BOUNDED WEAK SOLUTIONS TO THE THIN FILM MUSKAT PROBLEM VIA AN INFINITE FAMILY OF LIAPUNOV FUNCTIONALS

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Abstract. A countably infinite family of Liapunov functionals is constructed for the thin film Muskat problem, which is a second-order degenerate parabolic system featuring cross-diffusion. More precisely, for each $n \geq 2$ we construct an homogeneous polynomial of degree n, which is convex on $[0,\infty)^2$, with the property that its integral is a Liapunov functional for the problem. Existence of global bounded non-negative weak solutions is then shown in one space dimension.

1. Introduction

The thin film Muskat problem describes the dynamics of the respective heights of two immiscible fluids with different densities (ρ_-, ρ_+) and viscosities (μ_-, μ_+) in a porous media and reads

$$\partial_t f = \operatorname{div}\left(f \nabla \left[(1+R)f + Rg\right]\right) \quad \text{in} \quad (0,\infty) \times \Omega\,, \tag{1.1a}$$

$$\partial_t g = \mu R \operatorname{div}\left[g \nabla (f+g)\right] \quad \text{in} \quad (0,\infty) \times \Omega\,, \tag{1.1b}$$

supplemented with homogeneous Neumann boundary conditions

$$\nabla f \cdot \mathbf{n} = \nabla g \cdot \mathbf{n} = 0 \quad \text{on} \quad (0,\infty) \times \partial\Omega\,, \tag{1.1c}$$

and non-negative initial conditions

$$(f,g)(0) = (f^{in}, g^{in}) \quad \text{in} \quad \Omega\,. \tag{1.1d}$$

In (1.1), Ω is a bounded domain of $\mathbb{R}^d$, $d \geq 1$, with smooth boundary $\partial\Omega$, f and g denote the heights of the heavier and lighter fluids, respectively, and $R := \rho_+/(\rho_- - \rho_+)$ and $\mu := \mu_-/\mu_+$ are positive parameters depending solely on the densities $(\rho_- > \rho_+)$ and viscosities of the two fluids. We recall that (1.1) can be derived from the classical Muskat problem by a lubrication approximation [7,9,14].

From a mathematical viewpoint, the system (1.1) is a quasilinear degenerate parabolic system with full diffusion matrix and it is well-known that the analysis of cross-diffusion systems is in general rather involved. Fortunately, as noticed in [7], an important property of (1.1) is the availability of an energy functional. Specifically,

$$\mathcal{E}(f,g) := \frac{1}{2} \int_\Omega \left[f^2 + R(f+g)^2\right] \,\mathrm{d}x \tag{1.2}$$

is a non-increasing function of time along the trajectories of (1.1). In fact, a salient feature of (1.1), first observed in [11], is that it has, at least formally, a gradient flow structure for the energy $\mathcal{E}$ with respect to the 2-Wasserstein distance. This structure provides in particular a variational

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scheme to establish the existence of weak solutions to (1.1). Furthermore, as first noticed in [6] and subsequently used in [1, 11], the entropy functional

$$\mathcal{H}(f,g) := \int_\Omega \left[L(f) + \frac{1}{\mu} L(g) \right] \,\mathrm{d}x\,, \tag{1.3}$$

with $L(r) := r\ln r - r + 1 \ge 0$, $r \ge 0$, is also a non-increasing function of time along the trajectories of (1.1).

Building upon the above mentioned properties, the existence of non-negative global weak solutions (f,g) to (1.1) satisfying

$$(f,g) \in L_\infty((0,T), L_2(\Omega,\mathbb{R}^2)) \cap L_2((0,T), H^1(\Omega,\mathbb{R}^2))\,, \tag{1.4}$$

is shown in [6, 12] in one space dimension $d = 1$, see also [2, 3] for the existence of global weak solutions to a generalized version of (1.1) in the d-dimensional torus with periodic boundary conditions instead of the homogeneous Neumann boundary conditions (1.1c). Local existence and uniqueness of classical solutions to (1.1) with positive initial data are reported in [7], along with the local stability of spatially uniform steady states. As for the Cauchy problem when $\Omega = \mathbb{R}^d$ and $d \in \{1,2\}$, non-negative global weak solutions are constructed in [1, 11] for non-negative initial conditions $(f^{in}, g^{in}) \in L_1(\mathbb{R}^d,\mathbb{R}^2) \cap L_2(\mathbb{R}^d,\mathbb{R}^2)$ with finite second moments, exploiting the aforementioned gradient flow structure to set up a variational scheme, see also [10] for an extension to a multicomponent version of (1.1) in one space dimension. This approach is further developed in [15] to investigate the existence of non-negative global weak solutions to a broader class of quasilinear cross-diffusion systems.

In view of (1.4), weak solutions to (1.1) have rather low regularity. It is actually a general feature of cross-diffusion systems that classical regularity is hard to reach. In particular, the cross-diffusion structure impedes the use of bootstrap arguments which have proved efficient for triangular systems. A different route to improved regularity is to look for additional estimates and the purpose of this paper is to derive (formally) L_n-estimates for solutions to (1.1) for all integers $n \ge 2$, eventually leading to L_∞-estimates in the limit $n \to \infty$. This feature paves the way to the construction of non-negative global bounded weak solutions to (1.1) but, as explained below, we are only able to complete this construction successfully in one space dimension $d = 1$. The first main contribution of this paper is actually to show that, for each $n \ge 2$, there is an homogeneous polynomial Φ_n of degree n, which is non-negative and convex on $[0,\infty)^2$, and such that

$$u = (f,g) \longmapsto \int_\Omega \Phi_n(u)\,\mathrm{d}x \tag{1.5}$$

is a Liapunov functional for (1.1). More precisely, the first main result of this paper is the following.

Theorem 1.1. *Let $R > 0$, $\mu > 0$, and $u^{in} := (f^{in}, g^{in}) \in L_{\infty,+}(\Omega,\mathbb{R}^2)$. If $u = (f,g)$ is a sufficiently regular solution to (1.1) on $[0,\infty)$ with non-negative components, then*

(I1) *for all $t \ge 0$,*

$$\int_\Omega \Phi_1(u(t,x))\,\mathrm{d}x + \int_0^t \int_\Omega \left[|\nabla f|^2 + R|\nabla(f+g)|^2 \right](s,x)\,\mathrm{d}x\mathrm{d}s \le \int_\Omega \Phi_1(u^{in}(x))\,\mathrm{d}x\,, \tag{1.6}$$

where $\Phi_1(X) := L(X_1) + L(X_2)/\mu$, $X = (X_1, X_2) \in [0,\infty)^2$;

(I2) *for all $n \ge 2$ and all $t \ge 0$,*

$$\int_\Omega \Phi_n(u(t,x))\,\mathrm{d}x \le \int_\Omega \Phi_n(u^{in}(x))\,\mathrm{d}x\,, \tag{1.7a}$$

where Φ_n is the homogeneous polynomial of degree n given by

$$\Phi_n(X) := \sum_{j=0}^{n} a_{j,n} X_1^j X_2^{n-j}\,, \qquad X=(X_1,X_2)\in\mathbb{R}^2\,, \tag{1.7b}$$

with $a_{0,n} := 1$,

$$\begin{aligned} a_{j,n} &:= \binom{n}{j}\prod_{k=0}^{j-1}\frac{k+\alpha_{k,n}}{\alpha_{k,n}} > 0\,, \qquad 1\le j\le n\,,\\ \alpha_{k,n} &:= R[k+\mu(n-k-1)] > 0\,, \qquad 0\le k\le n-1\,. \end{aligned} \tag{1.7c}$$

In addition, Φ_n is convex on $[0,\infty)^2$;

(I3) *for $t\ge 0$,*

$$\|(f+g)(t)\|_\infty \le \frac{1+R}{R}\|f^{in}+g^{in}\|_\infty\,. \tag{1.8}$$

For $n=2$, Theorem 1.1 gives $(a_{0,2},a_{1,2},a_{2,2}) = (1,2,(1+R)/R)$. Therefore,

$$\Phi_2(X) = \frac{1+R}{R}X_1^2 + (X_1+X_2)^2\,, \qquad X\in\mathbb{R}^2\,,$$

and

$$\mathcal{E}(f,g) = \frac{R}{2}\int_\Omega \Phi_2((f,g))\ \mathrm{d}x\,,$$

so that we recover the time monotonicity of the energy from (1.7a) with $n=2$.

It seems worth pointing out that the availability of an infinite number of Liapunov functionals, eventually leading to L_∞-estimates, seems rather seldom for cross-diffusion systems and that we are not aware of other systems sharing this feature. Whether it is possible to extend the analysis performed in this paper to a broader class of cross-diffusion systems will be the subject of future research.

To construct the family of polynomials $(\Phi_n)_{n\ge2}$, we introduce $u=(f,g)$ and the mobility matrix

$$M(X) = (m_{jk}(X))_{1\le j,k\le 2} := \begin{pmatrix} (1+R)X_1 & RX_1\\ \mu RX_2 & \mu RX_2\end{pmatrix}\,, \qquad X\in\mathbb{R}^2\,, \tag{1.9}$$

so that (1.1a)-(1.1b) becomes

$$\partial_t u - \sum_{i=1}^{d}\partial_i\left(M(u)\partial_i u\right) = 0 \ \text{ in } \ (0,\infty)\times\Omega\,. \tag{1.10}$$

We then use the observation that, since $\Phi\in C^2(\mathbb{R}^2)$, (1.1c), (1.10), and the symmetry of the Hessian matrix $D^2\Phi(u)$ of Φ entail that

$$\frac{d}{dt}\int_\Omega \Phi(u)\ \mathrm{d}x + \sum_{i=1}^{d}\int_\Omega \langle D^2\Phi(u)M(u)\partial_i u, \partial_i u\rangle\ \mathrm{d}x = 0\,. \tag{1.11}$$

It readily follows from (1.11) that Φ provides a Liapunov functional for (1.10) as soon as the matrix $D^2\Phi(u)M(u)$ is symmetric and positive semidefinite. Using the ansatz (1.7b) for $\Phi=\Phi_n$ and the explicit form of the matrix M, we then compute $D^2\Phi_n(u)M(u)$ and require that it is a symmetric matrix, thereby obtaining (1.7c). Direct computations then show that the polynomial thus obtained is actually non-negative and convex on $[0,\infty)^2$, see section 4 and appendix A.

Having uncovered the estimates (l1)-(l3) at a somewhat formal level, it is tempting to exploit them to construct a bounded weak solution to (1.1) endowed with these properties. The difficulty we face here is the construction of an appropriate approximation of (1.1) having sufficiently smooth solutions for which the estimates (l1)-(l3) remain valid. In particular, boundedness of solutions to the approximation seems to be required to be able to compute the integral of $\Phi_n(u)$. Unfortunately, we have yet been unable to design an approximation scheme producing bounded solutions while preserving the structure leading to (l1)-(l3) that could work in arbitrary space dimensions and we only provide below the existence of a bounded weak solution to (1.1) in one space dimension $d = 1$.

