in the Susceptible-Infected-Susceptible (SIS) model, while it is impossible to achieve bistability in the traditional pairwise scheme [56]. All in all, findings indicate that the presence of higher-order interactions provides a general pathway to explosive phenomena. Yet, this marker of fragility of the collective behavior in higher-order systems is still awaiting formal proof.

Topological dynamical processes. Most of the research on dynamical processes on networks focused on the dynamics of node states, with interactions mediated by links. This is a natural and intuitive approach, because it describes the evolution of the most basic units of the system, coupled through the only possible (and simplest) interactions in networks [58]. The new framework of higher-order systems allows to transcend this approach. In fact, once we are able to represent higher-order interactions, it becomes possible to define couplings between interactions of different orders (nodes, and hyperedges or simplices). More importantly, we can associate state variables, not only to nodes, but also to hyperedges and simplices. For example, the state of an edge can influence the states of its two ending nodes, while contributing to and being influenced by the states of the higher-order interactions (e.g., a 3-hyperedge) it belongs to. In this way, higher-order dynamical systems become fully alive: what were before static interactions (i.e. edges, hyperedges, etc) are now active agents themselves, coupled to the rest of the system and evolving in time.

Recent results on simplicial oscillators are a particularly striking example of this phenomenon. Consider a Kuramoto model defined on a simplicial complex composed by nodes, edges and 2-simplices (Fig. 3a). In this case, phases are defined not only on nodes –as in the traditional description– but also on higher-order faces. Interestingly, the equations used in the classical formalism can be directly adapted to higher-order interactions, by substituting node incidence matrices with the appropriate higher-order analogues [7]. In simplicial complexes, these matrices correspond to boundary operators between interactions of orders differing by one, e.g. node and edges, or edges and 2-simplices, effectively providing a canonical mapping between phase dynamics of different orders. Interesting phenomena emerge already without adding further complications: the dynamics on 1-simplices (edges) displays a synchronization transition [57] that is only revealed when projected onto simplices of dimension higher (2-simplices, Fig. 3b) or lower (nodes, Fig. 3c); indeed, phase transitions appear in both projected dynamics. Even more interestingly, when the dynamics of the $(n-1)$- and $(n+1)$-simplices are coupled via the respective global order parameters, these transitions become explosive.

The Hodge decomposition provides a rationale for this behaviour in terms of the inner structure of higher-order states [29]. In fact, these can be decomposed in three components: a harmonic, a solenoidal and an irrotational components, corresponding respectively to the dynamics induced by the kernel of the higher-order Laplacian, and to those induced by the projection to simplices one dimension higher and lower. In this light, higher-order systems should be considered truly as collections of *topological signals*, i.e. timeseries associated to interactions at all orders, which lend themselves to analysis using tools at the interface between algebraic topology, differential geometry and discrete calculus [29, 59]. As examples of this paradigm, higher-order Laplacians were recently shown to improve the description of flow information on edges with respect to standard graph Laplacians –even for simplicial complexes containing only nodes and edges– [60], and to give the first formulation for signal processing on generic topological spaces[61].

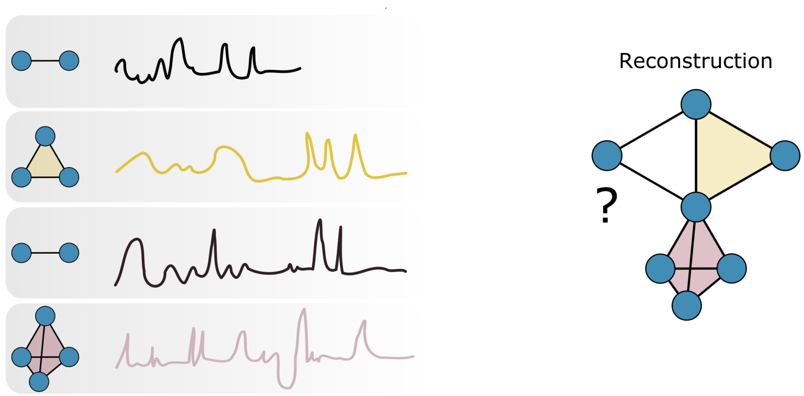

Figure 4. Inference of higher-order systems is still an open and challenging problem. While a few tools and concepts have been proposed, existing methods to extract signals associated to higher-order interactions from node-level measurements (e.g. cofluctuations of groups of brain regions or neuronal firing patterns), or to reconstruct higher-order structures from network-level observations or noisy measurements are still lacking. Early evidence points to the effectiveness of a combination of data-driven modeling efforts and Bayesian inference techniques, but the field is still taking its first steps.

