Closed ray affine manifolds

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Abstract

We consider closed manifolds that possess a so called rank one *ray structure*. That is a (flat) affine structure such that the linear part of the holonomy is given by products of a diagonal transformation and a commuting rotation.

We show that closed manifolds with a rank one ray structure are either complete or their developing map is a cover onto the complement of an affine subspace.

We prove, in the line of Markus conjecture, that if the rank one ray geometry has parallel volume, then closed manifolds are necessarily complete. Finally, we show that if the automorphism group of a closed manifold is non-compact then the manifold is complete.

1 Introduction

The class of affine manifolds is still rich of open questions. If we consider a closed affine manifold, *under which conditions is it (geodesically) complete? what happens when it is incomplete?*

In 1962, Markus [Mar62] conjectured that *a closed affine manifold with parallel volume should always be complete*. Although important cases of this conjecture are known [Smi77; FGH81; GH86; Fri86; Car89; JK04; Tho15], it is still open and remains a central challenge.

Carrière [Car89] made a study about affine manifolds that have a 1-*discompacity* holonomy group. This hypothesis means that a sequence of affine transformations should never have more than one singular value tending to zero. Carrière proved Markus conjecture under that condition. But in larger discompacity, it is still open.

The incompleteness of a manifold is also an interesting phenomenon. Among incomplete manifolds, some are nonetheless Kleinian: they can be obtained as the quotient of a non-maximal open. Some are close to be Kleinian and have their developing map that is a cover onto its image. (When the image is simply connected, it implies that it is in fact Kleinian.)

Fried [Fri80] has shown that incomplete closed similarity manifolds are always radiant: their developing map is a cover onto the complement of a single point.

Fried and Carrière have different objectives (one studies the incompleteness of similarity manifolds, the other shows the completeness of some closed affine manifolds). But they share a common point: the study of a dynamical system associated

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to the incompleteness of a geodesic curve. Carrière [Car89] refers to Fried [Fri80] for this consideration and therefore we call a *Fried dynamics* such a dynamical system.

The affine geometry is described by the affine group $\mathrm{GL}_n(\mathbf{R}) \rtimes \mathbf{R}^n$ acting on the space $\mathbf{R}^n$. In the isotropy group $\mathrm{GL}_n(\mathbf{R})$, we can consider the Cartan decomposition $\mathrm{GL}_n(\mathbf{R}) = KAK$. The subgroup A is diagonal and represents the singular values of a considered transformation. The subgroup K is $\mathrm{O}(n)$. The complexity of a Fried dynamics is closely related to the projection in A of the linear holonomy.

Fried supposes that A is reduced to $A_F = \{\lambda \text{ id}\}$ (the homotheties). Note that in this case, $K = \mathrm{O}(n)$ centralizes A_F. Carrière supposes that A is reduced to $A_C = \{\mathrm{diag}(\lambda, 1, \dots, 1, \lambda^{-1})\} \subset \mathrm{SL}_n(\mathbf{R})$ the set of diagonal transformations with at most one singular value tending to zero.

Note that in both cases, $\dim A_F = \dim A_C = 1$.

We can give an interpretation of both hypotheses. Fried's hypothesis gives the following property: with a diverging sequence $f_n \in A_F$ either tends to the null-application 0 or f_n^{-1} does. A subgroup $A_D \subset A$ verifying this property for any sequence is said to be a *dilation* subgroup and is necessarily 1-dimensionnal. In [Ale21a] we proved Fried's theorem for any dilation subgroup. Carrière's hypothesis allows the following: with a diverging sequence $f_n \in A_C$, an open ball tends (up to a subsequence) to a 1-codimensionnal ellipsoid. Again, this property can only be verified for any sequence if A_C has dimension 1.

We say that $K_R A_R \subset KAK$ is a *ray affine* isotropy if $K_R \subset K$ centralizes a subgroup $A_R \subset A$. We think that *ray geometries* $(K_R A_R \rtimes \mathbf{R}^n, \mathbf{R}^n)$ form an important class of affine geometries.

Those isotropies do not rely on either being a dilation group or being a 1-discompacity subgroup. For instance $A_R = \{\mathrm{diag}(\lambda, \lambda, \lambda^{-2})\} \subset \mathrm{SL}_3(\mathbf{R})$ is not a dilation subgroup and has 2-discompacity.

We propose to study the subgroups $K_R A_R \subset KAK$ when $A_R \subset A$ is any 1-dimensional subgroup. We will say in this case that $K_R A_R \rtimes \mathbf{R}^n = G_1$ has *rank one*. In general, the *rank* of the ray structure is the dimension of A_R.

The following result puts in a same perspective Fried's and Carrière's theorems.

Theorem (3.26)**.** *Let $(G_1, \mathbf{R}^n)$ be a rank one ray geometry. Let M be a closed $(G_1, \mathbf{R}^n)$-manifold. Then M is either complete or there exists an affine subspace $I \subset \mathbf{R}^n$ such that the developing map $D \colon \widetilde{M} \to \mathbf{R}^n - I$ is a cover.*

With further analysis of I (proposition 3.27), we can fully recover Fried's theorem. With a classic argument by Goldman and Hirsch [GH84], there can be no reducible closed affine manifold with parallel volume. By consequence, it implies a new case of Markus conjecture. (Note that there is a parallel volume if and only if $A_R \subset \mathrm{SL}_n(\mathbf{R})$.)

Corollary (4.1)**.** *Let $(G_1, \mathbf{R}^n)$ be a rank one ray geometry with parallel volume. Every closed $(G_1, \mathbf{R}^n)$-manifold is complete.*

With the perspective of Markus conjecture, the existence of a parallel volume is an interesting condition that may imply the completeness of a closed manifold. Another one is the non-compactness of the automorphism group.

A "vague general conjecture" by D'Ambra and Gromov [DG91, p. 24] states that *a sufficiently large automorphism group should suffice to classify manifolds.* We prove the following.

Theorem (4.3)**.** *Let* $(G_1, \mathbf{R}^n)$ *be a rank one ray geometry. Let* M *be a closed* $(G_1, \mathbf{R}^n)$*-manifold. If* $\mathrm{Aut}(M)$ *is non-compact then* M *is complete.*

Interestingly, there are counter-examples in higher rank ray geometries. We will give an example of a radiant manifold (therefore incomplete) having a rank two ray structure with an automorphism group that is non-compact.

Therefore, theorem 4.3 suggests that the completeness of closed manifolds with large automorphism groups is a phenomenon very special to rank one ray geometries. It strongly contrasts with Markus conjecture which could be true in any rank.

Other geometries Miner [Min90] generalized Fried's theorem to the Heisenberg space instead of $\mathbf{R}^n$ ([Ale21a] generalized it to every Carnot group). We expect the study of ray structures to not be too much dependent on $\mathbf{R}^n$ but rather on the choice of a nilpotent space, such as the Heisenberg space. In [Ale21b] we examine this generalization to 2-step nilpotent spaces.

Organization of the paper In section 2, we introduce ray structures and convexity arguments. In section 3, we study Fried dynamics: those are the dynamics associated to an incomplete geodesic. We show theorem 3.26 on the completeness and incompleteness of closed manifolds. In section 4 we show the completeness in the case of parallel volume (corollary 4.1) and in the case of an automorphism group acting non-properly (theorem 4.3).