Theorem 1.2. *Let $R > 0$, $\mu > 0$, $u^{in} := (f^{in}, g^{in}) \in L_{\infty,+}(\Omega, \mathbb{R}^2)$, and assume that $d = 1$ (so that Ω is a bounded interval of $\mathbb{R}$). Then, there is a bounded weak solution $u := (f, g)$ to (1.1) which satisfies:*

(p1) *for each $T > 0$,*

$$(f, g) \in L_{\infty,+}((0,T) \times \Omega, \mathbb{R}^2) \cap L_2((0,T), H^1(\Omega, \mathbb{R}^2)) \cap W_2^1((0,T), H^1(\Omega, \mathbb{R}^2)') \,; \tag{1.12}$$

(p2) *for all $\varphi \in H^1(\Omega)$ and $t \geq 0$,*

$$\int_\Omega (f(t,x) - f^{in}(x))\varphi(x)\ \mathrm{d}x + \int_0^t \int_\Omega f(s,x)\partial_x \left[(1+R)f + Rg\right](s,x) \cdot \partial_x\varphi(x)\, \mathrm{d}x\mathrm{d}s = 0 \tag{1.13a}$$

and

$$\int_\Omega (g(t,x) - g^{in}(x))\varphi(x)\ \mathrm{d}x + \mu R \int_0^t \int_\Omega g(s,x)\partial_x \left(f + g\right)(s,x) \cdot \partial_x\varphi(x)\, \mathrm{d}x\mathrm{d}s = 0\,; \tag{1.13b}$$

(p3) *and the properties (l1), (l2), and (l3) stated in Theorem 1.1.*

A key ingredient in the proof of Theorem 1.2 is the continuous embedding of $H^1(\Omega)$ in $L_\infty(\Omega)$, which readily provides the above mentioned boundedness of solutions to the approximation of (1.1) designed below and is of course only available in one space dimension. Besides, we employ a rather classical approximation approach, relying on a time implicit Euler scheme with constant time step $\tau \in (0,1)$ for the time discretization and a suitable modification of the mobility matrix to a non-degenerate one.

As a consequence of Theorem 1.2 and of the estimate (A.13), we obtain uniform L_∞-bounds for the height f of the denser fluid in the regime where $R \to 0$ and μ is fixed. Such an estimate has been used recently in [13, Corollary 1.4] when performing the singular limit $R \to 0$ (while μ is kept fixed or $\mu \to \infty$) in the thin film Muskat problem (1.1) in order to recover the porous medium equation $\partial_t f = \mathrm{div}\big(f \nabla f\big)$ in the limit.

Corollary 1.3. *If $R \max\{1, \mu\} \in (0, 1/(2e)]$, then the solution $u = (f, g)$ to (1.1) from Theorem 1.2 satisfies*

$$\|f(t)\|_\infty \leq \big(1 + e \max\{1, \mu\}\big) \|f^{in}\|_\infty + \|g^{in}\|_\infty \,, \qquad t \geq 0. \tag{1.14}$$

We provide the proof of Theorem 1.2 in sections 2 and 3 below. It involves three steps: we begin with the existence of a weak solution to the implicit time discrete scheme associated to (1.1) which satisfies a time discrete version of the estimates (1.7) and (1.8) (section 2). This result is achieved with an approximation procedure which is designed and studied in section 2.1, a technical lemma being postponed to appendix B. The next section 2.2 is devoted to the time discrete version of the estimates (1.7) and (1.8), the proof of the properties of the polynomials Φ_n, $n \geq 2$, being collected in appendix A. The proof of Theorem 1.2 is given in section 3 and is based on a compactness method. We finally prove Theorem 1.1 in section 4.

Notation. For $p \in [1,\infty]$, we denote by $\|\cdot\|_p$ the L_p-norm in $L_p(\Omega)$, $L_p(\Omega,\mathbb{R}^2) := L_p(\Omega)\times L_p(\Omega)$, and $H^1(\Omega,\mathbb{R}^2) := H^1(\Omega)\times H^1(\Omega)$. The positive cone of a Banach lattice E is denoted by E_+. Next, $\mathbf{M}_2(\mathbb{R})$ denotes the space of 2×2 real-valued matrices, $\mathbf{Sym}_2(\mathbb{R})$ is the subset of $\mathbf{M}_2(\mathbb{R})$ consisting of symmetric matrices, and $\mathbf{SPD}_2(\mathbb{R})$ is the set of symmetric and positive definite matrices in $\mathbf{M}_2(\mathbb{R})$. Finally, we denote the positive part of a real number $r\in\mathbb{R}$ by $r_+ := \max\{r,0\}$ and $\langle\cdot,\cdot\rangle$ is the scalar product on $\mathbb{R}^2$.

2. A time discrete scheme: $d=1$

Throughout this section, we assume that $d=1$ and Ω is a bounded open interval of $\mathbb{R}$. In order to construct bounded non-negative global weak solutions to the evolution problem (1.1) we introduce an implicit time discrete scheme, see (2.1a)-(2.1b), as well as a regularized version of this scheme, see (2.11). The aim of this section is to prove the existence of a bounded weak solution to the implicit time discrete scheme, as stated in Proposition 2.1.

Proposition 2.1. *Given $\tau>0$ and $U=(F,G)\in L_{\infty,+}(\Omega,\mathbb{R}^2)$, there is a weak solution $u=(f,g)$ with $u\in H^1_+(\Omega,\mathbb{R}^2)$ to*

$$\int_\Omega \Big[f\varphi + \tau f\partial_x\left[(1+R)f+Rg\right]\cdot\partial_x\varphi\Big]\ \mathrm{d}x = \int_\Omega F\varphi\ \mathrm{d}x\,, \qquad \varphi\in H^1(\Omega)\,, \tag{2.1a}$$

$$\int_\Omega \Big[g\psi + \tau\mu Rg\partial_x(f+g)\cdot\partial_x\psi\Big]\ \mathrm{d}x = \int_\Omega G\psi\ \mathrm{d}x\,, \qquad \psi\in H^1(\Omega)\,, \tag{2.1b}$$

which also satisfies

$$\int_\Omega \Phi_n(u)\ \mathrm{d}x \le \int_\Omega \Phi_n(U)\ \mathrm{d}x \tag{2.2}$$

for $n\ge 2$,

$$\|f+g\|_\infty \le \frac{1+R}{R}\|F+G\|_\infty\,, \tag{2.3}$$

and

$$\int_\Omega \Phi_1(u)\ \mathrm{d}x + \tau\int_\Omega \left[|\partial_x f|^2 + R|\partial_x(f+g)|^2\right]\ \mathrm{d}x \le \int_\Omega \Phi_1(U)\ \mathrm{d}x\,. \tag{2.4}$$

We fix $\tau>0$ and $U=(F,G)\in L_{\infty,+}(\Omega,\mathbb{R}^2)$. Recalling the definition (1.9) of the mobility matrix

$$M(X) = \begin{pmatrix} (1+R)X_1 & RX_1 \\ \mu RX_2 & \mu RX_2 \end{pmatrix}\,, \qquad X\in\mathbb{R}^2\,,$$

an alternative formulation of (2.1) reads

$$\int_\Omega \left[\langle u,v\rangle + \tau\langle M(u)\partial_x u,\partial_x v\rangle\right]\ \mathrm{d}x = \int_\Omega \langle U,v\rangle\ \mathrm{d}x\,, \qquad v\in H^1(\Omega,\mathbb{R}^2)\,. \tag{2.5}$$

Obviously, the mobility matrix $M(X)$ is in general not symmetric and the associated quadratic form

$$\mathbb{R}^2 \ni \xi = (\xi_1,\xi_2) \mapsto \sum_{j,k=1}^2 m_{jk}(X)\xi_j\xi_k \in \mathbb{R}$$

is not positive definite (even if $X\in[0,\infty)^2$), two features which complicate the analysis concerning the solvability of (2.5). Fortunately, as noticed in [4], the underlying gradient flow structure provides a way to transform (2.5) to an elliptic system with symmetric and positive semidefinite matrix. More precisely, we introduce the symmetric matrix S with constant coefficients

$$S := \begin{pmatrix} 1+R & R \\ R & R \end{pmatrix},$$

which is actually the Hessian matrix of $R\Phi_2/2$. Clearly, S belongs to $\mathbf{SPD}_2(\mathbb{R})$ and

$$\langle S\xi, \xi\rangle = \xi_1^2 + R\,(\xi_1+\xi_2)^2 \ge \frac{R}{1+2R}|\xi|^2\,, \qquad \xi\in\mathbb{R}^2\,. \tag{2.6}$$

Choosing Sv instead of $v \in H^1(\Omega,\mathbb{R}^2)$ as a test function in (2.5) and using the symmetry of S, lead to another alternative formulation of (2.1a)-(2.1b) (or (2.5)), which reads

$$\int_\Omega \left[\langle Su, v\rangle + \tau\langle SM(u)\partial_x u, \partial_x v\rangle\right]\,\mathrm{d}x = \int_\Omega \langle SU, v\rangle\,\mathrm{d}x\,, \qquad v\in H^1(\Omega,\mathbb{R}^2)\,. \tag{2.7}$$

We next observe that, for $X\in[0,\infty)^2$,

$$SM(X) = \begin{pmatrix} (1+R)^2X_1 + \mu R^2 X_2 & (1+R)RX_1 + \mu R^2 X_2 \\ (1+R)RX_1 + \mu R^2X_2 & R^2X_1 + \mu R^2 X_2 \end{pmatrix} \tag{2.8}$$

and

$$\langle SM(X)\xi, \xi\rangle = X_1((1+R)\xi_1 + R\xi_2)^2 + \mu R^2 X_2(\xi_1+\xi_2)^2 \ge 0\,. \tag{2.9}$$

Consequently, $SM(X)$ belongs to $\mathbf{SPD}_2(\mathbb{R})$ for all $X\in(0,\infty)^2$ and the formulation (2.7) seems more appropriate to study the solvability of (2.1a)-(2.1b). However, the matrix $SM(X)$ is still degenerate as $X_1\to 0$ or $X_2\to 0$, so that we first solve a regularized problem in the next section.