Finally, even when states for higher-order interactions are defined, the topological structure of the system –that is, the presence or absence of simplices and hyperedges– has typically been considered fixed in time (e.g. in neural codes [16]). However, in many systems the organization of the interactions changes over time [62]. It remains an open question how to define realistic models of topological co-evolution, where higher-order structure and higher-order dynamics evolve together under the effect of mutual feedback [63].

Inferring higher-order interactions from data. A crucial ingredient in modeling real systems is the reconstruction of higher-order interactions from data (Fig. 4). The vast majority of data available on network systems contains only records of pairwise interactions, even if the underlying rules rely on higher-order patterns. Naively attributing every observed dense subgraph in the pairwise network (e.g. triangles and larger cliques) to a putative higher-order interaction conflates the existence of an actual hyperedge with the coincidental accumulation of edges, due –for example– to community structure, homophily or a geometric embedding. Recent work [64] has demonstrated how it is possible to distinguish between hyperedges and combinations of lower-order edges by casting the problem as a Bayesian inference task, taking into account the parsimony of the resulting reconstruction. With such an approach, hyperedges are identified only if they are supported by statistical evidence. It remains an open problem how to generalize such approaches to include more realistic modelling assumptions, containing a tighter interplay with mesoscale structures and latent space embeddings [65].

Even when explicit hyperedge data are available, the same problem as with pairwise network data is still present, namely that a certain amount of errors and incompleteness is unavoidable, requiring us to reconstruct the object of study from uncertain observations [66, 67]. For hypergraph data, recent work [68] has proposed an approach based on comparisons with null models that is capable of filtering out hyperedges that are not statistically significant. More work is needed to provide uncertainty quantification on the analyses that are conditioned on the reconstruction, as well as leveraging more advanced techniques of hyperedge prediction to improve accuracy.

In addition to the reconstruction from direct but uncertain data, an important challenge is the inference of higher-order structures from *indirect* data such as time series, where no edges or hyperedges are directly measured, but only the dynamical behavior on the nodes that result from it. This is an important issue in many biological systems such as the brain, where diseases like Parkinson's and schizophrenia have been associated to dysfunctional brain connectivity [69, 70, 71], but direct network measurements are often not available. A common approach is to compute correlations [72] and measure synchronization [73] between time series. However, these approaches yield only a very unreliable understanding of the underlying system, since they cannot distinguish correlation from causation — i.e. two or more nodes can be highly correlated even if they do not share an edge or hyperedge. Another set of approaches consists in exploiting temporal correlations, for example the phase dynamics reconstruction given a set of multivariate time-series [74]. Originally devised for pairwise interactions only, the methodology has been generalised to account for small motifs of interacting units [75]. Interestingly, the development of new synchronization measures for triplets has made it possible to identify multi-body locking from experimental data, even when every pair of systems taken in isolation remains asynchronous [76]. In particular, this last approach can better differentiate between the physical connections and the effective ones, which are associated to the temporal influence of one node on another one: this can lead to more reliable network reconstruction methods [77, 78]. Finally, another yet-to-be-explored possibility is that of extending information-theoretical techniques, such as Granger causality [79] and transfer entropy [80], to account for the existence of multi-body interactions. Despite promising first steps in reconstructing higher-order interactions from static lower-order projections [64] and in multi-body information-theoretic quantities [81], it remains an open problem how to broaden this framework to consider fully higher-order interaction schemes. It also should be emphasized that reconstruction methods that are based only on temporal correlations still suffer from the problem of not being able to fully distinguish between direct and indirect causation (i.e. the existence of an actual edge/hyperedge between nodes vs. a longer path), as well as non-causal correlation. Ultimately, this is only possible in general if, instead of only observational data, we are allowed to make interventions [82]. Nevertheless, methods based on Bayesian inference of generative models are able to convey the uncertainty about the causal relationships [83]. An important future direction is to generalize such methods to contain higher-order interactions [84, 85, 86, 87, 88, 89, 90], possibly varying in time [91] and describing emergent higher-order geometry [92].