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2 Ray structures

Definition 2.1. *Let* $D\colon \widetilde{M} \to \mathbf{R}^n$ *be a local diffeomorphism. A curve* $\gamma\colon [0,1] \to \widetilde{M}$ *is* geodesic *if* $D(\gamma)$ *is a geodesic segment in* $\mathbf{R}^n$*.*

For any $p \in \widetilde{M}$*, define* $V_p \subset \mathrm{T}_p\widetilde{M}$ *the set of the vectors such that there exists a geodesic segment* $\gamma\colon [0,1] \to \widetilde{M}$ *with* $\gamma(0) = p$*,* $\gamma'(0) \in V_p$*. We say that* V_p *is the* visibility set *from (or of)* $p \in \widetilde{M}$*.*

Definition 2.2. *Let* $D\colon \widetilde{M} \to \mathbf{R}^n$ *be a local diffeomorphism. A subset* $C \subset \widetilde{M}$ *is* convex *if* D *is injective on* C *and* $D(C)$ *is convex. Let* $x \in \widetilde{M}$*, a point* $y \in \mathbf{R}^n$ *is* visible from x *if there exists a geodesic segment from* x *to* z *such that* $D(z) = y$*.*

Following Carrière [Car89] (see also [Ben60; Kos65]):

Proposition 2.3 ([Car89])**.** *Let* $D\colon \widetilde{M} \to \mathbf{R}^n$ *be a local diffeomorphism for any* $p \in \widetilde{M}$*.*

- *If* C_1, C_2 *are convex subsets in* $\widetilde{M}$ *and* $C_1 \cap C_2 \neq \emptyset$*, then* D *is injective on* $C_1 \cup C_2$*.*
- *The visible set* $V_p \subset \mathrm{T}_p\widetilde{M}$ *is open.*
- *Let* C *be convex and containing* $p \in \widetilde{M}$*. Then* $C \subset \exp_p(V_p)$*.*
- D *is injective on* $\exp_p(V_p)$ *for any* $p \in \widetilde{M}$*.*

- *If for $p \in \widetilde{M}$, $\exp_p(V_p)$ is convex then $\exp_p(V_p) = \widetilde{M}$.*

- *D is a diffeomorphism if and only if for any $p \in \widetilde{M}$, $V_p = \mathrm{T}_p\widetilde{M}$.*

Definition 2.4. *A manifold M is a (G,X)-manifold (or, equivalently, has a (G,X)-structure) for G a group acting (by analytic diffeomorphisms) on a space X if there exists (D,ρ) a pair of a local diffeomorphism $D\colon \widetilde{M} \to X$ (called the developing map) and a morphism $\rho\colon \pi_1(M) \to G$ (called the holonomy morphism) such that*

$$\forall \gamma \in \pi_1(M), \forall x \in \widetilde{M},\ D(\gamma \cdot x) = \rho(\gamma)D(x). \tag{1}$$

Let $\mathrm{Aff}(\mathbf{R}^n) = \mathbf{R}^n \rtimes \mathrm{GL}(\mathbf{R}^n)$. *We say that* $(\mathrm{Aff}(\mathbf{R}^n), \mathbf{R}^n)$ *is the* affine geometry *of* $\mathbf{R}^n$ *and if M has a* $(\mathrm{Aff}(\mathbf{R}^n), \mathbf{R}^n)$*-structure then it is an* affine manifold.

Definition 2.5. *Let M be an affine manifold. It is* complete *if $D\colon \widetilde{M} \to \mathbf{R}^n$ is a diffeomorphism.*

Definition 2.6. *A* ray geometry *on the space $\mathbf{R}^n$ is a model geometry $(G, \mathbf{R}^n)$ with $G = \mathbf{R}^n \rtimes KA$ and $KA \subset \mathrm{GL}(\mathbf{R}^n)$ such that the following conditions are verified.*

1. *The subgroup A is isomorphic to a multiplicative group $(\mathbf{R}_+^*)^r$, with r called the* rank *of the ray geometry. The subgroup K is compact and centralizes A.*
2. *There exists a fixed basis $(e_1, \dots, e_n)$ of $\mathbf{R}^n$ such that for an isomorphism $(\beta_1, \dots, \beta_r)\colon A \to (\mathbf{R}_+^*)^r$ there exist $d_{i,j} \in \mathbf{R}$ such that A consists of the transformations:*

$$\begin{pmatrix} \beta_1^{d_{1,1}} \cdots \beta_r^{d_{r,1}} & & \\ & \ddots & \\ & & \beta_1^{d_{1,n}} \cdots \beta_r^{d_{r,n}} \end{pmatrix}. \tag{2}$$

Limit sets Later one we will need to study sequences of subsets that have a limit.

Definition 2.7. *Let B_n be a sequence of subsets. Its* limit set, *denoted* $\lim B_n$, *is the set of the points* $\lim x_n$ *for sequences $x_n \in B_n$ (with x_n chosen for each B_n).*

Proposition 2.8. *Let $g_i \in \pi_1(M)$ be a sequence of transformations and $S \subset \widetilde{M}$ be closed and convex. Assume that $g_i S$ has a limit point $y \in \widetilde{M}$. Consider $B_\infty = \lim D(g_i S) =$* $\lim \rho(g_i)D(S)$ *the limit set in the developing map. Then B_∞ is closed, convex and there exists a closed and convex subset S_∞ containing y such that $D(S_\infty) = B_\infty$.*

Proof. Observe that by convexity, each $D(g_i S)$ is closed and convex. Therefore B_∞ is again closed. It is also convex. If two points x_1, x_2 belong to the limit set B_∞ then we consider their associated converging sequences. We can form geodesics in each $g_i S$ that joins the points from each sequence. The limit of those geodesics exists, is geodesic and connects x_1 and x_2.

Similarly, $S_\infty = \lim g_i S$ is closed and convex. We show its developing image covers B_∞. Let $y_i \in g_i S$ be a sequence tending to y. Let $z \in B_\infty$ be the limit of $D(z_i)$ for $z_i \in g_i S$. The geodesics γ_i from y_i to z_i are developed into $D(\gamma_i)$ and those tend to the geodesic from $D(y)$ to $D(z)$. This geodesic is compact (defined on $[0,1]$) and each γ_i is completely visible. Therefore $\lim \gamma_i$ corresponds to a visible geodesic issued from y. It necessarily ends at z by injectivity. □

3 Fried dynamics

Let M be an incomplete closed affine manifold. Choose any metric on M compatible with its topology. Denote $\pi\colon \widetilde{M} \to M$ its universal cover. If $x \in \widetilde{M}$ and $\gamma \subset \widetilde{M}$ is a geodesic issued from x, incomplete at $t = 1$, then the projection of γ in M is an open curve without any continuous completion at $t = 1$. Since M is closed, the projection $\pi(\gamma)$ has a recurrent point $y \in M$.

Let $U \subset M$ be a compact neighborhood of y, convex and trivializing the universal cover. For the choice of a decreasing sequence $\epsilon_i \to 0$ we can define t_i the time such that $\pi(\gamma(t_i))$ belongs to U and is at distance at most ϵ_i to $y \in M$. We ask that $\pi(\gamma)$ exits U between the times t_i and t_{i+1}. We have that $t_i \to 1$ since the geodesic is incomplete at $t = 1$.

We use the trivialization of the universal cover by U. It provides $U_i \subset \widetilde{M}$ such that $\gamma(t_i) \in U_i$. Let $y_i \in U_i \subset \widetilde{M}$ be the lifts of $y \in U \subset M$. Every U_i is again a compact convex neighborhood of y_i. Through the developing map $D\colon \widetilde{M} \to \mathbf{R}^n$, each $D(U_i)$ is compact, convex and they accumulate along the compactification $\overline{D(\gamma)}$ of $D(\gamma)$ at $t = 1$. Since the intersections of γ with the U_i's are transverse and disjoint along γ, the $D(U_i)$'s intersect transversally and disjointly $D(\gamma)$.

Definition 3.1. *Let $\gamma \subset \widetilde{M}$ be an incomplete geodesic at $t = 1$. Let $y \in M$ be a recurrent point of its projection into M. Let U be a convex, compact, trivializing neighborhood of y. Let $\epsilon_i \to 0$ be decreasing and let $t_i \to 1$ be an associated sequence of times such that (for the lifts $y_i \in U_i$ of $y \in U$) we have $\gamma(t_i) \in U_i$ and the distance between $\pi(\gamma(t_i))$ and y is lesser than ϵ_i. Define $g_{ji} \in \pi_1(M, y)$ the transformations of $\widetilde{M}$ verifiying*

$$g_{ji}(U_i) = U_j. \tag{3}$$

Those data define a Fried dynamics.

Note that a subsequence of the times $\{t_i\}$ (or the distances $\{\epsilon_i\}$) corresponds univocally to a subsequence of the pairs $\{y_i \in U_i\}$.

The transformations g_{ji} verify a cocycle property:

$$g_{ki} = g_{kj} g_{ji}. \tag{4}$$

For the affine geometry $(G, \mathbf{R}^n)$ considered, denote by T_{ji} the corresponding transformations by the holonomy morphism $T_{ji} = \rho(g_{ji}) \in G$.

Idea of the proof of theorem 3.26 To prove theorem 3.26 we use a strategy comparable to what Fried [Fri80] employed.

First, we study Fried dynamics. It is the most important step. Later we will introduce a convex subset S (with smooth boundary) that contains the invisible geodesic γ. The goal of this first step will be to be able to describe what is the shape of $T_{ji}^{-1}(S)$ when $j \gg i \to \infty$. This study is independent of the rank of the ray geometry.

