Approximation of exact controls for semilinear wave and heat equations through space-time methods

Arnaud Munch

Approximation of exact controls for semilinear wave and heat equations through space-time methods

Arnaud MÜNCH*

Abstract

We consider from the algorithmic and numerical viewpoints the exact controllability problems for linear and semilinear heat and wave equations. We notably report on some recent iterative approaches yielding to strongly convergent approximations of controlled solutions for semilinear equations. From the numerical perspective, we focus on the *control-then-discretize* strategy where the optimality system associated with each problem is solved within a space-time framework leading to strong convergence approximations with respect to the parameters of discretization. The role of global Carleman type estimates is emphasized in the robustness of the approaches. Numerical experiments in the one and two dimensional case illustrate our results of convergence.

AMS Classifications: 35K58, 93B05, 93E24.

Keywords: Control of semilinear PDEs, Fixed point theorem, Numerical approximation, Space-time discretization.

1 Introduction

Approximation of null controllability problems for partial differential equations is a delicate issue. In contrast with optimal control problems, the occurrence of a terminal constraint for the state of the equation makes the analysis non trivial, both at the theoretical but also at the numerical level. Thus, it is by now well-known since the pioneering works of Roland Glowinski in the nineties collected in [46] that the use of standard numerical schemes for hyperbolic equations may lead to divergent sequences of control as the discretization parameter goes to zero. This is due to spurious discrete high frequencies generated by the finite dimensional approximation. Similarly, for parabolic equations, the regularization phenomenon makes the approximation of controls badly conditioned and leads to highly oscillating behaviors. On the other hand, exact controllability results for semilinear equations, since the pioneering works of Enrique Zuazua [79] in the nineties, are usually based on non constructive fixed point arguments and therefore do not lead to method of approximations.

We focus here on the approximation of null distributed controls for semilinear wave and heat equation.

We first review some recent techniques that lead to robust numerical solution of null controllability problems associated with linear wave and heat equations. The methods are characterized by the fact that we approximate in finite dimension in space and time simultaneously. This is made possible by introducing an appropriate reformulation as an equation in a space of functions depending on the spatial and time variables which is then discretized and solved. In particular, we do not employ usual time-marching methods for the evolution equations. The well-posedness of these reformulations relies on so-called generalized observability inequalities, also refereed to as global Carleman estimates. The methods developed here to solve the optimality system associated with each controllability problem fall into the emergent strategy "control-then-discretize". In contrast with the classical reverse strategy "discretize-then-control", we emphasize that it

*Université Clermont Auvergne, CNRS, LMBP, F-63000 Clermont-Ferrand, France. e-mail: arnaud.munch@uca.fr.

leads to robust and strong convergent approximation with respect to the parameters of discretization. It is also notably appropriate for mesh adaptivity.

We also design, both for the wave and the heat case, a least-squares algorithm yielding sequences converging strongly and at least linearly to a controlled solution for the semilinear equation. Each element of the sequence is solution of a linearized controllability problem and therefore can be approximated numerically though a robust space-time formulation.

Section 2 is devoted to the wave equation and Section 3 is devoted to the heat equation. In both cases, we illustrate our results with numerical experiments performed with the software Freefem++ (see [48]). Section 4 concludes with some perspectives.

Notations In the text, Ω is a bounded domain of $\mathbb{R}^d$ ($d \geq 1$) with $\mathcal{C}^{1,1}$ boundary and $\omega \subset\subset \Omega$ is a non-empty open set. For any $T > 0$, we set $Q_T := \Omega \times (0,T)$, $q_T := \omega \times (0,T)$ and $\Sigma_T := \partial\Omega \times (0,T)$. The variable y is used for the state of the equation while the control is defined in term of the variable v. Moreover, f is the function defining the nonlinearity of the equation. Last, the variable C denotes a generic constant depending only on T, Ω, ω but not on any state variable.

2 The wave equation

This section is devoted to the linear and semilinear wave equations. We first recall some classical controllability results (Section 2.1 and Section 2.2), then explain how one may construct a sequence $(y_k, v_k)_{k\in\mathbb{N}}$ converging strongly to a controlled pair for the semilinear equation, based on a suitable linearization (Section 2.3). In Section 2.4, we discuss some methods of numerical approximation and we conclude with some numerical experiments in Section 2.5. We mainly focus on distributed controls although similar results are available for boundary controls.

2.1 Controllability results for the linear wave equation

The linear wave equation, completed with Dirichlet and initial conditions, reads as follows:

$$\begin{cases} \partial_{tt} y - \Delta y + Ay = v 1_\omega + F & \text{in } Q_T, \\ y = 0 \quad \text{on } \Sigma_T, \quad (y(\cdot\,,0), \partial_t y(\cdot\,,0)) = (u_0, u_1) & \text{in } \Omega. \end{cases} \tag{1}$$

Here, y is the state and $v \in L^2(q_T)$ is the control. We assume that the initial data (u_0, u_1) belongs to $\boldsymbol{V} := H^1_0(\Omega) \times L^2(\Omega)$, $F \in L^2(Q_T)$ and $A \in L^\infty(0,T; L^d(\Omega))$. Under these assumptions, (1) possesses a unique weak solution in $\mathcal{C}^0([0,T]; H^1_0(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$, see [63, 40].

The exact controllability problem for (1) in time T is formulated as follows:

$$\begin{gathered} \text{given } (u_0,u_1),(z_0,z_1) \in \boldsymbol{V}, \text{ find a control } v \in L^2(q_T) \text{ such that the weak solution to (1)} \\ \text{satisfies } (y(\cdot\,,T), \partial_t y(\cdot\,,T)) = (z_0,z_1). \end{gathered}$$

In other terms, through the action on the open subset ω of Ω, we want to steer the solution from the state (u_0, u_1) to the state (z_0, z_1). In view of the linearity of the system (1), it is equivalent to reach the zero target, i.e. take $(z_0, z_1) = (0, 0)$ leading the so-called *null controllability problem.*

Using *multiplier techniques,* this controllability problem was solved in the eighties in [63] in the case $A \equiv 0$, later generalized in [40] as follows.

Theorem 1. *[40, Theorem 2.2] For any $x_0 \in \mathbb{R}^d \backslash \overline{\Omega}$, let $\Gamma_0 := \{x \in \partial\Omega : (x - x_0)\cdot \nu(x) > 0\}$ and let $\mathcal{O}_\epsilon(\Gamma_0) := \{y \in \mathbb{R}^d : dist(y, \Gamma_0) \leq \epsilon\}$ for any $\epsilon > 0$. Assume*

35 $(\mathbf{H_0})$ *$T > 2 \max_{x\in\overline{\Omega}} |x - x_0|$ and $\omega \supseteq \mathcal{O}_\epsilon(\Gamma_0) \cap \Omega$ for some $\epsilon > 0$.*

Then (1) is exactly controllable in time T.

In Theorem 1, Γ_0 is the usual star-shaped part of Ω introduced in [63]. Using microlocal analysis, we recall that C. Bardos, G. Lebeau and J. Rauch proved in [6] that, in the class of $\mathcal{C}^\infty$ domains and for $A \equiv 0$, controllability holds if and only if (ω, T) satisfies the following geometric control condition: "*every ray of geometric optics that propagates in Ω and is reflected on Γ enters ω at a time $t < T$*".

Using duality arguments, Theorem 1 can be deduced from an observability estimate for the adjoint system. Thus, let us recall the following result, proved in [62].

Proposition 1. *[62, Theorem 2.1] Assume* $(\mathbf{H_0})$. *For any $A \in L^\infty(0,T;L^d(\Omega))$ and any $(\phi_0, \phi_1) \in \boldsymbol{H} := L^2(\Omega) \times H^{-1}(\Omega)$, the weak solution ϕ to*

$$\begin{cases} \partial_{tt}\phi - \Delta\phi + A\phi = 0 \quad \textit{in } Q_T, \\ \phi = 0 \quad \textit{on } \Sigma_T, \quad (\phi(\cdot\,,0), \partial_t\phi(\cdot\,,0)) = (\phi_0, \phi_1) \quad \textit{in } \Omega, \end{cases} \tag{2}$$

satisfies the following observability inequality, for some $C > 0$ only depending on Ω and T :

$$\|(\phi_0, \phi_1)\|_{\boldsymbol{H}} \leq C e^{C\|A\|^2_{L^\infty(0,T;L^d(\Omega))}} \|\phi\|_{L^2(q_T)}. \tag{3}$$

The inequality (3) is refereed as *an observability inequality* as the knowledge of ϕ on the subset q_T of Q_T allows to observe the full system. Among all admissible controls, we usually consider the control of minimal $L^2(q_T)$ norm which is unique and depends continuously on the data as follows.

Proposition 2. *Let $A \in L^\infty(0,T;L^d(\Omega))$, $F \in L^2(Q_T)$ and $(u_0,u_1), (z_0,z_1) \in \boldsymbol{V}$ be given. Assume* $(\mathbf{H_0})$. *Then the control of minimal $L^2(q_T)$ norm v together with the corresponding controlled weak solution y of* (1)*satisfy the following estimate, for some constant $C > 0$ only depending on Ω and T :*

$$\|v\|_{L^2(q_T)} + \|(y, \partial_t y)\|_{L^\infty(0,T;\boldsymbol{V})} \leq C\Big(\|F\|_{L^2(Q_T)} + \|(u_0,u_1)\|_{\boldsymbol{V}} + \|(z_0,z_1)\|_{\boldsymbol{V}}\Big) e^{C\|A\|^2_{L^\infty(0,T;L^d(\Omega))}}. \tag{4}$$

2.2 Controllability results for a semilinear wave equation

We consider now the following system for the semilinear wave equation:

$$\begin{cases} \partial_{tt}y - \Delta y + f(y) = v 1_\omega \quad \text{in } Q_T, \\ y = 0 \quad \text{on } \Sigma_T, \quad (y(\cdot\,,0), \partial_t y(\cdot\,,0)) = (u_0, u_1) \quad \text{in } \Omega. \end{cases} \tag{5}$$

Here, $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^1$ function such that $|f(r)| \leq C(1+|r|)\ln(2+|r|)$ for all $r \in \mathbb{R}$ and some $C > 0$. There exists a unique global weak solution to (5) in $\mathcal{C}^0([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega))$ (see [20]). Furthermore, imposing an adequate growth condition on f at infinity, the exact controllability problem has been solved in [62] and generalized in [40] to more general hyperbolic equations.

Theorem 2. *[62, Theorem 2.1] Let x_0, Γ_0 and $\mathcal{O}_\epsilon(\Gamma_0)$ be as in Theorem 1. Assume that* $(\mathbf{H_0})$ *holds. If f satisfies*

$(\mathbf{H_1})$ $\quad \limsup_{|r|\to\infty} \frac{|f(r)|}{|r|\ln^{1/2}|r|} = 0,$

then (5) *is exactly controllable in time T.*

Theorem 2 extends to the multi-dimensional case the result of [79] devoted to the one-dimensional case under the condition $\limsup_{|r|\to\infty} \frac{|f(r)|}{|r|\ln^2|r|} = 0$, later relaxed in [15] and [66]. The exact controllability for subcritical nonlinearities is obtained in [27] under the sign condition $rf(r) \geq 0$ for all $r \in \mathbb{R}$. This last assumption has been weakened in [51] to an asymptotic sign condition leading to a semi-global controllability result, in the sense that the final data (z_0, z_1) must be prescribed in a precise subset of $\boldsymbol{V}$.

The proof of Theorem 2 given in [62] is based on the fixed-point argument introduced in [78, 79] and the *a priori* estimate (4) for the linear wave equation (1). More precisely, it is shown that the operator $\Lambda : L^\infty(0,T;L^d(\Omega)) \mapsto L^\infty(0,T;L^d(\Omega))$, where $y_\xi := \Lambda(\xi)$ is the solution to the linear problem

$$\begin{cases}\partial_{tt}y_\xi - \Delta y_\xi + \widehat{f}(\xi)\, y_\xi = -f(0) + v_\xi 1_\omega & \text{in } Q_T,\\ y_\xi = 0 & \text{on } \Sigma_T,\\ (y_\xi(\cdot\,,0), \partial_t y_\xi(\cdot\,,0)) = (u_0,u_1) & \text{in } \Omega,\end{cases} \qquad \widehat{f}(r) := \begin{cases}\dfrac{f(r)-f(0)}{r} & r\neq 0,\\ f'(0) & r = 0\end{cases}, \tag{6}$$

and v_ξ is the minimal $L^2(q_T)$ norm control for which $(y_\xi(\cdot\,,T), y_{\xi,t}(\cdot\,,T)) = (z_0,z_1)$ has a fixed-point. The existence is obtained by using the Leray-Schauder's Theorem; in particular, under the growth assumption $(\mathbf{H_1})$, it is shown that there exists a positive constant $M = M(\|u_0,u_1\|_{\boldsymbol{V}}, \|z_0,z_1\|_{\boldsymbol{V}})$ such that Λ maps the ball $B_{L^\infty(0,T;L^d(\Omega))}(0,M)$ into itself.

2.3 Construction of a convergent sequence of state-control pairs for the semilinear system (5): a least-squares approach

We now discuss the explicit construction of a sequence $(v_k)_{k\in\mathbb{N}}$ that converges strongly to an exact control for (5). The controllability of nonlinear PDEs has attracted a large number of works in the last decades (see [26]). However, few are concerned with the computation of exact controls and the explicit construction of convergent approximations remains a challenge.

A first idea that comes to mind is to consider the Picard iterates $(y_k)_{k\in\mathbb{N}}$ associated with the operator Λ, defined by $y_{k+1} = \Lambda(y_k)$ for $k \geq 0$, starting from some $y_0 \in L^\infty(0,T;L^d(\Omega))$. The resulting sequence of controls $(v_k)_{k\in\mathbb{N}}$ fulfills the following property: $v_{k+1} \in L^2(q_T)$ is the control of minimal $L^2(q_T)$ norm for which the associated solution to

$$\begin{cases}\partial_{tt}y_{k+1} - \Delta y_{k+1} + \widehat{f}(y_k)\, y_{k+1} = -f(0) + v_{k+1}1_\omega & \text{in} \quad Q_T,\\ y_{k+1} = 0 \quad \text{on } \Sigma_T, \quad (y_{k+1}(\cdot\,,0), \partial_t y_{k+1}(\cdot\,,0)) = (y_0,y_1) & \text{in } \Omega\end{cases} \tag{7}$$

satisfies $(y_{k+1}(\cdot\,,T), \partial_t y_{k+1}(\cdot\,,T)) = (z_0,z_1)$. Such a strategy fails frequently, since the operator Λ is not in general a contraction, even if f is globally Lipschitz-continuous. We refer to [9] for a numerical evidence of the lack of convergence (see also [35] in a similar parabolic context).

