A Collatz Proof from The Book

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And why Tao almost had a proof of the non-existence of non-trivial cycles

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Monday 4th October, 2021

Abstract

Following the recent proof I co-authored in *Mathematics*, I finally give here a pure Peano-arithmetical demonstration that all Collatz orbits converge to 1. The point is to establish that the respective attractors of any pair of odd numbers, if we assumed they were separate, could always be finitely proven to collide upward, which is done by demonstrating they branch faster than the binary tree over odd numbers. This had me introduce two novel frameworks: Romanesco algebra and Dreamcatcher theory, which I discuss, along with Tao's fine-tuned "epsilon management" or what could then be called Poincaré-Denjoy-Tao theory, could crack other Collatz-like problems (e.g. the Juggler sequence). Here I just discuss how it could be used to independently prove there are no non-trivial cycles in Collatz, using only what Tao already had, and provide a new perspective on the Furstenberg $\times 2$ $\times 3$ conjecture.

1 Introduction

Sacralizing problems is a terrible disservice to render Humanity, in any field. Yet pusillanimous scholars and inventors tend to peer-pressure each other into doing just that: "let us, dear colleague, bask in the consensual impossibility of this and that". It is after all, a natural tendency of people to disguise a shortcoming under the formal declaration that a problem they could not solve is simply impossible. Allow a few generations to ferment on this social phenomenon and you have the absolute sacrality of some scientific problem, with the superstition, fostering both further learned helplessness on behalf of the new generations and unhealthy skepticism on behalf of editors, that most should not even dare tackle them. Flying was impossible, let alone at an industrial scale, until it rather suddenly wasn't, among others when some obscure Midwestern dealers in spare bicycle parts cracked it open while top academics they fortunately did not waste time reading (including Lord freaking Kelvin) had it repeatedly peer-reviewed that it was physical nonsense. It is no surprise that both John Nash and Paul Cohen despised thorough bibliographies on top open problems: they make you think like, they make your mind rhythmically walk the trail of those who failed, and whose ego tried to preserve itself by sacralizing the problem *post hoc*.

So the Syracuse problem[1] is impossible until it isn't. But if you thoroughly endeavor to fox-trot deep into the quicksand bibliography of those who failed, I don't expect you to succeed at, first,

[1]a.k.a. Collatz, but I have always found the Syracuse name to be immensely more inspiring, and as Pasteur beautifully wrote: "*The Greeks bequeathed to us one of the most beautiful words in our language–the word 'enthusiasm'–en theos–a god within. The grandeur of human actions is measured by the inspiration from which they spring. Happy is he who bears a god within, and who obeys it*"

kicking the problem off its ridiculous psychological pedestal, second, solving it as it is the basic work and intention of science. This is why, while I recently co-published a proof of the Collatz conjecture in *Mathematics* along with a beautiful team of German and one Indian scientist (who, thankfully, was, among other things, in charge of handling the thorough and homeworkey bibliography), this article will keep the references to the absolute minimum.

2 The basics

You want to define the attractor of any point in Syracuse, that is, the set of all odd numbers leading to it, with, for all intents and purpose, defining Syr(x) as "the next odd number in the forward orbit of x". The whole point is to use the intersection of the binary, ternary and quaternary trees over odd numbers as a coordinate system, that is, $2\mathbb{N}^* + 1$ endowed with operations $\{\cdot 2+1; \cdot 2-1; \cdot 3; \cdot 3+2; \cdot 3-2; \cdot 4+1; \cdot 4+3; \cdot 4-1; \cdot 4-3\}$. In practice though, you can mostly rely on the following actions:

Definition 2.1. Actions *G*, *V* and *S*: *For any natural number* a*,*

1. $G(a) := 2a - 1$
2. $S(a) := 2a + 1.$
3. $V(a) := 4a + 1 = G \circ S(a)$

Number theorists will be already familiar with these simple actions: that S(p) be prime if p is will define Sophie Germain primes, for example the famous {5;11;23;47} line-up which is the initial segment of the no less famous A083329 integer sequence which I show later is quite important in the Furstenberg $\times 2 \times 3$ conjecture[2]. The Euler result on Mersenne primes also involves action $S^2(x) = 4x + 3$, namely that if p and $2p + 1$ are Sophie Germain and p can be written $4x + 3$, then $2p + 1$ divides $2^p - 1$. Finally action V is a famous protagonist in Fermat's theorem on sums of squares: a prime p can be expressed as a sum of two squares if and only if $p = V(x)$.

To easily represent the base-3 world, you then define odd numbers depending on their final digit in this base:

Definition 2.2. Types *A*, *B*, and *C*:

1. *A number a is of type A if its base 3 representation ends with the digit 2.*
2. *A number b is of type B if its base 3 representation ends with the digit 0.*
3. *A number c is of type C if its base 3 representation ends with the digit 1.*

In homeworkey words, a number of type A belongs to residue class $[2]_3$, a number of type B belongs to $[0]_3$, and a number of type C belongs to $[1]_3$ in the ring $\mathbb{Z}/3\mathbb{Z}$. There is however a very practical reason to say "type A" rather than $[2]_3$, which is that you can now combine several

[2] I am in fact quite surprised Furstenberg be not even mentioned in the OEIS page of A083329

properties from the two worlds (bot binary and ternary) with less characters, and character parcimony is in turn the fundamental purpose of mathematical notation across history. For example:

Definition 2.3. *An A_g is a type A number that can be written $4x+1$ with x even. Their set is defined as* $24\mathbb{N}^*+17$

This A_g name is significantly more mind-ergonomic - namely, it needs less mind span to hold and use, than just $[17]_{24}$. The definition itself better reminds where they are exactly in the binary tree over odd numbers. For example, their most important property is that they are strictly decreasing under both Syr ($3A_g+1$ is always dividable by 4) and Syr^{-1} (being of type A, they always have a whole $(A+1)\cdot 2/3-1$ but they also are the only possible numbers ending in 1 in base 4 verifying $A_g \equiv S(A_g)$ under Rule 2 (see next paragraph).

If you further define $a \equiv b$ as just meaning "a and b have a common number in their orbit" then you get the following fundamental rules defining the attractor of any odd number in Syracuse. Although I formulated them in more detail in the *Mathematics* publication, this barebone formulation is in fact equivalent and sufficient to finish the proof because, in fact, all the rules can be inferred from Rule 3.

- **Rule 1**
  $\forall x$ odd, $V(x) \equiv (x)$
- **Rule 2**
  $\forall x \in \mathbb{N}$ if x is odd, then, $S^kV(x) \equiv S^{k+1}V(x)$ with k odd. If x is even $S^kV(x) \equiv S^{k+1}V(x)$ with k even.
- **Rule 3**
  $\forall n \in \mathbb{N}$, $\forall y \in \mathbb{N}$, $\forall x$ odd non dividable by 3, if $a \equiv y$ and $a = G(3^n x)$
  $\Rightarrow \bigwedge_{i=0}^{n} (S^i(G(3^{n-i}x)) \wedge S^{i+1}(G(3^{n-i}x))) \equiv y$

3 A Collatz proof from The Book

Theorem 3.1. *The Collatz dynamical system finitely maps any odd number it does not directly map to 1, 3 or 5 to an A_g or a $S(A_g)$ that it is equivalent to under Rule 2.*

This theorem is a consequence of the proof of Rule 2 in the *Mathematics* paper but one can quickly outline its proof, along with other important behaviors of the Collatz map that will underline its relation to base changes and to the Furstenberg $\times 2 \times 3$ conjecture:

1. If a number is written $x\underbrace{1\ldots1}_{\text{n}}$ in base 2, then it is finitely mapped to the result of operation G on the number written $y\underbrace{0\ldots0}_{\text{n}}$ in base 3 with $y=(x+1)/2$. This number is always of type A and precisely is always either an A_g or $S(A_g)$. Also note that this is the one and only way an orbit can rise in the Collatz dynamics.

2. If a number is written $z\underbrace{2\ldots2}_{\text{n}}1$ in base 4, then it is immediately mapped to a number written $x\underbrace{1\ldots1}_{\text{2n+1}}$ in base 2.

3. If a number is written $z\underbrace{2\ldots2}_{\text{n}}3$ in base 4, then it is immediately mapped to a number written $x\underbrace{0\ldots0}_{\text{2n+1}}1$ in base 2.

4. If a number is written $s\underbrace{0\ldots0}_{\text{2n+1}}1$ in base 2, then it is finitely mapped to $S(r\underbrace{0\ldots0}_{\text{n}})$ in base 3 with r as the base 3 representation of s.

5. If a number is written $v\underbrace{0\ldots0}_{\text{2n}}1$ in base 2, then it is finitely mapped to $V(w\underbrace{0\ldots0}_{\text{n}})$ in base 3 with w as the base 3 representation of v.

So solving the A_g solves Collatz, the first one being 17, which is the A_g of $7 \equiv 15$, and thus also of $9 \equiv 19$ and of $11 \equiv 23$. The second A_g is 41, which is the one of $27 \equiv 55$. One can single out the binary tree over the A_g numbers within the binary tree over $2\mathbb{N}^* + 1$ as $24\mathbb{N}^* + 17$ endowed with $\{\cdot 2 - 17; \cdot 2 + 7\}$. Let us now call this binary tree $\mathbb{A}_g$.

The whole point of the proof is to demonstrate that the growth factor of the attractor of any point within the binary tree over odd numbers, for a finite n, is always strictly greater than 2, and in fact great enough that there can only be one attractor in Collatz. From there indeed, once it is proven that any attractor branches faster than the binary tree itself, the pigeon-hole principle makes it easily demonstrable that any two pair of points on the same row of the tree (that is, any two points {a:b} such that $2^n < a < b < 2^{n+1}$) would always have a common point in their attractor if they were separate, which in turn solves Syracuse.

