Complexity of Finding Maximum Locally Irregular Induced Subgraphs

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Abstract

If a graph G is such that no two adjacent vertices of G have the same degree, we say that G is *locally irregular*. In this work we introduce and study the problem of identifying a largest induced subgraph of a given graph G that is locally irregular. Equivalently, given a graph G, find a subset S of $V(G)$ with minimum order, such that by deleting the vertices of S from G results in a locally irregular graph; we denote with $\mathrm{I}(G)$ the order of such a set S. We first examine some easy graph families, namely paths, cycles, trees, complete bipartite and complete graphs. However, we show that the decision version of the introduced problem is $\mathcal{NP}$-Complete, even for restricted families of graphs, such as subcubic planar bipartite, or cubic bipartite graphs. We then show that we can not even approximate an optimal solution within a ratio of $\mathcal{O}(n^{1-\frac{1}{k}})$, where $k \geq 1$ and n is the order the graph, unless $\mathcal{P}$=$\mathcal{NP}$, even when the input graph is bipartite.

Then, looking for more positive results, we turn our attention towards computing $\mathrm{I}(G)$ through the lens of parameterised complexity. In particular, we provide two algorithms that compute $\mathrm{I}(G)$, each one considering different parameters. The first one considers the size of the solution k and the maximum degree Δ of G with running time $(2\Delta)^k n^{\mathcal{O}(1)}$, while the second one considers the treewidth tw and Δ of G, and has running time $\Delta^{2tw} n^{\mathcal{O}(1)}$. Therefore, we show that the problem is FPT by both k and tw if the graph has bounded maximum degree Δ.

Since the algorithms we present are not FPT for graphs with unbounded maximum degree (unless we consider $\Delta + k$ or $\Delta + tw$ as the parameter), it is natural to wonder if there exists an algorithm that does not include additional parameters (other than k or tw) in its dependency. We answer negatively, to this question, by showing that our algorithms are essentially optimal. In particular, we prove that there is no algorithm that computes $\mathrm{I}(G)$ with dependence $f(k)n^{o(k)}$ or $f(tw)n^{o(tw)}$, unless the ETH fails.

1 Introduction

A graph G is said to be *locally irregular*, if every two adjacent vertices of G have different degrees. In this paper, we introduce and study the problem of finding a largest locally irregular induced subgraph of a given graph. This problem is equivalent to identifying what is the minimum number of vertices that must be deleted from G, so that what remains is a locally irregular graph.

Locally irregular graphs. The notion of locally irregular graphs was first introduced in [6]. The most interesting aspect of locally irregular graphs, comes from their connection to the so-called 1-2-3 Conjecture, proposed in [23]. Formally, the 1-2-3 Conjecture states that for almost

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every graph, we should be able to place weights from $\{1, 2, 3\}$ on the edges of that graph, so that the colouring, that assigns a colour to each vertex equal to the sum of the weights on its adjacent edges, is a proper vertex-colouring of the graph.

As we said earlier, the 1-2-3 Conjecture seems to have some very interesting links to locally irregular graphs. An obvious connection is that this conjecture holds for locally irregular graphs. Indeed, placing weight equal to 1 to all the edges of a locally irregular graph, suffices to produce a proper vertex-colouring, as each vertex receives a colour equal to its degree. Furthermore, there have been some steps towards proving that conjecture, which involve edge-decomposing a graph into a constant number of locally irregular subgraphs, *i.e.*, given G, find an edge-colouring of G using a constant number of colours, such that each colour induces a locally irregular subgraph of G. This is the main motivation behind [6], and it seems to remain interesting enough to attract more attention [8, 26, 31].

Note that the class of locally irregular graphs can be seen as an antonym to that of *regular*,*i.e.*, graphs such that all of their vertices have the same degree. It is important to state here that there exist several alternative such notions. This is mainly due to the very well known fact that there are no non-trivial *irregular* graphs, *i.e.*, graphs that do not contain two vertices (not necessarily adjacent) with the same degree (see [12]). Thus, the literature has plenty of slightly different definitions of irregularity (see for example [2, 12, 13, 21, 30]). One way to deal with the nonexistence of irregular graphs, is to define a notion of *local* irregularity. Intuitively, instead of demanding for all vertices of a graph to have different degrees, we are now considering each vertex v separately, and request that the vertices "around" v to verify some properties of irregularity. For example, the authors of [3] study graphs G such that for every vertex v of G, no two neighbours of v have the same degree. For an overview of other interesting notions of irregularity (local or otherwise), we refer the reader to [4].

Largest induced subgraph verifying specific properties. The problem we introduce belongs in a more general and well studied family of problems, which is about identifying a largest induced subgraph of a given graph that verifies a specific property Π. That is, given a graph $G = (V, E)$ and an integer k, is there a set $V' \subseteq V$ such that $|V'| \leq k$ and $G[V \setminus V']$ has the specified property Π? In our case, the property Π is "the induced subgraph is locally irregular". This generalised problem is indeed classic in graph theory, and it is known as the Induced Subgraph with Property Π (ISPΠ for short) problem in [22]. Unfortunately, it was shown in [25], that ISPΠ is a hard problem for any property Π that is *hereditary*, *i.e.*, all induced subgraphs of G verify Π if G itself verifies that property.

However, the ISPΠ problem remains interesting (one could say that it actually becomes more interesting) even if the property Π is not hereditary. Recently, the authors of [7] studied the problem for Π being "all vertices of the induced subgraph have odd degree", which clearly is not a hereditary property. Nevertheless, they showed that this is an $\mathcal{NP}$-hard problem, and they gave an FPT algorithm that solves the problem when parameterised by the rank-width. Also, the authors of [1, 5, 29] studied the ISPΠ problem, where Π is the rather natural property "the induced subgraph is d-regular", where d is an integer given in the input (recall that a graph is said to be *d-regular* if all of its vertices have the same degree d). In particular, in [5] it is shown that finding a largest (connected) induced subgraph that is d-regular, is $\mathcal{NP}$-hard to approximate, even when restricted on bipartite or planar graphs. The authors of [5] also provide a linear-time algorithm to solve this problem for graphs with bounded treewidth. In contrast, the authors of [1] take a more practical approach, as they focus on solving the problem for the particular values of $d = 1$ and $d = 2$, by using bounds from quadratic programming, Lagrangian relaxation and integer programming.

It is quite clear that, in some sense, the property that interests us lies on the opposite side of the one studied in [1, 5, 29]. However, both properties, "the induced subgraph is regular" and "the induced subgraph is locally irregular" are not hereditary. This means that we do not get an $\mathcal{NP}$-hardness result directly from [25]. Furthermore, the ISPΠ problem always admits an FPT algorithm, when parameterised by the size of the solution, if Π is a hereditary property (proven in [11, 24]), but for a non-hereditary one, this is not always true. Indeed in [29], the authors proved that when considering Π as "the induced subgraph is regular", the ISPΠ problem is W[1]-hard when parameterised by the size of the solution. That is, there should be no $f(k)n^c$ time algorithm for this problem, where c is a constant. For such problems, it is also interesting to see if there exists any algorithm with running time $n^{o(k)}$ or $f(k)n^{o(k)}$. The authors of [14, 15, 16] provide techniques that can be used to strongly indicate the non-existence of such algorithms, by applying them on a variety of W[1]-hard and W[2]-hard problems, such as the INDEPENDENT SET and the DOMINATING SET, parameterised by the size of their solutions. Usually these lower bounds are shown under the assumption of a weaker version of the EXPONENTIAL TIME HYPOTHESIS, which states that SAT can not be solved in time $2^{o(n+m)}$.

Our contribution. We begin in Section 2 by providing the basic notations and definitions that are going to be used throughout this paper. In Section 3, we deal with the complexity of the introduced problem. In particular, we show that the problem belongs in $\mathcal{P}$ if the input graph is a path, cycle, tree, complete bipartite or complete graph. We then prove that finding the maximum induced locally irregular subgraph of a given graph G is $\mathcal{NP}$-hard, even if G is restricted to being a subcubic planar bipartite, or a cubic bipartite graph.

As the problem we introduce seems to be computationally hard even for rather restricted families of graphs, we proceed by investigating its approximability. Unfortunately, we prove in Section 4 that for any bipartite graph G of order n and $k \geq 1$, there can be no polynomial time algorithm that finds an approximation of $\mathrm{I}(G)$ within ratio $\mathcal{O}(n^{1-\frac{1}{k}})$, unless $\mathcal{P}$=$\mathcal{NP}$. Nevertheless, we do manage to give a (simple) d-approximation algorithm for d-regular bipartite graphs.

We then decide to look into its parameterised complexity. In Section 5, we present two algorithms that compute $\mathrm{I}(G)$, each one considering different parameters. The first considers the size of the solution k and the maximum degree Δ of G, and and has running time $(2\Delta)^k n^{\mathcal{O}(1)}$, while the second considers the treewidth tw and Δ of G, and has running time $\Delta^{2tw} n^{\mathcal{O}(1)}$. Unfortunately, these algorithms can be considered as being FPT only if Δ is part of the parameter. In Section 5.3, we present two linear fpt-reductions which prove that the problem is W[2]-hard when parameterised only by the size of the solution and W[1]-hard when parameterised only by the treewidth. These reductions also show that we can not even have an algorithm that computes $\mathrm{I}(G)$ in time $f(k)n^{o(k)}$ or $\mathcal{O}^*(f(tw)n^{o(tw)})$, unless the ETH fails. The $\mathcal{O}^*$ notation is used to suppress polynomial factors in regards to n and tw.

2 Preliminaries

For notions and definitions on graph theory not explained here, we refer the reader to [19].

Let $G = (V, E)$ be a graph and $G' = (V', E')$ be a subgraph of G (*i.e.*, created by deleting vertices and/or edges of G). Recall first that the subgraph G' is *induced* if it can be created only by deleting vertices of G. That is, for each edge $uv \in E$, if $u, v \in V'$, then $uv \in E'$. For any vertex $v \in V$, let $N_G(v) = \{u \in V : uv \in E\}$ denote the *neighbourhood* of v in G, and let $d_G(v) = |N_G(v)|$ denote the *degree* of v in G. We also define $N_G[v] = N_G(v) \cup \{v\}$. Finally, for any $X \subseteq V$, we define $N_G[X] = \bigcup_{v \in X} N_G[v]$. Note that, whenever the graph G is clear from the context, we will omit the subscript and simply write $N(v)$, $d(v)$, $N[v]$ and $N[X]$.

One way to show that a problem can not be approximated within a certain ratio, is through a *gap reduction*. The goal of such a reduction is to show that it is $\mathcal{NP}$-hard to differentiate between instances that have a solution of size $\leq \alpha$ and those for which any solution has size $> \beta$. If such is the case, then we know that we cannot approximate the optimal solution within a ratio of $\frac{\beta}{\alpha}$, as otherwise we would get that $\mathcal{P}$=$\mathcal{NP}$.

Finally, recall that a *fixed parameter-tractable* (FPT for short) algorithm, is an algorithm with running time $f(k)n^{\mathcal{O}(1)}$, where f is a computable function and k is the considered parameter. We also make use of what is known as a *linear fpt-reduction*, a type of polynomial reduction such that the size of the parameter of the new problem is linear in regards to the size of the parameter of the original problem. Observe that if we have a linear fpt-reduction from a problem Q with parameter k to a problem Q' with parameter k' and the assumption that Q can not be solved in time $f(k)n_1^{o(k)}$ (where n_1 is the size of the input of Q), then we can conclude that there is no $f(k')n_2^{o(k')}$ time algorithm for Q (where n_2 is the size of the input of Q).

Let $G = (V, E)$ be a graph. We say that G is *locally irregular* if for every edge $uv \in E$, we have $d(u) \neq d(v)$. Now, let $S \subseteq V$ be such that $G[V \setminus S]$ is a locally irregular graph; any set

S that has this property is said to be an *irregulator of* G. For short, we will say that S is an $ir(G)$. Moreover, let $\mathrm{I}(G)$ be the minimum order that any $ir(G)$ can have. We will say that S is a *minimum* irregulator of G, for short S is an $ir^*(G)$, if S is an $ir(G)$ and $|S| = \mathrm{I}(G)$.

We also define the following notion, which generalises $ir(G)$. Let $G = (V, E)$ be a graph, $S, X \subseteq V$ and let $G' = G[V \setminus S]$. Now, let $S \subseteq V$ be such that, for each two neighbouring vertices u, v in $X \setminus S$, we have that $d_{G'}(u) \neq d_{G'}(v)$; any set S that has this property is said to be an *irregulator of* X *in* G, for short $ir(G, X)$. We define the notions of $ir^*(G, X)$ and $\mathrm{I}(G, X)$ analogously to the previous definitions.

We will now provide some lemmas and an observation that will be useful throughout this paper. In the three lemmas below, we investigate the relationship between $\mathrm{I}(G)$ and $\mathrm{I}(G, X)$.

Lemma 2.1. *Let* $G = (V, E)$ *be a graph and let* $X \subseteq V$*. Then* $\mathrm{I}(G, X) \leq \mathrm{I}(G)$*.*

Proof. Let S be an $ir^*(G)$, $G' = G[V \setminus S]$ and $X' = X \setminus S$. Observe that for each pair of vertices u, v such that $u \in X'$ and $v \in N_{G'}(u) \cap X'$, we have that $d_{G'}(u) \neq d_{G'}(v)$, since S is an $ir^*(G)$. It follows that S is also an irregulator of X in G, *i.e.* S is an $ir(G, X)$, and thus we have that $\mathrm{I}(G, X) \leq |S| = \mathrm{I}(G)$. □

Lemma 2.2. *Let* $G = (V, E)$ *be a graph and* $S, X \subseteq V$ *such that* S *is an* $ir^*(G, X)$*. Then,* $S \subseteq N[X]$ *and* $\mathrm{I}(G, X) = \mathrm{I}(G[N[X]], X)$*.*

Proof. Let S be an $ir^*(G, X)$, $S_1 = S \cap N[X]$ and $S_2 = S \setminus S_1$. It suffices to prove that S_1 is an $ir(G, X)$. Indeed, if $S_1 \subseteq S$ is an $ir(G, X)$, since S is an $ir^*(G, X)$, we can conclude that $S = S_1$ and that $S \subseteq N[X]$ (by definition of S_1).

Assume now that S_1 is not an $ir(G, X)$. Then there exists a pair of vertices u, v where uv is an edge in $G[X \setminus S_1]$ and $d_{G[V \setminus S_1]}(u) = d_{G[V \setminus S_1]}(v)$. Observe that $N[\{u, v\}] \subseteq N[X]$, and thus $N[\{u, v\}] \cap S_2 = \emptyset$. Therefore, $d_{G[V \setminus S]}(u) = d_{G[V \setminus S_1]}(u) = d_{G[V \setminus S_1]}(v) = d_{G[V \setminus S]}(v)$. This is a contradiction since S is an $ir^*(G, X)$.

Now, we will prove that $\mathrm{I}(G, X) = \mathrm{I}(G[N[X]], X)$. Let S be an $ir^*(G, X)$. Since $S \subseteq N[X]$ and any vertex $v \in X \setminus S$ has $N(v) \subseteq N[X]$, we have that $d_{G[V \setminus S]}(v) = d_{G[N[X] \setminus S]}(v)$. Thus, S is an $ir(G[N[X]], X)$ and $\mathrm{I}(G, X) \geq \mathrm{I}(G[N[X]], X)$. Now for the opposite direction, let S' be an $ir^*(G[N[X]], X)$. We will show that S' is also an $ir(G, X)$. Since for all $v \in X \setminus S'$, we have $d_{G[V \setminus S']}(v) = d_{G[N[X] \setminus S']}(v)$ (again because $N(v) \subseteq N[X]$) we have that S' is an $ir(G, X)$. Therefore, $\mathrm{I}(G, X) \leq \mathrm{I}(G[N[X]], X)$. □

Lemma 2.3. *Let* $G = (V, E)$ *be a graph, and* $X_1, \ldots, X_n \subseteq V$ *such that* $N[X_i] \cap N[X_j] = \emptyset$ *for every* $1 \leq i < j \leq n$*. Then* $\sum_{i=1}^{n} \mathrm{I}(G, X_i) \leq \mathrm{I}(G)$*.*

Proof. Let $X = \bigcup_{i=1}^{n} X_i$. For every $1 \leq i \leq n$, let S_i be an $ir^*(G, X_i)$ and $G'_i = G[V \setminus S_i]$, and let $S = \bigcup_{i=1}^{n} S_i$ and $G' = G[V \setminus S]$. Observe first that for every $i \neq j$, since $N[X_i] \cap N[X_j] = \emptyset$, we have that $S_i \cap S_j = \emptyset$ as well. Thus, $|S| = \sum_{i=1}^{n} |S_i|$.

We will now show that S is an $ir^*(G, X)$. Assume that there exists an S' such that $|S'| < |S|$ and S' is an $ir(G, X)$. Then, there exists a $k \leq n$ such that the set $S'_k = S' \cap N[X_k]$, is such that $|S'_k| < |S_k|$, as otherwise $|S'|$ can not be smaller than $|S|$. Observe that S'_k must be an $ir(G, X_k)$; this holds because for any vertex $u \in S' \setminus S'_k$, we know that $u \notin N[X_k]$. This is a contradiction since we have assumed that S_k is an $ir^*(G, X_k)$ and S'_k is an $ir(G, X_k)$ with size smaller than S_k. Therefore, S is an $ir^*(G, X)$, and the statement follows by Lemma 2.1. □

In this final lemma, we show that an irregulator of a graph is also an irregulator of any subset of the vertices of the graph, even if we only consider the neighbourhood of that subset.

