A scalable method for the analysis of networked linear systems with decentralized sampled-data control

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Abstract

In this paper, the stability analysis problem of networked linear systems with decentralized sampled-data controllers is considered. The networks under consideration are composed of interconnected identical systems. A dissipativity-based approach is utilized to analyze stability, grounded in an interconnection interpretation. A structure exploiting and scalable approach is employed to derive a sufficient stability condition for large-scale linear systems, which is independent of the number of subsystems. In the analysis, the effects of uncertainty on the system models, which include uncertainties that render the coupled systems non-identical, aperiodic sampling, and a switching network topology, are taken into account. Numerical examples are presented to demonstrate the main result and simulate the sampled-data system.

Keywords— Decentralized control ; Sampled-data control; Large-scale systems; LMI techniques

1. Introduction

Decentralized control is ubiquitous in modern control implementation schemes. This paradigm has seen an increasing number of developments in the last few years, due to the boom of Internet of Things (IoT) and large-scale cyber-physical systems [1, 2]. Each decentralized controller in the large scale system measures a part of the network state and acts on the system in accordance with its limited computation and communication capabilities. In this paper we are interested in the decentralized control problem for networks of (quasi-) identical systems. Typical examples are encountered in power systems [3, 4] and automated vehicular platoons [5].

The decentralized control architecture provides several advantages. Generally, no centralized and synchronous scheme is practical for coordinating the control actions in large scale networks. Decentralized controllers are preferred for such systems due to their practicality, ease of implementation, maintenance costs, and compatibility with various information sharing policies among the different actors involved [2, 5, 6]. However, the design of decentralized controllers is challenging since these controllers have to meet global objectives collectively while acting (and sensing) locally, without coordinating their action and using limited computational resources. The digital implementation of such control elements is usually asynchronous [2, 7, 8, 9]. While in the last decade a significant effort has been dedicated to the study of centralized network control schemes [10, 11, 12] and of systems with event-triggered controllers [13, 14, 15], the main problem today is to provide methods for the design of asynchronous sampled-data control architectures in large scale networks with decentralized controllers. In this paper, this topic is addressed for networks of (quasi-) identical systems with asynchronous (decentralized) sampled-data controllers. Two main theoretical challenges must be handled in this context.

The first challenge concerns the *asynchrony in sampling* among different controllers. This challenge is difficult to address since asynchrony in sampling times may be a source of poor performance and even instability. It has been recently shown in the literature that even for the case of simple linear time-invariant (LTI) systems, an asynchronous (decentralized) sampled-data control implementation may lead to an unstable behavior [15, 16] if particular attention is not payed. Unfortunately, although sampled-data control is a mature area in control theory, the particular stability problem for systems with asynchronously sampled controllers has so far received little attention. As far as we know, results exist only for the case of LTI systems [15, 16, 17, 18] and for linear time-delay systems [19].

The second theoretical challenge lies in the *dimensionality* of the problem. A fundamental problem in the analysis of decentralized controllers for networked systems consists of handling the large dimension, induced by the interconnection of many subsystems and controllers. Considering the entire network as one sampled-data

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system (i.e. applying directly the sampled-data approaches form [16, 17, 20]) would lead to stability conditions with a computational complexity that grows exponentially with the number of the subsystems, which would not be numerically tractable. The challenge is to derive *scalable* stability conditions in the sense that the computational cost to check these conditions should not depend on the number of subsystems.

Results addressing both of these theoretical challenges in the context of sampled-data networked systems are rare [19]. In this paper, this particular problem is addressed for networks of (quasi-) identical linear subsystems. In other words, we study the effect of asynchrony in sampling between local decentralized controllers on the Lyapunov stability of the overall networked system. The case of linear static decentralized controllers is considered. The main objective is to derive scalable Lyapunov stability conditions independent of the number of subsystems, thereby reducing the critical dimension for the numerical criteria to the dimension of individual subsystems. To the best of our knowledge, this problem setting is new in the literature.

There are a few conceptual similarities between the setting of this paper and that of other work in the literature. The problem of how large the asynchony (i.e. the clock offset) between actuator and sensor can be for a centralized control configuration, without compromising the stability of an LTI with linear static controller, was addressed in [8, 7, 21]. For linear plants, the analysis of decentralized event-triggered controllers (i.e. the case of controlled sampling) can be found in [13, 14]. In contrast, here we address the stability problem for the case of systems with arbitrary sampled controllers. Recently in [19], we have addressed the $\mathcal{L}_2$ stability problem for (time-invariant) networks of input-output time-delay subsystems with decentralized sampled-data controllers using a frequency domain approach. In the present paper, we propose a state-space method to address the Lyapunov stability problem. The main advantage of the state-space formulation is the fact that it easily allows to take into account time-varying network topologies and subsystems with polytopic uncertainties. From a technical point of view, the results presented in this paper use network structure exploiting ideas inspired by [22]. In the literature, sampled-data systems are modelled as time-delay systems [23, 24], hybrid systems [13, 25], discrete-time switched systems with varying parameters [12], feedback interconnections of systems [26, 27] (see the survey [28]). In this paper, the problem under study is addressed from a feedback interconnection point of view, building upon the results based on Dissipativity Theory and the study of the reset operator in [29, 30].

The main contributions of this paper are as follows. The major result is a dissipativity-based approach for the analysis of decentralized sampled-data linear static controllers in networks of (quasi-) identical LTI subsystems. We derive scalable Lyapunov stability conditions which are independent of the number of subsystems, thereby reducing the critical dimension for the numerical criteria to the dimension of individual subsystems. The proposed conditions are in the form of Linear Matrix Inequalities (LMI) of the order of the subsystems, which allow to analyze the stability of large scale network. For generality, we take into account imperfections that render the coupled subsystems non-identical, asynchronous operation of controllers, aperiodic sampling [31, 27] and switching network topologies [32, 33, 34].

The remainder of this paper is organized as follows. First, the sampled-data control problem of LTI system with time-invariant topology is presented in Section 2. In Section 3, the system is modelled as a feedback interconnection between a continuous-time closed-loop system (with no sampling) and an uncertainty operator representing the sampling. In Section 4, the stability analysis approach based on the dissipativity-based framework is presented. The results are extended to consider a switching network topology in Section 5 with a brief discussion on other possible extensions. Numerical examples are presented in Section 6 to verify the corresponding result using MATLAB (and YALMIP[35]) software. Finally, some concluding remarks are presented in Section 7.

We use the following notations throughout the paper. $\mathbb{N}$ is the set of all natural numbers and zero. $\mathbb{R}$ is the set of all real numbers. $\mathbb{R}^n$ is the n-dimensional real vector space. $\mathbb{Z}$, $\mathbb{Z}_0^+$, and $\mathbb{Z}^-$ are the sets of all integers, non-negative integers, and negative integers, respectively. $\mathbb{S}^{n\times n}$ is the set of all symmetric real-valued matrices of dimension $n \times n$, also $\mathbb{S}^{n\times n} \subset \mathbb{R}^{n\times n}$. The Kronecker product is denoted by $\otimes$. $\mathcal{L}_{2e}(a, b)$ is the extended $\mathcal{L}_2$-space of all square integrable and Lebesgue measurable functions defined on the interval (a, b) of appropriate dimension. We use $P > 0$ ($P < 0$) to denote that P is a positive (negative) definite matrix. A positive (negative) definite matrix has positive (negative) real part for all its eigenvalues. $\text{blkdiag}(P_1, P_2)$ is a matrix with the matrices P_1 and P_2 as its diagonal blocks while all the other elements are zero.

This article fits within two widely studied research areas by Prof. Vladimir L. Kharitonov: time-delay systems and robust stability analysis. Without Kharitonov's pioneering contributions to the mathematical foundations and analysis tools, this article would not have been possible. As the authors, we consider it a honor to contribute to this special issue.

2. Problem statement

Let us consider a system that satisfies the following nodal dynamics almost everywhere in time

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + \textstyle\sum_{j=1}^{n} a_{Mi,j} F x_j(t) + Bu_i(t) + \delta A_i(t) x_i(t), \\ x_i(0) &= x_{i,0} \in \mathbb{R}^{n_x}, \ \forall\, i \in \mathcal{N}, \end{aligned} \tag{1}$$

where $\mathcal{N} := \{1, ..., n\}$ with the number of subsystems $n \in \mathbb{N}$, $x_i \in \mathbb{R}^{n_x}$ is the state of the subsystem, $\dot{x}_i$ is the right-hand derivative of the subsystem state, $u_i \in \mathbb{R}^{n_u}$ is the input of the subsystem, $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, and $F \in \mathbb{R}^{n_x \times n_x}$ are constant real-valued matrices. Function $\delta A_i(t) \in \mathbb{R}^{n_x \times n_x}, t \in \mathbb{T}(= \mathbb{R}_0^+)$, represents the uncertainty at node i, and satisfies $||\delta A_i(t)||_2 \leq d_e, \ \forall\, i \in \mathcal{N}$ for some $d_e \in \mathbb{R}^+$. Since the uncertainties are allowed to be non-identical, we refer to systems of the form (1) as quasi-identical systems. Real number $a_{Mi,j}$ is the $(i,j)^{\text{th}}$ element of the constant (real-valued) connectivity matrix A_M. We consider the following assumption for the connectivity matrix.

Assumption 1 *(Undirected Graphs) The connectivity matrix, A_M, is a* real-symmetric *matrix, i.e. $A_M \in \mathbb{S}^{n \times n}$. Moreover, we assume that its spectrum belong to the interval* $[-1, 1]$.

The decentralized control loops are assumed to be asynchronous in the sense that each node has its own sampling mechanism. For each node i, we consider a monotonously increasing sampling sequence σ_i with bounded intervals,

$$\sigma_i = \{t_k^i\}_{k \in \mathbb{N}} \text{ with } t_{k+1}^i - t_k^i \in (0, h], \ k \in \mathbb{N}, \ \forall\, i \in \mathcal{N}, \tag{2}$$

where $h \in \mathbb{R}^+$. We assume that $t_0^i = 0, \ \forall\, i \in \mathcal{N}$. The outputs $(y_i(t) \in \mathbb{R}^{n_y}, \ \forall\, i \in \mathcal{N})$ from the system are sampled and held as follows

$$y_i(t) = Cx_i(t_k^i), \ \forall\, t \in [t_k^i, t_{k+1}^i), \ k \in \mathbb{N}, \ \forall\, i \in \mathcal{N}, \tag{3}$$

where $C \in \mathbb{R}^{n_y \times n_x}$. The controller is considered to be a static decentralized output feedback controller using the known, sampled version of state,

$$u_i(t) = Ky_i(t), \ \forall\, i \in \mathcal{N}, \tag{4}$$

where $K \in \mathbb{R}^{n_u \times n_y}$ is the real-valued constant feedback gain matrix. Let $x(t) = [x_1^\top(t) \ ... \ x_n^\top(t)]^\top$, $u(t) = [u_1^\top(t) \ ... \ u_n^\top(t)]^\top$, $y(t) = [y_1^\top(t) \ ... \ y_n^\top(t)]^\top$, and $\delta A(t) = \text{blkdiag}(\delta A_1(t), ..., \delta A_n(t))$. The main objective of this paper is to analyze the stability of system (1)-(4), by proposing an LMI stability criteria independent of the (fixed) number of nodes n. Throughout this paper, scalable Lyapunov like stability conditions are derived using LMI techniques.

