On non-autonomous maximal L p -regularity under Besov regularity in time in weighted spaces

Mahdi Achache

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September 21, 2021

Abstract

We consider the problem of maximal regularity for non-autonomous Cauchy problems

$$u'(t) + A(t)u(t) = f(t), \ \ t\text{-a.e.}, \ \ u(0) = u_0.$$

The time dependent operators $A(t)$ are associated with (time dependent) sesquilinear forms on a Hilbert space $\mathcal{H}$. We prove the maximal regularity result in temporally weighted L^p-spaces for $p > 2$ and other regularity properties for the solution of the previous problem under minimal regularity assumptions on the forms and the initial value u_0. Our main assumption is that $(\mathcal{A}(t))_{t\in[0,\tau]}$ are in the Besov space $B_p^{1-\frac{1}{p},2}$ with respect to the variable t and $u_0 \in (\mathcal{H}; D(A(0)))_{\theta,p}$, where $\theta = \frac{p-1-\beta}{p}$. Our results are motivated by boundary value problems.

keywords: Besov spaces, maximal regularity, non-autonomous evolution equations, sesquilinear forms. weighted spaces.

Contents

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1 Introduction

The present paper deals with maximal L^p-regularity for non-autonomous evolution equations in the setting of Hilbert spaces. Before explaining our results we introduce some notations and assumptions.
Let $(\mathcal{H}, (\cdot,\cdot), \|\cdot\|)$ be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. We consider another Hilbert space $\mathcal{V}$ which is densely and continuously embedded into $\mathcal{H}$. We denote by $\mathcal{V}'$ the (anti-) dual space of $\mathcal{V}$ so that

$$\mathcal{V} \hookrightarrow_d \mathcal{H} \hookrightarrow_d \mathcal{V}'.$$

We denote by $\langle , \rangle$ the duality $\mathcal{V}$-$\mathcal{V}'$ and note that $\langle \psi, v\rangle = (\psi, v)$ if $\psi, v \in \mathcal{H}$. Given $\tau \in (0,\infty)$ and consider a family of sesquilinear forms

$$\mathfrak{a} : [0,\tau] \times \mathcal{V} \times \mathcal{V} \to \mathbb{C}$$

such that

- [H1]: $D(\mathfrak{a}(t)) = \mathcal{V}$ (constant form domain),
- [H2]: $|\mathfrak{a}(t,u,v)| \le M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$ (uniform boundedness),
- [H3]: Re $\mathfrak{a}(t,u,u) + \nu\|u\|^2 \ge \delta\|u\|^2_{\mathcal{V}}$ $(\forall u \in \mathcal{V})$ for some $\delta > 0$ and some $\nu \in \mathbb{R}$ (uniform quasi-coercivity).

Here and throughout this paper, $\|\cdot\|_{\mathcal{V}}$ denotes the norm of $\mathcal{V}$.

To each form $\mathfrak{a}(t)$ we can associate two operators $A(t)$ and $\mathcal{A}(t)$ on $\mathcal{H}$ and $\mathcal{V}'$, respectively. Recall that $u \in \mathcal{H}$ is in the domain $D(A(t))$ if there exists $h \in \mathcal{H}$ such that for all $v \in \mathcal{V}$: $\mathfrak{a}(t,u,v) = (h,v)$. We then set $A(t)u := h$. The operator $\mathcal{A}(t)$ is a bounded operator from $\mathcal{V}$ into $\mathcal{V}'$ such that $\mathcal{A}(t)u = \mathfrak{a}(t,u,\cdot)$. The operator $A(t)$ is the part of $\mathcal{A}(t)$ on $\mathcal{H}$. It is a classical fact that $-A(t)$ and $-\mathcal{A}(t)$ are both generators of holomorphic semigroups $(e^{-rA(t)})_{r\ge 0}$ and $(e^{-r\mathcal{A}(t)})_{r\ge 0}$ on $\mathcal{H}$ and $\mathcal{V}'$, respectively. The semigroup $e^{-rA(t)}$ is the restriction of $e^{-r\mathcal{A}(t)}$ to $\mathcal{H}$. In addition, $e^{-rA(t)}$ induces a holomorphic semigroup on $\mathcal{V}$ (see, e.g., Ouhabaz [27, Chapter 1]).

A well known result by J.L. Lions asserts that the Cauchy problem

$$u'(t) + \mathcal{A}(t)u(t) = f(t), \; u(0) = u_0 \in \mathcal{H} \tag{1.1}$$

has maximal L_2-regularity in $\mathcal{V}'$, that is, for every $f \in L^2(0,\tau;\mathcal{V}')$ there exists a unique $u \in W^{1,2}(0,\tau;\mathcal{V}') \cap L^2(0,\tau;\mathcal{V})$ which satisfies (1.1) in the L^2-sense. The maximal regularity in $\mathcal{H}$ is however more interesting since when dealing with boundary value problems one cannot identify the boundary conditions if the Cauchy problem is considered in $\mathcal{V}'$. Maximal regularity in $\mathcal{H}$ differs considerably from the same property in $\mathcal{V}'$ and more difficult to prove. In this note we consider the question of maximal regularity for weighted L^p-spaces $L^p_\beta(0,\tau;\mathcal{H})$. The weights we consider are power weights in time.

Definition 1.1. *We say that the problem (1.1) has maximal L^p_β-regularity in $\mathcal{H}$, if for all $f \in L^p_\beta(0,\tau;\mathcal{H})$ there exists a unique $u \in W^{1,p}_\beta(0,\tau;\mathcal{H})$ which satisfies (1.1) in the L^p_β-sense.*

Thus all three functions $u', A(.)u$ and f are in $L^p_\beta(0,\tau;\mathcal{H})$, which is the reason for the terminology "maximal L^p_β-regularity" in $\mathcal{H}$. As a consequence, the solution is in the maximal L^p_β-regularity space, namely,

$$MR(p,\beta) := \{u \in W^{1,p}_\beta(0,\tau;\mathcal{H}) \cap L^p_\beta(0,\tau;\mathcal{V}) : A(.)u \in L^p_\beta(0,\tau;\mathcal{H})\}.$$

This is a Banach space for the norm

$$\|u\|^2_{MR(p,\beta)} = \|u'\|^2_{L^p_\beta(0,\tau;\mathcal{H})} + \|A(.)u\|^2_{L^p_\beta(0,\tau;\mathcal{H})}.$$

We define the corresponding trace space by $Tr(p,\beta) = \{u(0) : u \in MR(p,\beta)\}$ which is a Banach space for the norm

$$\|x\|_{Tr(p,\beta)} = \inf\{\|u\|_{MR(p,\beta)} : u \in MR(p,\beta), u(0) = x\}.$$

Note that $u \in MR(p,\beta)$ if and only if $u(0) \in Tr(p,\beta)$. Consequently, there are two tasks: Finding conditions on the form $\mathfrak{a}(.)$ that imply maximal L^p_β-regularity in $\mathcal{H}$, and then identifying the trace space $Tr(p,\beta)$.
Lions himself proved maximal L^2-regularity in $\mathcal{H}$ if the form $\mathfrak{a}(t,.,.)$ is symmetric for all $t \in [0,\tau]$ and $t \to \mathfrak{a}(t,u,v) \in C^1([0,\tau])$ for all $u,v \in \mathcal{V}$ (see [23]). The proof is based on a representation theorem of linear functionals due to himself and usually known in the literature as Lions's representation theorem. Using a different approach, maximal L^p-regularity was established in [28], assuming that $t \to \mathfrak{a}(t,u,v) \in C^\alpha([0,\tau])$ for all $u,v \in \mathcal{V}$, for some $\alpha > \frac{1}{2}$. This result was further improved in [20], where the Hölder condition is replaced by a weaker "Dini" condition for $\mathfrak{a}(.,.,.)$.
For maximal L^2-regularity, this result was improved to the fractional Sobolev regularity $t \to \mathcal{A}(t) \in H^{\frac{1}{2}+\alpha}([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))$ for $\alpha > \frac{1}{2}$ (see [13]). The proof is surprisingly elementary and based on the Lax-Milgram lemma. Furthermore, it is proved in [4] that maximal L^2-regularity holds if $t \to \mathcal{A}(t) \in H^{\frac{1}{2}}([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))$ (with some integrability conditions). Fackler [18], on the other hand was able to construct a symmetric non-autonomous form that is α-Hölder continuous for every $\alpha \le \frac{1}{2}$ but does not have maximal L^2-regularity in $\mathcal{H}$. Fackler [19], generalized the result in [13] for any $p \in (1,\infty)$ by assuming fractional Sobolev regularity. In fact, he proved that maximal L^p-regularity is satisfied if

(i) $\mathcal{A}(.) \in \dot{H}^{\frac{1}{2}+\epsilon}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))$ for $p \le 2$

(ii) $\mathcal{A}(.) \in \dot{W}^{\frac{1}{2}+\epsilon,p}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))$ for $p > 2$.