Lemma 2.4. *Let* $G = (V, E)$ *be a graph,* X *be a subset of* V *and* S *be an* $ir(G)$*. The set* $S \cap N[X]$ *is an* $ir(G, X)$ *and an* $ir(G[N[X]], X)$*.*

Proof. Let $S_X = S \cap N[X]$, $G' = G[V \setminus S]$ $G^* = G[V \setminus S_X]$. Assume that S_X is not an $ir(G, X)$. Then there exist two adjacent vertices v, u such that $\{v, u\} \subset X \setminus S_X$ and $d_{G^*}(u) = d_{G^*}(v)$. Since $S_X = S \cap N[X]$ we have that $N_{G[S]}(u) = N_{G[S_X]}(u)$ and $N_{G[S]}(v) = N_{G[S_X]}(v)$. Therefore $d_{G'}(u) = d_{G^*}(u) = d_{G^*}(v) = d_{G'}(u)$ which is a contradiction since G' is locally irregular. It remains to show that S_X is an $ir(G[N[X]], X)$. Note that for any vertex $v \in X \setminus S_X$ $N[v]$ is included in both G and $G[N[X]]$. Therefore, $d_{G^*}(v) = d_{G[N[X] \setminus S_X]}(v)$ since we have remove the same vertices from $N[X]$. So S_X is an $ir(G[N[X]], X)$. □

The following, almost trivial, observation, will be useful throughout the rest of the paper.

Observation 2.5. *Let $G = (V, E)$ be a graph and S be an $ir(G)$. Then, for each edge $uv \in E$, if $d(u) = d(v)$, then S contains at least one vertex in $N[\{u, v\}]$. Additionally, for a set $X \subseteq V$, let S^* be an $ir(G[N[X]], X)$. Then for each edge $uv \in E(G[X])$, if $d(u) = d(v)$, then S^* contains at least one vertex in $N[\{u, v\}]$.*

3 (Classic) complexity

In this section, we deal with the complexity of the problem we introduced. In Section 3.1 we present all the families of graphs for which we prove that $\mathrm{I}(G)$ is computed in polynomial time. Then in Section 3.2, we prove that, in general, computing $\mathrm{I}(G)$ is $\mathcal{NP}$-hard.

3.1 Polynomial Cases

Theorem 3.1. *Let G be a graph. If $G = K_n$, then $\mathrm{I}(G) = n - 1$. Also, if $G = K_{n,m}$ with $0 \leq n \leq m$, then $\mathrm{I}(G) \leq 1$ with the equality holding if and only if $n = m$.*

Proof. Let $G = (V, E)$. Assume that $G = K_n$, and let S be an $ir(G)$ with $|S| < n - 1$. Then $G' = G[V \setminus S]$ is a complete graph of order $n' > n - (n - 1) = 1$, and for any $n' \geq 2$, we have that $K_{n'}$ is not locally irregular, leading to a contradiction.

Observe that $K_{n,m}$, with $0 \leq n < m$, is locally irregular, and thus $\mathrm{I}(K_{n,m}) = 0$ in this case. Assume now that $G = K_{n,n}$ with $n \geq 1$. We have that $\mathrm{I}(G) \geq 1$ as $K_{n,n}$ is not locally irregular. Let L, R be the two bipartitions of V, with $|L| = n$ and $|R| = n$. Consider the set $S = \{v\}$, where v is any vertex of L. Clearly, after the deletion of v, the graph $G' = G[V \setminus S]$ is isomorphic to $K_{n-1,n}$ which is locally irregular. □

Theorem 3.2. *Let P_n be the path on n vertices, then*

$$\mathrm{I}(P_n) = \begin{cases} \lfloor \frac{n}{4} \rfloor, & \text{if } n \not\equiv 2 \bmod 4 \\ \lfloor \frac{n}{4} \rfloor + 1, & \text{if } n \equiv 2 \bmod 4 \end{cases}$$

Proof. We will begin our proof by examining the cases of P_1, P_2, P_3, and P_4. Observe first of all; that P_1 and P_3 are locally irregular graphs. It follows that $\mathrm{I}(P_1) = \mathrm{I}(P_3) = 0$.

On the other hand, it is also easy to check that P_2 is not locally irregular, but that deleting any one of its vertices suffices to turn it into P_1 (which is locally irregular). It follows that $\mathrm{I}(P_2) = 1$. We will now show that $\mathrm{I}(P_4) = 1$. Let $P_4 = v_1v_2v_3v_4$ and note that P_4 is not locally irregular (we have that $d(v_2) = d(v_3) = 2$). Moreover, deleting either v_1 or v_4 from P_4, results in the graph P_3, which is locally irregular. Thus $\mathrm{I}(P_4) = 1$. Observe moreover that any path on more than 4 vertices is not locally irregular.

We are now ready to continue with the proof. Let $n, k, d \in \mathbb{N}$, with $n \geq 5$, $n \equiv k \bmod 4$, $d = \lfloor \frac{n}{4} \rfloor$ and $G = P_n = v_1 \dots v_n$. We have the following two cases:

- $k \neq 2$. Consider the set $S = \{v_i : i \equiv 0 \bmod 4\}$. We have that $|S| = d$. Also, observe that the graph $G[V(G) \setminus S]$ has d connected components, each one of which is isomorphic to P_3, which are locally irregular, and a connected component isomorphic to P_k, where $k \in \{0, 1, 3\}$, which is also locally irregular (the graph P_0 is the null graph). It follows that S is an $ir(P_n)$ and that $\mathrm{I}(P_n) \leq |S| = d$. All that is left to show is that $\mathrm{I}(P_n) \geq d$. Let us assume that there exists a set S_0 that is an $ir(P_n)$ and $|S_0| < d$. Now observe that $G[V(G) \setminus S_0]$ contains at least one connected component isomorphic to P_m, with $m \geq 4$ This is a contradiction, since P_m is not locally irregular.

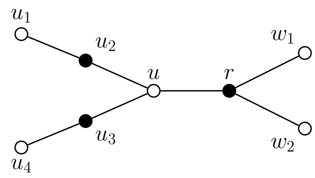


Figure 1: The gadget used in the proof of Theorem 3.5. The white and black vertices are used to denote vertices belonging to different bipartitions.

- $k = 2$. Consider the set $S = \{v_i : i \equiv 0 \text{ mod } 4\} \cup \{v_n\}$. We have that $|S| = d+1$. Similarly to the previous case, we have that $G[V(G)\setminus S]$ contains d connected components isomorphic to P_3 and one connected component isomorphic to P_1. Thus S is an $ir(P_n)$ and $\mathrm{I}(P_n) \leq |S| \leq d+1$. All that is left to show is that $\mathrm{I}(P_n) \geq d+1$. Observe that the arguments supporting that $\mathrm{I}(P_n) > d$ are the same as the previous case. So, we assume that there exists a set S_0 that is an $ir(P_n)$ and $|S_0| = d$. Observe that all the connected components of $G[V(G) \setminus S_0]$ are paths. Also, if there exists a connected component isomorphic to a P_m, with $m \geq 4$, then $G[V(G) \setminus S_0]$ is not locally irregular. So we may assume that all the connected components of $G[V(G) \setminus S_0]$ are isomorphic to a paths on at most 3 vertices. It follows that one of these components must be isomorphic to P_2, and P_2 is not locally irregular. This is a contradiction.

□

Corollary 3.3. *Let C_n be the cycle on $n \geq 3$ vertices, then $\mathrm{I}(C_n) = \mathrm{I}(P_{n-1}) + 1$.*

To explain the above statement, observe that for every vertex v belonging to the cycle $G = C_n$, we have that $d(v) = 2$. Thus, we know that $\mathrm{I}(C_n) \geq 1$ and that any S that is an $ir(G)$ contains at least one vertex, say vertex v. The statement follows by observing that the graph $G[V(G) \setminus v]$ is isomorphic to P_{n-1}.

Another interesting case is when G is a tree. In this case, we can calculate $\mathrm{I}(G)$ in polynomial time as a direct consequence of Theorem 5.3 (see Section 5). Indeed, in that theorem we provide an algorithm that computes $\mathrm{I}(G)$ in time $\Delta^{2tw} n^{\mathcal{O}(1)}$. The next corollary follows directly from the fact that trees have $tw = 1$ (by definition) and $\Delta = \mathcal{O}(n)$.

Corollary 3.4. *Let T be a tree. There exists a polynomial time algorithm that computes $\mathrm{I}(T)$.*

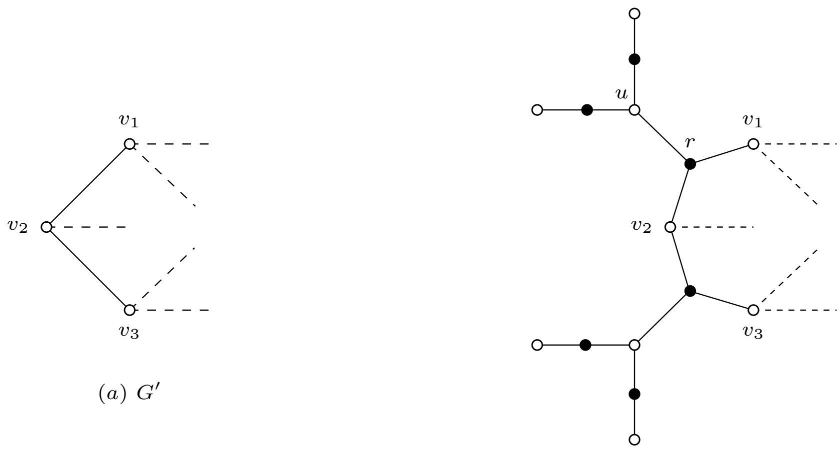
3.2 $\mathcal{NP}$-Hard Cases

We now show that finding a minimum irregulator of a graph is $\mathcal{NP}$-hard. Interestingly, this remains true even for quite restricted families of graphs, such as cubic (*i.e.*, 3-regular) bipartite graphs and subcubic (*i.e.*, of maximum degree at most 3) planar bipartite graphs.

Theorem 3.5. *Let G be graph and $k \in \mathbb{N}$. Deciding if $\mathrm{I}(G) \leq k$ is $\mathcal{NP}$-complete, even when G is a planar bipartite graph with maximum degree $\Delta \leq 3$.*

Proof. Since the problem is clearly in $\mathcal{NP}$, we will focus on proving it is also $\mathcal{NP}$-hard. The reduction is from the Vertex Cover problem, which remains $\mathcal{NP}$-complete when restricted to planar cubic graphs [28]. In that problem, a planar cubic graph G and an integer $k \geq 1$ are given as an input. The question is, whether there exists a vertex cover of G of order at most k. That is, whether there exists a set $VC \subseteq V(G)$ such that for every edge $uv \in E(G)$, at least one of u and v belongs in VC and $|VC| \leq k$.

Let G' be a planar cubic graph and $k \geq 1$ given as input for Vertex Cover. Let $|E(G')| = m$. We will construct a planar bipartite graph G as follows; we start with the graph G', and modify it by using multiple copies of the gadget, illustrated in Figure 1. Note that we will be following the naming convention illustrated in Figure 1 whenever we talk about the vertices of our gadgets. When we say that we *attach* a copy H of the gadget to the vertices v and v' of G', we mean that we add H to G', and we identify the vertices w_1 and w_2 to the vertices v and v' respectively. Now, for each edge $vv' \in E(G')$, attach one copy H of the gadget to the vertices v and v', and then delete



(a) G' (b) G

Figure 2: The construction in the proof of Theorem 3.5. The graph G' is the initial planar cubic graph, and G is the graph built during our reduction. In G, the white and black vertices are used to denote vertices belonging to different bipartitions.

the edge vv' (see Figure 2). Clearly this construction is achieved in linear time (we have added m copies of the gadget). Note also that the resulting graph G has $\Delta(G) = 3$ and that the planarity of G' is preserved since G is constructed by essentially subdividing the edges of G' and adding a tree pending from each new vertex. Also, G is bipartite. Indeed, observe that after removing the edges of $E(G')$, the vertices of $V(G')$ form an independent set of G. Furthermore, the gadget is bipartite, and the vertices w_1, w_2 (that have been identified with vertices of $V(G')$) belong to the same bipartition (in the gadget). Finally, for any $1 \leq i \leq m$, let H_i be the i^{th} copy of the gadget attached to vertices of G'. We will be using the vertices r^i and u^i to denote the copies of the vertices r and u (respectively) that also belong to H_i.

We are now ready to show that the minimum vertex cover of G' has size k' if and only if $\mathrm{I}(G) = k'$.

Let VC be a minimum vertex cover of G' and $|VC| = k'$. We will show that the set $S = VC$ is an $ir(G)$. Let $G^* = G[V(G) \setminus S]$. First, note that S contains only vertices of G'. Thus, for each i, the vertices of H_i except from r^i, which also remain in G^*, have the same degree in G' and in G^*. Also note that each vertex of G' is adjacent only to copies of r. It follows that it suffices to only consider the vertices r^i to show that VC is an $ir(G)$. Now, for any $1 \leq i \leq m$, consider the vertex r^i. Since VC is a vertex cover of G', for each edge $vv' \in E(G')$, VC contains at least one of v and v'. It follows that $d_{G^*}(r^i) \leq 2$. Note also that $N_{G^*}(r^i)$ contains the vertex $u^i \in V(H_i)$ and possibly one vertex $v \in V(G')$.

Also, since we only delete vertices in $V(H_i) \cap V(G')$, we have that $d_{G^*}(u^i) = 3 > d_{G^*}(r^i)$. In the case where $N_{G^*}(r^i)$ also contains a vertex $v \in V(G')$, the vertex v is adjacent only to vertices which do not belong in $V(G')$. Thus, $d_{G^*}(v) = d_G(v) = 3 > d_{G^*}(r^i)$. It follows that r^i has a different degree from all of its neighbours and that VC is an $ir(G)$.

Now, we prove that if $\mathrm{I}(G) = k'$ then there exists a vertex cover of size at most k'. Assume that $\mathrm{I}(G) = k'$ and let S be an $ir^*(G)$. Observe that since S is an $ir^*(G)$, S contains at least one vertex of H_i (for each $1 \leq i \leq m$). Let $X_i = V(H_i) \cap V(G')$. To construct a vertex cover VC of G' with $|VC| \leq k'$, we work as follows. For each $1 \leq i \leq m$:

1. for each vertex $v \in X_i$, if $v \in S$ then put v in VC. Then,

2. if $S \cap X_i = \emptyset$, put any one of the two vertices of X_i in VC.

Observe now that any vertex that is added to VC during step 1. of the above procedure, also belongs to S and any vertex that is added during step 2. of the above procedure corresponds to at least one vertex in S. It follows that $|VC| \leq k'$. Also note that VC contains at least one vertex of X_i, for each i, and that for each $uv \in E(G')$, there exists an i such that $V(X_i) = \{u, v\}$. Thus VC is indeed a vertex cover of G'.

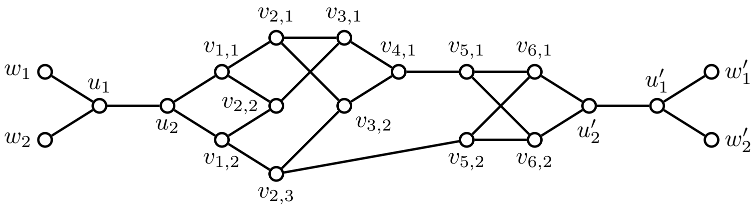


Figure 3: The gadget used in the construction of Theorem 3.6

Therefore G' has a minimum vertex cover of size k' if and only if $\mathrm{I}(G) = k'$. To complete the proof note that deciding if $\mathrm{I}(G) = k' < k$ for a given k, answers the question whether G' has a vertex cover of size less than k or not. □

In the following theorem we show that calculating $\mathrm{I}(G)$ is $\mathcal{NP}$-hard even if G is a cubic bipartite graph.

Theorem 3.6. *Let G be graph and $k \in \mathbb{N}$. Deciding if $\mathrm{I}(G) \leq k$ is $\mathcal{NP}$-complete even in cubic bipartite graphs.*

As the proof of Theorem 3.6 is rather involved, we first present the construction, as well as two lemmas that are going to be utilised in that proof.

Construction for Theorem 3.6 Assume that we have an instance (ϕ, X) of 2-Balanced 3-SAT comprised by a set $\{C_1, \ldots, C_m\}$ of clauses over the set of Boolean variables $\{x_1, \ldots, x_n\}$. We are going to create a cubic bipartite graph G. First we create a bipartite graph $F = (V, E)$ as follows: for each literal x_i ($\neg x_i$ resp.) in ϕ, add a *literal vertex* $v_{i,p}$ ($v_{i,n}$ resp.) in V, and for each clause C_j of ϕ, add a *clause vertex* c_j in V. Next, for each $1 \leq j \leq m$, add the edge $v_{i,p}c_j$ ($v_{i,n}c_j$ resp.) if the literal x_i ($\neg x_i$ resp.) appears in C_j according to ϕ. We create a copy $F' = (V', E')$ of F and we denote by $v'_{i,p}$, $v'_{i,n}$ and c'_j the copies of $v_{i,p}$, $v_{i,n}$ and c_j for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ respectively.