Definition 3.1. *The* ***growth factor*** *ρ of the attractor of an odd number a is the solution ρ to $\rho^n = N$ where N is the number of elements in the attractor of a that are strictly below $2^n \cdot (a+2)$*

Theorem 3.2. *Let a be any odd number greater than 1024. There is always $n < 1000$ such that the growth factor of the attractor of a to $2^n \cdot a$ is greater than 2.1 .*

Proof. Without any loss of generality thanks to Theorem 3.1, let us take $a \in \mathbb{A}_g$. By Rule 2 we have $a \equiv S(a)$ and thus two infinite series of equivalence by Rule 1:

$$\{a; V(a); ...; V^n(a)\} \tag{1}$$
$$\{S(a); V(S(a)); ...; V^n(S(a))\} \tag{2}$$

Let us simply call the first series $\mathbb{V}_a$ and the second one $\mathbb{V}_{S(a)}$. When counting elements to a finite n we will specifically call this finite segment $\mathbb{V}_a^n$

For any type A number a, V(a) is of type B, $V^2(a)$ of type C and $V^3(a)$ of type A so one point out of 3 in $\mathbb{V}_a$ is of type A, which mean 1 point out of every 6 powers of 2. Applying Rule 3 on these points generates another infinite series of points in the attractor of a, with x_1 defined as the result of ${}^{2(a+1)}/_3 - 1$ and x_2 as that of ${}^{2(S(a)+1)}/_3 - 1$. We may also define D(x):=64x+49

$$\{x_1; S(x_1); D(x_1); S(D(x_1))...; D^n(x_1); S(D^n(x_1))\} \tag{3}$$

$$\{x_2; S(x_2); D(x_2); S(D(x_2))...; D^n(x_2); S(D^n(x_2))\} \tag{4}$$

Let us call each of these infinite series $\mathbb{D}_{x_1}$ and $\mathbb{D}_{x_2}$

As with the impact of action V on types, action D also preserves the order "ABC", namely with $a \in \mathbb{A}_g$, D(a) is of type B, $D^2(a)$ of type C and $D^3(a)$ of type A, more precisely, it is always an A_g, so for the sake of simplicity we will retain $a \in \mathbb{A}_g \Rightarrow D^3(a) \in \mathbb{A}_g$. So in any $\mathbb{D}_x$ there is one A_g every 18 powers of 2.

At this point one may also note an important aspect on the distributions of A_g numbers and of $\mathbb{D}$ and $\mathbb{V}$ series among the binary tree over odd numbers, which is also very relevant to the Furstenberg $\times 2 \times 3$ conjecture. Namely, if you wrapped the binary tree over the unit circle (which we do in the next section), pushing each branch to infinity and thus with each possible branch defining a unique real angle, any iteration of actions V or D on any number would converge to only one limit angle, while action $24x + 17$ would not, because such action contains a multiplication by 3.

So, starting with a single A_g number and its $S(A_g)$ under Rule 2, we so far generated two $\mathbb{V}$ series, each generating two $\mathbb{D}$ series, each containing an infinity of new A_g numbers, namely one out of 18 powers of 2 in the binary tree.

Obviously this is not yet enough to prove a growth factor strictly above, say, 2.1, because all we have for now are at best ${}^{2n}/_{18}$ new A_g numbers in the attractor of our starting point, while up to n there are about ${}^{2^n}/_{24}$ new A_g in the binary tree starting from the same point, which is our standard of comparison, so one would need to multiply our initial finding by ${}^{3 \cdot 2^{n-3}}/_{n}$ to just even the score. Challenge accepted.

The good thing with action Syr^{-1} on the types A of a $\mathbb{V}$ series is that each element of the $\mathbb{D}$ series it generates now also has a unique $\mathbb{V}$ series of its own, allowing to repeat the process again and again. The repetition of course remains finite with n, because Syr^{-1} of a type A outputs either a type A, B or C in equal proportions thus even with a $\mathbb{V}^n$ starting with a type A, one may obtain a $\mathbb{D}^n$ or $\mathbb{D}^{n+1}$ (because Syr^{-1} is decreasing) beginning with a type B number, which is the worst possible case, in which the first new $\mathbb{D}$ series will be of length n-3 or n-2 over the binary tree. However, in determining exactly, for a given $\mathbb{V}^n$, what is the length m of the $\mathbb{D}^m$ obtained from it by Rule 3, we may define a precise **"splicing factor"** of Rule 3

Definition 3.2. *Splicing factor*

*Let $\mathbb{V}_x^n$ be any $\mathbb{V}$ series, the **splicing factor** λ of Rule 3 on it is the height of the first type A number in $\mathbb{V}_x^n$ starting from x included. If x is of type A $\lambda = 0$, if it is of type B $\lambda = 4$ and if it is of type C $\lambda = 2$*

One may note that within any $\mathbb{D}$ series types A, B and C are equifrequent, thus averaging λ to 2.

To summarize, from a single A_g number we obtained by Rule 1 and Rule 2 a pair of series $\mathbb{V}^n$ and $\mathbb{V}^{n-1}$, each turned by Rule 3 into a $\mathbb{D}^n$ series having again, by Rules 1 and 2, the following series of $\mathbb{V}$ series:

$$\{\mathbb{V}^n_x; \mathbb{V}^{n-1}_{S(x)}; \mathbb{V}^{n-6}_{D(x)}; \mathbb{V}^{n-7}_{S(D(x))} ... \mathbb{V}^{n-6k}_{D^k(x)}; \mathbb{V}^{n-(6k+1)}_{S(D^k(x))}\} \tag{5}$$

If we take n-1 to be dividable by 6, it is equivalent to having at least $^{(n-1)}/_6$ series $\mathbb{V}^n$, which yet again, by Rule 3, will give us the same number of $\mathbb{D}^{n-\lambda}$ series, each again exhibiting one A_g every 18 powers of 2.

We are now in a position to demonstrate that any A_g number, up to a certain n, will always have strictly more other A_g numbers in its attractor than $^{2^n}/_{24}$

Indeed, beginning with one $\mathbb{A}_g$ and thus obtaining two $\mathbb{D}_n$, we obtain for each of them the following multiplication pattern

$$\mathbb{D}_n \to \frac{n}{6}\mathbb{D}_{n-\lambda} \to \frac{n}{6}\frac{n-\lambda}{6}\mathbb{D}_{n-2\lambda} \to \frac{n}{6}\frac{n-\lambda}{6}\frac{n-2\lambda}{6}\mathbb{D}_{n-3\lambda} \to ... \tag{6}$$

Where each $\mathbb{D}_x$ so obtained has $\lfloor \frac{x}{18} \rfloor$ A_g numbers, giving us the following formula to establish the (minimal) amount of A_g in the attractor of any such number, up to n:

$$\lfloor \frac{2}{3} \sum_{k=0}^{\lfloor \frac{n}{\lambda} \rfloor} \prod_{i=0}^{k} \lfloor \frac{n-\lambda k}{6} \rfloor \rfloor \tag{7}$$

...which we now have to compare to $^{\rho^n}/_{24}$. For example, with $\lambda = 2$ and $n = 100$ we obtain $\rho \approx 2.7$. With $n = 129$ and $\lambda = 3$ we get $\rho \approx 2.113$, and even with the worst possible value of $\lambda = 4$ we get $\rho \approx 2.2675$ for $n = 400$, which again comes as no surprise since the multiplication process of $\mathbb{D}_n$ series under Rule 3 is a sum of λ-uple factorials normalized by a very manageable 6^k.

We therefore have that assuming any other independent attractor than that of 1 exists under the Collatz map over the binary tree over odd numbers, a contradiction arises that any two numbers pertaining to the same row must finitely have a common number in their respective attractor, which is impossible if they were separate in the first place .

$\square$

This demonstration is of course consistent with Figure 15 of our article in *Mathematics* which showed the proportion of numbers remaining to prove immediately when row n of the binary tree over odd numbers had been proven to converge to 1 under the three rules[3] implemented from $\mathbb{V}_1$ onward was decreasing exponentially. As for the limit growth factor of the three rules, we also obtained empirically that, started from $\mathbb{V}_1$ onward, whenever any full row n of the binary

[3]again, in the paper, we detailed the consequences of their combinations into two more rules, so there were actually five of them.

tree was completed (that is, fully proven to converge to 1), there were close to 3^{n+1} additional odd numbers proven above it.

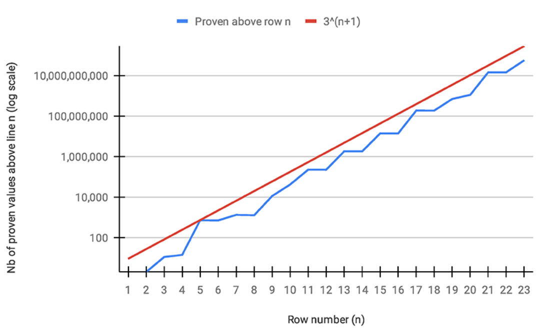

Figure 1: From the *Mathematics* paper: when the three rules are applied on odd numbers in their natural order starting from $1 \equiv 5 \equiv 3$, whenever they have proved the convergence of full row n of the binary tree, which they do one by one in their natural order, they have also proved the convergence of about 3^{n+1} additional numbers above.

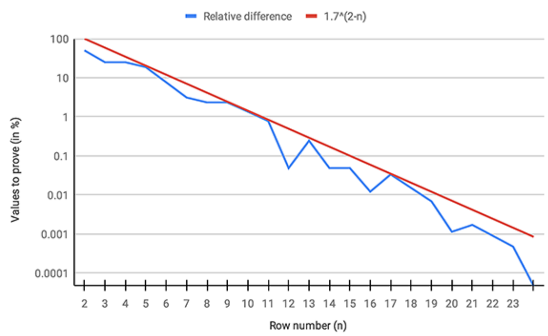

Figure 2: Consistent with Theorem 3.2: when row n has just been proven, the proportion of numbers remaining to prove in row $n+1$ decreases exponentially with an empirical upper bound of about 1.7^{2-n}, which first afforded me the then bold claim that *almost all orbits attain bounded values*. Both figures courtesy of Pierre Collet and Baptiste Rostalski.

This proof being both short and elementary, although its genesis fully relied on the systematic introduction of the multi-unary algebras I called "Romanescos" (see next part), I am ready to wager it is the most straightforward possible demonstration of the unity and totality of the attractor of 1 in Syracuse; the metamathematical question "what is the most *ad hoc* way of posing a problem" has always fascinated me, and I really believe multi-unary algebras are the most natural structures to arise in the study of discrete chaos *à la* Syracuse, but also beyond. Hence the next section.