The study of networked systems with higher-order interactions is still in its infancy, posing new challenges and opportunities for discoveries [10, 93, 94, 95]. Yet, it is also inspired by ideas from the past. For instance, earlier work had already considered systems of coupled cells where dependencies of different orders were encoded via particular graph structures [96], clarifying how higher-order symmetries affect synchronization [97, 98]. HOIs can also generate new insights on older problems where they emerge as effective theories. A paradigmatic example is that of networks of phase oscillators with higher-order interactions, that arise from the phase reduction of nonlinear oscillator systems [99, 100, 101]. As a consequence, understanding the dynamics of the phase reduced systems with higher-order interactions can also clarify the physics of the general higher-dimensional system [102, 103, 104, 105], in particular on the emergence of chaos [106] and metastable chimeras [107, 108]. Thus, in addition to providing an exciting way forward for network science, HOIs can also create opportunities for a wider dialogue on the physics of dynamical systems, preparing fertile ground for further potential crosspollination between the two fields.

From p-spin models [13, 109] to multilayer [110] and non-Markovian temporal networks [111] the past suggests that new phenomena may occur when more realistic patterns of interactions are considered. Overcoming previous limitations, new data and new theory are now informing our network models beyond pairwise interactions [10]. How will the physics of higher-order interactions look like? This is an exciting question for the whole community to resolve.

References

1. Barabási, A.-L. The network takeover. *Nat. Phys.* **8**, 14 (2011).

2. Boccaletti, S., Latora, V., Moreno, Y., Chavez, M. & Hwang, D.-U. Complex networks : Structure and dynamics. *Phys. Rep.* **424**, 175–308 (Fervier 2006).

3. Barrat, A., Barthelemy, M. & Vespignani, A. *Dynamical processes on complex networks* (Cambridge university press, 2008).

4. Vespignani, A. Modelling dynamical processes in complex socio-technical systems. *Nat. Phys.* **8**, 32–39 (2012).

5. Pastor-Satorras, R. & Vespignani, A. Epidemic spreading in scale-free networks. *Phys. Rev. Lett.* **86**, 3200 (2001).

6. Boguná, M., Pastor-Satorras, R. & Vespignani, A. Absence of epidemic threshold in scale-free networks with degree correlations. *Phys. Rev. Lett.* **90**, 028701 (2003).

7. Barahona, M. & Pecora, L. M. Synchronization in small-world systems. *Phys. Rev. Lett.* **89**, 054101 (2002).

8. Del Vicario, M. *et al.* The spreading of misinformation online. *Proc. Natl. Acad. Sci.* **113**, 554–559 (2016).

9. Patania, A., Vaccarino, F. & Petri, G. Topological analysis of data. *EPJ Data Sci.* **6**, 7 (2017).

10. Battiston, F. *et al.* Networks beyond pairwise interactions: structure and dynamics. *Phys. Rep.* **874**, 1–92 (2020).

11. Schneidman, E., Berry II, M. J., Segev, R. & Bialek, W. Weak pairwise correlations imply strongly correlated network states in a neural population. *Nature* **440**, 1007 (2006).

12. Schneidman, E., Still, S., Berry, M. J., Bialek, W. *et al.* Network information and connected correlations. *Phys. Rev. Lett.* **91**, 238701 (2003).

13. Yu, S. *et al.* Higher-order interactions characterized in cortical activity. *J. Neurosci.* **31**, 17514–17526 (2011).

14. Giusti, C., Pastalkova, E., Curto, C. & Itskov, V. Clique topology reveals intrinsic geometric structure in neural correlations. *Proc. Natl. Acad. Sci. U.S.A.* **112**, 13455–13460 (2015).

15. Gardner, R. J. *et al.* Toroidal topology of population activity in grid cells. *bioRxiv* (2021).

16. Ganmor, E., Segev, R. & Schneidman, E. Sparse low-order interaction network underlies a highly correlated and learnable neural population code. *Proc. Natl. Acad. Sci. U.S.A.* **108**, 9679–9684 (2011).

