# Enumerating Maximal Consistent Closed Sets in Closure Systems

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**Abstract.** Given an implicational base, a well-known representation for a closure system, an inconsistency binary relation over a finite set, we are interested in the problem of enumerating all maximal consistent closed sets (denoted by MCCEnum for short). We show that MCCEnum cannot be solved in output-polynomial time unless P = NP, even for lower bounded lattices. We give an incremental-polynomial time algorithm to solve MCCEnum for closure systems with constant Carathéodory number. Finally we prove that in biatomic atomistic closure systems MCCEnum can be solved in output-quasipolynomial time if minimal generators obey an independence condition, which holds in atomistic modular lattices. For closure systems closed under union (i.e., distributive), MCCEnum is solved by a polynomial delay algorithm [22, 25].

**Keywords:** Closure systems, implicational base, inconsistency relation, enumeration algorithm

## 1 Introduction

In this paper, we consider binary inconsistency relations (i.e., graphs) over implicational bases, a well-known representation for closure systems [7, 32]. More precisely, we seek to enumerate maximal closed sets of a closure system given by an implicational base that are consistent with respect to an inconsistency relation. We call this problem Maximal Consistent Closed Sets Enumeration, or MCCEnum for short.

This problem finds applications for instance in minimization of sub-modular functions [22] or argumentation frameworks [12]. It is moreover a particular case of dualization in closure systems given by an implicational bases, ubiquitous in computer science [7, 11, 15]. This latter problem however cannot be solved in output-polynomial time unless P = NP [3] even when the input implicational base has premises of size at most two [10]. When restricted to graphs and implicational bases with premises of size one, or posets equivalently, the problem can be solved in polynomial delay [22, 25].

More generally, inconsistency relations combined with posets appear also in event structures [28], representations of median-semilattices [4] or cubical

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complexes [2] in which the term *"inconsistency"* is used. Recently in [21, 22], the authors derive a representation for modular semi-lattices based on inconsistency and projective ordered spaces [20]. Furthermore, they characterize the cases where given an implicational base and an inconsistency relation, maximal consistent closed sets coincide with maximal independent sets of the inconsistency relation, seen as a graph.

In our contribution, we show first that enumerating maximal consistent closed sets cannot be solved in output-polynomial time unless $\mathsf{P} = \mathsf{NP}$, a surprising result which further emphasizes the hardness of dualization in lattices given by implicational bases [3, 10]. In fact, we show that this problem is already intractable for the well-known class of lower bounded lattices [1, 9, 16]. On the positive side, we show that when the maximal size of minimal generators is bounded by a constant, the problem can be solved in incremental-polynomial time. As a direct corollary, we obtain that MCCEnum can be solved efficiently in a several classes of convex geometries where this parameter, also known as the Carathéodory number, is constant [26]. Finally, we focus on biatomic atomistic closure systems [5,8]. We show that under an independence condition, the size of a minimal generator is logarithmic in the size of the groundset. As a consequence, we get a quasi-polynomial time algorithm for enumerating maximal consistent closed sets which can be applied to the well-known class of atomistic modular lattices [18, 20, 29, 31].

The rest of the paper is organized as follows. Section 2 gives necessary definitions about closure systems and implicational bases. In Section 3 we show that MCCEnum cannot be solved in output-polynomial time, in particular for lower bounded closure systems. In Section 4, we show that if the size of a minimal generator is bounded by a constant, MCCEnum can be solved efficiently. Section 5 is devoted to the class of biatomic atomistic closure systems. We conclude with open questions and problems in 6.

## 2 Preliminaries

All the objects considered in this paper are finite. Let $X$ be a set. We denote by $2^X$ its powerset. For any $n \in \mathbb{N}$, we write $[n]$ for the set $\{1, \dots, n\}$. We will sometimes use the notation $x_1 \dots x_n$ as a shortcut for $\{x_1, \dots, x_n\}$. The size of a subset $A$ of $X$ is denoted by $|A|$. If $\mathcal{H} = (X, \mathcal{E})$ is a hypergraph, we denote by $\mathsf{IS}(\mathcal{H})$ its independent sets (or stable sets). We write $\mathsf{MIS}(\mathcal{H})$ for its maximal independent sets. Similarly, if $G = (X, E)$ is a graph, its independent sets (resp. maximal independent sets) are written $\mathsf{IS}(G)$ (resp. $\mathsf{MIS}(G)$).

We recall principal notions on lattices and closure systems [18]. A mapping $\phi \colon 2^X \to 2^X$ is a *closure operator* if for any $Y, Z \subseteq X$, $Y \subseteq \phi(Y)$ (extensive), $Y \subseteq Z$ implies $\phi(Y) \subseteq \phi(Z)$ (isotone), and $\phi(\phi(Y)) = \phi(Y)$ (idempotent). We call $\phi(Y)$ the *closure* of $Y$. The family $\mathcal{F} = \{\phi(Y) \mid Y \subseteq X\}$ ordered by set-inclusion forms a *closure system* or *lattice*. A closure system $\mathcal{F} \subseteq 2^X$ is a set system such that $X \in \mathcal{F}$ and for any $F_1, F_2 \in \mathcal{F}$, $F_1 \cap F_2$ also belongs to $\mathcal{F}$. Elements of $\mathcal{F}$ are *closed sets* and we say that $F$ is *closed* if $F \in \mathcal{F}$. Each closure system $\mathcal{F}$ induces a unique closure operator $\phi$ such that $\phi(Y) = \bigcap\{F \in \mathcal{F} \mid Y \subseteq F\}$, for any $Y \subseteq X$. Thus, there is a one-to-one correspondence between closure systems and operators. Without loss of generality, we will assume that $\phi$ and $\mathcal{F}$ are *standard*: $\phi(\emptyset) = \emptyset$ and for any $x \in X$, $\phi(x) \setminus \{x\}$ is closed. Note that $\emptyset$ is thus the minimum element of $\mathcal{F}$, called the *bottom*. Similarly, $X$ is the *top* of $\mathcal{F}$.