A second idea is to use a method of the Newton kind to find a zero of the $\mathcal{C}^1$ mapping $\widetilde{\mathcal{F}} : Y \mapsto W$, defined by

$$\widetilde{\mathcal{F}}(y,v) := \Big(\partial_{tt}y - \Delta y + f(y) - v1_\omega, y(\cdot\,,0) - u_0, \partial_t y(\cdot\,,0) - u_1, y(\cdot\,,T) - z_0, \partial_t y(\cdot\,,T) - z_1\Big) \tag{8}$$

for some appropriates Hilbert spaces Y and W. Thus, starting from $(y_0,v_0) \in Y$, for each $k \geq 0$ we set $(y_{k+1}, v_{k+1}) = (y_k,v_k) - (Y_k,V_k)$ where V_k is the control of minimal $L^2(q_T)$ norm such that the solution to

$$\begin{cases}\partial_{tt}Y_k - \Delta Y_k + f'(y_k)\, Y_k = V_k\, 1_\omega + \partial_{tt}y_k - \Delta y_k + f(y_k) - v_k 1_\omega & \text{in } Q_T,\\ Y_k = 0 \quad \text{on } \Sigma_T, \quad Y_k(\cdot\,,0) = u_0 - y_k(\cdot\,,0),\ \partial_t Y_k(\cdot\,,0) = u_1 - \partial_t y_k(\cdot\,,0) & \text{in } \Omega\end{cases} \tag{9}$$

satisfies $Y_k(\cdot\,,T) = z_0 - y_k(\cdot\,,T)$ and $\partial_t Y_k(\cdot\,,T) = z_1 - \partial_t y_k(\cdot\,,T)$. As is well-known, the resulting sequence may fail to converge if the initial guess (y_0,v_0) is not close enough to a zero of $\widetilde{\mathcal{F}}$.

Given any initial data $(u_0,u_1) \in \boldsymbol{V}$, under assumptions on f that are slightly stronger than $(\mathbf{H_1})$ and $d \leq 3$, we can design an algorithm providing a sequence $(y_k,v_k)_{k\in\mathbb{N}}$ that converges to a controlled pair. Moreover, after a finite number of iterates, the convergence is super-linear. This is achieved by introducing a least-squares functional that measures how much a pair (y,v) is close to a controlled solution for (5) and, then by determining a particular convergent minimizing sequence. Following [9, 73], we define the Hilbert space

$$\mathcal{H} := \{(y,v) \in L^2(Q_T) \times L^2(q_T) : y \in \mathcal{C}^0([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega)),\ \partial_{tt}y - \Delta y \in L^2(Q_T)\},$$

which is endowed with the scalar product

$$\big((y,v),(\overline{y},\overline{v})\big)_{\mathcal{H}} := (y,\overline{y})_{L^2(Q_T)} + \big((y(\cdot\,,0),\partial_t y(\cdot\,,0)),(\overline{y}(\cdot\,,0),\partial_t\overline{y}(\cdot\,,0))\big)_{\boldsymbol{V}} \\ + (\partial_{tt}y-\Delta y,\partial_{tt}\overline{y}-\Delta\overline{y})_{L^2(Q_T)} + (v,\overline{v})_{L^2(q_T)}.$$

We then define the non-empty linear manifold

$$\mathcal{A} := \{(y,v)\in\mathcal{H} : (y(\cdot\,,0),\partial_t y(\cdot\,,0)) = (u_0,u_1),\ (y(\cdot\,,T),\partial_t y(\cdot\,,T)) = (z_0,z_1)\}$$

and the associated space $\mathcal{A}_0 := \{(y,v)\in\mathcal{H} : (y(\cdot\,,0),\partial_t y(\cdot\,,0)) = (0,0),\ (y(\cdot\,,T),\partial_t y(\cdot\,,T)) = (0,0)\}$ and consider the following non-convex extremal problem of the least-squares kind

$$\inf_{(y,v)\in\mathcal{A}} E(y,v), \qquad E(y,v) := \frac{1}{2}\big\|\partial_{tt}y-\Delta y+f(y)-v\,1_\omega\big\|^2_{L^2(Q_T)}. \tag{10}$$

The functional E is well-defined in $\mathcal{A}$: we check that there exists $C>0$ such that $E(y,v)\le C(1+\|(y,v)\|^3_{\mathcal{H}})$ for all $(y,v)\in\mathcal{A}$.

Main properties of the functional E The functional E is Gâteaux-differentiable over $\mathcal{A}$. Moreover, it is shown in [9] the following inequality.

Proposition 3. *[9, Proposition 3] Assume $(\mathbf{H_0})$ and let $d\le 3$. There exists $C=C(\omega,\Omega,T)>0$ such that*

$$\sqrt{E(y,v)} \le C\Big(1+\|f'(y)\|_{L^\infty(0,T;L^3(\Omega))}\Big)e^{C\|f'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}}\|E'(y,v)\|_{\mathcal{A}_0'}, \qquad \forall (y,v)\in\mathcal{A}. \tag{11}$$

Consequently, any *critical* point $(y,v)\in\mathcal{A}$ of E such that $\|f'(y)\|_{L^\infty(0,T;L^3(\Omega))}$ is finite is a zero for E, i.e. a solution to the controllability problem and any sequence $(y_k,v_k)_{k>0}$ satisfying $\|E'(y_k,v_k)\|_{\mathcal{A}_0'}\to 0$ as $k\to\infty$ for which $\|f'(y_k)\|_{L^\infty(0,T;L^3(\Omega))}$ is uniformly bounded is such that $E(y_k,v_k)\to 0$ as $k\to\infty$.

This property does not imply the convexity of the functional E (and *a fortiori* the strict convexity of E, which actually cannot hold in view of the multiple zeros for E). However, it shows that a minimizing sequence for E cannot be stuck in a local minimum. In order to construct a minimizing sequence for E, we formally look, for any $(y,v)\in\mathcal{A}$, for a pair $(Y^1,V^1)\in\mathcal{A}_0$ solving the following linear wave equation

$$\begin{cases}\partial_{tt}Y^1-\Delta Y^1+f'(y)\cdot Y^1 = V^1 1_\omega + \big(\partial_{tt}y-\Delta y+f(y)-v\,1_\omega\big) & \text{in } Q_T,\\ Y^1=0 \quad \text{on } \Sigma_T, \quad (Y^1(\cdot\,,0),\partial_t Y^1(\cdot\,,0)) = (0,0) & \text{in } \Omega.\end{cases} \tag{12}$$

Since (Y^1,V^1) belongs to $\mathcal{A}_0$, V^1 is a null control for Y^1. Among all the controls of this linear equation, we select the control of minimal $L^2(q_T)$ norm. In the sequel, we call the corresponding solution $(Y^1,V^1)\in\mathcal{A}_0$ the solution of *minimal control norm*. Then the derivative of E at $(y,v)\in\mathcal{A}$ in the direction (Y^1,V^1) satisfies $E'(y,v)\cdot(Y^1,V^1) = 2E(y,v)$ which allows to define a minimizing sequence for E.

Given $f\in\mathcal{C}^1(\mathbb{R})$ and $p\in(0,1]$, we introduce the following hypothesis:

$$(\overline{\mathbf{H}}_p) \quad [f']_p := \sup_{\substack{a,b\in\mathbb{R}\\ a\neq b}} \frac{|f'(a)-f'(b)|}{|a-b|^p} < +\infty$$

and set $\beta^\star(p) := \sqrt{\frac{p}{2C(2p+1)}}$ where $C>0$ (only depending on Ω and T) is the constant appearing in Proposition 2. The following result from [9] provides a constructive way to approximate a control for the semilinear wave equation (5).

Theorem 3. *[9, Theorem 2] Assume* $(\mathbf{H_0})$ *and let* $d \leq 3$. *Also, assume that* f' *satisfies* $(\overline{\mathbf{H}}_p)$ *for some* $p \in (0,1]$ *and*

$(\mathbf{H_2})$ *There exists* $\alpha \geq 0$ *and* $\beta \in [0, \beta^\star(p))$ *such that* $|f'(r)| \leq \alpha + \beta \ln^{1/2}(1+|r|)$ *for every* r *in* $\mathbb{R}$.

Then, for any initial and final data (u_0, u_1) *and* (z_0, z_1) *in* $\boldsymbol{V}$ *and any starting* $(y_0, v_0) \in \mathcal{A}$, *the sequence* $(y_k, v_k)_{k \in \mathbb{N}}$ *defined by*

$$\begin{cases} (y_{k+1}, v_{k+1}) = (y_k, v_k) - \lambda_k (Y_k^1, V_k^1), \quad k \in \mathbb{N}, \\ \lambda_k := argmin_{\lambda \in [0,1]} E\big((y_k, v_k) - \lambda (Y_k^1, V_k^1)\big), \end{cases} \tag{13}$$

where $(Y_k^1, V_k^1) \in \mathcal{A}_0$ *is the solution of minimal control norm of*

$$\begin{cases} \partial_{tt} Y_k^1 - \Delta Y_k^1 + f'(y_k) \cdot Y_k^1 = V_k^1 1_\omega + (\partial_{tt} y_k - \Delta y_k + f(y_k) - v_k 1_\omega) \quad \text{in } Q_T, \\ Y_k^1 = 0 \quad \text{on } \Sigma_T, \quad (Y_k^1(\cdot,0), \partial_t Y_k^1(\cdot,0)) = (0,0) \quad \text{in } \Omega \end{cases} \tag{14}$$

strongly converges to a pair $(\overline{y}, \overline{v}) \in \mathcal{A}$ *satisfying* (5) *and the condition* $(y(\cdot,T), \partial_t y(\cdot,T)) = (z_0, z_1)$, *for all* $(u_0, u_1), (z_0, z_1) \in \boldsymbol{V}$. *Moreover, the convergence is at least linear and at least of order* $p+1$ *after a finite number of iterations.*

Theorem 3 provides a new proof of the exact controllability of semilinear multi-dimensional wave equations which is moreover constructive, with an algorithm that converges unconditionally at least with order $p+1$.

Remark 1. The asymptotic condition $(\mathbf{H_2})$ on f' is slightly stronger than the assumption $(\mathbf{H_1})$ made in [40]: this is due to our linearization of (5), which concerns f', while the linearization (34) in [40] involves $r \to (f(r) - f(0))/r$. Remark however that the particular example $f(r) = a + br + cr \ln^{1/2}(1+|r|)$ with $a, b \in \mathbb{R}$ and $c > 0$ small enough (which is somehow the limit case in Theorem 2) satisfies $(\mathbf{H_2})$ as well as $(\overline{\mathbf{H}}_p)$ for any $p \in (0,1]$.

Remark 2. Defining $\mathcal{F} : \mathcal{A} \to L^2(Q_T)$ by $\mathcal{F}(y,v) := (\partial_{tt} y - \Delta y + f(y) - v\, 1_\omega)$, we have $E(y,v) = \frac{1}{2}\|\mathcal{F}(y,v)\|_2^2$ and we observe that, for $\lambda_k = 1$, the algorithm (13) coincides with the Newton algorithm associated with the mapping $\mathcal{F}$ (see (9)). This explains the super-linear convergence property in Theorem 3, in particular the quadratic convergence when $p = 1$. The optimization of the parameter λ_k allows a global convergence property of the algorithm and leads to the so-called damped Newton method applied to $\mathcal{F}$ (we refer to [28, Chapter 8])). As far as we know, the analysis of damped type Newton methods for PDEs has deserved very few attention in the literature (we mention [58, 76] in the context of fluids mechanics.)

Remark 3. Instead of the control of minimal L^2-norm, we may consider weighted costs involving both the state and the control. In the framework of boundary controllability, it is shown in [8] using global Carleman estimates (see [7]) that appropriate choices of the weights lead to convergent result with linear rate assuming only $(\mathbf{H_2})$. We also refer to Section 3, in particular Theorem 8 and 9, devoted to the heat equation where this is discussed with more details.

2.4 Numerical approximation of exact controls for the wave equation

We now discuss the approximation of exact controls for the wave equation. For brevity, we employ the notation $L_A \phi := \partial_{tt}\phi - \Delta\phi + A\phi$. According to the previous section and Theorem 3, a convergent numerical approximation of controls for the linear wave equation allows to construct a convergent numerical approximation of controls in the semilinear case as well. We therefore focus on the linear situation.

Without loss of generality, we assume that the target (z_0, z_1) vanishes and look for an approximation of the control of minimal $L^2(q_T)$-norm solution of

$$\inf_{v \in \mathcal{C}(u_0, u_1, T)} J(v), \qquad J(v) := \|v\|^2_{L^2(q_T)}$$

where $\mathcal{C}(u_0, u_1, T)$ denotes the non empty convex set of controls. Applying the Fenchel-Rockafellar duality theory (see [31]), the control of minimal $L^2(q_T)$ norm is expressed by $v = \phi 1_\omega$ where ϕ solves (2) with initial data $(\phi_0, \phi_1) \in \boldsymbol{H}$ and (ϕ_0, ϕ_1) solves the following extremal problem

$$\inf_{(\phi_0,\phi_1)\in \boldsymbol{H}} J^\star(\phi_0, \phi_1), \quad J^\star(\phi_0, \phi_1) := \frac{1}{2}\int_{q_T} |\phi|^2 + \int_{Q_T} F\phi + \langle \phi_1, u_0\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \langle \phi_0, u_1\rangle_{L^2(\Omega)}. \tag{15}$$

Here, $J^\star$ is the so-called conjugate functional associated with J. The observability inequality (3) for the variable ϕ leads to the well-posedness of this extremal problem. Compared with the initial minimization of J over exact null controls for (1), this equivalent problem does not make appear any terminal constraint and therefore can be solved through an iterative descent method: the conjugate gradient algorithm is usually employed (see [44]) since the so-called HUM operator related to $J^\star$ is coercive.

However, at the finite dimensional level (induced by the numerical approximation in time and space), (2) can not be in general solved exactly: in other words, the constraint $L_A\phi = 0$ in Q_T is not exactly satisfied what makes irrelevant the observability inequality (3). For some specific geometries, let us mention however spectral methods initially developed by F. Bourquin in [10] (then used in [56]) leading to precise convergence results. At least two possibilities appear in order to bypass this difficulty. The first one is to first reformulate the controllability problem at the finite dimensional level leading to so-called *discretize-then-control* strategy.

