Internal Waves under Ice Cover

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Abstract :

A theoretical model for propagation of internal waves under an ice cover is developed. The sea water is considered to be inviscid, non-rotating, and incompressible and the Brunt–Väisälä frequency is supposed to be constant. The ice is considered of uniform thickness, with constant values of Young's modulus, Poisson's ratio, density and compressive stress in the ice. The boundary conditions are such that the normal velocity at the bottom is zero and, at the undersurface of the ice, the linearized kinematical and dynamic boundary conditions are satisfied. We present and analyze explicit solutions for the internal waves under the ice cover and the dispersion equations. It is shown that when the frequency is near, but smaller than the Brunt–Väisälä frequency the ice deflections can be considerable. The theoretical results are compared with experimental data for the Arctic regions.

Key-words :

ice deflections; internal waves; Brunt–Väisälä frequency

1 Introduction

The influence of the ice cover on the propagation of internal waves is practically important, but poorly studied. It is a common assumption that the rigid lid approximation that filters out the surface mode and adequately describes the properties of internal waves should also be valid in the case, where the free surface is substituted by an ice cover. Since the vertical velocity at the surface is zero in the approximation, the internal waves cannot be recorded on the basis of fluctuations of the ice cover.

Such a conclusion, however, contradicts the data from previous observations [Czipott et al. (1991); Smirnov (1972); Smirnov, Savchenko (1972); Smirnov et al (2002)]. For example, oscillations of the ice cover recorded on the "Severnyi Polyus-20" ice drifting station in 1970 [Smirnov (1972)] cannot be related to surface flexural–gravity waves. They were interpreted as manifestations of internal waves in conditions with clearly manifested interleaving of waters in the Arctic Basin. Therefore, it is necessary to develop a more specific theory of internal waves applied to the waves in a stratified ocean covered with ice. In this paper we develop a theory of internal waves under an ice cover at constant Brunt–Väisälä frequency.

2 Formulation of the problem

Let us consider water motion under an ice cover, which is considered to be a thin elastic plate of constant thickness floating at the sea surface. We will consider wave motions as small deviations from the hydrostatic equilibrium state at zero background currents in the ocean. Then, the linearized system of equations of motion for non-rotating stratified fluid is written as [Pedlosky (2003)]:

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0}\nabla P\,;\ \frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\left(\frac{\partial P}{\partial z} + g\rho_w\right); \tag{1}$$

$$\nabla \cdot \mathbf{u} + \frac{\partial w}{\partial z} = 0\,;\ \frac{1}{\rho_0}\frac{\partial \rho_w}{\partial t} - \frac{N^2}{g} w = 0. \tag{2}$$

Here $\mathbf{u} = (u,\, v)$ is the vector of horizontal velocity (u is a component along the x axis, v is component along the y axis), w is vertical velocity (the z axis is directed vertically upwards), P is the pressure deviation from hydrostatic pressure, ρ_w is the water density deviation from the hydrostatic value $\rho_0(z)$, g is the gravity acceleration, $N(z) = \sqrt{-gd\rho_0(z)/\rho_0(z)dz}$ is the Brunt–Väisälä frequency, $\nabla = \left(\partial/\partial x,\, \partial/\partial y\right)$.

Equations (1) and (2) should be supplemented with boundary conditions at the bottom $z = -H(x,y)$ and at the lower surface of the ice $z = \eta(x,y,t)$, where $\eta(x,y,t)$ is the deflection of the ice cover. No flux condition $u\partial H/\partial x + v\partial H/\partial y + w = 0$ should be satisfied at the bottom, and kinematic $\partial\eta/\partial t = w$ and dynamical $P - g\int_0^\eta \rho_0(z)\,dz = P_a$ conditions should be satisfied at the ice surface $z = \eta(x,y,t)$, where $P_a = P_a(x,y,t)$ is the pressure at the water–ice boundary.

From hereon we will consider for simplicity that Brunt–Väisälä frequency N is constant. It is convenient to reduce equations (1) and (2) to one equation for pressure:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 P}{\partial z^2} + \frac{N^2}{g}\frac{\partial P}{\partial z}\right) + \left(\frac{\partial^2}{\partial t^2} + N^2\right)\Delta P = 0, \tag{3}$$

where $\Delta = \nabla^2$. This is the main equation of the problem.