An example from [12] shows that $\mathcal{A}(.) \in \dot{W}^{\frac{1}{2},p}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))$ for $p > 2$ is not enough to obtain maximal L^p-regularity. We refer to the recent papers [4] or [6] for more details and references.
Maximal L^p_β-regularity can be used to establish existence and uniqueness of solutions for quasilinear parabolic evolution equations. The choice of the weighted spaces has a big advantages. One of them is to reduce the necessary regularity for initial conditions of evolution equations. Time-weights can be used also to exploit parabolic regularization which is typical for quasilinear parabolic problems. Prüss and Simonett [29], proved maximal L^p_β-regularity for $\beta \in [0, p-1)$ in the autonomous case (i.e. $A(t) = A(0)$ for all $t \in [0,\tau]$) in a Banach space assuming that (1.1) has maximal L^p-regularity. In the present paper, we extend their results to $\beta \in [-1, p-1)$ and the result in [19] to the case of the weights spaces and assuming less regularity on the operators $\mathcal{A}(t)$ with respect to t for the case $p > 2$, which is our main motivation. We show maximal L^p_β-regularity on Hilbert spaces assuming Besov regularity regularity in time. Our main result shows that if $t \mapsto \mathcal{A}(t)$ is in the Besov space $B_p^{1-\frac{1}{p},2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))$ then maximal $L^p_\beta-$regularity in $\mathcal{H}$ is satisfied. We remark that $W^{\frac{1}{2}+\varepsilon,p} \subset B_p^{1-\frac{1}{p},2} \subset W^{\frac{1}{2},p}$ for all $\varepsilon > 0$. Then this regularity assumption is minimal and our results are the most general ones on this problem. The initial data u_0 is arbitrary in the interpolation space $(\mathcal{H}; D(A(0)))_{\theta,p}$ for $-1 < \beta < p-1$. Here, $\theta = \frac{p-1-\beta}{p}$.
In section 2, we start with basic properties of the weighted spaces, while in section 3 we prove several key estimates and develop the necessary tools for the proofs of the main results. The main results are proved in sections 5, 6 and several examples are given in section 7.

Notation.

- We denote by $\mathcal{L}(E,F)$ (or $\mathcal{L}(E)$) the space of bounded linear operators from E to F (from E to E). The spaces $L^p(a,b;E)$ and $W^{1,p}(a,b;E)$ denote respectively the Lebesgue and Sobolev spaces of function on (a,b) with values in E. $C^\alpha(a,b;E)$ denote the space of Hölder continuous functions of order α. Recall that the norms of $\mathcal{H}$ and $\mathcal{V}$ are denoted by $\|\cdot\|$ and $\|\cdot\|_{\mathcal{V}}$. The scalar product of $\mathcal{H}$ is $(\cdot,\cdot)$.
- We denote by C, C' or c... all inessential positive constants. Their values may change from line to line.
- On some cases we will use the notation $a \lesssim b$ to signify that there exists an inessential positive constant C such that $a \le Cb$.
- Finally, by $(E,F)_{\theta,p}, [E,F]_\theta, \theta \in (0,1), p \in (1,\infty)$ we denote the real and complex interpolation spaces respectively between E and F.

2 Properties of weighted spaces

In this section we briefly recall the definitions and we give the basic properties of vector-valued function spaces with temporal weights.

For $p \in (1, \infty)$ and $-1 < \beta < p-1$ we set $L^p_\beta(0,\tau;\mathcal{H}) := \{u : t \mapsto t^{\frac{\beta}{p}} u(t) \in L^p(0,\tau;\mathcal{H})\}$, endowed with the norm

$$\|u\|^p_{L^p_\beta(0,\tau,\mathcal{H})} := \int_0^\tau \|u(t)\|^p t^\beta \, dt.$$

In the case $p = \infty$, $L^\infty_\beta(0,\tau;\mathcal{H})$ is defined as follows

$$L^\infty_\beta(0,\tau;\mathcal{H}) := \{u \in L^1(0,\tau;\mathcal{H}) : \; s \to s^\beta u(s) \in L^\infty(0,\tau;\mathcal{H})\},$$

with norm $\|u\|_{L^\infty_\beta(0,\tau;\mathcal{H})} := \|s \mapsto s^\beta u(s)\|_{L^\infty(0,\tau;\mathcal{H})}$.
It is very seen that $L^p_\beta(0,\tau;\mathcal{H}) \hookrightarrow L^1(0,\tau;\mathcal{H})$. Indeed, for $u \in L^p_\beta(0,\tau;\mathcal{H})$ we find by Hölder's inequality

$$\int_0^\tau \|u(t)\| \, dt \le (\int_0^\tau t^{-\frac{\beta}{p-1}} \, dt)^{\frac{p-1}{p}} \|u\|_{L^p_\beta(0,\tau;\mathcal{H})}.$$

It clearly holds that $L^p(0,\tau;\mathcal{H}) \hookrightarrow L^p_\beta(0,\tau;\mathcal{H})$ for $\beta > 0$ and $L^p_\beta(0,\tau;\mathcal{H}) \hookrightarrow L^p(0,\tau;\mathcal{H})$ for $\beta < 0$.
We define the corresponding weighted Sobolev spaces

$$W^{1,p}_\beta(0,\tau;\mathcal{H}) := \{u \in W^{1,1}(0,\tau;\mathcal{H}) : \; u, u' \in L^p_\beta(0,\tau;\mathcal{H})\},$$

$$W^{1,p}_{\beta,0}(0,\tau;\mathcal{H}) := \{u \in W^{1,p}_\beta(0,\tau;\mathcal{H}) : \; u(0) = 0\},$$

which are Banach spaces for the norms, respectively,

$$\|u\|^2_{W^{1,p}_\beta(0,\tau;\mathcal{H})} := \|u\|^p_{L^p_\beta(0,\tau;\mathcal{H})} + \|u'\|^p_{L^p_\beta(0,\tau;\mathcal{H})},$$

$$\|u\|_{W^{1,p}_{\beta,0}(0,\tau;\mathcal{H})} := \|u'\|_{L^p_\beta(0,\tau;\mathcal{H})}.$$

Lemma 2.1 (Weighted Hardy inequality). *Let $p \in (1, \infty)$ and $\beta \in (-1, p-1)$. Then for all $f \in L^p_\beta(0,\tau,\mathcal{H})$,*

$$\int_0^\tau (\frac{1}{t} \int_0^t \|f(s)\| \, ds)^p t^\beta \, dt \lesssim \|f\|^p_{L^p_\beta(0,\tau;\mathcal{H})}.$$

Lemma 2.1 is proved in [31][Lemma 6].

Proposition 2.2. *We have the following properties*

1- For all $u \in L^p_\beta(0,\tau,\mathcal{H})$, $t \to v(t) = \frac{1}{t} \int_0^t u(s) \, ds \in L^p_\beta(0,\tau,\mathcal{H})$.

2- *We define the operator* $\Phi : L^p_\beta(0,\tau;\mathcal{H}) \to L^p(0,\tau;\mathcal{H})$, *such that* $(\Phi f)(t) = t^{\frac{\beta}{p}} f(t)$ *for* $f \in L^p_\beta(0,\tau;\mathcal{H})$ *and* $t \in [0,\tau]$. *Then* Φ *is an isometric isomorphism. We note also that* $\Phi \in \mathcal{L}(L^p(0,\tau;\mathcal{H}), L^p_{-\beta}(0,\tau;\mathcal{H}))$.

3- $L^{p'}_{-\frac{\beta}{p-1}}(0,\tau;\mathcal{H})$ *is the dual space of* $L^p_\beta(0,\tau;\mathcal{H})$ *by the duality defined in* $L^2(0,\tau;\mathcal{H})$.

4- *If* $u \in W^{1,p}_\beta(0,\tau;\mathcal{H})$, *we obtain that* u *has a continuous extension on* $\mathcal{H}$ *and*

$$W^{1,p}_\beta(0,\tau;\mathcal{H}) \hookrightarrow C([0,\tau];\mathcal{H}).$$

5- $C^\infty_c((0,\tau);\mathcal{H})$ *and* $C^\infty([0,\tau];\mathcal{H})$ *are dense in* $L^p_\beta(0,\tau;\mathcal{H})$.

Remark 2.3. *The restriction on* β *comes from several facts. The first one is the embedding* $L^p_\beta(0,\tau;\mathcal{H}) \hookrightarrow L^1(0,\tau;\mathcal{H})$. *The second one is due to Hardy' inequality and the third reason comes from the fact that functions in* $W^{1,p}_\beta(0,\tau;\mathcal{H})$ *have a well-defined trace in case that* $-1 < \beta < p-1$.

Proof. Prove ca svp et ajouter le lemme en bas a la proposition. □

Lemma 2.4. *Let* $\beta \in (-1, p-1)$ *and* $p \in (2,\infty)$. *Suppose that* $2(1+\beta) < p$, *then* $L^p_\beta(0,\tau;\mathcal{H}) \hookrightarrow L^2(0,\tau;\mathcal{H})$.

Proof. Let $f \in L^p_\beta(0,\tau;\mathcal{H})$. Hölder's inequality gives

$$\begin{aligned}
\int_0^\tau \|f(s)\|^2 ds &= \int_0^\tau s^{\frac{-2\beta}{p}} \|f(s)\|^2 s^{\frac{2\beta}{p}}\, ds \\
&\le \Big(\int_0^\tau s^{\frac{-2\beta}{p-2}}\, ds\Big)^{\frac{p-2}{p}} \|f\|^{\frac{2}{p}}_{L^p_\beta(0,\tau;\mathcal{H})} \\
&= \tau^{\frac{-2\beta+p-2}{p}} \|f\|^{\frac{2}{p}}_{L^p_\beta(0,\tau;\mathcal{H})}.
\end{aligned}$$

This finishes the proof. □

3 Preparatory lemmas

In this section we prove several estimates which will play an important role in the proof of the main results. We emphasize that one of the important points here is to prove estimates with constants which are independent of t. From now we assume without loss of generality that the forms are coercive, that is $[H3]$ holds with $\nu = 0$. The reason is that by replacing $A(t)$ by $A(t)+\nu$, the solution v of (1.1) is $v(t) = e^{-\nu t}u(t)$ and it is clear that $u \in W^{1,p}_\beta(0,\tau;\mathcal{H}) \cap L^p_\beta(0,\tau;\mathcal{V})$ if and only if $v \in W^{1,p}_\beta(0,\tau;\mathcal{H}) \cap L^p_\beta(0,\tau;\mathcal{V})$.