2.1. **A regularization of the time discrete scheme.** Let $\varepsilon\in(0,1)$ and define

$$M_\varepsilon(X) := (m_{\varepsilon,jk}(X))_{1\le j,k\le 2} := \varepsilon I_2 + M((X_{1,+}, X_{2,+}))\,, \qquad X\in\mathbb{R}^2\,. \tag{2.10}$$

Lemma 2.2. *Given $\tau>0$, $U=(U_1,U_2)\in L_{\infty,+}(\Omega,\mathbb{R}^2)$, and $\varepsilon\in(0,1)$, there is a weak solution $u_\varepsilon = (u_{1,\varepsilon}, u_{2,\varepsilon})\in H^1_+(\Omega,\mathbb{R}^2)$ to*

$$\int_\Omega \left[\langle u_\varepsilon, v\rangle + \tau\langle M_\varepsilon(u_\varepsilon)\partial_x u_\varepsilon, \partial_x v\rangle\right]\,\mathrm{d}x = \int_\Omega \langle U, v\rangle\,\mathrm{d}x\,, \qquad v\in H^1(\Omega,\mathbb{R}^2)\,. \tag{2.11}$$

Moreover,

$$\|u_{1,\varepsilon}+u_{2,\varepsilon}\|_\infty \le \frac{1+R}{R}\|U_1+U_2\|_\infty\,, \tag{2.12}$$

and, for $n\ge 2$,

$$\int_\Omega \Phi_n(u_\varepsilon)\,\mathrm{d}x \le \int_\Omega \Phi_n(U)\,\mathrm{d}x\,. \tag{2.13}$$

Proof. For each $\varepsilon\in(0,1)$, M_ε lies in $C(\mathbb{R}^2, \mathbf{M}_2(\mathbb{R}))$ and satisfies

$$\begin{aligned} m_{\varepsilon,11}(X) &\ge m_{\varepsilon,12}(X) = 0\,, \qquad X\in(-\infty,0)\times\mathbb{R}\,,\\ m_{\varepsilon,22}(X) &\ge m_{\varepsilon,21}(X) = 0\,, \qquad X\in\mathbb{R}\times(-\infty,0)\,. \end{aligned} \tag{2.14a}$$

In addition, it follows from (2.6), (2.8), (2.9), and (2.10) that $SM_\varepsilon(X)$ belongs to $\mathbf{SPD}_2(\mathbb{R})$ for all $X\in\mathbb{R}^2$ with

$$\langle SM_\varepsilon(X)\xi,\xi\rangle \ge \frac{\varepsilon R}{1+2R}|\xi|^2\,, \qquad \xi\in\mathbb{R}^2\,. \tag{2.14b}$$

According to the properties (2.14), we are now in a position to apply Lemma B.1 (with $A = S$ and $B = M_\varepsilon$) and deduce that there is a non-negative solution $u_\varepsilon \in H^1(\Omega, \mathbb{R}^2)$ to (2.11).

In the remaining part, we prove that u_ε satisfies both estimates (2.12) and (2.13). We begin with (2.13) and thus consider $n \geq 2$. Since Φ_n is a polynomial and $H^1(\Omega, \mathbb{R}^2)$ continuously embeds in $L_\infty(\Omega, \mathbb{R}^2)$, the vector field $D\Phi_n(u_\varepsilon)$ belongs to $H^1(\Omega, \mathbb{R}^2)$. We may then take $v = D\Phi_n(u_\varepsilon)$ in (2.11) to obtain

$$\int_\Omega \left[\langle u_\varepsilon - U, D\Phi_n(u_\varepsilon)\rangle + \tau\langle M_\varepsilon(u_\varepsilon)\partial_x u_\varepsilon, \partial_x(D\Phi_n(u_\varepsilon))\rangle\right]\ \mathrm{d}x = 0\,. \tag{2.15}$$

On the one hand, it follows from the convexity of Φ_n on $[0,\infty)^2$, see Proposition A.1 (a), that

$$\int_\Omega \langle u_\varepsilon - U, D\Phi_n(u_\varepsilon)\rangle\ \mathrm{d}x \geq \int_\Omega \left[\Phi_n(u_\varepsilon) - \Phi_n(U)\right]\ \mathrm{d}x\,. \tag{2.16}$$

On the other hand, owing to the symmetry of $D^2\Phi_n(u_\varepsilon)$,

$$\begin{aligned}\langle M_\varepsilon(u_\varepsilon)\partial_x u_\varepsilon, \partial_x(D\Phi_n(u_\varepsilon))\rangle &= \langle D^2\Phi_n(u_\varepsilon)M_\varepsilon(u_\varepsilon)\partial_x u_\varepsilon, \partial_x u_\varepsilon\rangle \\ &= \varepsilon\langle D^2\Phi_n(u_\varepsilon)\partial_x u_\varepsilon, \partial_x u_\varepsilon\rangle \\ &\quad + \langle D^2\Phi_n(u_\varepsilon)M(u_\varepsilon)\partial_x u_\varepsilon, \partial_x u_\varepsilon\rangle\,.\end{aligned}$$

Since both $D^2\Phi_n(u_\varepsilon)$ and $D^2\Phi_n(u_\varepsilon)M(u_\varepsilon)$ belong to $\mathbf{SPD}_2(\mathbb{R})$ by Proposition A.1, we conclude that

$$\langle M_\varepsilon(u_\varepsilon)\partial_x u_\varepsilon, \partial_x D\Phi_n(u_\varepsilon)\rangle \geq 0\,. \tag{2.17}$$

Combining (2.15), (2.16), and (2.17), we end up with

$$\int_\Omega \left[\Phi_n(u_\varepsilon) - \Phi_n(U)\right]\ \mathrm{d}x \leq 0\,,$$

and we have established (2.13). It next follows from (2.13) and Lemma A.2 that

$$\begin{aligned}\|u_{1,\varepsilon} + u_{2,\varepsilon}\|_n &\leq \left(\int_\Omega \Phi_n(u_\varepsilon)\ \mathrm{d}x\right)^{1/n} \leq \left(\int_\Omega \Phi_n(U)\ \mathrm{d}x\right)^{1/n} \\ &\leq \frac{1+R}{R}\|U_1 + U_2\|_n\,.\end{aligned}$$

Hence, letting $n \to \infty$ in the above inequality leads us to (2.12), and the proof is complete. □

We next derive estimates on $\partial_x u_\varepsilon$.

Lemma 2.3. *Let $\tau > 0$, $U \in L_{\infty,+}(\Omega, \mathbb{R}^2)$, and $\varepsilon \in (0,1)$. The weak solution $u_\varepsilon = (u_{1,\varepsilon}, u_{2,\varepsilon})$ constructed in Lemma 2.2 satisfies*

$$\int_\Omega \Phi_1(u_\varepsilon)\ \mathrm{d}x + \tau\int_\Omega \left[|\partial_x u_{1,\varepsilon}|^2 + R|\partial_x(u_{1,\varepsilon} + u_{2,\varepsilon})|^2\right]\ \mathrm{d}x \leq \int_\Omega \Phi_1(U)\ \mathrm{d}x\,.$$

Proof. Let $\eta \in (0,1)$. Recalling that $u_\varepsilon \in H^1_+(\Omega,\mathbb{R}^2)$ has non-negative components, we deduce that the vector field $(\ln(u_{1,\varepsilon}+\eta), \ln(u_{2,\varepsilon}+\eta)/\mu)$ belongs to $H^1(\Omega,\mathbb{R}^2)$, and we infer from (2.11) that

$$\begin{aligned} 0 = &\int_\Omega \left[(u_{1,\varepsilon}-U_1)\ln(u_{1,\varepsilon}+\eta) + \frac{1}{\mu}(u_{2,\varepsilon}-U_2)\ln(u_{2,\varepsilon}+\eta) \right]\ \mathrm{d}x \\ &+ \tau \int_\Omega \Big(m_{\varepsilon,11}(u_\varepsilon)\partial_x u_{1,\varepsilon} + m_{\varepsilon,12}(u_\varepsilon)\partial_x u_{2,\varepsilon} \Big) \frac{\partial_x u_{1,\varepsilon}}{u_{1,\varepsilon}+\eta}\ \mathrm{d}x \\ &+ \frac{\tau}{\mu} \int_\Omega \Big(m_{\varepsilon,21}(u_\varepsilon)\partial_x u_{1,\varepsilon} + m_{\varepsilon,22}(u_\varepsilon)\partial_x u_{2,\varepsilon} \Big) \frac{\partial_x u_{2,\varepsilon}}{u_{2,\varepsilon}+\eta}\ \mathrm{d}x\,. \end{aligned} \tag{2.18}$$

On the one hand, since $L'(r)=\ln r$, $r>0$, the convexity of L guarantees that

$$\begin{aligned} &\int_\Omega \left[(u_{1,\varepsilon}-U_1)\ln(u_{1,\varepsilon}+\eta) + \frac{1}{\mu}(u_{2,\varepsilon}-U_2)\ln(u_{2,\varepsilon}+\eta) \right]\ \mathrm{d}x \\ &\qquad \ge \int_\Omega \left[(L(u_{1,\varepsilon}+\eta) - L(U_1+\eta)) + \frac{1}{\mu}(L(u_{2,\varepsilon}+\eta) - L(U_2+\eta)) \right]\ \mathrm{d}x \\ &\qquad = \int_\Omega \Phi_1((u_{1,\varepsilon}+\eta, u_{2,\varepsilon}+\eta))\ \mathrm{d}x - \int_\Omega \Phi_1((U_1+\eta, U_2+\eta))\ \mathrm{d}x\,. \end{aligned}$$

Owing to the continuity of Φ_1 on $[0,\infty)^2$, letting $\eta\to 0$ in the above inequality gives

$$\begin{aligned} \liminf_{\eta\to 0} &\int_\Omega \left[(u_{1,\varepsilon}-U_1)\ln(u_{1,\varepsilon}+\eta) + \frac{1}{\mu}(u_{2,\varepsilon}-U_2)\ln(u_{2,\varepsilon}+\eta) \right]\ \mathrm{d}x \\ &\ge \int_\Omega \Phi_1(u_\varepsilon)\ \mathrm{d}x - \int_\Omega \Phi_1(U)\ \mathrm{d}x\,. \end{aligned} \tag{2.19}$$

On the other hand,

$$\begin{aligned} D(\eta) := &\ \tau \int_\Omega \Big(m_{\varepsilon,11}(u_\varepsilon)\partial_x u_{1,\varepsilon} + m_{\varepsilon,12}(u_\varepsilon)\partial_x u_{2,\varepsilon} \Big) \frac{\partial_x u_{1,\varepsilon}}{u_{1,\varepsilon}+\eta}\ \mathrm{d}x \\ &+ \frac{\tau}{\mu} \int_\Omega \Big(m_{\varepsilon,21}(u_\varepsilon)\partial_x u_{1,\varepsilon} + m_{\varepsilon,22}(u_\varepsilon)\partial_x u_{2,\varepsilon} \Big) \frac{\partial_x u_{2,\varepsilon}}{u_{2,\varepsilon}+\eta}\ \mathrm{d}x \\ = &\ \tau\varepsilon \int_\Omega \left(\frac{|\partial_x u_{1,\varepsilon}|^2}{u_{1,\varepsilon}+\eta} + \frac{|\partial_x u_{2,\varepsilon}|^2}{u_{2,\varepsilon}+\eta} \right)\ \mathrm{d}x \\ &+ \tau \int_\Omega \left[\frac{u_{1,\varepsilon}}{u_{1,\varepsilon}+\eta}|\partial_x u_{1,\varepsilon}|^2 + R|\partial_x u_{1,\varepsilon} + \partial_x u_{2,\varepsilon}|^2 \right]\ \mathrm{d}x \\ &- \tau R \int_\Omega \left[\left(1 - \frac{u_{1,\varepsilon}}{u_{1,\varepsilon}+\eta}\right) \partial_x u_{1,\varepsilon}\cdot \partial_x(u_{1,\varepsilon}+u_{2,\varepsilon}) \right]\ \mathrm{d}x \\ &- \tau R \int_\Omega \left[\left(1 - \frac{u_{2,\varepsilon}}{u_{2,\varepsilon}+\eta}\right) \partial_x u_{2,\varepsilon}\cdot \partial_x(u_{1,\varepsilon}+u_{2,\varepsilon}) \right]\ \mathrm{d}x \\ \ge &\ \tau \int_\Omega \left[|\partial_x u_{1,\varepsilon}|^2 + R|\partial_x(u_{1,\varepsilon}+u_{2,\varepsilon})|^2 \right]\ \mathrm{d}x \\ &- J_0(\eta) - J_1(\eta) - J_2(\eta)\,, \end{aligned} \tag{2.20}$$