Boundary condition at $z = -H(x,y)$ as a function of pressure P is:

$$\left(\frac{\partial^2}{\partial t^2} + N^2\right)\left(\frac{\partial P}{\partial x}\frac{\partial H}{\partial x} + \frac{\partial P}{\partial y}\frac{\partial H}{\partial y}\right) + \frac{\partial^3 P}{\partial z \partial t^2} = 0. \tag{4}$$

To proceed further we need a relation between the deflection of the ice cover $\eta(x,y,t)$ and the pressure $P\big|_{z=0}$ at the ice surface. From the kinematic condition, we get:

$$\left(\frac{\partial^2}{\partial t^2} + N^2\right)\eta = -\frac{1}{\rho_0(0)}\frac{\partial P}{\partial z}\bigg|_{z=0}. \tag{5}$$

From the dynamical condition in linear approximation, we have $P(x,y,0,\,t) - g\,\rho_0(0)\,\eta(x,y,t) = P_a$. If model ice as a thin horizontal elastic plate of constant thickness h, then we can find the pressure P_a at the lower boundary of the ice cover [Landau, Lifshits (1986); Liu, Mollo-Christensen (1988)]:

$$\frac{1}{\rho_0(0)}P_a = B\Delta^2\eta + Q\Delta\eta + M\frac{\partial^2\eta}{\partial t^2}, \tag{6}$$

where $B = \dfrac{Eh^3}{12(1-s^2)\rho_0(0)}$, $Q = \dfrac{Kh}{\rho_0(0)}$, $M = \dfrac{\rho_I h}{\rho_0(0)}$. Here, coefficient B is the cylindrical rigidity of ice, E is the Young's modulus of elasticity, K is the coefficient of ice compression, $\rho_I = \text{const}$ is the ice density. The terms proportional to B, M, and Q arise due to elastic properties of the ice, inertia forces, and compression–divergence forces applied to the ice cover, respectively. The characteristic values of these variables for ice are as follows [Liu, Mollo-

Christensen (1988)]: $E = 6\times10^9$ N/m^2; $s = 0.3$; $K = 10^6$ N/m^2; $\rho_0(0) = 1025$ kg/m^3; $\rho_I = 0.9\rho_0(0)$. For ice thickness $h = 1$ m, we have $B \approx 5\times10^6$ m^5/s^2, $Q \approx 10^3$ m^3/s^2, $M = 0.9$ m.

Taking equations (5) and (6) and the dynamical condition into account, we get the unique boundary condition at the lower ice surface in terms of pressure only:

$$\left[\left(\frac{\partial^2}{\partial t^2} + N^2\right)P + \left(g + B\Delta^2 + Q\Delta + M\frac{\partial^2}{\partial t^2}\right)\frac{\partial P}{\partial z}\right]_{z=0} = 0. \qquad (7)$$

A similar boundary condition was used in the study of edge waves under ice at a coast with a sloping beach [Muzylev (2006)].

3 Internal waves in an ocean of constant depth

Consider an ocean of constant depth in detail as a model example. In this case, internal waves under the ice cover obey the equation

$$\frac{\partial^4 P}{\partial t^2\,\partial z^2} + \left(\frac{\partial^2}{\partial t^2} + N^2\right)\Delta P = 0 \qquad (8)$$

with boundary conditions (7) and $\left.\partial P/\partial z\right|_{z=-H} = 0$. Term $N^2/(g\,\partial P/\partial z)$ in equation (3) in the majority of cases is small compared to $\partial^2 P/\partial z^2$, thus it is neglected in (8) (the Boussinesq approximation). If we do not use the Boussinesq approximation, no principal difficulties arise.

We seek the solution of the formulated problem in the form of a plane wave propagating in the horizontal direction $P(x, y, z, t) = e^{i(\mathbf{kr}-\omega t)}p(z)$, where $\mathbf{k} = (k_x, k_y)$ is the wave vector, $\mathbf{r} = (x, y)$, and ω is the frequency. Then we get a boundary value problem for function $p(z)$:

$$\frac{d^2 p}{d z^2} + \frac{N^2 - \omega^2}{\omega^2}k^2 p = 0\,; \qquad (9)$$

$$\left[\left(g + Bk^4 - Qk^2 - M\omega^2\right)\frac{d\,p}{d\,z} + \left(N^2 - \omega^2\right)p\right]_{z=0} = 0\,;\ \left.\frac{\partial p}{\partial z}\right|_{z=-H} = 0. \qquad (10)$$

Here $k = \sqrt{k_x^2 + k_y^2}$.

The boundary condition at $z = -H$ is satisfied if we determine the solution of equation (9) as $p(z) = a\cos\left[\lambda\left(H + z\right)\right]$, where $\lambda = \sqrt{\left(N^2 - \omega^2\right)k^2/\omega^2}$ and a is an amplitude. Then, from (10) we get the dispersion equation

$$\lambda\,\mathrm{tg}\,\lambda H = \frac{N^2 - \omega^2}{g + Bk^4 - Qk^2 - M\omega^2}. \qquad (11)$$

In the absence of ice, i.e. when $h = 0$ (in this case $B = Q = M = 0$), the dispersion relation (11) is transformed to the well-known dispersion equation for gravity waves in stratified ocean [Gill (1982); LeBlond, Mysak (1978); Orlova (2000); Pedlosky (2003)].