17. Merchan, L. & Nemenman, I. On the sufficiency of pairwise interactions in maximum entropy models of networks. *J. Stat. Phys.* **162**, 1294–1308 (2016).

18. Mayfield, M. M. & Stouffer, D. B. Higher-order interactions capture unexplained complexity in diverse communities. *Nat. Ecol. Evol.* **1**, 0062 (2017).

19. Cervantes-Loreto, A., Ayers, C., Dobbs, E., Brosi, B. & Stouffer, D. The context dependency of pollinator interference: how environmental conditions and species abundances impact floral visitation. *Authorea Prepr.* (2020).

20. Bairey, E., Kelsic, E. D. & Kishony, R. High-order species interactions shape ecosystem diversity. *Nat. Commun.* **7**, 12285 (2017).

21. Grilli, J., Barabás, G., Michalska-Smith, M. J. & Allesina, S. Higher-order interactions stabilize dynamics in competitive network models. *Nature* **548**, 210 (2017).

22. Ritz, A., Tegge, A. N., Kim, H., Poirel, C. L. & Murali, T. Signaling hypergraphs. *Trends Biotechnol.* **32**, 356–362 (2014).

23. Centola, D., Becker, J., Brackbill, D. & Baronchelli, A. Experimental evidence for tipping points in social convention. *Science* **360**, 1116–1119 (2018).

24. Milojević, S. Principles of scientific research team formation and evolution. *Proc. Natl. Acad. Sci.* **111**, 3984–3989 (2014).

25. Povh, B., Rith, K., Scholz, C. & Zetsche, F. Many-body systems in the strong interaction. In *Particles and Nuclei*, 281–303 (Springer, 1995).

26. Duck, I. Three-alpha-particle resonances via the fadeev equation. *Nucl. Phys.* **84**, 586–594 (1966).

27. Kim, H.-Y., Sofo, J. O., Velegol, D., Cole, M. W. & Lucas, A. A. van der waals forces between nanoclusters: Importance of many-body effects. *J. Chem. Phys.* **124**, 074504 (2006).

28. Zeiher, J. *et al.* Coherent many-body spin dynamics in a long-range interacting ising chain. *Phys. Rev. X* **7**, 041063 (2017).

29. Schaub, M. T., Benson, A. R., Horn, P., Lippner, G. & Jadbabaie, A. Random walks on simplicial complexes and the normalized Hodge Laplacian. *SIAM Rev.* **62**, 353–391 (2020).

30. Carletti, T., Battiston, F., Cencetti, G. & Fanelli, D. Random walks on hypergraphs. *Phys. Rev. E* **101**, 022308 (2020).

31. Millán, A. P., Torres, J. J. & Bianconi, G. Synchronization in network geometries with finite spectral dimension. *Phys. Rev. E* **99**, 022307 (2019).

32. Skardal, P. S. & Arenas, A. Abrupt desynchronization and extensive multistability in globally coupled oscillator simplexes. *Phys. Rev. Lett.* **122**, 248301 (2019).

33. Iacopini, I., Petri, G., Barrat, A. & Latora, V. Simplicial models of social contagion. *Nat. Commun.* **10**, 2485 (2019).

34. de Arruda, G. F., Petri, G. & Moreno, Y. Social contagion models on hypergraphs. *Phys. Rev. Res.* **2**, 023032 (2020).

35. Neuhäuser, L., Mellor, A. & Lambiotte, R. Multibody interactions and nonlinear consensus dynamics on networked systems. *Phys. Rev. E* **101**, 032310 (2020).

36. Alvarez-Rodriguez, U. *et al.* Evolutionary dynamics of higher-order interactions in social networks. *Nat. Hum. Behav.* **5**, 586–595 (2021).

37. Achlioptas, D., D'Souza, R. M. & Spencer, J. Explosive percolation in random networks. *Science* **323**, 1453–1455 (2009).

38. da Costa, R. A., Dorogovtsev, S. N., Goltsev, A. V. & Mendes, J. F. F. Explosive percolation transition is actually continuous. *Phys. Rev. Lett.* **105**, 255701 (2010).

39. Riordan, O. & Warnke, L. Explosive percolation is continuous. *Science* **333**, 322–324 (2011).

40. D'Souza, R. M., Gómez-Gardeñes, J., Nagler, J. & Arenas, A. Explosive phenomena in complex networks. *Adv. Phys.* **68**, 123–223 (2019).