Finally, we are going to connect the two graphs using n copies of the gadget graph H presented in Figure3. In particular, for each $i \in \{1, \ldots, n\}$ we create a copy H_i of H and we attach it to the vertices related to the variable x_i. That is, we add H_i to the graph, and we identify $v_{i,p}$, $v_{i,n}$ with the vertices w_1 and w_2 and $v'_{i,p}$, $v'_{i,n}$ with the vertices w'_1 and w'_2 respectively. This results in a cubic bipartite graph which we call G.

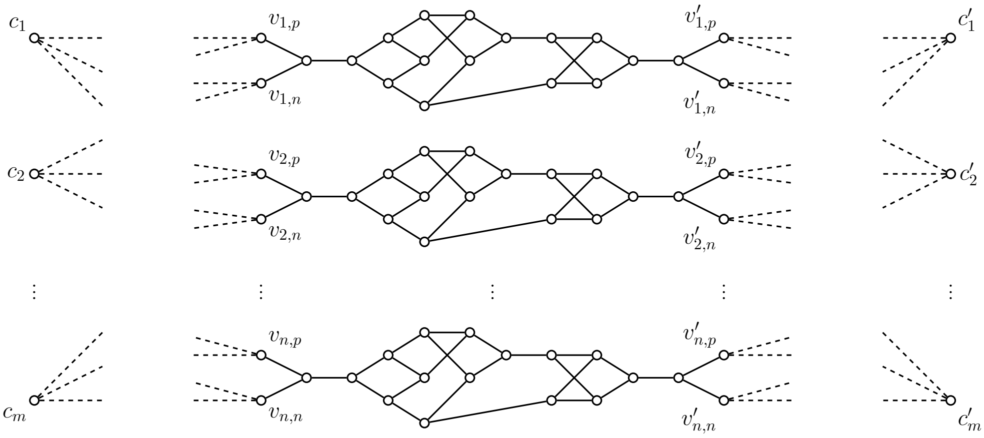


Figure 4: The cubic bipartite graph G, constructed in the proof of Theorem 3.6

Lemma 3.7. *Let $H = (V, E)$ be the graph in Figure 3 and let $V^* = V \setminus \{w_1, w_2, w'_1, w'_2\}$ then, the following holds:*

- $\mathrm{I}(H,V^*) = 3$
- *for any S that is an $ir^*(H,V^*)$ we have:*
  - $\{w_1, w_2, w'_1, w'_2\} \cap S = \emptyset$,
  - $\{u_1, u'_1\} \cap S = \emptyset$
  - $\{u_2, u'_2\} \cap S \neq \emptyset$
  - $\{u_2, u'_2\} \not\subseteq S$
- *for any S that is an $ir(H,V^*)$ and $|S| = 4$ we have:*
  - $|\{w_1, w_2, w'_1, w'_2\} \cap S| \leq 1$,
  - $|\{u_1, u'_1\} \cap S| \leq 1$

Proof. Consider the edges u_1u_2, $v_{3,1}v_{4,1}$ and $u'_2u'_1$. These edges have in common that both of their incident vertices have the same degree (equal to 3). It follows from Observation 2.5 that any $ir(H,V^*)$ contains at least one vertex in each one of the sets $S_1 = N(u_1u_2) = \{w_1, w_2, u_1, u_2, v_{1,1}, v_{1,2}\}$, $S_2 = N(v_{3,1}v_{4,1}) = \{v_{2,1}, v_{2,2}, v_{3,1}, v_{3,2}, v_{4,1}, v_{5,1}\}$ and $S_3 = N(u'_2u'_1) = \{v_{6,1}, v_{6,2}, u'_2, u'_1, w'_1, w'_2\}$. Finally, observe that the sets S_1, S_2 and S_3 are pairwise disjoint. It follows that $\mathrm{I}(H,V^*) \geq 3$. To show that $\mathrm{I}(H,V^*) = 3$, we provide the following list, which presents all the possible subsets of V that are irregulators of V^* in H and they have order 3:

- $\{u_2, v_{2,1}, v_{6,1}\}$
- $\{u_2, v_{2,1}, v_{6,2}\}$
- $\{u_2, v_{4,1}, v_{6,1}\}$
- $\{u_2, v_{4,1}, v_{6,2}\}$
- $\{v_{1,1}, v_{3,2}, u'_2\}$
- $\{v_{1,2}, v_{3,2}, u'_2\}$
- $\{v_{1,2}, v_{3,1}, u'_2\}$

It follows that $\mathrm{I}(H,V^*) = 3$. Note also that in all the sets presented in the above list, each set contains exactly one of the vertices u_2 and u'_2, and none of the vertices in $\{w_1, w_2, w'_1, w'_2, u_1, u'_1\}$. This suffices to validate the second item of the statement.

In the following list, we present all the possible subsets of V that are $ir(H,V^*)$ of order 4, validating the third item of the statement:

- $\{w_1, u_2, v_{2,1}, v_{6,1}\}$
- $\{w_1, u_2, v_{2,1}, v_{6,2}\}$
- $\{w_1, u_2, v_{4,1}, v_{6,1}\}$
- $\{w_1, u_2, v_{4,1}, v_{6,2}\}$
- $\{w_1, v_{2,2}, v_{2,3}, v_{6,2}\}$
- $\{w_1, v_{2,2}, v_{2,3}, v_{6,1}\}$
- $\{w_1, v_{2,2}, v_{2,1}, v_{6,2}\}$
- $\{w_1, v_{2,2}, v_{2,1}, v_{6,1}\}$
- $\{w_1, v_{2,2}, v_{4,1}, v_{6,2}\}$
- $\{w_1, v_{2,2}, v_{4,1}, v_{6,1}\}$
- $\{w_1, v_{2,3}, v_{2,1}, v_{6,2}\}$
- $\{w_1, v_{2,3}, v_{2,1}, v_{6,1}\}$
- $\{w_2, u_2, v_{2,1}, v_{6,1}\}$
- $\{w_2, u_2, v_{2,1}, v_{6,2}\}$
- $\{w_2, u_2, v_{4,1}, v_{6,1}\}$
- $\{w_2, u_2, v_{4,1}, v_{6,2}\}$
- $\{u_1, u_2, v_{2,1}, v_{6,2}\}$
- $\{u_1, u_2, v_{2,1}, v_{6,1}\}$
- $\{u_1, u_2, v_{4,1}, v_{6,2}\}$
- $\{u_1, u_2, v_{4,1}, v_{6,1}\}$
- $\{u_1, v_{1,1}, v_{3,2}, u'_2\}$
- $\{u_1, v_{1,2}, v_{3,2}, u'_2\}$
- $\{u_1, v_{1,2}, v_{3,1}, u'_2\}$
- $\{u_1, v_{3,2}, v_{3,1}, u'_2\}$
- $\{u_1, v_{3,1}, v_{5,2}, u'_2\}$
- $\{u_1, v_{3,1}, v_{5,2}, w'_1\}$
- $\{u_1, v_{3,1}, v_{5,2}, w'_2\}$
- $\{u_2, v_{2,2}, v_{2,3}, v_{6,2}\}$
- $\{u_2, v_{2,2}, v_{2,3}, v_{6,1}\}$
- $\{u_2, v_{2,2}, v_{2,1}, v_{6,2}\}$
- $\{u_2, v_{2,2}, v_{2,1}, v_{6,1}\}$
- $\{u_2, v_{2,2}, v_{4,1}, v_{6,2}\}$
- $\{u_2, v_{2,2}, v_{4,1}, v_{6,1}\}$
- $\{u_2, v_{2,3}, v_{2,1}, v_{6,2}\}$
- $\{u_2, v_{2,3}, v_{2,1}, v_{6,1}\}$
- $\{u_2, v_{2,3}, v_{4,1}, v_{6,2}\}$
- $\{u_2, v_{2,3}, v_{4,1}, v_{6,1}\}$
- $\{u_2, v_{2,3}, v_{4,1}, u'_1\}$
- $\{u_2, v_{2,1}, v_{4,1}, v_{6,2}\}$
- $\{u_2, v_{2,1}, v_{4,1}, v_{6,1}\}$
- $\{u_2, v_{2,1}, v_{6,2}, v_{6,1}\}$
- $\{u_2, v_{2,1}, v_{6,2}, u'_1\}$
- $\{u_2, v_{2,1}, v_{6,1}, u'_1\}$
- $\{u_2, v_{4,1}, v_{6,2}, v_{6,1}\}$
- $\{u_2, v_{4,1}, v_{6,2}, u'_1\}$
- $\{u_2, v_{4,1}, v_{6,1}, u'_1\}$
- $\{v_{1,1}, v_{1,2}, v_{3,2}, u'_2\}$
- $\{v_{1,1}, v_{1,2}, v_{3,1}, u'_2\}$
- $\{v_{1,1}, v_{1,2}, v_{5,1}, u'_2\}$
- $\{v_{1,1}, v_{1,2}, v_{5,1}, w'_1\}$
- $\{v_{1,1}, v_{1,2}, v_{5,1}, w'_2\}$
- $\{v_{1,1}, v_{3,2}, v_{3,1}, u'_2\}$
- $\{v_{1,1}, v_{3,2}, v_{5,1}, u'_2\}$
- $\{v_{1,1}, v_{3,2}, v_{5,1}, w'_1\}$
- $\{v_{1,1}, v_{3,2}, v_{5,1}, w'_2\}$
- $\{v_{1,1}, v_{3,2}, v_{6,2}, v_{6,1}\}$
- $\{v_{1,1}, v_{3,2}, v_{5,2}, u'_2\}$
- $\{v_{1,1}, v_{3,2}, v_{5,2}, w'_1\}$
- $\{v_{1,1}, v_{3,2}, v_{5,2}, w'_2\}$
- $\{v_{1,1}, v_{3,2}, u'_2, u'_1\}$
- $\{v_{1,1}, v_{3,1}, v_{5,2}, u'_2\}$
- $\{v_{1,1}, v_{3,1}, v_{5,2}, w'_1\}$
- $\{v_{1,1}, v_{3,1}, v_{5,2}, w'_2\}$
- $\{v_{1,1}, v_{5,1}, v_{6,2}, v_{5,2}\}$
- $\{v_{1,1}, v_{5,1}, v_{5,2}, v_{6,1}\}$
- $\{v_{1,1}, v_{5,1}, v_{5,2}, u'_2\}$

- $\{v_{1,1}, v_{5,1}, v_{5,2}, u'_1\}$
- $\{v_{1,1}, v_{5,1}, v_{5,2}, w'_1\}$
- $\{v_{1,1}, v_{5,1}, v_{5,2}, w'_2\}$
- $\{v_{1,1}, v_{3,2}, u'_2, w'_1\}$
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- $\{v_{2,2}, v_{4,1}, v_{6,2}, w_2\}$
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- $\{v_{1,2}, v_{3,2}, v_{3,1}, v_{6,2}\}$
- $\{v_{1,2}, v_{3,2}, v_{3,1}, v_{6,1}\}$
- $\{v_{1,2}, v_{3,2}, v_{3,1}, u'_2\}$
- $\{v_{1,2}, v_{3,2}, v_{5,1}, u'_2\}$
- $\{v_{1,2}, v_{3,2}, v_{5,1}, w'_1\}$
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- $\{v_{1,2}, v_{3,2}, v_{5,2}, u'_2\}$
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- $\{v_{1,2}, v_{3,2}, v_{5,2}, w'_2\}$
- $\{v_{1,2}, v_{3,2}, u'_2, u'_1\}$
- $\{v_{1,2}, v_{3,1}, v_{5,1}, u'_2\}$
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- $\{v_{1,2}, v_{3,1}, u'_2, w'_2\}$
- $\{v_{1,2}, v_{3,2}, u'_2, w'_1\}$
- $\{v_{1,2}, v_{3,2}, u'_2, w'_2\}$
- $\{v_{2,3}, v_{2,1}, v_{6,2}, w_2\}$
- $\{v_{2,3}, v_{2,1}, v_{6,1}, w_2\}$

□

At this point we would like to comment on the proof of Lemma 3.7. Indeed, one could prove that lemma by an extensive case analysis. We stress however that, even taking advantage of Observation 2.5, that case analysis would be extremely long and uninteresting. So, instead, we decided to check all possible $ir(H, V^*)$ of order 3 and 4 with the help of a computer program (running a simple exhaustive algorithm), which gave us the above lists.

Before we continue, we are going to give some notation that we are going to use in what follows. First, we will call V_C the set of all the clause vertices and V_X the set of all the literal vertices. Whenever it is not clear by the context, we will use $v(i)$ in order to talk about the copy of a vertex $v \in V(H)$ that belongs to H_i. Similarly, we will use V_i^* to denote the copy of V^* that is inside H_i. For all $i \in \{1, \dots, n\}$, let X_i be the set of literal vertices $\{v_{i,p}, v_{i,n}, v'_{i,p}, v'_{1,n}\}$, U_i be the set that contains the copies of u_1 and u'_1 that belong in H_i and $N_i = \bigcup_{v \in X_i} N(v) \setminus U_i$ the set of clause vertices are adjacent to a literal vertex of X_i.

Now, assume that we have an irregulator S of G (where G is the graph described in the previous construction). By Lemma 2.4 we have that $S \cap V(H_i)$ is an $ir(H_i, V_i^*)$ (since $H_i = G[N[V_i^*]]$) for all $i \in \{1, \dots, n\}$. Let $\mathbb{I} = \{j \mid j \in \{1, \dots, n\} \text{ and } |V(H_j) \cap S| = 3\}$. Then, for all $j \in \mathbb{I}$, the set $S \cap V(H_j)$ is an $ir^*(H_j, V_j^*)$ which means that it has the same properties as any $ir^*(H, V^*)$. Furthermore, by Lemma 3.7 we know that for all $v \in \{w_1, w_2, w'_1, w'_2, u_1, u'_1\}$ the copy of v in H_i is not included in $S \cap V(H_i)$. Additionally, one of the u_1, u'_1 has degree 2 and the other has degree 3 in the $G[V \setminus S]$. We call X_L the set of all the vertices $v_{j,p}$, $v_{j,n}$, $j \in \mathbb{I}$ such that the copy of u'_2 belonging to H_j is included to S. We call these the "left gadgets" and we denote by n_l the number of these gadgets. Similarly, we call X_R the set of all the vertices $v'_{j,p}$, $v'_{j,n}$, $j \in \mathbb{I}$ such that the copy of u_2 belonging to H_j is included to S. We call these the "right gadgets" and we denote by n_r the number of these gadgets. The gadgets H_i, $i \notin \mathbb{I}$, will be called *good* gadgets.

Now, we show that $\mathrm{I}(G) \geq 4n$, and we identify some additional properties of any minimum irregulator of G.

