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Florentina Nicolau, Witold Respondek, Jean-Pierre Barbot

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Florentina Nicolau*, Witold Respondek †, and Jean-Pierre Barbot ‡

Abstract

For observed mechanical systems, we study the problem of constructing flat inputs that are consistent with the mechanical structure of the systems. Such inputs will be called *mechanical flat inputs*. We show that contrary to flat inputs, that exist (almost) everywhere, in general, for the existence of *mechanical flat inputs*, additional structural conditions are needed. We provide necessary and sufficient conditions for the existence of mechanical inputs for observed mechanical systems with n degrees of freedom, where n is arbitrary, and $n-1$ measurements.

Keywords: Mechanical systems, flatness, flat inputs, mechanical flat inputs.

1 Introduction

In this paper, we consider observed mechanical systems, with n degrees of freedom, of the form

$$\mathcal{M}\colon \begin{cases} \dot{x}^i = v^i \\ \dot{v}^i = -\Gamma^i_{jk}(x)v^jv^k + d^i_j(x)v^j + e^i(x), \quad y = h(x),\, y \in \mathbb{R}^m, \\ \qquad 1 \le i \le n, \end{cases} \tag{1}$$

where $(x,v) = (x^1,\ldots,x^n,v^1,\ldots,v^n)^\top$ are local coordinates on the tangent bundle TX of the configuration manifold X and $y = (h_1(x),\ldots,h_m(x))^\top$ are (everywhere independent) outputs of the system. We use the Einstein summation convention, i.e., any expression containing a repeated index (in general, upper and lower) implies summation over that index up to n (whenever the summation is taken over another indexing set, we use the summation symbol). The expressions $\Gamma^i_{jk}(x)v^jv^k$ correspond to Coriolis and centrifugal terms. The terms $d^i_j(x)v^j$ correspond to forces linear with respect to velocities, like dissipative forces, and $e(x)$ represents an uncontrolled force (that can be potential or not). It is typically possible to control a mechanical system through external forces and *mechanical control systems* form an important class of control systems that has attracted a lot of attention because of their numerous applications. They form a natural bridge between mechanics and control theory and are studied, for instance, in [19, 1, 2, 20]. An important question is how and where controlled external forces should act in order to achieve a desired property for the resulting input-state-output system. A property that is very useful in applications (for instance, for trajectory tracking, constructive controllability or trajectory generation) is that of flatness (see, e.g., [3, 4, 14] and references therein). The problem of

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placing the actuators or the inputs in order to render an observed dynamical system flat has been introduced by [23, 24] who call those inputs *flat inputs*.

Flat inputs are objects dual to flat outputs and their construction can been seen as a dual problem to that of constructing a flat output. One of the motivations to construct a flat input for a given output is that with such an input, the tracking problem for that output can be solved with no need to calculate the zero dynamics [8, 22], but constructing flat inputs may be useful for other problems as well, like parameter identification [21] or private communication [17]. Similarly to the construction of a flat output (that can be seen as a problem of sensors placement in order to achieve flatness of the resulting input-state-output system) for which, from a technological point of view, it is not always possible to place the sensors exactly where we want them, for constructing flat inputs it may also be technologically difficult to place the actuators at the right place. This is often the case for mechanical systems for which the states variables are positions and velocities (denoted, resp., by x^i and v^i), and thus the derivative of a position always equals the corresponding velocity, i.e., we always have $\dot{x}^i = v^i$ and we cannot modify those equations by adding inputs. Therefore from a physical point of view, it is impossible to build an actuator that acts directly on the position.

The problem that we are studying in this paper is the construction of flat inputs that are consistent with the mechanical structure of the original dynamical system. More precisely, given the observed mechanical system $(\mathcal{M}, h)$, we want to find *mechanical* control vector fields $g_1, \dots, g_m$ (or equivalently, to place the actuators or the inputs) such that the mechanical control-affine system

$$\mathcal{M}_c : \begin{cases} \dot{x}^i = v^i \\ \dot{v}^i = -\Gamma^i_{jk}(x)v^j v^k + d^i_j(x)v^j + e^i(x) + \sum_{r=1}^m g^i_r(x)u_r \end{cases} \tag{2}$$

associated to $\mathcal{M}$, is flat with the original measurements $(h_1, \dots, h_m)^\top$ being a flat output. By mechanical control vector fields g_r we mean that they are consistent with the mechanical structure of $\mathcal{M}$, i.e., are of the form $g_r = \sum_{i=1}^n g^i_r(x)\frac{\partial}{\partial v^i}$, $1 \le r \le m$. The inputs u_r multiplying the mechanical control vector fields will be called *mechanical flat inputs,* and the vector fields $g_1, \dots, g_m$ will be called *mechanical flat-input control vector fields.*

The problem of constructing flat inputs consistent with the mechanical structure has been first considered in [24] where they are called physically realizable flat inputs. In [24], it is shown that a necessary condition for the flat inputs to be physically realizable (according to our definition, to be mechanical flat inputs) is that the distribution spanned by their associated flat-input control vector fields is contained in the vertical distribution of the mechanical system. A related question has then been studied in [7], where the output expression is redesigned, by allowing affine injections of the input and its derivatives, in order to compute physically realizable flat inputs for observable implicit nonlinear systems. In this paper, we work with outputs depending on positions only and we do not modify them to achieve flatness for the resulting mechanical control system. We notice first that if n (independent) functions $h_i(x)$, $1 \le i \le m = n$, are observed (as many as the degrees of freedom of the system), then we can always construct mechanical flat-input control vector fields, but if $m < n$, then even in the simplest possible case $(n, m) = (2, 1)$, structural conditions have to be satisfied for the existence of mechanical flat-input control vector fields. Therefore, the goal of the paper is to give necessary and sufficient conditions for the existence of mechanical flat-input control vector fields for observed mechanical systems with n degrees of freedom, with n arbitrary, and $n-1$ measurements (i.e., $m = n-1$). The paper is organized as follows. In Section 2, we recall the definition of flatness, formalize that of flat inputs and mechanical flat inputs, and present the first results of the paper. In Section 3, we give our main results and illustrate them by an example in Section 4.