3. Preliminaries

In this section, we present the preliminary concepts used in deriving scalable stability conditions. Let us first consider a continuous-time control signal assuming that there is no sampling,

$$\hat{u}_i(t) = KCx_i(t), \ \forall\, i \in \mathcal{N}, \tag{5}$$

for all $t > 0$. Notice that the actual control signal in (4) can be decomposed into a continuous-time signal $\hat{u}_i(t)$ and an error term corresponding to the sampling. We denote by $e_{s,i}(t)$ the error induced through sampling at the input for node i, that is,

$$e_{s,i}(t) = u_i(t) - \hat{u}_i(t), \tag{6}$$

where $u_i(t)$ is given in (4). In what follows, we rewrite the networked system as an input-output interconnection of a continuous time system (without sampling or uncertainties) and an error operator representing the sampling problem and uncertainties. Before re-modelling the system, we present some preliminaries on the ideal continuous-time network.

3.1. Scalable stability condition for the continuous-time system

In this subsection, scalable stability conditions of the order of the subsystems are provided. We consider a simple continuous-time interconnected system without any sampling process or uncertainty. That is, we consider the following network with identical nodal dynamics

$$\begin{cases} \dot{\tilde{x}}_i(t) = A\tilde{x}_i(t) + \sum_{j=1}^{n} a_{Mi,j} F \tilde{x}_j(t) + B\tilde{u}_i(t), \\ \tilde{y}_i(t) = C\tilde{x}_i(t), \\ \tilde{x}_i(0) = \tilde{x}_{i,0} \in \mathbb{R}^{n_x}, \ \forall\, i \in \mathcal{N}, \end{cases} \tag{7}$$

where $\tilde{x}_i(t)$ is the state, $\tilde{u}_i(t)$ is the continuous-time input, and $\tilde{y}_i(t)$ is the continuous-time measured output for the node i. The continuous-time controlled input is computed using the identical static controller K for each node, that is,

$$\tilde{u}_i(t) = K\tilde{y}_i(t),\ \forall\, i \in \mathcal{N}. \tag{8}$$

Then the continuous-time closed-loop system satisfies the following dynamical equation

$$\dot{\tilde{x}}(t) = (I \otimes A_{cl} + A_M \otimes F_{cl})\tilde{x}(t), \tag{9}$$

where $\tilde{x}(t) = [\tilde{x}_1^\top(t)\ ...\ \tilde{x}_n^\top(t)]^\top, A_{cl} = A + BKC$ and $F_{cl} = F$.

Before providing scalable stability conditions, we introduce some useful notations. Throughout the paper, we consider a polynomial matrix $\tilde{P} : [-1, 1] \to \mathbb{R}^{n_x \times n_x}$, that is,

$$\tilde{P}(\omega) = \tilde{P}_0 + \omega\tilde{P}_1 + \cdots + \omega^{n_a}\tilde{P}_{n_a}, \tag{10}$$

where $\tilde{P}_i \in \mathbb{R}^{n_x \times n_x},\ \forall\, i = 1, ..., n_a$, are the coefficient matrices and n_a is the order of the polynomial. Also, we consider an associated matrix valued function $\mathcal{P} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\cdot n_x \times n \cdot n_x}$, defined through

$$\mathcal{P}(\Omega) = I \otimes \tilde{P}_0 + \Omega \otimes \tilde{P}_1 + \cdots + \Omega^{n_a} \otimes \tilde{P}_{n_a}, \tag{11}$$

Then, we can state the following result regarding the stability of system (7)-(8).

Theorem 3.1 *If Assumption 1 holds and there exists a function $\tilde{P}(\lambda_a) = \tilde{P}(\lambda_a)^\top > 0$ of the form (10), such that the following LMI is satisfied,*

$$(A_{cl} + \lambda_a F_{cl})^\top \tilde{P}(\lambda_a) + \tilde{P}(\lambda_a)(A_{cl} + \lambda_a F_{cl}) < 0,\ \forall\ \lambda_a \in [-1, 1], \tag{12}$$

then system (7)-(8) is exponentially stable.

Proof. There exists a number $\alpha > 0$ such that

$$\tilde{P}(\lambda_a) \geq \alpha I,\ \ (A_{cl} + \lambda_a F_{cl})^\top \tilde{P}(\lambda_a) + \tilde{P}(\lambda_a)(A_{cl} + \lambda_a F_{cl}) \leq -\alpha I,\ \forall \lambda_a \in [-1, 1]. \tag{13}$$

We now consider the Lyapunov function

$$V(\tilde{x}) = \tilde{x}^\top \mathcal{P}(A_M)\tilde{x}. \tag{14}$$

Based on Assumption 1, there exists an orthogonal matrix, T, such that it can spectrally decompose the connectivity matrix to a real-valued diagonal matrix Λ, that is, $TA_MT^\top = \Lambda$. We note that $T \otimes I$ is also an orthogonal matrix and

$$\begin{aligned}\mathcal{P}(A_M) &= (T^\top T) \otimes \tilde{P}_0 + (T^\top \Lambda T) \otimes \tilde{P}_1 + \cdots + (T^\top \Lambda^{n_a} T) \otimes \tilde{P}_{n_a} \\ &= (T^\top \otimes I)\mathcal{P}(\Lambda)(T \otimes I),\end{aligned} \tag{15}$$

where

$$\mathcal{P}(\Lambda) = \begin{bmatrix} \tilde{P}(\lambda_{a1}) & & \\ & \ddots & \\ & & \tilde{P}(\lambda_{an}) \end{bmatrix}. \tag{16}$$

Letting $\tilde{z} = (T \otimes I)\tilde{x}$, such that $\|\tilde{z}\|_2 = \|\tilde{x}\|_2$, we get

$$V(\tilde{x}) = \tilde{z}^\top \mathcal{P}(\Lambda)\tilde{z} = \sum_{i=1}^{n} \tilde{z}_i^\top \tilde{P}(\lambda_{ai})\tilde{z}_i \geq \alpha \tilde{z}^T\tilde{z} = \alpha\|\tilde{x}\|_2^2. \tag{17}$$

The derivative of the Lyapunov function, along trajectories of the closed-loop system, satisfies

$$\begin{aligned}\dot{V}(\tilde{x}) &= \tilde{x}^T \left((I \otimes A_{cl} + A_M \otimes F_{cl})^\top \mathcal{P}(A_M) + \mathcal{P}(A_M)(I \otimes A_{cl} + A_M \otimes F_{cl})\right)\tilde{x} \\ &= \tilde{x}^\top \left((T^\top \otimes I)(I \otimes A_{cl} + \Lambda \otimes F_{cl})^\top \mathcal{P}(\Lambda)(T \otimes I) + (T^\top \otimes I)\mathcal{P}(\Lambda)(I \otimes A_{cl} + \Lambda \otimes F_{cl})(T \otimes I)\right)\tilde{x} \\ &= \tilde{z}^\top \left((I \otimes A_{cl} + \Lambda \otimes F_{cl})^\top \mathcal{P}(\Lambda) + \mathcal{P}(\Lambda)(I \otimes A_{cl} + \Lambda \otimes F_{cl})\right)\tilde{z}\end{aligned}$$

Using (13), we obtain

$$\dot{V}(\tilde{x}) \leq -\alpha\|\tilde{z}\|_2^2 = -\alpha\|\tilde{x}\|_2^2 \tag{18}$$

and the proof is complete. ◦

Remark 1 *Theorem 3.1 provides a scalable stability condition since all the information about the network structure is contained in Assumption 1. Hence if condition (12) along with the positivity of $\tilde{P}(\lambda_a)$ is satisfied, then* any *network of the considered subsystems satisfying Assumption 1 is exponentially stable. Furthermore, the condition in (12) is a sufficient condition for $(A_{cl} + \lambda_a F_{cl})$ to be Hurwitz, where λ_a can be interpreted as a bounded uncertainty corresponding to a possible value of an eigenvalue of the connectivity matrix A_M.*

The Lyapunov function (14), on which the proof of Theorem 3.1 is based, depends on connectivity matrix A_M. By restricting it to the form $V(\tilde{x}) = \tilde{x}^\top (I \otimes P)\tilde{x}$, we obtain the following corollary.

Corollary 3.2 *If Assumption (1) holds and there exists a $P = P^\top > 0$ such that the following LMI is satisfied,*

$$(A_{cl} + \lambda_a F_{cl})^\top P + P(A_{cl} + \lambda_a F_{cl}) < 0, \ \forall \ \lambda_a \in \{-1, 1\}, \tag{19}$$

then system (7)-(8) is exponentially stable.

Remark 2 *If the number of subsystems n is prescribed, then the LMI conditions in Corollary 3.2 provide stability conditions for any topology satisfying Assumption 1, with a Lyapunov function $\tilde{x}^T(I \otimes P)\tilde{x}$ that is independent of the topology. Using this property, it can be shown that stability is preserved if A_M is time-varying or switching between network topologies whose connectivity matrices satisfy Assumption 1 for almost all t.*

Remark 3 *Condition (19) can be interpreted as a sufficient condition for matrix $(A_{cl} + \lambda_a F_{cl}), \ \lambda_a \in [-1, 1]$, to be Hurwitz with common Lyapunov function $x^T P x, \ x \in \mathbb{R}^{n_s}$, in contrast to the parameterized Lyapunov function $x^T \tilde{P}(\lambda_a)x$ for (12). The consequence is that a polynomially parameter-dependent LMI is replaced by two standard LMIs, which are computationally more tractable, possibly at the price of increased conservatism.*

3.2. Sampled-data system remodelling

In this subsection, we provide a simpler model of the closed-loop system with asynchronous sampled-data control, (1)-(4), that can be used for the stability analysis. The main idea is to rewrite (1)-(4) as an interconnection of a nominal continuous-time system and an uncertainty operator representing the effect of sampling and parametric perturbations.

First, we recall the techniques used to handle the sampling induced error and norm-bounded uncertainty in the input-output framework. Then, we provide a new model of the closed loop system which can be handled easily in the dissipativity-based framework. The principle of the input-output interconnection method is to characterize the sampling and uncertainty induced perturbation by a single operator.