Lemma 3.8. *Let G be the graph constructed as described above, starting from an instance (ϕ, X) of 2-*Balanced *3-SAT with $|X| = n$. Then $\mathrm{I}(G) \geq 4n$. Furthermore, any set S that is an $ir^*(G)$, such that $|S| = 4n$, verifies the following:*

1. *there is no clause such that both its clause vertices c and c' belong to S.*
2. *$n_l + n_r = |S \cap V_C|$*
3. *$|S \cap V(H_i)| \leq 4$ for all $i \in \{1, \dots, n\}$*
4. *any clause vertex $c \in S$ has exactly two neighbours in $X_L \cup X_R$ and one neighbour that belongs to a good gadget.*
5. *for any c and c', corresponding to the same clause in ϕ, if $c \in S$ ($c' \in S$ resp.) then there exists a variable vertex $v \in S$ such that $v \in N_G(c')$ ($v \in N_G(c)$).*

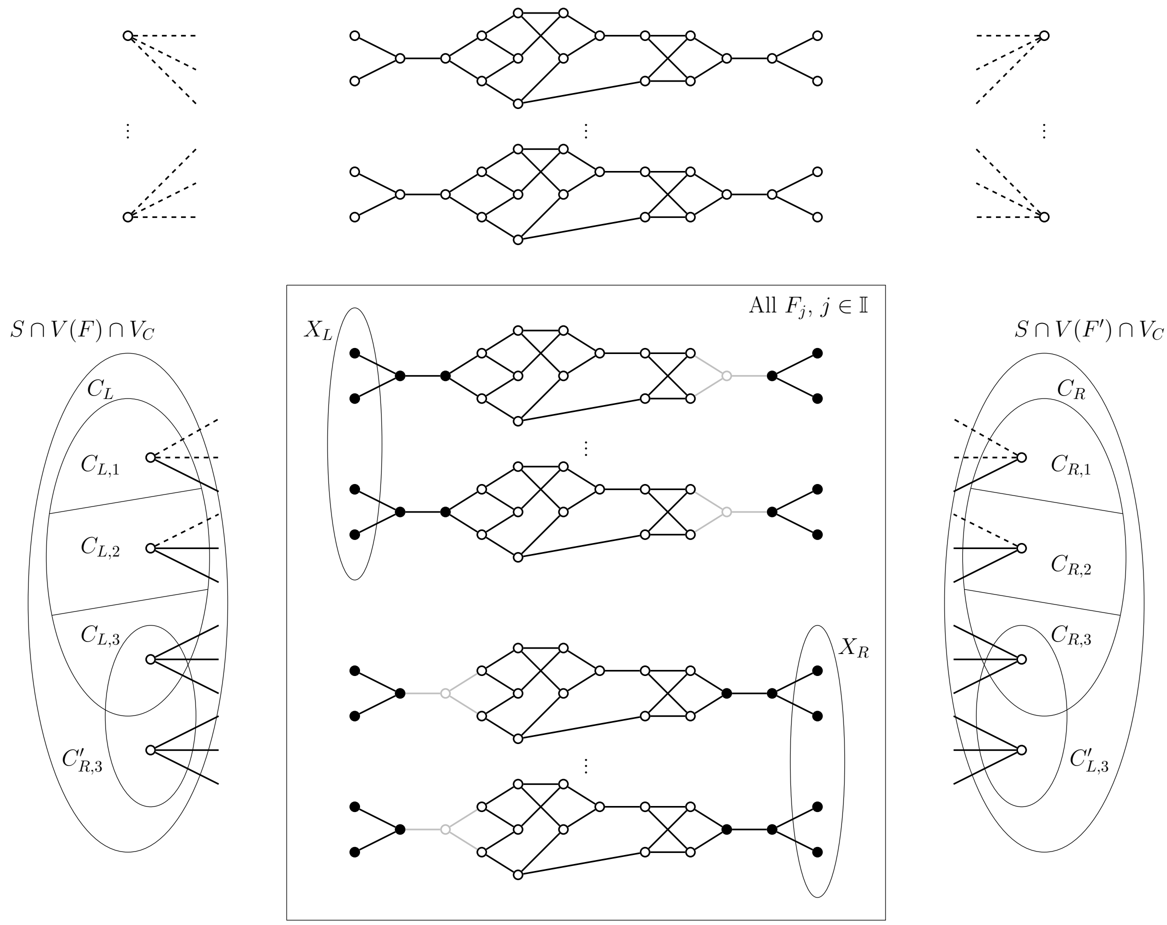


Figure 5: An illustration of the subsets of C_L and C_R, defined in the proof of Theorem 3.6. The edges incident to clause vertices: are dashed if they are incident to vertices of F_i for some $i \notin \mathbb{I}$; are not dashed if they are incident to vertices of F_j for some $j \in \mathbb{I}$. We know that the black vertices do not belong to S and that the light-grey vertices belong to S.

Proof. First we show that $\mathrm{I}(G) \geq 4n$. Let S be an $ir(G)$ and $G^* = G[V \setminus S]$. Let H_i be the copy of the gadget H that is attached to the literal vertices that correspond to the variable $x_i \in X$. Since $\mathrm{I}(H, V^*) = 3$, we have that $\mathrm{I}(G, V_i^*) = 3$ for all $i \in \{1, \ldots, n\}$. Observe that for any $i \in \mathbb{I}$, $|S \cap V(H_i)| = 3$ and $S \cap X_i = \emptyset$. We claim that S must include at least one vertex of the set N_i. Indeed, if this is not true then all the vertices in X_i have degree 3 in G^* and the same holds for one of the vertices in U_i which lead to a contradiction. It is also worth mentioning that for any $i \in \mathbb{I}$, for $(v, u) \in \{\{v_{i,p} v_{i,n}\} \times \{v'_{i,p} v'_{i,n}\}\}$ we have we have that $v \in X_L \iff u \notin X_R$ so $\{v, u\} \cap (X_L \cup X_R) \neq \emptyset$.

Before we continue, let us give some extra notation. Let $C_L \subseteq S$ (resp. $C_R \subseteq S$) be the set of clause vertices c such that $N_G(c) \cap X_L \neq \emptyset$ (resp. $N_G(c) \cap X_R \neq \emptyset$). Note that C_L is subset of the set of clause vertices of F ($C_L \subseteq V(F)$) and C_R is subset of the set of clause vertices of F' ($C_R \subseteq V(F')$). Furthermore, for $1 \leq l \leq 3$, let $C_{L,l}$ be the sets of vertices $c \in C_L$ such that $|N(c) \cap (\bigcup_{j \in \mathbb{I}} X_j)| = l$ and $C_{R,l}$ be the sets of vertices $c \in C_R$ such that $|N(c) \cap (\bigcup_{j \in \mathbb{I}} X_j)| = l$. In simple words, $C_{L,l}$ ($C_{R,l}$ resp.) are the clause vertices that belong to $S \cap F$ (to $S \cap F'$ resp.), which are adjacent to l left or right gadgets, with at least one of these being a left (right resp.) gadget. Finally, let $C'_{L,3}$ be the set $\{c'_i \mid c_i \in C_{L,3}\}$ and $C'_{R,3}$ be the set $\{c_i \mid c'_i \in C_{R,3}\}$. In Figure 5 we illustrate all information given above.

We will now show that $|S \cap V_C| \geq n_l + n_r$. To do that, we will preset some relations between the sets $C_L \cup C_R$, $X_L \cup X_R$ and the edges between them in G. Let E^* be the set containing exactly these edges. First, we calculate a lower bound for $|E^*|$. Observe that all vertices in X_L belong to a left gadget and are neighbours of a copy of u_1. Additionally, $d_{G^*}(u_1) = 3$ for any u_1 that belongs to a left gadget. Therefore, for all $v \in X_L$, $d_{G^*}(v) \leq 2$ as otherwise G^* is not locally irregular. This means that each $v \in X_L$ has at least one neighbour in S, and thus that there are at least $|X_L| = 2n_l$ edges between S and X_L. Furthermore, all these edges must be incident to vertices in C_L (by the definition of C_L). In the same way we can show that between S and X_R are at least $|X_R| = 2n_r$ edges all of them being incident to vertices in C_R. Therefore, $|E^*| \geq 2(n_l + n_r)$.

Now we are going to calculate an upper bound for $|E^*|$. To do that, we will have to split E^* into tree subsets, and treat them separately. For $1 \leq l \leq 3$, let E^*_l be the set of edges between $C_{L,l} \cup C_{R,l}$ and $X_L \cup X_R$. By the definition of $C_{L,1}$ and $C_{R,1}$ we know that each $c \in C_{L,1} \cup C_{R,1}$ has exactly one neighbour in $X_L \cup X_R$. It follows that $|E^*_1| \leq |C_{L,1} \cup C_{R,1}|$. Also, the vertices $c \in C_{L,2} \cup C_{R,2}$ have at most two neighbours in $X_L \cup X_R$. It follows that $|E^*_2| \leq 2|C_{L,2} \cup C_{R,2}|$. For the set $|E^*_3|$, an easy upper bound is $3|C_{L,3} \cup C_{R,3}|$. However, we are going to calculate a tighter upper bound. To do that, we will use the following claim.

Claim 3.9. *Both $C'_{R,3}$ and $C'_{L,3}$ are subsets of S.*

Proof. Let $c \in C'_{R,3}$. By the definition of $C'_{R,3}$, we know that there exists a $c' \in C_{R,3}$ and that c' is incident to a vertex $x \in X_R$ in G. Let H be the copy of the left gadget to which x belongs. Without loss of generality, assume that x is a copy of the vertex w'_1 of H. Therefore, in G, the vertex c is adjacent to the the copy of w_1 that belongs to H. Now, let z be the other clause vertex adjacent to w_1 in G. In order to decide if c belongs to S we need to consider two cases, the first being when $z \in S$ and the second when $z \notin S$. For the first case ($z \in S$), consider the vertex $u_1 \in H$. Since H is a left gadget, we have that $u_1 \notin S$, and that $d_{G^*}(u_1) = 2$. It follows that $c \in S$ as otherwise we would have that $d_{G^*}(w_1) = d_{G^*}(u_1) = 2$. So let $z \notin S$ (second case) and assume that $z \notin S$. It follows that $d_{G^*}(c) = 3$ as all of the neighbours of c in G are copies of vertices of X_R or of X_L (by the construction of $C'_{R,3}$) and none of these vertices belongs to S. Also, since $z \notin S$, we have that $d_{G^*}(w_1) = 3$ which is a contradiction since c is adjacent to w_1. So, for any $c \in C'_{R,3}$ we have that $c \in S$. With symmetric arguments we can show that any $c \in C'_{L,3}$ must belong to S. □

In order to calculate a tighter upper bound for E^*_3 we are going to use the sets $C_{L,3} \cup C'_{R,3}$ and $C_{R,3} \cup C'_{L,3}$. Observe that the set $C_{R,3} \cup C'_{L,3}$ contains exactly the copies of the vertices in $C_{L,3} \cup C'_{R,3}$. Now, due to symmetry, we have that each pair $(c_j, c'_j) \in (C_{L,3} \cup C'_{R,3}) \times (C_{R,3} \cup C'_{L,3})$ has exactly three neighbours in $X_L \cup X_R$ (if c_j has $\ell \leq 3$ neighbours in X_L then c'_j has $3 - \ell$ neighbours in X_R and *vice versa*). Therefore, between the sets $C_{L,3} \cup C'_{R,3} \cup C_{R,3} \cup C'_{L,3}$ and $X_L \cup X_R$ we have exactly $3|C_{L,3} \cup C'_{R,3}|$ edges in G. Since $C_{L,3} \cup C_{R,3} \subseteq C_{L,3} \cup C'_{R,3} \cup C_{R,3} \cup C'_{L,3}$ we have $|E^*_3| \leq 3|C_{L,3} \cup C'_{R,3}|$.

Combining the above upper bounds, we get that $|E^*| \leq |C_{L,1} \cup C_{R,1}| + 2|C_{L,2} \cup C_{R,2}| + 3|C_{L,3} \cup C'_{R,3}|$.

By combining the lower and the upper bounds for $|E^*|$, we get the following inequality:

$$2(n_l + n_r) \leq |C_{L,1} \cup C_{R,1}| + 2|C_{L,2} \cup C_{R,2}| + 3|C_{L,3} \cup C'_{R,3}| \quad (3.1)$$

Since the sets $C_{L,3} \cup C'_{R,3}$ and $C_{R,3} \cup C'_{L,3}$ are both subsets of S (by Claim 3.9), we have that $S \supseteq C_{L,1} \cup C_{L,2} \cup C_{L,3} \cup C'_{R,3} \cup C_{R,1} \cup C_{R,2} \cup C_{R,3} \cup C'_{L,3}$. Also, since the sets $C_{L,3} \cup C'_{R,3}$ and $C_{R,3} \cup C'_{L,3}$ have the same cardinalities, we get that:

$$|S \cap V_C| \geq |C_{L,1} \cup C_{R,1}| + |C_{L,2} \cup C_{R,2}| + 2|C_{L,3} \cup C'_{R,3}| \quad (3.2)$$

If we combine the two inequalities 3.1 and 3.2, we get that:

$$2(n_l + n_r) \leq 2|S \cap V_C| - |C_{L,1} \cup C_{R,1}| - |C_{L,3} \cup C'_{R,3}|$$

This gives us that $|S \cap V_C| \geq n_l + n_r$, from which follows that $|S| \geq 4n$.

Now, by taking a closer look at the previous inequalities, we can gain some extra information. Note that from this point we are considering only cases where we have an irregulator S such that $|S| = 4n$. Since for all $i \notin \mathbb{I}$ we have $|S \cap V(H_i)| \geq 4$, it follows that:

$$|S| = \sum_{i \notin \mathbb{I}} |S \cap V(H_i)| + \sum_{i \in \mathbb{I}} |S \cap V(H_i)| + |S \cap V_C| = \sum_{i \notin \mathbb{I}} |S \cap V(H_i)| + 3(n_l + n_r) + |S \cap V_C|.$$

Note that the number of the gadgets H_i with $i \notin \mathbb{I}$, is exactly $n - (n_l + n_r)$. Assume now that $|S \cap V_C| > (n_l + n_r)$. Then:

$$\begin{aligned} |S| &= \sum_{i \notin \mathbb{I}} |S \cap V(H_i)| + 3(n_l + n_r) + |S \cap V_C| \\ &\geq 4(n - n_l - n_r) + 3(n_l + n_r) + |S \cap V_C| && \text{since } \forall i \notin \mathbb{I} : |S \cap V(H_i)| \geq 4 \\ &> 4(n - n_l - n_r) + 4(n_l + n_r) && \text{since } |S \cap V_C| > (n_l + n_r) \\ &= 4n \end{aligned}$$

noindent which is a contradiction. Thus $|S \cap V_C| = (n_l + n_r)$. Now assume that there exists an $i \notin \mathbb{I}$ such that $|S \cap V(H_i)| > 4$. Then:

$$\begin{aligned} |S| &= \sum_{i \notin \mathbb{I}} |S \cap V(H_i)| + 3(n_l + n_r) + |S \cap V_C| \\ &\geq \sum_{i \notin \mathbb{I}} |S \cap V(H_i)| + 4(n_l + n_r) && \text{since } |S \cap V_C| = (n_l + n_r) \\ &> 4(n - n_l - n_r) + 3(n_l + n_r) + (n_l + n_r) && \text{since } \exists i \notin \mathbb{I} : |S \cap V(H_i)| > 4 \\ &= 4n \end{aligned}$$

This, again, is a contradiction. Therefore, $|S \cap V_C| = (n_l + n_r)$ and for all $i \notin \mathbb{I}$ we have $|S \cap V(H_i)| \leq 4$. In addition, for all $i \in \mathbb{I}$, we have that $|S \cap V(H_i)| = 3$. At this point we have shown the items 2. and 3. of the statement

Also, from eq. 3.1, we can deduce that both $C_{L,1} \cup C_{R,1}$ and $C_{L,3} \cup C'_{R,3}$ are empty sets as otherwise $|S \cap V_C| > n_l + n_r$. Furthermore, $C'_{R,3}$ being empty means that $C_{R,3}$ is empty as well.

Now we are going to show that $S \cap V_C = C_{L,2} \cup C_{R,2}$ and that there no clause vertex $c \in C_{L,2}$ such that its copy c' belongs to $C_{R,2}$ (which suffices to prove items 4. and 1.). We first show that $S \cap V_C = C_{L,2} \cup C_{R,2}$. Assume that $S \cap V_C \supset C_{L,2} \cup C_{R,2}$. Then, $|S \cap V_C| > |C_{L,2} \cup C_{R,2}|$. Using this we can modify eq. 3.1. This gives us $2(n_l + n_r) \leq 2|C_{L,2} \cup C_{R,2}| < 2|S \cap V_C|$ which contradicts the fact that $|n_l + n_r| = |S \cap V_C|$.

Assume now that there exists a set of clauses $S_c \subset C_{L,2}$ such that for each $c \in S_c$ we have that its copy c' belongs to $C_{R,2}$. Furthermore, let S_c be the largest such set and S'_c be the set $\{c' \mid c \in S_c\}$. We are going to calculate a new upper bound for the edges between $S \cap V_C$ and $X_L \cup X_R$ in G. Observe that, by construction, all the vertices in $C_{L,2} \cup C_{R,2}$ have exactly two neighbours in $\bigcup_{j \in \mathbb{I}} X_i$. Furthermore, the vertices in C_L have at least one neighbour in X_L and the vertices in C_R have at least one neighbour in X_R. So, we can conclude that all vertices in $S_c \cup S'_c$ have exactly one neighbour in $X_L \cup X_R$ and all vertices in $(C_{L,2} \cup C_{R,2}) \setminus (S_c \cup S'_c)$ have exactly two neighbours in $X_L \cup X_R$. In this case, the maximum number of edges between $S \cap V_C = C_{L,2} \cup C_{R,2}$ and $X_L \cup X_R$ in G is $|S_c \cup S'_c| + 2|(C_{L,2} \cup C_{R,2}) \setminus (S_c \cup S'_c)| = 2|S \cap V_C| - |S_c \cup S'_c|$. Since we have assumed that S_c is not empty we have $2(n_l + n_r) \leq 2|S \cap V_C| - |S_c \cup S'_c| < 2|S \cap V_C|$ which contradicts to the fact that $|S \cap V_C| = n_l + n_r$.

It remains to prove item 5. For that, consider a vertex $c \in S \cap V_C$. W.l.o.g. assume that $c \in C_L$. Recall that we have shown that $S \cap V_C = C_{L,2} \cup C_{R,2}$. Therefore, c' has two neighbours in left gadgets and one in a good gadget. Furthermore, since no literal vertex of a left gadget has been included in S we know that $d_{G^*}(c') \geq 2$. We will show the following claim:

Claim 3.10. $d_{G^*}(c') = 2$

Proof of the claim. Assume that c' has degree 3 in G^*. Now consider one of the neighbours of c' which belongs to a left gadget; let us call this neighbour v'. Since v' is a literal vertex of a left gadget and the vertex u'_1 of each left gadget has $d_{G^*}(u'_1) = 2$ we know that the degree of v' in G^* can be 0, 1 or 3 (because S is an $ir(G)$). Observe that u'_1 and c' do not belong to S so v' has at least two neighbours in G^*. This means that $d_{G^*}(v')$ can be only 3. This is a contradiction because G^* is a locally irregular graph, and the edge $v'c'$ belongs to G^* and $d_{G^*}(v') = d_{G^*}(c') = 3$. ⋄

Therefore c' can not have degree three in G^*, from which follows that one of its neighbours belong to S. This completes the proof of the lemma. □

Proof of Theorem 3.6

Proof. Since the problem is clearly in $\mathcal{NP}$, we will focus on proving it is also $\mathcal{NP}$-hard. The reduction is from 2-BALANCED 3-SAT, which was proven to be $\mathcal{NP}$-complete in [9]. Starting with a formula ϕ over a set of n variables X we create the graph $G = (V, E)$ described above (illustrated in Figure 4). We claim that $\mathrm{I}(G) = 4n$ if and only if ϕ is satisfiable.