The discretize-then-control strategy A possible strategy is to first discretize (5) and then determine a discrete control of minimal $L^2(q_T)$ norm by minimizing the associated discrete functional $J_h^\star$, where h stands for the discretization parameter. This has been the subject of numerous works and extended to many others PDEs, starting from the seminal contribution of R. Glowinski and J.-L. Lions [44] (see also [46]). The experiments there reveal that the convergence of the approach is very sensitive to the chosen approximation. Thus, if standard time marching convergent schemes are coupled with standard finite element approximations, the associated observability constant may not be uniformly bounded with respect to h, leading to a divergence of the discrete family of controls $(v_h)_{h>0}$ as h tends to zero. In practice, the conjugate gradient algorithm fails to converge as the discretization becomes finer. As conjectured in [44] and later analyzed (see [80] for a review), this is due to spurious high frequencies discrete modes which are not exactly controllable uniformly in h. This pathology can easily be avoided in practice by adding to the conjugate functional a regularized Tikhonov parameter; this leads to so called approximate controls, solving the control problem only up to a small remainder term:

$$\|y_h(\cdot, T), \partial_t y_h(\cdot, T)\|_{H_0^1(\Omega)\times L^2(\Omega)} \le \mathcal{O}(h^\alpha), \ \forall h > 0$$

where the real α is related to the order of the numerical scheme. This is sufficient for the applications but not fully satisfactory from a theoretical viewpoint. That is why several cures aiming to filter out the high frequencies have been proposed and analyzed, mainly for simple geometries (1d interval, unit square in 2d, etc) with finite differences schemes. The simplest but artificial approach is to eliminate the highest eigenmodes of a discrete approximation of the initial condition as discussed in one space dimension in [67], and extended in [64]. We also mention so called bi-grid methods (based on the projection of the discrete gradient of $J^\star$ on a coarse grid) proposed in [45] and analyzed in [50, 65] leading to convergence results. One may also design more elaborated discrete schemes avoiding spurious modes: we mention [43] based on a mixed reformulation of the wave equation analyzed later with finite difference schemes in [18, 19, 3] at the semi-discrete level and then extended in [69] to a full space-time discrete setting, leading to strong convergent results. For instance, in [69], the following scheme

$$\mathcal{D}_{\Delta t}(y_{\Delta_t,\Delta_x}) - \mathcal{D}_{\Delta x}(y_{\Delta_t,\Delta_x}) + \frac{1}{4}(\Delta_t^2 - \Delta_x^2)\mathcal{D}_{\Delta x}(\mathcal{D}_{\Delta t}(y_{\Delta_t,\Delta_x})) = 0$$

is proved to be uniformly controllable with respect to the discretization $h = (\Delta_t, \Delta_x)$ as it leads to a discrete family of controls converging strongly to a control for the wave equation as soon as the controllability T is large enough such that $T > 2\max(1, \Delta_t^2/\Delta_x^2)$ (see [69, Theorem 2.8]). Here, $\mathcal{D}_\eta(z)$ stands for the standard operator

$$\mathcal{D}_\eta(z)(r) = \frac{z(r+\eta) - 2z(r) + z(r-\eta)}{\eta^2}, \quad \forall r \in \mathbb{R}, \quad \forall \eta > 0$$

associated with the centered approximation of order two of the second derivative of any smooth function z.

The previous works, notably reviewed in [80, 34], fall within an approach that can be called "*discretize then control*" as they aim to control exactly to zero a finite dimensional approximation of the wave equation. A relaxed controllability approach is analyzed in [14] using a stabilized finite element method in space and leading for smooth two and three dimensional geometries to a strong convergent approximations (we refer to [14, Theorem 2.1]). The controllability requirement is imposed via appropriate penalty terms. We also mention [75] based on the Russel's *stabilization implies control* principle, extended in [25] and [47, 2] for least-squares based method.

The control-then-discretize strategy A second strategy allowing to bypass the issue of approximating the constraint $L_A\phi = 0$ is somehow to relax it by keeping the variable ϕ as the main variable into a space-time formulation. This leads to a "*control-then-discretize*" procedure, where the optimality system associated with problem (15) mixing the boundary condition in time and space and involving the primal and adjoint state is discretized within *a priori* a space-time approximation. The well-posedness of such system is achieved by using so called global or generalized observability inequalities (usually refereed to global Carleman inequality, see [7]).

To this purpose, we keep ϕ as the main variable and introduce the Hilbert space

$$\Phi := \big\{\phi \in L^2(Q_T); \phi \in \mathcal{C}([0,T]; L^2(\Omega)) \cap \mathcal{C}^1([0,T]; H^{-1}(\Omega), L_A\phi \in L^2(0,T; H^{-1}(\Omega))\big\},$$

endowed with the inner product $< \phi, \overline{\phi} >_\Phi := < \phi, \overline{\phi} >_{L^2(q_T)} + < L_A\phi, L_A\overline{\phi} >_{L^2(0,T;H^{-1}(\Omega))}$. We also introduce the subspace $W := \{\phi \in \Phi, L_A\phi = 0\}$ and remark that (15) is equivalent to the extremal problem $\min_{\phi \in W} J^\star(\phi)$ (using that ϕ is uniquely determined from $(\phi_0, \phi_1) \in L^2(\Omega) \times H^{-1}(\Omega)$). Since ϕ is now the main variable, we may address the linear constraint $L_A\phi = 0$ through a Lagrange multiplier $\lambda \in \mathcal{M} := L^2(0,T; H_0^1(\Omega))$, leading to the following equivalent saddle point problem

$$\sup_{\lambda \in L^2(0,T,H_0^1(\Omega))} \inf_{\phi \in \Phi} \mathcal{L}_r(\phi,\lambda), \quad \mathcal{L}_r(\phi,\lambda) := J^\star(\phi) + \int_0^T \langle \lambda, L_A\phi \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} + \frac{r}{2}\|L_A\phi\|^2_{L^2(0,T;H^{-1}(\Omega))} \tag{16}$$

for any augmentation parameter $r \geq 0$ and the following mixed formulation: for any $r \geq 0$, find $(\phi, \lambda) \in \Phi \times L^2(0,T; H_0^1(\Omega))$ such that

$$\begin{cases} a_r(\phi, \overline{\phi}) + b(\lambda, \overline{\phi}) = l(\overline{\phi}), & \forall \overline{\phi} \in \Phi, \\ b(\overline{\lambda}, \phi) = 0, & \forall \overline{\lambda} \in L^2(H_0^1(\Omega)), \end{cases}$$

with

$$\begin{cases} a_r : \Phi \times \Phi \to \mathbb{R}, \quad a_r(\phi, \overline{\phi}) := < \phi, \overline{\phi} >_{L^2(q_T)} + \displaystyle\int_{Q_T} F\overline{\phi} + r < L_A\phi, L_A\overline{\phi} >_{L^2(0,T;H^{-1}(\Omega)))}, \\ b : \mathcal{M} \times \Phi \to \mathbb{R}, \qquad b(\lambda, \phi) := \displaystyle\int_0^T < \lambda, L_A\phi >_{H_0^1(\Omega), H^{-1}(\Omega)}, \\ l : \Phi \to \mathbb{R}, \qquad l(\phi) := < \phi_1, u_0 >_{H^{-1}(\Omega), H_0^1(\Omega)} - < \phi_0, u_1 >_{L^2(\Omega)}. \end{cases} \tag{17}$$

It turns out that the Lagrange multiplier coincides with the controlled solution of the wave equation.

Theorem 4. *[17, Theorem 3.1] Assume* $(\mathbf{H_0})$ *and let* $r \geq 0$.

1. *The mixed problem* (17) *is well-posed and its unique solution* $(\phi, \lambda) \in \Phi \times L^2(0,T; H_0^1(\Omega))$ *is the unique saddle-point of the Lagrangian* $\mathcal{L}_r$ *defined in* (16).
2. *The optimal* ϕ *is the minimizer of* $J^\star$ *over* Φ*, while the optimal multiplier* λ *is the state of the controlled wave equation* (1) *in the weak sense (associated with the control* $\phi 1_\omega$*).*

The fundamental tool used to prove the well-posedness and notably the continuity of the linear form l is the following generalized observability inequality:

$$\|\phi(\cdot\,,0,),\partial_t\phi(\cdot\,,0)\|^2_{\boldsymbol{H}} \le C(\Omega,T,\|A\|_{L^\infty(0,T;L^d(\Omega))})\biggl(\|\phi\|_{L^2(q_T)}+\|L_A\phi\|_{L^2(0,T;H^{-1}(\Omega))}\biggr) \quad \forall\phi\in\Phi, \tag{18}$$

which can be easily deduced from the inequality (3) using the linearity of the equation (we refer for instance to [17]). With respect to (3), the main interest of (18) is that it remains true for any finite dimensional subspace $\Phi_h \subset \Phi$ parametrized with $h>0$ (with a constant independent of h). In other words, there is no need to prove any uniform property for some discrete observability constant.

The well-posedness of (17) is based on two properties (that should be preserved uniformly at the finite dimensional level):

(i) The coercivity of the bilinear form a over the kernel $\mathrm{Ker}\,(b)=\{\phi\in\Phi, b(\lambda,\phi)=0\ \forall\lambda\in\mathcal{M}\}$ of b and

(ii) The inf-sup property for b:

$$\exists\delta>0 \text{ s.t. } \inf_{\lambda\in\mathcal{M}}\sup_{\phi\in\Phi}\frac{b(\lambda,\phi)}{\|\lambda\|_{\mathcal{M}}\|\phi\|_\Phi}\ge\delta.$$

Let $\mathcal{T}:=\{\mathcal{T}_h, h>0\}$ be family of regular triangulations of the space-time domain Q_T such that $\overline{Q_T}=\cup_{K\in\mathcal{T}_h}K$. The family is indexed by $h=\max_{K\in\mathcal{T}_h}|K|$. The coercivity property of the bilinear form a remains true over $\Phi_h\times\Phi_h$ for any finite dimensional subspace $\Phi_h\subset\Phi$ as soon as the augmentation parameter r is strictly positive.

On the other hand, a discrete inf-sup property, uniformly with respect to the parameter h, is in general more delicate to obtain as it depends strongly on the discrete spaces $\mathcal{M}_h\subset\mathcal{M}$ and Φ_h used. For instance, if we define $\mathcal{M}_h:=\{p_h\in\mathcal{C}(\overline{Q_T}); (p_h)_{|K}\in\mathbb{P}_1(K)\ \forall K\in\mathcal{T}_h\}$ where $\mathbb{P}_k(K)$ denotes the space of polynomials of degree k and $\Phi_h:=\{p_h\in\mathcal{C}^1(\overline{Q_T}); (p_h)_{|K}\in HCT(K)\ \forall K\in\mathcal{T}_h\}$ where HCT denotes the Hsieh-Clough-Tocher composite finite element (see [23]), we numerically observe (by the inf-sup test, see [22]) that a discrete inf-sup property holds true when the parameter r is of order of h^2. This leads in practice to a convergent approximation of the control of minimal $L^2(q_T)$ norm. Remark that a $\mathcal{C}^1$ finite element is used in order to ensure that $L_A\phi_h$ belongs to $L^2(Q_T)$ for any $\phi_h\in\Phi_h$. The theoretical study of the behavior with respect to h of $\inf_{\lambda\in\mathcal{M}_h}\sup_{\phi\in\Phi_h}\frac{b(\lambda,\phi)}{\|\lambda\|_{\mathcal{M}}\|\phi\|_\Phi}$ is in general a difficult question. This is *a fortiori* true here since the constraint $L_A\phi\in L^2(Q_T)$ implies second derivates in time and space and involves $\mathcal{C}^1$ finite element. Hopefully, one may avoid it by stabilizing the mixed formulation with respect to the variable λ (see the seminal work [5]): this consists in adding to the Lagrangian some terms so as to get a coercivity property for the variable λ as well. This is notably employed in [68] devoted to the approximation of boundary controls for the wave equation, preliminary reformulated as a first order system. This reformulation as a first order system requires, within a conformal approximation, only $\mathcal{C}^0$ finite element (but needs to be stabilized whatever be the value of the augmentation parameter).

Stabilization methods may also be employed in the context of non-conformal approximations. In this respect, let us introduce $V_h^q=\{p_h\in\mathcal{C}(\overline{Q_T}); (p_h)_{|K}\in\mathbb{P}_q(K)\ \forall K\in\mathcal{T}_h\}$ and consider the discrete Lagrangian $\mathcal{L}_h: V_h^p\times V_h^q\to\mathbb{R}$, given by

$$\begin{aligned}\mathcal{L}_h(\phi_h,\lambda_h):=J^\star(\phi_h)+\frac{h^2}{2}\|L_A\phi_h\|^2_{L^2(Q_T)}+\frac{h}{2}\sum_{K\in\mathcal{T}_h}\int_{\partial K}[[\partial_\nu\phi_h]]^2+h^{-1}\int_{\Sigma_T}\phi_h^2-h^{-1}\int_{\Sigma_T}\lambda_h^2\\ +\int_{Q_T}(-\partial_t\phi_h\partial_t\lambda_h+\nabla\phi_h\cdot\nabla\lambda_h+A\phi_h\lambda_h)-\frac{h}{2}\sum_{K\in\mathcal{T}_h}\int_{\partial K}[[\partial_\nu\lambda_h]]^2-\frac{h^2}{2}\|L_A\lambda_h-\phi_h 1_\omega\|^2_{L^2(Q_T)},\end{aligned}$$

where $[[\partial_\nu\phi_h]]$ denotes the jump of the normal derivative of ϕ_h across the internal edges of the triangulation.

The terms $h^2\|L\phi_h\|_{L^2(Q_T)}$ and $-h^2\|L\lambda_h-\phi_h1_\omega\|^2_{L^2(Q_T)}$ play a symmetric role. Both vanish at the continuous level. On the other hand, the jump terms somehow aim to control the regularity of the approximation.

The discrete Lagrangian $\mathcal{L}_h$ admits a unique saddle-point. The well-posedness is based on a variant of the generalized observability inequality (18), where the $L^2(0,T;H^{-1}(\Omega))$ norm of $L_A\phi$ is replaced by the

$H^{-1}(Q_T)$ norm. Moreover, if the saddle-point (ϕ, λ) of $\mathcal{L}_r$ is smooth enough, the following approximation result holds true (we refer to [13, section 2] and also [12] in the closed context of data assimilation problems):

Theorem 5. *[13, Theorem 2.5] Assume* $(\mathbf{H_0})$. *Let* $p, q \geq 1$, $h > 0$ *and* $r \geq 0$. *Let* $(\phi_h, \lambda_h) \in V_h^q \times V_h^p$ *be the saddle point of* $\mathcal{L}_h$ *and assume that the saddle point* (ϕ, λ) *of* $\mathcal{L}_r$ *(see* (16)*) belongs to* $H^{q+1}(Q_T) \times H^{p+1}(Q_T)$. *Then, there exists a positive constant* C *independent of* h *such that*

$$\|\chi(\phi - \phi_h)\|_{L^2(Q_T)} \leq C(h^p \|\lambda\|_{H^{p+1}(Q_T)} + h^q \|\phi\|_{H^{q+1}(Q_T)}), \tag{19}$$

where χ *is a cut-off function of the form* $\chi(x,t) = \chi_0(x)\chi_1(t)$, *with* $\chi_0 \in \mathcal{C}_0^\infty(\omega)$ *and* $\chi_1 \in \mathcal{C}_0^\infty(0,T)$.

The regularity assumption on the optimal pair (ϕ, λ) notably holds true if the initial data (u_0, u_1) to be controlled are smooth and satisfy compatibility conditions at $\partial\Omega \times \{0\}$ (we refer to [33]).

To end this brief review on the *control-then-discretize* approach, we emphasize that, in order to avoid the delicate issue of the inf-sup condition, we can alternatively consider a cost that involves both the control and the state. Note that the minimizer of the functional $(y, v) \mapsto J(y, v) := \|y\|^2_{L^2(Q_T)} + \|v\|^2_{L^2(q_T)}$ over the control-state pair for (1) is given by $(y, v) = (-L_A\phi, \phi\, 1_\omega)$, where $\phi \in \Phi$ solves

$$a_{r=1}(\phi, \overline{\phi}) =< \partial_t\overline{\phi}(\cdot\,, 0), u_0 >_{H^{-1}(\Omega), H_0^1(\Omega)} - < \overline{\phi}(\cdot\,, 0), u_1 >_{L^2(\Omega)}, \quad \forall \overline{\phi} \in \Phi, \tag{20}$$

a well-posed problem in view of the generalized inequality (18). When a conformal and dense finite element approximation space $\Phi_h \subset \Phi$ is employed, Cea's Lemma yields $a_{r=1}(\phi_h - \phi, \phi_h - \phi) \to 0$ as $h \to 0$ and a strong convergent approximation $\phi_h 1_\omega$ of a control for (1) is obtained. Once ϕ_h is computed from the fourth-order in time and space elliptic problem (20), an approximation of the controlled solution is defined by $y_h := -L_A\phi_h$. We refer to [24] where this method is fully analyzed for the one-dimensional wave equation with $\mathcal{C}^1(Q_T)$ coefficients. We also refer to the recent work [8].