Let $\phi$ be a closure operator with corresponding closure system $\mathcal{F}$. Let $F_1, F_2 \in \mathcal{F}$. We say that $F_1$ and $F_2$ are *comparable* if $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$. They are *incomparable* otherwise. A subset $\mathcal{S}$ of $\mathcal{F}$ is an *antichain* if its elements are pairwise incomparable. If for any $F \in \mathcal{F}$, $F_1 \subset F \subseteq F_2$ implies $F = F_2$, we say that $F_2$ *covers* $F_1$, and denote it $F_1 \prec F_2$. An *atom* is a closed set covering the bottom $\emptyset$ of $\mathcal{F}$. Dually, a *co-atom* is a closed set covered by the top $X$ of $\mathcal{F}$. We denote by $\mathcal{C}(\mathcal{F})$ the set of co-atoms of $\mathcal{F}$. Let $M \in \mathcal{F}$. We say that $M$ is *meet-irreducible* in $\mathcal{F}$ if for any $F_1, F_2 \in \mathcal{F}$, $M = F_1 \cap F_2$ entails either $F_1 = M$ or $F_2 = M$. In this case, $M$ has a unique cover $M^*$ in $\mathcal{F}$. The set of meet-irreducible elements of $\mathcal{F}$ is denoted by $\mathcal{M}(\mathcal{F})$. Dually, $J \in \mathcal{F}$ is a *join-irreducible* element of $\mathcal{F}$ if for any $F_1, F_2 \in \mathcal{F}$, $J = \phi(F_1 \cup F_2)$ implies $J = F_1$ or $J = F_2$. Then, $J$ covers a unique element $J_*$ in $\mathcal{F}$. We denote by $\mathcal{J}(\mathcal{F})$ the join-irreducible elements of $\mathcal{F}$. When $\mathcal{F}$ and $\phi$ are standard, there is a one-to-one correspondence between $X$ and $\mathcal{J}(\mathcal{F})$ given by $\mathcal{J}(\mathcal{F}) = \{\phi(x) \mid x \in X\}$. Furthermore, $x_* = \phi(x)_* = \phi(x) \setminus x$. Consequently, we will identify $X$ with $\mathcal{J}(\mathcal{F})$.

Let $x \in X$. A *minimal generator* of $x$ is an inclusion-wise minimal subset $A_x$ of $X$ such that $x \in \phi(A_x)$. We consider $\{x\}$ as a trivial minimal generator of $x$. Following [26], the *Carathéodory number* $\mathrm{c}(\mathcal{F})$ of $\mathcal{F}$ is the least integer $k$ such that for any $A \subseteq X$ and any $x \in X$, $x \in \phi(A)$ implies the existence of some $A' \subseteq A$ with $|A'| \leq k$ such that $x \in \phi(A')$. At first, this notion was used for convex geometries, but its definition applies to any closure system. Moreover, the Carathéodory number of $\mathcal{F}$ is the maximal possible size of a minimal generator (see Proposition 4.1 in [26], which can be applied to any closure system). A *key* of $\mathcal{F}$ is a minimal subset $K \subseteq X$ such that $\phi(K) = X$. We denote by $\mathcal{K}$ the set of keys of $\mathcal{F}$. The number of keys in $\mathcal{K}$ is denoted by $|\mathcal{K}|$. It is well-known (see for instance [11]) that maximal independent sets $\mathsf{MIS}(\mathcal{K})$ of $\mathcal{K}$, viewed as a hypergraph over $X$, are exactly co-atoms of $\mathcal{F}$. We define arrow relations from [17]. Let $x \in X$ and $M \in \mathcal{M}(\mathcal{F})$. We write $x \uparrow M$ if $x \notin M$ but $x \in M^*$. Dually, we write $M \downarrow x$ if $x \notin M$ but $x_* \subseteq M$.

We move to implicational bases [7, 32]. An *implication* si an expression of the form $A \rightarrow B$ with $A, B \subseteq X$. We call $A$ the *premise* and $B$ the *conclusion*. A set $\Sigma$ of implications over $X$ is an *implicational base* over $X$. We denote by $|\Sigma|$ the number of implications in $\Sigma$. A subset $F \subseteq X$ *satisfies* or *models* $\Sigma$ if for any $A \rightarrow B \in \Sigma$, $A \subseteq F$ implies $B \subseteq F$. The family $\mathcal{F} = \{F \subseteq X \mid F \text{ satisfies } \Sigma\}$ is a closure system whose induced closure operator $\phi$ is the *forward chaining algorithm*. This procedure starts from any subset $Y$ of $X$ and constructs a sequence $Y = Y_0 \subseteq \cdots \subseteq Y_k = \phi(Y)$ of subsets of $X$ such that for any $i \in [k]$, $Y_i = Y_{i-1} \cup \{B \mid \exists A \rightarrow B \in \Sigma \; s.t. \; A \subseteq Y_{i-1}\}$. The algorithm stops when $Y_{i-1} = Y_i$.

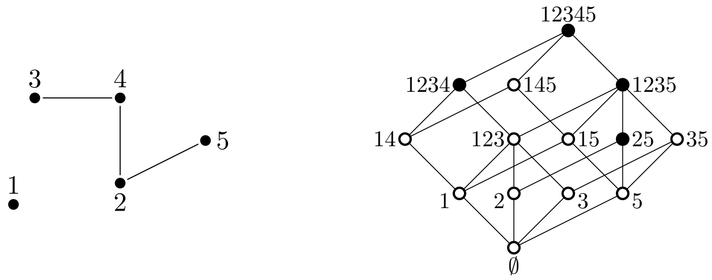


Fig. 1: On the left, a consistency-graph $G_c$ over $X = \{1, 2, 3, 4, 5\}$ with inconsistent pairs 34, 24 and 25. On the right, the closure system associated to $\Sigma = \{13 \rightarrow 2, 12 \rightarrow 3, 23 \rightarrow 1, 4 \rightarrow 1\}$. Black and white dots stand for inconsistent and consistent closed sets respectively. We have $\mathsf{maxCC}(\Sigma, G_c) = \{145, 123, 35\}$.

We now introduce our main problem. Following [2, 21, 22] we call an *inconsistency relation* any symmetric and irreflexive relation over $X$. Such a relation is sometimes called a *site* [4] or a *conflict relation* [28]. Usually, inconsistency relations need to satisfy more conditions in order to capture median or modular-semilattices [4, 21]. As we do not need further restrictions here, we can choose to model inconsistency as a graph $G_c = (X, E_c)$, and call it a *consistency-graph*. An edge $uv$ of $E_c$ represents an *inconsistent pair* of elements in $X$. A subset $Y$ which does not contain any inconsistent pair (i.e., an independent set of $G_c$) is called *consistent*. Let $\Sigma$ be an implicational base over $X$ and a $G_c = (X, E_c)$ consistency-graph. We denote by $\mathsf{maxCC}(\Sigma, G_c)$ the set of maximal consistent closed sets of $\mathcal{F}$, that is $\mathsf{maxCC}(\Sigma, G_c) = \max_{\subseteq}(\mathcal{F} \cap \mathsf{IS}(G_c))$. An example of implicational base along with a consistency-graph is given in Figure 1. Our problem is the following.