4 Romanesco algebra and Dreamcatcher theory

A Romanesco $\{m; n; p\}$ is the intersection of the m-ary, n-ary and p-ary trees over some number set (typically, $\mathbb{N}$). Each tree in turn is a collection of branches which are series of unary actions. For example, the Romanesco $\{2; 3\}$ on $2\mathbb{N}^* + 1$ which is the most fundamental algebraic object I designed and used to solve Syracuse is the set of odd numbers endowed with unary operations $\{\cdot 2 - 1; \cdot 2 + 1; \cdot 3 - 2; \cdot 3; \cdot 3 + 2\}$. It looks like this:

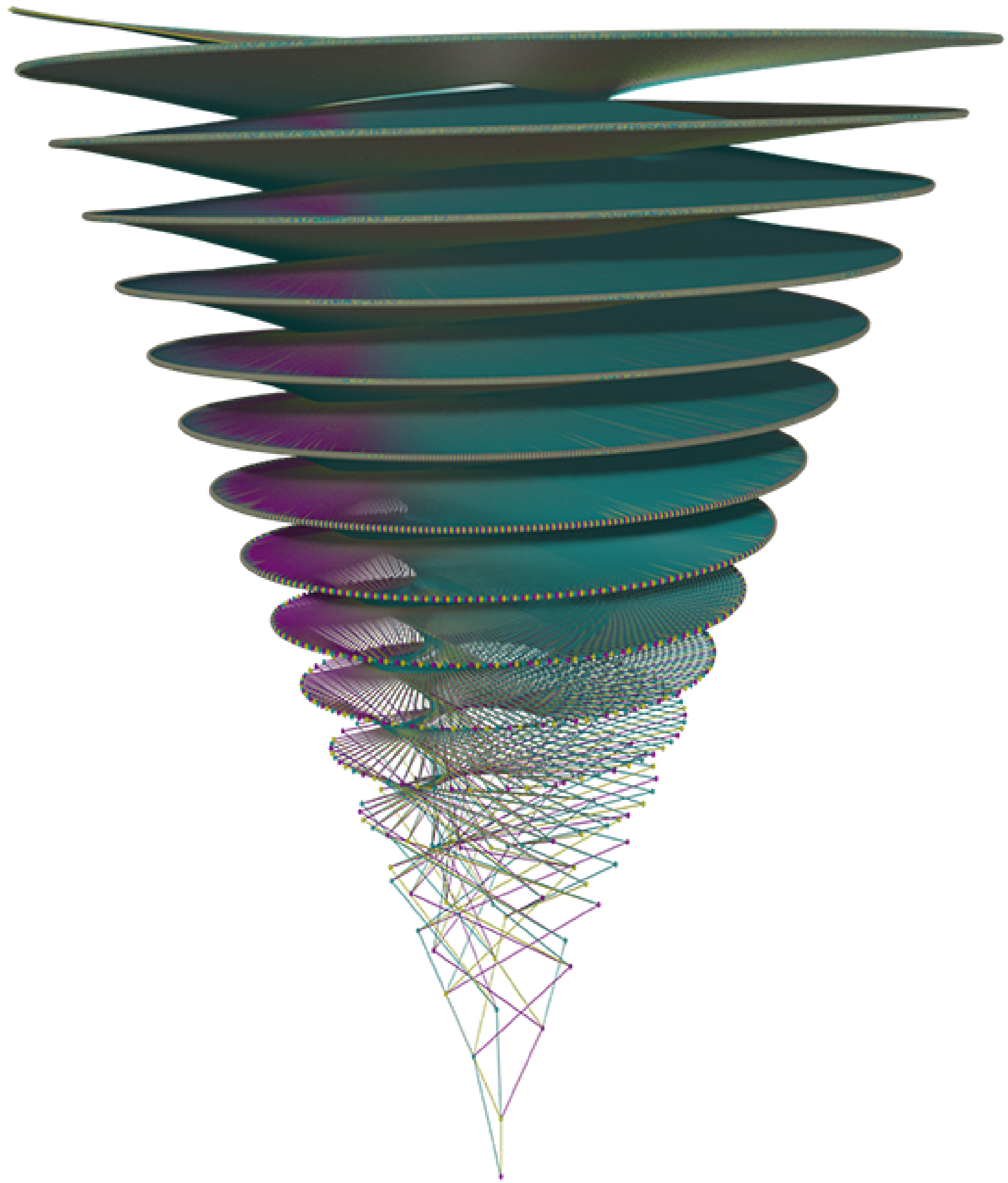

Figure 3: A representation of **Romanesco** $\{2;3\}$ over odd numbers up to 2^{18}. Type B numbers are the result of action $\cdot 3$ (yellow branches), type A numbers are the result of action $\cdot 3+2$ (teal branches) and type C numbers are the result of action $\cdot 3-2$ (purple branches). Figure courtesy of Max Henkel.

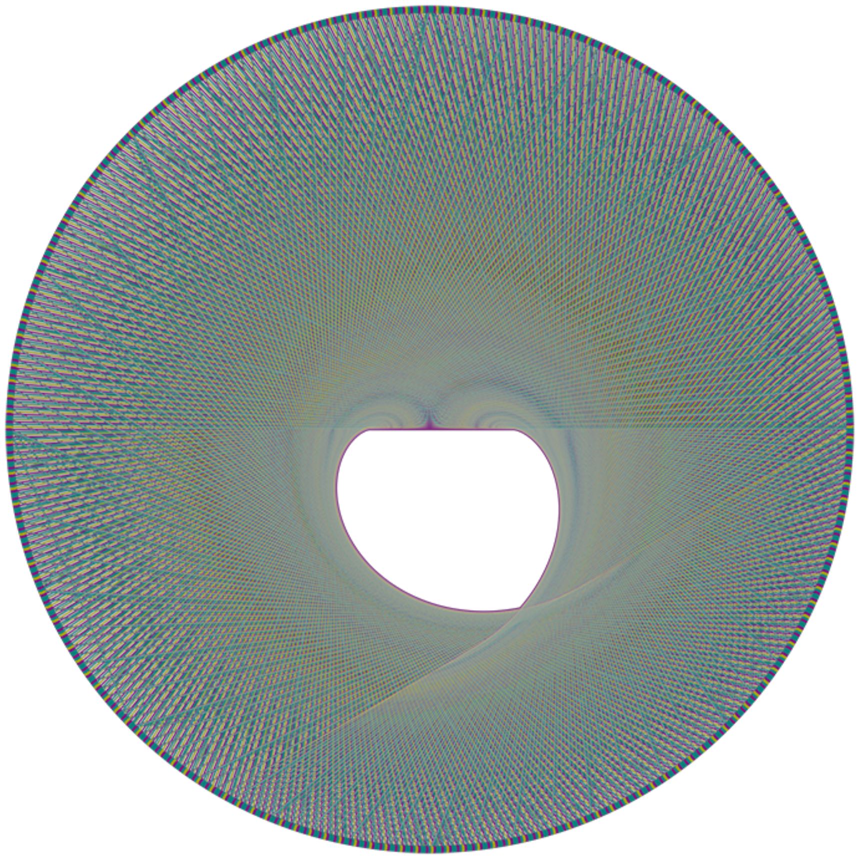

Figure 4: When Romanesco {2;3} is mapped on the unit circle by embedding the binary tree on it, the envelope of the ternary operations forms the **3-dreamcatcher in base 2**. Figure courtesy of Max Henkel.

It might look and feel just elementary to consider some unary operations on $\mathbb{N}$, but just as the very formulation of the Collatz map looks elementary yet displays complex emerging behaviors, the emerging properties of romanescos and dreamcatchers are not trivial at all, and they can shed new light on discrete chaos but also on p-adic arithmetic, ergodic theory and number theory at large. For example, Pablo Shmerkin has very well explained the fundamental intellectual motivation of such conjectures as Furstenberg's $\times 2 \times 3$ in this way:

Principle 4.1. ***(Furstenberg 1960)***
Expansions in bases 2 and 3 have no common structure.
More generally, this holds for bases p and q which are not powers of a common integer or, equivalently, $log(p)/log(q)$ is irrational.

And indeed, the complexity of changes from bases 2, 3 and 4 does capture all (and I mean **absolutely all**) of the chaoticity of the Syracuse dynamical system, which is in itself so high that Fabian Bocart demonstrated it could form a reliable proof of work in the cryptocurrency industry. Yet, the otherwise mysterious geometry of ternary operations over binary structures does appear in a much clearer way when represented say, in Romanesco {2;3} or in the 3-dreamcatcher in base 2. In particular, the extent of operation $\times 3$'s mixingness and its monotonous segments and singularities is very well captured by the shape of the 3-dreamcatcher, which can now also be further described, either as a parametric curve or as a concatenation of two bits of algebraic curves.

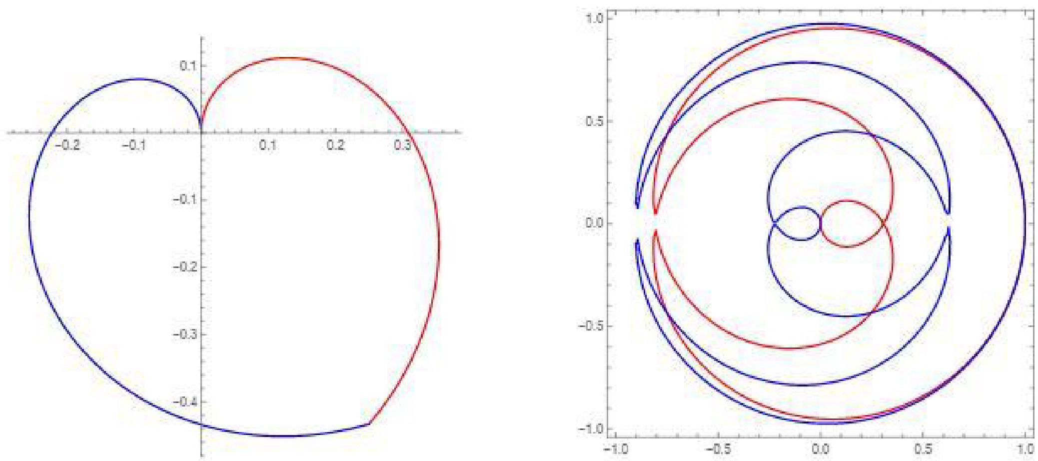

Figure 5: The 3-dreamcatcher in base 2 appears to be the concatenation of two bits of algebraic curves, further truncated by the x-axis (hence the flat top of figure 4). The two curves correspond to the parts of the unit circle where multiplication by 3 acts by scaling the angle by $0.75 + \textit{constant}$ (blue curve), and the other by 1.5 (red curve). Eldar Sultanow obtained the explicit formula (before the x-axis truncation) by computing the proper Sylvester matrix, which (theorem) is always possible for any p-dreamcatcher in base q as long as p and q are integers and may (conjecture) be so for any pair of reals too. The blue part appears to be an 8-degree algebraic curve: $-64y^8 - 256x^2y^6 + 112y^6 - 384x^4y^4 + 336x^2y^4 - 56y^4 - 256x^6y^2 + 336x^4y^2 - 112x^2y^2 + 7y^2 - 64x^8 + 112x^6 - 56x^4 + 7x^2 + x = 0$. The singularity at the bottom of the heart shape is limited by the mirror image of series $\mathbb{V}_1^n$ (which perfectly cuts the set of all branches of the binary tree over odd numbers into its first third) under the 5-7 axis, and is therefore at angle $^{2\tau}/_3$ on the unit circle