2 Definitions and motivations

Consider the control system $\Xi : \dot{\xi} = f(\xi) + \sum_{r=1}^{m} g_r(\xi)u_r$, where $\xi \in \mathbb{R}^N$, $u \in \mathbb{R}^m$ and $g_1, \ldots, g_m$ are independent.

Definition 2.1. Ξ is *flat* at $(\xi_0, u_0) \in \mathbb{R}^N \times \mathbb{R}^m$ if there exist a neighborhood $\mathcal{O}$ of (ξ_0, u_0) and m smooth functions $\varphi_r = \varphi_r(\xi)$, $1 \leq r \leq m$, defined in a neighborhood of ξ_0, with the following property: there exist an integer $s \geq 1$ and smooth maps $(\gamma, \delta) : \mathcal{O} \times \mathbb{R}^{m(s-1)} \to \mathbb{R}^N \times \mathbb{R}^m$ such that

$$\xi = \gamma(\varphi, \dot{\varphi}, \ldots, \varphi^{(s-1)}) \text{ and } u = \delta(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)}) \tag{3}$$

for any C^{s-1}-control $u(t)$ and corresponding trajectory $\xi(t)$ that satisfy $(\xi(t), u(t)) \in \mathcal{O}$, where the m-tuple $\varphi = (\varphi_1, \ldots, \varphi_m)^\top$ is called a *flat output.*

There exists a more general notion of flatness for which the functions φ_r may depend on the control and its successive time-derivatives up to a certain order q, i.e., $\varphi_r = \varphi_r(\xi, u, \dot{u}, \ldots, u^{(q)})$. We do not need this general notion since, in our study, all φ_r depend on the state ξ only and singularities depend on ξ and/or u (but never on derivatives of u). We send the reader to [7] for a generalized definition of flat inputs that allows dependence on the input and its derivatives.

Consider the dynamical system

$$\Sigma : \dot{\xi} = f(\xi), \quad y = h(\xi),$$

whose state $\xi \in \mathbb{R}^N$ (or more generally, ξ belongs on an N-dimensional manifold), together with the output $y = h(\xi) \in \mathbb{R}^m$. In order to emphasize the fact that the system is observed, we will use the notation (Σ, h). When we say that the dynamical system Σ is observed, this does not mean that Σ is necessarily observable with respect to the output h. The problem of constructing flat inputs consists of finding independent control vector fields $g_1, \ldots, g_m$ such that the control-affine system Σ_c, associated to Σ, and given by

$$\Sigma_c : \dot{\xi} = f(\xi) + \sum_{r=1}^{m} g_r(\xi)u_r,$$

is flat with respect to the original output $(h_1, \ldots, h_m)^\top$. In that case, we will say that the pair (Σ_c, h) is flat. The inputs $u_1, \ldots, u_m$ multiplying, resp., $g_1, \ldots, g_m$, are called in [23, 24] *flat inputs.* The vector fields $g_1, \ldots, g_m$ that render Σ flat will be called *flat-input control vector fields.* For the single-output case, according to [23], a flat input can be constructed if and only if the system Σ together with its output h is observable[1] (see [9, 13] for different notions of observability). The observable multi-output case has been discussed in [24], see also [6] for another approach based on the notion of unimodularity. In [15, 16] (see also [5]), the authors solved the unobservable case for which locally, around any point of an open and dense subset of the state space, we constructed flat-input control vector fields $g_1, \ldots, g_m$ such that the control-affine system Σ_c is flat with h being a flat output.

When comparing the problem of existence of flat inputs with that of verifying flatness for control systems, an interesting phenomenon can be noted: contrary to flat control systems that are very rare (the class of flat control systems is of codimension infinity among all control systems), any dynamical system (Σ, h) can be rendered flat (on an open and dense set) by adding suitable flat-input control vector fields (or equivalently, suitable flat inputs), see [16]. As explained in the introduction, similarly to the construction of a flat output and the corresponding sensor placement, from a technological point of view, for constructing flat inputs it is not always possible to

[1] Actually, according to [23], if and only if the codistribution $\mathcal{H}^{N-1} = \mathrm{span}\{\mathrm{d}L_f^{j-1}h(x), 0 \leq j \leq N-1\}$ is of full rank, thus implying local observability of (Σ, h), see [9]. We use that notion of observability in this paper.

place the actuators exactly where we want them. This is often the case for mechanical systems, where in general, the controls correspond to an action of forces or torques on the mechanical system that act on velocities only, enter the system in an affine way and multiply terms that depend on positions only. Therefore (and keeping in mind that a dynamical observed system (Σ, h) can always be rendered flat by adding suitable flat inputs), a natural question arises: for a given observed mechanical system $(\mathcal{M}, h)$, is it always possible to construct $g_1, \ldots, g_m$ that render $(\mathcal{M}, h)$ flat and, additionally, are consistent with the mechanical structure of the system, that is, are of the form $g_r = \sum_{i=1}^n g_r^i(x)\frac{\partial}{\partial v^i}$, $1 \le r \le m$? Such g_r's will be called *mechanical flat-input control vector fields* and the inputs $u_1, \ldots, u_m$ multiplying them will be called *mechanical flat inputs*.