It is also interesting to point out that the *control-then-discretize* approaches is notably well-suited for mesh adaptivity. We mention a growing interest for space-time (finite element) methods of approximation for the wave equation, initially advocated in [49] and more recently in [54], [1], [30], [29], [77].

2.5 Numerical illustrations

We first illustrate Theorem 5 in the one dimensional case. For simplicity, we take $A \equiv 0$ and $F \equiv 0$ in (1). The initial condition to be controlled is $(u_0, u_1) = (\sin(\pi x), 0) \in H^{k+1}(\Omega) \times H^k(\Omega)$ for all $k \in \mathbb{N}$ leading to regular controlled and adjoint solutions. The distributed control acts in $\omega \times (0,T)$ with $\omega = (a,b) = (0.1, 0.4)$ and $T = 2$. Precisely, the cut off functions are defined as $\chi_0(t) = \frac{e^{-\frac{1}{2t}} e^{-\frac{1}{2(T-t)}}}{e^{-\frac{1}{T}} e^{-\frac{1}{T}}}$ and $\chi_1(x) = \frac{e^{-\frac{1}{5(x-a)}} e^{-\frac{1}{5(b-x)}}}{e^{-\frac{2}{5(b-a)}} e^{-\frac{2}{5(b-a)}}} 1_{[a,b]}(x)$. Figure 1-left depicts the evolution of the relative error $\|\chi(\phi - \phi_h)\|_{L^2(Q_T)} / \|\chi\phi\|_{L^2(Q_T)}$ associated with $T = 2$ with respect to the parameter h for various pairs of (p, q). Remark that explicit solutions are not available in the distributed case: we define as "exact" solution (y, ϕ) the one of (16) from a fine and structured mesh (composed of 409 000 triangles and 205 261 vertices) corresponding to $h \approx 4.41 \times 10^{-3}$ and $(u_h, \phi_h) \in V_h^p \times V_h^q$ with $(p, q) = (3, 3)$. We observe rates close to 0.5, 2 and 3 for $(p, q) = (1, 1)$, $(p, q) = (2, 2)$ and $(p, q) = (3, 3)$ respectively, in agreement with Theorem 5. For comparison, Figure 1-right depicts the evolution of the relative error for $\chi_0(t) = 1$ and $\chi_1(x) = 1_{(a,b)}(x)$, i.e. when no regularization of the control support is introduced. We still observe the convergence with respect to the parameter h but with a reduced rate. For instance, for $(p, q) = (2, 2)$, the rate is close to 1.5. This highlights the influence of the cut off functions, including for very smooth initial conditions. We refer to [13, Section 5.1] for more details.

In order to enhance the robustness of the method, we also consider in the boundary case a stiff situation with discontinuous initial condition: $(u_0, u_1) = (4x 1_{(0,1/2)}, 0)$, $x \in (0, 1)$. We refer to [13, Theorem 4.6] for convergent results in the boundary cases. The corresponding control of minimal $L^2(0,T)$ with $T = 2$ acting at $x = 1$ is given by the trace of the corresponding solution : explicit computations using d'Alembert formula

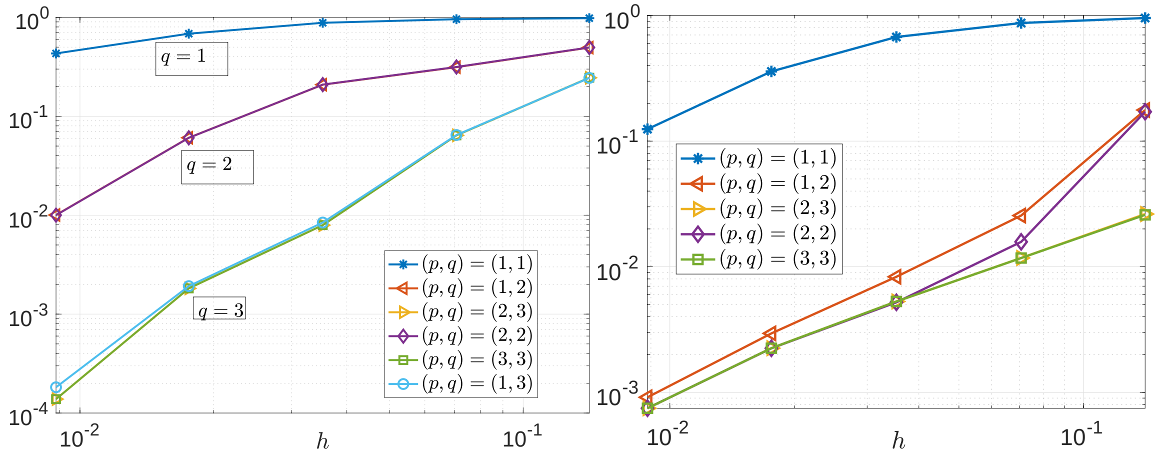

Figure 1: $\|\chi(\phi-\phi_h)\|_{L^2(Q_T)}/\|\chi\phi\|_{L^2(Q_T)}$ vs. h with (left) and without (right) regularization of the control support q_T.

leads to $v(t) = 2(1-t)1_{(1/2,3/2)}(t)$. The corresponding controlled solution is

$$y(x,t) = \begin{cases} 4x & 0 \le x+t < \frac{1}{2}, \\ 2(x-t) & -\frac{1}{2} < t-x < \frac{1}{2}, \quad x+t \ge \frac{1}{2}, \\ 0 & \text{elsewhere}. \end{cases}$$

The initial condition of the corresponding adjoint solution is $(\phi_0, \phi_1) = (0, -2x\, 1_{(0,1/2)}(x)) \in H^1(\Omega) \times L^2(\Omega)$ Both the variable ϕ and y develop singularities (where y and $\nabla\phi$ are discontinuous). Figure 2 depicts the evolution of $\|\partial_x \phi_h(1,\cdot) - v\|_{L^2(0,T)}/\|v\|_{L^2(0,T)}$ with respect to the discretization parameter h, leading to a rate of convergence close to $1/2$. We also emphasize that the space-time discretization formulation is appropriated for mesh adaptivity: using the space of approximation $V_h^1 \times V_h^2$, Figure 4-left (resp. right) depicts the mesh obtained after seven adaptative refinements based on the local values of the gradient of ϕ_h (resp. λ_h). Starting with a coarse mesh composed of 288 triangles and 166 vertices, the final mesh is composed with 13068 triangles and 6700 vertices. We refer to [13, Section 5] for numerical illustrations of Theorem 5 with smooth initial data.

The second experiment illustrates Theorem 3 devoted to a semilinear situation in the two dimensional case with $\Omega = (0,1)^2$ (we refer to [9] for more details). The final time is taken equal to $T = 3$ and the control domain ω is depicted in Figure 3. As for the initial and final conditions, we take $(u_0, u_1) \equiv (100\sin(\pi x_1)\sin(\pi x_2), 0)$ and $(z_0, z_1) \equiv (0,0)$, respectively. We refer to [3, 70] for numerical experiments in the two dimensional case. Moreover, for any real constant c_f, we consider the nonlinear function $f(r) = -c_f\, r \ln^{1/2}(2+|r|)$, for all $r \in \mathbb{R}$. Note that f satisfies $(\overline{\mathbf{H}}_p)$ for $p=1$ and $(\mathbf{H_2})$ for $|c_f|$ small enough. Remark that the unfavorable situation (for which the norm of the uncontrolled corresponding solution grows) corresponds to strictly positives values of c_f.

Table 1, Figures 5 and 6 show the results obtained for $c_f = 10$. The convergence is observed after 4 iterations. The optimal steps λ_k are very close to one. The main difference with lower values of c_f (for instance $c_f = 5$) is the behavior of the uncontrolled solution, which grows exponentially with respect to the time variable, as shown in Figure 5. As expected, this large value of c_f induces a large gap between the nonlinear and the linear controls.

We observe that the nonlinear control $v^\star$ – the limit of the sequence $(v_k)_{k\in\mathbb{N}}$ – acts stronger from the beginning, precisely in order to compensate the initial exponential growth of the solution outside the set ω. We also observe that the control reduces the oscillations of the corresponding controlled solution (in comparison with the solution to the linear equation). The effect of the nonlinear control on the system is measured through the relative error $\mathcal{E}_T := \frac{\|(y,\partial_t y)(\cdot\,,T;v^\star)\|_{\boldsymbol{V}}}{\|(y,\partial_t y)(\cdot\,,T;0)\|_{\boldsymbol{V}}}$ where $y(\cdot\,,T,v^\star)$ (resp. $y(\cdot\,,T,0)$) is the solution

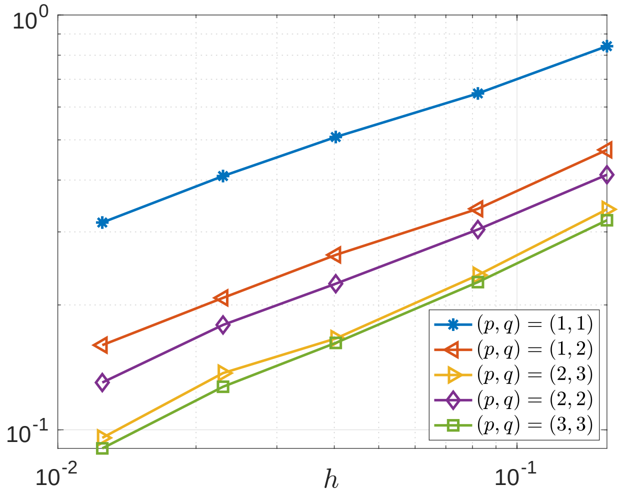

Figure 2: $\|\partial_x \phi_h(1,\cdot) - v\|_{L^2(0,T)}/\|v\|_{L^2(0,T)}$ with respect to h for different approximations.

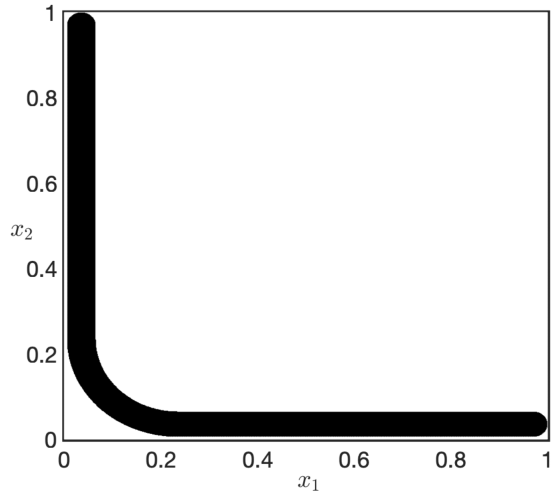

Figure 3: Control domain $\omega \subset \Omega = (0,1)^2$.

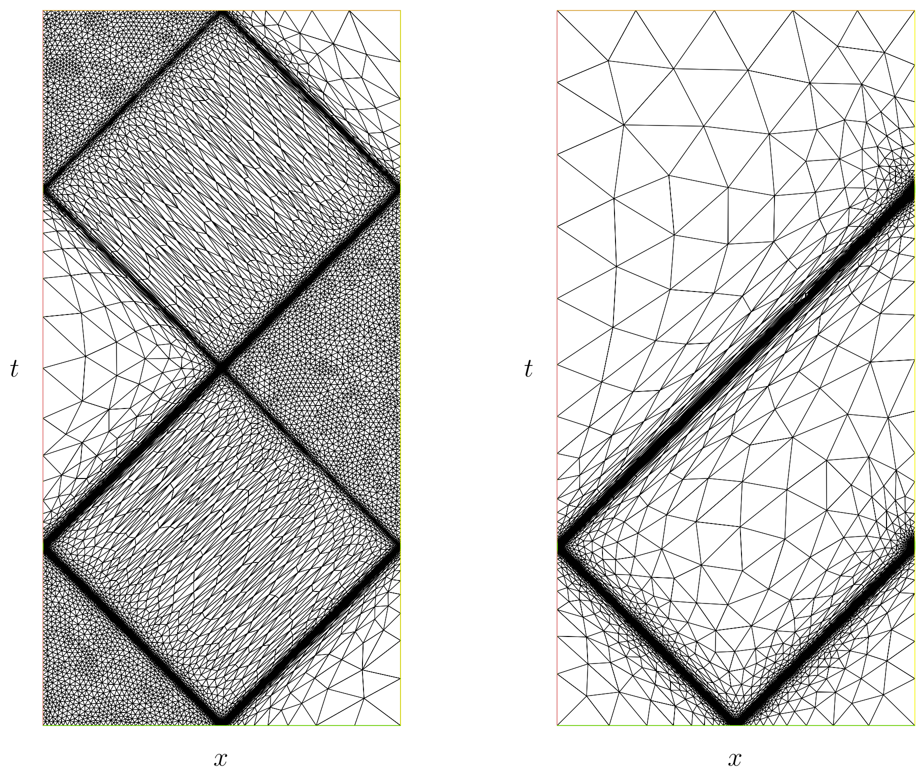

Figure 4: Locally refine spacetime meshes with respect to ϕ_h (left) and λ_h (right).

at time T of (5) with control equal to $v = v^\star$ (resp. $v = 0$). We obtain $\mathcal{E}_T \approx 5.83 \times 10^{-5}$. Larger values of c_f such $|c_f| > 40$ yield to first values of the sequence $(\lambda_k)_{k\in\mathbb{N}}$ far from one (as observed in [59] for the solution of the Navier-Stokes system with large values of the Reynolds number).

$\sharp$iterate k	$\sqrt{2E(y_k,v_k)}$	$\frac{\Vert y_k-y_{k-1}\Vert_{L^2(Q_T)}}{\Vert y_{k-1}\Vert_{L^2(Q_T)}}$	$\frac{\Vert v_k-v_{k-1}\Vert_{L^2_\chi(q_T)}}{\Vert v_{k-1}\Vert_{L^2_\chi(q_T)}}$	$\Vert y_k\Vert_{L^2(Q_T)}$	$\Vert v_k\Vert_{L^2_\chi(q_T)}$	λ_k
0	7.44×10^{2}	–	–	38.116	732.22	1
1	1.63×10^{2}	1.79×10^{0}	9.30×10^{-1}	58.691	667.602	1
2	1.62×10^{0}	8.42×10^{-2}	1.41×10^{-1}	60.781	642.643	1
3	1.97×10^{-3}	1.21×10^{-3}	4.66×10^{-3}	60.745	643.784	1
4	5.11×10^{-10}	6.43×10^{-7}	2.63×10^{-6}	60.745	643.785	–

Table 1: $c_f = 10$; Norms of (y_k, v_k) with respect to k defined by the algorithm (38).