Maximal Consistent closed-sets Enumeration (MCCEnum)
**Input:** An implicational base $\Sigma$ over $X$, a non-empty consistency-graph $G_c = (X, E_c)$.
**Output:** The set $\mathsf{maxCC}(\Sigma, G_c)$ of maximal consistent closed sets of $\mathcal{F}$ with respect to $G_c$.

Remark that $X$ is part of the input. If $G_c$ is empty, MCCEnum is easy to solve as $X$ is the unique element of $\mathsf{maxCC}(\Sigma, G_c)$. Hence, we will assume without loss of generality that $G_c$ is not empty. If $\Sigma$ is empty, then MCCEnum is equivalent to the enumeration of maximal independent sets of a graph which can be efficiently solved [23]. If premises of $\Sigma$ have size 1, the problem also reduces to maximal independent sets enumeration [22, 25]. In [21] the authors identify, for a fixed $\Sigma$, the consistency-graphs $G_c$ such that $\mathsf{MIS}(G_c) = \mathsf{maxCC}(\Sigma, G_c)$.

We conclude with a recall on enumeration algorithms [23]. Let $\mathcal{A}$ be an algorithm with input $x$ and output a set of solutions $R(x)$. We denote by $|R(x)|$ the number of solutions in $R(x)$. We assume that each solution in $R(x)$ has size

$\mathsf{poly}(|x|)$. The algorithm $\mathcal{A}$ is running in *output-polynomial* time if its execution time is bounded by $\mathsf{poly}(|x| + |R(x)|)$. It is *incremental-polynomial* if for any $1 \leq i \leq |R(x)|$, the time spent between the $i$-th and $i+1$-th output is bounded by $\mathsf{poly}(|x|+i)$, and the algorithm stops in time $\mathsf{poly}(|x|)$ after the last output. If the delay between two solutions output and after the last one is $\mathsf{poly}(|x|)$, $\mathcal{A}$ has *polynomial-delay.* Note that if $\mathcal{A}$ is running in incremental-polynomial time, it is also output-polynomial. Finally, we say that $\mathcal{A}$ runs in *output-quasipolynomial* time if is execution time is bounded by $N^{\mathsf{polylog}(N)}$ where $N = |x| + |R(x)|$.

## 3 Closure systems given by implicational bases

We show that MCCEnum cannot be solved in output-polynomial time unless $\mathsf{P} = \mathsf{NP}$. To do so, we use a reduction from the problem of enumerating co-atoms of a closure system.

Co-atoms Enumeration (CE)
**Input:** An implicational base $\Sigma_Y$ over $Y$.
**Output:** The co-atoms $\mathcal{C}(\mathcal{F}_Y)$ of the closure system $\mathcal{F}_Y$ associated to $\Sigma_Y$.

It is proved by Kavvadias et al. in [25] that CE admits no output-polynomial time algorithm unless $\mathsf{P} = \mathsf{NP}$. Our first step is to prove the following lemma.

**Lemma 1.** *Let $\Sigma_Y$ be an implicational base over $Y$. Let $X = Y \cup \{u, v\}$, $\Sigma = \Sigma_Y \cup \{Y \rightarrow uv\}$ and let $G_c = (X, E_c = \{uv\})$ be a consistency-graph. The following equality holds:*

$$\mathsf{maxCC}(\Sigma, G_c) = \bigcup_{C \in \mathcal{C}(\mathcal{F}_Y)} \{C \cup \{u\}, C \cup \{v\}\} \tag{1}$$

*Proof.* Let $C \in \mathcal{C}(\mathcal{F}_Y)$. We show that $C \cup \{u\}$ and $C \cup \{v\}$ are in $\mathsf{maxCC}(\Sigma, G_c)$. As no implication of $\Sigma$ has $u$ or $v$ in its premise, we have that $C \cup \{u\}$ and $C \cup \{v\}$ are consistent and closed with respect to $\Sigma$. Let $y \in Y \setminus C$. As $C$ is a co-atom of $\mathcal{F}_Y$, it must be that $\phi_Y(C \cup \{y\}) = Y$. As $Y \rightarrow uv$ is an implication of $\Sigma$, it follows that $uv \subseteq \phi(C \cup \{u, y\})$. Thus, for any $x \in X \setminus (C \cup \{u\})$, $\phi(C \cup \{u, x\})$ is inconsistent. We conclude that $C \cup \{u\} \in \mathsf{maxCC}(\Sigma, G_c)$. Similarly we obtain $C \cup \{v\} \in \mathsf{maxCC}(\Sigma, G_c)$.

Let $S \in \mathsf{maxCC}(\Sigma, G_c)$. We show that $S$ can be written as $C \cup \{u\}$ or $C \cup \{v\}$ for some co-atom $C$ of $\mathcal{F}_Y$. First, let $F$ be a consistent closed set in $\mathcal{F}$ such that $u \notin F$ and $v \notin F$. As $\Sigma$ has no implication with $u$ or $v$ in its premise, it follows that both $F \cup \{u\}$ and $F \cup \{v\}$ are closed and consistent. Hence, either $u \in S$ or $v \in S$. Without loss of generality, let us assume $u \in S$. Let $C = S \setminus \{u\}$. As $S \in \mathsf{maxCC}(\Sigma, G_c)$, it is closed with respect to $\Sigma_Y$ and does not contain $Y$. Thus, $C \in \mathcal{F}_Y$ and $C \subset Y$. Let $y \in Y \setminus C$. As $S \in \mathsf{maxCC}(\Sigma, G_c)$, it must be that $\phi(S \cup \{y\})$ contains the inconsistent pair $uv$ of $G_c$. Hence, $Y \subseteq \phi(S \cup \{y\})$ by construction of $\Sigma$. Consequently, we have that $Y = \phi_Y(C \cup \{y\})$ for any $y \in Y \setminus C$. Hence, we conclude that $C \in \mathcal{C}(\mathcal{F}_Y)$ as expected. □

Therefore, if there is an algorithm solving MCCEnum in output-polynomial time, it can be used to solve CE within the same running time using the reduction of Lemma 1. Consequently, we obtain the following theorem.

**Theorem 1.** *The problem* MCCEnum *cannot be solved in output-polynomial time unless* P = NP*.*

In fact, we can strengthen the preceding theorem by a careful analysis of the closure system used in the reduction in [25]. More precisely, we show that the problem remains untractable for lower bounded closure systems. These have been introduced with the doubling construction in [9] and then studied in [1, 6, 16]. A characterization of lower bounded lattices is given in [16] in terms of the $D$-relation. This relations relies on $\mathcal{J}(\mathcal{F})$ and we say that $x$ depends on $y$, denoted by $xDy$ (recall that we identified $X$ with $\mathcal{J}(\mathcal{F})$) if there exists a meet-irreducible element $M \in \mathcal{M}(\mathcal{F})$ such that $x \uparrow M \downarrow y$. A $D$-cycle is a sequence $x_1, \dots, x_k \subseteq X$ such that $x_1 D x_2 D \dots D x_k D x_1$.