with

$$
\begin{aligned}
J_0(\eta) &:= \tau \int_\Omega \left(1 - \frac{u_{1,\varepsilon}}{u_{1,\varepsilon}+\eta}\right) |\partial_x u_{1,\varepsilon}|^2 \ \mathrm{d}x\,, \\
J_1(\eta) &:= \tau R \int_\Omega \left[\left(1 - \frac{u_{1,\varepsilon}}{u_{1,\varepsilon}+\eta}\right) \partial_x u_{1,\varepsilon} \cdot \partial_x \left(u_{1,\varepsilon}+u_{2,\varepsilon}\right)\right] \ \mathrm{d}x\,, \\
J_2(\eta) &:= \tau R \int_\Omega \left[\left(1 - \frac{u_{2,\varepsilon}}{u_{2,\varepsilon}+\eta}\right) \partial_x u_{2,\varepsilon} \cdot \partial_x \left(u_{1,\varepsilon}+u_{2,\varepsilon}\right)\right] \ \mathrm{d}x\,.
\end{aligned}
$$

Now, $u_\varepsilon \in H^1(\Omega,\mathbb{R}^2)$ and, for $j\in\{1,2\}$,

$$
\begin{aligned}
&\lim_{\eta\to 0} \frac{u_{j,\varepsilon}}{u_{j,\varepsilon}+\eta} = 1 \ \text{ a.e. in } \ \{x\in\Omega \ : \ u_{j,\varepsilon}>0\}\,, \\
&\lim_{\eta\to 0} \left(1 - \frac{u_{j,\varepsilon}}{u_{j,\varepsilon}+\eta}\right) \partial_x u_{j,\varepsilon} = 0 \ \text{ a.e. in } \ \{x\in\Omega \ : \ u_{j,\varepsilon}=0\}\,,
\end{aligned}
$$

so that we infer from Lebesgue's dominated convergence theorem that

$$
\lim_{\eta\to 0} \left(J_0(\eta)+J_1(\eta)+J_2(\eta)\right) = 0\,. \tag{2.21}
$$

Combining (2.20) and (2.21) gives

$$
\liminf_{\eta\to 0} D(\eta) \ge \tau \int_\Omega \left[|\partial_x u_{1,\varepsilon}|^2 + R|\partial_x(u_{1,\varepsilon}+u_{2,\varepsilon})|^2\right] \ \mathrm{d}x\,. \tag{2.22}
$$

In view of (2.19) and (2.22), we may pass to the limit $\eta\to 0$ in (2.18) and obtain the stated inequality. □

2.2. **A time discrete scheme: existence.** Thanks to the analysis performed in the previous section, we are now in a position to take the limit $\varepsilon\to 0$ and prove Proposition 2.1.

Proof of Proposition 2.1. Consider $\tau>0$ and $U=(F,G)\in L_{\infty,+}(\Omega,\mathbb{R}^2)$. Given $\varepsilon\in(0,1)$, let $u_\varepsilon=(u_{1,\varepsilon},u_{2,\varepsilon})\in H^1_+(\Omega,\mathbb{R}^2)$ denote the weak solution to (2.11) provided by Lemma 2.2. It first follows from (2.12) and the componentwise non-negativity of u_ε that

$$
\max\{\|u_{1,\varepsilon}\|_\infty, \|u_{2,\varepsilon}\|_\infty\} \le \|u_\varepsilon\|_\infty \le \frac{1+R}{R}\|F+G\|_\infty\,. \tag{2.23}
$$

In view of the non-negativity of Φ_1, we infer from Lemma 2.3 that

$$
\int_\Omega \left[|\partial_x u_{1,\varepsilon}|^2 + R|\partial_x(u_{1,\varepsilon}+u_{2,\varepsilon})|^2\right] \ \mathrm{d}x \le \frac{1}{\tau}\int_\Omega \Phi_1(U)\ \mathrm{d}x\,.
$$

Hence, by (2.6),

$$
\|\partial_x u_\varepsilon\|_2^2 \le \frac{1+2R}{\tau R}\int_\Omega \Phi_1(U)\ \mathrm{d}x\,. \tag{2.24}
$$

Due to the compactness of the embedding of $H^1(\Omega)$ in $L_\infty(\Omega)$, we deduce from (2.23) and (2.24) that there is $u=(f,g)\in H^1_+(\Omega,\mathbb{R}^2)$ and a sequence $(u_{\varepsilon_j})_{j\ge 1}$ such that

$$
\begin{aligned}
&u_{\varepsilon_j} \rightharpoonup u \ \text{ in } \ H^1(\Omega,\mathbb{R}^2)\,, \\
&\lim_{j\to\infty} \|u_{\varepsilon_j}-u\|_\infty = 0\,.
\end{aligned} \tag{2.25}
$$

An immediate consequence of (2.12), (2.13), and (2.25) is that (f,g) satisfies (2.2) for $n\ge 2$ and (2.3). Moreover, another consequence of (2.25), along with Lemma 2.3 and a weak lower semicontinuity argument, is that (f,g) satisfies (2.4). Finally, owing to (2.25) and the boundedness of the coefficients of $M_\varepsilon(u_\varepsilon)$ due to (2.23), we may use Lebesgue's dominated convergence theorem to take the limit $j\to\infty$ in the identity (2.11) for u_{ε_j} and conclude that (f,g) satisfies (2.1), thereby completing the proof of Proposition 2.1. □

3. Existence of bounded weak solutions: $d=1$

This section is devoted to the proof of Theorem 1.2. To this end, we argue in a standard way and construct, starting from the initial condition $(f^{in},g^{in})\in L_{\infty,+}(\Omega,\mathbb{R}^2)$ and using Proposition 2.1, a family of piecewise constant functions $(u^\tau)_{\tau\in(0,1)}$. Specifically, we set $u^\tau(0):=u_0^\tau$ and

$$u^\tau(t)=u_l^\tau\,,\qquad t\in((l-1)\tau,l\tau]\,,\quad 1\le l\in\mathbb{N}\,, \tag{3.1}$$

where, given $\tau\in(0,1)$, the sequence $(u_l^\tau)_{l\ge 0}$ is defined as follows

$$\begin{aligned}
&u_0^\tau:=u^{in}=(f^{in},g^{in})\in L_{\infty,+}(\Omega,\mathbb{R}^2)\,,\\
&u_{l+1}^\tau=(f_{l+1}^\tau,g_{l+1}^\tau)\in H_+^1(\Omega,\mathbb{R}^2)\text{ is the solution to (2.1)}\\
&\text{with } U=u_l^\tau=(f_l^\tau,g_l^\tau)\text{ constructed in Proposition 2.1 for } l\ge 0\,.
\end{aligned} \tag{3.2}$$

In order to establish Theorem 1.2, we show that the family $(u^\tau)_{\tau\in(0,1)}$ converges along a subsequence $\tau_j\to 0$ towards a pair $u=(f,g)$ which fulfills all the requirements of Theorem 1.2.

Throughout this section, C and C_i, with $i\ge 0$, denote positive constants depending only on R, μ, and (f^{in},g^{in}). Dependence upon additional parameters will be indicated explicitly.

Proof of Theorem 1.1. Let $\tau\in(0,1)$ and let u^τ be defined in (3.1)-(3.2). Given $l\ge 0$, we infer from Proposition 2.1 that

$$\int_\Omega\Big[f_{l+1}^\tau\varphi+\tau f_{l+1}^\tau\partial_x\left[(1+R)f_{l+1}^\tau+Rg_{l+1}^\tau\right]\partial_x\varphi\Big]\,\mathrm{d}x=\int_\Omega f_l^\tau\varphi\,\mathrm{d}x\,,\qquad \varphi\in H^1(\Omega)\,, \tag{3.3a}$$

$$\int_\Omega\Big[g_{l+1}^\tau\psi+\tau\mu Rg_{l+1}^\tau\partial_x(f_{l+1}^\tau+g_{l+1}^\tau)\partial_x\psi\Big]\,\mathrm{d}x=\int_\Omega g_l^\tau\psi\,\mathrm{d}x\,,\qquad \psi\in H^1(\Omega)\,. \tag{3.3b}$$

Moreover,

$$\int_\Omega\Phi_n(u_{l+1}^\tau)\,\mathrm{d}x\le\int_\Omega\Phi_n(u_l^\tau)\,\mathrm{d}x \tag{3.4}$$

for $n\ge 2$ and we also have

$$\int_\Omega\Phi_1(u_{l+1}^\tau)\,\mathrm{d}x+\tau\int_\Omega\left[|\partial_x f_{l+1}^\tau|^2+R|\partial_x(f_{l+1}^\tau+g_{l+1}^\tau)|^2\right]\,\mathrm{d}x\le\int_\Omega\Phi_1(u_l^\tau)\,\mathrm{d}x\,. \tag{3.5}$$

It readily follows from (3.1), (3.2), (3.4), and (3.5) that, for $t>0$,

$$\int_\Omega\Phi_1(u^\tau(t))\,\mathrm{d}x+\int_0^t\int_\Omega\left[|\partial_x f^\tau(s)|^2+R|\partial_x(f^\tau+g^\tau)(s)|^2\right]\,\mathrm{d}x\mathrm{d}s\le\int_\Omega\Phi_1(u^{in})\,\mathrm{d}x\,, \tag{3.6}$$

and

$$\int_\Omega\Phi_n(u^\tau(t))\,\mathrm{d}x\le\int_\Omega\Phi_n(u^{in})\,\mathrm{d}x\,,\qquad n\ge 2\,. \tag{3.7}$$

An immediate consequence of (3.7) and Lemma A.2 is the estimate

$$\|f^\tau(t)+g^\tau(t)\|_n\le\frac{1+R}{R}\|f^{in}+g^{in}\|_n\,,\qquad n\ge 2\,,\ t>0\,.$$

Letting $n \to \infty$ in the above inequality gives

$$\|f^\tau(t) + g^\tau(t)\|_\infty \le C_1 := \frac{1+R}{R}\|f^{in} + g^{in}\|_\infty\,, \qquad t > 0\,. \tag{3.8}$$

Also, it readily follows from (2.6), (3.6), and the non-negativity of Φ_1 that

$$\begin{aligned}
&\frac{R}{1+2R}\int_0^t \int_\Omega \left(|\partial_x f^\tau(s)|^2 + |\partial_x g^\tau(s)|^2\right)\, \mathrm{d}x\mathrm{d}s \\
&\qquad \le \int_\Omega \Phi_1(u^\tau(t))\, \mathrm{d}x + \int_0^t \int_\Omega \left[|\partial_x f^\tau(s)|^2 + R|\partial_x (f^\tau + g^\tau)(s)|^2\right]\, \mathrm{d}x\mathrm{d}s \\
&\qquad \le \int_\Omega \Phi_1(u^{in})\, \mathrm{d}x\,.
\end{aligned}$$