Embedding some p-ary tree over odd numbers on the unit circle will generate either a Prüfer p-group or, if the branches are pushed to infinity, the circle group itself. When performing such an embedding, one associates a single angle to a single branch, which is in itself a single decomposition in base p. Therefore, the shape of the q-dreamcatcher in base p gives the precise mapping of one angle to another under operation $\cdot q$, which is equivalent to obtaining either the prefix of qp in base p (a p-ary prefix corresponds to a unique angle range or subtree or open) or its full decomposition (a single angle), as we will show later, this system easily suggests a certain class of infinitesimals because it formally separates an endless tail of trailing digits 1 in base 2 from one of trailing 0, from one of trailing 0 with some last digit 1 appended, thus implying each of the base representations are now unique, and indeed, they are now associated with a unique angle.

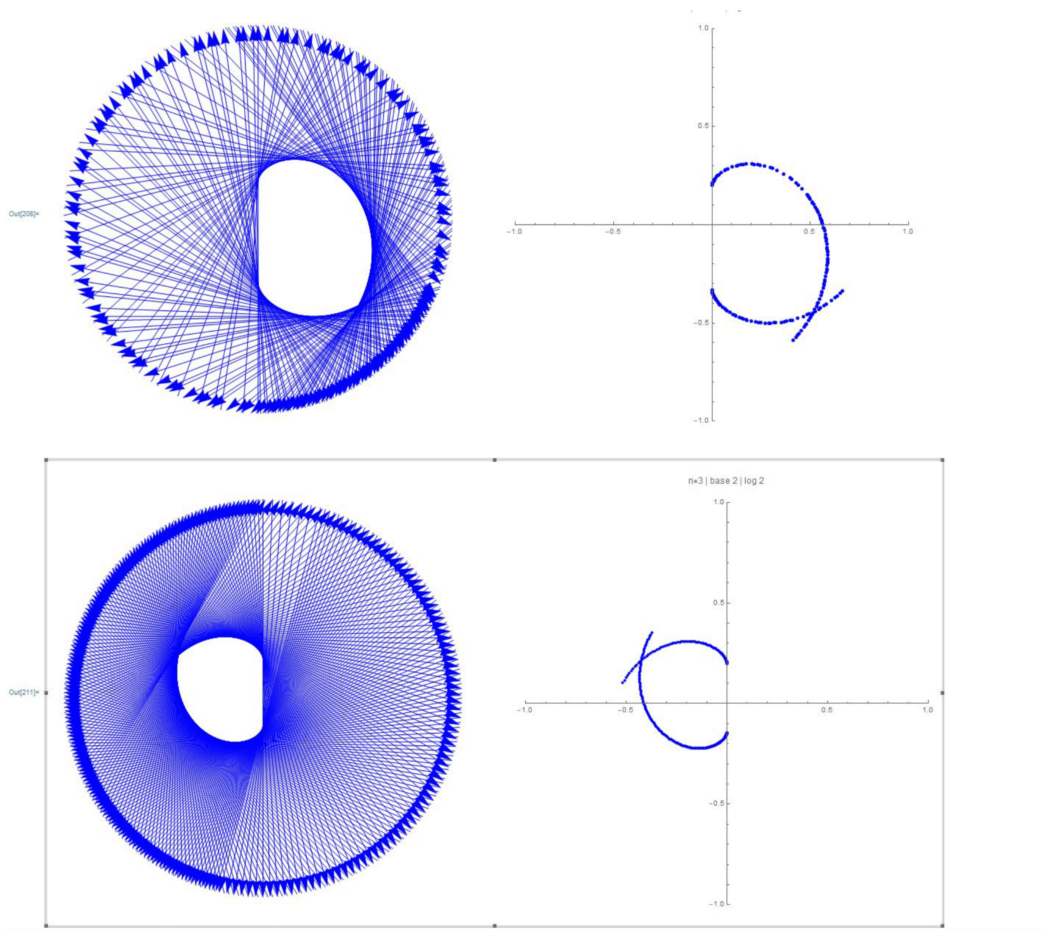

Figure 6: Up: 2-dreamcatcher in base 3, and (down) 3-dreamcatcher in base 2 (rotated). Depending on how p^n advances q^n the tangents of the envelope will be turning either clockwise or counterclockwise. More generally, dreamcatchers can be very useful in musical theory and transcendental number theory, especially in studying the approximations of powers of q by powers of p in finer details, but also in epsilon management and non-standard analysis (studying bases $1+\epsilon$). Figure courtesy of Max Henkel.

It turns out studying dreamcatchers has other potential applications. As they are envelopes generated by an unary application on the sets of p^{th} roots of unity, it is rather easy to relate them to Fourier transforms or to other number theoretical endeavors like p-adic arithmetic or complex multiplication[4]. For example, one can use them to great effect to map the known solutions to some difficult diophantine problems like the sums of three cubes and cubic quadruples, which then suddenly appear to have radically different behaviors:

[4]that their study could ultimately contribute to the *liebster Jugendtraum* is the firsts reason to call these objects *dreamcatchers*, the other one being their particular construction

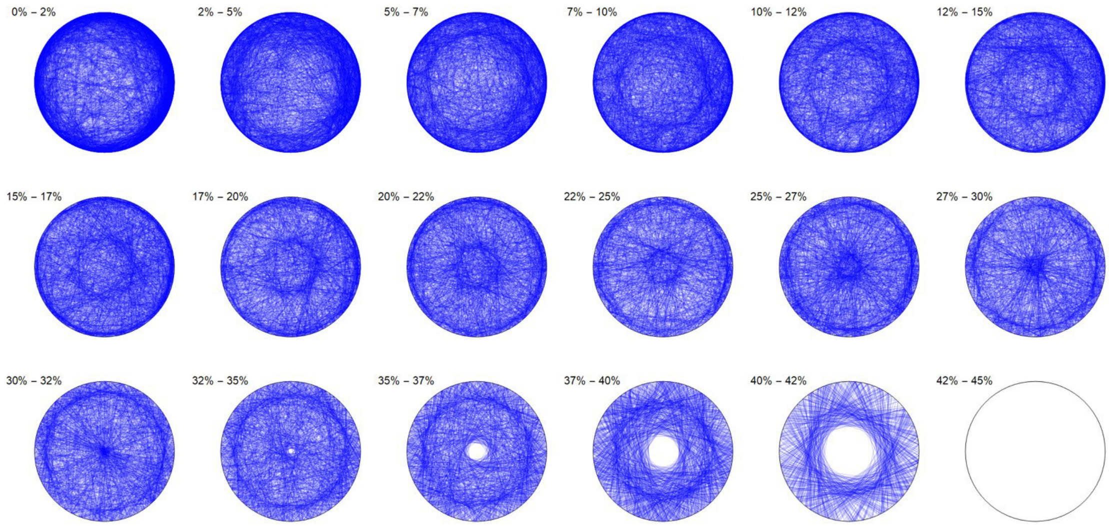

Figure 7: When mapped on the binary tree embedded on the unit circle, whole solutions {a;b;c} to equations of the form $a^3 + b^3 + c^3 = d^3$ (cubic quadruples) seem to behave ergodically, even when they are classified by the surface range of the triangle they generate (compared to the surface of the circle). One may excuse the presence of the impossible "42-45%" range which was initially left to check for errors. Figure courtesy of Max Henkel.

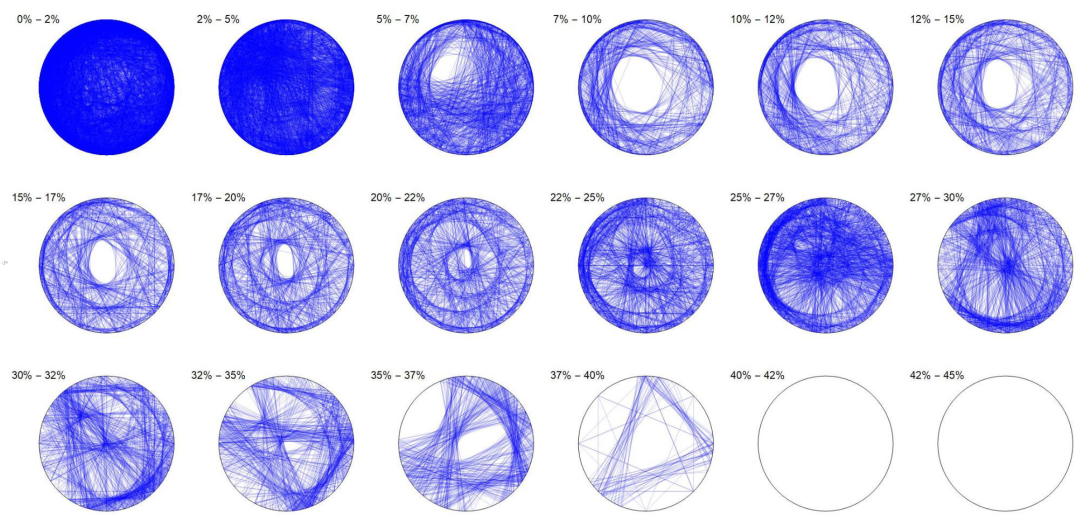

Figure 8: Such is however not at all the case of the known solutions to equations of the form $a^3+b^3+c^3 = d$ which then seem to have obvious "forbidden zones" which could further instruct more ambitious conjectures and research programs, for example: "there are no solutions forming an x-almost equilateral triangle in any base" or "there is a precise density distribution of solutions per surface range". Figure courtesy of Max Henkel.