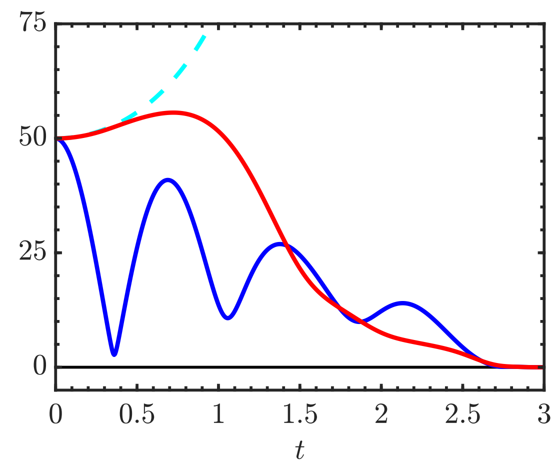

Figure 5: $c_f = 10$ – $\Vert y_4(\cdot,t)\Vert_{L^2(\Omega)}$ (red), $\Vert y_0(\cdot,t)\Vert_{L^2(\Omega)}$ (blue) and $\Vert y(\cdot,t;0)\Vert_{L^2(\Omega)}$ (cyan dashed) w.r.t. t.

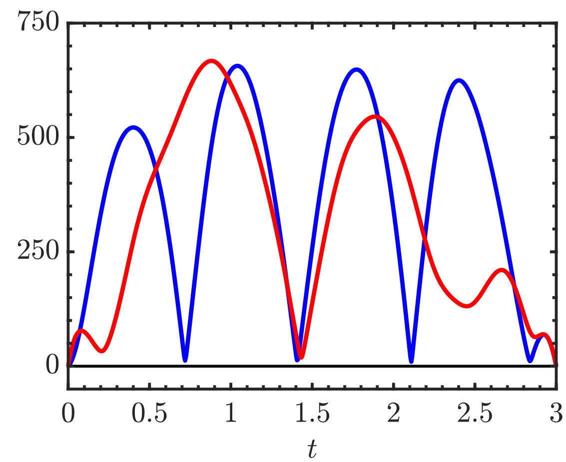

Figure 6: $c_f = 10$ – $\Vert v_4(\cdot,t)\Vert_{L^2_\chi(\omega)}$ (red), and $\Vert v_0(\cdot,t)\Vert_{L^2_\chi(\omega)}$ (blue) w.r.t. t.

3 The heat equation

In this section, we consider the heat equation, both in linear and semilinear regime and highlight that space-time approaches also lead to robust numerical approximation of exact controls. The approach is similar with the notable exception that it involves singular in time Carleman weights, in the framework proposed by Fursikov and Imanuvilov in [41]. This third section follows the same outline than the previous one.

3.1 Controllability results for the linear heat equation

As a preliminary step for a semilinear situation, we recall some controllability results for the linear heat equation with potential $A \in L^\infty(Q_T)$ and right hand side $F \in L^2(\rho_{0,s}, Q_T)$ for a precise weight $\rho_{0,s}$ parametrized by $s \in \mathbb{R}_+^\star$, that is defined in the sequel. More precisely, we are interested in the existence of a control v such that the solution z of

$$\begin{cases} \partial_t z - \Delta z + A z = v 1_\omega + F & \text{in } Q_T, \\ z = 0 \quad \text{on } \Sigma_T, \quad z(\cdot,0) = z_0 & \text{in } \Omega \end{cases} \tag{21}$$

satisfies

$$z(\cdot,T) = 0 \text{ in } \Omega. \tag{22}$$

We follow the usual strategy of [41] to construct a solution of the null controllability problem, using Carleman type estimates. Instead of using the classical estimates of [41], we use the one in [4] for which it is easier to deal with non zero initial data as the weight function does not blow up as $t \to 0$. For any $s \geq 0$, we consider the weight functions $\rho_s = \rho_s(x,t)$, $\rho_{0,s} = \rho_{0,s}(x,t)$ and $\rho_{1,s} = \rho_{1,s}(x,t)$ which are continuous, strictly positive and belong to $L^\infty(Q_{T-\delta})$ for any $\delta > 0$. Precisely, we use the weights introduced in [4]:

$(\rho_{0,s}, \rho_{1,s}, \rho_{2,s}, \rho_{3,s}) = (\xi^{-3/2}, \xi^{-1}, \xi^{-1/2}, \xi^{1/2})\rho_s$ where ρ_s and ξ are defined, for all $s \geq 1$ and $\lambda \geq 1$, as follows:

$$\rho_s(x,t) = e^{s\varphi(x,t)}, \quad \xi(x,t) = \theta(t)e^{\lambda\widehat{\psi}(x)}, \tag{23}$$

with $\theta \in \mathcal{C}^2([0,T))$ such that $\theta(0) = 1$ and $\theta(t) = (T-t)^{-1}$ for all $t \in [T-T_1, T]$ with $0 < T_1 < \min(\frac{1}{4}, \frac{3T}{8})$ and $\varphi \in \mathcal{C}^1([0,T))$ is defined by $\varphi(x,t) = \theta(t)\big(\lambda e^{12\lambda} - e^{\lambda\widehat{\psi}(x)}\big)$ with $\widehat{\psi} = \widetilde{\psi} + 6$, where $\widetilde{\psi} \in \mathcal{C}^1(\overline{\Omega})$ satisfies $\widetilde{\psi} \in (0,1)$ in Ω, $\widetilde{\psi} = 0$ on $\partial\Omega$ and $|\nabla\widetilde{\psi}(x)| > 0$ in $\overline{\Omega\backslash\omega}$. We emphasize that the weights blow up as $t \to T^-$ but not at $t = 0$ and that $\rho_{0,s}(x,t) = \xi^{-3/2}(x,t)\rho_s(x,t) \geq e^{3/2s}$ for all $(x,t) \in Q_T$.

3.1.1 Carleman estimates

The controllability property for the linear system (21) is a consequence of the following Carleman estimate, written to simplify in the one dimensional case :

Lemma 1. *[60, Lemma 2.1] Let $P_0 := \{q \in \mathcal{C}^2(\overline{Q_T}) \ : \ q = 0 \text{ on } \Sigma_T\}$. There exist $\lambda_0 \geq 1$ and $s_0 \geq 1$ such that for all $\lambda \geq \lambda_0$ and for all $s \geq \max(\|A\|^{2/3}_{L^\infty(Q_T)}, s_0)$, the following Carleman estimate holds*

$$\begin{aligned} \int_\Omega \rho_s^{-2}(0)|\partial_x p(0)|^2 + s^2\lambda^3 e^{14\lambda}\int_\Omega \rho_s^{-2}(0)|p(0)|^2 + s\lambda^2\int_{Q_T}\rho_{2,s}^{-2}|\partial_x p|^2 + s^3\lambda^4\int_{Q_T}\rho_{0,s}^{-2}|p|^2 \\ \leq C\int_{Q_T}\rho_s^{-2}|-\partial_t p - \partial_{xx}p + Ap|^2 + Cs^3\lambda^4\int_{q_T}\rho_{0,s}^{-2}|p|^2, \quad \forall p \in P_0. \end{aligned} \tag{24}$$

This estimate is deduced from the one obtained in [4, Theorem 2.5] devoted to the case $A \equiv 0$. In the sequel we assume that $\lambda = \lambda_0$. We then define and check that the bilinear form

$$(p,q)_P := \int_{Q_T}\rho_s^{-2}L_A^\star p\, L_A^\star q + s^3\lambda_0^4\int_{q_T}\rho_{0,s}^{-2}p\, q$$

where $L_A^\star q := -\partial_t q - \partial_{xx}q + Aq$ for all $q \in P_0$ is a scalar product on P_0 (see [36]). The completion P of P_0 for the norm $\|\cdot\|_P$ associated with this scalar product is a Hilbert space. By density arguments, (24) remains true for all $p \in P$, that is, for $\lambda = \lambda_0$,

$$\int_\Omega \rho_s^{-2}(0)|\partial_x p(0)|^2 + s^2\lambda_0^3 e^{14\lambda_0}\int_\Omega \rho_s^{-2}(0)|p(0)|^2 + s\lambda_0^2\int_{Q_T}\rho_{2,s}^{-2}|\partial_x p|^2 + s^3\lambda_0^4\int_{Q_T}\rho_{0,s}^{-2}|p|^2 \leq C\|p\|_P^2 \tag{25}$$

for all $s \geq \max(\|A\|^{2/3}_{L^\infty(Q_T)}, s_0)$. This inequality leads to the following result.

Lemma 2. *[60, Lemma 2.2] Let $s \geq \max(\|A\|^{2/3}_{L^\infty(Q_T)}, s_0)$. There exists a unique solution $p \in P$ of*

$$(p,q)_P = \int_\Omega z_0 q(0) + \int_{Q_T} Fq, \quad \forall q \in P. \tag{26}$$

This solution satisfies the following estimate (with $c := \|\varphi(\cdot,0)\|_{L^\infty(\Omega)}$)

$$\|p\|_P \leq Cs^{-3/2}\big(\|\rho_{0,s}\, F\|_{L^2(Q_T)} + e^{cs}\|z_0\|_{L^2(\Omega)}\big). \tag{27}$$

3.1.2 Application to controllability

Following closely [41], the previous lemma implies a controllability result for the linear system (21).

Theorem 6. *[60, Theorem 2.3] Assume $A \in L^\infty(Q_T)$, $s \geq \max(\|A\|^{2/3}_{L^\infty(Q_T)}, s_0)$, $F \in L^2(\rho_{0,s}, Q_T)$ and $z_0 \in L^2(\Omega)$. Let p the solution of (26). Then, the pair (z, v) defined by*

$$z = \rho_s^{-2} L_A^\star p \quad \text{ and } \quad v = -s^3 \lambda_0^4 \rho_{0,s}^{-2} p|_{q_T} \tag{28}$$

is a controlled pair and satisfies the following estimates

$$\|\rho_s\, z\|_{L^2(Q_T)} + s^{-3/2}\lambda_0^{-2}\|\rho_{0,s}\, v\|_{L^2(q_T)} \leq C s^{-3/2}\big(\|\rho_{0,s} F\|_{L^2(Q_T)} + e^{cs}\|z_0\|_{L^2(\Omega)}\big) \tag{29}$$

with $c := \|\varphi(\cdot, 0)\|_{L^\infty(\Omega)}$.

We refer to [38] for an estimate of the null control of minimal $L^2(q_T)$ norm (corresponding to $\rho_0 \equiv 1$ and $\rho = 0$) in the case $F \equiv 0$. Thus, the resolution of (26) leads in practice to a control for the linear problem. Moreover, following [41], we check that the pair (z, v) defined in (28) is the unique minimizer of the functional J defined as

$$J(z, v) := \frac{s^3\lambda_0^4}{2}\int_{Q_T} \rho_s^2|z|^2 + \frac{1}{2}\int_{q_T}\rho_{0,s}^2|v|^2 \tag{30}$$

over the set $\big\{(z, v) : \rho_s z \in L^2(Q_T),\ \rho_{0,s} v \in L^2(q_T),\ (z, v1_\omega) \text{ solves (21)-(22) in the transposition sense}\big\}$.

Before to discuss the numerical approximation of controls, we explain in the next section how we can construct, using the estimate of Theorem (6), convergence sequence of controlled pair in semilinear situation.

3.2 Controllability results for the semilinear heat equation

We now consider the null controllability problem for the following system for the semilinear heat equation:

$$\begin{cases} \partial_t y - \Delta y + f(y) = v 1_\omega & \text{in } Q_T \\ y = 0 \quad \text{on } \Sigma_T, \quad y(\cdot, 0) = u_0 & \text{in } \Omega \end{cases}, \tag{31}$$

with $u_0 \in L^2(\Omega)$, $v \in L^2(Q_T)$ and $f : \mathbb{R} \mapsto \mathbb{R}$. Recall that if f is locally Lipschitz-continuous and satisfies the condition $|f'(r)| \leq C(1+|r|^{4+d})$ for all $r \in \mathbb{R}$, then (31) possesses exactly one local in time solution. Moreover, in accordance with the results in [21, Section 5], under the growth condition $|f(r)| \leq C(1 + |r|\ln(1 + |r|))$ for all $r \in \mathbb{R}$ and some $C > 0$, the solutions to (31) are globally defined in $[0, T]$ and one has

$$y \in \mathcal{C}^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)). \tag{32}$$

Without the above growth condition, the solutions to (31) can blow up before $t = T$; in general, the blow-up time depends on f and the size of $\|u_0\|_{L^2(\Omega)}$. We refer to [53] and to [42, Section 2 and Section 5] for a survey on this issue.

System (31) is said to be *exactly controllable to trajectories* at time T if, for any $u_0 \in L^2(\Omega)$ and any globally defined bounded trajectory $y^\star \in \mathcal{C}^0([0, T]; L^2(\Omega))$ (corresponding to data $u_0^\star \in L^2(\Omega)$ and $f^\star \in L^2(q_T)$), there exist controls $f \in L^2(q_T)$ and associated states y that are again globally defined in $[0, T]$ and satisfy (32) and

$$y(x, T) = y^\star(x, T), \quad \forall x \in \Omega. \tag{33}$$

As for the wave equation, the uniform controllability strongly depends on the growth properties of the nonlinear function f at infinity. The following has been proven by Fernández-Cara and Zuazua in [39]:

Theorem 7. *[39, Theorem 1.2] Let $T > 0$ be given and $d \geq 1$. Assume that (31) admits at least one solution $y^\star$, globally defined in $[0, T]$ and bounded in Q_T. Assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is $\mathcal{C}^1$ and satisfies $|f'(r)| \leq C(1 + |r|^{4+d})$ for every $r \in \mathbb{R}$. If*

$$(\mathbf{H_4}) \quad \limsup_{|r|\to\infty} \frac{|f(r)|}{|r|\ln^{3/2}|r|} = 0,$$

then (31) is exactly controllable to $y^\star$ in time T.

Therefore, if $|f(r)|$ does not grow at infinity faster than $|r|\ln^p(1+|r|)$ for some $p<3/2$, then (31) is controllable. On the contrary, if f is too "super-linear" at infinity (specifically if $p>2$), then for some initial data the control cannot compensate the blow-up phenomenon occurring in $\Omega\backslash\overline{\omega}$ (see [39, Theorem 1.1]). The problem remains open when f behaves at infinity like $|r|\ln^p(1+|r|)$ with $3/2\leq p\leq 2$. In [55], Le Balc'h has proved the uniform controllability for $p\leq 2$ assuming that T is large enough and imposing sign conditions on f, notably that $f(r)>0$ for $r>0$ or $f(r)<0$ for $r<0$ (a condition not satisfied for $f(r)=-r\,\ln^p(1+|r|)$).

Theorem 7 is deduced in [39] from a null controllability result corresponding to the null trajectory, i.e. $y^\star\equiv 0$ corresponding to $v^\star\equiv 0, u_0^\star\equiv 0$ and assuming $f(0)=0$. The proof is based on a fixed-point method, initially introduced in [79] for a one-dimensional wave equation. Precisely, a stability result is shown for the operator $\Lambda: L^\infty(Q_T)\mapsto L^\infty(Q_T)$, where $y:=\Lambda(z)$ is a null controlled solution of the linear boundary value problem

$$\begin{cases}\partial_t y-\Delta y+y\,\widetilde{f}(z)=v1_\omega & \text{in } Q_T\\ y=0 \quad \text{on } \Sigma_T, \quad y(\cdot\,,0)=u_0 & \text{in } \Omega\end{cases}, \qquad \widetilde{f}(r):=\begin{cases}f(r)/r & r\neq 0\\ f'(0) & r=0\end{cases}. \tag{34}$$

Then, Kakutani's Theorem provides the existence of at least one fixed-point for the operator Λ, which is also a controlled solution for (31). The control of minimal $L^\infty(q_T)$ norm is considered in [39] leading to controlled solution in $L^\infty(Q_T)$.