Then, by using the convexity argument in proposition 2.8, we show how to construct from S a half-space of $\mathbf{R}^n$ that is completely visible. This step will depend on the fact that the ray geometry has rank 1.

Finally, we will show how the different half-spaces obtained by varying x and γ can be combined in $D(\widetilde{M})$. This will imply that the developing map avoids every invisible point, and therefore is a covering onto its image.

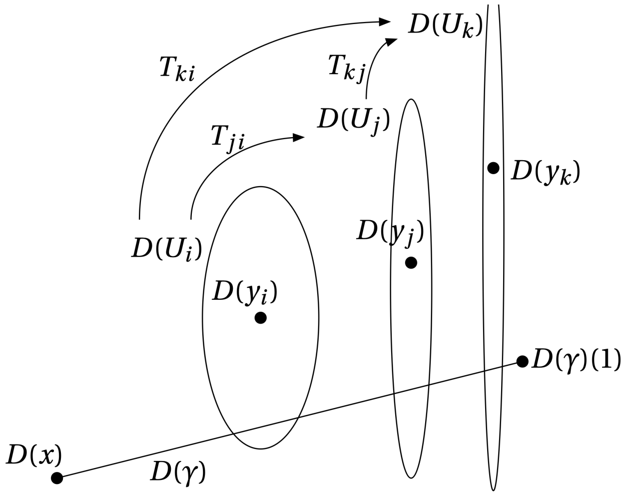

Figure 1: A Fried dynamics.

Now we return to the study of the Fried dynamics. We assume that $(G, \mathbf{R}^n)$ is a ray geometry. We have a basis $(e_1, \ldots, e_n)$ of $\mathbf{R}^n$ such that $A \subset KA \subset G$ acts diagonally. Once a base point of $\mathbf{R}^n$ is chosen, we can express any $T_{ji} \in G$ by

$$T_{ji}(x) = c_{ji} + f_{ji}(x), \tag{5}$$

with $c_{ji} \in \mathbf{R}^n$ and $f_{ji} \in KA$. Recall that a change of the base point is expressed by

$$T_{ji}(x) = T_{ji}(y - y + x) = (c_{ji} + f_{ji}(y)) + f_{ji}(-y + x). \tag{6}$$

It preserves the linear part f_{ji}. Decompose each f_{ji} into

$$f_{ji} = f_{ji,K} f_{ji,A} \tag{7}$$

with $f_{ji,K} \in K$ et $f_{ji,A} \in A$. Recall that K centralizes A, so both factors commute. The cocycle property on T_{ji} implies that each factor also verifies the cocycle relation:

$$f_{ki,K} = f_{kj,K} f_{ji,K}, \; f_{ki,A} = f_{kj,A} f_{ji,A} \tag{8}$$

Lemma 3.2. *Up to a subsequence of $\{y_i \in U_i\}$, we have*

$$\lim_{j\to\infty} \lim_{i\to\infty} f_{ji,K} = \mathrm{id}. \tag{9}$$

Let e_q be a basis vector. Denote by $\beta_{ji,q}$ the diagonal element of $f_{ji,A} \in A$ for the direction e_q. Up to a subsequence of $\{y_i \in U_i\}$, we have

$$\lim_{j\to\infty} \lim_{i\to\infty} \beta_{ji,q} = \omega_q \in \{0, 1, \infty\}. \tag{10}$$

Proof. Since $K \subset G$ is compact, we can suppose up to a subsequence of $\{y_i \in U_i\}$ that $f_{ji,K} \to L \in K$ when $j \gg i \to \infty$. But then the cocycle property implies $L^2 = L$ that can only be verified for $L = \mathrm{id}$. By an analogous argument, with the compactification of $\mathbf{R}_+$ into $\mathbf{R}_+ \cup \{\infty\}$ we have a limit $\beta_{ji,q} \to \omega_q$ that verifies $\omega_q^2 = \omega_q$, and it can only be true if $\omega_q \in \{0, 1, \infty\}$. □

It should be noted that if $\beta_{kj,q} \to \omega_q$ when $k \gg j \to \infty$, then we can obtain an information on how $\beta_{ki,q}$ can evolve when i is fixed and $k \to \infty$. Indeed, we have

$$\lim_{k\to\infty} \beta_{ki,q} = \lim_{k\to\infty}\lim_{j\to\infty} \beta_{ki,q} = \lim_{k\to\infty}\lim_{j\to\infty} \beta_{kj,q}\beta_{ji,q} = \omega_q \lim_{j\to\infty} \beta_{ji,q}. \tag{11}$$

If $\beta_{ki,q} \to r$ when $k \to \infty$ (up to a subsequence) then $r = \omega_q r$. If $\omega_q \in \{0,\infty\}$ it implies $r = \omega_q$.

Lemma 3.3. *Let $i > 0$ fixed. If $\beta_{ki,q} \to r$ and $\omega_q = 1$ then r must be a (finite) real positive number.*

Proof. Assume that $r = 0$, we show that $\omega_q = 0$. For any $j > i$ large enough, there exists $k > j$ such that $\beta_{ki,q} < \beta^2_{ji,q} < 1$. It implies that $\beta_{kj,q} = \beta_{ki,q}\beta^{-1}_{ji,q} < \beta_{ji,q} < 1$. Therefore $\beta_{kj,q}$ can only tend to $0 = \omega_q$. An analogous argument shows that if $r = \infty$ then $\omega_q = \infty$. □

Proposition 3.4. *There exists a subsequence of $\{y_i \in U_i\}$ such that:*

- *for $i > 0$ fixed, $f_{ji,K}$ converges;*
- *for $i > 0$ fixed, the sequence $\{\beta_{ji,q}\}$ is monotonic or constant for $j > i$.*

Once this proposition is proven, we will assume that we chose such a subsequence of $\{y_i \in U_i\}$.

Proof. For the first property, by compacity of K, we can assume that $f_{ji,K}$ converges since it must have an accumulation point in K.

For the last property, it can be done for a single $i_0 > 0$ up to a subsequence of $j > i_0$. Now let $i > i_0$. Then $\beta_{ji,q} = \beta_{ji_0,q}\beta^{-1}_{ii_0,q}$ and it must be again monotonic or constant. □

Note that the last property implies that $\beta_{ji,q} \to r$ and therefore the preceding discussion applies.

Definition 3.5. *We define a linear decomposition $E \oplus P \oplus F$ of $\mathbf{R}^n$ by deciding that $e_q \in E$ if $\omega_q = 0$, $e_q \in P$ if $\omega_q = 1$ and $e_q \in F$ if $\omega_q = \infty$.*

Since A acts diagonally and K centralizes A, we have that f_{ji} preserve the decomposition $E \oplus P \oplus F$.

Note that $\dim E > 0$ since $D(U_i)$ accumulates disjointly on $\overline{D(\gamma)}$. The other subspaces might be reduced to $\{0\}$.

If we chose any base point p, it gives three affine subspaces based at p:

$$\forall L \in \{E,F,P\} \forall p \in \mathbf{R}^n,\ L|_p := p + \exp(L). \tag{12}$$

We come back to M and discuss the choice of $U \subset M$. Choose $V \subset U$ a smaller neighborhood of y. Since we chose distances $\epsilon_i \to 0$, for $i \geq i_0$ large enough, every $\pi(\gamma(t_i))$ will also belong to V. Note that we can lift $V \subset M$ into $V_i \subset U_i \subset \widetilde{M}$.

Lemma 3.6. *For any $i > 0$, $g^{-1}_{ji}\gamma$ has y_i for limit point when $j \to \infty$.*

Proof. Let $i > 0$. Choose any neighborhood V_i of y_i contained in U_i. Then V_i corresponds to a neighborhood $V \subset U$ of y in M. Therefore for j large enough, V_j intersects γ and therefore $V_i = g^{-1}_{ji}V_j$ intersects $g^{-1}_{ji}\gamma$. □

Definition 3.7. *In M, let $U_1 = U$. Define for a sequence of $\{0 < r < 1\}$ tending to 0 a sequence $\{U_r\}$ of compact convex neighborhoods of $y \in M$. Assume that U_r is decreasing for the inclusion and that $U_r \to \{y\}$ when $r \to 0$. In $\widetilde{M}$, define $U_{1,i} = U_i$ and $U_{r,i}$ the lift of U_r such that $U_{r,i} \subset U_{1,i}$. In $D(\widetilde{M})$, define*

$$C_{r,i} = D(U_{r,i}). \tag{13}$$

Recall that $U_1 = U$ is a convex, compact and trivializing neighborhood of $y \in M$. Every $C_{r,i}$ is convex and compact in $\mathbf{R}^n$ since U_r is a convex, compact and trivializing neighborhood of y in M. By construction:

Lemma 3.8. *For any i, j and $r \geq 0$, $g_{ji} U_{r,i} = U_{r,j}$ and by consequence $T_{ji} C_{r,i} = C_{r,j}$.* □

As discussed before, since $U_r \subset U$ is a neighborhood of y, $D(\gamma)(1)$ is a limit point of $\{C_{r,i}\}$ since for $j > j_0$ large enough $D(\gamma(t_j))$ belongs to $C_{r,j}$.