When $\omega > N$, the parameter λ becomes purely imaginary and, in the case, equation (11) has only one real root $\omega_0(k)$ corresponding to the surface wave. Below, we are interested only in internal waves and put $\omega < N$. Equation (11) implicitly relates frequency $\omega_n(k)$, $n = 1, 2, \ldots$, and wavenumber k for different modes of internal gravity waves under an ice cover of constant thickness h. The dispersion curves (Fig. 1) do not practically differ from dispersion curves for internal waves in ice-free conditions.

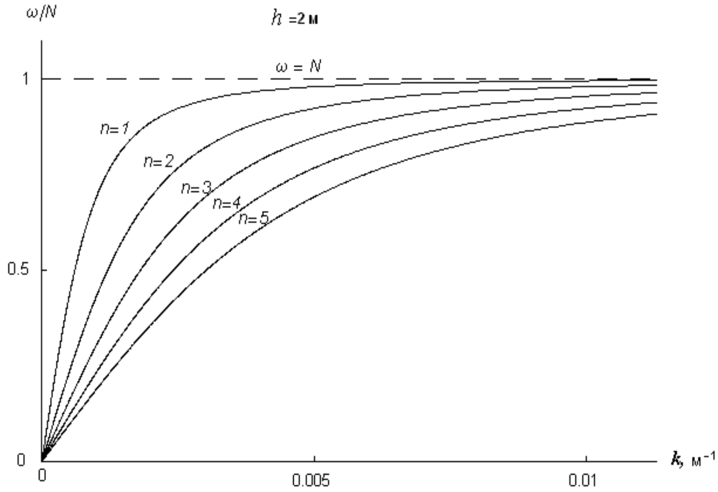

Fig. 1 – Dispersion curves $\omega_n(k)$ for internal gravity waves under an ice cover of constant thickness $h = 2\,\mathrm{m}$. The plots were made for an ocean with depth $H = 3000\,\mathrm{m}$ and Brunt–Väisälä frequency $N = 0.001\ \mathrm{s}^{-1}$.

However, it is not the case for the vertical velocity and deflection of the ice cover. We find vertical velocity from (1) and (2):

$$w(\omega,\, k;\, z) = -i\frac{a\,\omega}{\rho_0(0)}\frac{\lambda \sin\left[\lambda\left(H+z\right)\right]}{N^2-\omega^2}e^{i(\mathbf{kr}-\omega t)}. \tag{12}$$

In the rigid lid approximation, the vertical component of velocity w at the surface $z = 0$ is zero. Therefore, we get from (12) that $\sin\lambda H = 0$, so

$$\omega_n^2\left(k\right) = \frac{k^2H^2}{n^2\pi^2+k^2H^2}N^2. \tag{13}$$

At frequencies close to the Brunt–Väisälä frequency (i.e. at $\omega \approx N$), indefiniteness appears in relation (12).

The same situation appears in the consideration of the relation for deflection. Indeed, from equations (5) and (6) we get:

$$\eta(\omega,\, k) = \frac{a}{\rho_0(0)}\frac{\lambda\sin\lambda H}{N^2-\omega^2}e^{i(\mathbf{kr}-\omega t)} \quad \text{and}$$
$$\eta(\omega,\, k) = \frac{a}{\rho_0(0)}\frac{\cos\lambda H}{g+Bk^4-Qk^2-M\omega^2}e^{i(\mathbf{kr}-\omega t)}. \tag{14}$$

By virtue of the dispersion equation (12), these relations coincide.

Deflections η_n (n is mode number) of the ice cover caused by the internal waves as functions of normalized frequency $\sigma = \omega/N$ are shown in Fig. 2. We see that at the frequencies ω close to N, deflection of ice has a maximum. The maximum sharpness increases with decreasing ice cover thickness. The maximum is displaced to smaller frequencies with increasing mode number. One can expect that the oceanic internal waves under ice manifest themselves most clearly at the frequencies corresponding to such a maximum.