We start with the simplest situation: the answer to the above question is always positive for observed mechanical systems with n degrees of freedom, for which n (independent) functions $h_i(x)$, $1 \le i \le n$, are measured. In what follows, we denote by f the drift of $\mathcal{M}$, i.e., we have $\mathcal{M} : \dot{z} = f(z)$, where $z = (x, v)$. The value of a differential form $\omega = \omega_i^x(x, v)\mathrm{d}x^i + \omega_i^v(x, v)\mathrm{d}v^i$ on TX on a vector field $\eta = \eta_x^i(x, v)\frac{\partial}{\partial x^i} + \eta_v^i(x, v)\frac{\partial}{\partial v^i}$ on TX is $\langle\omega, \eta\rangle = \omega_i^x\eta_x^i + \omega_i^v\eta_v^i$[2]. For a distribution $\mathcal{G}$ on TX denote by $\mathcal{G}^\perp$ its annihilator, i.e., the codistribution given by $\mathcal{G}^\perp = \{\omega \in \Lambda^1(TX) : \langle\omega, g\rangle = 0, \forall g \in \mathcal{G}\}$, where $\Lambda^1(TX)$ is the space of smooth differentials one-forms on TX.

Proposition 2.1. *Consider the observed mechanical system $(\mathcal{M}, h)$ with $m = n$. Then $(\mathcal{M}, h)$ admits mechanical flat inputs around any $(x_0, v_0) \in TX$ and all choices of mechanical flat-input control vector fields $g_1, \ldots, g_n$ that render $(\mathcal{M}, h)$ flat are given by*

(i) $\langle \mathrm{d}h_i, g_j\rangle = 0$ *and* $\langle \mathrm{d}L_f h_i, g_j\rangle = D_{ij}(x)$, $1 \le i, j \le n$, *where* $(D_{ij}(x))$ *is any smooth invertible* $(n \times n)$*-matrix depending on positions only*

or, equivalently, by

(ii) *vector fields* $g_i = g_i(x)$, $1 \le i \le n$, *that generate the distribution* $\mathcal{G} = (\mathrm{span}\{\mathrm{d}h_i, 1 \le i \le n\})^\perp$.

The left-hand side of $\langle \mathrm{d}L_f h_i, g_j\rangle = D_{ij}(x)$ in (i) may a priori depend on (x, v) but the right-hand side implies, since $D_{ij} = D_{ij}(x)$, that it depends on x only. Notice also the condition $g_i = g_i(x)$, for $1 \le i \le n$, in (ii); without that assumption g_i's need not be mechanical. According to Proposition 2.1, if all n positions (or independent functions of them) are measured, then similarly to the results of [24, 16] for general systems, the observed mechanical system $(\mathcal{M}, h)$ always admits mechanical flat inputs and we can always construct mechanical control vector fields without any structural conditions. The above theorem reminds very much Theorem 3.3 of [16] and Theorem 4 of [24] treating the case when the considered dynamical system is observable with respect to h. That was to be expected since in the case when $m = n$, the mechanical system $\mathcal{M}$ is obviously observable with respect to h. It is however important to emphasize that the existence of mechanical flat-input control vector fields is not due to observability, but to the fact that we have as many measurements as degrees of freedom. This is illustrated by Proposition 2.2 below treating the case of mechanical systems with $n = 2$ and $m = 1$ (which is the simplest non trivial case). Before stating it, let us introduce some notations that will be used throughout.

Consider the observed mechanical system $(\mathcal{M}, h)$, given by (1), and suppose $m = n - 1$, that is, the system has n degrees of freedom and $n - 1$ independent functions $h_1(x), \ldots, h_{n-1}(x)$ of the positions are measured. For $1 \le i \le n$, we denote by a^i the functions

$$a^i(x, v) = -\Gamma_{jk}^i(x)v^j v^k + d_j^i(x)v^j + e^i(x), \tag{4}$$

[2] The summation convention is used for ω, η, and $\langle\omega, \eta\rangle$.

i.e., we have $\dot{v}^i = a^i(x, v)$, and recall that f denotes the drift of $\mathcal{M}$. To the output $(h_1(x), \dots, h_{n-1}(x))^\top$ we associate the following sequence of codistributions

$$\mathcal{H}^j = \operatorname{span}\{\mathrm{d}L_f^{q-1} h_i(x), 1 \le q \le j, 1 \le i \le n-1\}, \tag{5}$$

for $j \ge 1$. So, we have $\mathcal{H}^1 = \operatorname{span}\{\mathrm{d}h_i(x), 1 \le i \le n-1\}$ and $\mathcal{H}^2 = \operatorname{span}\{\mathrm{d}h_i(x), \mathrm{d}L_f h_i(x, v), 1 \le i \le n-1\}$ which are of constant rank $n-1$ and $2n-2$, resp.. We denote by $\mathcal{V}$ the vertical distribution associated to $\mathcal{M}$, that is,

$$\mathcal{V} = \operatorname{span}\{\frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^n}\}.$$

Since the functions $h_1(x), \dots, h_{n-1}(x)$ are supposed everywhere independent, we can always complete them by another function $\zeta(x)$ such that the extended point transformation $\Phi = (\phi, \mathrm{D}\phi(x)v)$, where

$$\phi = (h_1(x), \dots, h_{n-1}(x), \zeta(x))^\top,$$

defines a local (triangular) diffeomorphism that gives new coordinates

$$\tilde{x} = \phi(x), \quad \tilde{v} = \mathrm{D}\phi(x)v, \tag{6}$$

in which $\mathcal{M}$ takes the form $\dot{\tilde{x}}^i = \tilde{v}^i$, $\dot{\tilde{v}}^i = \tilde{a}^i(\tilde{x}, \tilde{v})$, $1 \le i \le n$, with $\tilde{a}^i$ quadratic with respect to the new velocities $\tilde{v}^i$, and which, dropping the "tildes", gives

$$\mathcal{M} : \begin{cases} \dot{x}^i = v^i \\ \dot{v}^i = a^i(x, v), \\ \quad 1 \le i \le n, \end{cases} \qquad \text{with} \quad h_i = x^i,\ 1 \le i \le n-1. \tag{7}$$

We call (x, v) of (7) to be $\mathcal{M}$-coordinates compatible with h.