41. Gómez-Gardenes, J., Gómez, S., Arenas, A. & Moreno, Y. Explosive synchronization transitions in scale-free networks. *Phys. Rev. Lett.* **106**, 128701 (2011).

42. Matamalas, J. T., Gómez, S. & Arenas, A. Abrupt phase transition of epidemic spreading in simplicial complexes. *Phys. Rev. Res.* **2**, 012049 (2020).

43. Chowdhary, S., Kumar, A., Cencetti, G., Iacopini, I. & Battiston, F. Simplicial contagion in temporal higher-order networks. *J. Phys.: Complex.* (2021).

44. St-Onge, G., Sun, H., Allard, A., Hébert-Dufresne, L. & Bianconi, G. Bursty exposure on higher-order networks leads to nonlinear infection kernels. *arXiv preprint arXiv:2101.07229* (2021).

45. Landry, N. W. & Restrepo, J. G. The effect of heterogeneity on hypergraph contagion models. *Chaos: An Interdiscip. J. Nonlinear Sci.* **30**, 103117 (2020).

46. St-Onge, G. *et al.* Influential groups for seeding and sustaining hypergraph contagions. *arXiv preprint arXiv:2105.07092* (2021).

47. Sun, H. & Bianconi, G. Higher-order percolation processes on multiplex hypergraphs. *arXiv preprint arXiv:2104.05457* (2021).

48. Kuramoto, Y. Self-entrainment of a population of coupled non-linear oscillators. In *International symposium on mathematical problems in theoretical physics*, 420–422 (Springer, 1975).

49. Skardal, P. S. & Arenas, A. Higher order interactions in complex networks of phase oscillators promote abrupt synchronization switching. *Commun. Phys.* **3**, 1–6 (2020).

50. Lucas, M., Cencetti, G. & Battiston, F. Multiorder laplacian for synchronization in higher-order networks. *Phys. Rev. Res.* **2**, 033410 (2020).

51. Gambuzza, L. *et al.* Stability of synchronization in simplicial complexes. *Nat. Commun.* **12**, 1–13 (2021).

52. Zhang, Y., Latora, V. & Motter, A. E. Unified treatment of synchronization patterns in generalized networks with higher-order, multilayer, and temporal interactions. *arXiv preprint arXiv:2010.00613* (2021).

53. de Arruda, G. F., Tizzani, M. & Moreno, Y. Phase transitions and stability of dynamical processes on hypergraphs. *Commun. Phys.* **4**, 1–9 (2021).

54. St-Onge, G., Thibeault, V., Allard, A., Dubé, L. J. & Hébert-Dufresne, L. Master equation analysis of mesoscopic localization in contagion dynamics on higher-order networks. *Phys. Rev. E* **103**, 032301 (2021).

55. Kuehn, C. & Bick, C. A universal route to explosive phenomena. *Sci. Adv.* **7**, eabe3824 (2021).

56. Cisneros-Velarde, P. & Bullo, F. Multi-group sis epidemics with simplicial and higher-order interactions. *arXiv:2005.11404* (2020).

57. Millán, A. P., Torres, J. J. & Bianconi, G. Explosive higher-order kuramoto dynamics on simplicial complexes. *Phys. Rev. Lett.* **124**, 218301 (2020).

58. Porter, M. & Gleeson, J. *Dynamical Systems on Networks: A Tutorial*. Frontiers in Applied Dynamical Systems: Reviews and Tutorials (Springer International Publishing, 2016).

59. Ghorbanchian, R., Restrepo, J. G., Torres, J. J. & Bianconi, G. Higher-order simplicial synchronization of coupled topological signals. *Commun. Phys.* **4**, 1–13 (2021).

60. Schaub, M. T. & Segarra, S. Flow smoothing and denoising: Graph signal processing in the edge-space. In *2018 IEEE Global Conference on Signal and Information Processing (GlobalSIP)*, 735–739 (IEEE, 2018).

61. Barbarossa, S. & Sardellitti, S. Topological signal processing over simplicial complexes. *IEEE Transactions on Signal Process.* (2020).