Therefore we have

$$\int_0^t \left(\|\partial_x f^\tau(s)\|_2^2 + \|\partial_x g^\tau(s)\|_2^2\right)\, \mathrm{d}s \le C_2 := \frac{1+2R}{R}\int_\Omega \Phi_1(u^{in})\, \mathrm{d}x\,, \qquad t > 0\,. \tag{3.9}$$

Next, for $l \ge 1$ and $t \in ((l-1)\tau, l\tau]$, we deduce from (3.3a), (3.8), and Hölder's inequality that, for $\varphi \in H^1(\Omega)$,

$$\begin{aligned}
\left|\int_\Omega \left(f^\tau(t+\tau) - f^\tau(t)\right)\varphi\, \mathrm{d}x\right| &= \left|\int_{l\tau}^{(l+1)\tau} \int_\Omega f^\tau(s)\partial_x\left[(1+R)f^\tau(s) + Rg^\tau(s)\right]\partial_x\varphi\, \mathrm{d}x\mathrm{d}s\right| \\
&\le \int_{l\tau}^{(l+1)\tau} \|f^\tau(s)\|_\infty \|\partial_x\left[(1+R)f^\tau(s) + Rg^\tau(s)\right]\|_2 \|\partial_x\varphi\|_2\, \mathrm{d}s \\
&\le C_1\|\partial_x\varphi\|_2 \int_{l\tau}^{(l+1)\tau} \|\partial_x\left[(1+R)f^\tau(s) + Rg^\tau(s)\right]\|_2\, \mathrm{d}s\,.
\end{aligned}$$

A duality argument then gives

$$\|f^\tau(t+\tau) - f^\tau(t)\|_{(H^1)'} \le C_1 \int_{l\tau}^{(l+1)\tau} \|\partial_x\left[(1+R)f^\tau(s) + Rg^\tau(s)\right]\|_2\, \mathrm{d}s\,, \quad t \in ((l-1)\tau, l\tau]\,, l \ge 1\,.$$

Now, for $L \ge 2$ and $T \in ((L-1)\tau, L\tau]$, the above inequality, along with Hölder's inequality, entails that

$$\begin{aligned}
\int_0^{T-\tau} \|f^\tau(t+\tau) - f^\tau(t)\|_{(H^1)'}^2\, \mathrm{d}t &\le \int_0^{(L-1)\tau} \|f^\tau(t+\tau) - f^\tau(t)\|_{(H^1)'}^2\, \mathrm{d}t \\
&= \sum_{l=1}^{L-1} \int_{(l-1)\tau}^{l\tau} \|f^\tau(t+\tau) - f^\tau(t)\|_{(H^1)'}^2\, \mathrm{d}t \\
&\le C_1^2\tau \sum_{l=1}^{L-1} \left(\int_{l\tau}^{(l+1)\tau} \|\partial_x\left[(1+R)f^\tau(s) + Rg^\tau(s)\right]\|_2\, \mathrm{d}s\right)^2 \\
&\le C_1^2\tau^2 \sum_{l=1}^{L-1} \int_{l\tau}^{(l+1)\tau} \|\partial_x\left[(1+R)f^\tau(s) + Rg^\tau(s)\right]\|_2^2\, \mathrm{d}s \\
&\le C_1^2\tau^2 \int_0^{L\tau} \|\partial_x\left[(1+R)f^\tau(s) + Rg^\tau(s)\right]\|_2^2\, \mathrm{d}s\,.
\end{aligned}$$

We then use (3.9) (with $t = L\tau$) and Young's inequality to obtain

$$\begin{aligned}\int_0^{T-\tau} \|f^\tau(t+\tau) - f^\tau(t)\|_{(H^1)'}^2 \, \mathrm{d}t &\le C_1^2\tau^2 \int_0^{L\tau} \left(2(1+R)^2\|\nabla f^\tau(s)\|_2^2 + 2R^2\|\nabla g^\tau(s)\|_2^2\right) \, \mathrm{d}s \\ &\le C_3\tau^2 \,, \end{aligned} \tag{3.10}$$

with $C_3 := 2(1+R)^2C_1^2C_2$. Similarly,

$$\int_0^{T-\tau} \|g^\tau(t+\tau) - g^\tau(t)\|_{(H^1)'}^2 \, \mathrm{d}t \le C_4\tau^2 \,, \tag{3.11}$$

with $C_4 := 2\mu^2R^2C_1^2C_2$.

Since $H^1(\Omega,\mathbb{R}^2)$ is compactly embedded in $L_2(\Omega,\mathbb{R}^2)$ and $L_2(\Omega,\mathbb{R}^2)$ is continuously embedded in $H^1(\Omega,\mathbb{R}^2)'$, we infer from (3.8), (3.9), (3.10), (3.11), and [5, Theorem 1] that, for any $T > 0$,

$$(u^\tau)_{\tau\in(0,1)} \;\text{ is relatively compact in }\; L_2((0,T)\times\Omega,\mathbb{R}^2) \,. \tag{3.12}$$

Owing to (3.8), (3.9), and (3.12), we may use a Cantor diagonal argument to find a function $u := (f, g)$ in $L_{\infty,+}((0,\infty)\times\Omega,\mathbb{R}^2)$ and a sequence $(\tau_j)_{j\ge1}$, $\tau_j \to 0$, such that, for any $T > 0$ and $p \in [1,\infty)$,

$$\begin{aligned} &u^{\tau_j} \longrightarrow u \quad \text{in} \quad L_p((0,T)\times\Omega,\mathbb{R}^2) \,, \\ &u^{\tau_j} \overset{*}{\rightharpoonup} u \quad \text{in} \quad L_\infty((0,T)\times\Omega,\mathbb{R}^2) \,, \\ &u^{\tau_j} \rightharpoonup u \quad \text{in} \quad L_2((0,T), H^1(\Omega,\mathbb{R}^2)) \,. \end{aligned} \tag{3.13}$$

In addition, the compact embedding of $L_2(\Omega,\mathbb{R}^2)$ in $H^1(\Omega,\mathbb{R}^2)'$, along with (3.7) with $n = 2$, (3.10), and (3.11), allows us to apply once more [5, Theorem 1] to conclude that

$$u \in C([0,\infty), H^1(\Omega,\mathbb{R}^2)') \,. \tag{3.14}$$

Let us now identify the equations solved by the components f and g of u. To this end, let $\chi \in W_\infty^1([0,\infty))$ be a compactly supported function and $\varphi \in C^1(\bar\Omega)$. In view of (3.3a), classical computations give

$$\begin{aligned} &\int_0^\infty \int_\Omega \frac{\chi(t+\tau)-\chi(t)}{\tau} f^\tau(t)\varphi \, \mathrm{d}x\mathrm{d}t + \left(\frac{1}{\tau}\int_0^\tau \chi(t)\, \mathrm{d}t\right)\int_\Omega f^{in}\varphi \, \mathrm{d}x \\ &\qquad = \int_0^\infty \int_\Omega \chi(t) f^\tau(t)\partial_x \left[(1+R)f^\tau(t) + Rg^\tau(t)\right]\partial_x\varphi \, \mathrm{d}x\mathrm{d}t \,. \end{aligned}$$

Taking $\tau = \tau_j$ in the above identity, it readily follows from (3.13) and the regularity of χ and φ that we may pass to the limit as $j \to \infty$ and conclude that

$$\begin{aligned} &\int_0^\infty \int_\Omega \frac{d\chi}{dt}(t) f(t)\varphi \, \mathrm{d}x\mathrm{d}t + \chi(0)\int_\Omega f^{in}\varphi \, \mathrm{d}x \\ &\qquad = \int_0^\infty \int_\Omega \chi(t)f(t)\partial_x\left[(1+R)f(t) + Rg(t)\right]\partial_x\varphi \, \mathrm{d}x\mathrm{d}t \,. \end{aligned} \tag{3.15}$$

Since $f\partial_x f$ and $f\partial_x g$ belong to $L_2((0,T)\times\Omega)$ for all $T > 0$ by (3.13), a density argument ensures that the identity (3.15) is valid for any $\varphi \in H^1(\Omega)$. We next use the time continuity (3.14) of f and a classical approximation argument to show that f solves (1.13a). A similar argument allows us to derive (1.13b) from (3.3b).

Finally, combining (3.13), (3.14), and a weak lower semicontinuity argument, we may let $j \to \infty$ in (3.6), (3.7), and (3.8) with $\tau = \tau_j$ to show that $u = (f, g)$ satisfies (1.6), (1.7a), and (1.8), thereby completing the proof. □

We end up this section with the proof of Corollary 1.3.

Proof of Corollary 1.3. Assume that $R\max\{1, \mu\} \in (0, 1/(2e)]$. Given an integer $m \geq 1$, we define the function $\xi : (0, 1/(2e)] \to \mathbb{R}$ by the formula

$$\xi(y) := \exp\left\{m\left[(1+y)\ln\left(1+\frac{1}{y}\right) - 1\right]\right\} - 1.$$

It then holds

$$y^m\xi(y) = (1+y)^m \exp\left\{m\left[y\ln\left(1+\frac{1}{y}\right) - 1\right]\right\} - y^m > \frac{(1+y)^m}{e^m} - y^m \geq \frac{1}{e^m} - \frac{1}{(2e)^m} \geq \frac{1}{2e^m}.$$

Consequently, the constant ν_n defined in Lemma A.2 satisfies

$$\nu_n > \frac{1}{2(eR\max\{1,\mu\})^{n-1}}, \qquad n \geq 2,$$

We then infer from Theorem 1.2 (p3), the above inequality, and (A.13) that, for $t > 0$ and $n \geq 2$,

$$\begin{aligned}\|f(t)\|_n^n &\leq \frac{1}{\nu_n}\int_\Omega \Phi_n((f(t), g(t)))\ \mathrm{d}x \leq \frac{1}{\nu_n}\int_\Omega \Phi_n((f^{in}, g^{in}))\ \mathrm{d}x \\ &\leq \frac{2(eR\max\{1,\mu\})^{n-1}}{R^n}\|(1+R)f^{in} + Rg^{in}\|_n^n.\end{aligned}$$

Hence,

$$\|f(t)\|_n \leq \left(\frac{2}{R}\right)^{1/n} (e\max\{1,\mu\})^{(n-1)/n}\|(1+R)f^{in} + Rg^{in}\|_n.$$

Taking the limit $n \to \infty$ in the above inequality gives

$$\|f(t)\|_\infty \leq e\max\{1,\mu\}\|(1+R)f^{in} + Rg^{in}\|_\infty,$$

and we use the upper bound $eR\max\{1,\mu\} \leq 1$ to obtain the desired estimate (1.14). □