Consider now the simplest non trivial case $(n, m) = (2, 1)$.

Proposition 2.2. *Consider the observed mechanical system $(\mathcal{M}, h)$, with $n = 2$ and $m = 1$. We have*

(FI) *$(\mathcal{M}, h)$ admits a flat input at (x_0, v_0) if and only if $\operatorname{rk} \mathcal{H}^4(x_0, v_0) = 4$, that is, $\mathcal{M}$ is locally observable with respect to h.*

(MFI) *The system $(\mathcal{M}, h)$ admits a mechanical flat input at (x_0, v_0) if and only if*

(C) *$(\mathcal{H}^3)^\perp \subset \mathcal{V}$ and $\operatorname{rk} \mathcal{H}^3(x_0, v_0) = 3$;*

or, equivalently,

(C)' *In any $\mathcal{M}$-coordinates (x,v) compatible with h, the function a^1 satisfies:*

$$\frac{\partial a^1}{\partial v^2} \equiv 0, \quad \frac{\partial a^1}{\partial x^2}(x_0, v_0) \neq 0, \tag{8}$$

and a^2 is any.

Moreover, any mechanical flat-input control vector field, for systems satisfying (C), is given by

$$L_g L_f^j h = 0, \ \textit{for } 0 \le j \le 2, \ \textit{and } L_g L_f^3 h = \gamma(x),$$

where $\gamma = \gamma(x)$ is any smooth function such that $\gamma(x_0) \neq 0$, and, in particular, by

$$g(x) = g^2(x) \frac{\partial}{\partial v^2}, \quad g^2(x_0) \neq 0,$$

for any system of the form (7) satisfying (C)'.

Under $(\mathcal{H}^3)^\perp \subset \mathcal{V}$, the condition $\operatorname{rk}\mathcal{H}^3(x_0, v_0) = 3$ of (MFI) is actually equivalent to $\operatorname{rk}\mathcal{H}^4(x_0, v_0) = 4$, showing that the observability rank conditions for (FI) and (MFI) are the same. The structural condition $\frac{\partial a^1}{\partial v^2} \equiv 0$ means $a^1 = \Gamma^1_{11}(x)v^1v^1 + d^1_1(x)v^1 + e^1(x)$. Condition (FI) simply says that in the single-output case, a flat input (there is only one since we have only one output) exists if and only if $(\mathcal{M}, h)$ is observable. So, if a system is observable, then it always admits a (not necessarily mechanical) flat input and for the flat input to be mechanical, the equivalent conditions (C) or (C)' of (MFI) have to be satisfied. Thus the above proposition shows that, even in the simplest case (mechanical systems with two degrees of freedom and one measurement), the problem of constructing mechanical flat-input control vector fields has non trivial solutions and that an additional structural condition is needed. Finally, observe that Proposition 2.2 completely describes the case $(n, m) = (2, 1)$. The goal of the paper is thus to give necessary and sufficient conditions for the existence of mechanical flat-input control vector fields for observed mechanical systems with n degrees of freedom, where $n \geq 3$ is arbitrary, and $n - 1$ measurements (i.e., $m = n - 1$).

3 Main results

Consider the observed mechanical system $(\mathcal{M}, h)$, given by (1), and suppose that $n \geq 3$ and $m = n - 1$. We denote by $\mathcal{G}_{n-1}$ the distribution

$$\mathcal{G}_{n-1} = (\mathcal{H}^2)^\perp \cap \mathcal{V}. \tag{9}$$

The distribution $\mathcal{G}_{n-1}$ is of constant rank one and we denote it by $\mathcal{G}_{n-1}$ because, as we will see below, it actually gives the only candidate for one of the mechanical flat-input control vector fields (that will be denoted by g_{n-1}). Theorem 3.1 provides necessary and sufficient conditions for the (local) existence of mechanical flat-input control vector fields $g_1, \ldots, g_{n-1}$ such that $(\mathcal{M}_c, h)$ is locally flat.

Theorem 3.1. *Consider the observed mechanical system $(\mathcal{M}, h)$, given by (1), together with $(h_1(x), \ldots, h_{n-1}(x))^\top$.*

(i) *There exist mechanical flat-input control vector fields $g_1, \ldots, g_{n-1}$ such that $(\mathcal{M}_c, h)$ is locally flat at $((x_0, v_0), u_0)$ if and only if there exists a differential one-form δ on TX of the form $\delta = \sum_{i=1}^{n-1} \delta_i(x)\mathrm{d}L_f h_i$ such that $\delta_\ell(x_0) \neq 0$, for a certain $1 \leq \ell \leq n - 1$, satisfying*

(C) *$\langle \delta, ad_f g_{n-1}\rangle = 0$ and $L_{g_{n-1}}\mu_\ell(x_0, v_0) \neq 0$, where $\mu_\ell = L_f^2 h_\ell + \sum_{i\neq\ell} \frac{\delta_i}{\delta_\ell} u_{i0}$, and $g_{n-1} = g^i_{n-1}(x)\frac{\partial}{\partial v^i}$ spans the distribution $\mathcal{G}_{n-1}$, given by (9), and is the vertical lift of a nonzero vector field $\mathfrak{g}_{n-1} = g^i_{n-1}(x)\frac{\partial}{\partial x^i}$ on X.*