Proposition 3.9. *Let $r > 0$. Any accumulation point of $\{C_{r,i}\}$ is a limit point.*

Proof. Choose any base point in $\mathbf{R}^n$. We know that $D(\gamma)(1)$ is a limit point. Let $y_j \in C_{r,j}$ be tending to $D(\gamma)(1)$. Each $y_j \in C_{r,j}$ can be written $c_{ji} + f_{ji}(x_{i,j})$ for $x_{i,j} \in C_{r,i}$. We can write any $z \in C_{r,i}$ as $x_{i,j} - x_{i,j} + z$ so that with $L_{r,i} = -x_{i,j} + C_{r,i}$ we have:

$$\begin{aligned} C_{r,j} = T_{ji}(C_{r,i}) &= c_{ji} + f_{ji}(C_{r,i}) & (14)\\ &= (c_{ji} + f_{ji}(x_{i,j})) + f_{ji}(-x_{i,j} + C_{r,i}) & (15)\\ &= y_j + f_{ji}(L_{r,i}). & (16) \end{aligned}$$

Therefore, since y_j converges, an accumulation point $z_{\sigma(j)} \to z$ of the sequence $\{C_{r,j}\}$ corresponds to an accumulation point $w_{\sigma(j)} \to w$ of $\{f_{ji}(L_{r,i})\}$. So we only need to prove that $\{f_{ji}(L_{r,i})\}$ has for limit set its set of accumulation points. Note that $0 \in L_{r,i}$ and $L_{r,i}$ is convex and compact. (The point $0 \in L_{r,i}$ corresponds in fact to y_j tending to $D(\gamma)(1)$.)

Let $i > 0$ and consider $j \geq j_0$. Then f_{ji} has its rotational part that converges and its diagonal factors that converges by monotonic or constant values. Recall that K centralizes A. So the action by the rotational part, $f_{ji,K}(L_{r,i})$, has a limit, say L_K. Now, the diagonal action is monotonic (or constant) in each direction, therefore any accumulation point of $f_{ji,A}(L_K)$ is in fact a limit point. □

Definition 3.10. *For any fixed value $r > 0$, let $C_{r,\infty}$ be the limit set of the sequence $\{C_{r,i}\}$.*

Note that when $r = 0$, each $C_{r,i}$ is reduced to $D(y_i)$. Those have no reason to accumulate to $D(\gamma)(1)$. This is why we asked r to be different from 0.

Definition 3.11. *Define*

$$C_{0,\infty} = \bigcap_{r>0} C_{r,\infty}. \tag{17}$$

This definition makes sense because the $C_{r,\infty}$ are decreasing for the inclusion when $r \to 0$.

Lemma 3.12. *All the sets $C_{r,\infty}$ and $C_{0,\infty}$ are convex. The set $C_{0,\infty}$ is nonempty (it contains $D(\gamma)(1)$) and is an affine subspace.*

Proof. Because every $C_{r,j}$ is convex, so are the limits $C_{r,\infty}$ and $C_{0,\infty}$. The fact that $D(\gamma)(1) \in C_{0,\infty}$ is clear. Now, observe that if $C_{0,\infty} = \{D(\gamma)(1)\}$ then the statement is true.

The set $C_{0,\infty}$ is closed and convex. To show that it is an affine subspace, we show that any maximal non-zero geodesic in $C_{0,\infty}$ is always open. Let $\gamma \subset C_{0,\infty}$. For any $r > 0$ there exists $z_i \in C_{r,i}$ and $w_i \in C_{r,i}$ such that the geodesic η_i from z_i to w_i tends to a geodesic containing (or equal) to γ. But then η_i is always contained in $C_{r+\epsilon,i}$ and can be extended to a strictly larger geodesic μ_i. Then μ_i tends to a geodesic in $C_{r+\epsilon,i}$ strictly containing γ. Because this can be done for any $\epsilon > 0$ small enough, γ is necessarily open if it is maximal. □

Lemma 3.13. *Choose $D(\gamma)(1)$ as base point of $\mathbf{R}^n$.*

$$\forall r > 0,\ C_{r,\infty} \subset (P \oplus F)|_{D(\gamma)(1)} \tag{18}$$

$$C_{0,\infty} = F|_{D(\gamma)(1)}. \tag{19}$$

Proof. Choose a $C_{r,i}$ for i large enough, so that T_{ji} tends to a transformation with a linear part with almost no rotation and diagonal elements tending to $\omega \in \{0,1,\infty\}$. The limit $T_{ji}C_{r,i} = C_{r,j} \to C_{r,\infty}$ can only contain geodesics in the directions where $\omega \neq 0$. So it is in $F \oplus P$ based at $D(\gamma)(1)$. If $r \to 0$, then the directions of P are contracted in $C_{r,\infty}$, so only F remains. □

Fixed point Now we return to the general study. The next step is to find an asymptotic fixed point for T_{ji}.

We have that for any $x \in \mathbf{R}^n$, there exists a unique decomposition $x = x_L + x_F$ with $x_L \in E \oplus P$ and $x_F \in F$.

Lemma 3.14. *Choose $D(\gamma)(1)$ as base point and decompose $T_{ji}(x) = c_{ji} + f_{ji}(x)$ with $f_{ji} \in KA$ and $c_{ji} \in \mathbf{R}^n$. Let $c_{ji} = c_{ji,L} + c_{ji,F}$ be the decomposition following $\mathbf{R}^n = L \oplus F$. Define $Q_{ji}(x) = c_{ji,F} + f_{ji}(x)$. Then*

$$\lim_{j\to\infty} T_{ji}(x) - Q_{ji}(x) = \lim_{j\to\infty} c_{ji,L} = 0 \tag{20}$$

and therefore this convergence is uniform for $x \in \mathbf{R}^n$.

Proof. For any $\epsilon > 0$, $T_{ji}(C_{\epsilon,i})$ has for limit $C_{\epsilon,\infty}$ when $j \to \infty$. Consider the linear decomposition $L \oplus F$ based at $D(\gamma)(1)$. When $\epsilon \to 0$ and $j \to \infty$, the coordinates of $T_{ji}(C_{\epsilon,i})$ all tend to 0 in the linear subspace $L \subset \mathbf{R}^n$. Indeed, $C_{0,\infty} = F|_{D(\gamma)(1)}$.

Note that $c_{ji,L}$ tends or not to 0 when $j \to \infty$ independently from the choice of ϵ.

Now decompose $T_{ji}(C_{\epsilon,i})$, and note that f_{ji} preserves the decomposition $L \oplus F$.

$$c_{ji} + f_{ji}(C_{\epsilon,i}) = c_{ji,L} + c_{ji,F} + f_{ji}(C_{\epsilon,i})_L + f_{ji}(C_\epsilon)_F \tag{21}$$

$$= \big(c_{ji,L} + f_{ji}(C_{\epsilon,i})_L\big) + \big(c_{ji,F} + f_{ji}(C_{\epsilon,i})_F\big) \tag{22}$$

The term $c_{ji,F} + f_{ji}(C_{\epsilon,i})_F$ is exclusively in F. Hence the coordinate in L is determined by $c_{ji,L} + f_{ji}(C_{\epsilon,i})_L$ and it must tend to 0 when $j \to \infty$ and $\epsilon \to 0$.

But for any $j > i$, f_{ji} acts as a contraction on $(C_{\epsilon,i})_L$. Hence, for any $j > i$ fixed, when $\epsilon \to 0$ we have $f_{ji}(C_{\epsilon,i})_L \to 0$. By consequence, when $j \to \infty$ and $\epsilon \to 0$, $c_{ji,L} \to 0$. It proves the lemma. □

Note that Q_{ji} has a fixed point on F, since F is preserved and f_{ji} acts by expansions on it.