Assume that ϕ is satisfiable and let $t : X \to \{T, F\}$ be a satisfying assignment. We construct an irregulator of G as follows: for each copy of H, select the vertices $u_1, v_{3,1}, v_{5,2}$. Then for each $i \in \{1, \dots, n\}$ we select the vertex $v'_{i,p}$ if $t(x_i) = T$ and the vertex $v'_{i,n}$ otherwise. Let S be the set of all of these vertices and $G^* = G(V \setminus S)$. We claim that S is an $ir^*(G)$. First observe that $|S| = 4n$ since we have select exactly 4 vertices from each copy of H. Furthermore, observe that, for each $i \in \{1, \dots, n\}$ we have that $S \cap V(H_i)$ is an $ir(G[V(H_i)], V_i^*)$. Thus, to show that S is an $ir(G)$, it suffices to show that there is no literal vertex in $G[V(G) \setminus S]$ that has the same degree as one of its neighbours (all the other pair of vertices either do not share an edge or they do not have the same degree because $S \cap V(H_i)$ is an $ir(G[V(H_i)], V_i^*)$). Observe that no literal vertex v, incident to a copy of u_1 in G, has been included in S. Therefore, any clause vertex c adjacent to any such literal vertex v, has degree 3 in G^*. Furthermore observe that for each such literal vertex v, the set S contains its neighbour u_1. Thus, any such literal vertex has degree 2 in G^* and only neighbours of degree 3. Now consider any literal vertex $v' \in V \setminus S$, incident to copies of u'_1. Since S does not include any vertex of V_C (recall that V_C is the set of all clause vertices in G) and any copy of u'_1, we have $d_{G^*}(v') = 3$. Furthermore, by the construction of S we have that $d_{G^*}(u'_1) = 2$ for all the copies of u'_1. It remains to show that the clause vertices $c' \in V_C \cap V(F')$ (the set of clause vertices to the right of the Figure 4) have $d_{G^*}(c') \leq 2$. Since S contains all the literal vertices that correspond to all the true literals under t, and t is a satisfying assignment, we know that S contains least one neighbour of c' for all $c' \in V_C \cap V(F')$. Since these vertices do not exist in G^*, we have $d_{G^*}(c') \leq 2$ for all $c' \in V_C \cap V(F')$. Therefore, S is an $ir(G)$. Additionally, since $|S| = 4n$ and we know that $I(G) \geq 4n$, we have that S is an $ir^*(G)$.

For the reverse direction, let S be an $ir^*(G)$ of size $4n$. We will show that ϕ is satisfiable. In particular we claim that

$$t(x_i) = \begin{cases} T & \text{if } \{v_{i,p}, v'_{i,p}\} \cap S \neq \emptyset, \\ F & \text{otherwise} \end{cases}$$

is a satisfying assignment of ϕ.

Since, by Lemma 3.7, we have that for all i, $|\{v_{i,p}, v'_{i,p}, v_{i,n}, v'_{i,n}\} \cap S \cap H_i| \leq 1$, it suffices to show that for each clause C_j we have $N(\{c_j, c'_j\}) \cap S \neq \emptyset$. Assume that there exists a clause C_j such that $N(\{c_j, c'_j\}) \cap S = \emptyset$. We distinguish two cases: $\{c_j, c'_j\} \cap S \neq \emptyset$ and $\{c_j, c'_j\} \cap S = \emptyset$.

Case 1. ($\{c_j, c'_j\} \cap S \neq \emptyset$) Let $c \in \{c_j, c'_j\} \cap S$; since the irregulator S has size $4n$, by Lemma 3.8, we know that there exists a literal vertex $v \in N(\{c_j, c'_j\}) \cap S$. This is a contradiction.

Case 2. ($\{c_j, c'_j\} \cap S = \emptyset$) Here, again, we are going to distinguish two cases; first if there exists a $k \in \mathbb{I}$ such that $N(\{c_j, c'_j\}) \cap V(H_k) \neq \emptyset$ and, second, if there is no such $k \in \mathbb{I}$.

Case 2(a). ($N(\{c_j, c'_j\}) \cap V(H_k) \neq \emptyset$) W.l.o.g. let H_k be a left gadget. Then, c'_j is adjacent to one of the vertices $v'_{k,p}$ or $v'_{k,n}$; let us call v' the neighbour of c'_j in H_k. Since v' belongs to a left gadget and $c'_j \notin S$, we have $d_{G^*}(v') \geq 2$ which is a contradiction. Indeed, v' is a literal vertex of a left gadget and we know that $d_{G^*}(u'_1) = 2$ in any left gadget. Furthermore $d_{G^*}(c'_j) = 3$. It follows that v' has the same degree as either c'_j (if $d_{G^*}(v') = 3$) or the vertex u'_1 belonging to H_k (if $d_{G^*}(v') = 2$).

Case 2(b). ($N(\{c_j, c'_j\}) \cap V(H_k) = \emptyset$ for all $k \in \mathbb{I}$) Let H_k be a gadget such that $N(\{c_j, c'_j\}) \cap V(H_k) \neq \emptyset$. By the hypothesis, $k \notin \mathbb{I}$. By Lemma 2.4 we have that $S_k = S \cap V(H_k)$ is an $ir(H_k, V_k^*)$. Furthermore, since $k \notin \mathbb{I}$, we have that $|S_k| = |S \cap V(H_k)| = 4$.

Therefore, we know that at most one of u_1 and u'_1 (of H_k) is included in S (since, by Lemma 3.7, at most one of them is included in S_k). W.l.o.g. assume that $u_1 \notin S$. Let $v \in \{v_{k,p}, v_{k,n}\}$ be the neighbour of c_j in H_k. Since $N_{G^*}(v) \supseteq \{c_j, u_1\}$ and $d_{G^*}(c_j) = 3$, it follows that $N_{G^*}(v) = \{c_j, u_1\}$ and that there exists a $q \neq j$ such that $c_q \in N_G(v)$ and $c_q \in S$.

We are going to show that c'_j has a neighbour $v' \in S$. Due to symmetry we know that the copies c'_j and c'_q have a common neighbour v' in H_k. Furthermore, because of Lemma 3.8 we know that two of the neighbours of c'_q belong to left gadgets and the third belongs to a good gadget.

Also, the neighbour that belongs to a good gadget must belong to S. Since v' belongs to a good gadget and it is a neighbour of c'_q, it follows that $v' \in S$. This contradicts the starting assumption that $N_G\{c_j, c'_j\} \cap S = \emptyset$.

This ends the proof of the theorem. □

4 (In)approximability

In the previous section we showed that computing $\mathrm{I}(G)$ is $\mathcal{NP}$-hard, even for graphs G belonging to quite restricted families of graphs. So the natural question to pose next, which we investigate in this section, is whether we can approximate $\mathrm{I}(G)$. Unfortunately, most of the results we present below are once again negative.

We start with a corollary that follows from the proof of Theorem 3.5 and the inapproximability of Vertex Cover in cubic graphs [17]:

Corollary 4.1. *Given a graph G, it is $\mathcal{NP}$-hard to approximate $\mathrm{I}(G)$ to within a ratio of $\frac{100}{99}$, even if G is bipartite and $\Delta(G) = 3$.*

Now, we are going to show that there can be no algorithm that approximates $\mathrm{I}(G)$ to within any decent ratio in polynomial time, unless $\mathcal{P}$=$\mathcal{NP}$, even if G is a bipartite graph (with no restriction on its maximum degree).

Theorem 4.2. *Let G be a bipartite graph of order N and $k \in \mathbb{N}$ be a constant such that $k \geq 1$. It is $\mathcal{NP}$-hard to approximate $\mathrm{I}(G)$ to within $\mathcal{O}(N^{1-\frac{1}{k}})$.*

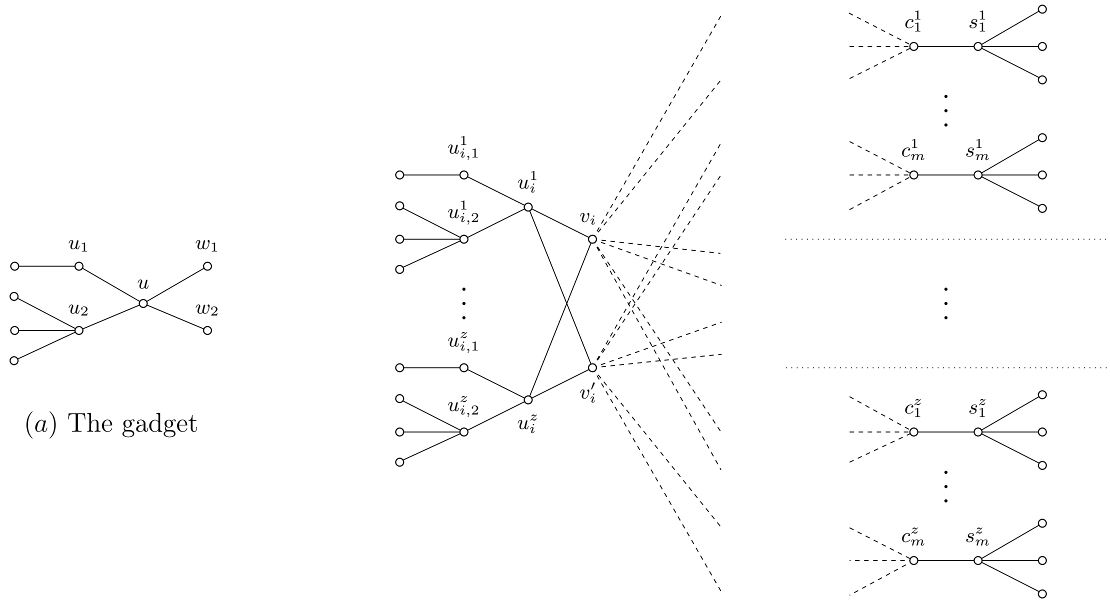
Proof. The proof is by a *gap producing reduction* from 2-Balanced 3-SAT, which was proven to be $\mathcal{NP}$-complete in [9]. In that problem, a 3CNF formula F is given as an input, comprised by a set C of clauses over a set of Boolean variables X. In particular, we have that each clause contains exactly 3 literals, and each variable $x \in X$ appears in F exactly twice as a positive and twice as a negative literal. The question is, whether there exists a truth assignment to the variables of X satisfying F.

Let F be a 3CNF formula with m clauses $C_1, \dots, C_m$ and n variables $x_1, \dots, x_n$ that is given as input to the 2-Balanced 3-SAT problem. Let $2k = k' + 1$. Based on the instance F, we are going to construct a bipartite graph $G = (V, E)$ where $|V| = \mathcal{O}(n^{k'+1})$ and

- $\mathrm{I}(G) \leq n$ if F is satisfiable
- $\mathrm{I}(G) > n^{k'}$ otherwise.

To construct $G = (V, E)$, we start with the following graph: for each literal x_i ($\neg x_i$ resp.) in F, add a *literal vertex* v_i (v'_i resp.) in V, and for each clause C_j of F, add a *clause vertex* c_j in V. Next, for each $1 \leq j \leq m$, add the edge $v_i c_j$ ($v'_i c_j$ resp.) if the literal x_i ($\neg x_i$ resp.) appears in C_j according to F. Observe that the resulting graph is bipartite, for each clause vertex c we have $d(c) = 3$ and for each literal vertex v we have $d(v) = 2$ (since in F, each variable appears twice as a positive and twice as a negative literal). To finish the construction of G, we will make use of the gadget shown in Figure 6(a), as well as some copies of S_5, the star on 5 vertices. When we say that we *attach* a copy H of the gadget to the vertices v_i and v'_i (for some $1 \leq i \leq n$), we mean that we add H to G, and we identify the vertices w_1 and w_2 to the vertices v_i and v'_i respectively. Now:

- for each $1 \leq i \leq n$, we attach $n^{k'}$ copies of the gadget to the vertices v_i and v'_i of G. For convenience, we will give unique names to the vertices corresponding to each gadget added that way. So, the vertex u^l_i (for $1 \leq l \leq n^{k'}$ and $1 \leq i \leq n$) is used to represent the vertex u of the l^{th} copy of the gadget attached to v_i and v'_i, and $u^l_{i,1}$ ($u^l_{i,2}$ resp.) is used to denote the vertex u_1 (u_2 resp.) of that same gadget. Then,
- for each $1 \leq j \leq m$, we add $n^{k'} - 1$ copies of the clause vertex c_j to G, each one of these copies being adjacent to the same literal vertices as c_j. For $1 \leq l \leq n^{k'}$, the vertex c^l_j is the l^{th} copy of c_j. Finally,



(a) The gadget

(b) The constructed graph G for $n^{k'} = z$

Figure 6: The construction in the proof of Theorem 4.2. In subfigure (b), we illustrate how each pair of literal vertices is connected to the rest of the graph. Whenever there is an upper index $1 \le l \le n^{k'}$ on a vertex, it is used to denote the l^{th} copy of that vertex. The dashed lines are used to represent the edges between the literal and the clause vertices.

- for each $1 \le j \le m$ and $1 \le l \le n^{k'}$, we add a copy of the star on 5 vertices S_5 to G and identify any degree-1 vertex of S_5 to c_j^l. Let s_j^l be the neighbour of c_j^l that also belongs to a copy of S_5.

Observe that the resulting graph G (illustrated in Figure 6(b)) remains bipartite and that this construction is achieved in polynomial time in regards to $n + m$.

From the construction of G, we know that for every $1 \le i \le n$, $d(v_i) = d(v_i') = \Theta(n^{k'})$. So, for sufficiently large n, the only pairs of adjacent vertices of G that have the same degrees are either the vertices u_i^l and $u_{i,2}^l$, or the vertices c_j^l and s_j^l (for every $1 \le i \le n$, $1 \le l \le n^{k'}$ and $1 \le j \le m$).

First, let F be a satisfiable formula and let t be a satisfying assignment of F. Also, let S be the set of literal vertices v_i (v_i' resp.) such that the corresponding literals x_i ($\neg x_i$ resp.) are assigned value *true* by t. Clearly $|S| = n$. We will also show that S is an $ir(G)$. Consider the graph $G' = G[V \setminus S]$. Now, for any $1 \le i \le n$, we have that either v_i or v_i', say v_i, belongs to the vertices of G'. Now for every $1 \le l \le n^k$, we have that $d_{G'}(u_i^l) = 3$, while $d_{G'}(u_{i,1}^l) = 2$ and $d_{G'}(u_{i,2}^l) = 4$ (since none of the neighbours of $u_{i,1}^l$ and $u_{i,2}^l$ belongs to S). Also, for every $1 \le j \le m$ and $1 \le l \le n^{k'}$, since t is a satisfying assignment of F, $N(c_j^l)$ contains at least one vertex in S. It follows that $d_{G'}(c_j^l) = 3 < 4 = d_{G'}(s_j^l)$. Finally, since S does not contain any neighbours of v_i, we have that $d_{G'}(v_i) = d_G(v_i) = \mathcal{O}(n^{k'})$. It follows that S is an $ir(G)$ and thus that $\mathrm{I}(G) \le n$.

Now let F be a non-satisfiable formula and assume that there exists an S that is an $ir(G)$ with $|S| \le n^{k'}$. As usual, let $G' = G[V \setminus S]$. Then:

1. For every $1 \le j \le m$, there exists a literal vertex v such that $v \in N(c_j^l)$ for every $1 \le l \le n^{k'}$. Assume that this is not true for a specific j. Then, since $d_G(c_j^l) = d_G(s_j^l) = 4$, for every $1 \le l \le n^{k'}$, we have that S contains at least one vertex in $N[\{c_j^l, s_j^l\}]$, which does not belong to the literal vertices. That is, S contains at least one (non-literal) vertex for each one of the $n^{k'}$ copies of c_j. Observe also that even if this is the case, S would also have to contain at least one more vertex to, for example, stop $u_{i,2}^1$ and u_i^1, from having the same degree in G'. It follows that $|S| > n^{k'}$, which is a contradiction.

2. For every $1 \leq i \leq n$, S does not contain both v_i and v'_i. Assume this is not true for a specific i. Then, for every $1 \leq l \leq n^{k'}$, we have that $d_{G'}(u_i^l) = d_{G'}(u_{i,1}^l) = 2$, unless S also contains an additional vertex of the gadgets attached to v_i and v'_i, for each one of the $n^{k'}$ such gadgets. It follows that $|S| \geq n^{k'}$. Since we have also assumed that for a specific i, both v_i and v'_i belong to S, we have that $|S| > n^{k'}$, a contradiction.