(ii) *For system (7), condition (C) reads as*

$$\sum_{i=1}^{n-1} \delta_i(x)\frac{\partial a^i}{\partial v^n} \equiv 0, \quad \textit{and} \tag{10}$$

$\frac{\partial \nu^\ell}{\partial x^n}(x_0, v_0) \neq 0$, *where $\nu^\ell = a^\ell + \sum_{i\neq\ell} b_i u_{i0}$ and $b_i(x) = -\frac{\delta_i}{\delta_\ell}$, for $1 \leq i \leq n - 1$, $i \neq \ell$.*

(iii) *For system $\mathcal{M}$ given by (7) and any δ satisfying (10), mechanical flat-input control vector fields $g_1, \ldots, g_{n-1}$ can be constructed algebraically by formula (12) below, and lead to the following control system $(\mathcal{M}_c, h)$ which is flat at $((x_0, v_0), u_0)$, and given, in $\mathcal{M}$-coordinates compatible with h, by*

$$\mathcal{M}_c : \begin{cases} \dot{x}^i = v^i & \dot{x}^{n-1} = v^{n-1} \\ \dot{v}^i = a^i(x, v) + u_i & \dot{v}^{n-1} = a^{n-1}(x, v) + \sum_{i=1}^{n-2} b_i(x)u_i \\ \quad 1 \leq i \leq n - 2, & \dot{x}^n \ = v^n \\ & \dot{v}^n \ = a^n(x, v) + u_{n-1}, \end{cases} \tag{11}$$

with $\frac{\partial(a^{n-1}-\sum_{i=1}^{n-2} b_i a^i)}{\partial v^n} \equiv 0$ *and* $\frac{\partial \nu^{n-1}}{\partial x^n}(x_0, v_0) \neq 0$*, where* ν^{n-1} *is defined as in (ii) and* ℓ *is (after a permutation) supposed to be* $\ell = n-1$*,* $b_i = -\frac{\delta_i}{\delta_{n-1}}$*,* $1 \leq i \leq n-2$*,* $h = (x_1, \ldots, x_{n-1})^\top$*, and*

$$g_i = \frac{\partial}{\partial v^i} + b_i(x)\frac{\partial}{\partial v^{n-1}},\ 1 \leq i \leq n-2, \quad \text{and} \quad g_{n-1} = \frac{\partial}{\partial v^n}. \tag{12}$$

Theorem 3.1 states that, in the case $m = n-1$, a necessary and sufficient condition for the existence of mechanical flat inputs is that the derivatives of the functions a^i (associated to the measured positions) with respect to the velocity v^n (whose corresponding position is not measured) are dependent over the ring of smooth functions depending on configurations only, see (ii) where condition (C) is given in $\mathcal{M}$-coordinates (x, v) compatible with h. In (i) condition (C) is expressed in an invariant way. The distribution $\mathcal{G}_{n-1}$ gives the only candidate for the mechanical vector field g_{n-1}. The functions b_i defining the mechanical flat-input control vector fields of $\mathcal{M}_c$, given by (11), and the functions δ_i defining the one-form δ, are obviously related: indeed, we take b_i as $b_i(x) = -\frac{\delta_i(x)}{\delta_{n-1}(x)}$, with $\delta_{n-1}(x)$ being assumed nonzero. The problem of constructing mechanical flat inputs reduces to the problem of computing the functions $\delta_i(x)$ for which a system of algebraic equations is provided in Subsection 3.1. Theorem 3.1 applies to both observable and non observable case (if $(\mathcal{M}, h)$ is observable, then introducing b_i keep observability; if $(\mathcal{M}, h)$ is not observable, then introducing b_i creates observability). The construction of the mechanical flat-input control vector fields $g_1, \ldots, g_{n-1}$ leading to $(\mathcal{M}_c, h)$ can be simplified for two particular cases treated in Propositions 3.1 and 3.2 below.

By the analysis of $\mathcal{M}$ in the form (7), we conclude that if we render it flat, then $n-2$ flat inputs, say $u_1, \ldots, u_{n-2}$, have to affect (permute h_i's if necessary) $\ddot{x}^1 = \dot{v}^1, \ldots, \ddot{x}^{n-2} = \dot{v}^{n-2}$, where $x^i = h_i$, $1 \leq i \leq n-2$, and the remaining output $h_{n-1} = x^{n-1}$ or, more generally, a function $\tilde{h}_{n-1} = \Psi(h_1, \ldots, h_{n-1})$, invertible with respect to h_{n-1}, has to provide information about v^{n-1} (with the help of $L_f\tilde{h}_{n-1}$), about x^n (with the help of $L_f^2\tilde{h}_{n-1}$) and about v^n (with the help of $L_f^3\tilde{h}_{n-1}$). In other words $\mathrm{d}x^n \in \mathcal{H}^3$, and an interesting property is whether there exists $\tilde{h}_{n-1}$ such that $\mathrm{d}x^n \in \mathcal{H}^2 + \mathrm{span}\{\mathrm{d}L_f^2\tilde{h}_{n-1}\}$, that is, in order to express x^n, we use $h_i = x^i$ and $v^i = \dot{x}^i$, for $1 \leq i \leq n-2$, as well as $\tilde{h}_{n-1} = \tilde{x}^n$, $\dot{\tilde{x}}^n$, and $\ddot{\tilde{x}}^n$, that means positions, velocities, and the second derivative of $\tilde{h}_{n-1}$ only. It follows that expression (3) involves $h_i^{(j)}$, $0 \leq j \leq 3$, $1 \leq i \leq n-2$, and $\tilde{h}_{n-1}^{(j)}$, $0 \leq j \leq 4$.