For $f \in L^p(0,\tau;\mathcal{H}), p \in (1,\infty)$ and for almost every $t \in [0,\tau]$ we define the operator L by

$$L(f)(t) := A(t) \int_0^t e^{-(t-s)A(t)} f(s)\, ds.$$

Note that in the autonomous case (i.e. $A(t) = A(0)$ for any $t \in [0,\tau]$) L is called the maximal regularity operator.
Our aim is to prove that $L \in \mathcal{L}(L^p_\beta(0,\tau;\mathcal{H}))$ for all $p \in (1,\infty)$. It is proved in [20] that L is bounded on $L^p(0,\tau;\mathcal{H})$ for all $p \in (1,\infty)$ provided $t \mapsto \mathfrak{a}(t,.,.)$ is C^ε for some $\varepsilon > 0$ (or similarly, $t \mapsto \mathcal{A}(t)$ is C^ε on $[0,\tau]$ with values in $\mathcal{L}(\mathcal{V},\mathcal{V}'))$. The proof for the case $p = 2$ is based on vector-valued pseudo-differential operators.

Lemma 3.1. *Assume that $t \mapsto \mathfrak{a}(t,.,.)$ is C^ε for some $\varepsilon > 0$. Then L is bounded on $L^p_\beta(0,\tau;\mathcal{H})$ for all $\beta \in (-1,p-1)$ and $p \in (1,\infty)$.*

Proof. Let $\beta \in (-1,p-1), p \in (1,\infty)$ and $f \in L^p_\beta(0,\tau;\mathcal{H})$. It is easy to see that $t \mapsto t^{\frac{\beta}{p}} f(t) \in L^p(0,\tau;\mathcal{H})$.
We split the integral into two parts to get

$$\begin{aligned}(Lf)(t) &= A(t) \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s)\, ds + A(t) \int_{\frac{t}{2}}^t e^{-(t-s)A(t)} f(s)\, ds \\ &:= I_1(t) + I_2(t).\end{aligned}$$

We begin by estimating the first one

$$\begin{aligned}\|I_1(t)\| = \|A(t) \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s)\, ds\| &\lesssim \int_0^{\frac{t}{2}} \frac{1}{t-s} \|f(s)\|\, ds \\ &\lesssim \frac{2}{t} \int_0^{\frac{t}{2}} \|f(s)\|\, ds.\end{aligned}$$

Hardy's inequality gives

$$\begin{aligned}&\int_0^\tau \|A(t) \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s)\, ds\|^p t^\beta\, dt \\ &\lesssim \int_0^\tau (\frac{2}{t} \int_0^{\frac{t}{2}} \|f(s)\|\, ds)^p t^\beta\, dt \\ &\lesssim \|f\|^p_{L^p_\beta(0,\tau;\mathcal{H})}.\end{aligned}$$

Now we estimate the second integral $I_2(t)$. Indeed,

$$\begin{aligned}t^{\frac{\beta}{2}} \|I_2(t)\| = t^{\frac{\beta}{2}} \|A(t) \int_{\frac{t}{2}}^t e^{-(t-s)A(t)} f(s)\, ds\| &\lesssim \|A(t) \int_0^t e^{-(t-s)A(t)} (\mathbb{1}_{(\frac{t}{2},t)} s^{\frac{\beta}{2}} f(s))\, ds\| \\ &= \|(Lg)(t)\|.\end{aligned}$$

Here, $g(s) = \mathbb{1}_{(\frac{t}{2},t)} s^{\frac{\beta}{2}} f(s)$. Since L is bounded on $L^p(0,\tau;\mathcal{H})$ we infer that $t \mapsto t^{\frac{\beta}{2}} I_2(t) \in L^p(0,\tau;\mathcal{H})$. Therefore, $L \in \mathcal{L}(L^p_\beta(0,\tau;\mathcal{H}))$ for all $p \in (1,\infty)$. $\square$

Proposition 3.2. *For $p-1 \le \beta$ the operator L is not bounded on $L^p_\beta(0,\tau;\mathcal{H})$ in general.*

Proof. Let $u \in \mathcal{H}$ and $g \in L^{p'}_{-\frac{\beta}{p-1}}(0,\tau;\mathcal{H}), p' = \frac{p}{p-1}$. Noting that

$$(L^*g)(t) = \int_t^\tau A(s)^* e^{-(s-t)A(s)^*} g(s)\,ds, t \in (0,\tau)$$

is the adjoint operator and $L \in \mathcal{L}(L^p_\beta(0,\tau;\mathcal{H}))$ if and only if $L^* \in \mathcal{L}(L^{p'}_{-\frac{\beta}{p-1}}(0,\tau;\mathcal{H}))$. Suppose that $A(s)^* = A(0)^*$ for all $s \in [0,\tau]$, then $(L^*g)(t) = \int_t^\tau A(0)^* e^{-(s-t)A(0)^*} g(s)\,ds$. Assume now that $t < 1 < \tau$ and take $g(s) = \mathbb{1}_{[1,\tau]}(s)u$, so

$$(L^*g)(t) = e^{-(1-t)A(0)^*}u - e^{-(\tau-t)A(0)^*}u,$$

which converges to $e^{-A(0)^*}u - e^{-\tau A(0)^*}u$ as $t \to 0$.
We claim that

$$e^{-A(0)^*}u - e^{-\tau A(0)^*}u \neq 0,$$

then

$$\begin{aligned}\|L^*g\|^{p'}_{L^{p'}_{-\frac{\beta}{p-1}}(0,\tau;\mathcal{H})} &\ge \|L^*g\|^{p'}_{L^{p'}_{-\frac{\beta}{p-1}}(0,1;\mathcal{H})}\\ &= \int_0^1 \|e^{-(1-t)A(0)^*}u - e^{-(\tau-t)A(0)^*}u\|^{p'} \frac{dt}{t^{\frac{\beta}{p-1}}} = \infty.\end{aligned}$$

Now, suppose that $e^{-A(0)^*}u - e^{-\tau A(0)^*}u = 0$. Thus

$$e^{-A(0)^*}u = e^{-(2\tau-1)A(0)^*}u.$$

Using induction, for all $n \in \mathbb{N}$ we obtain

$$e^{-A(0)^*}u - e^{-(n(\tau-1)+1)A(0)^*}u = 0.$$

By letting $n \to \infty$, it follows that $e^{-A(0)^*}u = 0$. Hence, $e^{-tA(0)^*}u = 0$ for all $t \ge 1$, and we deduce that $u = 0$ by an application of the isolated point theorem and the analyticity of the semigroup. $\square$

Lemma 3.3. *Let $f \in L^p_\beta(0,\tau;\mathcal{H})$ and $p > 2, \beta \in (-1,p-1)$. We have $t^{\frac{\beta}{p}-\frac{1}{2}+\frac{1}{p}}(L_1 f)(t) \in \mathcal{V}$, where*

$$(L_1 f)(t) := \int_0^t e^{-(t-s)A(t)} f(s)\,ds,\ t \in [0,\tau].$$

As a consequence for $2(\beta+1) \le p, (L_1 f)(t) \in \mathcal{V}$ for all $t \in [0,\tau]$.

Proof. Write

$$t^{\frac{\beta}{p}}(L_1 f)(t) = t^{\frac{\beta}{p}} \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s)\, ds + t^{\frac{\beta}{p}} \int_{\frac{t}{2}}^{t} e^{-(t-s)A(t)} f(s)\, ds. \tag{3.1}$$

A straightforward computation gives

$$\begin{aligned} \|t^{\frac{\beta}{p}} \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s)\, ds\|_{\mathcal{V}} &\lesssim t^{\frac{\beta}{p}} \int_0^{\frac{t}{2}} \|e^{-(t-s)A(t)}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} \|f(s)\|\, ds \\ &\lesssim t^{\frac{\beta}{p}-\frac{1}{2}} (\int_0^{\frac{t}{2}} s^{-\frac{p'\beta}{p}}\, ds)^{\frac{1}{p'}} \|f\|_{L^p_\beta(0,\tau;\mathcal{H})} \\ &\lesssim t^{\frac{1}{p'}-\frac{1}{2}} \|f\|_{L^p_\beta(0,\tau;\mathcal{H})} \\ &= t^{\frac{1}{2}-\frac{1}{p}} \|f\|_{L^p_\beta(0,\tau;\mathcal{H})}. \end{aligned}$$

Here, $p' = \frac{p}{p-1}$ is the conjugate of p. For the second term in RHS of (3.1),

$$\begin{aligned} \|t^{\frac{\beta}{p}} \int_{\frac{t}{2}}^{t} e^{-(t-s)A(t)} f(s)\, ds\|_{\mathcal{V}} &\lesssim \int_{\frac{t}{2}}^{t} \|e^{-(t-s)A(t)}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} s^{\frac{\beta}{p}} \|f(s)\|\, ds \\ &\lesssim \int_{\frac{t}{2}}^{t} \frac{1}{(t-s)^{\frac{1}{2}}} s^{\frac{\beta}{p}} \|f(s)\|\, ds \\ &\lesssim t^{\frac{1}{2}-\frac{1}{p}} \|f\|_{L^p_\beta(0,\tau;\mathcal{H})}. \end{aligned}$$

This shows the result. □

Proposition 3.4. *Let $p > 2$ and $-1 < \beta < p-1$. Assume that $\int_0^\tau \frac{\|\mathcal{A}(t)-\mathcal{A}(0)\|^p_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t^{\frac{p}{2}}}\, dt < \infty$. Then for all $u_0 \in (\mathcal{H}; D(A(0)))_{\theta,p}$, we have*

$$t \to (Fu_0)(t) = t^{\frac{\beta}{p}} A(t) e^{-tA(t)} u_0 \in L^p(0,\tau;\mathcal{H}).$$

Proof. Write

$$\begin{aligned} (Fu_0)(t) &= t^{\frac{\beta}{p}} A(t) e^{-tA(t)} u_0 \\ &= t^{\frac{\beta}{p}} (A(t)e^{-tA(t)} u_0 - A(0) e^{-tA(0)} u_0) \\ &+ t^{\frac{\beta}{p}} A(0) e^{-tA(0)} u_0. \end{aligned}$$