Recall the sampled-data control input in (4), u_i, and the ideal, continuous-time version in (5), $\hat{u}_i$. The operator has as argument the derivative of the continuous-time input without sampled-data control ($KC\dot{x}_i(t)$, to describe the sampling error) and the state ($x_i(t)$, to describe the norm-bounded uncertainty). Let us consider $e_{s,i}(t)$ as the error induced due to sampling at node i. Notice that

$$u_i(t) = \hat{u}_i(t_k^i), \ \forall \ t \in [t_k^i, t_{k+1}^i), \tag{20}$$

leading to

$$e_{s,i}(t) = \hat{u}_i(t_k^i) - \hat{u}_i(t), \ \forall \ t \in [t_k^i, t_{k+1}^i), \tag{21}$$

that is,

$$e_{s,i}(t) = -\int_{t_k^i}^{t} z_{s,i}(t), \ \forall \ t \in [t_k^i, t_{k+1}^i), \tag{22}$$

where $\hat{u}_i(t) = KCx_i(t), \ \forall \ i \in \mathcal{N}$ and $z_{s,i}(t) = \dot{\hat{u}}_i(t), \ \forall \ t \in [t_k^i, t_{k+1}^i), \ k \in \ \mathbb{N}, \ \forall \ i = 1, ..., n$. Then the sampling induced perturbation can be written as $e_{s,i}(t) = (\Delta_{s,i} z_{s,i})(t)$, where the operator $\Delta_{s,i} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is defined as a reset integrator, i.e.

$$\Delta_{s,i} : \begin{cases} \dot{w}_i(t) & = z_{s,i}(t), \ \forall \ t \in [t_k^i, t_{k+1}^i), \\ w_i(t_k^i) & = 0, \ \forall \ k \in \mathbb{N}, \\ e_{s,i}(t) & = -w_i(t), \ \forall \ t \geq 0, \ \forall \ i \in \mathcal{N}. \end{cases} \tag{23}$$

The operator has $\frac{d}{dt} \ \hat{u}_i(t)$ as an argument. Similarly, the error due to uncertainties can be written as

$$e_{u,i}(t) = \delta A_i(t) x_i(t), \ \forall \ i \in \mathcal{N}. \tag{24}$$

Then, the uncertainty induced perturbation can be written as $e_{u,i}(t) = (\Delta_{u,i} z_{u,i})(t)$, where $z_{u,i}(t)$ is the state $x_i(t)$ and the operator $\Delta_{u,i} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is defined as

$$e_{u,i}(t) = (\Delta_{u,i} z_{u,i})(t) := \delta A_i(t) z_{u,i}(t), \ \forall \ t \geq 0, \ \forall \ i \in \mathcal{N}. \tag{25}$$

Consider $z(t) = [z_1^\top(t) \ ... \ z_n^\top(t)]^\top$, $z_i(t) = [z_{s,i}^\top(t) \ z_{u,i}^\top(t)]^\top$, $e(t) = [e_1^\top(t) \ ... \ e_n^\top(t)]^\top$, and $e_i(t) = [e_{s,i}^\top(t) \ e_{u,i}^\top(t)]^\top$, $\forall \ i \in \mathcal{N}$, now we define the uncertainty operator $\Delta : z \to e$ by

$$e(t) = (\Delta z)(t) := \begin{bmatrix} (\Delta_1 z_1)(t) \\ \vdots \\ (\Delta_n z_n)(t) \end{bmatrix}, \text{ where } e_i(t) = (\Delta_i z_i)(t) := \begin{bmatrix} (\Delta_{s,i} z_{s,i})(t) \\ (\Delta_{u,i} z_{u,i})(t) \end{bmatrix}. \tag{26}$$

Now we introduce the following proposition to recast the closed-loop system as a feedback interconnection of a nominal continuous-time system and the uncertainty operator defined above.

Proposition 3.3 *Consider system (1)-(4) and the operator introduced by (20)-(26). In an input-output interconnection form, the closed-loop system (1)-(4) can be rewritten as a feedback interconnection of the continuous-time system*

$$G: \begin{cases} \dot{x}(t) = (I \otimes A_{cl} + A_M \otimes F_{cl})x(t) + (I \otimes B_{cl})e(t) \\ z(t) = \Bigg(I \otimes \underbrace{\begin{bmatrix} KCA_{cl} \\ I \end{bmatrix}}_{C_{cl1}} + A_M \otimes \underbrace{\begin{bmatrix} KCF_{cl} \\ 0 \end{bmatrix}}_{C_{cl2}}\Bigg)x(t) + \Bigg(I \otimes \underbrace{\begin{bmatrix} KCB_{cl} \\ 0 \end{bmatrix}}_{D_{cl}}\Bigg)e(t), \end{cases} \tag{27}$$

where $A_{cl} = A + BKC$, $B_{cl} = [B\ I]$, $F_{cl} = F$, and the sampling and uncertainty induced operator Δ is given by (26).

Proof. Since the controlled inputs are piecewise constant and there is no feed-through, the state $x_i(t)$ is piecewise continuously differentiable. That is, for any $t_2 > t_1 \geq 0$, we can express $x_i(t_2) - x_i(t_1) = \int_{t_1}^{t_2} \dot{x}_i(\theta)\mathrm{d}\theta$, $\forall\, i \in \mathcal{N}$. Then, we can rewrite (6) as

$$e_{s,i}(t) = -\int_{t_k^i}^{t} KC\dot{x}_i(\theta)\mathrm{d}\theta,\ \forall\, t \in [t_k^i, t_{k+1}^i), k \in \mathbb{N},\ \forall\, i \in \mathcal{N}. \tag{28}$$

Notice that the system in (1), represented using nodal dynamics, can be re-written as an interconnected system of the form

$$\dot{x}(t) = (I \otimes A)x(t) + (A_M \otimes F)x(t) + (I \otimes B)u(t) + \delta A(t)x(t), \tag{29}$$

where $u(t) = [u_1^\top(t)\ \dots\ u_n^\top(t)]^\top$, $x(t) = [x_1^\top(t)\ \dots\ x_n^\top(t)]^\top$, and $\delta A(t) := \mathrm{blkdiag}(\delta A_1(t), \dots, \delta A_n(t))$. Furthermore, (29) and (3)-(4) can be expressed as

$$\begin{aligned} \dot{x}(t) &= (I \otimes A)x(t) + (A_M \otimes F)x(t) + (I \otimes B)\left(\begin{bmatrix} \hat{u}_1(t) \\ \vdots \\ \hat{u}_n(t) \end{bmatrix} + \begin{bmatrix} e_{s,1}(t) \\ \vdots \\ e_{s,n}(t) \end{bmatrix}\right) + \begin{bmatrix} e_{u,1}(t) \\ \vdots \\ e_{u,n}(t) \end{bmatrix} \\ &= (I \otimes A)x(t) + (A_M \otimes F)x(t) + (I \otimes BKC)x(t) + (I \otimes B)\begin{bmatrix} e_{s,1}(t) \\ \vdots \\ e_{s,n}(t) \end{bmatrix} + \begin{bmatrix} e_{u,1}(t) \\ \vdots \\ e_{u,n}(t) \end{bmatrix}, \end{aligned} \tag{30}$$

with output variables

$$\begin{bmatrix} z_{u,1}(t) \\ \vdots \\ z_{u,n}(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \begin{bmatrix} z_{s,1}(t) \\ \vdots \\ z_{s,n}(t) \end{bmatrix} = (I \otimes KC)\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}, \tag{31}$$

and with input variables

$$\begin{bmatrix} e_{u,1}(t) \\ \vdots \\ e_{u,n}(t) \end{bmatrix} = \begin{bmatrix} (\Delta_{u,1}z_{u,1})(t) \\ \vdots \\ (\Delta_{u,n}z_{u,n})(t) \end{bmatrix}, \quad \begin{bmatrix} e_{s,1}(t) \\ \vdots \\ e_{s,n}(t) \end{bmatrix} = \begin{bmatrix} (\Delta_{s,1}z_{s,1})(t) \\ \vdots \\ (\Delta_{s,n}z_{s,n})(t) \end{bmatrix}. \tag{32}$$

Also, we have

$$\begin{aligned} \begin{bmatrix} z_{s,1}(t) \\ \vdots \\ z_{s,n}(t) \end{bmatrix} &= (I \otimes KC)\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} \\ &= (I \otimes KC)\left((I \otimes (A + BKC))x(t) + (A_M \otimes F)x(t) + (I \otimes B)\begin{bmatrix} e_{s,1}(t) \\ \vdots \\ e_{s,n}(t) \end{bmatrix} + \begin{bmatrix} e_{u,1}(t) \\ \vdots \\ e_{u,n}(t) \end{bmatrix}\right). \end{aligned} \tag{33}$$

Hence, the proof is complete. ◦

Remark 4 *Proposition 3.3 remodels system (1)-(4) as an interconnection of an augmented continuous-time system and an uncertainty operator. This is an essential step to derive the scalable stability LMI conditions that are independent of the number of subsystems (n).*

4. Dissipativity-based framework for sampled-data control

In this section, we propose an approach to analyze stability of the interconnected system using dissipativity theory (see [36, 37, 38] and the references therein). We put the sampling sequences in chronological order:

$$\gamma = \{t_s\}_{s\in\mathbb{N}},\ t_0 = 0,\ t_{s+1} = \min_{\forall\ i\in\mathcal{N},\ k\in\mathbb{N}} \{t_k^i \mid t_k^i > t_s\},\ \forall\ s \in \mathbb{N}. \tag{34}$$

Additionally, the coefficient

$$k_{i,s} = \max\{k \in \mathbb{N} \mid t_k^i \leq t_s\}, \tag{35}$$

represents the index of the last sample sent to the controller by sensor i before time t_s. Then, $t^i_{k_{i,s}}$ represents the last time instance at which the sensor i sent a measurement before time t_s. For a detailed explanation on the notation and sequence, we refer to [17]. Now we introduce the dissipativity function to include the uncertainty and sampling aspect, presented in previous sections, as follows.