Proposition 3.1. *Consider the observed mechanical system* $(\mathcal{M}, h)$*, given by (1), together with* $(h_1(x), \ldots, h_{n-1}(x))^\top$*. The following conditions are equivalent:*

(i) *There exist mechanical flat-input control vector fields* $g_1, \ldots, g_{n-1}$ *such that* $(\mathcal{M}_c, h)$ *is locally flat at* (x_0, v_0) *and admits a flat output* $(\varphi_1, \ldots, \varphi_{n-1})$ *verifying* $\mathrm{span}\{\mathrm{d}\varphi_i, 1 \leq i \leq n-1\} = \mathrm{span}\{\mathrm{d}h_i, 1 \leq i \leq n-1\}$*, and for which expression (3) involves* $\varphi_\ell^{(j)}$*,* $0 \leq j \leq 4$*, for a certain* $1 \leq \ell \leq n-1$*, and* $\varphi_i^{(j)}$*,* $0 \leq j \leq 3$ *only, for* $i \neq \ell$*.*

(ii) *There exists a differential one-form form* $\delta = \sum_{i=1}^{n-1} \delta_i(x)\mathrm{d}L_f h_i$ *satisfying (C) in Theorem 3.1 and moreover, such that the differential one-form* $\omega = \sum_{i=1}^{n-1} \delta_i(x)\mathrm{d}h_i$ *on* X*, associated to* δ*, fulfills* $\omega \wedge \mathrm{d}\omega = 0$*.*

(iii) *There exists a map* $\Psi = \Psi(h_1, \ldots, h_{n-1})$*, invertible with respect to* h_ℓ*, for a certain* $1 \leq \ell \leq n-1$*, such that* $\psi(x) = \Psi(h_1(x), \ldots, h_{n-1}(x))$*, in* $\mathcal{M}$*-coordinates compatible with* h*, satisfies*

$$\sum_{i=1}^{n-1} \frac{\partial \psi}{\partial x^i}\frac{\partial a^i}{\partial v^n} \equiv 0 \quad \text{and} \quad \frac{\partial L_f^2\psi}{\partial x^n}(x_0, v_0) \neq 0.$$

(iv) *There exists a map* $\Psi = \Psi(h_1, \ldots, h_{n-1})$*, invertible with respect to* h_ℓ*, for a certain* $1 \leq \ell \leq n-1$ *(say, without loss of generality,* $\ell = n-1$*), such that in the coordinates in which* $x^1 = h_1(x), \ldots, x^{n-2} = h_{n-2}(x)$*,* $x^{n-1} = \psi(x)$*, where* $\psi(x) = \Psi(h_1(x), \ldots, h_{n-1}(x))$*, the system* $\mathcal{M}$ *takes the form*

$$
\begin{cases}
\dot{x}^i = v^i & \dot{x}^{n-1} = v^{n-1} \\
\dot{v}^i = a^i(x,v) & \dot{v}^{n-1} = a^{n-1}(x,v), \\
\quad 1 \le i \le n-2, & \dot{x}^n \ = v^n \\
 & \dot{v}^n \ = a^n(x,v)
\end{cases}
\tag{13}
$$

where $\frac{\partial a^{n-1}}{\partial v^n} \equiv 0$, $\frac{\partial a^{n-1}}{\partial x^n}(x_0, v_0) \neq 0$, *and* $h = (x^1, \ldots, x^{n-2}, h_{n-1}(x^1, \ldots, x^{n-1}))^\top$.

Moreover, if $\mathcal{M}$ *satisfies any of the equivalent conditions (i)-(iv), then mechanical flat-input control vector fields* $g_1, \ldots, g_{n-1}$ *are given for* (13), *by*

$$
g_i = \frac{\partial}{\partial v^i},\ 1 \le i \le n-2, \quad \text{and} \quad g_{n-1} = \frac{\partial}{\partial v^n},
\tag{14}
$$

i.e., modify only the equations for v^i, $1 \le i \le n-2$, *as* $\dot{v}^i = a^i(x,v) + u_i$, *and for* v^n *as* $\dot{v}^n = a^n(x,v) + u_{n-1}$.

Proposition 3.1 is analogous to Theorem 3.1, but its conditions are more restrictive than those of Theorem 3.1. To compute ψ we have to solve a system of PDE's (we actually have $\frac{\partial \psi}{\partial x^i} = \delta_i(x)$) and therefore integrability conditions have to be satisfied, see Subsection 3.2. The integrability conditions are expressed as $\omega \wedge \mathrm{d}\omega = 0$ and guarantee the existence of ψ and allow to avoid calculating ψ. The function ψ is, in fact, another candidate for one of the flat output components. Indeed, for $(\mathcal{M}_c, h)$ obtained from (13) by adding the mechanical flat-input control vector fields given by (14), it is clear that $(h_1, \ldots, h_{n-2}, \psi) = (x^1, \ldots, x^{n-2}, x^{n-1})$ is also a flat output, and, moreover, it is actually such that expression (3) uses derivatives up to order three of h_i, for $1 \le i \le n-2$, and up to order four of ψ (so $(h_1, \ldots, h_{n-2}, \psi)$ satisfies the same conditions as the flat output φ of Proposition 3.1(i)). The mechanical flat-input control vector fields constructed with the help of Proposition 3.1 have only $n-1$ non zero components (which is the minimal possible). Proposition 3.1 covers also the case $n = 2$ and $m = 1$ (compare it to Proposition 2.2), and it provides a structural sufficient condition (far from being trivial) for the existence of mechanical flat inputs, but also a regularity condition. The flatness singularity $\frac{\partial a^{n-1}}{\partial x^n}(x^0, v^0) = 0$ depends on the function ψ (we have $a^{n-1} = L_f^2 \psi$), so different choices of ψ (if they exist) lead, in general, to different singularities. Finally, remark that the systems $\mathcal{M}$, for which Proposition 3.1 applies, are necessarily observable with respect to h.