62. Holme, P. & Saramäki, J. Temporal networks. *Phys. Rep.* **519**, 97–125 (2012).

63. Gross, T. & Blasius, B. Adaptive coevolutionary networks: a review. *J. Royal Soc. Interface* **5**, 259–271 (2008).

64. Young, J.-G., Petri, G. & Peixoto, T. P. Hypergraph reconstruction from network data. *Commun. Phys.* **4**, 1–11 (2021).

65. Saebi, M., Ciampaglia, G. L., Kaplan, L. M. & Chawla, N. V. Honem: learning embedding for higher order networks. *Big Data* **8**, 255–269 (2020).

66. Newman, M. E. J. Network structure from rich but noisy data. *Nat. Phys.* 542–545 (2018).

67. Peixoto, T. P. Reconstructing networks with unknown and heterogeneous errors. *Phys. Rev. X* 041011 (2018).

68. Musciotto, F., Battiston, F. & Mantegna, R. N. Detecting informative higher-order interactions in statistically validated hypergraphs. *arXiv preprint arXiv:2103.16484* (2021).

69. Bohland, J. W. *et al.* A proposal for a coordinated effort for the determination of brainwide neuroanatomical connectivity in model organisms at a mesoscopic scale. *PLoS Comput. Biol* **5**, e1000334 (2009).

70. Bassett, D. S. & Sporns, O. Network neuroscience. *Nat. Neurosci.* **20**, 353 (2017).

71. Glomb, K. *et al.* Computational models in electroencephalography. *Brain Topogr.* 1–20 (2021).

72. Skudlarski, P. *et al.* Measuring brain connectivity: diffusion tensor imaging validates resting state temporal correlations. *NeuroImage* **43**, 554—561 (2008).

73. Tass, P. *et al.* Detection of n:m Phase Locking from Noisy Data: Application to Magnetoencephalography. *Phys. Rev. Lett.* **81**, 3291–3294 (1998).

74. Kralemann, B., Cimponeriu, L., Rosenblum, M., Pikovsky, A. & Mrowka, R. Phase dynamics of coupled oscillators reconstructed from data. *Phys. Rev. E* **77**, 066205 (2008).

75. Kralemann, B., Pikovsky, A. & Rosenblum, M. Reconstructing phase dynamics of oscillator networks. *Chaos* **21**, 025104 (2011).

76. Kralemann, B., Pikovsky, A. & Rosenblum, M. Detecting triplet locking by triplet synchronization indices. *Phys. Rev. E* **87**, 052904 (2013).

77. Kralemann, B., Pikovsky, A. & Rosenblum, M. Reconstructing effective phase connectivity of oscillator networks from observations. *New J. Phys.* **16**, 085013 (2014).

78. Pikovsky, A. Reconstruction of a random phase dynamics network from observations. *Phys. Lett. A* **382**, 147–152 (2018).

79. Barnett, L., Barrett, A. B. & Seth, A. K. Granger causality and transfer entropy are equivalent for gaussian variables. *Phys. Rev. Lett.* **103**, 238701 (2009).

80. Schreiber, T. Measuring information transfer. *Phys. Rev. Lett.* **85**, 461 (2000).

81. Rosas, F. E., Mediano, P. A., Gastpar, M. & Jensen, H. J. Quantifying high-order interdependencies via multivariate extensions of the mutual information. *Phys. Rev. E* **100**, 032305 (2019).

82. Pearl, J. *Causality* (Cambridge university press, 2009).

83. Peixoto, T. P. Network reconstruction and community detection from dynamics. *Phys. Rev. Lett.* **123**, 128301 (2019).

84. Bianconi, G., Darst, R. K., Iacovacci, J. & Fortunato, S. Triadic closure as a basic generating mechanism of communities in complex networks. *Phys. Rev. E* **90**, 042806 (2014).

85. Courtney, O. T. & Bianconi, G. Generalized network structures: The configuration model and the canonical ensemble of simplicial complexes. *Phys. Rev. E* **93**, 062311 (2016).

86. Chodrow, P. S. Configuration models of random hypergraphs. *J. Complex Netw.* **8**, DOI: 10.1093/comnet/cnaa018 (2020).

87. Chodrow, P. S., Veldt, N. & Benson, A. R. Generative hypergraph clustering: From blockmodels to modularity. *Sci. Adv.* **7**, eabh1303, DOI: 10.1126/sciadv.abh1303 (2021). Publisher: American Association for the Advancement of Science Section: Research Article.