Lemma 4.1 *Consider Δ_i in (26), let $\tilde{R}^\top = \tilde{R} > 0$, $\tilde{Y}^\top = \tilde{Y} > 0$, and*

$$\begin{aligned}\mathfrak{J}(z_i(t), e_i(t)) = e_i(t)^\top \begin{bmatrix}\tilde{R} & 0\\ 0 & I\end{bmatrix} e_i(t) - z_i(t)^\top \begin{bmatrix}h^2\tilde{R} & 0\\ 0 & d_e^2 I\end{bmatrix} z_i(t)\\ + e_i(t)^\top \begin{bmatrix}\tilde{Y} & 0\\ 0 & 0\end{bmatrix} z_i(t) + z_i(t)^\top \begin{bmatrix}\tilde{Y} & 0\\ 0 & 0\end{bmatrix} e_i(t),\end{aligned} \tag{36}$$

$\forall\ i \in \mathcal{N}$, where h and d_e bound the sampling interval and additive uncertainty, see Section 2. Then we have the following inequality,

$$\int_{t_k^i}^{t} \mathfrak{J}(z_i(\theta), (\Delta_i z_i)(\theta))\mathrm{d}\theta \leq 0,\ \forall\ t \in [t_k^i, t_{k+1}^i),\ k \in \mathbb{N},\ \forall\ i \in \mathcal{N}. \tag{37}$$

Proof. The proof is presented in the appendix. ∘

The goal of Lemma 4.1 is to characterize $\mathcal{L}_2$ bound properties of the sampling and uncertainty errors which are characterized by Δ in (26). This will be used for the scalable stability criteria. The following proposition presents a generic stability condition for system (1)-(4).

Proposition 4.2 *Consider sequence γ in (34), sampled-data system (1)-(4), and the feedback interconnection of (27) and (26). Furthermore, consider a continuous function $\mathfrak{J}(z_i(\theta), e_i(\theta))$ of the form (36), with $\tilde{R}^\top = \tilde{R} > 0$, $\tilde{Y}^\top = \tilde{Y} > 0$.*
Assume that

1. *Along the solution of (27) and (26), there exists a differentiable positive definite storage function $V : \mathbb{R}^{n\cdot n_x} \to \mathbb{R}$ and class $\mathcal{K}$ functions β_1 and β_2 satisfying*

$$\beta_1(|x(t)|^2) \leq V(x(t)) \leq \beta_2(|x(t)|^2),\ \forall\ x \in \mathbb{R}^{n\cdot n_x},\ \forall\ i \in \mathcal{N},\ \forall\ t \geq 0. \tag{38}$$

2. *The following condition is satisfied for the interconnection of (27) and (26)*

$$\dot{V}(x(t)) < \textstyle\sum_{i=1}^{n} \mathfrak{J}(z_i(t), e_i(t)),\ \forall\ t \in [t_s, t_{s+1}). \tag{39}$$

Then the closed-loop system of (1)-(4) is Lyapunov stable.

Proof. The proof is presented in the appendix. ∘

Proposition 4.2 is a generic result with storage function $V(x(t))$ which considers the entire system. In the following theorem, we use Proposition 4.2 to derive scalable conditions for the stability of system (1)-(4) using feedback interconnection model (26)-(27).

Theorem 4.3 *Consider the feedback interconnection of (27) and (26) satisfying Assumption 1, notations (10)-(11),*

$$\begin{aligned}\tilde{A}_{cl}(\lambda_a) = A_{cl} + \lambda_a F_{cl},\ \tilde{C}_{cl}(\lambda_a) = C_{cl1} + \lambda_a C_{cl2},\\ L = \begin{bmatrix}h^2\tilde{R} & 0\\ 0 & d_e^2 I\end{bmatrix},\ R = \begin{bmatrix}\tilde{R} & 0\\ 0 & I\end{bmatrix},\ Y = \begin{bmatrix}\tilde{Y} & 0\\ 0 & 0\end{bmatrix},\end{aligned} \tag{40}$$

where h and d_e in L are associated with the sampling and the parametric uncertainty, respectively. If there exist $\tilde{R} = \tilde{R}^\top > 0,\ \tilde{P}(\lambda_a) = \tilde{P}(\lambda_a)^\top > 0,\ \tilde{Y} = \tilde{Y}^\top > 0$, such that the LMI

$$\begin{bmatrix} \tilde{A}_{cl}(\lambda_a)^\top \tilde{P}(\lambda_a) + \tilde{P}(\lambda_a)\tilde{A}_{cl}(\lambda_a) & \tilde{P}(\lambda_a)B_{cl} \\ * & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}_{cl}(\lambda_a)^\top L\tilde{C}_{cl}(\lambda_a) & \tilde{C}_{cl}(\lambda_a)^\top LD_{cl} - \tilde{C}_{cl}(\lambda_a)^\top Y \\ * & D_{cl}^\top LD_{cl} - YD_{cl} - D_{cl}^\top Y \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ * & -R \end{bmatrix} < 0 \tag{41}$$

is feasible $\forall\ \lambda_a \in [-1, 1]$, then the closed-loop system of (1)-(4) is (Lyapunov) stable, with $V(x) = x^T \mathcal{P}(A_M)x$ as storage function.

Proof. The proof is presented in the appendix. ◦

Remark 5 *Theorem 4.3 provides a scalable stability condition for the interconnected sampled-data system (1)-(4). The information on the network structure satisfying Assumption 1 is captured in the (eigenvalue) parameter λ_a. The stability condition in (41) is of the order of the subsystem. It relies on the storage function containing the matrix polynomial defined in (11).*

The presence of cross terms with parameters, such as $\tilde{A}_{cl}(\lambda_a)^\top \tilde{P}(\lambda_a)$ and $\tilde{C}_{cl}(\lambda_a)^\top L\tilde{C}_{cl}(\lambda_a)$ in (41), increases the order of the LMIs with respect to the parameter λ_a, which is a disadvantage in case standard approaches would be used to turn the parameter-dependent LMIs into sufficient standard LMIs (such as Pólya's relaxation and sum-of-squares based techniques). This can be avoided by using the descriptor method [23], leading to the following corollary from Thoerem 4.3.

Corollary 4.4 *Consider the feedback interconnection of (27) and (26), Assumption 1, and notations (10)-(11) and (42). If there exist $\tilde{R} = \tilde{R}^\top > 0,\ \tilde{P}(\lambda_a) = \tilde{P}(\lambda_a)^\top > 0,\ \tilde{Y} = \tilde{Y}^\top > 0, P_2 \in \mathbb{R}^{n_x \times n_x}$, and $P_3 \in \mathbb{R}^{n_x \times n_x}$, such that the LMI*

$$\begin{bmatrix} \tilde{A}_{cl}(\lambda_a)^\top P_2 + P_2^\top \tilde{A}_{cl}(\lambda_a) + d_e^2 I & \tilde{P}(\lambda_a) - P_2^\top + \tilde{A}_{cl}(\lambda_a)^\top P_3 & P_2^\top B_{cl} \\ * & -P_3 - P_3^\top + h^2 (KC)^\top \tilde{R} KC & P_3^\top B_{cl} - (KC)^\top [\tilde{Y}\ 0] \\ * & * & -R \end{bmatrix} < 0 \tag{42}$$

is satisfied $\forall\ \lambda_a \in [-1, 1]$. Then the closed-loop system of (1)-(4) is (Lyapunov) stable, with $V(x) = x^T \mathcal{P}(A_M)x$ as storage function.

Proof. The proof is presented in the appendix. ◦

5. Additional structure on the storage function

As mentioned earlier, the results presented in this paper are not restricted to LTI systems with a time-invariant topology. This is also evident from Corollary 3.2, where the candidate matrix for the storage function has additional structure. Suppose the topology is switching, then the nodal dynamics of the system changes from (1) to

$$\begin{cases} \dot{x}_i(t) &= Ax_i(t) + \sum_{j=1}^n a_{Mi,j}(t)Fx_j(t) + Bu_i(t) + \delta A_i(t)x_i(t), \\ x_i(0) &= x_{i,0}\ \in\ \mathbb{R}^{n_x},\ \forall\ i \in \mathcal{N},\ n \in \mathbb{N}, \end{cases} \tag{43}$$

where $a_{Mi,j}(t)$ is the $(i,j)^{\text{th}}$ element of the time-varying connectivity matrix $A_M(t) \in \mathbb{S}^{n\times n}$ and $A_M(t)$ is piecewise constant. As before, $\delta A_i(t)$ denotes the non-identical uncertainty at node i, and $||\delta A_i(t)||_2 \le d_e,\ \forall\ i \in \mathcal{N}$. Let us consider a finite set of connectivity matrices $\mathcal{A} := \{A_{M,1}, ..., A_{M,q}\}, \mathcal{A} \subset \mathbb{S}^{n\times n}$, where the connectivity matrix $A_{M,j}$ satisfies Assumption 1 $\forall\ j = 1, ..., q$. We consider the following assumption for $A_M(t)$.

Assumption 2 *The connectivity matrix, $A_M : \mathbb{T} \to \mathcal{A}$, arbitrarily switches between* real-symmetric *connectivity matrices in $\mathcal{A}$, with a monotonously increasing switching sequence $\sigma := \{\kappa_m\}_{m\in\mathbb{N}}$, that is,*

$$A_M(t) = A_M^{\kappa_m},\ \forall\ t \in [\kappa_m, \kappa_{m+1}),\ A_M^{\kappa_m} \in \mathcal{A},\ \forall\ m \in \mathbb{N},\ \kappa_0 = 0. \tag{44}$$

Considering the same sampling methodology and uncertainty as in the previous sections (see (6) and (24)), the closed-loop system of (43) and (2)-(4) can be rewritten as a feedback interconnection of the continuous-time system

$$\hat{G} : \begin{cases} \dot{x}(t) = (I \otimes A_{cl} + A_M(t) \otimes F_{cl})x(t) + (I \otimes B_{cl})e(t), \\ z(t) = \Bigg(I \otimes \underbrace{\begin{bmatrix} KCA_{cl} \\ I \end{bmatrix}}_{C_{cl1}} + A_M(t) \otimes \underbrace{\begin{bmatrix} KCF_{cl} \\ 0 \end{bmatrix}}_{C_{cl2}}\Bigg)x(t) + \Bigg(I \otimes \underbrace{\begin{bmatrix} KCB_{cl} \\ 0 \end{bmatrix}}_{D_{cl}}\Bigg)e(t), \end{cases} \tag{45}$$

and the sampling and uncertainty induced operator $\Delta : z \to e$ defined in (26). Notice that Theorem 4.3 can be modified to provide results for the switching system (43) and (2)-(4), provided that $V(x) = x(I \otimes P)x$ is chosen as a *common* storage function. Using the descriptor method[23], we arrive at the following theorem by adapting Corollary 4.4 to systems with switching topologies.