Now we consider the unobservable case for which the defect of observability is maximal.

Proposition 3.2. *Consider the observed mechanical system* $(\mathcal{M}, h)$, *given by* (1), *together with* $(h_1(x), \ldots, h_{n-1}(x))^\top$, *and suppose that the rank of the observability codistribution* $\mathcal{H} = \operatorname{span}\{\mathrm{d}L_f^j h_i, 1 \le i \le n-1, j \ge 0\}$ *is constant and equals* $2n-2$, *i.e.,* $\mathcal{H} = \mathcal{H}^2$. *Then in* $\mathcal{M}$*-coordinates compatible with* h, *we have* $\dot{x}^i = v^i$, $\dot{v}^i = a^i(x,v)$, *with* $\frac{\partial a^i}{\partial x^n} = \frac{\partial a^i}{\partial v^n} \equiv 0$, *for all* $1 \le i \le n-1$. *For that form, we construct mechanical* $g_1, \ldots, g_{n-1}$ *such that* $(\mathcal{M}_c, h)$ *is flat at any* $((x_0, v_0), u_0)$, *with* $u_{10} \neq 0$, *and is given by*

$$
\mathcal{M}_c : \begin{cases}
\dot{x}^i = v^i & \dot{x}^{n-1} = v^{n-1} \\
\dot{v}^i = a^i(x,v) + u_i & \dot{v}^{n-1} = a^{n-1}(x,v) + x^n u_1, \\
\quad 1 \le i \le n-2, & \dot{x}^n \ = v^n \\
 & \dot{v}^n \ = a^n(x,v) + u_{n-1},
\end{cases}
\tag{15}
$$

with $\frac{\partial a^i}{\partial x^n} = \frac{\partial a^i}{\partial v^n} \equiv 0$, $1 \le i \le n-1$, $h = (x^1, \ldots, x^{n-1})^\top$.

Proposition 3.2 applies for the unobservable case for which there are two unobserved variables x^n and v^n in the well chosen $\mathcal{M}$-coordinates compatible with h. In that case, mechanical flat inputs always exist without structural conditions. Moreover, if $\operatorname{rk} \mathcal{H} = 2n-2$ everywhere, there is no singularity in the state space, i.e., $h = (x^1, \ldots, x^{n-1})^\top$ is a local flat output around any

$(x_0, v_0) \in TX$. Notice, however, that the system exhibits a singularity in the control space given by $u_{10} = 0$ that is unavoidable.

3.1 Computation of the functions δ_i. Consider the system $\mathcal{M}$ in $\mathcal{M}$-coordinates compatible with h. Since all $a^i(x,v)$ are quadratic with respect to the velocities (see (4)), the condition $\sum_{i=1}^{n-1} \delta_i(x) \frac{\partial a^i}{\partial v^n} = 0$ translates into a polynomial of degree one with respect to v^i, $1 \le i \le n$, (whose coefficients are thus functions of x only) being identically zero. This leads to the following system of $n+1$ algebraic equations with unknowns $\delta_1(x), \ldots, \delta_{n-1}(x)$: $\sum_{i=1}^{n-1} \delta_i(x)\Gamma^i_{jn}(x) = 0$, $1 \le j \le n-1$, $\sum_{i=1}^{n-1} \delta_i(x)\Gamma^i_{nn}(x) = 0$, $\sum_{i=1}^{n-1} \delta_i(x) d^i_n(x) = 0$, which may or may not have nontrivial solutions (we used the symmetry of Christoffel symbols $\Gamma^i_{nj} = \Gamma^i_{jn}$).

3.2 Computation of the function ψ. The existence of ψ is guaranteed by $\omega \wedge \mathrm{d}\omega = 0$. If we need to compute it, we deduce that ψ has to satisfy the following system of PDE's: $\sum_{i=1}^{n-1} \frac{\partial \psi}{\partial x^i}(x)\Gamma^i_{jn}(x) = 0$, $1 \le j \le n-1$, $\sum_{i=1}^{n-1} \frac{\partial \psi}{\partial x^i}(x)\Gamma^i_{nn}(x) = 0$, $\sum_{i=1}^{n-1} \frac{\partial \psi}{\partial x^i}(x) d^i_n(x) = 0$, which has a nontrivial solution if and only if $\omega \wedge \mathrm{d}\omega = 0$.

4 Example

Consider the following mechanical system with 4 degrees of freedom that can be seen as a combination of a two-link manipulator and a single link manipulator with joint elasticity [12, 18]:

$$\mathcal{M}: \begin{cases} \dot{x}^i = v^i & \dot{x}^i = v^i \\ \dot{v}^i = -\Gamma^i_{jk}(x) v^j v^k + e^i(x), & \dot{v}^i = e^i(x), \\ \quad 1 \le i \le 2, & \quad 3 \le i \le 4, \end{cases} \tag{16}$$