3.3 Construction of two sequences converging to a controlled pair for (31)

We now discuss how we can design sequences $(y_k,v_k)_{k\in\mathbf{N}}$ converging to a controlled pair for (31). The motivation, difficulties and ideas are very similar to the ones explained in Section 2.3 dedicated to the wave equation. The arguments for the heat equation are however a bit more technical since singular Carleman weights appear in the various estimates.

As for the wave equation, one may consider for any initial guess in $L^\infty(Q_T)$ the Picard iterates associated with the operator Λ. Numerical experiments reported in [35, Section 4] exhibit the non convergence of the sequence $(y_k)_{k\in\mathbb{N}}$ for some initial conditions large enough, related to the fact that the operator Λ is not contracting.

In the one-dimensional case, a least-squares type approach, based on the minimization over $Z:=L^2((T-t)^{-1},Q_T)=\{z:(T-t)^{-1}z\in L^2(Q_T)\}$ of the functional $\mathcal{R}:Z\to\mathbb{R}^+$ defined by $\mathcal{R}(z):=\|z-\Lambda(z)\|_Z^2$ has been introduced in [35]. Assuming $u_0\in L^\infty(\Omega)$, $\widetilde{f}\in\mathcal{C}^1(\mathbb{R})$ and $(\widetilde{f})'\in L^\infty(\mathbb{R})$, it is proved (see [35, Proposition 3.2]) that $\mathcal{R}\in\mathcal{C}^1(Z;\mathbb{R}^+)$ and that, for some constant $C>0$

$$\big(1-C\|(\widetilde{f})'\|_{L^\infty(\mathbb{R})}\|u_0\|_{L^\infty(\Omega)}\big)\sqrt{\mathcal{R}(z)}\leq\|\mathcal{R}'(z)\|_{L^2(Q_T)}\quad\forall z\in Z$$

implying that if $\|(\widetilde{f})'\|_{L^\infty(\mathbb{R})}\|u_0\|_{L^\infty(\Omega)}$ is small enough, then any critical point for $\mathcal{R}$ is a fixed point for Λ (see [35, Proposition 3.2]). In particular, taking u_0 small in L^∞ makes of no relevance the behavior of $\tilde{f}$ at infinity, as it enters in the framework of local controllability results. Under such smallness assumption on the data, numerical experiments (see [35, Section 4]) display the convergence of gradient based minimizing sequences for $\mathcal{R}$ and a better behavior than the ones associated with the Picard iterates for Λ.

Similarly, we can employ a Newton type method to find a zero of the mapping $\widetilde{\mathcal{F}}:Y\mapsto W$ defined by

$$\widetilde{\mathcal{F}}(y,v)=(\partial_t y-\Delta y+f(y)-v1_\omega, y(\cdot\,,0)-u_0)\quad\forall(y,v)\in Y, \tag{35}$$

where the Hilbert space Y and W are defined as follows

$$Y:=\big\{(y,v):\rho_s y\in L^2(Q_T),\ \rho_{0,s}(\partial_t y-\Delta y)\in L^2(Q_T), y=0 \text{ on } \Sigma_T,\ \rho_{0,s}v\in L^2(q_T)\big\}$$

and $W:=L^2(\rho_{0,s},Q_T)\times L^2(\Omega)$ for some appropriates weights. Here, $L^2(\rho_{0,s},Q_T)$ stands for the space $\{z:\rho_{0,s}z\in L^2(Q_T)\}$. It is shown in [35, Section 3.3] that, if $f\in\mathcal{C}^1(\mathbb{R})$ and $f'\in L^\infty(\mathbb{R})$, then $\widetilde{\mathcal{F}}\in\mathcal{C}^1(Y;W)$.

This enables to derive the Newton iterative sequence. Starting from (y_0,v_0) in Y, we set, for each $k\geq 0$, $(y_{k+1},v_{k+1})=(y_k,v_k)-(Y_k,V_k)$, where V_k is a null control for the system

$$\begin{cases}\partial_t Y_k-\Delta Y_k+f'(y_k)\,Y_k=V_k\,1_\omega+\partial_t y_k-\Delta y_k+f(y_k)-v_k 1_\omega & \text{in } Q_T,\\ Y_k=0 \quad \text{on } \Sigma_T, \quad Y_k(\cdot\,,0)=u_0-y_k(\cdot\,,0) & \text{in } \Omega\end{cases} \tag{36}$$

and $Y_k(\cdot,T)=-y_k(\cdot,T)$. Numerical experiments in [35, Section 4] exhibit however the lack of convergence of the Newton method for large values of $\|u_0\|_{L^2(\Omega)}$.

3.3.1 A least-squares approach related to a Newton type linearization

Let us introduce, for each $s\geq s_0$, the vector space

$$\mathcal{A}_{0,s}:=\{(y,v):\rho_s y\in L^2(Q_T),\ \rho_{0,s}v\in L^2(q_T),\rho_{0,s}(\partial_t y-\Delta y)\in L^2(Q_T),\ y(\cdot\,,0)=0 \text{ in } \Omega,\ y=0 \text{ on } \Sigma_T\},$$

where ρ_s, $\rho_{1,s}$ and $\rho_{0,s}$ are defined in (23). Endowed with the scalar product

$$\big((y,v),(\overline{y},\overline{v})\big)_{\mathcal{A}_{0,s}}:=\big(\rho_s y,\rho_s\overline{y}\big)_{L^2(Q_T)}+\big(\rho_{0,s}v,\rho_{0,s}\overline{v}\big)_{L^2(q_T)}+\big(\rho_{0,s}(\partial_t y-\Delta y),\rho_{0,s}(\partial_t\overline{y}-\Delta\overline{y})\big)_{L^2(Q_T)},$$

$\mathcal{A}_{0,s}$ is a Hilbert space. Let us also consider the linear manifold

$$\mathcal{A}_s:=\{(y,v):\rho_s\, y\in L^2(Q_T),\ \rho_{0,s}v\in L^2(q_T),\rho_{0,s}(\partial_t y-\Delta y)\in L^2(Q_T),\ y(\cdot\,,0)=u_0 \text{ in } \Omega,\ y=0 \text{ on } \Sigma_T\}.$$

We endow $\mathcal{A}_s$ with the same scalar product. If $(y,v)\in\mathcal{A}_s$, then $y\in\mathcal{C}^0([0,T];L^2(\Omega))$. Moreover, the property $\rho_s y\in L^2(Q_T)$ implies that $y(\cdot\,,T)=0$ so that the null controllability requirement is incorporated in the spaces $\mathcal{A}_{0,s}$ and $\mathcal{A}_s$. For any fixed $s\geq 0$, we consider the following non-convex extremal problem:

$$\inf_{(y,v)\in\mathcal{A}_{0,s}} E_s(y,v), \quad E_s(y,v):=\frac{1}{2}\big\|\rho_{0,s}\big(\partial_t y-\Delta y+f(y)-v\,1_\omega\big)\big\|^2_{L^2(Q_T)}. \tag{37}$$

We check that $\rho_{0,s}f(y)\in L^2(Q_T)$ for any $(y,f)\in\mathcal{A}_s$, so that E_s is well-defined. Assuming slightly stronger assumption on f than in Theorem 7, a strong convergent approximation of a controlled pair is obtained:

Theorem 8. *[60, Theorem 4.3] Let $T>0$ be given. Let $d=1$. Assume that (31) admits at least one solution $y^\star$, globally defined in $[0,T]$ and bounded in Q_T associated with $v^\star\in L^2(\rho_{0,s},q_T)$ and s large enough. Assume that $f\in\mathcal{C}^1(\mathbb{R})$ satisfies $(\overline{\mathbf{H}}_p)$ from some $p\in[0,1]$ (introduced in page 5) and the growth condition*

$$(\mathbf{H_1'})\quad \exists\alpha>0,\ s.t.\ |f'(r)|\leq(\alpha+\beta^\star\ln(1+|r|))^{3/2}, \quad \forall r\in\mathbb{R}$$

for some $\beta^\star=\beta^\star(y^\star)>0$ small enough. Then, for any $u_0\in H^1_0(\Omega)$ and any starting $(y_0,v_0)\in\mathcal{A}_s$, the sequence $(y_k,v_k)_{k\in\mathbb{N}}\in\mathcal{A}_s$ defined as follows:

$$\begin{cases}(y_0,v_0)\in\mathcal{A}_s, \quad (y_{k+1},v_{k+1})=(y_k,v_k)-\lambda_k(Y^1_k,V^1_k), \quad k\geq 0,\\ \lambda_k=\mathit{argmin}_{\lambda\in[0,1]}E_s\big((y_k,v_k)-\lambda(Y^1_k,V^1_k)\big),\end{cases} \tag{38}$$

where $(Y^1_k,F^1_k)\in\mathcal{A}_{0,s}$ is the minimal controlled pair solution (with respect to the cost J, see (30)) of

$$\begin{cases}\partial_t Y^1_k-\Delta Y^1_k+f'(y_k)Y^1_k=V^1_k 1_\omega+\partial_t y_k-\Delta y_k+f(y_k)-v_k 1_\omega & \text{in} \quad Q_T,\\ Y^1_k=0 \ \text{on} \ \Sigma_T, \quad Y^1_k(\cdot\,,0)=0 \ \text{in} \ \Omega,\end{cases} \tag{39}$$

converges strongly to a controlled pair for (31) satisfying (33). Moreover, after a finite number of iterations, the convergence is of order at least $1+p$.

The hypothesis on f are stronger here than in Theorem 7: it should be noted however that the function $f(r)=a+br+\beta r\ln(1+|r|)^{3/2}$, $a,b\in\mathbb{R}$ which is somehow the limit case in $(\mathbf{H_4})$ satisfies $(\mathbf{H_1'})$ and $(\overline{\mathbf{H}}_\mathbf{1})$.

On the other hand, Theorem 8 devoted to the one dimensional case is constructive, contrary to Theorem 7. A similar construction is performed in a multi-dimensional case with $d \leq 3$ in [57] assuming that f is globally Lipschitz. The minimizing sequence for E_s constructed in [57, 60] are related to the operator $\Lambda_N : \mathcal{A} \to \mathcal{A}$ defined by $y = \Lambda_N(z)$ controlled solution of

$$\begin{cases} \partial_t y - \Delta y + f'(z)y = v 1_\omega + f'(z)z - f(z) & \text{in} \quad Q_T \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = u_0 & \text{in} \quad \Omega \end{cases}, \tag{40}$$

through the control v of minimal $L^2(\rho_0(s), q_T)$ norm. The analysis in [60] makes use of global L^2 Carleman estimates as initially introduced in this context in [41]. The arguments used in the proof take their roots in the works [58, 59], concerned with the approximation of the solution to Navier-Stokes-like problems through least-square methods; see also [61, 71], inspired in the seminal contribution [11].

We also emphasize that the $L^2(Q_T)$ norm in E_s indicates that we are looking for regular weak solutions to the parabolic equation (31). We refer to [57], devoted to the case $f' \in L^\infty(\mathbb{R})$ and $d \leq 3$, where the $L^2(0, T; H^{-1}(\Omega))$ norm is considered leading to weaker solutions.

The analysis in [60] indicates that the parameter s plays a crucial role: a large value of this parameter ensures convergence properties. This is the also the case in the following section where a different method based on a simpler linearization is discussed.

3.3.2 Influence of the parameter s on a simpler linearization

The following extension is proved in [32] based on simpler linearization.

Theorem 9. *[32, Theorem 8] Let $T > 0$ be given. Let $d \leq 5$ and $u_0 \in L^\infty(\Omega)$. Assume that f is locally Lipschitz-continuous and satisfies $(\mathbf{H_1'})$ for $\beta^\star$ small enough. There exist s and R large enough such that, for any $y_0 \in \mathcal{C}_R(s) := \{y \in L^\infty(Q_T) : \|y\|_{L^\infty(Q_T)} \leq R, \|\rho_{0,s} y\|_{L^2(Q_T)} \leq R^{1/2}\}$, the sequence $(y_k)_{k\in\mathbb{N}}$ given by*

$$\begin{cases} \partial_t y_k - \Delta y_k = v_k 1_\omega - f(y_{k-1}) & \textit{in} \quad Q_T, \\ y_k = 0 \quad \textit{on} \quad \Sigma_T, \quad y_k(\cdot\,, 0) = u_0 & \textit{in} \quad \Omega, \end{cases} \tag{41}$$

where $v_k \in L^2(\rho_{0,s}, q_T)$ is such that (y_k, v_k) minimizes J (see (30)), remain in $\mathcal{C}_R(s)$ and converge strongly to a controlled solution for (31).

To prove this result, we proceed as follows:

- First, we introduce, for each $\widehat{y} \in L^2(\rho_{0,s}, Q_T) \cap L^\infty(Q_T)$, the following corresponding linear null controllability problem: find v such that the solution to

$$\begin{cases} \partial_t y - \Delta y = v 1_\omega - f(\widehat{y}), & \text{in} \quad Q_T, \\ y = 0, \quad \text{on} \quad \Sigma_T, \quad y(\cdot\,, 0) = u_0, & \text{in} \quad \Omega \end{cases} \tag{42}$$

  satisfies $y(\cdot\,, T) = 0$.

- Then, we consider the mapping Λ_s that associates to each $\widehat{y}$ the solution to (42) with the control $v 1_\omega$ furnished by Theorem 6 (for $A \equiv 0$ and $F = -f(\widehat{y})$) and prove that for s large enough, the operator Λ_s is a contraction.