**Theorem 2.** *(Reformulated from Corollary 2.39, [16]) A closure system $\mathcal{F}$ is lower bounded if and only if it contains no $D$-cycle.*

**Corollary 1.** *The problem* MCCEnum *cannot be solved in output-polynomial time unless* P = NP*, even in lower bounded closure systems.*

*Proof.* Consider a positive 3-CNF over $n$ variables and $m$ clauses

$$\psi(x_1, \dots, x_n) = \bigwedge_{i=1}^{m} C_i = \bigwedge_{i=1}^{m} (x_{i,1} \vee x_{i,2} \vee x_{i,3})$$

Let $Y = \{x_1, \dots x_n, y_1, \dots, y_m, z\}$ and consider the following sets of implications:

- $\Sigma_1 = \{x_{i,k} x_{j,k} \rightarrow z \mid i \in [m], k \in [3]\}$,
- $\Sigma_2 = \{y_i \rightarrow z \mid i \in [m]\}$,
- $\Sigma_3 = \{x_{i,k} z \rightarrow y_i \mid i \in [m], k \in [3]\}$.

And let $\Sigma_Y = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. In [25] the authors show that CE is already intractable for these instances.

Therefore, applying the reduction from Lemma 1, we obtain that MCCEnum cannot be solved in output-polynomial time in the following case: $X = Y \cup \{u, v\}$, $\Sigma = \Sigma_Y \cup \{Y \rightarrow uv\}$, $G_c = (X, E_c = \{uv\})$.

Let us show that $\mathcal{F}$, the closure system associated to $\Sigma$, is indeed lower bounded. We proceed by analysing the $D$-relation. Observe first that $\mathcal{F}$ is standard. We begin with $u, v$. Let $t \in X \setminus \{u\}$ and $M \in \mathcal{M}(\mathcal{F})$ such that $t \uparrow M$. As no premise of $\Sigma$ contains $u$, it must be that $u \in M$. Hence for any $t \in X \setminus \{u\}$, $t$ does not depend on $u$. Applying the same reasoning on $v$, we obtain that no $D$-cycle can contain $u$ or $v$. Let $x_i \in X$, $i \in [n]$. As $x_i$ is the conclusion of no implication in $\Sigma$, we have that the unique meet-irreducible element $M_i$ satisfying $x_i \uparrow M_i$ is $X \setminus x_i$. Therefore, there is no element in $X \setminus \{x_i\}$ on which $x_i$ depends, so that no $D$-cycle can contain $x_i$, for any $i \in [n]$. Let us move to $z$. As $y_j \rightarrow z \in \Sigma$ for any $j \in [m]$, we have ${y_j}_* = \phi(y_j)_* = \{z\}$. Hence, $zDy_j$ cannot hold since $M \downarrow y_j$ implies $z \in M$, for any $M \in \mathcal{M}(\mathcal{F})$. Thus, $z$ only depends on some of the $x_i$'s, $i \in [n]$, and no $D$-cycle can contain $z$ either.

Henceforth, the only possible $D$-cycles must be contained in $\{y_1, \dots, y_m\}$. We show that for any $i, k \in [m]$, $y_i D y_k$ does not hold. For any $y_i$, $i \in [m]$, we have ${y_i}_* = \{z\}$ as $y_i \rightarrow z \in \Sigma$. Hence, a meet-irreducible element $M_i$ satisfying $y_i \uparrow M_i \downarrow y_k$ must contain $z$. Let $F \in \mathcal{F}$ be any closed set satisfying $y_i \notin F$ but $z \in F$. Assume there exists some $y_k$ such that $y_k \notin F$. Then $F \cup \{y_k\} \in \mathcal{F}$, as $y_k \rightarrow z$ is the only implication having $y_k$ in its premise, and $z \in F$. Therefore, it must be that for any $M_i \in \mathcal{M}(\mathcal{F})$ such that $z \in M_i$ and $y_i \notin M_i$, $\{y_1, \dots, y_m\} \setminus \{y_i\} \subseteq M_i$ is verified, so that $y_i \uparrow M_i \downarrow y_k$ is not possible. As a consequence $y_i D y_k$ cannot hold, for any $i, k \in [m]$. We conclude that $\mathcal{F}$ has no $D$-cycles and that it is lower bounded by Theorem 2. □

Therefore, there is no algorithm solving MCCEnum in output-polynomial time unless $\mathsf{P} = \mathsf{NP}$ even when restricted to lower bounded closure systems. In the next section, we consider classes of closure systems where MCCEnum can be solved in incremental-polynomial time.

## 4 Minimal generators with bounded size

Let $\Sigma$ be an implicational base over $X$ and $G_c$ a non-empty consistency-graph. Observe that $\mathsf{IS}(G_c) \cup \{X\}$ is a closure system where a set $F \subseteq X$ is closed if and only if $F = X$ or it is an independent set of $G_c$. From this point of view, elements of $\mathsf{maxCC}(\Sigma, G_c)$, are those maximal proper subsets of $X$ that are both closed in $\mathcal{F}$ and $\mathsf{IS}(G_c) \cup \{X\}$. Consequently, the maximal consistent closed sets of $\mathcal{F}$ with respect to $G_c$ are exactly the co-atoms of $\mathcal{F} \cap (\mathsf{IS}(G_c) \cup \{X\})$. Now, if we can guarantee that $\mathcal{K}$, the keys of $\mathcal{F} \cap (\mathsf{IS}(G_c) \cup \{X\})$, has polynomial size with respect to $\Sigma$, $X$ and $G_c$, we can derive an incremental-polynomial time algorithm computing $\mathsf{maxCC}(\Sigma, G_c)$ in two steps:

1. Compute the set of keys $\mathcal{K}$ which has polynomial size with respect to $X$,
2. Compute $\mathsf{MIS}(\mathcal{K}) = \mathsf{maxCC}(\Sigma, G_c)$.