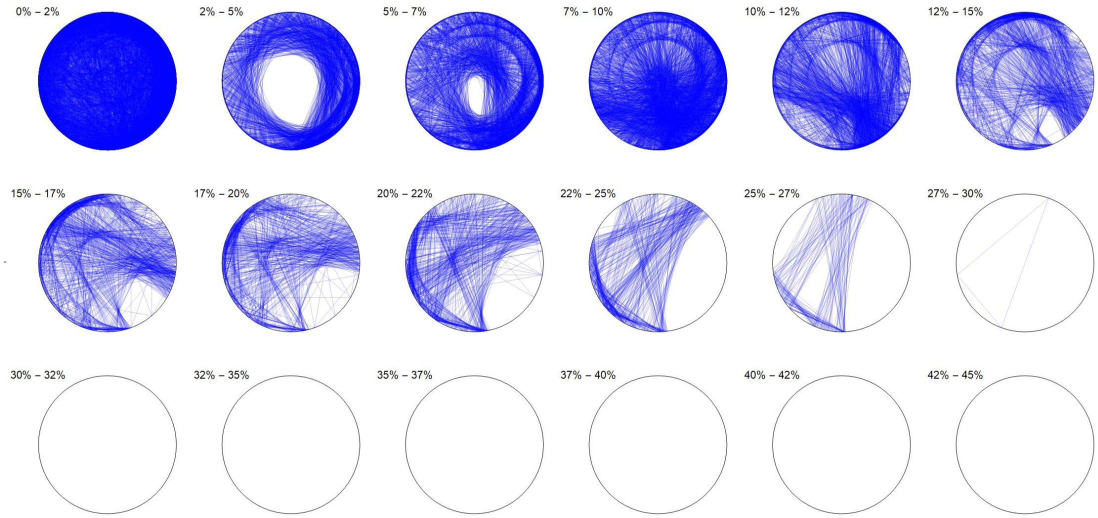

Figure 9: The previous figure, now in base 3 (ternary tree embedded in the unit circle, forming a Prüfer 3-group as we remain in the discrete case). One interesting extension of ergodic theory could consist in studying not measures but bases that are ergodic for a certain distribution or even, in applied mathematics, a certain dataset, which could be searched through artificial intelligence. One possible conjecture is that there is no real base (so even including, say, the phinary, pi-ary, e-ary or log(2)-ary bases) that could morph the distribution of the solutions to the sums of three cubes into that of the cubic quadruples in base 2. Morphisms (their existence or not, their properties and algebraic behaviors) of envelopes obtained in certain bases are typical questions of dreamcatcher theory. For data science, studying and classifying or even predicting data patterns that would produce such and such envelope or envelope spectrum (a superposition of envelopes) would be a fun and, I wager, fertile endeavour (a multidimensional data point could also be represented as an n-gon). Figure courtesy of Max Henkel.

So initially dreamcatcher theory may be developed to shed new light on diophantine arithmetic and to endeavor a finer theory of base conversions, leading to new insight on the Furstenberg $\times 2 \times 3$ conjecture. What I want to focus on here however is how dreamcatchers could be used to inform more advanced solutions to the Syracuse problem, and in particular, a separate proof that no non-trivial cycle can exist in the dynamical system. This approach owes a lot to Poincaré-Denjoy theory, but it specifically requires a new tool coming, among others, from Baker and Tao, which is epsilon management.

5 Poincaré-Denjoy-Tao Theory: Tao almost had it

Terence Tao is one of the bests proponents of epsilon management, all the while, with the Green-Tao theorem, also being a proponent and master of what Matheus called the "*remarkable effectiveness of ergodic theory in number theory*". Undergraduate mathematics will often stop, and condition most to stop, at demonstrating such orbit is dense or that such set is perfect. And indeed, 3^p approximations of powers of 2 are dense. But this is not enough at all. With

Baker, and in fact, the very foundation of transcendental number theory, the point is to first ask "how dense?" and then to establish a solid theory of comparative density. I argue this is what is most needed to obtain an independent proof that no non-trivial cycle can exist in Collatz, and that this proof itself could actually lead to a different solution of the Syracuse problem, less elementary but much richer theoretically with possible applications in many other hard dynamical maps (eg. The Juggler sequence).

One of the reasons epsilon management immediately becomes interesting in dreamcatcher theory is that, when you wrap a p-ary tree over the unit circle, you immediately define a very precise smallest possible epsilon, which is action +1. For example, if you wrap the binary tree over odd numbers action +1 will turn $2^n - 1$ into 1, and any $2^n x - 1$ into x. There is no difference in the field of real numbers between the angle occupied by $S^n(x)$ and that of x+1 when n goes to infinity, because there is no real number to distinguish 0.9999... from 1 even though it is the smallest natural epsilon to distinguish a minimal distance between two consecutive reals. With the binary tree over odd numbers, it naturally becomes (exactly half) the distance between two consecutive branches, that is, for any x, there are exactly two smallest epsilons between $S^n(G(x))$ and $G^n(S(x))$ namely between $G(x)\underbrace{1\ldots1}_{\text{n}}$ and $x\underbrace{0\ldots0}_{\text{n}}1$ in base 2 when n goes to infinity, with $S^n(G(x)) + \epsilon = x = G^n(S(x)) - \epsilon$

Now we obviously know that iterated multiplications by 3 are dense and loopless in the binary tree over odd numbers, but from this trivial piece of knowledge, one can achieve more elaborate considerations by summoning non-archimedean fields, namely: how ϵ-far from their own path will these iterations ever go, which is a typical point of transcendental number theory, as we know from Baker that ${}^{log2}/_{log3}$ is transcendental. Also, one fundamental purpose of embedding the binary tree over odd numbers on the unit circle is that divisions by 2 from an even number will always leave angles invariant. Thus, what action +1 does to any $x > 1$ in the coordinate system of the base 2 dreamcatcher of odd numbers is "send the number to the first smaller value clockwise", for example any Mersenne goes to 1, and any "anti-Mersenne" $2^n + 1$ goes to $2^{n-1} + 1$ *e.g.* 17 goes to 9. Can this drift be enough to open iterations of $\times 3$ to possible loops? Section 3 of this paper already says no of course, but what would really be interesting methodologically, is to demonstrate it through some *ad hoc* epsilon management, which would form a definite case study for the field. This endeavour I call Poincaré-Denjoy-Tao theory, "theory" because it goes way beyond just cracking Collatz.

Even though Collatz is chaotic the 3-dreamcatcher in base 2 well-behaves[5] the +1 action for you as its impact on the angle is perfectly predictable, even after iterations. It always corresponds to a rotation of ${}^{\tau}/_{l}$ where l is the length of the base 2 representation of the starting number. If there are no non-trivial cycles in Collatz, it means that even though operation $\times 3$ is dense in the circle, it is never "+1-dense" (except for the trivial cycle $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$, which becomes the identity $1 \rightarrow 1$ in the 3-dreamcatcher in base 2 as then Syr(1) does not even reach another element of $\mathbb{V}_1$ than 1 itself) because it is everywhere out of reach of this rotation, of this particular ϵ angle. One way of demonstrating that there are no possible non-trivial cycles in Collatz, would be to demonstrate that no x can come any close - much less intersect - to its $\mathbb{V}_x$ which is equivalent to saying it can't reach $\frac{4^n(3x+1)-1}{3}$ for any n, because the only way you could possibly loop in Collatz is true Rule 1.

To sum it all up, **what happens when you start properly blending Baker-Tao and**

[5] "best-behaves" maybe? So far I can't see a better way of well-behaving the consequences of action +1 in the Collatz map, but as Tao reminds in *spending symmetry* ignorance is not an argument

Poincaré-Denjoy is that you can rigorously establish definitive interdiction zones around each step of a given orbit which needless to say is an extremely powerful tool in chaos theory at large. Wherever a certain point flies within the circle, it will come to interdict some infinitesimal angle (which exact span entirely relies on transcendental number theory and careful epsilon management) around every point it touches, hopefully depleting enough angles to achieve a proof of convergence: any orbit $Orb(x)$ won't be able to step into such and such collection of opens until reaching $y < x$ is the only option left. Note that while the fundamental theorem of arithmetic guarantees the trivial $|3^p - 2^q| > 0$, we also already have:

$$3^p - 2^q = 3^p(1 - 3^{q(\frac{log2}{log3} - \frac{p}{q}})\tag{8}$$

And if the Tao-Collatz conjecture is true (a.k.a the "weak Collatz conjecture") then

$$2^p > 3^q \Rightarrow 2^p - 3^q \geqslant (1+\epsilon)^q\tag{9}$$

...meaning the base infinitesimal to fit many times between 2^p and 3^q is some q-compounded non-archimedean interest (I believe a simple theory of non-armichedean compound interests could form a very accessible introduction to epsilon-management and transcendental number theory).

The added value of Denjoy is that now that actions $\cdot p$ in base q can be defined as diffeomorphisms of the unit circle with an irrational rotation number, one can further open their study to diagram-chasing through their topological conjugacy. Think of dreamcatchers as a class of adapters from number theory and arithmetic topology to ergodic theory. It is not a coincidence that René Thom (of Zeeman-Catastrophe Theory fame[6]) wrote the theory of envelopes should be mandatory in undergraduate mathematics.

6 Some thoughts on the Furstenberg $\times 2 \times 3$ conjecture

They both are open problems about chaotic dynamical systems involving multiplications by 2 and 3, yet the Syracuse problem and the Furstenberg $\times 2$ $\times 3$ conjecture - even though the latter has always been related to problems of diophantine aproximation - never show up together in the literature. To me, this is just plain peer-pressured nonsense (a.k.a "guildshit"). Furstenberg proved in 1960 that the unit circle under iterated $\times 2$ mod 1 and $\times 3$ mod 1 has no (infinite) non-trivial closed invariant set, which is a consequence of the relative "mixingness" of operation $\times 3$ over a binary representation, which in itself forms the core of the inflation propensity of Collatz orbits, itself, finally, so pseudorandom it can form a proof of work (Bocart 2017), so I mean of course the two problems are deeply related and should be studied together and it is my opinion that Figure 6 of this paper, if properly theorized, will offer very valuable insight into both the Furstenberg "principle" in general and the $\times 2$ $\times 3$ conjecture in particular.