Theorem 5.1 *Consider system (43) and (2)-(4), its feedback interconnection model (45) and (26), Assumption 2, and*

$$\tilde{A}_{cl}(\lambda_a) = A_{cl} + \lambda_a F_{cl},\ R = \begin{bmatrix} \tilde{R} & 0 \\ 0 & I \end{bmatrix}. \tag{46}$$

If there exist $\tilde{R} = \tilde{R}^\top > 0,\ P = P^\top > 0,\ \tilde{Y} = \tilde{Y}^\top > 0, P_2 \in \mathbb{R}^{n_x \times n_x}$, *and* $P_3 \in \mathbb{R}^{n_x \times n_x}$, *such that the LMI*

$$\begin{bmatrix} \tilde{A}_{cl}(\lambda_a)^\top P_2 + P_2^\top \tilde{A}_{cl}(\lambda_a) + d_e^2 I & P - P_2^\top + \tilde{A}_{cl}(\lambda_a)^\top P_3 & P_2^\top B_{cl} \\ * & -P_3 - P_3^\top + h^2 (KC)^\top \tilde{R} KC & P_3^\top B_{cl} - (KC)^\top [\tilde{Y}\ 0] \\ * & * & -R \end{bmatrix} < 0 \tag{47}$$

is satisfied $\forall\ \lambda_a \in \{-1, 1\}$, *where* h *and* d_e *are associated with the sampling interval and the parametric uncertainty, respectively, then the closed loop system of (43) and (2)-(4) is Lyapunov stable.*

Proof. The proof follows the same steps as the one of Corollary 4.4 (see the Appendix). ∘

The stability condition in (47) is independent of both the number of subsystems (n) and the number of (switching) network topologies (q). Also, notice that the matrices A_{cl} and F_{cl} appear in an affine way in (47). Therefore, the conditions can also be adapted to consider systems of the form (43) where system matrices A and B are subjected to polytopic uncertainties.

Further generalizations concern networks of quasi-identical Linear Parameter-Varying (LPV) systems. Here the system model takes the form of a linear system, but the matrices may depend on so-called scheduling parameters, which can typically be measured. LPV systems may be used to model various practical applications such as multi-source electric vehicles [39], wafer stages [40], servo systems [41], active noise and vibration control [42], active suspension systems [43], and overhead cranes [44]. In the adopted network setting, the LMIs for stability would not only be dependent on eigenvalue parameter λ_a, but also on the scheduling parameters, whose values and possibly variation are bounded. It is possible to let the Lyapunov matrix depend on the scheduling parameter, which may reduce conservatism. Additionally, it is possible to include scheduling parameter in the controller resulting in a gain-scheduled control law. However, including gain-scheduling parameter in this manner typically results in a higher-order dependence of the LMI on the parameters. As already noted, different techniques exist to provide *sufficient* conditions for the feasibility of a polynomially parameter-dependent LMI in terms of (a finite number of) standard LMIs. See [45, 46, 47] for an application to LPV systems.

6. Numerical examples

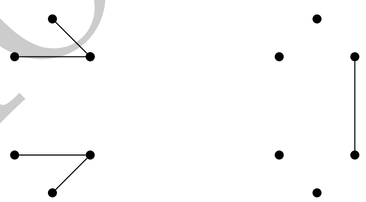

Figure 1: Two clusters of nodes with communication constraints requiring to switch communication links to save energy.

Example 1: Consider the following numerical example for (43) with the following coefficient matrices

$$A = \begin{bmatrix} -3 & -1 \\ 2.8 & 4 \end{bmatrix},\ B = \begin{bmatrix} 2 \\ -2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 \end{bmatrix},\ K = \begin{bmatrix} 3 \end{bmatrix},\ F = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \tag{48}$$

and the connectivity matrices considered are

$$A_{M,1} = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \end{bmatrix},\ A_{M,2} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{49}$$

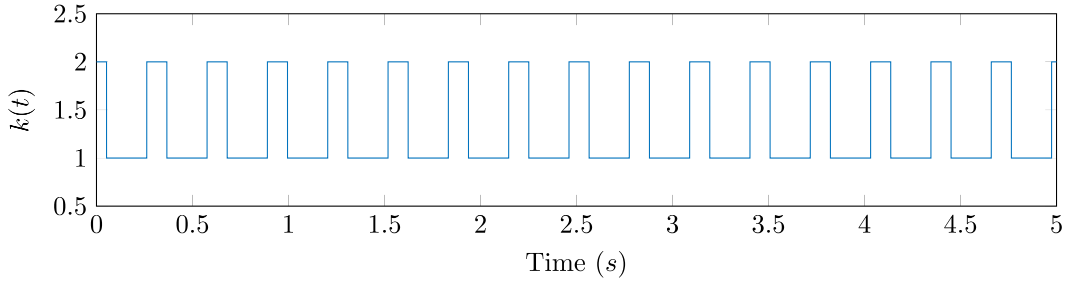

Figure 2: The topology was made to switch according to the above piecewise constant switching signal $k(t)$ such that $A_M(t) = A_{M,k(t)}$ based on the information from (49).

whose spectrum belong to $[-1, 1]$. Also, we consider the values for upper-bounds as $d_e = 0.1$ (corresponding to the parametric uncertainty) and $h = 0.12$ (corresponding to the sampling intervals). The LMI (47) from Theorem 5.1 is feasible $\forall\ \lambda_a \in [-1, 1]$ with the matrices

$$P = \begin{bmatrix} 0.1441 & 0.2119 \\ 0.2119 & 0.3557 \end{bmatrix},\ P_2 = \begin{bmatrix} 0.0948 & 0.0249 \\ 0.1476 & 0.0758 \end{bmatrix},\ P_3 = \begin{bmatrix} 0.0552 & 0.0669 \\ 0.0970 & 0.1311 \end{bmatrix}, \tag{50}$$

$\tilde{Y} = 0.0014$ and $\tilde{R} = 0.1897$. Therefore, system (43), whose matrices are defined by (48), is stable for any switching network topology (whose connectivity matrices have their spectrum in $[-1, 1]$) with a sampled-data controller implementation (that respects the upper-bounds) for any number of nodes. To illustrate this, we consider two network topologies with 6 nodes switching between the configurations displayed in Figure 1. A simulation-based study was performed for the corresponding system with information on the topology switching given in Figure 2. The study was performed for $x(0^+) = 0.1 \cdot \bar{1}$, where $\bar{1}$ is used to represent a vector of appropriate dimension with all its elements equal to 1. The corresponding state $x(t)$ and sampled output $y(t)$ for the switching networked system was obtained as shown in Figures 3-4. Let us now consider a scenario

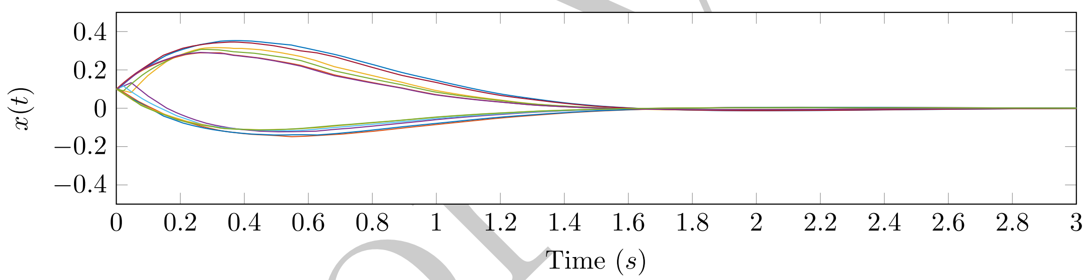

Figure 3: The above figure illustrates the stability of the networked system with switching topology (with connectivity matrices in (49), $k(t)$ in Figure 2, and $n = 6$) by means of the trajectory of the system state $x(t)$.

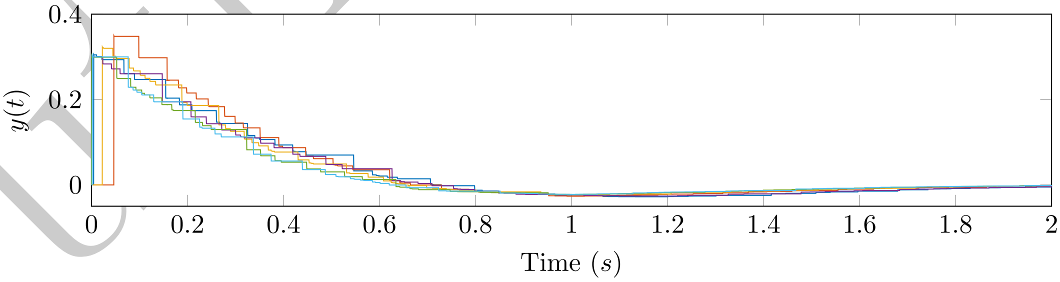

Figure 4: The above figure shows the sampled output signal $y(t)$ (with connectivity matrices in (49), $k(t)$ in Figure 2, and $n = 6$) that is used for generating the controlled input, which results in the trajectory for the system state $x(t)$ shown in Figure 3. The aperiodic sampling at asynchronous sensors used for this simulation respects the upper-bound limit of h (≤ 0.12).

representing large-scale systems, with the number of systems much larger than 6. System (43), with numerical data (48), is used to model a large-scale interconnected system with the total number of subsystems $n = 10000$. Recall that the stability condition from Theorem 5.1 is independent of the number of subsystems, the number

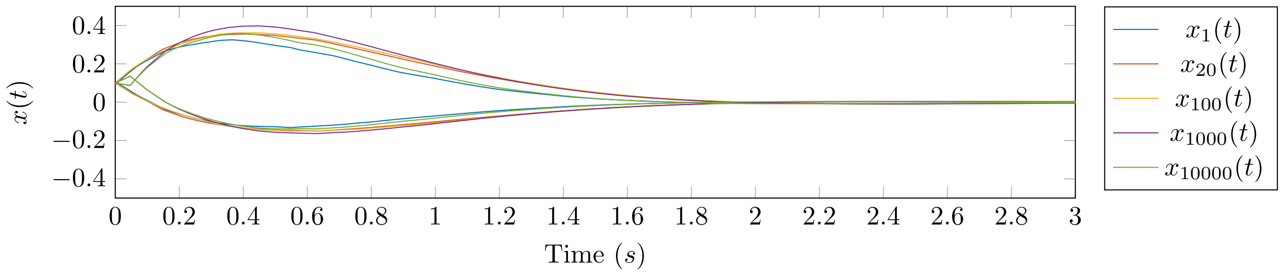

Figure 5: The above figure illustrates the stability of the networked system with switching topology (with connectivity matrices in (51), $k(t)$ in Figure 2, and $n = 10000$) by means of the trajectory of the system state $x(t)$.