Lemma 3.15. *Denote $q_{ji} \in F$ the fixed point of Q_{ji}. For $i > 0$, q_{ji} converges when $j \to \infty$ and we denote $q_i = \lim q_{ji}$. Also, $D(y_i) \in E|_{q_i}$.*

Proof. Choose q_{ji} as base point of $\mathbf{R}^n$ when we consider the transformation T_{ji}. We start by showing that $D(y_i)$ is an accumulation point of $E|_{q_{ji}}$.

Suppose that $D(y_i)$ is not an accumulation point of the stable subspace $(E \oplus P)|_{q_{ji}}$. Then there exists $\epsilon > 0$ so that $C_{\epsilon,i}$ avoids every $(E \oplus P)|_{q_{ji}}$ for j large enough. But then $T_{ji} C_{\epsilon,i} = C_{\epsilon,j}$ escapes to infinity and cannot accumulate at $D(\gamma)(1)$, impossible.

If $D(y_i)$ is an accumulation point of $(E \oplus P)|_{q_{ji}}$ then $D(y_i)$ is an accumulation point of $E|_{q_{ji}}$ since $C_{\epsilon,j}$ accumulates at $F|_{q_{ji}} = C_{0,\infty}$ and therefore has arbitrarily small coordinates along P.

Now we show that q_{ji} does not tend to infinity. Since $E|_{q_{ji}}$ accumulates at $D(y_i)$, it comes that q_{ji} must asymptotically be at the intersection of $E|_{D(y_i)} = \lim E|_{q_{ji}}$ with $C_{0,\infty} = F|_{q_{ji}}$. But this intersection is reduced to a single point, so $q_{ji} \to q_i$. □

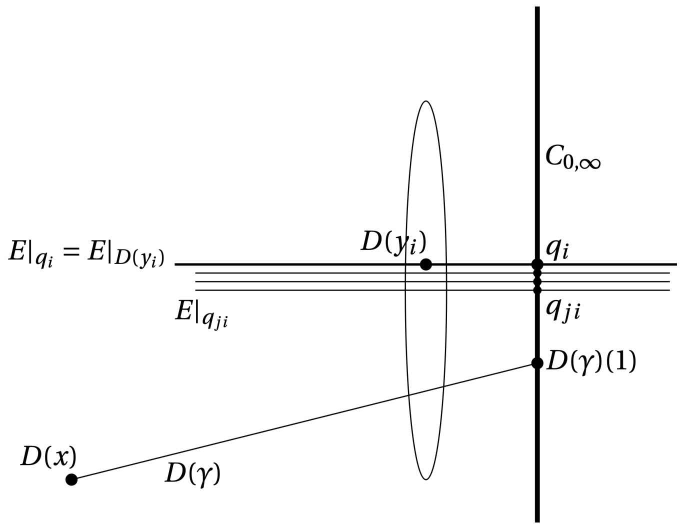

Figure 2: The dynamics of Q_{ji}.

Asymptotic dynamics Now we can determine the limits of the orbits in positive time, and apply proposition 2.8.

Lemma 3.16. *Let $z \in E|_{q_i}$ and V a neighborhood of z. Then $\{T_{ji}(V)\}$ has $F|_{q_i}$ in its limit set.*

Proof. It is the case for $Q_{ji}(V)$ since V intersects every $E|_{q_{ji}}$ for j large enough, and we know that $T_{ji} - Q_{ji} \to 0$. □

Proposition 3.17. *Let $S \subset \widetilde{M}$ be a convex containing the incomplete geodesic γ. Assume that $D(S)$ has a smooth boundary at $D(\gamma)(1)$. Let $i > 0$. We have the following properties.*

- *The orbit $T_{ji}^{-1}(D(S))$ tends to a product $E_{+,i} \times P_c$ of a half-space $E_{+,i} \subset E|_{q_i}$ and a neighborhood $P_c \subset P|_{q_i}$ of the origin.*

- *The product $E_{+,i} \times P_c$ is visible from y_i.*
- *The boundary of $E_{+,i}$ is described by the limit of $T_{ji}^{-1}(\mathrm{T}_{D(\gamma)(1)} D(S)) \cap E|_{q_i}$.*

Proof. To prove this statement, we apply the preceding lemma with proposition 2.8 which allows to prove visibility of the limit of convex subsets.

For $i > 0$ fixed, we consider the convex subsets $T_{ji}^{-1}(D(S))$. We already know that $D(y_i)$ is a limit point since $\gamma \subset S$ (see lemma 3.6). Therefore the limit of $T_{ji}^{-1} D(S)$ is a closed convex subset visible from y_i.

Consider $z \in D(S)$ and write $z = z_E + z_P + z_F$ with $z_P \in P$ and $z_F \in F$. Then

$$T_{ji}^{-1}(z) = T_{ji}^{-1}(z_E) + f_{ji}^{-1}(z_P) + f_{ji}^{-1}(z_F). \tag{23}$$

Note that, up to a rotation, $f_{ji}^{-1}(z_P) \to r z_P$ (with $r > 0$ as in lemma 3.3) and $f_{ji}^{-1}(z_F) \to 0$.

This observation shows that the limit of $T_{ji}^{-1} D(S)$ has vanishing coordinates on F and almost identical coordinates on P (up to a rotation), say P_c.

Now in the coordinate of E, a point $z \in E|_{q_i}$ is a limit point of $T_{ji}^{-1}(D(S))$ if any neighborhood V of z has $T_{ji}(V)$ accumulating at $D(\gamma)(1)$ from inside $D(S)$. This property is described by the relative position of $T_{ji}(V)$ to $\mathrm{T}_{D(\gamma)(1)} D(S)$. It describes a half-space $E_{+,i}$ in $E|_{q_i}$.

Therefore the limit is $E_{+,i} \times P_c$ and the listed properties follow. □

To get a better description of $E_{+,i}$ inside $E|_{q_i}$, we must explain how to approximate T_{ji}^{-1} relatively to Q_{ji}^{-1}. Note that *a priori*

$$T_{ji}^{-1}(x) - Q_{ji}^{-1}(x) = -f_{ji}^{-1}(c_{ji,L}) \tag{24}$$

and might not be tending to zero, or even stay bounded.

The study of $-f_{ji}^{-1}(c_{ji,L})$ is technical. With the two following lemmas, we will later prove that $-f_{ji}^{-1}(c_{ji,L}) \to 0$ when $j \gg i \to \infty$.

Lemma 3.18. *Assume that $E_{+,i}$ does not contain q_i. For any $i > 0$, there exists $M > 0$ such that for any $j > i$,*

$$c_{ji,L} = f_{ji}(b_{ji,E}) + b_{ji,P} \tag{25}$$

with $b_{ji,E} \in E$, $b_{ji,P} \in P$ verifying $b_{ji,P} \to 0$ and $\|b_{ji,E}\| < M$.

Proof. We consider the orbit of $D(\gamma)$. We know that $T_{ji}^{-1} D(\gamma)$ has in its limit set $D(y_i) \in E_{+,i}$. Furthermore,

$$T_{ji}^{-1}(D(\gamma)) = -f_{ji}^{-1}(c_{ji,L}) + Q_{ji}^{-1}(D(\gamma)) \tag{26}$$

and $Q_{ji}^{-1}(D(\gamma))$ tends to an half-line Δ based at $q_i \in E|_{q_i}$. Therefore, in order for $T_{ji}^{-1}(D(\gamma))$ to accumulate at $D(y_i) \in E|_{q_i}$, one must have that $-f_{ji}^{-1}(c_{ji,L})$ is either bounded or the sum of a P-component tending to zero and a point along the opposite half-line $-\Delta$ escaping to infinity. In the latter case, it would imply that $q_i = \Delta(0)$ belongs to $E_{+,i}$. □

We will prove in the case of a rank one ray geometry that $E_{+,i}$ never contains q_i. So in fact $b_{ji,E}$ is bounded. The next lemma explains that in fact T_{ji}^{-1} is as well approximated by Q_{ji}^{-1} as $i > 0$ gets large.