Choose a contour Γ in the positive half-plane and write by the holomorphic functional calculus for the sectorial operators $A(t), A(0)$

$$A(t)e^{-tA(t)} - A(0)e^{-tA(0)} = \frac{1}{2\pi i} \int_\Gamma \lambda e^{-t\lambda} (\lambda I - A(t))^{-1} \Big(\mathcal{A}(t) - \mathcal{A}(0)\Big) (\lambda I - A(0))^{-1}\, d\lambda.$$

Consequently,

$$\begin{aligned}
&\|A(t)e^{-tA(t)}u_0 - A(0)e^{-tA(0)}u_0\| \\
&\le C\int_0^\infty |\lambda| e^{-t|\cos\gamma||\lambda|}\|(\lambda I-\mathcal{A}(0))^{-1}\|_{\mathcal{L}((\mathcal{H},D(A(0)))_{\theta,p};\mathcal{V})} \\
&\times \|(\lambda I-\mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}';\mathcal{H})}\, d|\lambda| \|\mathcal{A}(t)-\mathcal{A}(0)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \\
&\times \|u_0\|_{(\mathcal{H},D(A(0)))_{\theta,p}}.
\end{aligned}$$

Therefore

- for $2(\beta+1) < p$, we have by Lemma 5.3 that $(\mathcal{H}, D(A(0)))_{\theta,p} \hookrightarrow \mathcal{V}$. Hence,

$$\|(A(t)e^{-tA(t)} - A(0)e^{-tA(0)})u_0\| \le C\frac{\|\mathcal{A}(t)-\mathcal{A}(0)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t^{-\beta+\frac{p}{2}}}\|u_0\|_{(\mathcal{H},D(B(0)))_{\theta,p}}.$$

- for $p < 2(\beta+1)$, Lemma 5.3 shows that $(\mathcal{H}, D(A(0)))_{\theta,p} = (\mathcal{H},\mathcal{V})_{2\theta,p}$. Then

$$\|(A(t)e^{-tA(t)} - A(0)e^{-tA(0)})u_0\| \le C\frac{\|\mathcal{A}(t)-\mathcal{A}(0)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t^{\frac{1}{p}}}\|u_0\|_{(\mathcal{H},\mathcal{V})_{2\theta,p}}.$$

- for $p = 2(\beta+1)$, we obtain $(\mathcal{H}, D(A(0)))_{\theta,p} = (\mathcal{H}, D(A(0)))_{\frac{1}{2},p}$. Using Lemma 5.3 we get for all $\varepsilon > 0$

$$\|(A(t)e^{-tA(t)} - A(0)e^{-tA(0)})u_0\| \le C\frac{\|\mathcal{A}(t)-\mathcal{A}(0)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t^{\frac{1}{p}+\varepsilon}}\|u_0\|_{(\mathcal{H},D(A(0)))_{\frac{1}{2},p}}.$$

On the other hand, since $A(0)$ is invertible, it is well-known that $t \to A(0)e^{-tA(0)}u_0 \in L^p_\beta(0,\tau;\mathcal{H})$ if and only if $u_0 \in (\mathcal{H}, D(A(0)))_{\theta,p}$ (see e.g. [25] [Proposition 5.1.1]) and

$$\int_0^\infty \|A(0)e^{-tA(0)}u_0\|^p t^\beta dt \le \|u_0\|^p_{(\mathcal{H},D(A(0)))_{\theta,p}}. \tag{3.2}$$

Then for $2 < p \neq 2(\beta+1)$ (or $p = 2(\beta+1)$ we get by taking $\varepsilon < \frac{1}{2} - \frac{1}{p}$),

$$\|Fu_0\|_{L^p_\beta(0,\tau;\mathcal{H})} \le C\Big[\Big(\int_0^\tau \frac{\|\mathcal{A}(t)-\mathcal{A}(0)\|^p_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t^{\frac{p}{2}}}\, dt\Big)^{\frac{1}{p}} + 1\Big]\|u_0\|_{(\mathcal{H},D(A(0)))_{\theta,p}} < \infty.$$

□

The next lemma shows the quadratic estimate for $A(t)$ with constant independent of t. This lemma was proved in [5] or in [2]. Quadratic estimates are an important tool in harmonic analysis and we will use them at several places in the proofs of maximal regularity.

Lemma 3.5. *Let $x \in \mathcal{H}$ and $t \in [0,\tau]$. We have*

$$\int_0^\tau \|A(t)^{\frac{1}{2}} e^{-sA(t)}x\|^2\, ds \le c\|x\|^2, \tag{3.3}$$

where c is a positive constant independent of t.

4 Maximal regularity for autonomous problems

In this section we are interested in the regularity of the following problem

$$\begin{cases} u'(t) + \mathcal{A}(0)u(t) = f(t), \ t\text{-a.e.} \\ \\ u(0) = u_0. \end{cases} \tag{4.1}$$

Theorem 4.1. *Let $f \in L^p_\beta(0, \tau, \mathcal{H}), p \in (1, \infty)$ and $u_0 \in (\mathcal{H}; D(A(0)))_{\theta,p}$ for $-1 < \beta < p - 1$. Here, $\theta = \frac{p-1-\beta}{p}$. Then (**??**) has maximal L^p_β-regularity in $\mathcal{H}$. Moreover, $Tr(p, \beta) = (\mathcal{H}; D(A(0)))_{\theta,p}, u \in C([0, \tau]; (\mathcal{H}; D(A(0)))_{\theta,p})$ and*

$$\|u\|_{W^{1,p}_\beta(0,\tau;\mathcal{H})\cap C([0,\tau];(\mathcal{H};D(A(0)))_{\theta,p})} + \|A(0)u\|_{L^p_\beta(0,\tau;\mathcal{H})} \le C\Big[\|u_0\|_{(\mathcal{H};D(A(0)))_{\theta,p}} + \|f\|_{L^p_\beta(0,\tau;\mathcal{H})}\Big],$$

where C is a positive constant independent of u_0 and f.
For $p - 1 \le \beta$, maximal L^p_β−regularity may fails.

As mentioned in the introduction, this theorem was proved by Prüss and Simonett [29] but they only consider the case $0 \le \beta < p - 1$.

Proof. Since $A(0)$ is a generator of an analytic semigroup in $\mathcal{H}$, it is well known that by the variation of constants formula the solution of problem (4.1) is given by

$$u(t) = e^{-tA(0)}u_0 + \int_0^t e^{-(t-s)A(0)} f(s)\, ds.$$

Thus,

$$\begin{aligned} A(0)u(t) &= A(0)e^{-tA(0)}u_0 + A(0)\int_0^t e^{-(t-s)A(0)} f(s)\, ds \\ &:= (F_1u_0)(t) + (Lf)(t). \end{aligned}$$

Hence, to prove that $u \in W^{1,p}_\beta((0, \tau; \mathcal{H}) \cap L^p_\beta(0, \tau; D(A(0)))$ it is enough to show that Lf and F_1u_0 are bounded in $L^p_\beta(0, \tau, \mathcal{H})$. Lemma 3.1 shows that $L \in \mathcal{L}(L^p_\beta(0, \tau, \mathcal{H}))$ and $F_1u_0 \in L^p_\beta(0, \tau, \mathcal{H})$ by (3.2).
Note that for $u_0 = 0$ and $\beta \ge p-1$ we have $A(0)u = Lf$. Using the example in Proposition 3.2, we get that $A(0)u \notin L^p_\beta(0, \tau, \mathcal{H})$. Therefore maximal L^p_β-regularity may fails in the case $\beta \ge p - 1$.
Next, we prove that $u(s) \in (\mathcal{H}; D(A(0)))_{\theta,p}$ for all $s \in [0, \tau]$, where $u \in$

$W^{1,p}_\beta(0,\tau,\mathcal{H}) \cap L^p_\beta(0,\tau,D(A(0)))$. First we consider the case $s = 0$. We have

$$\begin{aligned}
\|u(0)\|^p_{(\mathcal{H};D(A(0)))_{\theta,p}} &= \int_0^1 \|t^{\frac{\beta}{p}} A(0)e^{-tA(0)}u(0)\|^p\,dt + \|u(0)\|^p \\
&\lesssim \int_0^1 \|t^{\frac{\beta}{p}} A(0)e^{-tA(0)}(u(0)-u(t))\|^p\,dt + \|u(0)\|^p \\
&+ \int_0^1 \|t^{\frac{\beta}{p}} A(0)e^{-tA(0)}u(t)\|^p\,dt \\
&\lesssim \int_0^1 t^\beta (\frac{1}{t}\int_0^t \|\dot{u}(l)\|\,ds)^p\,dl \\
&+ \int_0^\tau t^\beta \|A(0)u(t)\|^p\,dt + \|u(0)\|^2 \\
&\lesssim \|\dot{u}\|^p_{L^p_\beta(0,\tau;\mathcal{H})} + \|A(.)u\|^p_{L^p_\beta(0,\tau;\mathcal{H})} + \|u(0)\|^2. \qquad (4.2)
\end{aligned}$$

Now, we prove the result for all $s \in]0,\tau]$. Indeed, let $t \in]0,\tau[$ and set

$$v(t) := \begin{cases} u(t+s), & t \in [0,\tau-s]. \\ u(\frac{\tau}{s}(\tau-t)), & t \in [\tau-s,\tau]. \end{cases}$$