To identify cases where $\mathcal{K}$ has polynomial size with respect to $\Sigma, X$ and $G_c$, the first step is to characterize its elements. To do so, we have to guarantee that a set $Y \subset X$ contains a key of $\mathcal{K}$ whenever $Y$ or $\phi(Y)$ is inconsistent with respect to $G_c$. Looking at $G_c$ is sufficient to distinguish between consistent and inconsistent closed sets of $\mathcal{F}$. However, there may be consistent (non-closed) sets $Y$ such that $\phi(Y)$ contains an edge of $G_c$. These will not be seen by just considering $G_c$. Thus, if $uv$ is the edge of $G_c$ contained in $\phi(Y)$, we deduce that there must be a minimal generator $A_u$ of $u$ contained in $Y$, possibly $A_u = \{u\}$. Similarly, $Y$ contains a minimal generator $A_v$ of $v$. In particular, keys in $\mathcal{K}$ will share the following property.

**Proposition 1.** *Let $K \in \mathcal{K}$. Then there exists $uv \in E_c$, a minimal generator $A_u$ of $u$, and a minimal generator $A_v$ of $v$ such that $K = A_u \cup A_v$.*

*Proof.* Let $K \in \mathcal{K}$. By assumption, $\phi(K)$ contains an edge $uv$ of $G_c$. Thus, there exists minimal generators $A_u$ of $u$ and $A_v$ of $v$ such that $A_u \cup A_v \subseteq K$. Assume that $A_u \cup A_v \subset K$ and let $x \in K \setminus (A_u \cup A_v)$. As $u \in \phi(A_u)$ and $v \in \phi(A_v)$, we get $uv \in \phi(K \setminus \{x\})$, a contradiction with the minimality of $K$. □

*Example 1.* We consider $\Sigma$, $X$ and $G_c$ of Figure 1. We have that $\phi(135) = 1235$ is inconsistent as it contains 25. However 135 is consistent with respect to $G_c$. For this example, we will have $\mathcal{K} = \{135, 34, 24, 25\}$. Note that 135 can be decomposed following Proposition 1 as the minimal generator 13 of 2, and 5 as a trivial minimal generator for itself.

Remark that $E_c \nsubseteq \mathcal{K}$ in the general case, as there may be an implication $u \rightarrow v$ in $\Sigma$ for some inconsistent pair $uv \in E_c$. Thus $u$ is a key which satisfies Proposition 1 with $A_u = A_v = \{u\}$. It also follows from Proposition 1 that $\mathrm{c}(\mathcal{F})$ plays an important role for MCCEnum. When no restriction on $\mathrm{c}(\mathcal{F})$ holds, $\mathcal{K}$ can have exponential size with respect to $\Sigma$ and $G_c$. The next example drawn from [25] illustrates this exponential growth.

*Example 2.* Let $X = \{x_1, \dots, x_n, y_1, \dots, y_n, u, v\}$ and $\Sigma = \{x_i \rightarrow y_i \mid i \in [n]\} \cup \{y_1 \dots y_n \rightarrow uv\}$. The consistency-graph is $G_c = (X, \{uv\})$. The set of non-trivial minimal generators of $u$ and $v$ is $\{z_1 \dots z_n \mid z_i \in \{x_i, y_i\}, i \in [n]\}$. Moreover, minimal generators of $u$ and $v$ are also the keys of $\mathcal{F} \cap (\mathsf{IS}(G_c) \cup \{X\})$. Thus, $|\mathcal{K}| = 2^n$, which is exponential with respect to $\Sigma$ and $G$. Observe that $\Sigma$ is acyclic [19, 32]: for any $x, y \in X$ if $y$ belongs to some minimal generator of $x$, then $x$ is never contained in a minimal generator of $y$.

Hence, computing $\mathsf{maxCC}(\Sigma, G_c)$ through the intermediary of $\mathcal{K}$ is in general impossible in output-polynomial time. In fact, this exponential blow up occurs even for small classes of closure systems where the Carathéodory number $\mathrm{c}(\mathcal{F})$ is unbounded. In Example 2 for instance, the closure system induced by $\Sigma$ is acyclic [19, 32], a particular case of lower boundedness [1].

On the other hand, let us assume now that $\mathrm{c}(\mathcal{F})$ is bounded by some constant $k \in \mathbb{N}$. By Proposition 1, every key in $\mathcal{K}$ has at most $2 \times k$ elements. As a consequence we show in the next theorem that the two-steps algorithm we described can be conducted in incremental-polynomial time.

**Theorem 3.** *Let $\Sigma$ be an implicational base over $X$ with induced $\mathcal{F}$, and $G_c = (X, E_c)$ a consistency-graph. If $\mathrm{c}(\mathcal{F}) \leq k$ for some constant $k \in \mathbb{N}$, the problem* MCCEnum *can be solved in incremental-polynomial time.*

*Proof.* The set of keys $\mathcal{K}$ can be computed in incremental-polynomial time with respect to $\mathcal{K}$, $\Sigma$, $X$ and $G_c$ using the algorithm of Lucchesi and Osborn [27] with input $\Sigma' = \Sigma \cup \{uv \rightarrow X \mid uv \in E_c\}$. Observe that the closure system associated to $\Sigma'$ is exactly $\mathcal{F} \cap \{\mathsf{IS}(G_c) \cup \{X\}\}$. Indeed, a consistent closed set of $\mathcal{F}$ models $\Sigma'$ and a subset $F \subseteq X$ which satisfies $\Sigma'$ must also satisfy $\Sigma$ and being an independent set of $G_c$ if $F \subset X$. Note that $\mathcal{K}$ is then computed in time $\mathsf{poly}(|\Sigma| + |X| + |G_c| + |\mathcal{K}|)$. As the total size of $\mathcal{K}$ is bounded by $|X|^{2k}$ by Proposition 1, we get that $\mathcal{K}$ is computed in time $\mathsf{poly}(|\Sigma| + |X| + |G_c|)$.

Then, we apply the algorithm of Eiter and Gottlob [13] to compute $\mathsf{MIS}(\mathcal{K}) = \mathsf{maxCC}(\Sigma, G_c)$ which runs in incremental polynomial time. Since $\mathcal{K}$ has polynomial size with respect to $|X|$, the delay between the $i$-th and $(i+1)$-th solution of $\mathsf{maxCC}(\Sigma, G_c)$ output is bounded by $\mathsf{poly}(|X|^{2k}+i)$, that is $\mathsf{poly}(|X|+i)$. Furthermore, the delay after the last output is also bounded by $\mathsf{poly}(|X|^{2k}) = \mathsf{poly}(|X|)$. As the time spent before the first solution output is $\mathsf{poly}(|\Sigma| + |X| + |G_c|)$, the whole algorithm has incremental delay as expected. □

To conclude this section, we show that Theorem 3 applies to various classes of closure systems present in the theory of convex geometries [26].

A closure system $\mathcal{F}$ is *distributive* if for any $F_1, F_2 \in \mathcal{F}$, $F_1 \cup F_2 \in \mathcal{F}$. Implicational bases of distributive closure systems have premises of size one [18].