3.4 Numerical approximation of exact controls for the heat equation

Approximations of null controls for the linear heat equation is a delicate issue: we mention the seminal work [16] dealing with the control of minimal L^2-norm which is very oscillatory near the final time $t = T$ and therefore difficult to construct and implement for real life applications (see also [52, 74] where this is discussed at length). On the other hand, as discussed in [36, 37], introduction of Carleman weights in the cost functional J leads - within *the control-then-discretize* strategy - to a robust method and strong convergent approximations with respect to the discretization parameter. Precisely, in view of Theorem 6, one have to approximate the solution $p \in P$ of the second order in time and fourth order in space variational formulation (26). A conformal parametrized approximation, say P_h of P, leads to the finite dimensional problem : find $p_h \in P_h$ solution of

$$(p_h, \overline{p}_h)_P = \int_{Q_T} F\overline{p}_h + \int_\Omega u_0 \overline{p}_h(0) \quad \forall \overline{p}_h \in P_h. \tag{43}$$

If the family $(P_h)_{h>0}$ is dense in P, Cea type lemma implies the convergence $\|p_h - p\|_P \to 0$ as $h \to 0$. From p_h, an approximation of the controlled state is then given by $(y_h, v_h) := (\rho_s^{-2} L^\star p_h, -s^3 \rho_{0,s}^{-2} p_h \, 1_\omega)$. In order to solve (43), it is very convenient to preliminary perform the change of variable

$$m = \rho_{0,s}^{-1} p, \quad z = \rho_s^{-1} L^\star p$$

so that $z = \rho_s^{-1} L^\star(\rho_{0,s} m)$ and $y = \rho_s^{-1} z$ and then replace the formulation (43) by the equivalent and well-posed following mixed formulation: find $(z, m, \eta) \in L^2(Q_T) \times \rho_{0,s}^{-1} P \times L^2(Q_T)$ solution of

$$\begin{cases} \displaystyle\int_{Q_T} z\overline{z} + s^3 \int_{q_T} m\,\overline{m} + \int_{Q_T} (T-t)^{1/2} \eta\big(\overline{z} - \rho_s^{-1} L^\star(\rho_{0,s}\overline{m})\big) \\ \displaystyle\qquad\qquad = -\int_{Q_T} \rho_{0,s} F \overline{m} + \int_\Omega \rho_{0,s}(0) u_0 \overline{m}(0), \quad \forall (\overline{m}, \overline{z}) \in \rho_s^{-1} P \times L^2(Q_T), \\ \displaystyle\int_{Q_T} (T-t)^{1/2} \overline{\eta}\big(z - \rho_s^{-1} L^\star(\rho_{0,s} m)\big) = 0, \quad \forall \overline{\eta} \in L^2(Q_T). \end{cases} \tag{44}$$

η_k stands as a Lagrange multiplier for the constraint $z - \rho_s^{-1} L^\star(\rho_{0,s} m) = 0$ in Q_T. For every $m \in \rho_s^{-1} P$, we check that

$$-\rho_s^{-1} L^\star(\rho_{0,s} m) = (g_1(\theta, \varphi) + g_2(\theta, \varphi)) m + \theta^{-3/2}(\partial_t m + \Delta m) + g_3(\theta, \varphi) \cdot \nabla m$$

with

$$\begin{cases} g_1(\theta, \varphi) := \rho_s^{-1} \partial_t \rho_{0,s} = \partial_t(\theta^{-3/2}) + \theta^{-3/2} s(\partial_t \varphi), \\ g_2(\theta, \varphi) := \theta^{-3/2}(s\Delta\varphi + s^2(\nabla\varphi)^2), \quad g_3(\theta, \varphi) := \rho_s^{-1} \nabla \rho_{0,s} = \theta^{-3/2} s \nabla \varphi. \end{cases}$$

We observe that g_2 is singular like $(T-t)^{-1/2}$ for $t \geq T - T_1$ and therefore introduce the function $(T-t)^{1/2}$ in (44). The equivalent formulation (44) instead of (43) allows, first to eliminate the exponential singularity of the coefficients for t close to T and second to obtain simultaneously the control and the controlled solution. We refer to [36, 37] where experiments are discussed in detail and emphasize the robustness of the approximation. We also refer to [72] for some numerical evidences of the robustness of the method with respect to the parameter h associated with the cost $J(v) = \|\rho_{0,s} v\|^2_{L^2(q_T)}$.

3.5 Numerical illustrations

We illustrate the convergence stated in Theorem 9 by computing the sequence $(y_k, v_k)_{k \in \mathbb{N}^\star}$ solution of (41) and minimizing for each k the functional J_s defined in (30) with s large enough. We consider the one dimensional setting with $\Omega = (0,1)$. We take $T = 1/2$ and consider data for which the uncontrolled solution of (31) blows up before T. Moreover, in order to reduce the decay of the solution of (31) when $f \equiv 0$, we replace the term $-\Delta y$ in (31) by $-\nu \Delta y$ with $\nu = 10^{-1}$. We consider the nonlinear even function $f(r) = c_f\big(\alpha + \beta \ln(1 + |r|)\big)^{3/2} r$ with $\alpha = \beta = 1$ and $c_f < 0$. As for the initial condition to be controlled, we consider $u_0(x) = c_{u_0} \sin(\pi x)$ parametrized by $c_{u_0} > 0$. We use a mesh composed of 29132 vertices and 14807 triangles corresponding to $h \approx 1.17 \times 10^{-2}$. The sequence $(y_k, v_k)_{k \in \mathbb{N}}$ is initialized with the state-control pair (y_0, v_0) corresponding to the controlled trajectory of the linear heat equation with initial datum u_0 and zero source term) and is computed until the following criterion is satisfied $\frac{\|\rho_{0,s}(y_{k+1} - y_k)\|_{L^2(Q_T)}}{\|\rho_{0,s} y_k\|_{L^2(Q_T)}} \leq 10^{-6}$. We shall denote by $k^\star$ the lowest integer k for which it holds true.

For $\omega = (0.2, 0.8)$, $c_{u_0} = 10$ and $c_f = -5$, Figure 7-left depicts the evolution of the relative error $\frac{\|\rho_{0,s}(y_{k+1}-y_k)\|_{L^2(Q_T)}}{\|\rho_{0,s}y_k\|_{L^2(Q_T)}}$ with respect to the parameter of iteration k for $s \in \{1,2,3,4\}$. In agreement with the theoretical results, the convergence is observed for s large enough, here $s \geq 2$. Moreover, the rate increases with s: the convergence is observed after $k^\star$ iterations equal to $48, 17, 13$ for $s = 2, 3$ and 4 respectively. Figure 7-right depicts the ratio $\frac{\|\rho_{0,s}(\Lambda_s(y_k)-\Lambda_s(y_{k-1}))\|_{L^2(Q_T)}}{\|\rho_{0,s}(y_k-y_k)\|_{L^2(Q_T)}}$ highlighting the lack of contracting property of Λ_s for $s = 1$. Figure 8 depicts the evolution of the $L^2(\Omega)$ norm of the control and corresponding controlled solution with respect to the time variable for $s = 2, 3, 4$. In view of the behavior of the weights, large values of s concentrate the action of the control close to the initial time and leads to large $L^\infty(q_T)$ norm of the control (see Table 2). Figure 9 and Figure 10 depict the control and corresponding controlled solution in Q_T for these values of s.

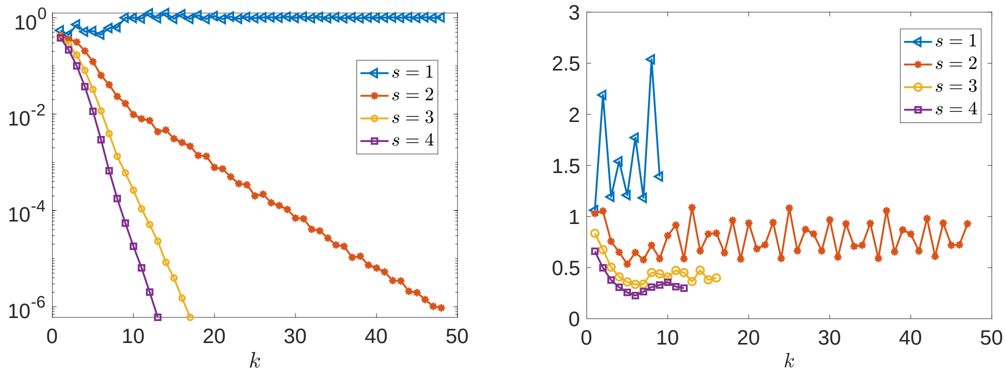

Figure 7: $\frac{\|\rho_{0,s}(y_{k+1}-y_k)\|_{L^2(Q_T)}}{\|\rho_{0,s}y_k\|_{L^2(Q_T)}}$ (**Left**) and $\frac{\|\rho_{0,s}(y_{k+1}-y_k)\|_{L^2(Q_T)}}{\|\rho_{0,s}(y_k-y_{k-1})\|_{L^2(Q_T)}}$ (**Right**) w.r.t. k for $s \in \{1,2,3,4\}$.

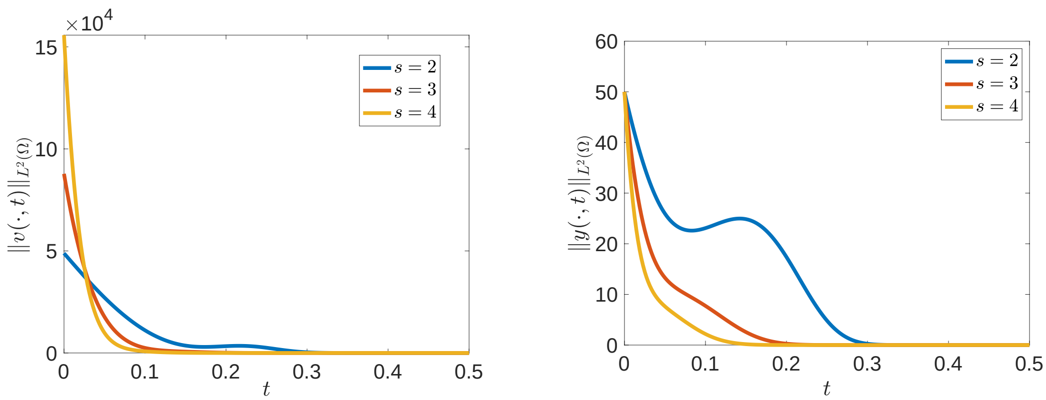

Figure 8: $\|v_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ and $\|y_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ w.r.t. $t \in [0,T]$ for $c_{u_0} = 10$, $c_f = -5$ and $s \in \{2,3,4\}$.

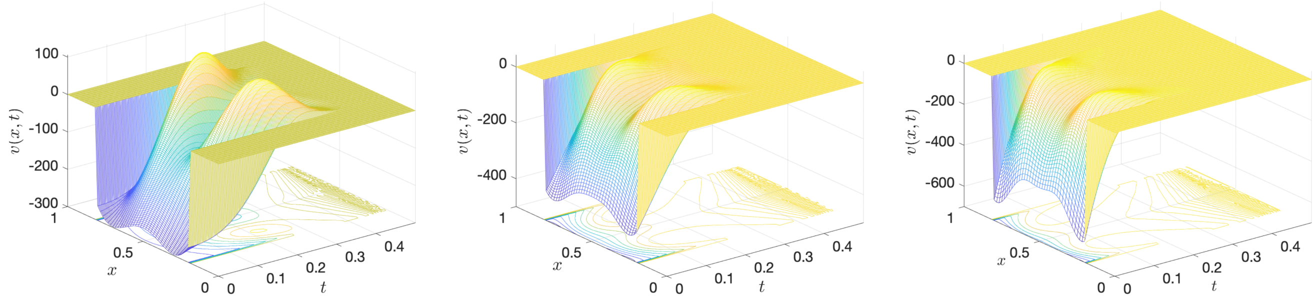

Figure 9: The control $v_{k^\star}$ in Q_T for $c_{u_0} = 10$, $c_f = -5$ and $s \in \{2, 3, 4\}$.

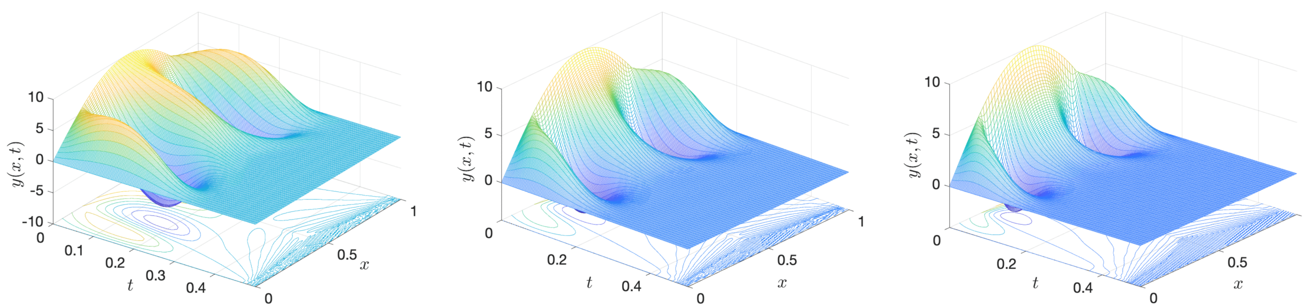

Figure 10: The controlled solution $y_{k^\star}$ in Q_T for $c_{u_0} = 10$, $c_f = -5$ and $s \in \{2, 3, 4\}$.

s	$\Vert y_{k^\star}\Vert_{L^2(Q_T)}$	$\Vert\rho_s y_{k^\star}\Vert_{L^2(Q_T)}$	$\Vert v_{k^\star}\Vert_{L^2(q_T)}$	$\Vert\rho_{0,s} v_{k^\star}\Vert_{L^2(q_T)}$	$\Vert v_{k^\star}\Vert_{L^\infty(q_T)}$	$k^\star$
2	2.43	80.50	58.24	208.52	297.56	48
3	1.415	86.53	51.30	463.69	414.93	17
4	1.108	173.17	52.83	1366.08	605.20	13
5	0.931	429.07	57.04	4328.61	889.05	11

Table 2: $c_{u_0} = 10$; $c_f = -5$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. s.

Table 3 provides some norms of the solution for $s = 3$ with respect to the fineness h of the triangular mesh used and highlights the stability of the approximation. The large degree equal to 3 of the approximation space induced by the composite finite element HCT makes the convergence fast with respect to h. We also observe that the number of iterations to reach the convergence of the sequence $(y_k)_{k\geq 0}$ is independent of h.

h	$\Vert y_{k^\star}\Vert_{L^2(Q_T)}$	$\Vert\rho_s y_{k^\star}\Vert_{L^2(Q_T)}$	$\Vert v_{k^\star}\Vert_{L^2(q_T)}$	$\Vert\rho_{0,s} v_{k^\star}\Vert_{L^2(q_T)}$	$\Vert v_{k^\star}\Vert_{L^\infty(q_T)}$	$k^\star$
0.1562	1.47841	90.9285	51.4646	469.008	420.345	18
0.0760	1.46148	87.9869	51.2379	465.822	419.42	17
0.0441	1.45521	87.0578	51.0243	464.527	416.886	17
0.0208	1.45056	86.2678	51.0448	463.253	414.223	17
0.0117	1.45203	86.5628	51.1068	463.723	415.114	17

Table 3: $c_{u_0} = 10$; $c_f = -5$; $s = 3$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. h.

4 Perspectives

Within the approach *control-then-discretize strategy*, we have emphasized, both for the wave and heat equation, the ability of variational space-time formulations to get robust finite dimensional approximation of exact controls. The space-time framework makes easier both the numerical analysis and the numerical implementation than classical methods within the approach *discretize-then-control.* Moreover, it is very appropriate for (space-time) mesh adaptivity, allowing a notable reduction of the computational cost. Then, we have defined strongly convergent sequences to control-state pairs for semilinear wave and heat equation. In both cases, the convergence is ensured assuming an asymptotic growth condition on the first derivative of the nonlinear function. Numerical experiments, within the space-time methods introduced in the first part, confirm the theoretical results. In both parts, the main tool is a parametrized global Carleman inequality allowing precise estimates of the state-control pair in term of the data. As emphasized for the heat equation in sections 3.3.1 and 3.3.2 an appropriate choice of the Carleman parameters guarantees contracting properties for some fixed point application. This is also true for wave type equations; we refer to the recent work [8] where a constructive convergent sequence to boundary controls for semilinear wave equation is designed. Actually, since global Carleman inequalities are now available for many equations and systems, the methods presented here can very likely be extended to other situations involving notably nonlinearity with gradient terms and arising in fluid and solid mechanics. We mention the Burgers equation and the Navier-Stokes equation in incompressible regime which are now under investigation.