Since $v \in W^{1,p}_\beta(0,\tau,\mathcal{H}) \cap L^p_\beta(0,\tau,D(A(0)))$,

$$v(0) = u(s) \in (\mathcal{H};D(A(0)))_{\theta,p}.$$

For the case $s = \tau$, we take $v(t) = u(\tau - t)$. This shows that $u(s) \in (\mathcal{H};D(A(0)))_{\theta,p}$, for all $s \in [0,\tau]$.
For $0 \le s \le l \le t \le \tau$, we set $v(l) = e^{-(t-l)A(0)}u(l)$. This yields

$$\begin{aligned}
u(t) - u(s) &= v(s) - u(s) + \int_s^t v'(l)\,dl \\
&= (e^{-(t-s)A(0)} - I)u(s) + \int_s^t e^{-(t-l)A(0)} f(l)\,dl. \qquad (4.3)
\end{aligned}$$

Observe that $e^{-(t-s)A(0)}$ is strongly continuous on $(\mathcal{H};D(A(0)))_{\theta,p}$. In particular, this ensures that

$$\|(e^{-(t-s)A(0)} - I)u(s)\|_{(\mathcal{H};D(A(0)))_{\theta,p}} \to 0, \text{ as } t \to s.$$

The estimate (4.2) for the case $u_0 = 0$ gives that

$$\|\int_s^t e^{-(t-l)A(0)} f(l)\,dl\|_{(\mathcal{H};D(A(0)))_{\theta,p}} \lesssim \|f\|_{L^p_\beta(s,t;\mathcal{H})}.$$

It follows that u is right continuous on $(\mathcal{H};D(A(0)))_{\theta,p}$.
Now, set $v(l) = e^{-(l-s)A(0)}u(r)$, for $0 \le s \le l \le t$.
Then

$$\begin{aligned}
u(s) - u(t) &= v(t) - u(t) - \int_s^t v'(l)\,dl \\
&= (e^{-(t-s)A(0)} - I)u(t) - \int_s^t e^{-(l-s)A(0)}(f(l) - 2A(0)u(l))\,dl.
\end{aligned}$$

The same argument shows that u is left continuous on $(\mathcal{H}; D(A(0)))_{\theta,p}$. Thus, $u \in C([0,\tau]; (\mathcal{H}; D(A(0)))_{\theta,p})$. This completes the proof. $\square$

5 Maximal regularity for non-autonomous problems.

In this section we focus with non-autonomous maximal L^p_β-regularity for $p > 2$.
We start by recalling the definition of vector-valued Besov space.

Definition 5.1. *Let X be a Banach space, $p, q \in [1, \infty]$ and $\alpha \in (0,1)$. A Bochner measurable function $f : [0,\tau] \to X$, is in the homogeneous Besov space $\dot{B}^{\alpha,p}_q(0,\tau;X)$ if*

$$\|f\|^q_{\dot{B}^{\alpha,p}_q(0,\tau;X)} := \int_0^\tau (\int_0^t \frac{\|f(t)-f(s)\|^p_X}{|t-s|^{p\alpha+\frac{p}{q}}}\,ds)^{\frac{q}{p}}\,dt < \infty.$$

A function $f \in L^p(0,\tau;X)$ is in the Besov space $B^{\alpha,p}_q(0,\tau;X)$ if

$$\|f\|^q_{B^{\alpha,p}_q(0,\tau;X)} := \|f\|^q_{L^p(0,\tau;X)} + \|f\|^q_{\dot{B}^{\alpha,p}_q(0,\tau;X)} < \infty.$$

We refer the reader to section 6 for more details about this spaces type.

Maximal L^p_β-regularity may fail even for ordinary non-autonomous equation, letting $\mathcal{H} = \mathbb{R}$.

Example 5.2. *Consider $\phi(t) = t^{-\frac{\beta+1}{p}}$ and $p \in (1,\infty), \beta \in (-1, p-1)$. Then $\phi \in L^q_\beta(0,\frac{1}{2})$ for all $q \in (1,p[$. Set $a(t) := |\phi(t-\frac{1}{4})|$. Consequently, $a \in L^q_\beta(0,\frac{1}{2})$ for all $q \in (1,p[$ but $a \notin L^p_\beta(0,\frac{1}{2})$.*
Consider now the ordinary non-autonomous equation

$$u'(t) + a(t)u(t) = 1,\ u(0) = 0.$$

Then by variation of constants formula, $u(t) = \int_0^t e^{-\int_t^s a(r)\,dr}\,ds$. Since $a(r) \geq 0$,

$$\begin{aligned} |a(t)u(t)| &= a(t)\int_0^t e^{-\int_t^s a(r)\,dr}\,ds \\ &\geq a(t)\int_0^t e^{-\int_0^1 a(r)\,dr}\,ds \\ &= e^{-\int_0^1 a(r)\,dr}\,ta(t). \end{aligned}$$

Therefore, $\|a(.)u\|_{L^p_\beta(0,\frac{1}{2})} \geq C\|t \mapsto ta(t)\|_{L^p_\beta(0,\frac{1}{2})} = \infty$.
On the other hand, if we replace the constant function 1 by $f \in L^q_\beta(0,\frac{1}{2})$ we

infer

$$\begin{aligned}|a(t)u(t)| &= a(t)\left|\int_0^t e^{-\int_t^s a(r)\,dr} f(s)\,ds\right| \\ &\leq a(t)\int_0^t |f(s)|\,ds \\ &\leq a(t)t^{-\frac{\beta}{q}+\frac{1}{q'}}\|f\|_{L^q_\beta(0,\frac{1}{2})} \\ &\leq a(t)\|f\|_{L^q_\beta(0,\frac{1}{2})}.\end{aligned}$$

This shows that maximal L^q_β−regularity holds for all $q \in]1+\beta, p[$. Notice however, this example is not a counterexample to the questions we raise, since our standing hypothesis [H2] is not satisfied here.

It is known that $-A(t)$ is sectorial operator and generates a bounded holomorphic semigroup on $\mathcal{H}$. The same is true for $-\mathcal{A}(t)$ on $\mathcal{V}'$. From [27] [Theorem 1.52 and Theorem 1.55], we have the following lemma which points out that the constants involved in the estimates are uniform with respect to t.

Lemma 5.3. *Let $\gamma = \frac{\pi}{2} - \arctan(\frac{M}{\delta})$ and $t \in [0,\tau]$. For all $\theta > \frac{1}{2}$ and $p \in (1,\infty)$ we infer that $(\mathcal{H}, D(A(t)))_{\theta;p} \hookrightarrow \mathcal{V}$. In the case $\theta < \frac{1}{2}$, we have $(\mathcal{H}, D(A(t)))_{\theta;p} = (\mathcal{H},\mathcal{V})_{2\theta;p}$. Moreover,*

(1) $\|e^{-tA(t)}\|_{\mathcal{L}((\mathcal{H},D(A(t)))_{\theta;p},\mathcal{V})} \lesssim \frac{1}{t^{\frac{1}{2}-\theta}}$.

(2) $\|(\lambda + A(t))^{-1}\|_{\mathcal{L}((\mathcal{H},D(A(t)))_{\theta;p},\mathcal{V})} \lesssim \frac{1}{|\lambda|^{\frac{1}{2}+\theta}}$, where $\lambda \in \Sigma_{\pi-\alpha}$ and $\alpha < \gamma$.

Proof. Applying [10][Theorem 4.6.1], we get for all $\theta \in [0, \frac{1}{2}[$

$$(\mathcal{H}, D(A(t)))_{\theta;2} = (\mathcal{H},\mathcal{V})_{2\theta;2} = [\mathcal{H},\mathcal{V}]_{2\theta}. \tag{5.1}$$

The reiteration theorem for the real method [32][1.10.3, Theorem 2] or the property of power of positive operator [25][Theorem 4.3.11] shows that,

$$(\mathcal{H}, D(A(t)))_{\theta,p} = (\mathcal{H}, [\mathcal{H}, D(A(t))]_{\frac{\sqrt{2\theta}}{2}})_{\sqrt{2\theta},p}. \tag{5.2}$$

Combining (5.1) and (5.2), we obtain

$$\begin{aligned}(\mathcal{H}, D(A(t)))_{\theta,p} &= (\mathcal{H}, [\mathcal{H},\mathcal{V}]_{\sqrt{2\theta}})_{\sqrt{2\theta},p} \\ &= (\mathcal{H},\mathcal{V})_{2\theta,p}.\end{aligned}$$

Let $\theta > \frac{1}{2}$ and $p \in (1,\infty)$. [5] [Lemma 3.15] shows that for all $\gamma \in (0, \frac{1}{2})$

$$(\mathcal{H}, D(A(0)))_{\frac{1}{2}+\gamma;2} \hookrightarrow \mathcal{V}.$$

Hence, for any $\varepsilon < (\theta - \frac{1}{2})$,

$$(\mathcal{H}, D(A(0)))_{\theta;p} = (\mathcal{H}, D(A(0)))_{\frac{1}{2}+(\theta-\frac{1}{2});p} \hookrightarrow (\mathcal{H}, D(A(0)))_{\frac{1}{2}+(\theta-\frac{1}{2})-\varepsilon;2} \hookrightarrow \mathcal{V}.$$

Due to [4] [Lemma 3.1], we obtain $\|e^{-tA(t)}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} \lesssim \frac{1}{t^{\frac{1}{2}}}$ for all $t > 0$ and $\|e^{-tA(t)}\|_{\mathcal{L}(\mathcal{V})} \lesssim 1$. So an interpolation argument gives for all $\theta \in [0, \frac{1}{2}[$,

$$\|e^{-tA(t)}\|_{\mathcal{L}((\mathcal{H},D(A(t)))_{\theta;p},\mathcal{V})} \lesssim \frac{1}{t^{\frac{1}{2}-\theta}}.$$

We prove the estimate (2) similarly. □

In the following result, $p > 2$ and $\mathcal{A} \in C^{\varepsilon}([0,\tau]; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$.