Now, without resorting to the Prüfer p-groups or roots of unity, one can still represent iterations of the dyadic transformation (the $\times 2$ mod 1 map) and the triadic transformation (the $\times 3$ mod

[6]also, that one mathematician who blew Salvador Dali's mind so much the mad painter ended up believing Perpignan station was the freaking center of the Universe and also painted a few pieces for him

1 map) as some sorts of billiards within the unit circle. These modular multiplications will generate beautiful caustics that are most often - and I deplore it - thought of as just inspiring pieces of recreational mathematics. They really are not.

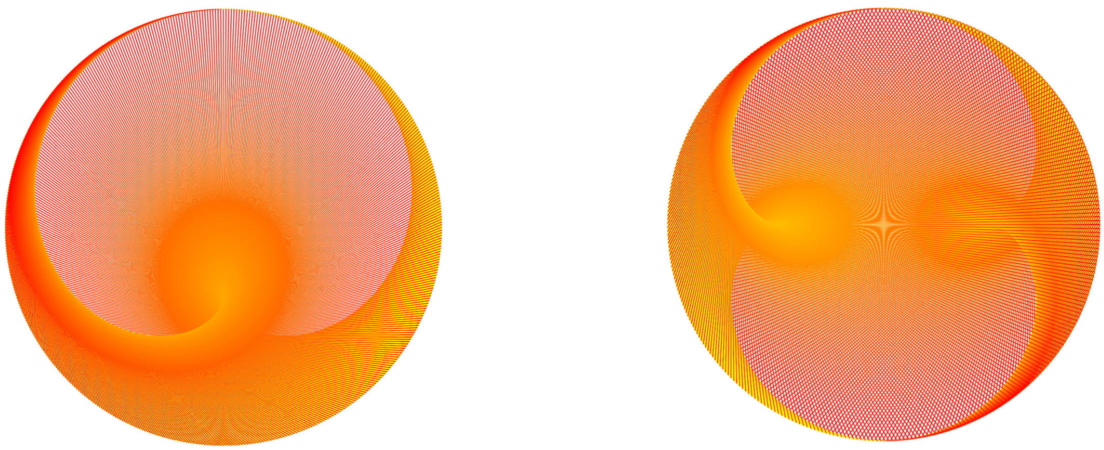

Figure 10: Left: the caustic generated by map $\times 2$ mod 1 on the unit circle; value 0 is now on top (because it is the only decent way to represent the Alliance Starbird you uncultured hog you). Right: the caustic generated by $\times 3$ mod 1, equivalent to a light source placed at infinity. Both figures courtesy of Max Henkel.

Geometrically, the single most fundamental property that operation $\times 2$ will contribute will be to force the presence of a caustic on point 0.5 (angle π), which just cannot happen with any 3^n.

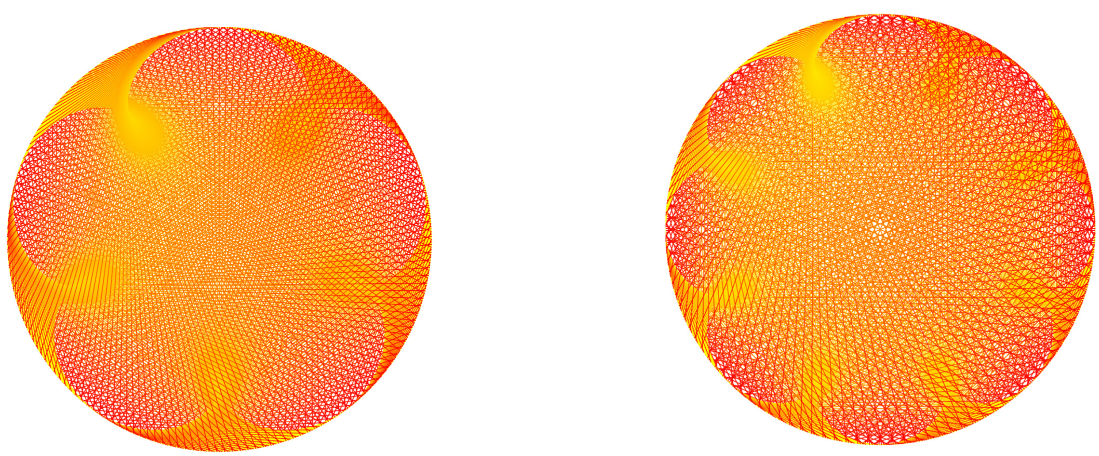

Figure 11: Ccustic generated by $\times 6$ mod 1 (Left) and $\times 9$ mod 1. Operation $\times 2$ is heavily symmetry-breaking on operation $\times 3$. The caustic of $\times 3^n$ mod 1 will have exactly $2 \cdot (3^n - 1)$ axis of symmetry, because each of its $3^n - 1$ caustics is facing another of the same size, whereas in any form $2^n \cdot 3^m$ one can now only have $2^n \cdot 3^m - 1$ axis of symmetry, which is decisive in such dynamical systems having no infinite non-trivial closed invariant set. A system $\times p$ mod 1 where p is of the form 3^n will have $2(p-1)$ axis of symmetry while if p is of any form $2^n \cdot 3^m$ it will have $p - 1$ axis of symmetry.

So we know what $\times 2$ does to a 3^n system: it forces it to always have a caustic at position 0.5 (or $\pi/2$) and breaks the symmetry that the caustics of any 3^n system must always face each other. If the symmetry group of the 3^n system is $C_{2\cdot(3^n-1)}$ that of $2\cdot 3^n$ is $C_{(2\cdot 3^n)-1}$. Now what does action $\times 3$ interdict in any 2^n system? Essentially, it forbids the number of caustics to be dividable by 3, and in fact, forces it to belong to $[2]_3$ ("type A"), but even more, precisely *forces it to belong to the image of the Mersenne branch ($\mathbb{M}$) under iterated (up to monotony) Collatz transforms*, namely the order of their rotation group always belongs to:

$$Syr^{\infty}(\mathbb{M}) = \mathbb{S}_5; \mathbb{S}_{17}; \mathbb{S}_{53}; \mathbb{S}_{161}; ...; \mathbb{S}_{G(3^{\infty})}. \tag{10}$$

... with $\mathbb{S}_5$ being OEIS A083329. Unlike with $\mathbb{A}_g$ there is no $x\mathbb{N}+y$ with $x, y \in \mathbb{N}$ to identify this sequence. Also, let us whack out some recreational conjecture of advanced Poincaré-Denjoy-Tao theory :

Conjecture 6.1. *Of all the k-gons distributing the caustics of any system $2^n 3^m$ on the unit circle, the only straightedge and compass constructible ones are k=5 and k=17.*

So back to the Furstenberg conjecture: any modular multiplication ax mod 1 in the unit circle will leave an envelope that is a finite equidistribution of caustics which centers are the vertices of the regular (a-1)-gon with a vertex at $\frac{0.5}{a-1}$ which for any prime a, generates a particular family of envelopes for $a^n x$ mod 1 that will allow a large diversity of ergodic measures. However, the Furstenberg conjecture implies that when you are only allowed certain specific series to pick from, you collapse the set of ergodic Borel probability measures to the translation-invariant one or to atomic ones. The case of 2 is marking because it is the only possible case in which the mirrors (the caustics) will never be facing each other, which obviously has many important implications in terms of cyclic trajectories and thus the $\times 2 \times 3$ mod 1 case belongs to this specific family of rotation groups with odd orders. In general, crossing two multiplicatively independent integers a and b will always ensure that $ab-1$ is never dividable by any of them. Is this the only property we should be relying on? If yes, this would give the following loose conjecture:

Conjecture 6.2. *Some divisibility constraint on the order of the rotation group of the envelope forces any ergodic probability measure to be atomic or translation-invariant*

In any case, we already have from Furstenberg that...

Theorem 6.3. *If a and b are multiplicatively independent whole numbers, then for any irrational x, then $\{a^n b^m x, n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}\backslash\mathbb{Z}$*

...so obviously $e^{2i\pi a^n b^m x}$ is dense in $\mathbb{T}$ (how dense by the way?).

A still loose argument I want to share here regarding the possibility that certain families of regular polygons, in distributing the caustics on the unit circle, be forcing any non-atomic invariant Borel measure to be translation invariant is based on the possibly simplest proof of the translation invariance of the Lebesgue measure (as shared on Omar Antolín Camarena's webpage).

Lemma 6.4. *The Lebesgue measure is the only measure on $\mathbb{R}^n$ defined on all Borel subsets, invariant under translations and such that the measure of $[0;1)$ is 1.*

Proof. Let μ be a measure satisfying those conditions. Using translation invariance and additivity we get successively that

1. $\mu(\Pi[0; m_i)) = \Pi m_i$ for positive integers m_i
2. $(\Pi n_i) \cdot \mu(\Pi[0; \frac{m_i}{n_i})) = \Pi m_i$ for positive integers m_i and n_i
3. μ is the Lebesgue measure for any product of half-open intervals with rational endpoints

□

Rational endpoints are important in our case because it is easy to prove that the beginning and end of any caustic in any p^n envelope are rational. When $T = 2x$ mod 1 the invariance of the Lebesgue measure is ensured by that any interval S of measure $\mu_s < \frac{1}{2}$ will have 2 other non-overlapping intervals $T_1^{-1}(S)$ and $T_2^{-1}(S)$ mapping to it, symmetrically distributed, and each of measure $\frac{1}{2}$. So here are the take-home constraints that I believe should be banded together to solve the Furstenberg $\times 2 \times 3$ conjecture

1. the envelope of any "ditryadic" transformation $d = 2^n 3^m$ mod 1 in S^1 is made of $2^n 3^m - 1$ caustics equally distributed on the circle, with one centered on 0.5.
2. so any interval $[\frac{k}{d}; \frac{k+1}{d})$ is mapped to by d segments $[\frac{k}{d^2}; \frac{k+1}{d^2})$ of caustics and no caustic is facing the center of another
3. in base 3, the number of caustics is an element of $\mathbb{S}_{12_3}\underbrace{2\ldots2}_{\text{k}}$ with k $\in \mathbb{N}^*$ which is never dividable by neither 2 nor 3.