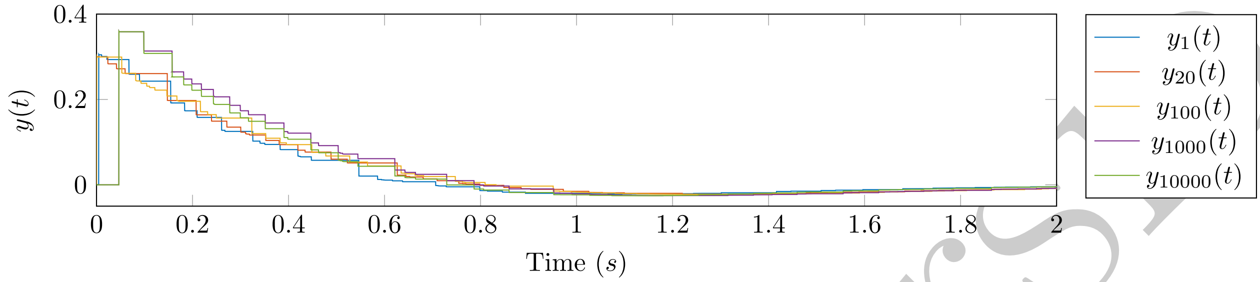

Figure 6: The above figure shows the sampled output signal $y(t)$ (with connectivity matrices in (51), $k(t)$ in Figure 2, and $n = 10000$) that is used for generating the controlled input, which results in the trajectory for the system state $x(t)$ shown in Figure 5. The aperiodic sampling at asynchronous sensors used for this simulation respects the upper-bound limit of h (≤ 0.12).

of (switching) network topologies, and the switching sequence. For this scenario, we consider the system to be switching between a bi-directional ring topology and a series topology. That is, the communication link between the first system and the last system is switching on and off. Then the connectivity matrices of dimension 10000 are

$$A_{M,1} = \begin{bmatrix} 0 & 0.5 & 0 & \dots & 0 & 0 \\ 0.5 & 0 & 0.5 & \dots & 0 & 0 \\ 0 & 0.5 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0.5 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0.5 \\ 0 & 0 & 0 & \dots & 0.5 & 0 \end{bmatrix}, \; A_{M,2} = \begin{bmatrix} 0 & 0.5 & 0 & \dots & 0 & 0.5 \\ 0.5 & 0 & 0.5 & \dots & 0 & 0 \\ 0 & 0.5 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0.5 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0.5 \\ 0.5 & 0 & 0 & \dots & 0.5 & 0 \end{bmatrix}. \tag{51}$$

Recall that the eigenvalues of both connectivity matrices in (51) lie in the real interval $[-1, 1]$. Once again, the switching signal is assumed to be the function $k(t)$ displayed in Figure 2 for the simplicity of representation. The study was performed for $x(0^+) = 0.1 \cdot \bar{1}$. The corresponding state $x(t)$ and sampled output $y(t)$ for the switching networked system (large-scale system with $n = 10000$) was obtained as shown in Figures 5-6.

Example 2: Let us consider a network of friction-less carts that each balance an inverted pendulum, and are interconnected via identical springs to each other[48, 49, 50, 51]. Additionally, they are confined to a fixed space wherein the first and the last carts are connected to the wall using the same spring (see Figure 7). By linearizing the equations of motions around the equilibrium at the origin (see [48, 49, 50, 51] for more details), we obtain a linear state-space model of form (43) with coefficient matrices as follows.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2k_E}{M_E} & 0 & -\frac{m_E g_E}{M_E} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2k_E}{M_E l_E} & 0 & \frac{(m_E+M_E)g_E}{M_E l_E} & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \frac{1}{M_E} \\ 0 \\ -\frac{1}{M_E l_E} \end{bmatrix}, \; C = I_{4\times4}, \; F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{2k_E}{M_E} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2k_E}{M_E l_E} & 0 & 0 & 0 \end{bmatrix}, \tag{52}$$

where $M_E = 1$ kg is the mass of the individual cart, $m_E = 0.05$ kg is the mass of the pendulum's bob which is connected to the cart using a mass-less rod of length $l = 1$ m, $k = 1$ N/m is the spring constant, and the acceleration due to gravity is assumed to be $g = 9.8$ m/s^2. We consider identical local state-feedback controllers characterized by

$$K = \begin{bmatrix} 23.7007 & 16.3009 & 82.4139 & 25.9097 \end{bmatrix}. \tag{53}$$

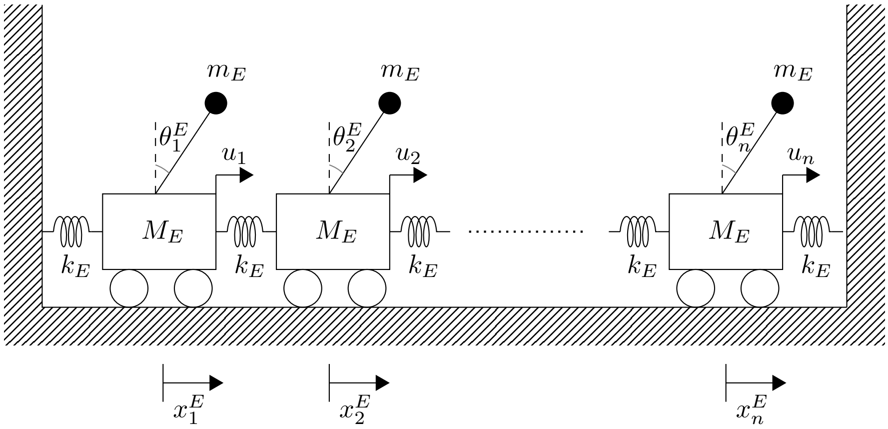

Figure 7: Schematic representation of interconnected pendula for a series type network with a large number of carts (n).

Moreover, we consider the values for upper-bounds as $d_e = 0.04$ (corresponding to the parametric uncertainty) and $h = 0.04$ (corresponding to the sampling intervals). Again, we use the LMI (47) from Theorem 5.1 to assess the stability of the system and the LMI is feasible $\forall\ \lambda_a \in [-1, 1]$ with the matrices

$$P = \begin{bmatrix} 0.6731 & 0.3101 & 1.3040 & 0.4017 \\ 0.3101 & 0.2490 & 0.7314 & 0.3282 \\ 1.3040 & 0.7314 & 3.0685 & 0.9722 \\ 0.4017 & 0.3282 & 0.9722 & 0.4436 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.0007 & -0.9996 & 0.0011 & 0.0011 \\ -0.1190 & -0.0884 & -5.3171 & -0.1406 \\ 0.0011 & 0.0007 & 0.0020 & -0.9983 \\ 0.1181 & 0.0876 & -4.4866 & 0.1396 \end{bmatrix}, \tag{54}$$

$$P_3 = \begin{bmatrix} 1.0001 & 0.0001 & -0.0003 & 0.0001 \\ 0.0012 & 0.5035 & 0.0044 & 0.4986 \\ 0.0001 & 0.0002 & 0.9996 & 0.0002 \\ -0.0011 & 0.4966 & -0.0043 & 0.5015 \end{bmatrix}, \tag{55}$$

$\tilde{Y} = 0.0001$, and $\tilde{R} = 0.0152$. Note that the stability condition from Theorem 5.1 is independent of the number of subsystems, the number of (switching) network topologies, and the switching sequence. That is, the set up described in this example is stable for any switching network topology (whose connectivity matrices have their spectrum in $[-1, 1]$), any number of subsystems n, and sampled-data controller implementation (that respects the upper-bounds). Let us consider $n = 5000$, and the network adjacency matrix corresponding to the series network ($A_{M,1}$) in (51). The networked system was simulated in MATLAB. The angles for a few selected carts are shown in Figure 8, where $\theta_i^E(t)$ is the angle from the verticle line for the mass-less rod of cart i at time t.

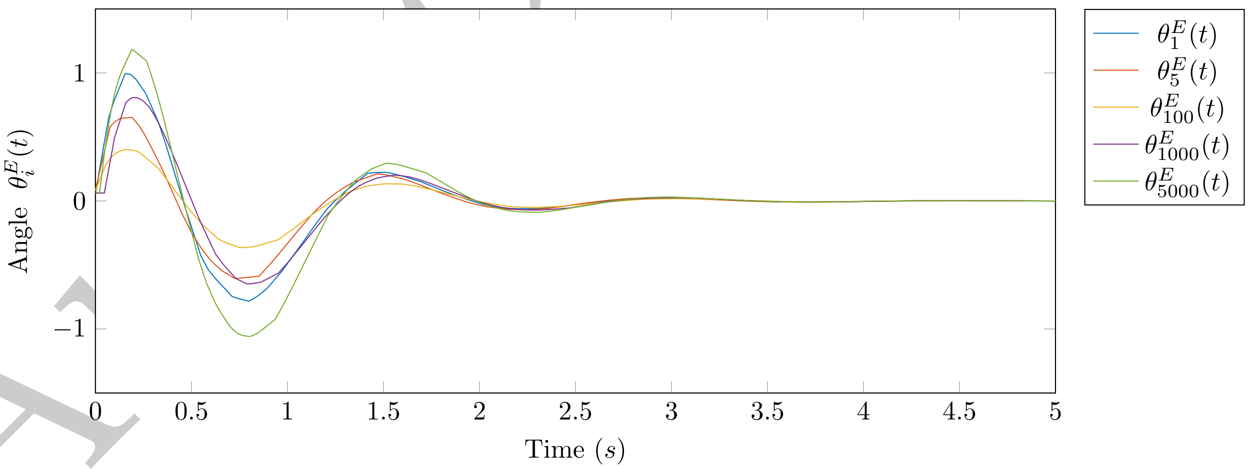

Figure 8: The above figure shows the angle $\theta_i^E(t)$ of the interconnected inverted pendula (with connectivity matrices in (51), $k(t)$ in Figure 2, and $n = 5000$). The aperiodic sampling at asynchronous sensors used for this simulation respects the upper-bound limit of h (≤ 0.12).

7. Conclusion

In this paper, a structure exploiting and scalable approach was presented to conclude about the stability of large-scale linear systems. These systems may be composed of quasi-identical subsystems with sampled-data control and (switching) network topologies. The underlying principle relied on decoupling the sufficient criterion for Lyapunov stability of the entire system in the dissipativity-based framework, resulting in a stability criterion involving (only) a parameterised LMI whose size corresponds to the dimension of one subsystem. Additionally, the effectiveness of the presented method was illustrated using numerical examples.

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APPENDIX

A. Proof of Lemma 1

First we adapt a lemma from [17] and [52] which are prerequisites to prove Lemma 4.1.

Lemma A.1 *Consider the notations of operators (6)-(26), $\tilde{R}^\top = \tilde{R} > 0$. We have the following inequality*

$$\int_{t_k^i}^{t} (\Delta_i z_i)(\theta)^\top (\Delta_i z_i)(\theta) - z_i(\theta)^\top \begin{bmatrix} h^2 I & 0 \\ 0 & d_e^2 I \end{bmatrix} z_i(\theta) \mathrm{d}\theta \leq 0,\ \forall\ t \in [t_k^i, t_{k+1}^i),\ k \in \mathbb{N},\ \forall\ i \in \mathcal{N}. \tag{56}$$

It follows that there exist a function $\mathfrak{J}_1$ such that

$$\mathfrak{J}_1(z_i(t), e_i(t) = e_i(t)^\top \begin{bmatrix} \tilde{R} & 0 \\ 0 & I \end{bmatrix} e_i(t) - z_i(t)^\top \begin{bmatrix} h^2 \tilde{R} & 0 \\ 0 & d_e^2 I \end{bmatrix} z_i(t), \tag{57}$$

which satisfies assumption (37), where $e_i(t) = (\Delta_i z_i)(t)\ \forall\ i \in \mathcal{N}$ defined in (26).