Proposition 5.4. *Let $f \in L^p_\beta(0,\tau;\mathcal{H})$ and $u_0 \in (\mathcal{H}; D(A(0)))_{\theta,p}$ where $\theta = \frac{p-1-\beta}{p}$. There exists a unique u such that $u \in L^\infty(0,\tau;\mathcal{V})$ if $2(1+\beta) < p$ and $u \in L^\infty_\alpha(0,\tau;\mathcal{V})$ otherwise, where $\alpha = \frac{\beta}{p} - \frac{1}{2} + \frac{1}{p}$ and u satisfies*

$$\begin{cases} u'(t) + \mathcal{A}(t)u(t) = f(t), \ t\text{-a.e.} \\ u(0) = u_0. \end{cases} \tag{P}$$

Proof. Let $f \in L^p_\beta(0,\tau;\mathcal{H})$ and $u_0 \in (\mathcal{H}; D(A(0)))_{\theta,p}$. Set $v(s) = e^{-(t-s)A(t)}u(s)$. Since $u(t) = e^{-tA(t)}u_0 + \int_0^t v'(s)\,ds$,

$$\begin{aligned} u(t) &= e^{-tA(t)}u_0 + \int_0^t e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s)\,ds + \int_0^t e^{-(t-s)A(t)}f(s)\,ds \\ &:= (Mu_0)(t) + (M_1u)(t) + (L_1f)(t). \end{aligned} \tag{5.3}$$

We consider two cases. The first one is when $p \leq 2(\beta+1)$ and the second is $2(\beta+1) < p$. For $2(\beta+1) < p$ we have by Lemma 2.4 that $L^p_\beta(0,\tau;\mathcal{H}) \hookrightarrow L^2(0,\tau;\mathcal{H})$ and due to Lemma 5.3, $(\mathcal{H}; D(A(0)))_{\theta,p} \hookrightarrow \mathcal{V}$. Hence, a direct application of [4][Proposition 4.5] gives $u \in L^\infty(0,\tau;\mathcal{V})$. Now, we consider the first case. Using Lemmas 3.3, 5.3, we get that $t^\alpha(Mu_0)(t)$ and $t^\alpha(L_1f)(t)$ are bounded in $\mathcal{V}$ for all $t \in [0,\tau]$. Now, we show that $M_1u \in L^\infty_\alpha(0,\tau;\mathcal{V})$, for all $u \in L^\infty_\alpha(0,\tau;\mathcal{V})$.
We write

$$\begin{aligned} (M_1u)(t) &= \int_0^{\frac{t}{2}} e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s)\,ds \\ &\quad + \int_{\frac{t}{2}}^t e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s)\,ds \\ &:= (M_{11}u)(t) + (M_{12}u)(t). \end{aligned}$$

Taking $x \in \mathcal{V}'$, we obtain by Cauchy-schwarz inequality and the analyticity of the semigroup $s \mapsto e^{-sA(t)^*}$,

$$
\begin{aligned}
&|((M_{12}u)(t), x)_{\mathcal{V}' \times \mathcal{V}}| \\
&= |\int_{\frac{t}{2}}^{t} (e^{-\frac{(t-s)}{2}A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s), A(t)^{*\frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)^*} A(t)^{*-\frac{1}{2}} x)\, ds| \\
&\le (\int_{\frac{t}{2}}^{t} \|e^{-\frac{(t-s)}{2}A(t)}\|^2_{\mathcal{L}(\mathcal{V}',\mathcal{H})} \|(\mathcal{A}(t) - \mathcal{A}(s))u(s)\|^2_{\mathcal{V}'}\, ds)^{\frac{1}{2}} \\
&\times (\int_{\frac{t}{2}}^{t} \|A(t)^{*\frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)^*} A(t)^{*-\frac{1}{2}} x\|^2\, ds)^{\frac{1}{2}} \\
&\lesssim (\int_{\frac{t}{2}}^{t} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t-s} s^{-2\alpha}\, ds)^{\frac{1}{2}} \|u\|_{L^\infty_\alpha(0,\tau;\mathcal{V})}.
\end{aligned}
$$

Hence,

$$t^\alpha \|(M_{12}u)(t)\|_{\mathcal{V}} \lesssim t^\varepsilon \|\mathcal{A}(.)\|_{C^\varepsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|u\|_{L^\infty_\alpha(0,\tau;\mathcal{V})}.$$

Now, we estimate the norm of $(M_{11}v)(t)$ in $\mathcal{V}$ as follows

$$
\begin{aligned}
&t^\alpha \|(M_{11}v)(t)\|_{\mathcal{V}} \\
&\lesssim t^\alpha \int_0^{\frac{t}{2}} \|e^{-\frac{(t-s)}{2}A(t)}\|_{\mathcal{L}(\mathcal{V}',\mathcal{V})} \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} s^{-\alpha}\, ds \|s \mapsto s^\alpha u(s)\|_{L^\infty(0,\frac{t}{2};\mathcal{V})} \\
&\lesssim t^\alpha \int_0^{\frac{t}{2}} \frac{s^{-\alpha}}{(t-s)^{1-\varepsilon}}\, ds \sup_{s\in[0,\frac{t}{2}]} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{(t-s)^\varepsilon} \|s \mapsto s^\alpha u(s)\|_{L^\infty(0,\frac{t}{2};\mathcal{V})}.
\end{aligned}
$$

Note that

$$t^\alpha \int_0^{\frac{t}{2}} \frac{s^{-\alpha}}{(t-s)^{1-\varepsilon}}\, ds = t^\varepsilon \int_0^{\frac{1}{2}} \frac{l^{-\alpha}}{(1-l)^{1-\varepsilon}}\, dl.$$

Therefore

$$
\begin{aligned}
&t^\alpha \|(M_{11}v)(t)\|_{\mathcal{V}} \\
&\lesssim t^\epsilon \|\mathcal{A}\|_{C^\epsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|u\|_{L^\infty_\alpha(0,\frac{t}{2};\mathcal{V})}.
\end{aligned}
$$

Choosing τ small enough, $M_1 \in \mathcal{L}(L^\infty_\alpha(0,\tau;\mathcal{V}))$, with norm $\|M_1\|_{\mathcal{L}(L^\infty_\alpha(0,\tau;\mathcal{V}))} < 1$. Therefore, $(I - M_1)$ is invertible in $L^\infty_\alpha(0,\tau;\mathcal{V})$. Hence,

$$u = (I - M_1)^{-1}(Mu_0 + L_1 f) \in L^\infty_\alpha(0,\tau;\mathcal{V}).$$

For arbitrary τ we proceed analogously as [4][Proposition 4.5]. We split $[0,\tau]$ into a finite number of subintervals with small sizes and proceed exactly as in the previous proof. Finally, we stick the solutions of each interval to get the desired result. This finishes the proof. □

For simplicity of notation, we set

$$E := B_p^{1-\frac{1}{p},2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}')).$$

The following is our main result in this section.

Theorem 5.5. *Assume that $\mathcal{A}(.) \in E$. Then for all $f \in L^p_\beta(0,\tau;\mathcal{H})$ and $u_0 \in (\mathcal{H}; D(A(0)))_{\theta,p}$ with $p > 2$, there exists a unique $u \in W^{1,p}_\beta(0,\tau;\mathcal{H})$ such that $A(.)u \in L^p_\beta(0,\tau;\mathcal{H})$ be the solution to (P).*

As mentioned in the introduction, since by Propositions 7.1, 7.2, $W^{\frac{1}{2}+\varepsilon,p} \subset B_p^{1-\frac{1}{p},2} \subset W^{\frac{1}{2},p}$, then the regularity assumption in time for $\mathcal{A}(.)$ (or the forms $\mathfrak{a}(.)$) is minimal and our results are the most general ones on this problem. Notice however, $B_p^{1-\frac{1}{p},2} \subset W^{\frac{1}{2},2}$ by Proposition 7.2. Hence, an important case left open is that of $\mathcal{A}(.) \in W^{\frac{1}{2},2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))$, which we are not able to answer at the moment.

Proof. For $0 \le s \le t \le \tau$, we set $v(s) = e^{-(t-s)A(t)}u(s)$. We remark that $v(t) = u(t), v(0) = e^{-tA(t)}u(0)$ and

$$v'(s) = e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s) + e^{-(t-s)A(t)}f(s).$$

Since $v(t) = v(0) + \int_0^t v'(s)\,ds$,

$$u(t) = e^{-tA(t)}u(0) + \int_0^t e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s)\,ds + \int_0^t e^{-(t-s)A(t)}f(s)\,ds.$$

Therefore

$$\begin{aligned} A(t)u(t) &= A(t)e^{-tA(t)}u(0) + A(t)\int_0^t e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s)ds \\ &\quad + A(t)\int_0^t e^{-(t-s)A(t)}f(s)ds \\ &= (Ru_0)(t) + (Su)(t) + (Lf)(t). \end{aligned}$$

We shall prove that $A(.)u \in L^p_\beta(0,\tau;\mathcal{H})$. Noting that $L \in \mathcal{L}(L^p_\beta(0,\tau;\mathcal{H}))$ by Lemma 3.1 and $Ru_0 \in L^p_\beta(0,\tau;\mathcal{H})$ thanks to Lemmas 3.4, 7.3. Then it remains only to prove that $Su \in L^p_\beta(0,\tau;\mathcal{H})$.
Consider the case $2(\beta+1) < p$. Applying Proposition 5.4 we get that $u \in L^\infty(0,\tau;\mathcal{V})$ is the unique solution to (P). Take $g \in L^{p'}(0,\tau;\mathcal{H})$, where