The third point is not trivial at all, for this kind of pegging in the base 2 and 3 representations of the numbers of caustics in the envelope of the dynamical system is the essential reason behind it not having any invariant Cantor set. For example, one may remember that the regular Cantor set can be seen as that of all real number in [0;1] which ternary representation does not contain digit 1 (which also shows why it is continuous). 1 is in the Cantor set because it is indistinguishable from 0.222... unless you would go non-archimedean and epsilon-manage a non-standard Cantor set. So the mixingness of pegging the number of caustics of the envelope to $2^n 3^m - 1$, numbers that can never be divided by either 2 or 3 leads us to a consideration that is extremely close to that of the Collatz map, where Mersenne numbers are so important because they always define the longest possible monotonous increase under the map at a given row of the binary tree, and they do so because at any such row, they have the longest "trailing digits 1" by definition, and finally those trailing digits define how many consecutive times you can apply action 3x+1 without the result being dividable by 4.

So the third point above means that you take 5 (101_2) and allow an indefinite number of trailing digits 1 to it in base 2, then allow an indefinite number of digits 2 in base 3 to any of those numbers, and the envelope of the dynamical system in S^1 can only have so many caustics, which is a very precise subset out of all the possible distributions of a whole number of equally spaced caustics on the unit circle.

Also note that even though base 2 to 3 conversions are wildly chaotic (hence the Collatz and Furstenberg conjectures in the first place), you can always define some perfectly precise **suffix periodicity** for them. For example,

- As the Mersenne numbers, in their natural order, are Type C $\to$ Type B (eg. $1 \to 3$) the periodic 1-digit suffix of their base 3 representation is $1 \to 0$ so it has period 2
- the 2-digits suffix is $01 \to 10 \to 21 \to 20 \to 11 \to 00$ so it has period 6
- the 3-digits suffix has period 18
- the n-digits suffix has period $2 \cdot 3^{n-1}$

So we can already predict arbitrary long suffixes, while the 2-dreamcatcher in base 3 will also give us arbitrary long prefixes. This goes way beyond the Furstenberg conjecture by the way, I am convinced it could expose some critical vulnerabilities in the otherwise massive difficulty (but let's not start sacralizing another problem just yet) of proving there are infinitely many Mersenne primes, based on which conspiracies you obtain from prime powers of 2 and the suffix they reach. The whole point of Romanesco Algebra is to give a geometric meaning to diophantine conspiracies in the first place, by studying how multiplicatively independent iterated linear sequences wrap around each other, which is of interest in studying distributions of prime numbers among certain sequences and in giving a geometric meaning to base changes. The point of Dreamcatcher theory and of a Poincaré-Denjoy-Tao theory is then to provide a precise dynamical system and ergodic theoretic framework to study the chaoticity of base changes, define interdiction zones and possibly, through - more precisely - ergodic Ramsey theory (which is what Furstenberg used to prove his 1977 theorem, and could be argued, is also the spirit of the Green-Tao theorem), define sufficient conditions for the existence or non-existence of progressions, loops and properties. Ultimately, there is a great interest in bringing the $\times 2 \times 3$ conjecture back to its original motivation of studying the chaoticity of conversions between multiplicatively independent bases (the 2;3 case being the most fundamental) and their arithmetic "avalanches" in particular, that is, their self-organized criticality under some iterated operations which Romanesco and Dreamcatcher theory begins by limiting to the case of unary algebras. One way of seeing my "Collatz Proof from The Book" is that it essentially demonstrates the 3x+1 dynamical system is self-organizedly critical by studying the branching factor of the attractor of any of its points, which in turn is a result of the mixingness of operation $\cdot 3$, itself the fundamental property behind the ditryadic map having no non-trivial closed invariant set. **Simply put: base 3 transforms self-organize into base 2 avalanches all the time**.

We may also extend the Furstenberg conjecture by attempting to be more systematic in studying how the geometric and arithmetic properties of the envelopes of certain iterated systems in the unit circle will be forcing a certain measure rigidity.

- Any $\times 2$ system generates a Mersenne number of caustics
- Any $\times 3$ system generates a "Base 3 Mersenne" of caustics, namely a number that can be written 2...2 in base 3
- Any $\times 2 \times 3$ system generates $Syr^{\infty}(\mathbb{M}) = \mathbb{S}_5; \mathbb{S}_{17}; \mathbb{S}_{53}; \mathbb{S}_{161}; ...; \mathbb{S}_{G(3^{\infty})}$ caustics which we may call a "Collatz-Mersenne" number.

But then what rigidity would be implied when the number of caustics is picked, say, from any $\mathbb{G}_7; \mathbb{G}_{19}; \mathbb{S}_{55}; \mathbb{S}_{163}; ...; \mathbb{G}_{S(3^\infty)}$, meaning you would be starting with the AntiMersennes and then branch them with trailing digits 1 in base 3. Would the Lebesgue measure also be the only invariant one? Would this claim be equivalent to the Furstenberg conjecture? In a way, I find this intellectual process quite similar to asking, say, whether the infinity of Mersenne primes would be equivalent to the infinity of AntiMersenne primes.

To summarize, base-change dynamical systems could use their *ad hoc* algebraic structures, which I intended with Romanesco algebra. Transformations iterating trailing digits in different bases generalize Bernoulli shifts or Baker maps (would the trailing digit appending be bidirectional), where the Cantor function, fractals with a transcendental Hausdorff dimension (e.g $^{Log(2)}/_{Log(3)}$ for the Cantor set) and p-adic arithmetic already offer much insight. I introduced Dreamcatcher theory with the intended program of endowing it with non-archimedean extensions (Poincaré-Denjoy-Tao theory) to be able to better grasp and pose otherwise difficult problems regarding certain transformations of number series. For example, defining Romanesco $\{2; 3; 4\}$ in the minimalistic way of calling it any (finite or not) word from operations $\{S; G; A; B; C; V\}$ starting from number 1, where operations A (resp. B and C) are defined as "adding an end digit 2 (resp 0, 1) in base 3 and V as "adding an end digit 1 in base 4", one may ask "how often does an A series starting from x return to inside the binary tree rooted in x?" or even more precise ones: in which smallest trees rooted in elements of $\mathbb{V}_x$? From there we can study the sturmian sequence, say, starting from 5 (to stay with the Furstenberg conjecture) and note "1" whenever a point of this series falls between $2^n + 1$ and $3 \cdot 2^n - 1$ both included, and "0" when it doesn't

$$5(1) - 17(1) - 53(0) - 161(1) - 485(0) - 1457(1) - 4373(1)... \tag{11}$$

And note for example that 4373 is just far enough from $\mathbb{V}_{17}$ as well as 161 is far enough from $\mathbb{V}_5$, and it is a founding purpose of Poincaré-Denjoy-Tao theory to demonstrate there is a mathematical restraining order forbidding any A, B, C series (either pure, or even possibly combined) starting from x to get closer than a certain limit to $\mathbb{V}_x$ (which again is the basis to demonstrate there are no non-trivial cycles in Collatz). As a Romanesco is endowed with a natural metric (composed of words from the unary operations of the tree, eg. A,B,C) one could call this particular problem one of "dendricab geometry", namely, the equivalent of taxicab geometry but in a tree. Adding PDT theory to it, one could tick off entire "dendriboxes", namely, finite trees rooted in some y that would be interdicted after iterations of some ternary operations, and this could probably generate a "Collatz sponge", reminiscent to a Menger Sponge or a Sierpinski triangle, composed of the binary tree over odd numbers of which infinite series of dendriboxes have been removed. The rigorous definition of the size of the interdicted $\mathbb{V}_y$ and the dendriboxes rooted in their elements would require the most careful epsilon management, but once done, I would be very curious to see representations of those Collatz sponges. Epistemologically, bringing some methods of non-standard analysis (epsilon management) to discrete dynamical systems by embedding them in an object which is itself bridging $\mathbb{N}$ and $\mathbb{R}$ (the binary tree over odd numbers has a continuous amount of branches to infinity) and then defining traversal series of some comparatively infinitesimal sub-volumes of the tree (the dendriboxes) with the stated purpose of obtaining measures and fractal dimensions for their complementary sets I suspect could lead to important new tools in the ergodic theory of numbers.

Another point of interest lies in the geometry of Dreamcatchers and how they bridge considerations involving p-adic arithmetic and algebraic curves. For example, the 3-dreamcatcher has a non-trivial singularity (the trivial one being in $\{0;0\}$) corresponding to the equality of the two algebraic curves composing it (the red and the blue one, one of degree eight and one of degree six), and which precise position is the infinite iteration of $S \circ G$ starting from number 5 ; for example 75 itself is cutting interval $[65;85]$ in two equal parts. Since on the dreamcatcher $S \circ G$ is equivalent to "remove a quarter of the angle from itself" if we start with number 1 on the left and set it to angle 0, we have number 5 which is angle $({}^{\tau}/_{4})$ then number 19 so angle $({}^{\tau}/_{4} - {}^{\tau}/_{4^2})$ then 75 $({}^{\tau}/_{4} - {}^{\tau}/_{4^2} - {}^{\tau}/_{4^3})$ etc. thus giving us ${}^{1}/_{4} - {}^{1}/_{4} \cdot \sum_{k=1}^{\infty} {}^{1}/_{4^k} = {}^{1}/_{4} - {}^{1}/_{4} \cdot {}^{1}/_{3} = {}^{1}/_{6}$

Similarly, $\mathbb{V}_1$ and $\mathbb{V}_3$ are cutting the binary tree embedded in the unit circle into perfect thirds because $V = G \circ S$ is equivalent to : "add a quarter of the angle to itself" thus again $\sum_{k=1}^{\infty} {}^{1}/_{4^k} = {}^{1}/_{3}$. Here too, bringing together difficult polynomials, envelopes, p-adic arithmetics, base p representations and simple Dirichlet series would seem at best recreational to the obtuse scholar, but it really isn't. I am convinced a theory of Dreamcatcher multiplications could expose vulnerabilities in open problems such as the infinity of Mersenne primes, namely, the properties of divisors of prime numbers of digits 1 in base 2. For example, $\mathbb{V}_1$ corresponds to the image of the Mersenne numbers dividable by 3 in their natural orders, converging to a single angle on the 3-dreamcatcher in base 2, which symmetry against the 5-7 axis also solves one crossing point of the two algebraic curves (the non-trivial singularity of the dreamcatcher), and this process we can now iterate for all Mersenne numbers.