Let $P = (X, \leq)$ be a partially ordered set, or poset. A subset $Y \subseteq X$ is *convex* in $P$ if for any triple $x \leq y \leq z$, $x, z \in Y$ implies $y \in Y$. The family $\{Y \subseteq X \mid Y \text{ is convex in } P\}$ is known to be closure system over $X$ [8, 24].

Let $G = (X, E)$ be a graph. We say that $G$ is *chordal* if every it has no induced cycle of size $\geq 4$. A *chord* in a path from $x$ to $y$ is an edge connecting to non-adjacent vertices of the path. A subset $Y$ of $X$ is *monophonically convex* in $G$ if for every pair $x, y$ of elements in $Y$, every $z \in X$ which lies on a chordless path from $x$ to $y$ is in $Y$. The family $\{Y \subseteq X \mid Y \text{ is monophonically convex in } G\}$ is a closure system [14, 26].

Finally, let $X \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, be a finite set of points, and denote by $\mathsf{ch}(Y)$ the *convex hull* of $Y$. The set system $\{\mathsf{ch}(Y) \mid Y \subseteq X\}$ forms a closure system [26] usually known as an *affine convex geometry.*

**Corollary 2.** *Let $\Sigma$ be an implicational base over $X$ and $G_c = (X, E_c)$.* MCCEnum *can be solved in incremental-polynomial time in the following cases:*

- *$\mathcal{F}$ is distributive,*
- *$\mathcal{F}$ is the family of convex subsets of a poset,*
- *$\mathcal{F}$ is the family of monophonically convex subsets of a chordal graph,*
- *$\mathcal{F}$ is an affine convex geometry in $\mathbb{R}^k$ for a fixed constant $k$.*

*Proof.* Distributive lattices have Carathéodory number 1 as they can be represented by implicational bases with singleton premises. The family of convex subsets of a poset has Carathéodory number 2 [24] (Corollary 13). The family of monophonically convex subsets of a chordal graphs have Carathéodory number at most 2 [14] (Corollary 3.4). The Carathéodory number of an affine convex geometry in $\mathbb{R}^k$ is $k - 1$ (see for instance [26], p. 32). □

In the distributive case, the algorithm can perform in polynomial delay using the algorithm of [23] since $\mathcal{K}$ will be a graph by Proposition 1. This connects with previous results on distributive closure systems by Kavvadias et al. [25].

## 5 Biatomic atomistic closure systems

In this section, we are interested in biatomic atomistic closure systems. Namely, we show that when minimal generators obey an independence condition, the size of $X$ is exponential with respect to $\mathrm{c}(\mathcal{F})$. To do so, we show that in biatomic atomistic closure systems, each subset of a minimal generator is itself a minimal generator. This result applies to atomistic modular closure systems, which can be represented by implications with premises of size at most 2 [31]. This suggests that MCCEnum becomes more difficult when implications have binary premises.

First, we need to define atomistic biatomic closure systems. let $\mathcal{F}$ be a closure system over $X$ with associated closure operator $\phi$. We say that $\mathcal{F}$ is *atomistic* if for any $x \in X$, $\phi(x) = \{x\}$. Equivalently, $\mathcal{F}$ is atomistic if its join-irreducible elements equal its atoms. Note that in a standard closure system, an atom is a singleton element. Biatomic closure systems have been studied by Birkhoff and Bennett in [5,8]. We reformulate their definition in terms of closure systems. A closure system $\mathcal{F}$ is *biatomic* if for every closed sets $F_1, F_2 \in \mathcal{F}$ and any atom $\{x\} \in \mathcal{F}$, $x \in \phi(F_1 \cup F_2)$ implies the existence of atoms $\{x_1\} \subseteq F_1$, $\{x_2\} \subseteq F_2$ such that $x \in \phi(x_1 x_2)$. In atomistic closure systems in particular, the biatomic condition applies to every element of $X$. Hence the next property of biatomic atomistic closure systems.

**Proposition 2.** *Let $\mathcal{F}$ be a biatomic atomistic closure system. Let $F \in \mathcal{F}$ and $x, y \in X$ with $x, y \notin F$. If $y \in \phi(F \cup \{x\})$, then there exists an element $z \in F$ such that $y \in \phi(xz)$.*

*Proof.* In atomistic closure systems, every element of $X$ is closed, therefore we apply the definition to the closed sets $F$ and $\{x\}$. □

We will also make use of the following folklore result about minimal generators. We give a proof for self-containment.

**Proposition 3.** *If $A_x$ is a minimal generator of $x \in X$, then $\phi(A) \cap A_x = A$ for any $A \subseteq A_x$.*

*Proof.* First, we have that $A \subseteq \phi(A) \cap A_x$ as $A \subseteq \phi(A)$ and $A \subseteq A_x$. Now suppose that there exists $a \in \phi(A) \cap A_x$ such that $a \notin A$. Then, $a \in \phi(A_x \setminus \{a\})$ as $A \subseteq A_x \setminus \{a\}$. Hence, $\phi(A_x) = \phi(A_x \setminus \{a\})$ and $x \in \phi(A_x \setminus \{a\})$, a contradiction with $A_x$ being a minimal generator of $x$. □

Our first step is to show that in a biatomic atomistic closure system, if $A_x$ is a minimal generator for some $x \in X$, then every non-empty subset $A$ of $A_x$ is itself a minimal generator for some $y \in X$. We prove this statement in Lemmas 2 and 3. Recall that an element $x \in X$ is a (trivial) minimal generator of itself.

**Lemma 2.** *Let $x \in X$ and let $A_x$ be a minimal generator of $x$ with size $k \geq 2$. Then for any $a_i \in A_x$, $i \in [k]$, there exists $y_i \in X$ such that $A_x \setminus \{a_i\}$ is a minimal generator of $y_i$.*

*Proof.* Let $A_x = \{a_1, \ldots, a_k\}$ be a minimal generator of $x$ such that $k \geq 2$. Then, for any $a_i \in A_x$, $i \in [k]$, we have $a_i \notin \phi(A_x \setminus \{a_i\})$ by Proposition 3. However, we have $x \in \phi(\{a_i\} \cup \phi(A_x \setminus \{a_i\})) = \phi(A_x)$. Thus, by Proposition 2, there must exists $y_i \in \phi(A_x \setminus \{a_i\})$ such that $x \in \phi(a_i y_i)$.