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Owing to Proposition A.1, the proof of Theorem 1.1 is a simple application of the scheme described in (1.10) and (1.11). Indeed, let $u = (f, g)$ be a sufficiently regular solution to (1.1) on $[0, \infty)$ and $n \geq 2$. It follows from the alternative form (1.10) of the system (1.1a)-(1.1b) and the boundary conditions (1.1c) that

$$\frac{d}{dt}\int_\Omega \Phi_n(u)\ \mathrm{d}x + \sum_{i=1}^d \int_\Omega \langle D^2\Phi_n(u)M(u)\partial_i u, \partial_i u\rangle\ \mathrm{d}x = 0.$$

According to (A.3) and Proposition A.1 (b), we infer from the componentwise non-negativity of u and the continuity of $D^2\Phi_n M$ that

$$\langle D^2\Phi_n(u)M(u)\partial_i u, \partial_i u\rangle \geq 0 \quad \text{in} \quad (0,\infty)\times\Omega, \qquad 1 \leq i \leq d.$$

Consequently,

$$\frac{d}{dt}\int_\Omega \Phi_n(u)\ \mathrm{d}x \leq 0, \qquad t > 0,$$

and (1.7a) is proved. In particular, thanks to (A.13), we have shown that, for $t > 0$ and $n \geq 2$,

$$\|f(t)+g(t)\|_n \leq \left(\int_\Omega \Phi_n(u(t))\ \mathrm{d}x \right)^{1/n} \leq \left(\int_\Omega \Phi_n(u^{in})\ \mathrm{d}x \right)^{1/n} \leq \frac{1+R}{R} \|f^{in}+g^{in}\|_n \,.$$

Taking the limit $n \to \infty$ in the above inequality gives (1.8). Finally, to establish the inequality (1.6), we use (1.1) to compute the time derivative of

$$\int_\Omega \Phi_1((f(t)+\eta, g(t)+\eta))\ \mathrm{d}x$$

and argue as in the proof of Lemma 2.3 to derive (1.6). □

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Appendix A. Properties of the polynomials Φ_n, $n \geq 2$

In this section, we establish some important properties of the polynomials Φ_n, $n \geq 2$, defined in (1.7b) and (1.7c), which lead to Theorem 1.1 according to the scheme outlined in the Introduction, see (1.10)-(1.11), and are extensively used in Section 2, see the proof of Lemma 2.2. Let thus $n \geq 2$. To begin with, we recall that $a_{0,n} = 1$,

$$a_{j,n} = \binom{n}{j} \prod_{k=0}^{j-1} \frac{k+\alpha_{k,n}}{\alpha_{k,n}} \,, \qquad 1 \leq j \leq n \,, \tag{A.1a}$$

where

$$\alpha_{k,n} = R[k+\mu(n-k-1)] = \mu R(n-1) + R(1-\mu)k > 0 \,, \qquad 0 \leq k \leq n-1 \,, \tag{A.1b}$$

and

$$\Phi_n(X) := \sum_{j=0}^{n} a_{j,n} X_1^j X_2^{n-j} \,, \qquad X = (X_1, X_2) \in \mathbb{R}^2 \,. \tag{A.2}$$

Also, the mobility matrix $M \in C^\infty(\mathbb{R}^2, \mathbf{M}_2(\mathbb{R}))$ is defined in (1.9) by

$$M(X) := \begin{pmatrix} (1+R)X_1 & RX_1 \\ \mu R X_2 & \mu R X_2 \end{pmatrix} \,, \qquad X \in \mathbb{R}^2 \,.$$

The aim of this section is twofold. On the one hand, we establish the convexity of Φ_n on $[0,\infty)^2$ and actually show that its Hessian matrix $D^2\Phi_n \in C^\infty(\mathbb{R}^2, \mathbf{Sym}_2(\mathbb{R}))$, defined as usual by

$$D^2\Phi_n(X) = \begin{pmatrix} \partial_1^2 \Phi_n(X) & \partial_1\partial_2\Phi_n(X) \\ \partial_1\partial_2\Phi_n(X) & \partial_2^2\Phi_n(X) \end{pmatrix} \,, \qquad X \in \mathbb{R}^2 \,,$$

is positive definite on $[0,\infty)^2 \setminus \{(0,0)\}$. On the other hand, we prove that the matrix

$$S_n(X) := D^2\Phi_n(X) M(X) \,, \qquad X \in \mathbb{R}^2 \,, \tag{A.3}$$

belongs to $\mathbf{Sym}_2(\mathbb{R})$ and actually lies in $\mathbf{SPD}_2(\mathbb{R})$ for $X \in (0,\infty)^2$.

Proposition A.1. *Let $n \ge 2$.*

(a) *The polynomial Φ_n is non-negative and convex on $[0,\infty)^2$. Moreover, we have:*
   (a1) *The gradient $D\Phi_n(X)$ belongs to $[0,\infty)^2$ provided that $X \in [0,\infty)^2$;*
   (a2) *The Hessian matrix $D^2\Phi_n(X)$ belongs to $\mathbf{SPD}_2(\mathbb{R})$ for all $X \in [0,\infty)^2 \setminus \{(0,0)\}$.*

(b) *Given $X \in \mathbb{R}^2$, the matrix $S_n(X)$ defined in (A.3) is symmetric. In addition, it holds that $S_n(X) \in \mathbf{SPD}_2(\mathbb{R})$ for all $X \in (0,\infty)^2$.*

Proof. The proof in the case $n = 2$ is a simple exercise. Let now $n \ge 3$. We first note that (A.1) implies that $(a_{j,n})_{1\le j\le n}$ satisfies the following recursion formula

$$a_{j+1,n} = \frac{(n-j)(j+\alpha_{j,n})}{(j+1)\alpha_{j,n}} a_{j,n}\,, \qquad 0 \le j \le n-1\,, \tag{A.4}$$

from which we deduce that

$$a_{j,n} > 0\,, \qquad 0 \le j \le n\,. \tag{A.5}$$

In particular, Φ_n is non-negative on $[0,\infty)^2$ and, since

$$\partial_1\Phi_n(X) = \sum_{j=0}^{n-1} (j+1) a_{j+1,n} X_1^j X_2^{n-j-1}\,, \qquad X \in \mathbb{R}^2\,,$$

$$\partial_2\Phi_n(X) = \sum_{j=0}^{n-1} (n-j) a_{j,n} X_1^j X_2^{n-j-1}\,, \qquad X \in \mathbb{R}^2\,,$$

the gradient $D\Phi_n(X)$ belongs to $[0,\infty)^2$ for $X \in [0,\infty)^2$, which proves (a1).

Convexity of Φ_n on $[0,\infty)^2$. The convexity of Φ_n on $[0,\infty)^2$ is a consequence of the property (a2) which we establish now. Let $X \in [0,\infty)^2$. We then have

$$\partial_1^2\Phi_n(X) = \sum_{j=1}^{n-1} j(j+1) a_{j+1,n} X_1^{j-1} X_2^{n-j-1} = \sum_{j=0}^{n-2} (j+1)(j+2) a_{j+2,n} X_1^j X_2^{n-j-2}\,,$$

$$\partial_1\partial_2\Phi_n(X) = \sum_{j=1}^{n-1} j(n-j) a_{j,n} X_1^{j-1} X_2^{n-j-1} = \sum_{j=0}^{n-2} (j+1)(n-j-1) a_{j+1,n} X_1^j X_2^{n-j-2}\,,$$

$$\partial_2^2\Phi_n(X) = \sum_{j=0}^{n-2} (n-j)(n-j-1) a_{j,n} X_1^j X_2^{n-j-2}\,.$$

It then readily follows from (A.5) that the Hessian matrix $D^2\Phi_n(X)$ has a non-negative trace

$$\mathrm{tr}(D^2\Phi_n(X)) := \partial_1^2\Phi_n(X) + \partial_2^2\Phi_n(X) \ge 0\,, \qquad X \in [0,\infty)^2\,. \tag{A.6}$$

Next,

$$\begin{aligned} \det(D^2\Phi_n(X)) &= \partial_1^2\Phi_n(X)\partial_2^2\Phi_n(X) - [\partial_1\partial_2\Phi_n(X)]^2 \\ &= \sum_{j=0}^{n-2}\sum_{k=0}^{n-2} (j+1)(n-k-1) A_{j,k} X_1^{j+k} X_2^{2n-j-k-4}\,, \end{aligned} \tag{A.7}$$

where

$$A_{j,k} := (j+2)(n-k) a_{j+2,n} a_{k,n} - (n-j-1)(k+1) a_{j+1,n} a_{k+1,n}\,, \qquad 0 \le j,k \le n-2\,.$$

We now simplify the above formula for $A_{j,k}$ and first use (A.4) to replace $a_{j+2,n}$ and $a_{k+1,n}$ and subsequently the definition (A.1b) of $\alpha_{k,n}$, thereby obtaining

$$\begin{aligned}
A_{j,k} &= (n-j-1)(n-k)\frac{j+1+\alpha_{j+1,n}}{\alpha_{j+1,n}}a_{j+1,n}a_{k,n} - (n-j-1)(n-k)\frac{k+\alpha_{k,n}}{\alpha_{k,n}}a_{j+1,n}a_{k,n} \\
&= \mu R(n-1)\frac{(n-j-1)(n-k)(j+1-k)}{\alpha_{j+1,n}\alpha_{k,n}}a_{j+1,n}a_{k,n}
\end{aligned} \tag{A.8}$$

for $0 \le j,k \le n-2$. In particular,

$$A_{k-1,j+1} = -A_{j,k}\,, \qquad 0 \le j \le n-3\,,\ 1 \le k \le n-2\,. \tag{A.9}$$

It then follows from (A.7) and (A.8) that

$$\begin{aligned}
2\det(D^2\Phi_n(X)) &= \sum_{j=0}^{n-2}\sum_{k=0}^{n-2}(j+1)(n-k-1)A_{j,k}X_1^{j+k}X_2^{2n-j-k-4} \\
&\quad + \sum_{l=1}^{n-1}\sum_{i=-1}^{n-3} l(n-i-2)A_{l-1,i+1}X_1^{i+l}X_2^{2n-i-l-4} \\
&= \sum_{j=0}^{n-2}\sum_{k=0}^{n-2}(j+1)(n-k-1)A_{j,k}X_1^{j+k}X_2^{2n-j-k-4} \\
&\quad + \sum_{j=-1}^{n-3}\sum_{k=1}^{n-1} k(n-j-2)A_{k-1,j+1}X_1^{j+k}X_2^{2n-j-k-4} \\
&= \sum_{j=0}^{n-3}\sum_{k=1}^{n-2}(j+1)(n-k-1)A_{j,k}X_1^{j+k}X_2^{2n-j-k-4} \\
&\quad + \sum_{k=0}^{n-2}(n-1)(n-k-1)A_{n-2,k}X_1^{n-2+k}X_2^{n-k-2} \\
&\quad + \sum_{j=0}^{n-3}(j+1)(n-1)A_{j,0}X_1^{j}X_2^{2n-j-4} \\
&\quad + \sum_{j=0}^{n-3}\sum_{k=1}^{n-2} k(n-j-2)A_{k-1,j+1}X_1^{j+k}X_2^{2n-j-k-4} \\
&\quad + \sum_{k=1}^{n-1} k(n-1)A_{k-1,0}X_1^{k-1}X_2^{2n-k-3} \\
&\quad + \sum_{j=0}^{n-3}(n-1)(n-j-2)A_{n-2,j+1}X_1^{j+n-1}X_2^{n-j-3}\,.
\end{aligned}$$