3. For every $1 \leq i \leq n$, S contains at least one of v_i and v'_i. Assume this is not true for a specific i. Then, for every $1 \leq l \leq n^{k'}$, we have that $d_{G'}(u_i^l) = d_{G'}(u_{i,2}^l) = 4$, unless S also contains an additional vertex of the gadgets attached to v_i and v'_i, for each one of the $n^{k'}$ such gadgets. Even if this is the case, S would also have to contain at least one more vertex to, for example, stop c_1^1 and S_1^1 from having the same degree in G'. It follows that $|S| > n^{k'}$, which is a contradiction.

So from items 2. and 3. above, it follows that for each $1 \leq i \leq n$, S contains exactly one of v_i and v'_i. Now consider the following truth assignment: we assign the value *true* to every variable x_i if the corresponding literal vertex v_i belongs in S, and value *false* to every other variable. Now, from item 1. above, it follows that each clause C_j contains either a positive literal x_i which has been set to *true*, or a negative literal $\neg x_i$ which has been set to *false*. Thus F is satisfied, which is a contradiction.

Up to this point, we have shown that there exists a graph $G = (V, E)$ with $|V(G)| = N = O(n^{k'+1})$ where

- $\mathrm{I}(G) \leq n$ if F is satisfiable
- $\mathrm{I}(G) > n^{k'}$ otherwise.

Therefore, we have that $\mathrm{I}(G)$ is not $\mathcal{O}(n^{k'-1})$ approximable in polynomial time unless $\mathcal{P}$=$\mathcal{NP}$.

Now, since $N = |V(G)| = \Theta(n^{k'+1})$ and $2k = k' + 1$ we have $\mathcal{O}(n^{k'-1}) = \mathcal{O}(N^{\frac{k'-1}{k'+1}}) = \mathcal{O}(N^{1-\frac{2}{k'+1}}) = \mathcal{O}(N^{1-\frac{1}{k}})$. This ends the proof of this theorem. □

Now, we consider the case where G is regular bipartite graph. Below we present an upper bound to the size of $\mathrm{I}(G)$. This upper bound is then used to obtain a (simple) Δ-approximation of an optimal solution.

Theorem 4.3. *For any d-regular bipartite graph $G = (L, R, E)$ of order n we have that $\mathrm{I}(G) \geq n/2d$.*

Proof. Let S be an $ir^*(G)$ and $G' = G[(L \cup R) \setminus S]$. We distinguish two cases according to if S is a subset of one of the bipartitions L or R, or if S contains at least one vertex from each bipartition.

Let us first deal we the first case and assume, w.l.o.g, that $S \subseteq L$. If $|S| = |L|$ then the theorem holds (since $|L| = |R|$). Therefore, we consider the case $|S| < |L|$. Observe that any vertex $v \in R$ must have $d_{G'}(v) < d$. Indeed, since $S \subseteq L$, we know that for any vertex $u \in N_{G'}(v)$, we have $d_{G'}(v) = d$ and S is an irregulator. It follows that $N(S) = R$ so $d|S| \geq n/2$ which gives us $\mathrm{I}(G) \geq n/2d$.

Now, we consider the second case (S contains at least one vertex from each bipartition). Let $L_S = S \cap L$ and $R_S = S \cap R$. We partition L (R respectively) in to three sets: the $L_S = S \cap L$ ($R_S = S \cap R$ resp.), the $L_{d-1} = \{u \mid u \in L \setminus S \text{ and } d_{G'}(u) < d\}$ ($R_{d-1} = \{u \mid u \in R \setminus S \text{ and } d_{G'}(u) < d\}$) and the $L_d = L \setminus (L_S \cup L_{d-1})$ ($R_d = L \setminus (R_S \cup R_{d-1})$ resp.). Note that for all $u \in L_d \cup R_d$ we have $d_{G'}(u) = d$. Therefore all the vertices in L_d have exactly d neighbours in R_{d-1} (in both G and G') and all the vertices in R_d have exactly d neighbours in L_{d-1}. Furthermore, since $d_{G'}(u) < d$ for all $u \in L_{d-1} \cup R_{d-1}$, we know that each $u \in L_{d-1}$ has at least one neighbour in R_S and each $u \in R_{d-1}$ has at least one neighbour in L_S. Now we are going to find some upper bounds on the number of vertices in L_{d-1} and L_d.

Since each vertex $u \in L_{d-1}$ has at least one neighbour in R_S, we have that $|L_{d-1}| \leq d|R_S|$. Similarly we can show that $|R_{d-1}| \leq d|L_S|$.

Let E^* be the set of edges between L_d and R_{d-1}. Since, any vertex of L_d have exactly d neighbours in R_{d-1} we know that $|L_d| = |E^*|/d$. Since each vertex of R_{d-1} has at least one neighbours in L_S, it has at most $d-1$ edges in the E^*. Therefore $|E^*| \leq (d-1)|R_{d-1}| \leq d(d-1)|L_S|$. This gives us that $|L_d| \leq (d-1)|L_S|$.

Now, observe that $|L| = |L_S| + |L_{d-1}| + |L_d| \leq |L_S| + d|R_S| + (d-1)|L_S| = d(|L_S| + |R_S|)$. So, since $S = L_S \cup R_S$, we have that $\mathrm{I}(G) \geq n/2d$. □

Now recall that in any bipartite graph G, any bipartition of G is a vertex cover of G. Also observe that any vertex cover of a graph G, is also an irregulator of G. Indeed, deleting the vertices of any vertex cover of G, leaves us with an independent set, which is locally irregular. The next corollary follows from these observations and Theorem 4.3:

Corollary 4.4. *For any d-regular bipartite graph $G = (L, R, E)$, any of the sets L and R is a d-approximation of $ir^*(G)$.*

5 Parameterised complexity

As the problem of computing a minimal irregulator of a given graph G seems to be rather hard to solve, and even to approximate, we focused our efforts towards finding parameterised algorithms that can solve it. In Section 5.1 we present an FPT algorithm that calculates $\mathrm{I}(G)$ when parameterised by the size of the solution and Δ, the maximum degree of the graph. Then, we turn our attention towards graphs that are "close to being trees", that is graphs of bounded treewidth. Indeed, in Section 5.2 we provide an FPT algorithm that finds a minimum irregulator of G, when parameterised by the treewidth of the input graph and by Δ. Observe that both of our algorithms have to consider Δ as part of the parameter if they are to be considered as FPT. The natural question to ask at this point is whether we can have an FPT algorithm, when parameterised only by the size of the solution, or the treewidth of the input graph. In Section 5.3, we give a strong indication towards the negative answer for both cases, proving that, in some sense, the algorithms provided in Sections 5.1 and 5.2 are optimal.

5.1 FPT by the size of the solution and Δ

Let us first present the following lemma:

Lemma 5.1. *Let $G = (V, E)$ be a graph such that, G is not locally irregular, and S be an $ir^*(G)$. Furthermore let $G_v = (V', E')$ be the graph $G[V \setminus \{v\}]$ for a vertex $v \in S$. Then $\mathrm{I}(G_v) = \mathrm{I}(G) - 1$.*

Proof. First observe that $S' = S \setminus \{v\}$ must be an $ir(G_v)$ as $G_v[V' \setminus S'] = G[V \setminus S]$. It follows that $\mathrm{I}(G_v) \leq \mathrm{I}(G) - 1$. Assume that $\mathrm{I}(G_v) < \mathrm{I}(G) - 1$. Then these exists an S'' such that $|S''| < \mathrm{I}(G) - 1$ and S'' is an $ir(G_v)$. Since $G_v[V' \setminus S''] = G[V \setminus (S'' \cup \{v\})]$, we have that $S'' \cup \{v\}$ is an $ir(G)$ and $|S'' \cup \{v\}| = |S''| + 1 < \mathrm{I}(G)$. This is a contradiction. □

We are now ready to show the following:

Theorem 5.2. *For a given graph $G = (V, E)$ with $|V| = n$ and maximum degree Δ, and for $k \in \mathbb{N}$, there exists an algorithm that decides if $\mathrm{I}(G) \leq k$ in time $(2\Delta)^k n^{\mathcal{O}(1)}$.*

Proof. In order to decide if $\mathrm{I}(G) \leq k$ we are going to use a recursive algorithm. The algorithm has input (G, k), where $G = (V, E)$ is a graph and $k \geq 0$ is an integer. The basic idea of this algorithm, is to take advantage of Observation 2.5. We present the exact procedure in Algorithm 1.

Now, let us argue about the correctness and the efficiency of this algorithm. We claim that for any graph $G = (V, E)$ and any integer $k \geq 0$, Algorithm 1 returns yes if $\mathrm{I}(G) \leq k$ and no otherwise. Furthermore, the number of steps that the algorithm requires, is $f(k, n) = (2\Delta)^k n^{\mathcal{O}(1)}$, where $n = |V|$. We will prove this by induction on k.

Base of the induction ($k = 0$)**:** Here, we only need to check if G is locally irregular. Algorithm 1 does this in line 1 and returns yes if it is (line 2) and no otherwise (line 4). Furthermore, we can check if G is locally irregular in polynomial time. So, the claim is true for the base.

Induction hypothesis ($k = k_0 \geq 0$)**:** We assume that we have a $k_0 \geq 0$ such that Algorithm 1 can decide if any graph G with n vertices and maximum degree Δ has $\mathrm{I}(G) \leq k_0$ in $f(k_0, n) = (k_0 + 1)(2\Delta)^{k_0} n^{\mathcal{O}(1)}$ steps.

Algorithm 1 [IsIrregular(G, k) decision function]

Input: A graph $G = (V, E)$ and an integer $k \geq 0$.
Output: Is $\mathrm{I}(G) \leq k$ or not?

1: **if** G is irregular **then**
2:     **return** *yes*
3: **else if** $k = 0$ **then**
4:     **return** *no*
5: **else** ▷ $k > 0$ and G is not irregular
6:     $ans \leftarrow no$
7:     find an edge $vu \in E$ such that $d_G(v) = d_G(u)$
8:     **for all** $w \in N_G[\{u, v\}]$ **do**
9:         set $G_w = G[V \setminus \{w\}]$
10:         **if** IsIrregular($G_w, k-1$) returns *yes* **then**
11:             $ans \leftarrow yes$
12:     **return** ans

Induction step ($k = k_0 + 1$)**:** Let $G = (V, E)$ be a graph. If G is locally irregular then $\mathrm{I}(G) = 0$ and Algorithm 1 answers correctly (in line 2). Assume that G is not locally irregular; then there exist an edge $vu \in E$ such that $d_G(v) = d_G(v)$. Now, let S be an $ir^*(G)$. It follows from Observation 2.5 that S must include at least one vertex $w \in N_G[\{v, u\}]$. Since Algorithm 1 considers all the vertices in $N_G[\{v, u\}]$, at some point it also considers the vertex $w \in S \cap N_G[\{v, u\}]$. Now, observe that for any $x \in S$, the set $S_x = S \setminus \{x\}$ is an $ir^*(G_x)$, where $G_x = G[V \setminus \{x\}]$. Furthermore, by Lemma 5.1, we have $\mathrm{I}(G_x) \leq k - 1 = k_0$ iff $\mathrm{I}(G) \leq k$. By the induction hypothesis, we know that the algorithm answers correctly for all the instances (G_x, k_0). Thus, if $\mathrm{I}(G) \leq k = k_0 + 1$, there must exist one instance (G_w, k_0), where $w \in S \cap N_G[\{v, u\}]$, for which the Algorithm 1 returns yes. Therefore the algorithm answers for $(G, k_0 + 1)$ correctly. Finally, this process request $n^{\mathcal{O}(1)}$ steps in order to check if the graph is locally irregular and $2\Delta f(k-1, n-1)$ steps (by induction hypothesis) in order to check if for any graph G_x we have $\mathrm{I}(G_x) \leq k - 1 = k_0$ (where $x \in N[\{u, v\}]$). So, the algorithm decides in $n^{\mathcal{O}(1)} + 2\Delta f(k-1, n-1) \leq n^{\mathcal{O}(1)} + 2\Delta k (2\Delta)^{k-1}(n-1)^{\mathcal{O}(1)} \leq n^{\mathcal{O}(1)} + k(2\Delta)^k n^{\mathcal{O}(1)} \leq (k+1)(2\Delta)^k n^{\mathcal{O}(1)}$ steps. Finally, note that $k \leq n - 1$, and the result follows. □

5.2 FPT by Treewidth and Δ

Theorem 5.3. *For a given a graph $G = (V, E)$ and a nice tree decomposition of G, there exists an algorithm that returns $\mathrm{I}(G)$ in time $\Delta^{2tw} n^{\mathcal{O}(1)}$, where tw is the treewidth of the given decomposition and Δ is the maximum degree of G.*

Proof. As the techniques we are going to use are standard, we are sketching some of the introductory details. For more details on tree decompositions (definition and terminology) see [20]. We are going to perform dynamic programming on the nodes of the given nice tree decomposition (see [10] for the definition of a nice tree decomposition). For a node t of the given tree decomposition of G, we denote by B_t the bag of this node and by $B_t^{\downarrow}$ the set of vertices of the graph that appears in the bags of the nodes of the subtree with t as a root. Observe that $B_t \subseteq B_t^{\downarrow}$.

The idea behind our algorithm, is that for each node t we store all the sets $S \subseteq B_t^{\downarrow}$ such that S is an $ir(G, B_t^{\downarrow} \setminus B_t)$. We will also store the necessary "conditions" (explained more in what follows) such that if there exists a set S', where $S' \setminus S \subseteq V \setminus B_t^{\downarrow}$, that meets these conditions, then S' is an $ir(G, B_t^{\downarrow})$. Observe that if we manage to do such a thing for every node of the tree decomposition, then we can find $\mathrm{I}(G)$. To do so, it suffices to check the size of all the irregulators we stored for the root r of the tree decomposition, which also meet the conditions we have set. In that way, we can find a set S that is an $ir(G, B_r^{\downarrow} \setminus B_r)$, satisfies our conditions and is of minimum order, and since $B_r^{\downarrow} = V$, this set S is a minimum irregulator of G and $\mathrm{I}(G) = |S|$.

Let us now present the actual information we are keeping for each node. Assume that t is a node of the tree decomposition and $S \subseteq B_t^{\downarrow}$ is an irregulator of $B_t^{\downarrow} \setminus B_t$ in G, *i.e.*, S is an $ir(G, B_t^{\downarrow} \setminus B_t)$.

For this S we want to remember which vertices of B_t belong to S as well as the degrees of the vertices $v \in B_t \setminus S$ in $G[B_t^{\downarrow} \setminus B_t]$. This can be done by keeping a table D of size $tw+1$ where, if $v \in B_t \setminus S$ we set $D(v) = d_{G[B_t^{\downarrow} \setminus B_t]}(v)$ and if $v \in B_t \cap S$ we set $D(v) = \emptyset$ (slightly abusing the notation, by $D(v)$ we mean the position in the table D that corresponds to the vertex v). Like we have already said, we are going to keep some additional information about the conditions that could allow these sets to be extended to irregulators of $B_t^{\downarrow}$ in G if we add vertices of $V \setminus B_t^{\downarrow}$. For that reason, we are also going to keep a table with the "*target degree*" of each vertex; in this table we assign to each vertex $v \in B_t \setminus S$ a degree d_v such that, if there exists S' where $S' \setminus S \subseteq V \setminus B_t^{\downarrow}$ and for all $v \in B_t \setminus S$ we have $d_{G[V \setminus S']}(v) = d_v$, then S is an $ir(G, B_t^{\downarrow})$. This can be done by keeping a table T of size $tw+1$ where for each $v \in B_t \setminus S$ we set $T(v) = i$, where i is the target degree, and for each $v \in B_t \cap S$ we set $T(v) = \emptyset$. Such tables T will be called *valid* for S in B_t. Finally, we are going to keep the set $X = S \cap B_t$ and the value $min = |S|$. Note that the set X does not gives us any extra information, but we keep it as it will be useful to refer to it directly.

To sum up, for each node t of the tree decomposition of G, we keep a set of quadruples (X, D, T, min), each quadruple corresponding to a valid combination of a set S that is an $ir(G, B_t^{\downarrow} \setminus B_t)$ and the target degrees for the vertices of $B_t \setminus S$. Here it is important to say that when treating the node B_t, for every two quadruples (X_1, D_1, T_1, min_1) and (X_2, D_2, T_2, min_2) such that for all $v \in B_t$ we have that $D_1(v) = D_2(v)$ and $T_1(v) = T_2(v)$ (this indicates that $X_1 = X_2$ as well), then we are only going to keep the quadruple with the minimum value between min_1 and min_2 as we will prove that this is enough in order to find $\mathrm{I}(G)$.