We may begin with the simple task of cutting the binary tree into perfect Mersenneth domains, just as $\mathbb{V}_1$ already cuts it into its first third, which is easy because $\sum_{k=1}^{\infty} {}^{1}/_{(2^n)^k} = {}^{1}/_{(2^n-1)}$. From there Romanesco algebra finishes the task by defining any first Mersenneth of the tree as the iteration of the appending of a certain base 2 suffix which is strictly equal to the n-th AntiMersenne. Namely: 9 is 1001 in base 2 so the first 7th of the binary tree is found by just iterating concatenations of 1001 in this base. 17 is 10001 so the first 15th of the tree is found by iterating 10001... Studying the caudal singularity of Mersenne Dreamcatchers in base 2 is one way of attempting to study the distribution of Mersenne primes (or Sophie Germain primes) from an epistemological angle similar to that of the euclidian constructibility of regular polygons. Also, by symmetry, this simple endeavour will always point to the one singularity of the Mersenne dreamcatcher in base 2, solving infinitely many (very) hard equations on the way. Since any p-dreamcatcher is "sylvesterizable" (has a Sylvester matrix for each of the two algebraic curves composing it) does it mean this could also be used to break the ground for a general theory of polynomial resolution, not with radicals obviously, but with the more advanced envelopes instead, or to put it provocatively: could envelopes be the new radicals? Is it not, beyond mathematically recreational, mathematically interesting *per se*? Would it not deserve to be called a theory?

Let me conclude this part with some other crazy question to set the tone for the last section, namely since p-dreamcatchers in base q can be seen as twisted, "degenerate" forms of the regular cardioid, could there be a fractal curve, obtained from some iterative dynamical system obviously, that would behave like the Mandelbrot set but with dreamcatchers instead of cardioids? And if it turns out to be possible with the same dreamcatcher in all the curve, could it be so with any possible series of them?

7 Some mad ideas to wrap it up

Science is about being absolutely systematic, because that Truth be Singular implies that Truth be Total, as only absolute totality may be singular. I have never really been a fan of the Popper criterion and I understand why some scholars have railed its most radical users as "popperazzi", because Popper's refutability has been unjustly sacralized. The Popper criterion is good for politics, it is essentially fiduciary but as it blatantly fails for all of metamathematics, beginning as early as with second order logic where defining undisputable falsification is simply impossible, it is a very limited one indeed. I prefer to guide science with the criterion of systematicity instead: is real science what is truly systematic, especially more so when it comes to mathematics. One should explore the neighbourhood of existing paradigms, their mirror images, their contradictions, etc. Cohen has justly brought the concepts of field extension to logic in inventing forcing, and I think it should be further expanded to noetics (or "conceptics"), namely the discipline of studying the universe closed under all possible mental operations.

Once refutability is replaced with systematicity or totality, one may still adjoin the criterion of aesthetics, which generalizes Occam's razor in an antifragile way (the razor is very easy to fool locally because it is very predictable and it is thus fragile in the sense of Taleb) with some aesthetical intuition remaining "grasping the most with the least". So totality is the ocean, aesthetics is the meta-scientific lighthouse, because art is long and life is short. Aesthetics can remain baudelairian for any mathematician: you may and should find beauty in pathologies and mathematics' flowers of evil have spawned many marvels already (eg. Monstrous Moonshine). There is already enough in each one's particular interpretation of the aesthetical criterion to define mortal schools and guilds with their limitations (eg. Pythagora's hatred of irrationals, Berkeley's skepticism towards infinitesimals or Kronecker and later Poincaré's disdain for Cantor's genius) so why go beyond? I have a lot of respect for those baudelairian mathematicians who fascinates themselves with pathologies, they find beauty in the beasts and sometimes they end up well-behaving them in grand ways so we should encourage and cultivate their attitude which is one of the creams of sciences. Beauty elicits attention and devotion, and this is what one needs for epic success. Sacralizing problems in the bad way though, conditions young scholars into never being able to see the beauty, and only being able to see the beast: a waste of human mental life and the castration of enthusiasm, which scholarly fools yet adore practicing.

I have studied multi-unary algebras with the desire of being systematic (my impression was that they had not been developed with the same thoroughness as the binary ones) and have let my personal sense of aesthetic guide me through an attempt at theorizing them towards the well-behaving of some beastly beauty: the Syracuse problem. Those names: Romanescoes, Dreamcatchers, are way too cute not to entice outright rape by the scholarly perverse confined in their mental prison, those fools who cherish the belief that cuteness, enthusiam and *élan* should be banned from the serious science - after all, won't we call the opposite of a professional a literal *amateur*, namely, "the one who likes"? Now as blasé as they are, reviewers want to cowardly believe being a pro is being a bore, in a psychological phenomenon akin to hazing: once hazed, one either courageously accepts it was fundamentally for nothing and decides they will be the last, or one hazes the next generation, turns hazing into a glorious tradition, so they can give meaning to their otherwise meaningless, but passed, suffering. Thus a majority of scholarly fools feel the burning, subconscious compulsion to castrate enthusiasm wherever they see it, and they hate that someone would make their theory cute, by giving it the names that would inspire them the most. Making an invention or a theory cute is just practical neuroscience though: what is cute effortlessly attracts your time and attention; if babies were not cute, Humanity, with its epitome of the K-strategy, would have gone plain extinct. Now inventors and scientists must breed their theory and work into acceptance, even tough some, like Wegener, will have to fend child-eating fools their entire life (and indeed plate tectonic had to wait till the nineteen-sixties to be widely accepted when Wegener died in 1930); this requires constant nursing, the protracted, spontaneous offering of attention and time, and for this you better make your theory cute, above all, to yourself, because you are the one nursing it. Just as no bored fool would ever peer-review the Mouse at Xerox (yes they invented it, just as much as they spoiled it), or "fractals" as Mandelbrot would admit himself (which is why he worked at IBM in the first place: have fun finding what hardcore bourbakist would discreetly say of him in the nineteen-fifties), no blasé lobotomee would validate such enthusiastic names as Romanescoes and Dreamcatchers, so I thankfully publish them in this ArXiv piece. Once you define totality as the only absolute scientific criterion, unencumbered mental freedom is not an option anymore.

In the epistemological scheme of things though, the Totality criterion also forces us to ask those dreadful questions of alternative history and study their application to contemporary situations. This I call the "Toltech effect", with a -tech for technology, that is, epistemological vicariance. Though it was impressive for its area, Toltec metallurgy was nowhere close to the contemporary European one, but this of course can only be relative. How do we know however, that our "mentallurgy", the depth and heat and focus of our mental life has reached the full potential of any of its eras? A lazy answer is that we cannot, but I believe the Totality criterion to be the best way to avoid a Toltech effect in the noosphere: how can mental vicariance split you from great ideas if you abolish mountain ranges in the realm of ideas in the first place? Oceans, depths, cordilleras, are mental-made in the noosphere, while in the geosphere, they are not, and explain why in geopolitics, "geography is destiny"; as the human mind also has congenital blind spots and limitations, there is some sort of congenital topography in its noosphere as well, but the Totality criterion calls for a systematic exploration of it. how do you really know you haven't missed something elementary in your civilisational epistemology? The Romans could have had the industrial revolution if they had searched for it: it was within the radius of their current technology, just vetoed by their rigid socio-political structure, so they just did not collectively search for it.

Case in point: Romanescoes and Dreamcatchers are elementary constructions in nature, they could have easily been thought by Archimedes himself, he could have studied and obtained fundamental theorems on them[7], and so could have Al Khwarizmi, Omar Khayyam, Marin Mersenne, Pascal and Fermat. There is no Romanesco theorem I could find in my entire life that would not have been found by Euler or Gauss had they had been offered a practical picture of them, and the very thought that Galois and Wantzel, whose early passing was a mathematical catastrophe, could have had as many large colored printed maps of Romaneso $\{2;3;4\}$ at their disposal as they'd needed gives me ASMR chills. Of course, for the working Romanesco algebraist, a zoomable picture is a must, and modern tablets are best fit for it, though I am sure some would be very prompt to argue against what they would see as some pathetic mental crutch - intellectual-yet-idiots often despise mental ergonomics in the first place, but there is something much deeper to this possible debate.

It is funny indeed that Sir Michael Atiyah would make such a claim while being fluent in Arabic and well-knowing that *Al-Jabr*, which could be poetically translated as "He who mends", is a divine name in Islam, but after having seen myself what happens when one attempts giving up

[7] I pick him because he was the best of his time. Bonus points for being born and passing in Syracuse and having his modernized profile on the freaking Fields Medal; the man is basically the Chairman of the Fielderal Reserve in perpetuity, with his face forever minted on that mathematical prestige money that never goes "brrr".

geometry in studying the Furstenberg and Syracuse conjectures, I cannot agree more with it:

> *"Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine."* –Sir Michael Atiyah

...because it is naturally much more mind-ergonomic geometry allows our brain to handle wider abstract objects; it enlarges our mental span, and as such, it can be illuminating indeed. So, in a way, "let none but geometers enter here", and now that we are sincerely committed to bringing geometry back, we may, among so many things ...

- tropicalize dreamcatchers : unsurprisingly, the (Min, +) algebra makes them much simpler
- characterize moduli spaces of dreamcatchers
- establish a complete theory of dreamcatcher multiplication solving the angle of any multiplication (say the famous $23 \times 89 = 2047$, with the clear intent of dragging the question of the infinity of the Mersenne primes into a realm epistemologically similar to that of the euclidian constructibility of regular polygons) and beyond, any polynomial.
- obtain an epistemological equivalent to Fourier theory to decompose infinite superpositions of envelopes with applications in diophantine algebra, for example in the study of Figures 8 and 9.
- characterize the rotation number of any p-dreamcatcher in base q, and under which conditions it would be algebraic. Also which geometric properties do all the dreamcatchers with the same rotation number share?
- extend dreamcatchers to complex numbers with tangents now representing automorphisms of the unit sphere.

...and beyond, obtain - through geometry still - an abstract generalization (which in the context of this section, really means "systematisation" or "totalisation") of unary algebras and Prüfer group automorphisms by forcing some envelopes onto p-groups and defining the resulting algebraic operation. For example: what is the operation obtained by rotations and symmetries of the 3-dreamcatcher in base 2? Could the properties of these exotic " $\times 3^*$ " operations shed new light on some diophantine problems? Ultimately of course, the point is to achieve a (geometric) form of Diophantine Galois theory...

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Also, that one non-listed video I made quick-and-dirty on my youtube channel right after my 35th birthday to share the essence of this "Proof from The Book" with the German team https://youtu.be/YZ_5wpw2eTU