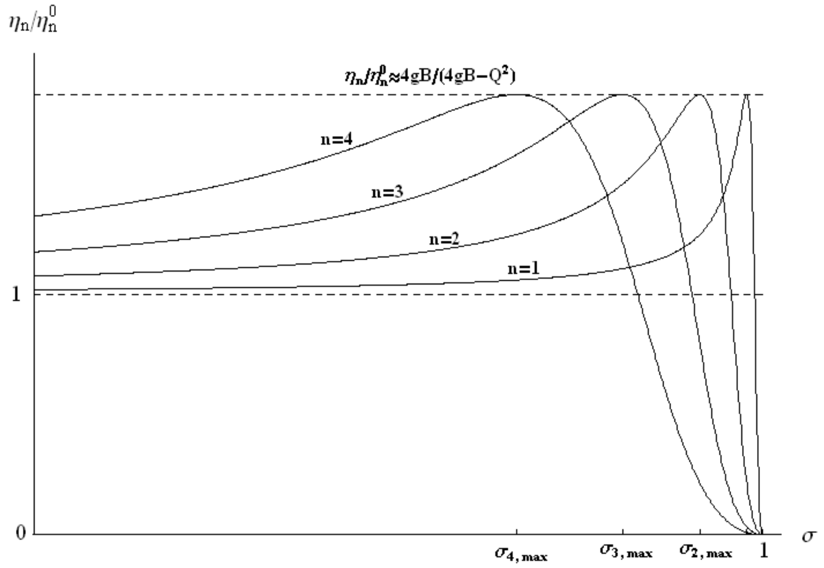

Fig. 2 – Deflections η_n caused by internal gravity waves under ice cover as a function of normalized frequency $\sigma = \omega / N$. Each profile is normalized by amplitude η_n^0 at $\omega = 0$.

Since the rigid lid approximation describes well the dispersion curves of internal waves under ice, we will use it to simplify the second equation (14). Then

$$\eta_n(\omega, k) \approx \frac{a}{\rho_0(0)} \frac{(-1)^n}{g + Bk^4 - Qk^2 - M\omega_n^2(k)} e^{i(\mathbf{kr} - \omega_n t)}, \tag{15}$$

where $\omega_n^2(k)$ is determined by relation (13). It follows from (15) that the wavenumber k_{max}, at which the extreme flexure is observed, can be found from equation:

$$2Bk_{max}^2 - Q = MN^2H^2 \frac{n^2\pi^2}{\left(n^2\pi^2 + k_{max}^2 H^2\right)^2}. \tag{16}$$

Since, in real conditions, the right part of (16) is small, we get $k_{max} \approx \sqrt{Q/2B}$. Owing to (13), the corresponding frequency $\omega_{n,max}$ and phase velocity $c_{n,max}$ are equal to

$$\omega_{n,max} \approx NH\sqrt{\frac{Q}{2n^2\pi^2 B + QH^2}}; \quad c_{n,max} \approx NH\sqrt{\frac{2B}{2n^2\pi^2 B + QH^2}}, \quad n = 1, 2, \dots. \tag{17}$$

It is easy to see that for all values of parameters of the problem considered, $\omega_{n,\,max} < N$.

It is interesting that the wave characteristics for maximal flexure do not depend on the gravity acceleration *g*, but depend only on the elastic properties of the ice surface, Brunt–Väisälä frequency, and ocean depth.

4 Comparison of the theoretical results with the data of observations

The developed model of internal waves under ice is simplified and schematic. However, it predicts the possibility of recording oceanic internal waves using measurements of the oscillations of the ice cover.

Let us compare numerical values of the characteristics of ice cover oscillations based on the results of the suggested model with the data of observations in the Arctic Ocean [Smirnov et al (2002)]. From the data reported in Smirnov et al (2002), it is difficult to calculate Brunt–Väisälä frequency, thus we will assume that it is equal to 0.005 s^{-1}. Then, in the case of compact Arctic fields that are 3 m thick, observations indicate that the period of waves is equal to 24 min. From the theoretical formulae (17), we get that $T_{max} = 2\pi / \omega_{max} = 22\ \mathrm{min}$ for the first mode. For the ice in the Central Arctic Basin, whose thickness reaches 7 m, the wave period according to the observations can range from 3 to 80 min. If we assume that the Brunt–Väisälä frequency is in the range from 0.002 to 0.05 s^{-1}, the theory allows us to determine that $4\ \mathrm{min} < T_{max} < 93\ \mathrm{min}$ for the first mode. We see that field and theoretical results are quite comparable.

5 Conclusions

A theoretical model describing propagation of internal waves under an ice cover in an ocean of constant depth was developed. The Brunt–Väisälä frequency was assumed to be constant. According to the obtained results, the oscillations of the ice surface with a frequency close, but smaller than the Brunt–Väisälä frequency can have amplitudes sufficient for recording internal waves. In particular, this means that application of the rigid lid approximation for vertical velocity in the problems with ice can lead to wrong conclusions. The comparison between the theory and observations showed that their agreement is satisfactory.

Acknowledgments

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