Claim 5.4. *Assume that for a node t, we have two sets S_1 and S_2 that are both $ir(G, B_t^{\downarrow} \setminus B_t)$, and that T is a target table that is common to both of them. Furthermore, assume that $(X_1, D_1, T, |S_1|)$ and $(X_2, D_2, T, |S_2|)$ are the quadruples we have to store for S_1 and S_2 respectively (both respecting T), with $D_1(v) = D_2(v)$ for every $v \in B_t$. Then for any set $S \subseteq V \setminus B_t^{\downarrow}$ such that $d_{G[V \setminus (S_1 \cup S)]}(v) = T(v)$ for all $v \in B_t$, we also have that $d_{G[V \setminus (S_2 \cup S)]}(v) = T(v)$ for all $v \in B_t$.*

Proof of the claim. Assume that we have such an S for S_1, let v be a vertex in B_t and $H = G[v \cup ((V \setminus B_t^{\downarrow}) \setminus S)]$ (observe that H does not depend on S_1 or S_2). Since $d_{G[V \setminus (S_1 \cup S)]}(v) = T(v)$, we know that in the graph H, v has exactly $T(v) - D_1(v)$ neighbours (as $D_1(v) = d_{G[B_t^{\downarrow} \setminus S_1)]}(v)$). Now, since $D_1(v) = D_2(v) = d_{G[B_t^{\downarrow} \setminus S_2]}(v)$ we have that $d_{G[V \setminus S_2 \cup S]}(v) = T(v)$. Therefore, the claim holds. ◇

Simply put, Claim 5.4 states that for any two quadruples $Q_1 = (X, D, T, min_1)$ and $Q_2 = (X, D, T, min_2)$, any extension S of S_1 is also an extension of S_2 (where S_1 and S_2 are the two sets that correspond to Q_1 and Q_2 respectively). Therefore, in order to find the minimum solution, it is sufficient to keep the quadruple that has the minimum value between min_1 and min_2.

Now we are going to explain how we create all the quadruples (X, D, T, min) for each type of node in the tree decomposition. First we have to deal with the Leaf Nodes. For a Leaf node t we know that $B_t = B_t^{\downarrow} = \emptyset$. Therefore, we have only one quadruple (X, D, T, min), where the size of both D and T is zero (so we do not need to keep any information in them), $S = \emptyset$ and $min = |S| = 0$.

Now let t be an Introduce node; assume that we have all the quadruples (X, D, T, min) for its child c and let v be the introduced vertex. By construction, we know that v is introduced in B_t and thus it has no neighbours in $B_t^{\downarrow} \setminus B_t$. It follows that if $S \subseteq B_c^{\downarrow}$ is an irregulator for $B_c^{\downarrow} \setminus B_c$, then both S and $S \cup \{v\}$ are irregulators for $B_t^{\downarrow} \setminus B_t$ in G. Furthermore, there is no set $S \subseteq B_t^{\downarrow} \setminus \{v\}$ that is an irregulator of $B_t^{\downarrow} \setminus B_t$ and is not an irregulator of $B_c^{\downarrow} \setminus B_c$. So, we only need to consider two cases for the quadruples we have to store for c; if v belongs in the under-construction irregulator of $B_t^{\downarrow} \setminus B_t$ in G or not.

Case 1. (v is in the irregulator): Observe that for any S that is an $ir(G, B_c^{\downarrow} \setminus B_c)$, which is stored in the quadruples of B_c, for every $u \in B_c \setminus S$, we have that $d_{G[B_c^{\downarrow} \setminus S]}(u) = d_{G[B_t^{\downarrow} \setminus (S \cup \{v\})]}(u)$. Moreover, for any target table T which is valid for S in c, the target table T' is valid for $S \cup \{v\}$ in t, where T' is almost the same as T, the only difference being that T' also contains the information about v, *i.e*, $T'(v) = \emptyset$. So, for each quadruple (X, D, T, min) in c, we need to create one quadruple $(X \cup \{v\}, D', T', min + 1)$ for t, where D' is the almost the same as D, except that it also contains the information about v, *i.e.*, $D'(v) = \emptyset$.

Case 2. (v is not in the irregulator): Let $q = (X, D, T, min)$ be a stored quadruple of c and S be the corresponding $ir(G, B_c^{\downarrow} \setminus B_c)$. We will first explain how to construct D' of t, based on q. Observe that the only change between $G[B_c^{\downarrow} \setminus S]$ and $G[B_t^{\downarrow} \setminus S]$, is that in the latter there exist some new edges from v to some of the vertices of B_c. Therefore, for each vertex $u \in B_c \setminus X$ we set $D'(u) = D(u)+1$ if $u \in N[v]$ and $D'(u) = D(u)$ otherwise. Finally, for the introduced vertex v, we set $D'(v) = |N(v) \cap (B_c \setminus X)|$. We will now treat the target degrees for t. Observe that the target degrees for each vertex in $B_t \setminus \{v\}$ are the same as in T, since v only has edges incident to vertices in B_t. Now, we only need to decide which are the valid targets for v. Since $d_{G[B_t^{\downarrow} \setminus S]}(v) = D'(v)$, we know that for every target t', we have that $D'(v) \leq t' \leq \Delta$. Furthermore, we can not have the target degrees of v to be the same as the targets of one of its neighbours in B_c (these values are stored in T), as, otherwise, any valid target table T' of t would lead to adjacent vertices in B_t having the same degree. Let $\{t_1, \dots, t_k\} \subset \{D(v), \dots, \Delta\}$ be an enumeration of all the valid targets for v (i.e. $t_i \neq T(u)$ for all $u \in N[v] \cap B_c \setminus X$). Then, for each quadruple (X, D, T, min) in c, and for each $i = 1, \dots, k$, we need to create the quadruple (X, D', T_i, min), such that $T_i(u) = T(u)$ for all $u \in B_c$ and $T_i(v) = t_i$. In total, we have $k \leq \Delta$ such quadruples.

Now, let us explain how we deal with the Join nodes. Assume that t is a Join Node with c_1 and c_2 as its two children in the tree decomposition. Here, it is important to mention that $B_{c_1} = B_{c_2}$ and $(B_{c_1}^{\downarrow} \setminus B_{c_1}) \cap (B_{c_2}^{\downarrow} \setminus B_{c_2}) = \emptyset$. Assume that there exists an irregulator S of $B_t^{\downarrow} \setminus B_t$ in G, a valid target table T of S, and let (X, D, T, min) be the quadruple we need to store in t for this pair (S, T). Observe that this pair (S, T) is valid for both c_1 and c_2, so we must already have stored at least one quadruple in each node. Let $X \subseteq B_t$ and a target table T such that (X, D_1, T, min_1) and (X, D_2, T, min_2) are stored for c_1 and c_2 respectively. We create the quadruple (X, D, T, min) for t by setting $D(u) = D_1(u) + D_2(u) - d_{G[B_t \setminus X]}(u)$ for all $u \in B_t \setminus X$, $D(u) = \emptyset$ for all $u \in X$ and $min = min_1 + min_2 - |X|$. Observe that these are the correct values for the $D(u)$ and min, as otherwise we would count $d_{G[B_t \setminus X]}(u)$ and $|X|$ twice. Finally, we need to note that we do not store any quadruple (X, D, T, min) we create for the Join Note such that $D(u) > T(u)$ for a vertex $u \in B_t \setminus X$. This is because for such quadruples, the degree of vertex u will never be equal to any of the target degrees we have set, as it can only increase when we consider any of the ancestor (*i.e.* parent, grantparent etc.) nodes of t.

Finally, we need to treat the Forget nodes. Let t be a Forget node, c be the its child and v be the forgotten vertex. Assume that we have to store in t a quadruple (X, D, T, min). Then, since $X = B_t \cap S$ for an irregulator S of B_t in G, we know that in c we must have already stored a quadruple (X', D', T', min') such that, $X' = S \cap B_c$, $D'(u) = D(u)$ for all $u \in B_c$, $T'(u) = T(u)$ for all $u \in B_c$ and $min' = min$. Therefore, starting from the stored quadruples in c, we can create all the quadruples of t. For each quadruple (X', D', T', min') in c, we create at most one quadruple (X, D, T, min) for t by considering two cases; the forgotten vertex v_f belongs to X' or not.

Case 1. (v belongs to X'): then the quadruple (X, D, T, min) is almost the same as $(X', D'$ $, T', min')$, with the following differences: $X = X' \setminus \{v\}$, $min = min'$, $D(u) = D'(u)$ and $T(u) = T'(u)$ for all $u \in B_t$ and the tables D and T do not include any information for v as this vertex does not belong to B_t anymore.

Case 2. (v does not belong to X'): we will first check if $D'(v_f) = T'(v_f)$ or not. This is important because the degree of the v will never again be considered by our algorithm, and thus its degree will remain unchanged. So, if $D'(v_f) = T'(v_f)$, we create the quadruple (X, D, T, min) where $X = X'$, $min = min'$, $D(u) = D'(u)$ and $T(u) = T'(u)$ for all $u \in B_t$ and the tables D and T do not include any information for v.

For the running time, observe that the number of nodes of a nice tree decomposition is $\mathcal{O}(tw \cdot n)$ and all the other calculations are polynomial in $n + m$. Thus we only need to count the different quadruples in each node. Now, for each vertex v, we either include it in X or we have $\Delta+1$ options for the value $D(u)$ and $\Delta + 1 - i$ for the value $T(u)$ if $D(u) = i$. Also, for sufficiently large Δ, we have that $1 + \sum_{i=0}^{\Delta}(\Delta + 1 - i) < \Delta^2$. Furthermore, the set X and the value min do not increase the number of quadruples because $X = \{u \mid D(u) = \emptyset\}$ and from all quadruples (X, D_1, T_1, min_1), (X, D_2, T_2, min_2) such that $D_1(u) = D_2(u)$ and $T_1(u) = T_2(u)$ for all $u \in B_t$, we only keep one of them (by Claim 5.4).

In total, the number of different quadruples in each node is Δ^{2tw}, and therefore the algorithm decides in $\Delta^{2tw} n^{\mathcal{O}(1)}$ time. □

It is worth noting that the algorithms of Theorem 5.2 and 5.3 can be used in order to also return an $ir^*(G)$.

5.3 W-Hardness

In this section we show a strong indication that there can be no FPT algorithm that calculates an optimal irregulator of the input graph G, when parameterised by just the size of the solution or the treewidth of G. To do so, we present two linear-fpt reductions. The first is from the Dominating Set problem, when parameterised by the size of the solution, and the second is from the List Colouring problem, when parameterised by the treewidth of the graph.

Theorem 5.5. *Let G be a graph and $k \in \mathbb{N}$. Deciding if $\mathrm{I}(G) \leq k$ is $\mathrm{W}[2]$-hard, when parameterised by k.*

Proof. The reduction is from the Dominating Set problem, which was shown to be W[2]-complete when parameterised by the size of the solution (*e.g.* in [18]). In that problem, a graph $H = (V, E)$ and an integer k are given as input. The question asked, is whether there exists a set $D \subseteq V$ of order at most k (called a *dominating set of* H), such that $V = N[D]$.

Let $H = (V, E)$ be a graph and $k \in \mathbb{N}$. We will construct a graph $G = (V', E')$ such that H has a dominating set of order at most k if and only if G has an irregulator of order at most k. We begin by defining an arbitrary enumeration of the vertices of V. That is, $V = \{v_1, \dots, v_n\}$. The graph G is built starting from a copy of the graph H. To avoid any confusion in what is to follow, we will always use H to denote the original graph, and $G|_H = G[\{v'_1, \dots, v'_n\}]$ to denote the copy of H that lies inside G (where the indices of the v'_is' are the same as the indices of the corresponding v_is'). Then, for each $1 \leq i \leq n$, we attach the necessary number of pending vertices (meaning vertices of degree 1) to the vertex v'_i, so that the degree of v'_i becomes equal to $i \cdot n$. Finally, for each v'_i, let u'_i be one of its newly attached pending vertices, and attach the necessary number of new pending vertices to u'_i, so that its degree becomes equal to that of v'_i. The resulting graph is G. To be clear, for every vertex v of G, we either have that $v = v'_i$ or $v = u'_i$, or that v is a vertex pending from v'_i or u'_i (for some $1 \leq i \leq n$). Note also that for each $1 \leq i \leq n$, we have that $d_G(v'_i) = d_G(u'_i) = i \cdot n$.

Now let D be a dominating set of H, with $|D| = m \leq k$, and let D' be the subset of V' that corresponds to the vertices of D. That is, $D' = \{v'_i \in V' : v_i \in D\}$. We claim that the graph $G' = G[V' \setminus D']$ is locally irregular. Indeed, for every $1 \leq i \leq n$, let $\alpha(i)$ be the number of neighbours of v_i that belong to D. Observe that since D is a dominating set of H, we have that $1 \leq \alpha(i) \leq n-1$. Now, for every vertex v'_i in V', we have that either $v'_i \in D'$, in which case v'_i does not belong to G', or $d_{G'}(v'_i) = d_G(v'_i) - \alpha(i) < d_{G'}(u'_i)$. Moreover, for every $1 \leq i < j \leq n$, if $v'_i, v'_j \notin D'$, we have that $d_G(v'_j) - d_G(v'_i) \geq n$, and thus $d_{G'}(v'_j) - d_{G'}(v'_i) = d_G(v'_j) - \alpha(j) - d_G(v'_i) + \alpha(i) \geq n + \alpha(i) - \alpha(j) \geq 2$. Finally, every pending vertex l of G' is attached to either u'_i or v'_i, which have degree (in G') strictly larger than 1. It follows that D' is an irregulator of G with $|D'| = m \leq k$, and thus $\mathrm{I}(G) \leq k$.

For the other direction, assume that $\mathrm{I}(G) \leq k$ and let S be an $ir(G)$, with $|S| = k$, and $G' = G[V' \setminus S]$. For each $1 \leq i \leq n$, let $S_i = N[v'_i] \cup N(u'_i)$. We claim that for every i, we have $S \cap S_i \neq \emptyset$. Assume that this is not true, *i.e.*, that there exists an i_0 such that $S_{i_0} \cap S = \emptyset$. Then, by deleting the vertices of S from G, the degrees of v'_{i_0} and u'_{i_0} remain unchanged. Formally, we have that $d_{G'}(v'_{i_0}) = d_G(v'_{i_0}) = d_G(u'_{i_0}) = d_{G'}(u'_{i_0})$. This is a contradiction since S is an irregulator of G. Now, we consider the set S', defined as follows:

- Start with $S' = S$.
- For each i, while there exists a vertex $v \in S_i \cap S'$ such that $d_G(v) = 1$ or $v = u'_i$, remove v from S' and add v'_i to S'.

Clearly, we have that S' only contains vertices from $V(G|_H)$ and that $|S'| \leq |S| = k$. Also, from the construction of S', for every i, we have that $S_i \cap S' \neq \emptyset$. It follows that for every vertex v'_i, we either have $v'_i \in S'$ or there exists a vertex $v \in N(v'_i) \cap V(G|_H)$ such that $v \in S'$. Going back to H, let $D = \{v_i : v'_i \in S'\}$. It is clear that D is a dominating set of H of order at most k. This finishes our reduction.

Finally, note that throughout the above described reduction, the value of the parameter of the two problems is the same (in both of them, the parameter has the value k). Moreover, the construction of the graph G is achieved in polynomial time in regards to n. These observations conclude our proof. $\square$

Theorem 5.6. *Let G be a graph with treewidth tw, and $k \in \mathbb{N}$. Deciding if $\mathrm{I}(G) = k$ is $W[1]$-hard when parameterised by tw.*

Proof. We will present a reduction from the LIST COLOURING problem: the input consists of a graph $H = (V, E)$ and a list function $L : V \to \mathcal{P}(\{1, \dots, k\})$ that specifies the available colours for each vertex $u \in V$. The goal is to find a proper colouring $c : V \to \{1, \dots, k\}$ such that $c(u) \in L(u)$ for all $u \in V$. When such a colouring exists, we say that (H, L) is a *yes-instance* of LIST COLOURING. This problem is known to be $W[1]$-hard when parameterised by the treewidth of H [20].

Now, starting from an instance (H, L) of LIST COLOURING, we will construct a graph $G = (V', E')$ (see Figure 7 (a)) such that:

- $|V'| = \mathcal{O}(|V|^6)$,
- $tw(G) = tw(H)$ and
- $\mathrm{I}(G) = nk$ if and only if (H, L) is a yes-instance of LIST COLOURING.

Before we start with the construction of G, let us give the following observation.

Observation 5.7. *Let (H, L) be an instance of* LIST COLOURING *where $H = (V, E)$ and there exists a vertex $u \in V$ such that $|L(u)| > d(u)$. Then the instance $(H[V \setminus \{u\}], L')$, where $L'(v) = L(v)$ for all $v \in V \setminus \{u\}$, is a yes-instance of* LIST COLOURING *if and only if (H, L) is a yes-instance of* LIST COLOURING.

Indeed, observe that for any vertex $u \in V$, by any proper colouring c of H, $c(u)$ only has to avoid $d(u)$ colours. Since $|L(u)| > d(u)$, we will always have a spare colour to use on u that belongs in $L(u)$. From the previous observation, we can assume that in our instance, for all $u \in V$, we have $|L(u)| \leq d(u)$. Furthermore, we can deduce that $k \leq n(n-1)$ as the degree of any vertex is at most $n-1$. Finally, let us denote by $\overline{L}(u)$ the set $\{0, 1, \dots, k\} \setminus L(u)$. It is important to note here that for every $u \in V$, the list $L(u)$ contains at least one element belonging in $\{1, \dots, k\}$. It follows that $\overline{L}(u)$ also contains at least one element, the colour 0. To sum up, we have that $1 \leq |\overline{L}(u)| \leq k$.