88. Courtney, O. T. & Bianconi, G. Weighted growing simplicial complexes. *Phys. Rev. E* **95**, 062301 (2017).

89. Kovalenko, K. *et al.* Growing scale-free simplices. *Commun. Phys.* **4**, 1–9 (2021).

90. Peixoto, T. P. Disentangling homophily, community structure and triadic closure in networks. *arXiv:2101.02510 [physics, stat]* (2021). ArXiv: 2101.02510.

91. Cencetti, G., Battiston, F., Lepri, B. & Karsai, M. Temporal properties of higher-order interactions in social networks. *Sci. reports* **11**, 1–10 (2021).

92. Bianconi, G. & Rahmede, C. Emergent hyperbolic network geometry. *Sci. Rep.* **7**, 1–9 (2017).

93. Torres, L., Blevins, A. S., Bassett, D. S. & Eliassi-Rad, T. The why, how, and when of representations for complex systems. *arXiv:2006.02870* (2020).

94. Bick, C., Gross, E., Harrington, H. A. & Schaub, M. T. What are higher-order networks? *arXiv:2104.11329* (2021).

95. Benson, A. R., Gleich, D. F. & Higham, D. J. Higher-order network analysis takes off, fueled by classical ideas and new data. *arXiv preprint arXiv:2103.05031* (2021).

96. Golubitsky, M. & Stewart, I. Nonlinear dynamics of networks: the groupoid formalism. *Bull. Am. Math. Soc* **43**, 305–364 (2006).

97. Stewart, I., Golubitsky, M. & Pivato, M. Symmetry groupoids and patterns of synchrony in coupled cell networks. *SIAM J. on Appl. Dyn. Syst.* **2**, 609–646 (2003).

98. Golubitsky, M., Stewart, I. & Török, A. Patterns of synchrony in coupled cell networks with multiple arrows. *SIAM J. on Appl. Dyn. Syst.* **4**, 78–100 (2005).

99. Nakao, H. Phase reduction approach to synchronisation of nonlinear oscillators. *Contemp. Phys.* **57**, 188–214 (2016).

100. Pietras, B. & Daffertshofer, A. Network dynamics of coupled oscillators and phase reduction techniques. *Phys. Rep.* (2019).

101. Matheny, M. H. *et al.* Exotic states in a simple network of nanoelectromechanical oscillators. *Science* **363**, eaav7932 (2019).

102. Komarov, M. & Pikovsky, A. Dynamics of multifrequency oscillator communities. *Phys. Rev. Lett.* **110**, 134101 (2013).

103. Ashwin, P. & Rodrigues, A. Hopf normal form with S_N symmetry and reduction to systems of nonlinearly coupled phase oscillators. *Phys. D* **325**, 14–24 (2016).

104. Ashwin, P., Bick, C. & Burylko, O. Identical phase oscillator networks: Bifurcations, symmetry and reversibility for generalized coupling. *Front Appl. Math. Stat.* **2**, 7 (2016).

105. León, I. & Pazó, D. Phase reduction beyond the first order: The case of the mean-field complex Ginzburg-Landau equation. *Phys. Rev. E* **100**, 012211 (2019).

106. Bick, C., Ashwin, P. & Rodrigues, A. Chaos in generically coupled phase oscillator networks with nonpairwise interactions. *Chaos* **26**, 094814 (2016).

107. Bick, C. Heteroclinic switching between chimeras. *Phys. Rev. E* **97**, 050201 (2018).

108. Bick, C. Heteroclinic dynamics of localized frequency synchrony: Heteroclinic cycles for small populations. *J Nonlin Sci* (2019).

109. Kirkpatrick, T. R. & Thirumalai, D. Dynamics of the structural glass transition and the p-spin—interaction spin-glass model. *Phys. Rev. Lett.* **58**, 2091 (1987).

110. De Domenico, M., Granell, C., Porter, M. A. & Arenas, A. The physics of spreading processes in multilayer networks. *Nat. Phys.* **12**, 901–906 (2016).

111. Lambiotte, R., Rosvall, M. & Scholtes, I. From networks to optimal higher-order models of complex systems. *Nat. Phys.* 1 (2019).