Lemma 3.19. *Assume that for any i, $E_{+,i}$ never contains q_i. Then*

$$\lim_{k\to\infty}\lim_{j\to\infty} b_{kj,E} = 0. \tag{27}$$

Proof. The cocycle relation $T_{kj}T_{ji} = T_{ki}$ gives:

$$T_{ji} = f_{ji}(b_{ji,E}) + b_{ji,P} + c_{ji,F} + f_{ji} \tag{28}$$

$$T_{kj}T_{ji} = f_{kj}(b_{kj,E}) + f_{kj}(f_{ji}(b_{ji,E})) + b_{kj,P} + f_{kj}(b_{ji,P}) + c_{kj,F} + f_{kj}(c_{ji,F} + f_{ji}) \tag{29}$$

$$= f_{ki}(b_{ki,E}) + b_{ki,P} + c_{ki,F} + f_{ki} \tag{30}$$

and it implies by identification:

$$f_{ki}(b_{ki,E}) = f_{kj}(b_{kj,E}) + f_{kj}(f_{ji}(b_{ji,E})) \tag{31}$$

$$= f_{kj}(b_{kj,E}) + f_{ki}(b_{ji,E}) \tag{32}$$

$$b_{ki,E} - b_{ji,E} = f_{ji}^{-1}(b_{kj,E}) \tag{33}$$

and since $\|b_{ki,L} - b_{ji,L}\| < 2M$, we have that $b_{kj,E}$ must tend to zero since otherwise $f_{ji}^{-1}(b_{kj,E})$ would escape to infinity. □

Rank one ray manifolds

We examine closed manifolds with a rank one ray geometry. For those manifolds, the holonomy takes its values in $G_1 = \mathbf{R}^n \rtimes KA_1$, where A_1 has dimension 1.

Lemma 3.20. *If $(G_1, \mathbf{R}^n)$ is a rank one ray geometry, then there exists a decomposition $L_1 \oplus L_2 \oplus L_3$ such that for any Fried dynamics, $P = L_3$ and $\{E, F\} = \{L_1, L_2\}$.*

Proof. Because the rank is one, each direction e_i has in A_1 a single diagonal factor β^{d_i} with degree $d_i \in \mathbf{R}$. Let L_1 be generated by the vectors e_i such that $d_i < 0$, L_2 when $d_i > 0$ and L_3 when $d_i = 0$. Note that if β is exchanged with β^{-1} then L_1 becomes L_2 and conversely (L_3 is unchanged).

If $E \oplus P \oplus F$ is a Fried dynamics, then we only need to show that $P = L_3$ since then it is clear that $\{E, F\}$ must be equal to $\{L_1, L_2\}$. But in rank one, if a diagonal factor β^{d_i} tends to 1 correspondingly to a direction $e_i \in P$, then either $d_i = 0$ or $\beta \to 1$. If $d_i = 0$ then $e_i \in L_3$. If $\beta \to 1$ then it is true globally on $\mathbf{R}^n$, showing that $E = \{0\}$, impossible. □

A consequence of this observation is that P is independent from the dynamics, since $d_i = 0$ is a condition on $A = A_1$. Therefore, from a Fried dynamics to another one E and F are either the same or exchanged. Also, every direction in P is completely visible by the following lemma.

Lemma 3.21. *The direction vector of an incomplete geodesic $D(\gamma)$ has a non-vanishing coordinate along E in the linear decomposition $\mathbf{R}^n = E \oplus P \oplus F$ associated to its Fried dynamics.*

Proof. Note that $D(U_i)$ accumulates disjointly on $\overline{D(\gamma)}$. Indeed, we asked that $\pi(\gamma)$ exits U in M between the times t_i and t_{i+1}. So it is again the case in the developing map: $D(\gamma)$ exits $D(U_i)$ before entering into $D(U_{i+1})$.

But if the direction vector of $D(\gamma)$ vanishes on E then the infinite number of subsets $D(U_j) = T_{ji}D(U_i)$ cannot intersect the relatively compact geodesic $D(\gamma)$ in such a way. Indeed, $D(\gamma)$ is relatively compact and by intersecting along $D(\gamma)$, $T_{ji}(U_i) \cap D(\gamma)$ would not tend to a point, and therefore can not let $D(\gamma)$ exits U_j before entering into $T_{ki}(U_i) \cap D(\gamma)$. □

Proposition 3.22. *Let $S \subset \widetilde{M}$ be a convex containing γ. Assume that $D(S)$ has smooth boundary at $D(\gamma)(1)$. Let $i > 0$. The subspace $H_i = E_{+,i} \times (P \oplus F)|_{y_i}$ is visible from y_i (this is a half-space of $\mathbf{R}^n$).*

Proof. By proposition 3.17, $E_{+,i}$ is a visible from y_i. Since the directions in P are always complete, the product $E_{+,i} \times P|_{y_i}$ is fully visible from y_i. We show that we can extend $E_{+,i} \times P|_{y_i}$ to a visible open H_i containing $E_{+,i} \times (P \oplus F)|_{y_i}$.

Let $K \subset (E \oplus P|_{y_i})$ be convex and visible from y_i. Since the visible space from y_i is open, there exists an open convex $W \subset F$ such that $K \times W$ is visible and convex. We can extend K to $E_{+,i} \times P|_{y_i}$. Indeed, otherwise, there exists η an incomplete geodesic parallel to $E_{+,i} \times P|_{y_i}$ with endpoint in $K \times \{w\}$ and with $w \in W$. Because we are in rank one, its limit $C'_{0,\infty}$ must be parallel to $F|_{y_i}$. Therefore it intersects $E_{+,i} \times P|_{y_i}$ and is simultaneously invisible since it is in $C'_{0,\infty} \cap (K \times W)$ and visible since $E_{+,i} \times P|_{y_i}$ is, a contradiction.

Now, $(E|_+ \times P|_{y_i}) \times W$ can be extended to H_i such that it contains $E_{+,i} \times P|_{y_i} \times F|_{y_i} = E_{+,i} \times (P \oplus F)|_{y_i}$. Indeed, apply T_{ji} for $j \to \infty$ (note that $T_{ji}(x+y) = T_{ji}(x) + f_{ji}(y)$):

$$\begin{aligned} \lim_{j\to\infty} T_{ji}((E|_+ \times P|_{y_i}) \times W) &= (E|_+ \times P|_{y_i}) \times \lim_{j\to\infty} f_{ji}(W) &\quad (34)\\ &= (E|_+ \times P|_{y_i}) \times F|_{y_i}. \end{aligned}$$

□

Lemma 3.23. *The half-spaces H_i tend to a half-space denoted H_x when $i \to \infty$. For $i \to \infty$, q_i gets closer to $\overline{H_i}$.*

Proof. For each H_i, note that $D(\gamma)(1) \in q_i + F$. Since $D(\gamma)(1)$ is invisible, H_i cannot contain q_i in its interior. Therefore lemma 3.19 applies. When $i \to \infty$, by the study of $c_{ji,L}$:

$$\lim_{j\to\infty} T_{ji}^{-1}(D(\gamma)(1)) = \lim_{j\to\infty} -f_{ji}^{-1}(c_{ji,L}) + q_{ji} - f_{ji}^{-1}(q_{ji}) = q_i \quad (35)$$

is a point of $\overline{H_x}$. □

Lemma 3.24. *Let $x \in \widetilde{M}$. Let $D(S) \subset D(\widetilde{M})$ be the maximal Euclidean open ball such that $x \in S$ and S is visible from x. Then $H_x = \lim H_i$ contains x.*

Proof. Consider the Euclidean metric that makes $(e_1, \ldots, e_n)$ orthonormal. We can always consider the family of open balls from $D(x)$ that are visible from x. There is a maximal one S such that $\gamma \subset S$ is incomplete at $t = 1$.

The set H_x is a product $E_+ \times (P \oplus F)$ with E_+ determined by $\lim T_{ji}^{-1}(S)$. Therefore the point x belongs to H_x if we can show that $T_{ji}(x)$ belongs to $S + (P \oplus F)$ if $j \gg i$ are large enough.