Owing to (A.5) and (A.8), $A_{l,0} > 0$ and $A_{n-2,l} > 0$ for $0 \le l \le n-2$, so that the terms in the above identity involving a single sum are non-negative. Therefore, using the symmetry property (A.9) and retaining in the last two sums only the terms corresponding to $k = 1$ and $j = n-3$, respectively, we get

$$\begin{aligned}2\det(D^2\Phi_n(X)) &\geq \sum_{j=0}^{n-3}\sum_{k=1}^{n-2}\left[(j+1)(n-k-1)-k(n-j-2)\right]A_{j,k}X_1^{j+k}X_2^{2n-j-k-4}\\ &\qquad + (n-1)A_{n-2,n-2}X_1^{2n-4} + (n-1)A_{0,0}X_2^{2n-4}\\ &= \sum_{j=0}^{n-3}\sum_{k=1}^{n-2}(n-1)(j+1-k)A_{j,k}X_1^{j+k}X_2^{2n-j-k-4}\\ &\qquad + (n-1)A_{n-2,n-2}X_1^{2n-4} + (n-1)A_{0,0}X_2^{2n-4}\,.\end{aligned}$$

Since

$$(n-1)(j+1-k)A_{j,k} = \mu R(n-1)^2\frac{(n-j-1)(n-k)(j+1-k)^2}{\alpha_{j+1,n}\alpha_{k,n}}a_{j+1,n}a_{k,n} \geq 0$$

for $0\leq j,k\leq n-2$ by (A.1b), (A.5), and (A.8), we conclude that

$$\det(D^2\Phi_n(X)) \geq (n-1)A_{n-2,n-2}X_1^{2n-4} + (n-1)A_{0,0}X_2^{2n-4}\,, \qquad X\in[0,\infty)^2\,. \tag{A.10}$$

Since $A_{0,0}>0$ and $A_{n-2,n-2}>0$, we have thus established that, for each $X\in[0,\infty)^2\setminus\{(0,0)\}$, the symmetric matrix $D^2\Phi_n(X)$ has non-negative trace and positive determinant, so that it is positive definite. This proves (a2).

Symmetry of $S_n(X)$. Let $X\in\mathbb{R}^2$. It follows from (A.3) that

$$\begin{aligned}[S_n(X)]_{11} &= (1+R)X_1\partial_1^2\Phi_n(X) + \mu R X_2\partial_1\partial_2\Phi_n(X)\\ &= (1+R)\sum_{j=1}^{n-1}j(j+1)a_{j+1,n}X_1^jX_2^{n-j-1} + \mu R\sum_{j=0}^{n-2}(j+1)(n-j-1)a_{j+1,n}X_1^jX_2^{n-j-1}\,,\\ [S_n(X)]_{12} &= RX_1\partial_1^2\Phi_n(X) + \mu R X_2\partial_1\partial_2\Phi_n(X)\\ &= R\sum_{j=1}^{n-1}j(j+1)a_{j+1,n}X_1^jX_2^{n-j-1} + \mu R\sum_{j=0}^{n-2}(j+1)(n-j-1)a_{j+1,n}X_1^jX_2^{n-j-1}\,,\\ [S_n(X)]_{21} &= (1+R)X_1\partial_1\partial_2\Phi_n(X) + \mu R X_2\partial_2^2\Phi_n(X)\\ &= (1+R)\sum_{j=1}^{n-1}j(n-j)a_{j,n}X_1^jX_2^{n-j-1} + \mu R\sum_{j=0}^{n-2}(n-j)(n-j-1)a_{j,n}X_1^jX_2^{n-j-1}\,,\\ [S_n(X)]_{22} &= RX_1\partial_1\partial_2\Phi_n(X) + \mu R X_2\partial_2^2\Phi_n(X)\\ &= R\sum_{j=1}^{n-1}j(n-j)a_{j,n}X_1^jX_2^{n-j-1} + \mu R\sum_{j=0}^{n-2}(n-j)(n-j-1)a_{j,n}X_1^jX_2^{n-j-1}\,.\end{aligned}$$

It then holds

$$[S_n(X)]_{12} = Rn(n-1)a_{n,n}X_1^{n-1} + \sum_{j=1}^{n-2}(j+1)\alpha_{j,n}a_{j+1,n}X_1^jX_2^{n-j-1} + \mu R(n-1)a_{1,n}X_2^{n-1}\,.$$

Using the recursion formula (A.4) and the definition (A.1b) of $\alpha_{j,n}$, we get

$$\begin{aligned}
[S_n(X)]_{12} &= R(n-1)\frac{n-1+\alpha_{n-1,n}}{\alpha_{n-1,n}} a_{n-1,n} X_1^{n-1} + \sum_{j=1}^{n-2} (n-j)(j+\alpha_{j,n}) a_{j,n} X_1^j X_2^{n-j-1} \\
&\qquad + \mu R n(n-1) a_{0,n} X_2^{n-1} \\
&= (1+R)(n-1) a_{n-1,n} X_1^{n-1} + (1+R) \sum_{j=1}^{n-2} j(n-j) a_{j,n} X_1^j X_2^{n-j-1} \\
&\qquad + \mu R \sum_{j=1}^{n-2} (n-j)(n-j-1) a_{j,n} X_1^j X_2^{n-j-1} + \mu R n(n-1) a_{0,n} X_2^{n-1} \\
&= [S_n(X)]_{21}\,,
\end{aligned}$$

so that $S_n(X) \in \mathbf{Sym}_2(\mathbb{R})$.

Positive definiteness of $S_n(X)$. Let $X \in [0,\infty)^2$. It readily follows from (A.5) that $[S_n(X)]_{11} \ge 0$ and $[S_n(X)]_{22} \ge 0$, hence

$$\mathrm{tr}(S_n(X)) \ge 0\,. \tag{A.11}$$

Moreover, (A.3) and (A.10) imply that

$$\det(S_n(X)) = \det(D^2\Phi_n(X))\det(M(X)) = \mu R X_1 X_2 \det(D^2\Phi_n(X)) \ge 0\,. \tag{A.12}$$

Consequently, $S_n(X)$ is a positive semidefinite symmetric matrix for each $X \in [0,\infty)^2$. Moreover, if $X \in (0,\infty)^2$, then $\det(S_n(X)) > 0$ by (A.10) and (A.12), so that $S_n(X) \in \mathbf{SPD}_2(\mathbb{R})$. This completes the proof of (b). □

We next derive lower and upper bounds for Φ_n, $n \ge 2$.

Lemma A.2. *Given $n \ge 2$, we have*

$$\nu_n X_1^n + (X_1+X_2)^n \le \Phi_n(X) \le \frac{[(1+R)X_1 + RX_2]^n}{R^n}\,, \qquad X \in [0,\infty)^2\,, \tag{A.13}$$

where ν_n is the positive number defined by

$$\nu_n := \exp\left\{(n-1)\left[(1+R\max\{1,\mu\})\ln\left(1+\frac{1}{R\max\{1,\mu\}}\right) - 1\right]\right\} - 1 > 0\,.$$

Proof. On the one hand, since the function

$$\chi(z) := \frac{\mu R + [1+R(1-\mu)]z}{\mu R + R(1-\mu)z}\,, \qquad z \in [0,1]\,,$$

is increasing, we deduce from (A.1) that, for $1 \le j \le n$,

$$a_{j,n} = \binom{n}{j} \prod_{k=0}^{j-1} \chi\left(\frac{k}{n-1}\right) \le \binom{n}{j} [\chi(1)]^j = \left(\frac{1+R}{R}\right)^j \binom{n}{j}\,.$$

The upper bound in (A.13) is then a straightforward consequence of the above inequality.

On the other hand, in order to estimate $\Phi_n(X)$, $X \in [0,\infty)^2$, from below we infer from (A.1a) that

$$a_{j,n} \ge \binom{n}{j}\,, \qquad 0 \le j \le n-1\,.$$

When estimating the coefficient $a_{n,n}$ from below we need to be more subtle and proceed as follows:

$$\begin{aligned} a_{n,n} &= \prod_{k=0}^{n-1} \frac{k+\alpha_{k,n}}{\alpha_{k,n}} = \prod_{k=1}^{n-1} \left(1 + \frac{k}{R[k+\mu(n-k-1)]}\right) \\ &\ge \prod_{k=1}^{n-1} \left(1 + \frac{k}{R\max\{1,\mu\}(n-1)}\right) . \end{aligned}$$

Now,

$$\begin{aligned} \ln\left(\prod_{k=1}^{n-1}\left(1+\frac{k}{R\max\{1,\mu\}(n-1)}\right)\right) &= \sum_{k=1}^{n-1} \ln\left(1+\frac{k}{R\max\{1,\mu\}(n-1)}\right) \\ &\ge (n-1)\sum_{k=1}^{n-1}\int_{(k-1)/(n-1)}^{k/(n-1)} \ln\left(1+\frac{x}{R\max\{1,\mu\}}\right)\ \mathrm{d}x \\ &= (n-1)\int_0^1 \ln\left(1+\frac{x}{R\max\{1,\mu\}}\right)\ \mathrm{d}x \\ &= (n-1)\left[(1+R\max\{1,\mu\})\ln\left(1+\frac{1}{R\max\{1,\mu\}}\right)-1\right] \end{aligned}$$

and, taking into account that

$$(1+x)\ln\left(1+\frac{1}{x}\right) > 1 \quad \text{for } x>0\,,$$

we end up with

$$a_{n,n} \ge \exp\left\{(n-1)\left[(1+R\max\{1,\mu\})\ln\left(1+\frac{1}{R\max\{1,\mu\}}\right)-1\right]\right\} = 1+\nu_n\,.$$

We thus have

$$\Phi_n(X) \ge \nu_n X_1^n + \sum_{j=0}^n \binom{n}{j} X_1^j X_2^{n-j} = \nu_n X_1^n + (X_1+X_2)^n\,,$$

and the proof is complete. □

Appendix B. An auxiliary elliptic system

In this appendix, we establish Lemma B.1, which is an important argument in the proof of Lemma 2.2. Let $\tau>0$, $B=(b_{jk})_{1\le j,k\le 2} \in C(\mathbb{R}^2,\mathbf{M}_2(\mathbb{R}))$, and $A=(a_{jk})_{1\le j,k\le 2}\in \mathbf{SPD}_2(\mathbb{R})$ satisfy $AB(X)\in\mathbf{SPD}_2(\mathbb{R})$ for all $X\in\mathbb{R}^2$ and assume that there is $\delta_1>0$ with the property

$$\langle AB(X)\xi,\xi\rangle \ge \delta_1|\xi|^2\,, \qquad (X,\xi)\in\mathbb{R}^2\times\mathbb{R}^2\,. \tag{B.1}$$