Proof. Consider $\Delta_{s,i}$ in (23), $z_{s,i} \in \mathcal{L}_{2e}$, and $t \in [t_k^i, t_{k+1}^i),\ k \in \mathbb{N},\ \forall\ i \in \mathcal{N}$. From Lemma 3 in [17], for the uncertainty operators corresponding to the sampling error, we have

$$\int_{t_k^i}^{t} (\Delta_{s,i} z_{s,i})(\theta)^\top R_i (\Delta_{s,i} z_{s,i})(\theta) - h^2 z_{s,i}(\theta)^\top R_i z_{s,i}(\theta)\ \mathrm{d}\theta \leq 0, \tag{58}$$

with $R_i > 0$. Here, we can fix $R_i = \tilde{R},\ \forall\ i \in \mathcal{N}$. Consider $\Delta_{u,i}$ in (25) and $z_{u,i} \in \mathcal{L}_{2e}$. By definition, $||\delta A_i(t)||_2 \leq d_e,\ \forall\ t \geq 0$. Hence, for the uncertainty operators corresponding to the norm-bounded uncertainty, we have

$$\int_{t_k^i}^{t} (\Delta_{u,i} z_{u,i})(\theta)^\top (\Delta_{u,i} z_{u,i})(\theta) - d_e^2 z_{u,i}(\theta)^\top z_{u,i}(\theta)\ \mathrm{d}\theta \leq 0,\ \forall\ t \in [t_k^i, t_{k+1}^i),\ k \in \mathbb{N},\ \forall\ i \in \mathcal{N}. \tag{59}$$

Then from (58) and (59), we have

$$\int_{t_k^i}^{t} (\Delta_i z_i)(\theta)^\top (\Delta_i z_i)(\theta) - z_i(\theta)^\top \begin{bmatrix} h^2 \tilde{R} & 0 \\ 0 & d_e^2 I \end{bmatrix} z_i(\theta) \mathrm{d}\theta \leq 0,\ \forall\ t \in [t_k^i, t_{k+1}^i),\ k \in \mathbb{N},\ \forall\ i \in \mathcal{N}. \tag{60}$$

Hence, the lemma has been proved. ∘

Now, we recall Lemma 5.3 from [52], which corresponds to the anti-passivity property, and then we present the proof for Lemma 4.1.

Lemma A.2 *Consider the reset integrator in (23), then, we have the following inequality for any $\tilde{Y} > 0$*

$$\int_{t_k^i}^{t} e_i(t)^\top \begin{bmatrix} \tilde{Y} & 0 \\ 0 & 0 \end{bmatrix} z_i(t) + z_i(t)^\top \begin{bmatrix} \tilde{Y} & 0 \\ 0 & 0 \end{bmatrix} e_i(t) \mathrm{d}\theta \leq 0,\ \forall\ t \in [t_k^i, t_{k+1}^i),\ k \in \mathbb{N},\ \forall\ i \in \mathcal{N}. \tag{61}$$

Proof of Lemma 4.1. Using the results from Lemmas A.1-A.2, it follows that there exists a functions $\mathfrak{J}$,

$$\begin{aligned} \mathfrak{J}(z_i(t), e_i(t)) = e_i(t)^\top \begin{bmatrix} \tilde{R} & 0 \\ 0 & I \end{bmatrix} e_i(t) - z_i(t)^\top \begin{bmatrix} h^2 \tilde{R} & 0 \\ 0 & d_e^2 I \end{bmatrix} z_i(t) \\ + e_i(t)^\top \begin{bmatrix} \tilde{Y} & 0 \\ 0 & 0 \end{bmatrix} z_i(t) + z_i(t)^\top \begin{bmatrix} \tilde{Y} & 0 \\ 0 & 0 \end{bmatrix} e_i(t) \end{aligned} \tag{62}$$

that satisfies (37). Hence, the proof is complete. ∘

B. Proof of Proposition 2

To prove the proposition, we adapt the main result from [17]. Consider $s \in \mathbb{N}$ and $t \in [t_s, t_{s+1})$. Integrating (39) over time, we have

$$V(x(t)) - V(x(t_s)) < \int_{t_s}^{t} \sum_{i=1}^{n} \mathfrak{J}(z_i(\theta), e_i(\theta)) \mathrm{d}\theta, \tag{63}$$

which leads to

$$V(x(t)) < V(x(t_s)) + \sum_{i=1}^{n} \int_{t_s}^{t} \mathfrak{J}(z_i(\theta), e_i(\theta)) \mathrm{d}\theta. \tag{64}$$

Also, we have

$$V(x(t_s)) < V(x(t_{s-1})) + \sum_{i=1}^{n} \int_{t_{s-1}}^{t_s} \mathfrak{J}(z_i(\theta), e_i(\theta))\mathrm{d}\theta, \tag{65}$$

$\forall\ s \in \mathbb{N}$. By recursive substitution of (65) in (64), we get

$$V(x(t)) < V(x(t_0)) + \sum_{i=1}^{n} \sum_{j=1}^{s} \int_{t_{j-1}}^{t_j} \mathfrak{J}(z_i(\theta), e_i(\theta))\mathrm{d}\theta + \sum_{i=1}^{n} \int_{t_s}^{t} \mathfrak{J}(z_i(\theta), e_i(\theta))\mathrm{d}\theta. \tag{66}$$

By construction of the sequences, we can rewrite the above inequality as

$$V(x(t)) < V(x(t_0)) + \sum_{i=1}^{n} \sum_{k=0}^{k_{i,s}-1} \int_{t_k^i}^{t_{k+1}^i} \mathfrak{J}(z_i(\theta), e_i(\theta))\mathrm{d}\theta + \sum_{i=1}^{n} \int_{k_{i,s}}^{t} \mathfrak{J}(z_i(\theta), e_i(\theta))\mathrm{d}\theta, \tag{67}$$

that is, using (37) we have

$$V(x(t)) < V(x(t_0)). \tag{68}$$

Finally, assumption (38) leads to

$$\begin{aligned} |x(t)| &< \beta_1^{-1}(V(x(t_0)) \\ &< \beta_1^{-1}(\beta_2(|x(t_0)|)) \\ &:= \beta(|x(t_0)|),\ \forall\ t > t_0, \end{aligned} \tag{69}$$

which concludes the proof. ◦

C. Proof of Theorem 2

We know that (41) is feasible $\forall\ \lambda_a \in [-1, 1]$. First, we prove the following preliminary lemma.

Lemma C.1 *The feasibility of the LMI condition in (41) is sufficient for the feasibility of the following inequality*

$$\begin{aligned} 2x(t)^\top(\mathcal{P}(A_M))((I \otimes A_{cl} + A_M \otimes F_{cl})x(t) + (I \otimes B_{cl})e(t)) \\ < e(t)^\top(I \otimes R)e(t) - \zeta(t, A_M)^\top(I \otimes L)\zeta(t, A_M) \\ + e(t)^\top(I \otimes Y))\zeta(t, A_M) + \zeta(t, A_M)^\top(I \otimes Y)e(t), \\ \zeta(t, A_M) = (I \times C_{cl1} + A_M \otimes C_{cl2})x(t) + (I \otimes D_{cl})e(t). \end{aligned} \tag{70}$$

Proof. Recall that A_M satisfies Assumption 1. From (41), we have

$$\begin{aligned} &\begin{bmatrix} (I \otimes A_{cl} + \Lambda \otimes F_{cl})^\top\mathcal{P}(\Lambda) + \mathcal{P}(\Lambda)(I \otimes A_{cl} + \Lambda \otimes F_{cl}) & \mathcal{P}(\Lambda)(I \otimes B_{cl}) \\ * & 0 \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{C}_{cl}(\Lambda)^\top(I \otimes L)\tilde{C}_{cl}(\Lambda) & \tilde{C}_{cl}(\Lambda)^\top(I \otimes L)(I \otimes D_{cl}) - \tilde{C}_{cl}(\Lambda)^\top(I \otimes Y) \\ * & (I \otimes D_{cl})^\top(I \otimes L)(I \otimes D_{cl}) - (I \otimes Y)(I \otimes D_{cl}) - (I \otimes D_{cl})^\top(I \otimes Y) \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ * & -I \otimes R \end{bmatrix} < 0, \end{aligned} \tag{71}$$

where

$$\tilde{C}_{cl}(\Lambda) = (I \otimes C_{cl1} + \Lambda \otimes C_{cl2}).$$

This is because the diagonal matrix Λ has elements belonging to the interval $[-1, 1]$. By multiplying left with blkdiag$(T^\top \otimes I, T^\top \otimes I)$ and right with its transpose[1], we get

$$\begin{aligned} &\begin{bmatrix} (I \otimes A_{cl} + A_M \otimes F_{cl})^\top\mathcal{P}(A_M) + \mathcal{P}(A_M)(I \otimes A_{cl} + A_M \otimes F_{cl}) & \mathcal{P}(A_M)(I \otimes B_{cl}) \\ * & 0 \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{C}_{cl}(A_M)^\top(I \otimes L)\tilde{C}_{cl}(A_M) & \tilde{C}_{cl}(A_M)^\top(I \otimes L)(I \otimes D_{cl}) - \tilde{C}_{cl}(A_M)^\top(I \otimes Y) \\ * & (I \otimes D_{cl})^\top(I \otimes L)(I \otimes D_{cl}) - (I \otimes Y)(I \otimes D_{cl}) - (I \otimes D_{cl})^\top(I \otimes Y) \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ * & -I \otimes R \end{bmatrix} < 0, \end{aligned} \tag{72}$$

Recall that $(T^\top \otimes I)\mathcal{P}(\Lambda)(T \otimes I) = \mathcal{P}(A_M)$. Consider the quadratic function $V(x(t)) = x^\top(t)(\mathcal{P}(A_M))x(t)$, which is dependent on the topology. By multiplying (72) from the left hand side with $[x^\top(t)\ e^\top(t)]^\top$ and the

[1] see Theorem 3.1 for more detailed explanation on the transformation.

right hand side with its transpose, we get (70) and, hence, the proof is complete. ◦

Proof of Theorem 4.3. Using the result from Lemma C.1, we can write for the interval $[t_k^i, t_{k+1}^i)$ that

$$\begin{aligned}\dot{V}(x(t)) &< e(t)^\top (I \otimes R)e(t) - z(t)^\top (I \otimes L)z(t) + e(t)^\top (I \otimes Y)z(t) \\ &\quad + z(t)^\top (I \otimes Y)e(t) \\ &< \sum_{i=1}^{n} (e_i(t)^\top R e_i(t) - z_i(t)^\top L z_i(t) + e_i(t)^\top Y z_i(t) + z_i(t)^\top Y e_i(t)) \\ &< \sum_{i=1}^{n} \mathfrak{I}(z_i(t), e_i(t)).\end{aligned} \tag{73}$$

Rewriting the above equation and integrating over the time intervals, we obtain (25). The quadratic function $V(x(t)) = x(t)^\top(\mathcal{P}(A_M))x(t)$ satisfies (23) with the functions $\beta_1(|x|) = \lambda_{\min}(\mathcal{P}(A_M))|x|^2$ and $\beta_2(|x|) = \lambda_{\max}(\mathcal{P}(A_M))|x|^2$. Hence, using the result of Proposition 4.2, the proof is complete. ◦

D. Proof of Corollary 2

Both Theorem 4.3 and Corollary 4.4 rely on using the final result of Proposition 4.2 to show that their respective numerical criteria are sufficient for Lyapunov stability of the closed-loop system (1)-(4). First, we derive the following lemma and then we present the proof of the corollary.