with $x \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^4$, where x^1, x^2 denote the angles of the first and the second link, resp., and x^3, x^4 denote the angles of the first and the second motor shaft. The only non-zero Christoffel symbols are Γ^i_{jk}, for $1 \le i, j, k \le 2$; the functions $e^1(x)$ and $e^2(x)$ depend explicitly on all positions x^i, for $1 \le i \le 4$, while $e^3(x)$ (resp., $e^4(x)$) is a function of x^1 and x^3 (resp., x^2 and x^4) only, see [18] for their exact expressions. The controlled two-link manipulator with joint elasticity $(\mathcal{MS}) : \dot{z} = f(z) + u_1 g_1(x) + u_2 g_2(x)$, with $z = (x, v)$, f the drift of (16), and control vector fields $g_1 = \frac{1}{J_1}\frac{\partial}{\partial v^3}$ and $g_2 = \frac{1}{J_2}\frac{\partial}{\partial v^4}$, has been considered in [18] from a point of view of static feedback linearization (see [11, 10]); it has been shown that $(\mathcal{MS})$ is static feedback linearizable with linearizing outputs x^1, x^2, but the transformed linear system is not mechanical since the linearizing diffeomorphism is not mechanical and, moreover, that there are no linearizing outputs defining a mechanical diffeomorphism (i.e., $(\mathcal{MS})$ is not mechanical static feedback linearizable, see [18] for a formal definition). It is thus natural to investigate whether there exist other mechanical control vector fields (equivalently, mechanical flat inputs) that render $\mathcal{M}$ flat and, in particular, mechanical static feedback linearizable. For the system to fall into the class considered in this paper (that is, n degrees of freedom and $n-1$ outputs), we need 3 outputs. We study two cases: we first consider both linearizing outputs of [18], i.e., $h_1 = x^1$, $h_2 = x^2$, and complete them by $h_3 = x^3$, and then we study the case $h_1 = x^1$, $h_2 = x^3$ and $h_3 = x^4$.

Suppose first that $h_1 = x^1$, $h_2 = x^2$, $h_3 = x^3$. Notice that the functions a^i are such that: $a^i = a^i(x, v^1, v^2)$, for $1 \le i \le 2$, and $a^i = e^i(x)$, for $3 \le i \le 4$. Proposition 3.1 is obviously verified (simply take $\psi = h_1 = x^1$) and we define the mechanical $g_1 = \frac{\partial}{\partial v^2}$, $g_2 = \frac{\partial}{\partial v^3}$, and $g_3 = \frac{\partial}{\partial v^4}$ that lead to the mechanical control system

$$\mathcal{M}_c : \begin{cases} \dot{x}^2 = v^2 & \dot{x}^3 = v^3 & \dot{x}^1 = v^1 \\ \dot{v}^2 = a^2(x, v^1, v^2) + u_1, & \dot{v}^3 = e^3(x) + u_2, & \dot{v}^1 = a^1(x, v^1, v^2) \\ & & \dot{x}^4 = v^4 \\ & & \dot{v}^4 = e^4(x) + u_3, \end{cases}$$

with $h = (x^1, x^2, x^3)^\top$ a flat output at any (x_0, v_0) such that $\frac{\partial a^1}{\partial x^4}(x_0, v_0) \neq 0$. Other constructions are possible based on Theorem 3.1. For instance, if we work around a point such that $\frac{\partial a^1}{\partial x^4}(x_0, v_0) = 0$ (that is a flatness singularity for the above form), we can avoid that singularity by constructing the mechanical control system as follows (which is of the form (11) with $b_1 = x_4$, $b_2 = 0$, equivalently, with $\delta_1 = x_4$, $\delta_2 = 0$, $\delta_3 = 1$):

$$\mathcal{M}'_c : \begin{cases} \dot{x}^i = v^i & \dot{x}^3 = v^3 \\ \dot{v}^i = a^i(x, v^1, v^2) + u_i, & \dot{v}^3 = e^3(x^1, x^3) + x^4 u_1, \\ \quad 1 \leq i \leq 2, & \dot{x}^4 = v^4 \\ & \dot{v}^4 = e^4(x) + u_3, \end{cases}$$

which is flat with $h = (x^1, x^2, x^3)^\top$ a flat output around any $(x_0, v_0) \in TX$ and $u_{10} \neq 0$ (notice the new singularity in the control space). For both mechanical control systems $\mathcal{M}_c$ and $\mathcal{M}'_c$ proposed above, a simple calculus shows that their linearizability distribution $\mathcal{D}^1 = \mathrm{span}\{g_i, ad_f g_i, 1 \leq i \leq 3\}$ is not involutive, hence none of them is static feedback linearizable (and in particular, not mechanical static feedback linearizable), although flat. Finally, observe that since no function a^i depends on v^4, we can construct a mechanical control system $(\mathcal{M}_c, h)$, with $h = (x^1, x^2, x^3)^\top$, by choosing any functions $b_i(x)$ (or equivalently, any one-form δ) in Theorem 3.1.

Consider now the case when $h_1 = x^1$, $h_2 = x^3$ and $h_3 = x^4$. We apply Proposition 3.1 with $\psi = h_3 = x^4$, and define the mechanical $g_1 = \frac{\partial}{\partial v^1}$, $g_2 = \frac{\partial}{\partial v^3}$, and $g_3 = \frac{\partial}{\partial v^2}$ that lead to the mechanical control system

$$\mathcal{M}''_c : \begin{cases} \dot{x}^1 = v^1 & \dot{x}^3 = v^3 & \dot{x}^4 = v^4 \\ \dot{v}^1 = a^1(x, v^1, v^2) + u_1, & \dot{v}^3 = e^3(x) + u_2, & \dot{v}^4 = e^4(x^2, x^4), \\ & & \dot{x}^2 = v^2 \\ & & \dot{v}^2 = a^2(x, v^1, v^2) + u_3, \end{cases}$$

with $h = (x^1, x^3, x^4)^\top$ a flat output at any $(x_0, v_0) \in TX$ (since we always have $\frac{\partial e^2}{\partial x^4} \neq 0$, [18]). It can be easily verified that $\mathcal{M}''_c$ is now static feedback linearizable and moreover, that the linearizing diffeomorphism is compatible with the mechanical structure yielding a linear mechanical control system, the measurements $h = (x^1, x^3, x^4)^\top$ being also the mechanical linearizing outputs.

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