References

[1] Paola F. Antonietti, Ilario Mazzieri, and Francesco Migliorini. A space-time discontinuous Galerkin method for the elastic wave equation. *J. Comput. Phys.*, 419:109685, 26, 2020.

[2] Ernesto Aranda and Pablo Pedregal. A variational method for the numerical simulation of a boundary controllability problem for the linear and semilinear 1D wave equations. *J. Franklin Inst.*, 351(7):3865–3882, 2014.

[3] M. Asch and A. Münch. Uniformly controllable schemes for the wave equation on the unit square. *J. Optim. Theory Appl.*, 143(3):417–438, 2009.

[4] Mehdi Badra, Sylvain Ervedoza, and Sergio Guerrero. Local controllability to trajectories for non-homogeneous incompressible Navier-Stokes equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(2):529–574, 2016.

[5] Helio J. C. Barbosa and Thomas J. R. Hughes. The finite element method with Lagrange multipliers on the boundary: circumventing the Babuška-Brezzi condition. *Comput. Methods Appl. Mech. Engrg.*, 85(1):109–128, 1991.

[6] Claude Bardos, Gilles Lebeau, and Jeffrey Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30(5):1024–1065, 1992.

[7] Mourad Bellassoued and Masahiro Yamamoto. *Carleman estimates and applications to inverse problems for hyperbolic systems.* Springer Monographs in Mathematics. Springer, Tokyo, 2017.

[8] Kuntal Bhandari, Jérôme Lemoine, and Arnaud Münch. Exact boundary controllability of 1d semilinear wave equation through a constructive approach. *Submitted. https://hal.archives-ouvertes.fr/hal-03599036.*

[9] Arthur Bottois, Jérome Lemoine, and Arnaud Münch. Constructive exact control of semilinear multi-dimensional wave equations. *Submitted. https://hal.archives-ouvertes.fr/UMR6620/hal-03112309.*

[10] Frédéric Bourquin. Approximation theory for the problem of exact controllability of the wave equation with boundary control. In *Second International Conference on Mathematical and Numerical Aspects of Wave Propagation (Newark, DE, 1993)*, pages 103–112. SIAM, Philadelphia, PA, 1993.

[11] M. O. Bristeau, O. Pironneau, R. Glowinski, J. Periaux, and P. Perrier. On the numerical solution of nonlinear problems in fluid dynamics by least squares and finite element methods. I. Least square formulations and conjugate gradient. *Comput. Methods Appl. Mech. Engrg.*, 17(18)(part, part 3):619–657, 1979.

[12] Erik Burman, Ali Feizmohammadi, Arnaud Münch, and Lauri Oksanen. Space time stabilized finite element methods for a unique continuation problem subject to the wave equation. *ESAIM Math. Model. Numer. Anal.*, 55(suppl.):S969–S991, 2021.

[13] Erik Burman, Ali Feizmohammadi, Arnaud Münch, and Lauri Oksanen. Spacetime finite element methods for control problems subject to the wave equation. *Submitted*, 2021.

[14] Erik Burman, Ali Feizmohammadi, and Lauri Oksanen. A fully discrete numerical control method for the wave equation. *SIAM J. Control Optim.*, 58(3):1519–1546, 2020.

[15] Piermarco Cannarsa, Vilmos Komornik, and Paola Loreti. One-sided and internal controllability of semilinear wave equations with infinitely iterated logarithms. *Discrete Contin. Dyn. Syst.*, 8(3):745–756, 2002.

[16] C. Carthel, R. Glowinski, and J.-L. Lions. On exact and approximate boundary controllabilities for the heat equation: a numerical approach. *J. Optim. Theory Appl.*, 82(3):429–484, 1994.

[17] Carlos Castro, Nicolae Cîndea, and Arnaud Münch. Controllability of the linear one-dimensional wave equation with inner moving forces. *SIAM J. Control Optim.*, 52(6):4027–4056, 2014.

[18] Carlos Castro and Sorin Micu. Boundary controllability of a linear semi-discrete 1-D wave equation derived from a mixed finite element method. *Numer. Math.*, 102(3):413–462, 2006.

[19] Carlos Castro, Sorin Micu, and Arnaud Münch. Numerical approximation of the boundary control for the wave equation with mixed finite elements in a square. *IMA J. Numer. Anal.*, 28(1):186–214, 2008.

[20] Thierry Cazenave and Alain Haraux. Équations d'évolution avec non linéarité logarithmique. *Ann. Fac. Sci. Toulouse Math. (5)*, 2(1):21–51, 1980.

[21] Thierry Cazenave and Alain Haraux. *An introduction to semilinear evolution equations*, volume 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1990 French original by Yvan Martel and revised by the authors.

[22] D. Chapelle and K.-J. Bathe. The inf-sup test. *Comput. & Structures*, 47(4-5):537–545, 1993.

[23] Philippe G. Ciarlet. Interpolation error estimates for the reduced Hsieh-Clough-Tocher triangle. *Math. Comp.*, 32(142):335–344, 1978.

[24] Nicolae Cîndea, Enrique Fernández-Cara, and Arnaud Münch. Numerical controllability of the wave equation through primal methods and Carleman estimates. *ESAIM Control Optim. Calc. Var.*, 19(4):1076–1108, 2013.

[25] Nicolae Cîndea, Sorin Micu, and Marius Tucsnak. An approximation method for exact controls of vibrating systems. *SIAM J. Control Optim.*, 49(3):1283–1305, 2011.

[26] Jean-Michel Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[27] B. Dehman and G. Lebeau. Analysis of the HUM control operator and exact controllability for semilinear waves in uniform time. *SIAM J. Control Optim.*, 48(2):521–550, 2009.

[28] Peter Deuflhard. *Newton methods for nonlinear problems*, volume 35 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2011. Affine invariance and adaptive algorithms, First softcover printing of the 2006 corrected printing.

[29] Willy Dörfler, Stefan Findeisen, and Christian Wieners. Space-time discontinuous galerkin discretizations for linear first-order hyperbolic evolution systems. *Computational Methods in Applied Mathematics*, 16(3):409–428, 2016.

[30] Serge Dumont, Franck Jourdan, and Tarik Madani. 4D remeshing using a space-time finite element method for elastodynamics problems. *Math. Comput. Appl.*, 23(2):Paper No. 29, 18, 2018.

[31] Ivar Ekeland and Roger Témam. *Convex analysis and variational problems*, volume 28 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, english edition, 1999. Translated from the French.

[32] Sylvain Ervedoza, Jérôme Lemoine, and Arnaud Münch. Constructive exact control of semilinear heat equation. *Submitted. https://hal.archives-ouvertes.fr/hal-03350534*.

[33] Sylvain Ervedoza and Enrique Zuazua. A systematic method for building smooth controls for smooth data. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1375–1401, 2010.

[34] Sylvain Ervedoza and Enrique Zuazua. *Numerical approximation of exact controls for waves*. SpringerBriefs in Mathematics. Springer, New York, 2013.

[35] Enrique Fernández-Cara and Arnaud Münch. Numerical null controllability of semi-linear 1-D heat equations: fixed point, least squares and Newton methods. *Math. Control Relat. Fields*, 2(3):217–246, 2012.

[36] Enrique Fernández-Cara and Arnaud Münch. Strong convergent approximations of null controls for the 1D heat equation. *SeMA J.*, 61:49–78, 2013.

[37] Enrique Fernández-Cara and Arnaud Münch. Numerical exact controllability of the 1D heat equation: duality and Carleman weights. *J. Optim. Theory Appl.*, 163(1):253–285, 2014.

[38] Enrique Fernández-Cara and Enrique Zuazua. The cost of approximate controllability for heat equations: the linear case. *Adv. Differential Equations*, 5(4-6):465–514, 2000.

[39] Enrique Fernández-Cara and Enrique Zuazua. Null and approximate controllability for weakly blowing up semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17(5):583–616, 2000.

[40] Xiaoyu Fu, Jiongmin Yong, and Xu Zhang. Exact controllability for multidimensional semilinear hyperbolic equations. *SIAM J. Control Optim.*, 46(5):1578–1614, 2007.

[41] A. V. Fursikov and O. Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[42] Victor A. Galaktionov and Juan L. Vázquez. The problem of blow-up in nonlinear parabolic equations. volume 8, pages 399–433. 2002. Current developments in partial differential equations (Temuco, 1999).

[43] R. Glowinski, W. Kinton, and M. F. Wheeler. A mixed finite element formulation for the boundary controllability of the wave equation. *Internat. J. Numer. Methods Engrg.*, 27(3):623–635, 1989.

[44] R. Glowinski and J.-L. Lions. Exact and approximate controllability for distributed parameter systems. In *Acta numerica, 1995*, Acta Numer., pages 159–333. Cambridge Univ. Press, Cambridge, 1995.

[45] Roland Glowinski, Chin Hsien Li, and Jacques-Louis Lions. A numerical approach to the exact boundary controllability of the wave equation. I. Dirichlet controls: description of the numerical methods. *Japan J. Appl. Math.*, 7(1):1–76, 1990.

[46] Roland Glowinski, Jacques-Louis Lions, and Jiwen He. *Exact and approximate controllability for distributed parameter systems*, volume 117 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2008. A numerical approach.

[47] M. Gunzburger, L. S. Hou, and L. Ju. A numerical method for exact boundary controllability problems for the wave equation. *Comput. Math. Appl.*, 51(5):721–750, 2006.

[48] F. Hecht. New development in Freefem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.

[49] Gregory M. Hulbert and Thomas J. R. Hughes. Space-time finite element methods for second-order hyperbolic equations. *Comput. Methods Appl. Mech. Engrg.*, 84(3):327–348, 1990.

[50] Liviu I. Ignat and Enrique Zuazua. Convergence of a two-grid algorithm for the control of the wave equation. *J. Eur. Math. Soc. (JEMS)*, 11(2):351–391, 2009.

[51] Romain Joly and Camille Laurent. A note on the semiglobal controllability of the semilinear wave equation. *SIAM J. Control Optim.*, 52(1):439–450, 2014.

[52] Stéphane Labbé and Emmanuel Trélat. Uniform controllability of semidiscrete approximations of parabolic control systems. *Systems Control Lett.*, 55(7):597–609, 2006.

[53] A. A. Lacey. Global blow-up of a nonlinear heat equation. *Proc. Roy. Soc. Edinburgh Sect. A*, 104(1-2):161–167, 1986.

[54] Ulrich Langer and Olaf Steinbach, editors. *Space-Time Methods*. De Gruyter, 2019.

[55] Kévin Le Balc'h. Global null-controllability and nonnegative-controllability of slightly superlinear heat equations. *J. Math. Pures Appl. (9)*, 135:103–139, 2020.

[56] Gilles Lebeau and Maëlle Nodet. Experimental study of the HUM control operator for linear waves. *Experiment. Math.*, 19(1):93–120, 2010.

[57] Jérôme Lemoine, Irene Marín-Gayte, and Arnaud Münch. Approximation of null controls for semilinear heat equations using a least-squares approach. *ESAIM Control Optim. Calc. Var.*, 27:Paper No. 63, 28, 2021.

[58] Jérôme Lemoine and Arnaud Münch. A fully space-time least-squares method for the unsteady Navier-Stokes system. *J. Math. Fluid Mech.*, 23(4):Paper No. 102, 38, 2021.

[59] Jérôme Lemoine and Arnaud Münch. Resolution of the implicit Euler scheme for the Navier-Stokes equation through a least-squares method. *Numer. Math.*, 147(2):349–391, 2021.

[60] Jérôme Lemoine and Arnaud Münch. Constructive exact control of semilinear 1d heat equation. *Mathematical Control and Related Fields, doi: 10.3934/mcrf.2022001*, 2022.

[61] Jérôme Lemoine, Arnaud Münch, and Pablo Pedregal. Analysis of continuous H^{-1}-least-squares approaches for the steady Navier-Stokes system. *Appl. Math. Optim.*, 83(1):461–488, 2021.

[62] Liangyu Li and Xu Zhang. Exact controllability for semilinear wave equations. *J. Math. Anal. Appl.*, 250(2):589–597, 2000.

[63] J.-L. Lions. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, volume 8 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1988. Contrôlabilité exacte. [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

[64] Pierre Lissy and Ionel Rovenţa. Optimal filtration for the approximation of boundary controls for the one-dimensional wave equation using a finite-difference method. *Math. Comp.*, 88(315):273–291, 2019.

[65] Paola Loreti and Michel Mehrenberger. An Ingham type proof for a two-grid observability theorem. *ESAIM Control Optim. Calc. Var.*, 14(3):604–631, 2008.

[66] Patrick Martinez and Judith Vancostenoble. Exact controllability in "arbitrarily short time" of the semilinear wave equation. *Discrete Contin. Dyn. Syst.*, 9(4):901–924, 2003.

[67] Sorin Micu. Uniform boundary controllability of a semi-discrete 1-D wave equation. *Numer. Math.*, 91(4):723–768, 2002.

[68] Santiago Montaner and Arnaud Münch. Approximation of controls for linear wave equations: a first order mixed formulation. *Math. Control Relat. Fields*, 9(4):729–758, 2019.

[69] Arnaud Münch. A uniformly controllable and implicit scheme for the 1-D wave equation. *M2AN Math. Model. Numer. Anal.*, 39(2):377–418, 2005.

[70] Arnaud Münch. Optimal design of the support of the control for the 2-D wave equation: a numerical method. *Int. J. Numer. Anal. Model.*, 5(2):331–351, 2008.

[71] Arnaud Münch. A least-squares formulation for the approximation of controls for the Stokes system. *Math. Control Signals Systems*, 27(1):49–75, 2015.

[72] Arnaud Münch and Diego A. Souza. A mixed formulation for the direct approximation of L^2-weighted controls for the linear heat equation. *Adv. Comput. Math.*, 42(1):85–125, 2016.

[73] Arnaud Münch and Emmanuel Trélat. Constructive exact control of semilinear 1d wave equations by a least-squares approach. *SIAM J. Control Optim.*, 60(2):652–673, 2022.

[74] Arnaud Münch and Enrique Zuazua. Numerical approximation of null controls for the heat equation: ill-posedness and remedies. *Inverse Problems*, 26(8):085018, 39, 2010.

[75] Pablo Pedregal, Francisco Periago, and Jorge Villena. A numerical method of local energy decay for the boundary controllability of time-reversible distributed parameter systems. *Stud. Appl. Math.*, 121(1):27–47, 2008.

[76] Pierre Saramito. A damped Newton algorithm for computing viscoplastic fluid flows. *J. Non-Newton. Fluid Mech.*, 238:6–15, 2016.

[77] Olaf Steinbach and Marco Zank. *A Stabilized Space–Time Finite Element Method for the Wave Equation*, pages 341–370. Springer International Publishing, Cham, 2019.

[78] Enrique Zuazua. Exact controllability for the semilinear wave equation. *J. Math. Pures Appl. (9)*, 69(1):1–31, 1990.

[79] Enrique Zuazua. Exact controllability for semilinear wave equations in one space dimension. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(1):109–129, 1993.

[80] Enrique Zuazua. Propagation, observation, and control of waves approximated by finite difference methods. *SIAM Rev.*, 47(2):197–243, 2005.