Now, we present the three gadgets we are going to use in the construction of G. First, we have the "forbidden colour gadget" H_i, which is a star with i leaves (see Figure 7(c)). When we say that we attach a copy of H_i on a vertex v of a graph G, we mean that we add H_i to G and we identify the vertices v and w_2 (where here and in what follows, we are using the naming illustrated in Figure 7 when talking about the vertices w_1, w_2, w_3, v_1 and v_2). The second, will be the "degree gadget", which is presented in Figure 7(b). Finally, we have the "horn gadget", which is a path on three vertices (see Figure 7(d)). We define the operation of attaching these two gadgets on a vertex v of a graph G similarly to how we defined this operation for the forbidden colour gadget (each time using the appropriate w_1 or w_3, according to if it is a degree or a horn gadget respectively).

In order to construct G, we start from a copy of H. Let us use $G|_H$ to denote the copy of H that lies inside of G and, for each vertex $u \in V$, let u' be its copy in V'. We will call the set of these vertices U. That is, $U = \{v \in V(G|_H)\}$. Then, we are going to attach several copies of each gadget to u', for each vertex $u' \in U$. We start by attaching k copies of the degree gadget to each vertex $u' \in U$. Then, for each $u \in V$ and each $i \in \overline{L}(u)$, we attach one copy of the forbidden colour gadget H_{2n^3-i} to the vertex u'. Finally, for each $u' \in U$, we attach to u' as many copies of the horn gadget as are needed, in order to have $d_G(u') = 2n^3$.

Before we continue, observe that, for sufficiently large n, we have attached more than n^3 horn gadgets to each vertex of U. Indeed, before attaching the horn gadgets, each vertex $u' \in U$ has $d_G(u) \leq n-1$ neighbours in U, k neighbours from the degree gadgets and at most $k < n^2$ neighbours from the forbidden colour gadgets (recall that $|\overline{L}(u)| \leq k$). We will now show that $|V'| = \mathcal{O}(n^6)$. For that purpose, let us calculate the number of vertices in all the gadgets attached to a single vertex $u' \in U$. First, we have $5k < 5n^2$ vertices in the degree gadgets. Then, we have

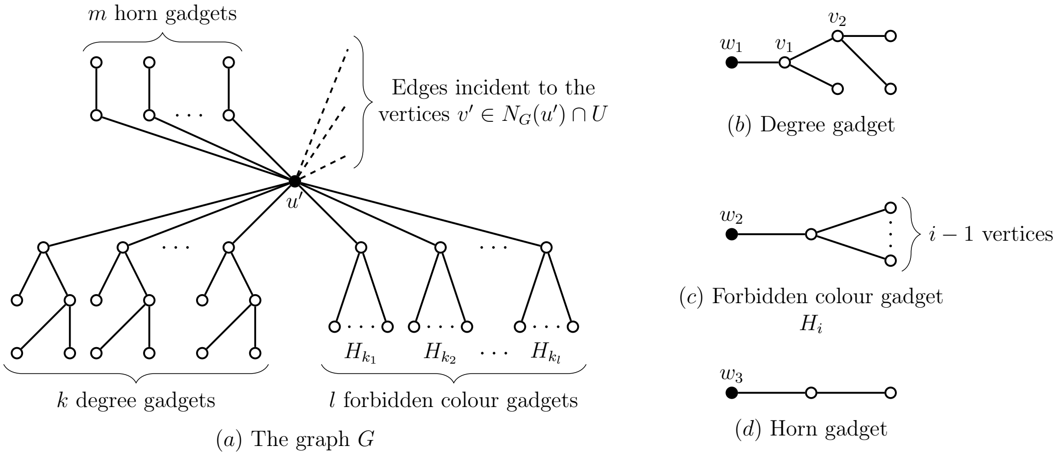


Figure 7: In (a) we illustrate the construction of G, as it is described in the proof of Theorem 5.6. The black vertex represents every vertex that belongs in U. For the specific vertex u' shown in the figure, we have that $\overline{L}(u) = \{c_1, \ldots, c_l\}$ and $k_i = n^3 - c_i$ for all $i = 1, \ldots, l$. We also have that $m = 2n^3 - d_G(u) - k - l$.

less than $4n^3$ vertices in the horn gadgets (as we have less that $2n^3$ such gadgets). Finally, we have at most $k < n^2$ forbidden colour gadgets, each one of which containing at most $2n^3$ vertices. So, for each vertex $u' \in U$, we have at most $2n^5 + 4n^3 + 5n^2$ vertices in the gadgets attached to u'. Therefore, we have $|V'| = \mathcal{O}(n^6)$.

Before we prove that $\mathrm{I}(G) \leq nk$ if and only if (H, L) is a yes-instance of List Colouring, we need to argue about two things. First, about the treewidth of the graph G and second, about the minimum value of $\mathrm{I}(G)$. Since our construction only attaches trees to each vertex of $G|_H$ (and recall that a tree has a treewidth of 1 by definition), we know that $tw(G) = tw(G|_H) = tw(H)$. As for $\mathrm{I}(G)$, we will show that it has to be at least equal to nk. For that purpose we have the following two claims.

Claim 5.8. *Let S be an $ir(G)$ and $S \cap U \neq \emptyset$. Then $|S| > n^3$.*

Proof of the claim. Let $u' \in S \cap U$. By construction, G contains more than n^3 horn gadgets that are attached to u'. Therefore, by deleting u', we create more than n^3 copies of the P_2 graph, each one of which forces us to include at least one of its vertices in S. Hence, $|S| > n^3$. ◇

Claim 5.9. *Let S be an $ir(G)$ and $S \cap U = \emptyset$. Then $|S| \geq nk$. In particular, S includes at least one vertex from each copy of the degree gadget used in the construction of G.*

Proof of the claim. Let D be a copy of the degree gadget, attached to some vertex $u' \in U$. Observe that we have $d_G(v_1) = d_G(v_2)$. It follows by Observation 2.5, that S contains at least one vertex v in $N[\{v_1, v_2\}]$, and since $u' \notin S$, this v is a vertex other than w_1. The result follows from the fact that the same arguments hold for any degree gadget attached to any vertex of U (recall that $|U| = n$ and we have attached k copies of the degree gadget to each one of the vertices of U). Hence, $|S| \geq nk$. ◇

By the previous two claims, we conclude that $\mathrm{I}(G) \geq nk$. We are ready to show that, if (H, L) is a yes-instance of List Colouring, then there exists a set $S \subseteq V'$ such that S is an $ir(G)$ and $|S| = nk$. Let c be a proper colouring of H such that $c(u) \in L(u)$ for all $u \in V$. We will construct an $ir(G)$ as follows. For each $u \in V$, we partition (arbitrarily) the k degree gadgets attached to the vertex u' to $c(u)$ "good" and $(k - c(u))$ "bad" degree gadgets. For each good degree gadget, we add the copy of the vertex v_1 of that gadget to S and for each bad degree gadget we add the copy of the vertex v_2 of that gadget to S. This process creates a set S of size nk, as it includes k distinguished vertices for each vertex $u' \in U$.

Now we need to show that S is an $ir(G)$. Let $G' = G[V' \setminus S]$; observe that each vertex $u' \in U$ has degree $d_{G'}(u') = 2n^3 - c(u)$. Therefore, u' does not have the same degree as any of its neighbours that do not belong in U. Indeed, for every $v \in N_{G'}(u') \setminus U$, we have that $d_{G'}(v) \in \{1,2\}$ (if v belongs to a bad degree or a horn gadget) or $d_{G'}(v) \in \{2n^3 - i : i \in \overline{L}(u)\}$ (if v belongs to a forbidden colour gadget). Furthermore, since c is a proper colouring of H, for all $uv \in E$, we have that $c(u) \neq c(v)$. This gives us that for any edge $u'v' \in E'$ with $u', v' \in U$, we have that $d_{G'}(u') = 2n^3 - c(u) \neq 2n^3 - c(v) = d_{G'}(v')$.

So, we know that for every vertex $u' \in U$, there is no vertex $w \in N_{G'}(u')$ such that $d_{G'}(u') = d_{G'}(w)$. It remains to show that, in G', there exist no two vertices belonging to the same gadget, which have the same degrees. First of all, we have that S does not contain any vertex from any of the horn and forbidden colour gadgets, nor from U. Thus any adjacent vertices belonging to these gadgets have different degrees. Last, it remains to check the vertices of the degree gadgets. Observe that for any copy of the degree gadget, S contains either v_1 or v_2. In both cases, after the deletion of the vertices of S, any adjacent vertices belonging to any degree gadget have different degrees. Therefore, S is an $ir(G)$ of order nk and since $\mathrm{I}(G) \geq nk$ we have that $\mathrm{I}(G) = nk$.

Now, for the opposite direction, assume that there exists a set $S \subseteq V'$ such that S is an $ir^*(G)$ and $|S| = nk$. Let $G' = (V'', E'')$ be the graph $G[V' \setminus S]$. It follows from Claim 5.8 and Claim 5.9, that $S \cap U = \emptyset$ and that S contains exactly one vertex from each copy of the degree gadget in G and no other vertices. Consider now the colouring c of H defined as $c(u) = 2n^3 - d_{G'}(u')$. We will show that c is a proper colouring for H and that $c(u) \in L(u)$. First, we have that c is a proper colouring of H. Indeed, for any edge $uv \in E$, there exists an edge $u'v' \in E''$ (since $S \cap U = \emptyset$). Since G' is locally irregular we have that $d_{G'}(u') \neq d_{G'}(v')$, an thus $c(u) \neq c(v)$. It remains to show that $c(u) \in L(u)$ for all $u \in V$. First observe that, during the construction of G, we attached exactly k degree gadgets to each $u' \in U$. It follows that $d_{G'}(u') = 2n^3 - j$ and $c(u) = j$ for a $j \in \{0, 1, \dots, k\}$. It is sufficient to show that $j \notin \overline{L}(u)$. Since S contains only vertices from the copies of the degree gadgets, we have that each $u' \in U$ has exactly one neighbour of degree $2n^3 - i$ for each $i \in \overline{L}(u)$ (this neighbour is a vertex of the H_i forbidden colour gadget that was attached to u'). Furthermore, for all $u' \in U$, since G' is locally irregular, we have that $d_{G'}(u') \neq 2n^3 - i$ for all $i \in \overline{L}(u)$. Equivalently, $d_{G'}(u') = 2n^3 - j$ for any $j \in L(u)$. Thus, $c(u) \in L(u)$ for all $u \in V$. □

Note that the reductions presented in the proofs of Theorem 5.5 and Theorem 5.6 are linear fpt-reductions. Additionally we know that

- there is no algorithm that answers if a graph G of order n has a Dominating Set of size at most k in time $f(k)n^{o(k)}$ unless the ETH fails [27] and
- there is no algorithm that answers if an instance (G, L) of the List Colouring is a yes-instance in time $\mathcal{O}^*(f(tw)n^{o(tw)})$ unless the ETH fails [20].

So, the following corollary holds.

Corollary 5.10. *Let G be a graph of order n and assume the ETH. For $k \in \mathbb{N}$, there is no algorithm that decides if $\mathrm{I}(G) \leq k$ in time $f(k)n^{o(k)}$. Furthermore, assuming that G has treewidth tw, there is no algorithm that computes $\mathrm{I}(G)$ in time $\mathcal{O}^*(f(tw)n^{o(tw)})$.*

6 Conclusion

In this work we introduce the problem of identifying the largest locally irregular induced subgraph of a given graph. There are many interesting directions that could be followed for further research. An obvious one is to investigate whether the problem of calculating $\mathrm{I}(G)$ remains $\mathcal{NP}$-hard for other, restricted families of graphs. The first candidate for such a family would be the one of chordal graphs. On the other hand, there are some interesting families, for which the problem of computing an optimal irregulator could be decided in polynomial time, such as split graphs. Also, it could be feasible to conceive approximation algorithms for regular bipartite graphs, which have a better approximation ratio than the (simple) algorithm we present. The last aspect we find intriguing, is to study the parameterised complexity of calculating $\mathrm{I}(G)$ when considering other parameters, like the size of the minimum vertex cover of G, with the goal of identifying a parameter that suffices, by itself, in order to have an FPT algorithm. Finally, it is worth investigating whether calculating $\mathrm{I}(G)$ could be done in FPT time (parameterised by the size of the solution) in the case where G is a planar graph.

References

[1] A. Agra, G. Dahl, T. A. Haufmann, and S. J. Pinheiro. The k-regular induced subgraph problem. *Discretete Applied Mathematics*, 222:14–30, 2017.

[2] Y. Alavi, A. Boals, G. Chartrand, O. Oellermann, and P. Erdős. K-path irregular graphs. *Congressus Numerantium*, 65, 01 1988.

[3] Y. Alavi, G. Chartrand, F. R. K. Chung, P. Erdös, R. L. Graham, and O. R. Oellermann. Highly irregular graphs. *Journal of Graph Theory*, 11(2):235–249, 1987.

[4] A. Ali, G. Chartrand, and P. Zhang. *Irregularity in Graphs*. Springer briefs in mathematics. Springer, 2021.

[5] Y. Asahiro, H. Eto, T. Ito, and E. Miyano. Complexity of finding maximum regular induced subgraphs with prescribed degree. *Theoretical Computer Science*, 550:21–35, 2014.

[6] O. Baudon, J. Bensmail, J. Przybyło, and M. Wozniak. On decomposing regular graphs into locally irregular subgraphs. *European Journal of Combinatorics*, 49:90–104, 2015.

[7] R. Belmonte and I. Sau. On the complexity of finding large odd induced subgraphs and odd colorings. *Algorithmica*, 83(8):2351–2373, 2021.

[8] J. Bensmail, M. Merker, and C. Thomassen. Decomposing graphs into a constant number of locally irregular subgraphs. *European Journal of Combinatorics*, 60:124–134, 2017.

[9] P. Berman, M. Karpinski, and A. D. Scott. Approximation hardness of short symmetric instances of MAX-3SAT. *Electronic Colloquium on Computational Complexity*, (049), 2003.

[10] H. L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoretical Computer Science*, 209(1):1–45, 1998.

[11] L. Cai. Fixed-parameter tractability of graph modification problems for hereditary properties. *Information Processing Letters*, 58(4):171–176, 1996.

[12] G. Chartrand, P. Erdös, and O. Oellermann. How to define an irregular graph. *The College Mathematics Journal*, 19, 01 1988.

[13] G. Chartrand, M. Jacobon, J. Lehel, O. Oellermann, S. Ruiz, and F. Saba. Irregular networks. *Congressus Numerantium*, 64, 01 1986.

[14] J. Chen, B. Chor, M. Fellows, X. Huang, D. W. Juedes, I. A. Kanj, and G. Xia. Tight lower bounds for certain parameterized np-hard problems. *Information and Computation*, 201(2):216–231, 2005.

[15] J. Chen, X. Huang, I. A. Kanj, and G. Xia. On the computational hardness based on linear fpt-reductions. *Journal of Combinatorial Optimization*, 11(2):231–247, 2006.

[16] J. Chen, X. Huang, I. A. Kanj, and G. Xia. Strong computational lower bounds via parameterized complexity. *Journal of Computer and System Sciences*, 72(8):1346–1367, 2006.

[17] M. Chlebík and J. Chlebíková. Complexity of approximating bounded variants of optimization problems. *Theor. Comput. Sci.*, 354(3):320–338, 2006.

[18] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015.

[19] R. Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.

[20] R. G. Downey and M. R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013.

[21] A. M. Frieze, R. J. Gould, M. Karonski, and F. Pfender. On graph irregularity strength. *Journal of Graph Theory*, 41(2):120–137, 2002.

[22] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.

[23] M. Karoński, T. Łuczak, and A. Thomason. Edge weights and vertex colors. *Journal of Combinatorial Theory*, 91:151–157, 05 2004.

[24] S. Khot and V. Raman. Parameterized complexity of finding subgraphs with hereditary properties. *Theoretical Computer Science*, 289(2):997–1008, 2002.

[25] J. M. Lewis and M. Yannakakis. The node-deletion problem for hereditary properties is np-complete. *Journal of Computer and System Sciences*, 20(2):219–230, 1980.

[26] C. N. Lintzmayer, G. O. Mota, and M. Sambinelli. Decomposing split graphs into locally irregular graphs. *Discrete Applied Mathematics*, 292:33–44, 2021.

[27] D. Lokshtanov, D. Marx, and S. Saurabh. Lower bounds based on the exponential time hypothesis. *Bulletin of the European Association for Theoretical Computer Science*, 105:41–72, 2011.

[28] B. Mohar. Face covers and the genus problem for apex graphs. *J. Comb. Theory, Ser. B*, 82(1):102–117, 2001.

[29] H. Moser and D. M. Thilikos. Parameterized complexity of finding regular induced subgraphs. *Journal of Discrete Algorithms*, 7(2):181–190, 2009.

[30] J. Przybyło. Irregularity strength of regular graphs. *Electronic Journal of Combinatorics*, 15, 06 2008.

[31] J. Przybyło. On decomposing graphs of large minimum degree into locally irregular subgraphs. *Electronic Journal of Combinatorics*, 23(2):2–31, 2016.