$p' = \frac{p}{p-1}$ is the conjugate of p. We have

$$
\begin{aligned}
&|(.^{\frac{\beta}{p}} Su, g)_{L^2(0,\tau;\mathcal{H})}| \\
&\leq \tau^{\frac{\beta}{p}} \int_0^\tau |((Su)(t), g(t))|\, dt \\
&= \tau^{\frac{\beta}{p}} \int_0^\tau |\int_0^t \langle (\mathcal{A}(t) - \mathcal{A}(s))u(s), A(t)^* e^{-(t-s)A(t)^*} g(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\, ds|\, dt \\
&= \tau^{\frac{\beta}{p}} \int_0^\tau |\int_0^t |\langle (\mathcal{A}(t) - \mathcal{A}(s))u(s), A(t)^* e^{-(t-s)A(t)^*} g(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\, ds|\, dt \\
&\lesssim \int_0^\tau \int_0^t \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \|e^{-\frac{(t-s)}{2}A(t)^*}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} \\
&\times \|A(t)^{*\frac{1}{2}} e^{-\frac{(t-s)}{4}A(t)^*}\|_{\mathcal{L}(\mathcal{H})} \|A(t)^{*\frac{1}{2}} e^{-\frac{(t-s)}{4}A(t)^*} g(t)\|\, ds\, dt \|u\|_{L^\infty(0,\tau;\mathcal{V})}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&|(.^{\frac{\beta}{p}} Su, g)_{L^2(0,\tau;\mathcal{H})}| \\
&\lesssim_{(a)} \int_0^\tau \int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t-s} \|A(t)^{*\frac{1}{2}} e^{-\frac{(t-s)}{4}A(t)^*} g(t)\|\, ds\, dt \|u\|_{L^\infty(0,\tau;\mathcal{V})} \\
&\leq_{(b)} \int_0^\tau (\int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{(t-s)^2}\, ds)^{\frac{1}{2}} \\
&\times (\int_0^t \|A(t)^{\frac{1}{2}*} e^{-(t-s)A(t)^*} g(t)\|^2\, ds)^{\frac{1}{2}}\, dt \|u\|_{L^\infty(0,\tau;\mathcal{V})} \\
&\lesssim_{(c)} \int_0^\tau (\int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{(t-s)^2}\, ds)^{\frac{1}{2}} \\
&\times \|g(t)\|\, dt \|u\|_{L^\infty(0,\tau;\mathcal{V})} \\
&\lesssim_{(d)} (\int_0^\tau (\int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{(t-s)^2}\, ds)^{\frac{p}{2}}\, dt)^{\frac{1}{p}} \|u\|_{L^\infty(0,\tau;\mathcal{V})} \|g\|_{L^{p'}(0,\tau;\mathcal{H})} \\
&= \|\mathcal{A}(.)\|_E \|u\|_{L^\infty(0,\tau;\mathcal{V})} \|g\|_{L^{p'}(0,\tau;\mathcal{H})},
\end{aligned}
$$

where we used in (a) the analyticity of the semigroup $s \mapsto e^{-sA(t)^*}$, Lemma 3.5 in (c) and Hölder's inequality in (b) and (d). Therefore, $Su \in L^p_\beta(0,\tau;\mathcal{H})$ and hence $A(.)u \in L^p_\beta(0,\tau;\mathcal{H})$.
Now, consider the case $p \leq 2(1+\beta)$. Due to Proposition 5.4, $u \in L^\infty_\alpha(0,\tau;\mathcal{V})$.

For $g \in L^{p'}(0,\tau;\mathcal{H})$ we infer that

$$\begin{aligned}
&|(.^{\frac{\beta}{p}} Su, g)_{L^2(0,\tau;\mathcal{H})}| \\
&= |\int_0^\tau t^{\frac{\beta}{p}} \int_0^t \langle (\mathcal{A}(t)-\mathcal{A}(s))u(s), A(t)^* e^{-(t-s)A(t)^*} g(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\, ds\, dt| \\
&\leq |\int_0^\tau t^{\frac{\beta}{p}} \int_0^{\frac{t}{2}} \langle (\mathcal{A}(t)-\mathcal{A}(s))u(s), A(t)^* e^{-(t-s)A(t)^*} g(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\, ds\, dt| \\
&+ |\int_0^\tau t^{\frac{\beta}{p}} \int_{\frac{t}{2}}^t \langle (\mathcal{A}(t)-\mathcal{A}(s))u(s), A(t)^* e^{-(t-s)A(t)^*} g(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\, ds\, dt| \\
&:= I_1 + I_2.
\end{aligned}$$

For I_2 we find similarly to (**??**)

$$I_2 \lesssim \|\mathcal{A}(.)\|_E \|u\|_{L^\infty_\alpha(0,\tau;\mathcal{V})} \|g\|_{L^{p'}(0,\tau;\mathcal{H})}.$$

Concerning I_1,

$$\begin{aligned}
I_1 &\lesssim \int_0^\tau t^{\frac{\beta}{p}} \int_0^{\frac{t}{2}} \frac{s^{-\alpha}}{(t-s)^{\frac{3}{2}-\varepsilon}} \|g(t)\|\, ds\, dt \\
&\times \|\mathcal{A}\|_{C^\varepsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|s \mapsto s^\alpha u(s)\|_{L^\infty(0,\tau;\mathcal{V})} \\
&\lesssim \int_0^\tau t^{\frac{\beta}{p}-\frac{3}{2}+\varepsilon} \int_0^{\frac{t}{2}} s^{-\alpha}\, ds \|g(t)\|\, dt \|\mathcal{A}\|_{C^\varepsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|s \mapsto s^\alpha u(s)\|_{L^\infty(0,\tau;\mathcal{V})} \\
&= \int_0^\tau t^{-\frac{1}{p}+\varepsilon} \|g(t)\|\, dt \|\mathcal{A}\|_{C^\varepsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|u\|_{L^\infty_\alpha(0,\tau;\mathcal{V})} \\
&\lesssim \|g\|_{L^{p'}(0,\tau,\mathcal{H})} \|\mathcal{A}\|_{C^\varepsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|u\|_{L^\infty_\alpha(0,\tau;\mathcal{V})}.
\end{aligned}$$

Combining theses estimates we get that $Su \in L^p_\beta(0,\tau;\mathcal{H})$ and so $A(.)u \in L^p_\beta(0,\tau;\mathcal{H})$. Since $u' = f - A(.)u$ then $u \in W^{1,p}_\beta(0,\tau;\mathcal{H})$ and this completes the proof. □

Proposition 5.6. *Assume that $\mathcal{A}(.) \in E$. Then for all $g \in L^p(0,\tau;\mathcal{H}), p > 2$ and $-1 < \beta < p-1$ there exists a unique $v \in W^{1,p}(0,\tau;\mathcal{H})$ such that $A(.)v \in L^p(0,\tau;\mathcal{H})$ be the solution of the singular equation*

$$\begin{cases} v'(t) + \mathcal{A}(t)v(t) + \frac{\beta}{p}\frac{v(t)}{t} = g(t),\ t\text{-}a.e. \\ \\ v(0) = 0. \end{cases} \tag{5.4}$$

Proof. We set $f(t) = t^{\frac{\beta}{p}} g(t)$ with $t \in [0,\tau]$, so $f \in L^p_\beta(0,\tau;\mathcal{H})$. Let $u \in W^{1,p}_{\beta,0}(0,\tau;\mathcal{H})$ be the unique solution to the problem

$$\begin{cases} u'(t) + \mathcal{A}(t)u(t) = f(t),\ t\text{-a.e.} \\ \\ u(0) = 0. \end{cases} \tag{5.5}$$

Now, set $v = t^{-\frac{\beta}{p}} u$. Then $v \in W^{1,p}(0,\tau;\mathcal{H})$ and v is the unique solution to problem (5.4). □

6 Applications

This section is devoted to some applications of the results given in the previous sections. We give examples illustrating the theory without seeking for generality. Here we study maximal L^p_β-regularity for $p > 2$ and $-1 < \beta < p-1$.

– ***Elliptic operators on*** $\mathbb{R}^n$***.***

Let $\mathcal{H} := L^2(\mathbb{R}^n)$ and $\mathcal{V} := H^1(\mathbb{R}^n)$ and define the sesquilinear forms

$$\mathfrak{a}(t,u,v) := \sum_{k,l=1}^{n} \int_{\mathbb{R}^n} c_{kl}(t,x) \partial_k u \, \overline{\partial_l v} \, dx, \quad u, v \in \mathcal{V}.$$

We define the operator $P(t) \in \mathcal{L}(\mathcal{V},\mathcal{H})$ by $P(t)u := \sum_{j=1}^n b_j(t)\partial_j u$, where $u \in \mathcal{V}$ and $t \in [0,\tau]$.

We assume that the matrix $C(t,x) = (c_{kl}(t,x))_{1\le k,l \le n}$ satisfies the usual ellipticity condition. Next we assume that $C \in B_p^{1-\frac{1}{p},2}(0,\tau; L^\infty(\mathbb{C}^{n^2}))$ and $b_j \in L^p(0,\tau; L^\infty(\mathbb{R}^n))$ where $j \in \{1, ..., n\}$. We note that

$$\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \le M' \|C(t,.) - C(s,.)\|_{L^\infty(\mathbb{C}^{n^2})}$$

for some constant M'. This implies that $\mathcal{A} \in B_p^{1-\frac{1}{p},2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))$.