By the preceding study, with $D(\gamma)(1)$ as base point,

$$\begin{aligned} T_{ji} &= f_{ji}(b_{ji,E}) + b_{ji,P} + Q_{ji}, &\quad (36)\\ Q_{ji} &= q_{ji} - f_{ji}(q_{ji}) + f_{ji}. &\quad (37) \end{aligned}$$

We want to show that $T_{ji}(x)$ belongs to $S + (P \oplus F)$, so in fact we prove that the E-coordinate of $T_{ji}(x)$ belongs to the euclidean ball (in E) $\| - x_E + y_E\|_E < \|x\|$. With $y = T_{ji}(x)$ (note that $b_{ji,P}$ and q_{ji} belong to $P \oplus F$) we obtain:

$$-x_E + T_{ji}(x)_E = -x_E + f_{ji}(b_{ji,E}) + f_{ji}(x_E). \quad (38)$$

We can now estimate on each subspace on which f_{ji} acts by (almost) homotheties (note that the rotational part tends to the identity). On a coordinate e_m of E, f_{ji} acts like $\lambda_{ji}^{d_m}$, hence on the coordinate e_m:

$$\|f_{ji}(b_{ji,E}) + x_E - f_{ji}(x_E)\|_{e_m} \le \|f_{ji}(b_{ji,E})\|_{e_m} + \|x_E - f_{ji}(x_E)\|_{e_m} \tag{39}$$

$$\le \lambda_{ji}^{d_m} \|b_{ji,E}\|_{e_m} + (1-\lambda_{ji}^{d_m})\|x_E\|_{e_m} \tag{40}$$

Since $b_{ji,E} \to 0$, the inequality $\|f_{ji}(b_{ji,E}) + x_E - f_{ji}(x_E)\|_{e_m} < \|x_E\|_{e_m}$ is verified on each coordinate e_m, so the required (global) inequality on E is verified by summing the squares. □

For any $x \in \widetilde{M}$, we choose $S \subset \widetilde{M}$ the maximal convex open subset such that $D(S)$ is a Euclidean open ball. By applying the construction to the point x, the convex S and an incomplete geodesic $\gamma \subset S$ (which exists by maximality of S) we obtain an half-space $H_x = \lim H_i$ and $D(x) \in H_x$. This choice is now assumed.

Lemma 3.25. *Let $I \subset \partial H_x$ be the invisible set from the interior of H_x. (Note that $(F \oplus P)|_{D(\gamma)(1)} \subset I$.) Then I does not depend on $x \in \widetilde{M}$ and I is an affine subspace.*

Proof. Let $D(q) \in \partial H_x$ be visible from $x \in \mathscr{H}_x$. Consider H_q a half-space corresponding to $D(q)$ and containing $D(q)$.

Both half-spaces $D^{-1}(H_x)$ and $D^{-1}(H_q)$ are convex and intersect. Therefore the developing map is injective on the union. The invisible set from H_x must be invisible from H_q and vice-versa. But under a large T_{ji} (that leaves asymptotically stable H_x) it can only be possible if I is common to both since $T_{ji} H_q$ would contain in its interior invisible points of H_q. (See figure 3.)

It shows also that I is an affine subspace because it is the intersection of a finite number of half-spaces. □

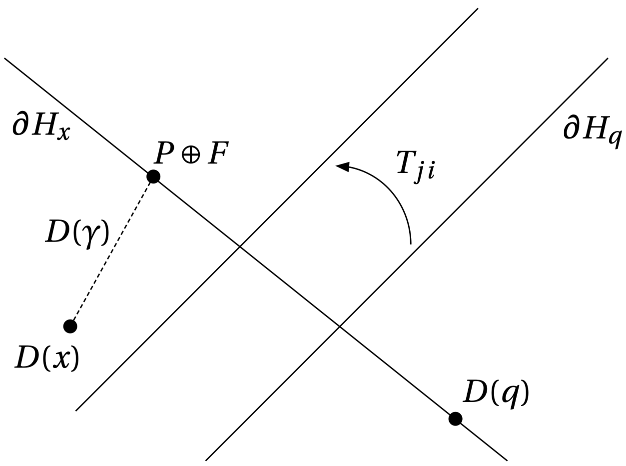

Figure 3: The invisible subspace I.

Theorem 3.26. *Let $(G_1, \mathbf{R}^n)$ be a rank one ray geometry. Let M be a closed $(G_1, \mathbf{R}^n)$-manifold. Then M is either complete or there exists an affine subspace $I \subset \mathbf{R}^n$ such that the developing map $D\colon \widetilde{M} \to \mathbf{R}^n - I$ is a cover.*

Proof. The affine subspace I is constant and contains every $D(\gamma)(1)$ for any incomplete geodesic $\gamma \subset S$ in the maximal Euclidean open ball of any $x \in \widetilde{M}$. We show it implies that $D\colon \widetilde{M} \to \mathbf{R}^n - I$ is a covering map.

Let $\delta\colon [0,1] \to \mathbf{R}^n - I$ be a path. Choose $x \in \widetilde{M}$ such that $D(x) = \delta(0)$. We need to prove that δ can be lifted to a path in $\widetilde{M}$ based at x. We can assume that δ can be lifted for $t < 1$. We show that is can be lifted at $t = 1$. Since $\delta(1) \notin I$, there exists a point $\delta(s)$ with $s < 1$ such that the maximal Euclidean open ball based at $\delta(s)$ and avoiding I contains $\delta(t)$ for $s \le t \le 1$. But then for the lift at time s, the corresponding open ball S is convex and allows to lift δ for $s \le t \le 1$. □

To be more precise on the nature of I, we use a discreteness argument very close to what Matsumoto [Mat92] proposed (see also [Ale21a, end of sec. 4]).

Proposition 3.27. *Let $(G_1, \mathbf{R}^n)$ be a rank one ray geometry such that for any Fried dynamics $F = \{0\}$. The invisible subspace avoided by the developing map is $I = P|_w$ for any Fried dynamics associated to an invisible geodesic ending at $w \in I$.*

Lemma 3.28 ([Mat92, pp. 214-215])**.** *The subset $\Delta \subset \Gamma$ constituted by the transformations T_{ji} for every Fried dynamics such that T_{ji} has almost no rotation generates a discrete subgroup of G_1.*

Proof. If $D\colon \widetilde{M} \to \mathbf{R}^n - I$ is a cover onto a simply connected complement, it is certainly true since the whole holonomy group $\Gamma \supset \langle\Delta\rangle$ would be discrete. We need to treat the case where I has codimension 2.

Consider G_1' the stabilizer of I. Assume for simplicity that $0 \in I$. Note that I must be stable under $A \subset G_1$. Then $\Gamma \subset G_1'$. Consider also the cover $Q \times \mathbf{R} \to \mathbf{R}^n - I$ consisting in taking a half-hyperplane Q (stable by $A \subset G_1'$) with I as boundary and rotating it around I. The transformation group of this cover is $G_1' \times \mathbf{R}$. The developing map $D\colon \widetilde{M} \to \mathbf{R}^n - I$ (that is a cover) is lifted to a diffeomorphism $D\colon \widetilde{M} \to Q \times \mathbf{R}$. In $Q \times \mathbf{R}$ the new holonomy group $\widetilde{\Gamma}$ is discrete. Now, if T_{ji} has almost no rotation, then T_{ji} preserves almost P and therefore $\widetilde{T_{ji}} \simeq T_{ji} \times \{0\}$, showing that $\langle\Delta\rangle \simeq \langle\widetilde{\Delta}\rangle$ is discrete. □

Proof of the proposition. The hypothesis $F = \{0\}$ implies that an asymptotic fixed point q_i for T_{ji} is necessarily equal to $D(\gamma)(1)$ since $q_i \in F|_{D(\gamma)(1)} = \{D(\gamma)(1)\}$.

To prove the proposition, we show that $I \cap E|_{D(\gamma)(1)}$ is always reduced to $D(\gamma)(1)$. It will imply $I = P|_{D(\gamma)(1)}$.

Assume that T_{ji} are the transformation of the Fried dynamics based at the initial $D(\gamma)(1)$. Let R be any other transformation T_{mn} but for a different Fried dynamics associated to a geodesic with its endpoint in $E|_{D(\gamma)(1)}$ but different from $D(\gamma)(1)$. (It can be chosen so if, and only if, $I \neq P|_{D(\gamma)(1)}$.)

By the preceding lemma the subgroup $\langle R, \{T_{ji}\}\rangle$ must be discrete. Now consider

$$G_j = T_{ji} R T_{ji}^{-1} \tag{41}$$

for an $i > 0$ fixed and large enough. We show that G_j must converge without being constant, a contradiction.

Assume for simplicity that $D(\gamma)(1) = 0$. For $i > 0$ fixed, write $T_{ji}(x) = c_{ji} + f_{ji}(x)$ with $c_{ji} \to 0$ and write also $R(x) = b + h(x)$. Then the linear part of G_j is given by $f_{ji} h f_{ji}^{-1}$ and therefore must converge in $KA \subset G_1$.

Now, the translational part is given by $c_{ji} + f_{ji}(b) - f_{ji} h f_{ji}^{-1}(c_{ji})$. Each term must converge and therefore so does the sum.