Let us show that $A_x \setminus \{a_i\}$ is a minimal generator of $y_i$. Assume for contradiction this is not the case. As $y_i \in \phi(A_x \setminus \{a_i\})$, there must be a proper subset $A$ of $A_x \setminus \{a_i\}$ which is a minimal generator for $y_i$. Note that since $A_x$ has at least 2 elements, at least one proper subset of $A_x \setminus \{a_i\}$ exists. As $A \subset A_x \setminus \{a_i\}$, there exists $a_j \in A_x$, $a_j \neq a_i$, such that $a_j \notin A$. Therefore, $A \subseteq A_x \setminus \{a_j\}$ and $\phi(A) \subseteq \phi(A_x \setminus \{a_j\})$. More precisely, $y_i \in \phi(A)$ and hence $y_i \in \phi(A_x \setminus \{a_j\})$. However, we also have that $a_i \in \phi(A_x \setminus \{a_j\})$, and since $x \in \phi(a_i y_i)$, we must have $x \in \phi(A_x \setminus \{a_j\})$, a contradiction with $A_x$ being a minimal generator of $x$. Thus, we deduce that $A_x \setminus \{a_i\}$ is a minimal generator for $y_i$, concluding the proof. □

In the particular case where $A_x$ has only two elements, say $a_1$ and $a_2$, then $A_x \setminus \{a_1\} = \{a_2\}$ and the element $a_2$ is a trivial minimal generator of itself. By using inductively Lemma 2 on the size of $A_x$, one can derive the next straightforward lemma.

**Lemma 3.** *Let $\mathcal{F}$ be a biatomic atomistic closure system. Let $A_x$ be a minimal generator of some $x \in X$. Then, for any $A \subseteq A_x$ with $A \neq \emptyset$, there exists $y \in X$ such that $A$ is a minimal generator of $y$.*

Thus, for a given minimal generator $A_x$ of $x$, any non-empty subset $A$ of $A_x$ is associated to some $y \in X$. We show next than when $A_x$ also satisfies an independence condition, $A$ will be the unique subset of $A_x$ associated to $y$. Following [18], we reformulate the definition of independence in an atomistic closure system $\mathcal{F}$, but restricted to its atoms. A subset $Y$ of $X$ is *independent* in $\mathcal{F}$ if for any $Y_1, Y_2 \subseteq Y$, $\phi(Y_1 \cap Y_2) = \phi(Y_1) \cap \phi(Y_2)$.

**Lemma 4.** *Let $\mathcal{F}$ be a biatomic atomistic closure system. Let $A_x$ be an independent minimal generator of $x \in X$, and let $A$ be a non-empty subset of $A_x$. Then, there exists $y \in X$ such that $A$ is the unique minimum subset of $A_x$ satisfying $y \in \phi(A)$.*

*Proof.* Let $A_x$ be an independent minimal generator of $x \in X$, and let $A$ be a non-empty subset of $A_x$. By Lemma 3, there exists $y \in X$ such that $A$ is a minimal generator for $y$, which implies $y \in \phi(A)$.

To prove that $A$ is the unique minimum subset of $A_x$ such that $y \in \phi(A)$, we show that for any $B \subseteq A_x$ such that $A \nsubseteq B$, $y \in \phi(B)$ cannot hold. Consider $B \subseteq A_x$ with $A \nsubseteq B$ and suppose that $y \in \phi(B)$. Note that $B$ must exist as the empty set is always a possible choice. Since $y \in \phi(A)$, we have $y \in \phi(A) \cap \phi(B)$. Furthermore, $\phi(A \cap B) \subset \phi(A)$ as $A \cap B \subset A$ and $\phi(A \cap B) \cap A_x = A \cap B$ by Proposition 3. Moreover, $A_x$ is independent, so that $\phi(A) \cap \phi(B) = \phi(A \cap B)$. Hence, $y \in \phi(A \cap B) \subset \phi(A)$, a contradiction with $A$ being a minimal generator of $y$. □

Thus, when $A_x$ is independent, each non-empty subset $A$ of $A_x$ is the unique minimal generator of some $y$ being included in $A_x$. As a consequence, we obtain the following theorem.

**Theorem 4.** *Let $\mathcal{F}$ be a biatomic atomistic closure system. If for any $x \in X$ and any minimal generator $A_x$ of $x$, $A_x$ is independent, then* $\mathrm{c}(\mathcal{F}) \leq \lceil \log_2(|X|+1) \rceil$.

*Proof.* Let $A_x$ be a minimal generator of $x$, $x \in X$ such that $\mathrm{c}(\mathcal{F}) = |A_x|$. As $A_x$ is a minimal generator, $\phi(A) \neq \phi(A')$ for any distinct $A, A' \subseteq A_x$, due to Proposition 3. Furthermore $A_x$ is independent by assumption. Thus, by Lemma 4, for each non-empty subset of $A$, there exists $y \in X$ such that $A$ is the unique minimum subset of $A_x$ with $y \in \phi(A)$. Consequently, $X$ must contain at least $2^{|A_x|} - 1$ elements in order to cover each non-empty subset of $A_x$, that is $2^{|A_x|} - 1 \leq |X|$, which can be rewritten as $|A_x| = \mathrm{c}(\mathcal{F}) \leq \lceil \log_2(|X|+1) \rceil$ as required. □

Now let $\mathcal{F}$ be a biatomic atomistic closure system on $X$ given by some implicational base $\Sigma$ and let $G_c = (X, E_c)$ be a consistency-graph. Assume that every minimal generator is independent. By Theorem 4, we have that $|X|$ has exponential size with respect to $\mathrm{c}(\mathcal{F})$, and by Proposition 1, it must be that the size of a key in $\mathcal{K}$ cannot exceed $2 \times \lceil \log_2(|X|+1) \rceil$. Thus, with respect to $\Sigma$, $G_c$ and $X$, $\mathcal{K}$ will have size quasi-polynomial in the worst case. Using the same algorithm as in Section 4, we obtain the next theorem.