We are now allowed to apply Theorem 5.5. We obtain maximal L^p_β-regularity and apriori estimate for the parabolic problem

$$\begin{cases} u'(t) + A(t)u(t) + P(t)u(t) = f(t) \\ u(0) = 0. \end{cases}$$

That is, for every $f \in L^p_\beta(0,\tau; L^2(\mathbb{R}^n))$ there is unique solution

$$u \in W^{1,p}_\beta(0,\tau; L^2(\mathbb{R}^n)) \cap L^p_\beta(0,\tau; H^1(\mathbb{R}^n)).$$

– ***Schrödinger operators with time-dependent potentials.***

Let $0 \le m_0 \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $m : [0,\tau] \times \mathbb{R}^n \to \mathbb{R}$ be a measurable function for which there exist positive constants α_1, α_2 and M such that for a.e. x and all $t \in [0,\tau]$

$$\alpha_1 m_0(x) \le m(t,x) \le \alpha_2 m_0(x).$$

We define the form

$$\mathfrak{a}(t,u,v) := \int_{\mathbb{R}^n} \nabla u \nabla v dx + \int_{\mathbb{R}^n} m(t,x) uv \, dx,$$

with domain

$$\mathcal{V} := \{u \in H^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} m_0(x)|u|^2 dx < \infty\}.$$

It is clear that $\mathcal{V}$ is a Hilbert space for the norm $\|u\|_{\mathcal{V}}$ given by

$$\|u\|_{\mathcal{V}}^2 = \int_{\mathbb{R}^n} |\nabla u|^2 \, dx + \int_{\mathbb{R}^n} m_0(x)|u|^2 \, dx.$$

In addition, $\mathfrak{a}$ is $\mathcal{V}$-bounded and coercive. Its associated operator on $L^2(\mathbb{R}^n)$ is formally given by

$$A(t) = -\Delta + m(t,.)$$

with domain

$$D(A(t)) := \{u \in \mathcal{V} : \; -\Delta u + m(t,.)u \in L^2(\mathbb{R}^n)\}.$$

Next we assume that $t \to m(t,.)m_0(.)^{-1} \in B_p^{1-\frac{1}{p},2}(0,\tau; L^\infty(\mathbb{R}^n))$, with $p > 2$.
We have

$$\begin{aligned}
&\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \\
&= \sup_{\|u\|_{\mathcal{V}}=1, \|v\|_{\mathcal{V}}=1} |\mathfrak{a}(t,u,v) - \mathfrak{a}(s,u,v)| \\
&\leq \sup_{\|u\|_{\mathcal{V}}=1, \|v\|_{\mathcal{V}}=1} \int_{\mathbb{R}^n} |m(t,x) - m(s,x)||u||v| dx \\
&\leq \|(m(t,.) - m(s,.))m_0^{-1}(.)\|_{L^\infty(\mathbb{R}^n)} \sup_{\|u\|_{\mathcal{V}}=1, \|v\|_{\mathcal{V}}=1} \int_{\mathbb{R}^n} m_0(x)|u||v| dx \\
&\leq \|(m(t,.) - m(s,.))m_0^{-1}(.)\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}$$

Then we get $\mathcal{A} \in B_p^{1-\frac{1}{p},2}(0,\tau; \mathcal{L}(\mathcal{V},\mathcal{V}'))$.
Given $f \in L^p_\beta(0,\tau; L^2(\mathbb{R}^n))$, we apply Theorem 5.5 and obtain a unique solution $u \in W^{1,p}_\beta(0,\tau; L^2(\mathbb{R}^n)) \cap L^p_\beta(0,\tau; \mathcal{V})$ of the evolution equation

$$\begin{cases} u'(t) - \Delta u(t) + m(t,.)u(t) = f(t) \text{ a.e.} \\ u(0) = 0. \end{cases}$$

7 Appendix

Let X be a Banach space and consider $p \in [1, +\infty)$. If $\mathfrak{D}_{\max}$ is the differentiation operator on $L^p(0,\tau; X)$ with maximal domain i.e.

$$\begin{aligned} D(\mathfrak{D}_{\max}) &:= W^{1,p}(0,\tau; X) \\ \mathfrak{D}_{\max} f &:= f', \end{aligned}$$

then we have $(X, D(\mathfrak{D}_{\max}))_{\theta,q} = B_q^{\theta,p}(0,\tau;X) (\theta \in (0,1), 1 \leq q < \infty)$. In particular,

$$(X, D(\mathfrak{D}_{\max}))_{\theta,p} = B_p^{\theta,p}(0,\tau;X) = W^{\theta,p}(0,\tau;X) \ (\theta \in (0,1)).$$

Let $\mathfrak{D}$ be the restriction of $\mathfrak{D}_{\max}$ on $L^p(0,\tau;X)$ to the domain $D(\mathfrak{D}) := \{f \in W^{1,p}(0,\tau;X) : f(0) = 0\}$. Then

$$(X, D(\mathfrak{D}))_{\theta,q} = B_{q,(0)}^{\theta,p}(0,\tau;X) = \{f \in B_q^{\theta,p}(0,\tau;X) : f(0) = 0\},$$

for $\theta > \frac{1}{p}$. For $\theta < \frac{1}{p}$ we get $(X, D(\mathfrak{D}))_{\theta,q} = (X, D(\mathfrak{D}_{\max}))_{\theta,q} = B_q^{\theta,p}(0,\tau;X)$. We recall that the operator $\mathfrak{D}$ is sectorial of angle $\frac{\pi}{2}$, while $\mathfrak{D}_{\max}$ is not sectotrial. In fact $\sigma(\mathfrak{D}_{\max}) = \mathbb{C}$. For the case where $\mathfrak{D}$ is the differentiation operator on $C(0,\tau;X)$ with domain

$$D(\mathfrak{D}) := \{f \in C^1(0,\tau;X) : f(0) = 0\},$$

we have $(X, D(\mathfrak{D}))_{\theta,\infty} = C_{(0)}^{\theta}(0,\tau;X) = \{f \in C^{\theta}(0,\tau;X) : f(0) = 0\}$. For more details and references see [11] [Section 2], [25] [Example 1.9, Exercises 5 and 6, p.18] and [32] [Theorem, p.204].

Proposition 7.1. *Let $p \in [1,\infty)$ and $1 \leq q_1 \leq q_2 < \infty$. We have*

$$B_{q_1}^{\theta,p}(0,\tau;X) \hookrightarrow B_{q_2}^{\theta,p}(0,\tau;X).$$

Moreover, if $\theta > \frac{1}{p}$ we find for all $q \in [1,\infty), 0 < \varepsilon \leq \theta - \frac{1}{p}$

$$B_q^{\theta,p}(0,\tau;X) \hookrightarrow C^{\theta-\frac{1}{p}-\varepsilon}(0,\tau;X).$$

Proof. The first statement follows immediately by the inclusion properties of the real interpolation spaces. Let $\theta > \frac{1}{p}, q \in [1,\infty)$ and $0 < \varepsilon \leq \theta - \frac{1}{p}$. The inclusion properties of the real interpolation spaces ([25] [Proposition 1.1.4]) gives $B_q^{\theta,p}(0,\tau;X) \hookrightarrow B_p^{\theta-\varepsilon,p}(0,\tau;X) = W^{\theta-\varepsilon,p}(0,\tau;X)$. Now we use [31][Theorem 29] to get the desired result. □

Set

$$E := B_p^{1-\frac{1}{p},2}(0,\tau;X).$$

Proposition 7.2. *Let $p > 2$. We have*

$$W^{\frac{1}{2}+\varepsilon,p}(0,\tau;X) \subset E \subset W^{\frac{1}{2},2}(0,\tau;X), \textit{ for all } \varepsilon > 0.$$

Proof. To prove this statement, it is enough to use Hölder's inequality. □

Lemma 7.3. *Let X be a Banach space and $p > 2$. Suppose that $\mathcal{A} \in E$. Then*

$$\sup_{t\in[0,\tau]} \int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_X^p}{(t-s)^{\frac{p}{2}}}\, ds \lesssim \|\mathcal{A}(.)\|_E^p.$$

Proof. Let $p > 2$ and $t \in [0, \tau]$. We set for $l \in [0, t]$, $g^t(l) = [\mathcal{A}(t) - \mathcal{A}(t-l)]$, then

$$\begin{aligned}
\int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_X^p}{(t-s)^{\frac{p}{2}}}\, ds &= \int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(t-l)\|_X^p}{l^{\frac{p}{2}}}\, dl \\
&= \int_0^t \frac{\|g^t(l)\|_X^p}{l^{\frac{p}{2}}}\, dl \\
&= \int_0^t \frac{\| \int_0^l [g^t(l) - g^t(r)]\, dr + \int_0^l g^t(r)\, dr\|_X^p}{l^{\frac{p}{2}+p}}\, dl \\
&\leq \int_0^t \frac{1}{l^{\frac{p}{2}+p}} (\int_0^l \|[g^t(l) - g^t(r)]\|_X\, dr)^p\, dl \\
&+ \int_0^t \frac{1}{l^{\frac{p}{2}+p}} (\int_0^l \|g^t(r)\|_X dr)^p\, dl \\
&\leq \int_0^t \frac{1}{l^p} (\int_0^l \|[g^t(l) - g^t(r)]\|_X^2\, dr)^{\frac{p}{2}}\, dl \\
&+ \int_0^t \frac{1}{l^{\frac{p}{2}+p}} (\int_0^l \|g^t(r)\|_X\, dr)^p\, dl \\
&\leq \int_0^t (\int_0^l \frac{\|[g^t(l) - g^t(r)]\|_X^2}{(l-r)^2}\, dr)^{\frac{p}{2}}\, dl \\
&+ \int_0^t \frac{1}{l^{\frac{p}{2}+p}} (\int_0^l \|g^t(r)\|_X\, dr)^p\, dl.
\end{aligned}$$

For the first inequality we used the inequality of Minkowski.
For the second one and before the last we used Holder's inequality.
Now we use Hardy's inequality, namely

$$\int_0^t \frac{1}{r^{(1+\frac{1}{2})p}} (\int_0^r \|g^t(l)\|_X\, dl)^p\, dr \leq \frac{1}{(1+\frac{1}{2}-\frac{1}{p})^p} \int_0^t \|g^t(l)\|_X^p \frac{dl}{l^{\frac{p}{2}}}$$

to get

$$\sup_{t\in[0,\tau]} \int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_X^p}{(t-s)^{\frac{p}{2}}}\, ds \leq C(p) \|\mathcal{A}(.)\|_E^p,$$

where $C(p) = \frac{(1+\frac{1}{2}-\frac{1}{p})^p}{(1+\frac{1}{2}-\frac{1}{p})^p - 1}$. □

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