Since $A\in\mathbf{SPD}_2(\mathbb{R})$, there is also $\delta_2>0$ such that

$$\langle A\xi,\xi\rangle \ge \delta_2|\xi|^2\,, \qquad \xi\in\mathbb{R}^2\,. \tag{B.2}$$

Here, Ω is a bounded interval of $\mathbb{R}$ ($d=1$) and we recall that, in that specific case, $H^1(\Omega)$ embeds continuously in $L_\infty(\Omega)$, so that there is $\Lambda>0$ with

$$\|z\|_\infty \le \Lambda\|z\|_{H^1}\,, \qquad z\in H^1(\Omega)\,. \tag{B.3}$$

Lemma B.1. *Given $U \in L_2(\Omega, \mathbb{R}^2)$, there is a solution $u \in H^1(\Omega, \mathbb{R}^2)$ to the nonlinear elliptic equation*

$$\int_\Omega [\langle u, v\rangle + \tau \langle B(u)\partial_x u, \partial_x v\rangle] \, \mathrm{d}x = \int_\Omega \langle U, v\rangle \, \mathrm{d}x \,, \qquad v \in H^1(\Omega, \mathbb{R}^2) \,. \tag{B.4}$$

Moreover, if

$$\begin{aligned} b_{11}(X) \ge b_{12}(X) = 0 \,, \qquad & X \in (-\infty, 0) \times \mathbb{R} \,, \\ b_{22}(X) \ge b_{21}(X) = 0 \,, \qquad & X \in \mathbb{R} \times (-\infty, 0) \,, \end{aligned} \tag{B.5}$$

and $U(x) \in [0, \infty)^2$ for a.a. $x \in \Omega$, then $u(x) \in [0, \infty)^2$ for a.a. $x \in \Omega$.

Proof. To set up a fixed point scheme, we define $\delta_0 := \min\{\tau\delta_1, \delta_2\}$ and introduce the compact and convex subset $\mathcal{K}$ of $L_2(\Omega, \mathbb{R}^2)$ defined by

$$\mathcal{K} := \left\{ u \in H^1(\Omega, \mathbb{R}^2) \ : \ \|u\|_{H^1} \le \frac{\|AU\|_2}{\delta_0} \right\} \,, \tag{B.6}$$

the compactness of $\mathcal{K}$ being a straightforward consequence of the compactness of the embedding of $H^1(\Omega, \mathbb{R}^2)$ in $L_2(\Omega, \mathbb{R}^2)$. According to (B.3),

$$\|u\|_\infty \le \frac{\Lambda \|AU\|_2}{\delta_0} \,, \qquad u \in \mathcal{K} \,. \tag{B.7}$$

We now consider $u \in \mathcal{K}$ and define a bilinear form b_u on $H^1(\Omega, \mathbb{R}^2)$ by

$$b_u(v, w) := \int_\Omega [\langle Av, w\rangle + \tau \langle AB(u)\partial_x v, \partial_x w\rangle] \, \mathrm{d}x \,, \qquad (v, w) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega, \mathbb{R}^2) \,.$$

Owing to (B.1) and (B.2),

$$b_u(v, v) \ge \delta_0 \|v\|_{H^1}^2 \,, \qquad v \in H^1(\Omega, \mathbb{R}^2) \,, \tag{B.8}$$

while the continuity of B and the boundedness (B.7) of u guarantee that

$$|b_u(v, w)| \le b_u^* \|v\|_{H^1} \|w\|_{H^1} \,, \qquad (v, w) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega, \mathbb{R}^2) \,,$$

where

$$b_u^* := 2 \max_{1 \le j,k \le 2} \{|a_{jk}|\} \left(1 + 2\tau \max_{1 \le j,k \le 2} \{\|b_{jk}(u)\|_\infty\} \right) \,.$$

We then infer from Lax-Milgram's theorem that there is a unique $\mathcal{V}[u] \in H^1(\Omega, \mathbb{R}^2)$ such that

$$b_u(\mathcal{V}[u], w) = \int_\Omega \langle AU, w\rangle \, \mathrm{d}x \,, \qquad w \in H^1(\Omega, \mathbb{R}^2) \,. \tag{B.9}$$

In particular, taking $w = \mathcal{V}[u]$ in (B.9) and using (B.8) and Hölder's inequality give

$$\delta_0 \|\mathcal{V}[u]\|_{H^1}^2 \le b_u(\mathcal{V}[u], \mathcal{V}[u]) \le \|AU\|_2 \|\mathcal{V}[u]\|_2 \le \|AU\|_2 \|\mathcal{V}[u]\|_{H^1} \,.$$

Consequently,

$$\|\mathcal{V}[u]\|_{H^1} \le \frac{\|AU\|_2}{\delta_0} \quad \text{and} \quad \mathcal{V}[u] \in \mathcal{K} \,. \tag{B.10}$$

We now claim that the map $\mathcal{V}$ is continuous on $\mathcal{K}$ with respect to the norm-topology of $L_2(\Omega, \mathbb{R}^2)$. Indeed, consider a sequence $(u_j)_{j \ge 1}$ in $\mathcal{K}$ and $u \in \mathcal{K}$ such that

$$\lim_{j \to \infty} \|u_j - u\|_2 = 0 \,.$$

Upon extracting a subsequence (not relabeled), we may assume that

$$\lim_{j\to\infty} u_j(x) = u(x) \ \text{ for a.a. } \ x \in \Omega\,,$$

so that the continuity of B and (B.7) ensure that

$$\lim_{j\to\infty} B(u_j(x)) = B(u(x)) \ \text{ for a.a. } \ x \in \Omega \tag{B.11a}$$

and

$$\max\left\{\|B(u)\|_\infty, \sup_{j\ge 1}\{\|B(u_j)\|_\infty\}\right\} \le \max_{|X|\le \Lambda\|AU\|_2/\delta_0}\{|B(X)|\}\,. \tag{B.11b}$$

It also follows from (B.10) and the compactness of the embedding of $H^1(\Omega,\mathbb{R}^2)$ in $L_2(\Omega,\mathbb{R}^2)$ that there is $v\in H^1(\Omega,\mathbb{R}^2)$ such that, after possibly extracting a further subsequence,

$$\lim_{j\to\infty}\|\mathcal{V}[u_j]-v\|_2 = 0 \qquad \text{and} \qquad \mathcal{V}[u_j] \rightharpoonup v \ \text{ in } \ H^1(\Omega,\mathbb{R}^2)\,. \tag{B.12}$$

Since

$$\int_\Omega \langle AB(u_j)\partial_x\mathcal{V}[u_j], \partial_x w\rangle \ \mathrm{d}x = \int_\Omega \langle \partial_x\mathcal{V}[u_j], AB(u_j)\partial_x w\rangle \ \mathrm{d}x\,,$$

due to the symmetry of $AB(X)$ for $X\in\mathbb{R}^2$, it readily follows from (B.11), (B.12), and Lebesgue's dominated convergence theorem that we may pass to the limit $j\to\infty$ in the variational identity (B.9) for $\mathcal{V}[u_j]$ and conclude that

$$b_u(v,w) = \int_\Omega \langle AU, w\rangle \ \mathrm{d}x\,, \qquad w\in H^1(\Omega,\mathbb{R}^2)\,,$$

that is, $v=\mathcal{V}[u]$. We have thus shown that any subsequence of $(\mathcal{V}[u_j])_{j\ge1}$ has a subsequence that converges to $\mathcal{V}[u]$, which proves the claimed continuity of the map $\mathcal{V}$. We are therefore in a position to apply Schauder's fixed point theorem, see [8, Theorem 11.1] for instance, and conclude that the map $\mathcal{V}$ has a fixed point $u\in\mathcal{K}$. In particular, the function u satisfies

$$b_u(u,w) = \int_\Omega \langle AU, w\rangle \ \mathrm{d}x\,, \qquad w\in H^1(\Omega,\mathbb{R}^2)\,.$$

Now, given $v\in H^1(\Omega,\mathbb{R}^2)$, the function $w=A^{-1}v$ also belongs to $H^1(\Omega,\mathbb{R}^2)$ and we infer from the above identity and the symmetry of A that

$$\begin{aligned}\int_\Omega \langle U, v\rangle \ \mathrm{d}x &= \int_\Omega \langle AU, w\rangle \ \mathrm{d}x = b_u(u,w) = b_u(u,A^{-1}v)\\ &= \int_\Omega \left[\langle u,v\rangle + \tau\langle B(u)\partial_x u, \partial_x v\rangle\right] \ \mathrm{d}x\,.\end{aligned}$$

We have thus constructed a solution $u\in H^1(\Omega,\mathbb{R}^2)$ to (B.4).

We now turn to the non-negativity-preserving property and assume that $U(x)\in[0,\infty)^2$ for a.a. $x\in\Omega$. Let $u\in H^1(\Omega,\mathbb{R}^2)$ be a solution to (B.4) and set $\varphi := -u$. Then $(\varphi_{1,+},\varphi_{2,+})$ belongs to $H^1(\Omega,\mathbb{R}^2)$ and it follows from (B.4) that

$$\begin{aligned}&\int_\Omega \left(\varphi_1\varphi_{1,+} + \varphi_2\varphi_{2,+} + \tau\sum_{j,k=1}^2 b_{jk}(u)\partial_x\varphi_k\partial_x(\varphi_{j,+})\right) \ \mathrm{d}x\\ &\qquad\qquad = -\int_\Omega \left(U_1\varphi_{1,+} + U_2\varphi_{2,+}\right) \ \mathrm{d}x \le 0\,.\end{aligned} \tag{B.13}$$

We now infer from (B.5) that

$$\begin{aligned} b_{11}(u)\partial_x\varphi_1\partial_x\varphi_{1,+} &= b_{11}(u)\mathbf{1}_{(-\infty,0)}(u_1)|\partial_x u_1|^2 \ge 0\,,\\ b_{12}(u)\partial_x\varphi_2\partial_x\varphi_{1,+} &= b_{12}(u)\mathbf{1}_{(-\infty,0)}(u_1)\partial_x u_1\partial_x u_2 = 0\,,\\ b_{21}(u)\partial_x\varphi_1\partial_x\varphi_{2,+} &= b_{21}(u)\mathbf{1}_{(-\infty,0)}(u_2)\partial_x u_1\partial_x u_2 = 0\,,\\ b_{22}(u)\partial_x\varphi_2\partial_x\varphi_{2,+} &= b_{22}(u)\mathbf{1}_{(-\infty,0)}(u_2)|\partial_x u_2|^2 \ge 0\,, \end{aligned}$$

so that the second term on the left-hand side of (B.13) is non-negative. Consequently, (B.13) gives

$$\int_\Omega \left(|\varphi_{1,+}|^2 + |\varphi_{2,+}|^2\right)\, \mathrm{d}x \le 0\,,$$

which implies that $\varphi_{1,+} = \varphi_{2,+} = 0$ a.e. in Ω. Hence, $u(x) \in [0,\infty)^2$ for a.a. $x \in \Omega$ and the proof of Lemma B.1 is complete. □

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