**Theorem 5.** *Let $\Sigma$ be an implicational base of a biatomic atomistic closure system $\mathcal{F}$ over $X$ and $G_c$ a consistency-graph. If for any $x \in X$ and any minimal generator $A_x$ of $x$, $A_x$ is independent, then* MCCEnum *can be solved in output-quasipolynomial time.*

*Proof.* For clarity, we put $n = |X|$ and $k$ as the total size of the output $\mathsf{MIS}(\mathcal{K})$. $\mathcal{K}$ can be computed in incremental-polynomial time with the algorithm in [27]. Furthermore, by Theorem 4, the total size of $\mathcal{K}$ is bounded by $n^{\log(n)}$. Thus, this first step runs in time $\mathsf{poly}(|\Sigma| + |G_c| + n + n^{\log(n)})$, which is bounded by $\mathsf{poly}(|\Sigma| + |G_c| + n)^{\log(n)}$ being quasipolynomial in the size of $\Sigma$, $G_c$, $\mathcal{K}$ and $X$. To compute $\mathsf{MIS}(\mathcal{K}) = \mathsf{maxCC}(\Sigma, G_c)$ we use the algorithm of Fredman and Khachiyan [15] whose running time is bounded by $(n^{\log(n)} + k)^{o(\log(n^{\log(n)}+k))}$. In our case, we can derive the following upper bounds:

$$\begin{aligned}(n^{\log(n)} + k)^{o(\log(n^{\log(n)}+k))} &\leq (k+n)^{\log(n) \times o(\log(k+n)^{\log(n)})} \\ &\leq (k+n)^{O(\log^3(k+n))}\end{aligned}$$

Thus, the time needed to compute $\mathsf{MIS}(\mathcal{K})$ from $\mathcal{K}$ is output-quasipolynomial in the size of $X$ and $\mathsf{maxCC}(\Sigma, G_c)$. Consequently, the running time of the whole algorithm is bounded by

$$\mathsf{poly}(|\Sigma| + |G_c| + n)^{\log(n)} + (k+n)^{O(\log^3(k+n))}$$

which is indeed quasipolynomial in the size of the input $\Sigma$, $X$, $G_c$ and the output $\mathsf{MIS}(\mathcal{K}) = \mathsf{maxCC}(\Sigma, G_c)$. □

To conclude this section, we show that atomistic modular closure systems [18,29] satisfy conditions of Theorem 5. Recall that a closure system $\mathcal{F}$ is modular if for any $F_1, F_2, F_3 \in \mathcal{F}$, $F_1 \subseteq F_2$ implies $\phi(F_1 \cup (F_2 \cap F_3)) = \phi(F_1 \cup F_3) \cap F_2$. It was proved for instance in [5] (Theorem 7) that atomistic modular closure systems are biatomic. To show that any minimal generator is independent, we make use of the following result.

**Theorem 6.** *(Reformulated from [18], Theorem 360) Let $\mathcal{F}$ be a modular closure system. A subset $A = \{a_1, \ldots, a_k\} \subseteq X$ is independent if and only if $\phi(a_1) \cap \phi(a_2) = \phi(a_1 a_2) \cap \phi(a_3) = \cdots = \phi(a_1 \ldots a_{k-1}) \cap \phi(a_k) = \emptyset$.*

**Proposition 4.** *Let $\mathcal{F}$ be an atomistic modular closure system. Let $A_x$ be a minimal generator of some $x \in X$. Then $A_x$ is independent.*

*Proof.* Let $A_x = \{a_1, \ldots, a_k\}$ be a minimal generator for some $x \in X$. Then, by Proposition 3, $\phi(a_1 \ldots a_i) \cap A_x = a_1 \ldots a_i$ for any $i \in [k]$. Furthermore, $\phi(a) = \{a\}$ for any $a \in X$ since $\mathcal{F}$ is atomistic. Thus we conclude that $\phi(a_1 \ldots a_i) \cap \phi(a_{i+1}) = \emptyset$ for any $i \in [k-1]$ as $a_{i+1} \notin a_1 \ldots a_i$. It follows by Theorem 6 that $A_x$ is indeed independent. □

**Corollary 3.** *Let $\Sigma$ be an implicational base over $X$ and $G_c = (X, E_c)$. Then* MCCEnum *can be solved in output-quasipolynomial time if:*

- *$\mathcal{F}$ is biatomic atomistic and has Carathéodory number* 2 *(including convex subsets of a poset and monophonically convex sets of a chordal graph),*
- *$\mathcal{F}$ is atomistic modular.*

*Proof.* For the first statement, note that in an atomistic closure system with Carathéodory number 2, any minimal generator $A_x$ contains exactly two elements $a_1, a_2$. Since $\mathcal{F}$ is atomistic, $a_1$ and $a_2$ are closed and the independence of $A_x$ follows.

If $\mathcal{F}$ is atomistic modular, biatomicity follows from [5] (Theorem 7), and independence from Proposition 4. □

*Remark 1.* For atomistic modular closure systems, the connection between the size of $X$ and the Carathéodory number may also be derived from counting arguments on subspaces of vector spaces [30].

## 6 Conclusion

In this paper we proved that given a consistency-graph over an implicational base, the enumeration of maximal consistent closed sets is impossible in output-polynomial time unless $\mathsf{P} = \mathsf{NP}$. Moreover, we showed that this problem, called MCCEnum, is already intractable for the well-known class of lower bounded closure systems. On the positive side, we proved that when the size of a minimal generator is bounded by a constant, the enumeration of maximal consistent closed sets can be conducted in incremental polynomial time. This result covers various classes of convex geometries. Finally, we proved that in biatomic atomistic closure systems, MCCEnum can be solved in output-quasipolynomial time provided minimal generators obey an independence condition. This applies in particular to atomistic modular closure systems. In Figure 2, we summarize our results in the hierarchy of closure systems.

For future research, we would like to understand which properties or paramaters of closure systems make the problem intractable or solvable in output-polynomial time. For instance, we have seen that a bounded Carathéodory number gives an incremental-polynomial time algorithm, while lower boundedness makes the problem intractable. Another question is the following: is the problem still hard if the closure system is given by a context (equivalently, its meet-irreducible elements)? The question is particularly interesting for classes such as semidistributive closure systems where we can compute the context in polynomial time in the size of an implicational base.

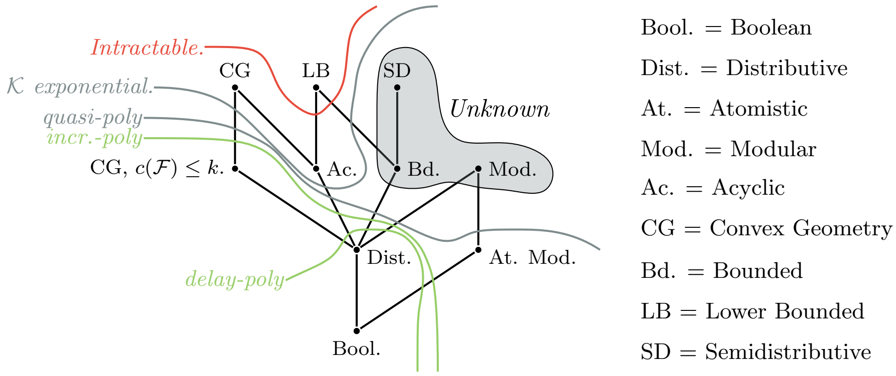


Fig. 2: The complexity of MCCEnum in the hierarchy of closure systems

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