Lemma D.1 *The feasibility of the LMI condition in (42) is sufficient for the feasibility of the following set of conditions*

$$\begin{aligned} & x(t)^\top(\mathcal{P}(A_M))\dot{x}(t) + \dot{x}(t)^\top(\mathcal{P}(A_M))x(t) \\ & \quad + 2(x(t)^\top (I \otimes P_2)^\top + \dot{x}(t)^\top (I \otimes P_3)^\top)((I \otimes A_{cl} + A_M \otimes F_{cl})x(t) + I \otimes B_{cl}e(t) - \dot{x}(t)) \\ & \quad < e(t)^\top (I \otimes R)e(t) - z(t)^\top (I \otimes L)z(t) + e(t)^\top (I \otimes Y)z(t) \\ & \qquad + z(t)^\top (I \otimes Y)e(t). \end{aligned} \tag{74}$$

Proof. Due to convexity, the LMI condition in (42) is also feasible $\forall\ \lambda_a \in [-1, 1]$. Then, from (42) we have

$$\begin{bmatrix} \Xi_1(\Lambda) & \Xi_2(\Lambda) & \Xi_3 \\ * & \Xi_4 & \Xi_5 \\ & * & \Xi_6 \end{bmatrix} < 0, \tag{75}$$

where

$$\begin{aligned} & \Xi_1(\Lambda) = (I \otimes A_{cl} + \Lambda \otimes F_{cl})^\top (I \otimes P_2) + (I \otimes P_2)^\top (I \otimes A_{cl} + \Lambda \otimes F_{cl}) \\ & \quad + I \otimes d_e^2 I, \\ & \Xi_2(\Lambda) = \mathcal{P}(\Lambda) - (I \otimes P_2)^\top + (I \otimes A_{cl} + \Lambda \otimes F_{cl})^\top (I \otimes P_3), \\ & \Xi_3 = (I \otimes P_2^\top)(I \otimes B_{cl}), \\ & \Xi_4 = -I \otimes P_3 - I \otimes P_3^\top + (I \otimes (KC)^\top)(I \otimes h^2 \tilde{R})(I \otimes KC), \\ & \Xi_5 = (I \otimes P_3^\top)(I \otimes B_{cl}) - (I \otimes KC)(I \otimes [\tilde{Y}\ 0]), \\ & \Xi_6 = (I \otimes -R), \end{aligned} \tag{76}$$

since the diagonal matrices Λ have elements belonging to the interval $[-1, 1]$. By multiplying left with $\mathrm{blkdiag}(T^\top \otimes I, T^\top \otimes I, I)$ and right with its transpose (using the idea from the proof of Corollary 3.2) we get,

$$\begin{bmatrix} \Xi_1(A_M) & \Xi_2(A_M) & \Xi_3 \\ * & \Xi_4 & \Xi_5 \\ & * & \Xi_6 \end{bmatrix} < 0. \tag{77}$$

Recall that $(T^\top \otimes I)\mathcal{P}(\Lambda)(T \otimes I) = \mathcal{P}(A_M)$. Also, $(T^\top \otimes I)\Xi_k(\Lambda)(T \otimes I) = \Xi_k(A_M)$, whereas $(T^\top \otimes I)\Xi_l(T \otimes I) = \Xi_l$, $k = 1, 2$, and $l = 3, ..., 6$. Consider the quadratic function $V(x(t)) = x^\top(t)(\mathcal{P}(A_M))x(t)$. Multiplying (77) from left with $[x(t)^\top \dot{x}(t)^\top\ e(t)^\top]^\top$ and right with its transpose, we obtain (74) and hence the proof is complete. ◦

Proof of Corollary 4.4. We recall the notations in (34)-(35), then using the descriptor method we know

$$0 = 2(x(t)^\top (I \otimes P_2)^\top + \dot{x}(t)^\top P_3^\top)((I \otimes A_{cl} + A_M \otimes F_{cl})x(t) + I \otimes B_{cl}e(t) - \dot{x}(t)), \tag{78}$$

$\forall\ t \in [t_s, t_{s+1}),\ s \in \mathbb{N}$ (see definition of the sequence in (34)). From Lemma D.1 and (78), we have

$$\begin{aligned}\dot{V}(x(t)) &< e(t)^\top (I \otimes R) e(t) - z(t)^\top (I \otimes L) z(t) + e(t)^\top (I \otimes Y) z(t) \\ &\quad + z(t)^\top (I \otimes Y) e(t) \\ &< \sum_{i=1}^{n} \mathfrak{J}(y_i(t), e_i(t)),\end{aligned} \tag{79}$$

$\forall\ t \in [t_s, t_{s+1}),\ s \in \mathbb{N}$. The above equation is same as (73). Hence, using the result of Proposition 4.2, the proof is complete. ◦

E. Proof of Theorem 3

The feasibility of (47) provides a stability condition for any connectivity matrix in $\mathcal{A}$ and $I \otimes P$ is the common storage function candidate matrix. From Assumption 2, we know that all connectivity matrices in the finite set $\mathcal{A}$ satisfy Assumption 1. That is, there exist orthogonal matrices ($T_1 T_1^\top = I, ..., T_q T_q^\top = I$) such that they can spectrally decompose the connectivity matrices to real-valued diagonal matrices ($\Lambda_1, ..., \Lambda_q$), respectively, that is,

$$\begin{aligned} T_1 A_{M,1} T_1^\top &= \Lambda_1, \\ &\vdots \\ T_q A_{M,q} T_q^\top &= \Lambda_q. \end{aligned} \tag{80}$$

Also, consider the following notation

$$L = \begin{bmatrix} h^2 \tilde{R} & 0 \\ 0 & d_e^2 I \end{bmatrix},\ R = \begin{bmatrix} \tilde{R} & 0 \\ 0 & I \end{bmatrix},\ Y = \begin{bmatrix} \tilde{Y} & 0 \\ 0 & 0 \end{bmatrix}. \tag{81}$$

Now we derive the following lemma.

Lemma E.1 *The feasibility of the LMI condition in (47) is sufficient for the feasibility of the following set of inequalities*

$$\begin{aligned} & x(t)^\top (I \otimes P) \dot{x}(t) + \dot{x}(t)^\top (I \otimes P) x(t) \\ &\quad + 2(x(t)^\top (I \otimes P_2)^\top + \dot{x}(t)^\top (I \otimes P_3)^\top)((I \otimes A_{cl} + A_{M,j} \otimes F_{cl}) x(t) + I \otimes B_{cl} e(t) - \dot{x}(t)) \\ &\quad < e(t)^\top (I \otimes R) e(t) - z(t)^\top (I \otimes L) z(t) + e(t)^\top (I \otimes Y) z(t) \\ &\qquad + z(t)^\top (I \otimes Y) e(t),\ j = 1, ..., q. \end{aligned} \tag{82}$$

Proof. Due to convexity, the LMI condition in (42) is also feasible $\forall\ \lambda_a \in [-1, 1]$. Then, from (47) we have

$$\begin{bmatrix} \Xi_1(\Lambda_j) & \Xi_2(\Lambda_j) & \Xi_3 \\ * & \Xi_4 & \Xi_5 \\ & * & \Xi_6 \end{bmatrix} < 0,\ j = 1, ..., q, \tag{83}$$

where,

$$\begin{aligned} \Xi_1(\Lambda_j) &= (I \otimes A_{cl} + \Lambda_j \otimes F_{cl})^\top (I \otimes P_2) + (I \otimes P_2)^\top (I \otimes A_{cl} + \Lambda_j \otimes F_{cl}) \\ &\quad + I \otimes \alpha P + I \otimes d_e^2 I, \\ \Xi_2(\Lambda_j) &= I \otimes P - (I \otimes P_2)^\top + (I \otimes A_{cl} + \Lambda_j \otimes F_{cl})^\top (I \otimes P_3), \\ \Xi_3 &= (I \otimes P_2^\top)(I \otimes B_{cl}), \\ \Xi_4 &= -I \otimes P_3 - I \otimes P_3^\top + (I \otimes C_K^\top)(I \otimes h^2 \tilde{R})(I \otimes C_K), \\ \Xi_5 &= (I \otimes P_3^\top)(I \otimes B_{cl}) - (I \otimes C_K)(I \otimes [\tilde{Y}\ 0]), \\ \Xi_6 &= (I \otimes -R), \end{aligned} \tag{84}$$

since the diagonal matrices $\Lambda_j,\ j = 1, ..., q$, have elements belonging to the interval $[-1, 1]$. By multiplying left with $\text{blkdiag}(T_j^\top \otimes I, T_j^\top \otimes I, I)$ and right with its transpose (using the idea from the proof of Theorem 3.1) we get,

$$\begin{bmatrix} \Xi_1(A_{M,j}) & \Xi_2(A_{M,j}) & \Xi_3 \\ * & \Xi_4 & \Xi_5 \\ & * & \Xi_6 \end{bmatrix} < 0,\ j = 1, ..., q. \tag{85}$$

Consider the quadratic function $V(x(t)) = x^\top(t)(I \otimes P)x(t)$. Multiplying (77) from left with $[x(t)^\top \dot{x}(t)^\top\ e(t)^\top]^\top$ and right with its transpose, we obtain (74) and hence the proof is complete. ◦

Proof of Theorem 5.1. The proof uses the result of Lemma E.1 and then follows the same steps as the proof of Corollary 4.4. ◦