Therefore G_j converge, but cannot be constant. Indeed, otherwise, the translational part converge to $f_{ji}(b)$ (since $c_{ji} \to 0$) and should be constant. But $f_{ji}(b)$ cannot be constant since $b \neq 0$ and $b \in E$, since R is associated to a geodesic with endpoint different from $D(\gamma)(1)$ but in E. □

Example 1 Fried's theorem [Fri80] and other generalizations on $\mathbf{R}^n$ [Ale21a], all depend dynamically on the hypothesis $F = P = \{0\}$. The preceding proposition shows that an incomplete manifold is radiant.

Example 2 We can easily construct examples with $F = \{0\}$ and $P \neq \{0\}$. Let $K = \{e\}$ and $A = \{(\lambda x, y)\}$ for $\lambda > 0$, $x \in \mathbf{R}^k$ and $y \in \mathbf{R}^m$. Then let Γ be a subgroup generated by $(2x, y)$ and a lattice on $\mathbf{R}^m \ni y$. Then $\mathbf{R}^k - \{0\} \times \mathbf{R}^m$ quotiented by this subgroup gives a product of a radiant manifold with a Euclidean manifold. Here, $I = \{0\} \times \mathbf{R}^m$.

Note In the case of Carrière [Car89], the hypothesis of 1-discompacity implies necessarily $I = (P \oplus F)|_{D(\gamma)(1)}$ by dimensionality. Now the same construction given by the last proposition can be applied by taking $T_{ji}^{-1} H T_{ji}$: it furnishes a convergent sequence of transformations (a contradiction). Therefore (for such ray geometries) it does not require an additional argument such that the irreducibility given by Goldman-Hirsch [GH84].

Open question Is it true that for any rank one ray geometry we always have $I = P|_{D(\gamma)(1)}$? It would show the completeness of the structures having a non-zero F for any Fried dynamics. In the next section, we show the completeness for the structures having a parallel volume (for those, F can never be zero), but we believe that this dynamic-geometric property on I is intriguing.

4 Reducibility of incomplete manifolds

Markus conjecture A first consequence of theorem 3.26 is about Markus conjecture [Mar62]. This conjectures states that *closed manifolds with parallel volume are complete.*

The fact that an incomplete manifold has its holonomy that preserves I implies that the holonomy is reducible. But by Goldman-Hirsch [GH84], the holonomy of a closed manifold with parallel volume can never be reducible.

Corollary 4.1. *Let $(G_1, \mathbf{R}^n)$ be a rank one ray geometry with parallel volume. Every closed $(G_1, \mathbf{R}^n)$-manifold is complete.*

The automorphism group It is a vague conjecture [DG91] that geometric manifolds with large automorphism groups should be classifiable.

Definition 4.2. *Let M be a (G, X)-manifold. An* automorphism $f: M \to M$ *is a diffeomorphism such that if $\widetilde{f}: \widetilde{M} \to \widetilde{M}$ is any lift then there exists a unique $\chi(\widetilde{f}) \in N(\Gamma)$ in the normalizer of the holonomy group, such that $D(\widetilde{f}(x)) = \chi(\widetilde{f}) D(x)$.*

By unicity of $\chi(\widetilde{f})$, if $\widetilde{f}_1$ and $\widetilde{f}_2$ are two lifts of f then $\widetilde{f}_2 = g \widetilde{f}_1$ for $g \in \pi_1(M)$ and therefore $\chi(\widetilde{f}_2) = \rho(g) \chi(\widetilde{f}_1)$.

The hypothesis that $\chi(\widetilde{f})$ normalizes Γ follows from the fact that if $\widetilde{f}$ lifts f then it must preserves the fibers $\pi_1(M)$ of $\widetilde{M}$ and therefore

$$\chi(\widetilde{f})\Gamma \cdot D(x) = D(\widetilde{f}(\pi_1(M)\cdot x)) = D(\pi_1(M)\cdot \widetilde{f}(x)) = \Gamma \cdot \chi(\widetilde{f})D(x). \tag{42}$$

Theorem 4.3. *Let $(G_1, \mathbf{R}^n)$ be a rank one ray geometry. Let M be a closed $(G_1, \mathbf{R}^n)$-manifold. If* $\mathrm{Aut}(M)$ *is non-compact then M is complete.*

A non-compact automorphism group acts non-properly on a closed manifold M The fact that $\mathrm{Aut}(M)$ acts non-properly on M is equivalent to the existence of $x_n \to x$ in M and diffeomorphisms f_n such that $f_n(x_n) \to y$, and such that in $\Gamma\backslash N(\Gamma)$, lifts of f_n escape every compact. So we can assume that $x_n \to x$ in $\widetilde{M}$, $g_n \in N(\Gamma)$ with Γg_n escaping every compact of $\Gamma\backslash N(\Gamma)$, and $y \in \widetilde{M}$ such that $\Gamma g_n D(x_n) \to \Gamma D(y)$.

Proof. Assume that M is not complete, then by theorem 3.26, we get that $D \colon \widetilde{M} \to \mathbf{R}^n - I$ is a cover. Therefore the holonomy Γ preserves I and so does the normalizer $N(\Gamma)$. Both must be subgroups of $I \rtimes KA_1$. We show that $\mathrm{Aut}(M)$ must act properly.

Choose a base point $p \in I$ which is the asymptotic fixed point of a Fried dynamics and consider the decomposition $I \oplus V = \mathbf{R}^n$ with $V \subset E$.

Assume that $\mathrm{Aut}(M)$ acts non-properly. As noted, let $g_n \in N(\Gamma)$ be escaping every compact of $\Gamma\backslash N(\Gamma)$, let $x_n \to x_\infty$ and y such that $\Gamma g_n D(x_n) \to \Gamma D(y)$. Write $g_n(x) = c_n + f_n(x)$ with $c_n \in I$ and $f_n \in KA_1$. The Fried dynamics $T_{ji} \in \Gamma$ considered can be written $T_{ji}x = c_{ji} + f_{ji}(x)$. Note that

$$T_{ji}g_n(x) = c_{ji} + f_{ji}(c_n) + f_{ji}f_n(x). \tag{43}$$

It shows that we can assume $f_{ji}f_n$ bounded in KA_1 since A_1 has rank one (up to exchange T_{ji} with T_{ji}^{-1}).

Hence there exists $\gamma_n \in \Gamma$ such that $\gamma_n g_n = h_n$ has its KA_1-factor that converges, up to a subsequence. The convergence $\Gamma g_n D(x_n) \to \Gamma D(y)$ says that there exists $\eta_n \in \Gamma$ such that $\eta_n h_n D(x_n) \to D(y)$.

Write $\eta_n(x) = b_n + q_n(x)$ and $h_n(x) = c_n + r_n(x)$ we have by construction $r_n \to r$ and

$$(\eta_n h_n)(D(x_n)) = b_n + q_n(c_n) + q_n r_n(D(x_n)) \to D(y). \tag{44}$$

We again have a decomposition in $I \oplus V = \mathbf{R}^n$. The term $b_n + q_n(c_n)$ must belong to I since $c_n \in I$. So the V-coordinate $((\eta_n h_n)(D(x_n)))_V$ tending to $D(y)_V$ is determined by $(q_n r_n(D(x_n)))_V$. But since $D(y)_V \neq 0$ and $V \subset E$, it shows that q_n itself must converge to $q \in KA_1$ (note that $r_n(D(x_n)) \to r(D(x_\infty))$).

The I-coordinate $((\eta_n h_n)(D(x_n)))_I$ tending to $D(y)_I$ has the same limit as the I-factor of $b_n + q_n(c_n) + qr(D(x_n))$. Since $D(y)_I$ and $D(x_\infty)$ are both finite, $b_n + q(c_n)$ must converge.

Therefore $\eta_n h_n$ converges in $I \rtimes KA_1$, contradicting the fact that Γg_n escapes every compact of $\Gamma\backslash N(\Gamma)$. □

In higher rank We give a relatively generic example showing that in higher rank this phenomenon can no longer be true. Consider the rank two ray geometry given by the diagonal action of $(\beta_1 x, \beta_1\beta_2 y)$. Then consider the radiant manifold $\mathbf{R}^2 - \{0\}/\langle(2x, 2y)\rangle$. Let $f(x,y) = (x/2, y)$ and $p = (x, 1)$. Then $f^n(p) \to (0,1)$ and it corresponds in M to an automorphism acting non-properly. The radiant manifold being incomplete, it gives a counter-example in rank two.

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